Pseudo-Riemannian VSI spaces II

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Abstract
In this paper we consider pseudo-Riemannian spaces of arbitrary signature for which all of the polynomial curvature invariants vanish (VSI spaces). Using an algebraic classification of pseudo-Riemannian spaces in terms of the boost-weight decomposition, we first show more generally that a space which is not characterized by its invariants must possess the $\mathbf{S}_1^G$-property. As a corollary, we then show that a VSI space must possess the $\mathbf{N}^G$-property (these results are the analogues of the alignment theorem, including corollaries, for Lorentzian spacetimes). As an application we classify all 4D neutral VSI spaces and show that these belong to one of two classes: (1) those that possess a geodesic, expansion-free, shear-free, and twist-free null congruence (Kundt metrics), or (2) those that possess an invariant null plane (Walker metrics). By explicit construction we show that the latter class contains a set of VSI metrics which have not previously been considered in the literature.

PACS numbers: 02.40.−k, 02.40.Hw, 02.10.Hh

1. Introduction

In this paper we will consider an arbitrary-dimensional pseudo-Riemannian space of signature $(k, k+m)$. We will investigate when such a space has a degenerate curvature structure; in particular, we shall determine criteria for when a space, or tensor, has all vanishing polynomial curvature invariants (VSI space). Recall that a polynomial curvature invariant is defined as the polynomial invariants of the components of the curvature tensors. Previously, the VSI spaces for Lorentzian metrics have been studied [1] and it was shown that these comprise a subclass of the degenerate Kundt metrics [2]. Here, we will see that Kundt-like metrics also play a similar role for pseudo-Riemannian VSI metrics of arbitrary signature; however, we will see that another class of metrics arises in the pseudo-Riemannian case, namely the Walker metrics [3]. In order to obtain these results, we will utilize invariant theory to obtain important properties of the structure of tensors having degenerate invariants. In particular, tensors not characterized by their invariants will be shown to possess the $\mathbf{S}_1^G$-property, while in the VSI case they necessarily must possess the $\mathbf{N}^G$-property. We will use this fact to construct a new set of four-dimensional (4D) Walker metrics with vanishing curvature invariants of neutral signature.

Walker metrics are metrics possessing an invariant null plane and have been studied in various contexts [3, 4]. Here we will show that they also play a role in the classification of VSI metrics. Indeed, we will give a new class of VSI metrics which has not been considered before. These metrics are related to a bigger class of Walker metrics with a degenerate curvature structure. The curvature structure of these metrics is distinct from the Kundt metrics known from the Lorentzian case. One of the consequences of this district feature is that we need to consider invariants containing up to four derivatives. Indeed, interestingly, there is a family of Walker metrics which is VSI_3, but not VSI_4: perhaps the simplest member of this family is

$$\mathrm{d}s^2 = 2\,\mathrm{d}u(\mathrm{d}v + V\,\mathrm{d}u) + 2\,\mathrm{d}U(\mathrm{d}V + av^4\,\mathrm{d}U), \tag{1}$$

where a is a constant. This peculiar property of being VSI_3 but not VSI_4 has no analogue in the Lorentzian case[1].

First we will review some of the techniques used in this paper. Then we will provide the general result for tensors (or spaces) not being characterized by its invariants. This result is the analogue of the alignment theorem in the Lorentzian-signature case [5]. Then, as a corollary, we will state the important VSI case. We will then use this VSI result to consider the 4D neutral case in detail.

1.1. Boost-weight decomposition

Let us first review the boost-weight classification, originally used to study degenerate metrics in Lorentzian geometry [6], in the pseudo-Riemannian case [7]. We will assume the manifold is of dimension $(2k+m)$ and of signature $(k, k+m)$. We first introduce a suitable (real) null frame such that the metric can be written as

$$\mathrm{d}s^2 = 2(\boldsymbol{\ell}^1\boldsymbol{n}^1 + \ldots + \boldsymbol{\ell}^I\boldsymbol{n}^I + \ldots + \boldsymbol{\ell}^k\boldsymbol{n}^k) + \delta_{ij}\boldsymbol{m}^i\boldsymbol{m}^j, \tag{2}$$

where the indices $i = 1, \ldots, m$.

Let us consider the k independent boosts which forms an Abelian subgroup of the group $SO(k, k+m)$:

$$\begin{aligned}
(\boldsymbol{\ell}^1, \boldsymbol{n}^1) &\mapsto (e^{\lambda_1}\boldsymbol{\ell}^1, e^{-\lambda_1}\boldsymbol{n}^1)\\
(\boldsymbol{\ell}^2, \boldsymbol{n}^2) &\mapsto (e^{\lambda_2}\boldsymbol{\ell}^2, e^{-\lambda_2}\boldsymbol{n}^2)\\
&\vdots\\
(\boldsymbol{\ell}^k, \boldsymbol{n}^k) &\mapsto (e^{\lambda_k}\boldsymbol{\ell}^k, e^{-\lambda_k}\boldsymbol{n}^k).
\end{aligned} \tag{3}$$

This action will be considered pointwise at the manifold.

For a tensor T, we can then consider the boost weights of this tensor, $\mathbf{b} \in \mathbb{Z}^k$, as follows. If we consider the components of T with respect to the above-mentioned null frame, then if a component $T_{\mu_1\ldots\mu_n}$ transforms as

$$T_{\mu_1\ldots\mu_n} \mapsto e^{(b_1\lambda_1 + b_2\lambda_2 + \cdots + b_k\lambda_k)}T_{\mu_1\ldots\mu_n},$$

then we will say the component $T_{\mu_1\ldots\mu_n}$ is of boost weight $\mathbf{b} \equiv (b_1, b_2, \ldots, b_k)$. We can now decompose a tensor into boost weights; in particular,

$$T = \sum_{\mathbf{b}\in\mathbb{Z}^k}(T)_{\mathbf{b}},$$

[1] In the Lorentzian case, VSI_2 implies VSI [1], while VSI_1 Kundt implies VSI [2].

where $(T)_{\mathbf{b}}$ means the projection onto the components of boost weight $\mathbf{b}$. The projections $(T)_{\mathbf{b}}$ are the eigentensors of a set of commuting operators (the infinitesimal generators of the boosts) with integer eigenvalues. For example, a tensor $P = A\boldsymbol{\ell}^I\boldsymbol{n}^J\boldsymbol{m}^i\boldsymbol{m}^j$ with $I \neq J$ and A is some scalar has boost weight $\mathbf{b} = (b_1, \dots, b_k)$ where $b_I = -1$, $b_J = 1$, other $b_i = 0$. Indeed, writing out a totally covariant tensor T using the basis in (2), the boost weight is given by $\mathbf{b} = (b_I)$ where $b_I = \#(\boldsymbol{n}^I) - \#(\boldsymbol{\ell}^I)$.

By considering tensor products, the boost weights obey the following additive rule:

$$(T \otimes S)_{\mathbf{b}} = \sum_{\tilde{\mathbf{b}}+\hat{\mathbf{b}}=\mathbf{b}} (T)_{\tilde{\mathbf{b}}} \otimes (S)_{\hat{\mathbf{b}}}. \tag{4}$$

We also note that the metric g is of boost weight 0, i.e. $g = (g)_0$; hence, raising and lowering indices of a tensor do not change the boost weights.

1.2. The $\mathbf{S}_i$- and $\mathbf{N}$-properties

Let us consider a tensor, T, and list a few conditions that the tensor components may fulfil [7, 8]:

Definition 1.1. *We define the following conditions:*

(B1) $(T)_{\mathbf{b}} = 0$*, for all* $\mathbf{b} = (b_1, b_2, b_3, \dots, b_k)$*,* $b_1 > 0$.
(B2) $(T)_{\mathbf{b}} = 0$*, for all* $\mathbf{b} = (0, b_2, b_3, \dots, b_k)$*,* $b_2 > 0$.
(B3) $(T)_{\mathbf{b}} = 0$*, for all* $\mathbf{b} = (0, 0, b_3, \dots, b_k)$*,* $b_3 > 0$.
$\vdots$
(Bk) $(T)_{\mathbf{b}} = 0$*, for all* $\mathbf{b} = (0, 0, \dots, 0, b_k)$*,* $b_k > 0$.

Definition 1.2. *We will say that a tensor T possesses the $\mathbf{S}_1$-property if and only if there exists a null frame such that condition (B1) above is satisfied. Furthermore, we say that T possesses the $\mathbf{S}_i$-property if and only if there exists a null frame such that conditions (B1)–(Bi) above are satisfied.*

Definition 1.3. *We will say that a tensor T possesses the $\mathbf{N}$-property if and only if there exists a null frame such that conditions (B1)–(Bk) in definition 1.1 are satisfied, and*

$$(T)_{\mathbf{b}} = 0, \quad \text{for} \quad \mathbf{b} = (0, 0, \dots, 0, 0).$$

Let us also recall the following result [7, 8]:

Proposition 1.4. *For tensor products, we have*

(1) Let T and S possess the $\mathbf{S}_i$- and $\mathbf{S}_j$-properties, respectively. Assuming, with no loss of generality, that $i \leqslant j$, then $T \otimes S$ possesses the $\mathbf{S}_i$-property.
(2) Let T and S possess the $\mathbf{S}_i$- and $\mathbf{N}$-properties, respectively. Then $T \otimes S$ possesses the $\mathbf{S}_i$-property. If $i = k$, then $T \otimes S$ possesses the $\mathbf{N}$-property.
(3) Let T and S both possess the $\mathbf{N}$-property. Then $T \otimes S$, and any contraction thereof, possesses the $\mathbf{N}$-property.

We extend this and define a set of related conditions which will prove useful to us. Consider a tensor, T, that does not necessarily meet any of the conditions above. However, since the boost weights $\mathbf{b} \in \mathbb{Z}^k \subset \mathbb{R}^k$, we can consider a linear $GL(k)$ transformation, $G : \mathbb{Z}^k \mapsto \Gamma$, where Γ is a lattice in $\mathbb{R}^k$. Now, if there exists a G such that the transformed boost weights, $G\mathbf{b}$, satisfy (some) of the conditions in definition 1.1, we will say, correspondingly, that T possesses the $\mathbf{S}_i^G$-property. Similarly, for the $\mathbf{N}^G$-property.

If we have two tensors T and S both possessing the $\mathbf{S}_i^G$-property, with the same G, then when we take the tensor product

$$(T \otimes S)_{G\mathbf{b}} = \sum_{G\hat{\mathbf{b}}+G\tilde{\mathbf{b}}=G\mathbf{b}} (T)_{G\hat{\mathbf{b}}} \otimes (S)_{G\tilde{\mathbf{b}}}.$$

Therefore, the tensor product will also possess the $\mathbf{S}_i^G$-property, with the same G. This will be useful later when considering degenerate tensors and metrics with degenerate curvature tensors. Note also that the $\mathbf{S}_i^G$-property reduces to the $\mathbf{S}_i$-property for $G = I$ (the identity).

1.3. Tensors not characterized by its invariants

Another useful concept is the question when a tensor/spacetime is 'characterized by its invariants'. Henceforth, by *invariants* we will always mean the *polynomial invariants*. Such have been discussed in several papers both in the Lorentzian case and in the more general case [9, 10].

We will now recall some of the definitions and concepts from invariant theory, see, e.g., [11–13]. For a tensor T, we define the action of the semi-simple group $G = O(k, k+m)$ on the components of T as follows. For simplicity, assume that the components of T have been lowered: $T_{a_1\dots a_p}$. We form the N-tuple consisting of the components of T as $X = [T_{a_1a_2\dots a_p}] \in \mathbb{R}^N$. The action corresponds to a frame rotation and explicitly, if we consider the matrix $g = (M^a_b) \in O(k, k+m)$, acting as a frame rotation $g\omega = \{M^a_1\boldsymbol{e}_a, \dots, M^a_n\boldsymbol{e}_a\}$, the frame rotation induces an action on X through the tensor structure of the components:

$$g(X) = \left[M^{b_1}_{a_1}\dots M^{b_p}_{a_p}T_{b_1\dots b_p}\right].$$

The (real) orbit $\mathcal{O}(X)$ is now defined by

$$\mathcal{O}(X) \equiv \{g(X) \in \mathbb{R}^N \big| g \in O(k, k+m)\} \subset \mathbb{R}^N.$$

We can then extend this definition to a direct sum of vectors, $T = T^{(1)} \oplus \dots \oplus T^{(q)}$. The action $g(X)$ on the components is then extended through the standard direct sum representation of the group G acting on the direct sum of tensors.

In the case of a pseudo-Riemannian space, T is a direct sum of the curvature tensors

$$T = \text{Riem} \oplus \nabla\text{Riem} \oplus \nabla\nabla\text{Riem} \oplus \dots \oplus \nabla^{(K)}\text{Riem}$$

up to some sufficiently high order K.

Definition 1.5. *A tensor T (or pseudo-Riemannian space) is characterized by its invariants if and only if the corresponding orbit $\mathcal{O}(X)$ is topologically closed in $\mathbb{R}^N$ with respect to the standard Euclidean topology.*

The motivation for this definition is given in [5]—essentially, the set of closed orbits

$$\mathfrak{C} = \{\mathcal{O}(X) \subset V \big| \mathcal{O}(X) \text{ closed}\}$$

is parameterized by the invariants, possibly up to a complex rotation (indeed, the complexified orbits are parameterized uniquely, the real orbits intersect these a finite number of times)[2].

For more details on these issues we would refer the reader to [11–13, 5].

[2] In [12] they denote this set as $V//G$.

2. Pseudo-Riemannian metrics not characterized by its invariants

A tensor, T, satisfying the $\mathbf{S}_i^G$-property or $\mathbf{N}^G$-property is not generically determined by its invariants in the sense that there may be another tensor, T', with precisely the same invariants. The $\mathbf{S}_i^G$-property thus implies a certain *degeneracy* in the tensor.

Indeed,

Theorem 2.1. *A tensor T is not characterized by its invariants if and only if it possesses (at least) the $\mathbf{S}_1^G$-property.*

Proof. Assume that T is not characterized by its invariants; i.e. the corresponding orbit is not closed. Using the results of Richardson–Slodowy [12], there then exists a $\mathcal{X} \in \mathfrak{B}$, where $\mathfrak{B}$ is the vector subspace of the Lie algebra $\mathfrak{so}(k, k+m)$ consisting of symmetric matrices (so that $\mathfrak{so}(k, k+m) = \mathfrak{B} \oplus \mathfrak{K}$, where $\mathfrak{K}$ is the Lie algebra of the maximal compact subgroup), such that $\exp(\tau\mathcal{X})(T) \to p$. We note that the maximal compact subgroup of $SO(k, m+k)$ is $K \cong SO(k) \times SO(m+k)$, which we represent as $g = (g_1, g_2) \in SO(k) \times SO(m+k)$. The $\mathcal{X}$ can be represented as

$$\mathcal{X} = \begin{bmatrix} \mathbf{0_k} & S \\ S^t & \mathbf{0}_{m+k} \end{bmatrix}, \tag{5}$$

where S is a $k \times (k+m)$ matrix. The transformation, $g^{-1}\mathcal{X}g$ induces a transformation of S according to $g_1^{-1}Sg_2$, $(g_1, g_2) \in SO(k) \times SO(m+k)$. Thus by the singular value decomposition, we can always find a $g \in K$ such that S is diagonal $S = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$. This therefore corresponds to a pure boost; specifically, by applying a null frame, $\mathcal{X}$ will represent the boost given in equation (3). Henceforth, let us represent $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_k)$ as a vector. Then if the tensor T is decomposed using the corresponding boost-weight components relative to the null frame; i.e. $T = \sum_{\mathbf{b}} (T)_{\mathbf{b}}$, we can write

$$\exp(\tau\mathcal{X})(T)_{\mathbf{b}} = \exp(\tau\mathbf{b} \cdot \boldsymbol{\lambda})(T)_{\mathbf{b}}, \tag{6}$$

where $\mathbf{b} \cdot \boldsymbol{\lambda} = \sum_{i=1}^n b_i\lambda_i$. In the limit $\tau \to \infty$, $\exp(\tau\mathcal{X})(T)$ has to approach p which is finite: hence, if $(T)_{\mathbf{b}} \neq 0$ we obtain the requirement $\mathbf{b} \cdot \boldsymbol{\lambda} \leqslant 0$. In particular,

$$\begin{aligned} \exp(\tau\mathbf{b} \cdot \boldsymbol{\lambda})(T)_{\mathbf{b}} &\to (T)_{\mathbf{b}}, \quad \mathbf{b} \cdot \boldsymbol{\lambda} = 0, \\ \exp(\tau\mathbf{b} \cdot \boldsymbol{\lambda})(T)_{\mathbf{b}} &\to 0, \quad \mathbf{b} \cdot \boldsymbol{\lambda} < 0, \end{aligned} \tag{7}$$

all other $(T)_{\mathbf{b}}$ must be zero

$$(T)_{\mathbf{b}} = 0, \quad \mathbf{b} \cdot \boldsymbol{\lambda} > 0. \tag{8}$$

Using a $G \in O(k)$ transformation in boost-weight space, we can align $\boldsymbol{\lambda}$ with the first basis vector so that $G\boldsymbol{\lambda} = |\boldsymbol{\lambda}|(1, 0, 0, \ldots, 0)$. Thus the requirement equation (8) implies that T fulfils the $\mathbf{S}_1^G$-property. □

2.1. The VSI properties

For the VSI spaces we now obtain an important corollary:

Theorem 2.2. *For a tensor T in pseudo-Riemannian space, the following is equivalent:*

(1) T has only vanishing polynomial invariants (VSI).
(2) Any operator constructed from T (by raising/lowering indices, contractions and tensor products) is nilpotent.
(3) T possesses the $\mathbf{N}^G$-property.

Proof. The proof of 1 $\Leftrightarrow$ 2 follows from [9]. Furthermore, 3 $\Rightarrow$ 1 follows from this work also. Left to prove is thus 1 $\Rightarrow$ 3.

From the proof of theorem 2.1 we see that tensors having all vanishing invariants must either have closed orbits, or have a limit which approaches an element in this closed orbit. We note that the zero tensor $\tilde{T} = 0$ has a closed orbit, and since the complex orbit consists of only the zero element, the zero tensor must be the unique tensor which has closed (real) orbits. Thus, we can choose the limit in the proof to be $p = 0$. This implies that equation (8) turns into the stronger requirement

$$(T)_{\mathbf{b}} = 0, \quad \mathbf{b} \cdot \boldsymbol{\lambda} \geqslant 0. \tag{9}$$

By the same transformation matrix G, we write $G\boldsymbol{\lambda} = |\lambda|(1, 0, 0, \ldots, 0)$ and the $\mathbf{N}^G$-property follows. □

3. 4D neutral space: all VSI metrics

The even-dimensional case with signature (k, k) (i.e. $m = 0$) is called the *neutral* case. Let us consider the 4D neutral case which is of particular interest (see, e.g., [14, 15]); in particular, we will use the above theorem to find all neutral VSI spaces of dimension 4. Such spaces have been studied before; however, only spaces satisfying the **N**-property were investigated. Although it was noted that the $\mathbf{N}^G$-property was sufficient for VSI, this possibility was not investigated in detail. Indeed, we will show that there are VSI spaces satisfying the $\mathbf{N}^G$-property, but not the **N**-property thus establishing a new class of VSI spacetimes. We also derive all such metrics and show that they are all Walker metrics possessing an invariant null plane.

In 4D neutral signature we thus obtain two classes of metric, the Kundt metrics and the Walker metrics. These will be reviewed in what follows. We will also utilize the work of Law [4] where all the spin coefficients of 4D neutral space were investigated. Using Law's notation, we adopt the slightly modified null frame $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \tilde{\boldsymbol{m}}) \equiv (\boldsymbol{\ell}^1, \boldsymbol{n}^1, \boldsymbol{\ell}^2, -\boldsymbol{n}^2)$ so that metric (2) can be written as

$$\mathrm{d}s^2 = 2\boldsymbol{\ell}\boldsymbol{n} - 2\boldsymbol{m}\tilde{\boldsymbol{m}}. \tag{10}$$

In the neutral case, this frame is purely real. With respect to such a frame, Law defined the spin coefficients which we will use in proving the main theorem. In [4] Law writes the spin coefficients in terms of $\kappa, \rho, \sigma, \tau, \epsilon, \alpha, \beta, \gamma$, and their tilded $(\tilde{\kappa}, \tilde{\rho}, \ldots)$, primed $(\kappa', \rho', \ldots)$, and primed-tilded $(\tilde{\kappa}', \tilde{\rho}', \ldots)$ counterparts. All these spin coefficients are real. For example, the covariant derivatives of the frame vector ℓ^a can be written as

$$\begin{aligned}
\ell^b\nabla_b\ell^a &= (\epsilon + \tilde{\epsilon})\ell^a + \tilde{\kappa}m^a + \kappa\tilde{m}^a, \\
\tilde{m}^b\nabla_b\ell^a &= (\alpha + \tilde{\beta})\ell^a + \tilde{\sigma}m^a + \rho\tilde{m}^a, \\
m^b\nabla_b\ell^a &= (\tilde{\alpha} + \beta)\ell^a + \tilde{\rho}m^a + \sigma\tilde{m}^a, \\
n^b\nabla_b\ell^a &= (\gamma + \tilde{\gamma})\ell^a + \tilde{\tau}m^a + \tau\tilde{m}^a, \\
&\ldots \text{etc.}
\end{aligned} \tag{11}$$

We refer to [4], in particular, equations (2.10) and (2.11) therein, for details.

3.1. Invariant null planes: Walker metrics

Here, we will consider the 4D neutral spaces which possess an invariant null plane. Such metrics are known as *Walker metrics*.

Consider two orthogonal null vectors $\boldsymbol{\ell}$ and $\boldsymbol{m}$. These span an invariant null plane iff

$$\nabla_a(\boldsymbol{\ell} \wedge \boldsymbol{m}) = k_a(\boldsymbol{\ell} \wedge \boldsymbol{m}), \tag{12}$$

for a vector k_a. Using [4] this immediately implies the vanishing of certain spin coefficients

$$\kappa = \rho = \sigma = \tau = 0.$$

Indeed, one can see that the vanishing of these spin coefficients implies the existence of an invariant null plane (hence, it is a Walker metric).

Furthermore, Walker [3] showed that the requirement of an invariant two-dimensional (2D) null plane implies that the (Walker) metric can be written in the canonical form:

$$\mathrm{d}s^2 = 2\,\mathrm{d}u(\mathrm{d}v + A\,\mathrm{d}u + C\,\mathrm{d}U) + 2\,\mathrm{d}U(\mathrm{d}V + B\,\mathrm{d}U), \tag{13}$$

where A, B and C are functions that may depend on all of the coordinates.

In particular, this implies that we can choose a frame such that [4]

$$\kappa = \rho = \sigma = \tau = \epsilon = \beta = 0, \quad \alpha' = \gamma' = \rho' = \tau' = 0, \tag{14}$$

$$\tilde{\kappa} = \tilde{\rho} = \tilde{\alpha} = \tilde{\epsilon} = 0, \quad \tilde{\beta}' = \tilde{\gamma}' = \tilde{\sigma}' = \tilde{\tau}' = 0. \tag{15}$$

We note that $\tilde{\sigma}$ needs not be zero, and hence, these Walker metrics need not be Kundt spacetimes (see below).

3.2. Pseudo-Riemannian Kundt metrics

In the Lorentzian case the Kundt metrics play an important role for degenerate metrics, and VSI metrics in particular [1]. Their pseudo-Riemannian analogues also play an important role for pseudo-Riemannian spaces of arbitrary signature [8, 15].

We define the pseudo-Riemannian Kundt metrics in a similar fashion, namely:

Definition 3.1. *A pseudo-Riemannian Kundt metric is a metric which possesses a non-zero null vector $\boldsymbol{\ell}$ which is geodesic, expansion-free, twist-free and shear-free.*

This implies that, in terms of the spin coefficients defined in [4], a space is Kundt if and only if there exists a frame such that

$$\tilde{\kappa} = \kappa = \tilde{\rho} = \rho = \tilde{\sigma} = \sigma = 0. \tag{16}$$

Therefore, we will consider metrics of the form (which is equivalent to the above definition)

$$\mathrm{d}s^2 = 2\,\mathrm{d}u[\mathrm{d}v + H(v,u,x^C)\,\mathrm{d}u + W_A(v,u,x^C)\,\mathrm{d}x^A] + g_{AB}(u,x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B \tag{17}$$

(here, the indices A, B range over the null indices $I = 2, 3$. The metric (17) possesses a null vector field $\boldsymbol{\ell}$ obeying[3]

$$\ell_{\mu;\nu} = L_{11}\ell_\mu\ell_\nu + L_{1i}\ell_{(\mu}m^i_{\nu)} + \tilde{L}_{1i}\ell_{(\mu}\tilde{m}^i_{\nu)},$$

and consequently it is geodesic, non-expanding, shear-free and non-twisting. Since this is a pseudo-Riemannian space of signature $(2, 2)$, the transverse metric

$$\mathrm{d}s_1^2 = g_{AB}(u,x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B,$$

will be of signature $(1, 1)$.

[3] If, in addition $L_{1i} = \tilde{L}_{1i} = 0$, the vector ℓ_μ is also recurrent (hence, Walker), and if $L_{1i} = \tilde{L}_{1i} = L_{11} = 0$, then ℓ_μ is covariantly constant.

3.3. The 4D neutral VSI theorem

Let us now state an important result regarding the determination of all 4D VSI metrics.

Theorem 3.2. *A 4D neutral VSI metric is of one (or both) of the following types:*

(1) A Walker metric possessing an invariant 2D null plane.
(2) A Kundt metric.

In order to prove this theorem one needs to consider theorem 2.1 and consider the covariant derivatives $\nabla^{(N)}$(Riemann). We will prove the theorem using two different methods, one is the more indirect method using the one-parameter family of boosts $B_\tau = e^{\tau\mathcal{X}}$, the other is the direct method by explicitly computing the covariant derivatives. These two illustrate two conceptually different methods and both provide us with separate information about the underlying structure of these spaces. For example, while the first is a more 'elegant' proof, the second gives some information of how many derivatives are necessary and provides with more details about the various special cases.

3.3.1. The boost method. Let us employ the frame which is aligned with the family of boosts $B_\tau = \exp(\tau\mathcal{X})$ providing us with the limit in theorem 2.1. This is a pointwise action but consider a point p and assume this is regular[4] implying that there exists a neighbourhood U such that the algebraic structure of the space does not change over U. Consider now a compact $K \subset U$ neighbourhood of p. The boost B_τ acts pointwise; however, since K is compact, we can assume that the B_τ does not depend on the point in K. Thus, with respect to the adapted frame, the boost will be constant over K:

$$\boldsymbol{\ell} \mapsto e^{-\tau\lambda_1}\boldsymbol{\ell}, \quad \mathbf{n} \mapsto e^{\tau\lambda_1}\mathbf{n}, \quad \mathbf{m} \mapsto e^{-\tau\lambda_2}\mathbf{m}, \quad \tilde{\mathbf{m}} \mapsto e^{\tau\lambda_2}\tilde{\mathbf{m}}$$

Note that such a boost will transform the curvature tensors at p as follows:

$$\exp(\tau\mathcal{X})(T)_{\mathbf{b}} = \exp(\tau\mathbf{b}\cdot\boldsymbol{\lambda})(T)_{\mathbf{b}}. \tag{18}$$

Now, in relation to the ϵ-property [10], we have that this boost manifests the limit:

$$X = \tilde{X} + N.$$

Furthermore, since K is compact, $||N||$ will have a maximum, $N_{\max}$, over K so that $||N|| \leqslant N_{\max}$; consequently,

$$||X - \tilde{X}|| \leqslant N_{\max}.$$

In the VSI case, $\widetilde{X} = 0$, so that $X = N$ and the ϵ-property implies that the components can be arbitrary close to flat space.

Consider now the action of the boost B_τ. The vector N is a direct sum of tensorial objects implying that, since it must be of type III, or simpler, there is an $a > 0$ such that

$$||B_\tau(N)|| \leqslant e^{-a\tau}||N|| \leqslant e^{-a\tau}N_{\max}.$$

We can assume that the neighbourhood U is a coordinate patch and map U into $\mathbb{R}^4$ with p at the origin. Then we can assume that the compact neighbourhood $K \subset \mathbb{R}^4$. We now consider the $X = N$ as a set of differential equations on U as follows:

Express the components of the Riemann tensor (relative to the adapted frame) in terms of the spin coefficients $\Gamma^\mu{}_{\alpha\beta}$ in the standard way:

$$R^\mu{}_{\alpha\beta\nu} = \boldsymbol{\partial}_\nu\left(\Gamma^\mu{}_{\alpha\beta}\right) - \boldsymbol{\partial}_\beta\left(\Gamma^\mu{}_{\alpha\nu}\right) + (\Gamma\star\Gamma)^\mu{}_{\alpha\beta\nu}\,, \tag{19}$$

[4] In the sense of [16], the number of independent Cartan invariants does not change at p.

where $\Gamma \star \Gamma$ indicates the quadratic terms in the spin coefficients. Similarly, the covariant derivatives can also be expressed using the spin coefficients

$$\nabla R = \nabla R(\partial\partial\Gamma, \partial\Gamma, \Gamma), \quad \nabla\nabla R = \nabla\nabla R(\partial\partial\partial\Gamma, \partial\partial\Gamma, \partial\Gamma, \Gamma), \quad \text{etc.}$$

We thus replace the left-hand side of $X = N$ with a PDE

$$\text{Pde}[\Gamma] = N. \tag{20}$$

The relation between the frame $\boldsymbol{\partial}_\alpha$ and Γ is given via

$$[\boldsymbol{\partial}_\alpha, \boldsymbol{\partial}_\beta] = -(\Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha})\boldsymbol{\partial}_\mu. \tag{21}$$

Equations (20) and (21) provide us with a set of PDEs and integrability conditions over the neighbourhood U in terms of the functions $\Gamma^\mu{}_{\alpha\beta}$. We can now consider the 'boosted' set of equations

$$\text{Pde}[\widehat{\Gamma}] = B_\tau(N) \tag{22}$$

over U. This gives us a one-parameter family of equations. Since $B_\tau(N)$ can be made arbitrary small, this can be seen as a perturbation of a PDE describing flat space. Let us now consider the Cartan equivalence problem [17] which will give us a more direct perturbation. Let us make sure that we consider sufficient number of derivatives in X to satisfy the Cartan bound. Consider the point p. For every τ there is an inverse boost so that the $B_\tau(N)$ is mapped onto $X = N$. Considering the boost that leaves the point p fixed, the equivalence principle implies that there exists a diffeomorphism ϕ_τ that maps K onto $\phi_\tau(K)$, leaving p fixed, and induces (through ϕ_τ^*) the boost B_τ acting on the tangent space at p. The diffeomorphism does not necessarily map K into itself. Consider an increasing sequence τ_n such that $\tau_n \to \infty$, and define $K_n = \phi_{\tau_n}(K)$, which is compact. In particular, K_n is closed and $p \in K_n$. This implies further that $p \in K \cap \left(\bigcap_n K_n\right)$ (and closed).

Note that the set $K \cap (\bigcap_n K_n)$ may not be a neighbourhood, indeed, in many cases it may be a single line. Thus, the limiting procedure may result in a mere pointwise result at p causing the functions $\widehat{\Gamma}$ to not necessarily have the right functional dependence in the limit $\tau \to \infty$ over K. Thus in the limit we should only consider the value of Γ restricted to the set $K \cap \left(\bigcap_n K_n\right)$. On the other hand, for τ_n finite, the result applies to a neighbourhood.

It is thus more appropriate to consider the following perturbed PDE:

$$\text{Pde}[\widehat{\Gamma}] = B_\tau(\phi_\tau^*(N)), \tag{23}$$

where $\phi^*(N)$ should be thought of as acting on the components of N as functions; i.e. if $N_{a\ldots b}$ is a component, then $\phi^*(N_{a\ldots b}) = N_{a\ldots b} \circ \phi$ [17].

Assuming we are considering a certain metric g, we know that there exists a set of equations to this PDE. In particular, there is a continuous family of solutions $\widehat{\Gamma}(\tau)$ which solves equation (22). Moreover, over the compact region K_n, since this is a perturbed PDE which implies that it satisfies a Cauchy property, namely there exists an increasing sequence $\tau_n \to \infty$, such that for any $\epsilon > 0$, there exists M such that

$$n, m \geqslant M \Rightarrow ||\widehat{\Gamma}(\tau_n) - \widehat{\Gamma}(\tau_m)|| < \epsilon. \tag{24}$$

The diffeomorphism ϕ_τ acts as follows on the connection [16, 17]: if $\boldsymbol{\Omega}$ is the connection form, then $\tilde{\phi}_\tau^*\boldsymbol{\Omega} = \widehat{\boldsymbol{\Omega}}$, where $\tilde{\phi}_\tau$ is the induced transformation on the frame bundle and $\widehat{\boldsymbol{\Omega}}$ is the transformed connection, we obtain over U:

$$\widehat{\Gamma}^\mu{}_{\alpha\beta} = (M^{-1})^\mu{}_\nu \left[M^\gamma{}_\alpha \phi_t^*(\Gamma^\nu{}_{\gamma\delta}) + M^\gamma{}_{\alpha,\delta}\right] M^\delta{}_\beta.$$

Furthermore, since $p = \phi_\tau(p)$, we have $\Gamma^\mu_{\gamma\delta} = \phi_\tau^*(\Gamma^\nu{}_{\gamma\delta})$ at p. Moreover, in the aforementioned frame, we have $M^1{}_{1,\mu} = -M^2{}_{2,\mu}$, $M^3{}_{3,\mu} = -M^4{}_{4,\mu}$, while all other components of $M^\gamma{}_{\alpha,\delta}$ are zero.

Equation (24) implies that the connection coefficients can be chosen to be arbitrary close to flat space. Componentwise we have $|\widehat{\Gamma}^{\alpha}{}_{\beta\gamma}(\tau_n)-\widehat{\Gamma}^{\alpha}{}_{\beta\gamma}(\tau_m)| < \epsilon$. Since some of the components of the connection transforms as tensor components under the boost, if the component has boost weight **b**, we obtain

$$\begin{aligned} \left|\widehat{\Gamma}^{\alpha}{}_{\beta\gamma}(\tau_n) - \widehat{\Gamma}^{\alpha}{}_{\beta\gamma}(\tau_m)\right| &= \left|\exp[\mathbf{b}\cdot\boldsymbol{\lambda}\tau_n]\Gamma^{\alpha}{}_{\beta\gamma} - \exp[\mathbf{b}\cdot\boldsymbol{\lambda}\tau_m]\Gamma^{\alpha}{}_{\beta\gamma}\right| \\ &= \exp[\mathbf{b}\cdot\boldsymbol{\lambda}\tau_n]\left|\Gamma^{\alpha}{}_{\beta\gamma} - \exp[\mathbf{b}\cdot\boldsymbol{\lambda}(\tau_m-\tau_n)]\Gamma^{\alpha}{}_{\beta\gamma}\right| \\ &< \epsilon. \end{aligned} \tag{25}$$

If we fix m, then it is clear that

$$\mathbf{b}\cdot\boldsymbol{\lambda} \leqslant 0, \quad \text{or} \quad \Gamma^{\alpha}{}_{\beta\gamma} = 0 \quad \text{for} \quad \mathbf{b}\cdot\boldsymbol{\lambda} > 0.$$

This is valid for an arbitrary point $p \in U$; hence, it is valid everywhere in the neighbourhood.

We can now consider the connection coefficients that transform tensorially, and consider the various cases. By a simple geometric argument, we obtain

(1) $\tilde{\kappa} = \kappa = \tilde{\rho} = \rho = \tilde{\sigma} = \sigma = 0$, and hence Kundt; *or*
(2) $\tilde{\kappa} = \tilde{\rho} = \tilde{\sigma} = \tilde{\tau} = 0$, and hence, a Walker space possessing an invariant null 2-plane.

3.3.2. The direct method. Before we embark on the direct method let us remind ourselves of some useful identities and formulae. The covariant derivative of a tensor T has the formal structure

$$\nabla T = \partial T - \sum \Gamma \star T, \tag{26}$$

where the ∂T indicates the partial derivative piece, and the $\Gamma \star T$ indicates the algebraic piece where Γ are the spin coefficients. Furthermore, also useful are the second Bianchi identity and the generalized Ricci identity

$$R_{ab(cd;e)} = 0, \tag{27}$$

$$[\nabla_a, \nabla_b]\, T_{c_1\ldots c_k} = \sum_{i=1}^{k} T_{c_1\ldots d\ldots c_k} R^{d}{}_{c_i ab}, \tag{28}$$

which enable us to permute covariant derivatives up to algebraic terms. We note that all the algebraic terms are of lower order in derivatives of T.

Assuming that T fulfils the $\mathbf{N}^G$-property, there are therefore two potential ways the covariant derivative ∇T of the tensor can violate the $\mathbf{N}^G$-property; namely through the components of the partial derivatives propagating the components of T across the $\mathbf{b}\cdot\boldsymbol{\lambda} = 0$ line in boost-weight space, and the algebraic terms. At every level of covariant derivatives, we can thus first permute the derivatives as much as possible, and then impose the necessary conditions on the remaining components. Thus we ensure that the $\mathbf{N}^G$-property is valid at every lower derivative so that when using the Ricci identity, it does not involve $\mathbf{N}^G$-property breaking terms through the algebraic piece.

Let us first split the Riemann tensor into its irreducible parts R, S_{ab}, W^{+}_{abcd} and W^{-}_{abcd}. For a VSI space, $R = 0$ so the trace-free Ricci tensor, S_{ab}, is equal to the Ricci tensor $S_{ab} = R_{ab}$.

Then consider a non-zero Ricci tensor. By considering $R_{ac}R^{c}_{b}$ or higher powers if necessary, we can assume the Ricci tensor is of the form (brackets mean symmetrization)

$$R = a\boldsymbol{\ell}\boldsymbol{\ell} + b(\boldsymbol{\ell}\tilde{\boldsymbol{m}}) + c\tilde{\boldsymbol{m}}\tilde{\boldsymbol{m}}. \tag{29}$$

We need to compute the derivatives $\nabla^{(k)}R_{ab}$. The various cases depend on the components a, b and c and let us consider these in turn.

$ac \neq 0$. Here, we can boost so that a and c are both constants. Computing first $\nabla_a R$, some of the components are proportional to

$$\begin{aligned}&(-1,2) : a\tilde{\sigma},\ (0,1) : a\tilde{\kappa},\\ &(1,0) : c\tilde{\sigma},\quad (2,-1) : c\tilde{\kappa};\end{aligned}$$

consequently, by the $\mathbf{N}^G$-property, $\tilde{\kappa} = \tilde{\sigma} = 0$. Computing $\nabla_b\nabla_a R$ we obtain similarly $\tilde{\rho} = \tilde{\tau} = 0$. Thus, this is a Walker space.

$ab \neq 0$, $c = 0$. Here, we can boost so that a and b are both constants. Considering the first derivative, $\nabla_a R$, we obtain (among others) the components

$$\begin{aligned}&(-1,2) : a\tilde{\sigma},\ (0,1) : a\tilde{\kappa},\\ &(0,1) : b\tilde{\sigma},\quad (1,0) : b\tilde{\kappa},\ (0,-1) : b\tilde{\rho};\end{aligned}$$

hence, there are two possibilities $\tilde{\kappa} = \tilde{\rho} = 0$, or $\tilde{\kappa} = \tilde{\sigma} = 0$. By computing $\nabla_b\nabla_a R$, we quickly obtain $\tilde{\rho} = 0$. Thus we need to consider the two cases $\tilde{\sigma} \neq 0$, and $\tilde{\sigma} = 0$.

From the second derivative, and Law's equation (3.4) in [4], we obtain the conditions

$$\tilde{\sigma}\rho = \tilde{\sigma}\sigma = \tilde{\sigma}\kappa = \tilde{\tau}\kappa = \tau\tilde{\sigma} + \rho\tilde{\tau} = 0. \tag{30}$$

If $\tilde{\sigma} \neq 0$, then $\rho = \kappa = \sigma = \tau = 0$, and consequently Walker.

Assume then $\tilde{\sigma} = 0$. If $\tilde{\tau} = 0$, then the space is again Walker. Left to consider is therefore $\tilde{\tau} \neq 0$ and $\tilde{\kappa} = \tilde{\sigma} = \tilde{\rho} = 0$. From the equations above, we thus obtain $\kappa = \rho = 0$ also. If $\sigma = 0$, then the space is Kundt. We need thus to check if $\sigma \neq 0$. By computing $\nabla^{(3)}R$ and $\nabla^{(4)}R$, we obtain numerous constraints from the requiring the $\mathbf{N}^G$-property. Most of these are the same as the Bianchi identity. Imposing these and some algebraic conditions on the spin coefficients, we obtain the following b.w. (0, 0)-component to be

$$R_{22;4311} = 12\tilde{\tau}^3\sigma b.$$

By the $\mathbf{N}^G$-property this component has to vanish which is contradictory to the assumptions given above. Hence, the space has to be either Walker or Kundt.

$b \neq 0$, $a = c = 0$. Here, we note that there is a discrete symmetry which flips boost-weight space with respect to the line $b_1 - b_2 = 0$. Using this symmetry, the case here essentially reduces to the case $ab \neq 0$ above. Thus also here the $\mathbf{N}^G$-property implies Walker or Kundt.

$a \neq 0$, $b = c = 0$. Lastly we need to consider the case when only a is non-zero. First we look at $\nabla^{(2)}R$. Using the symmetry $(b_1, b_2) \mapsto (b_1, -b_2)$ we obtain the conditions

$$\tilde{\kappa} = \tilde{\sigma} = \tilde{\rho} = \rho = 0. \tag{31}$$

In addition, the vanishing of the (0, 0) components implies $\kappa\tilde{\tau} = 0$. If $\tilde{\tau} = 0$, then the space is Walker. Assume thus $\tilde{\tau} \neq 0$, implying $\kappa = 0$.

In addition, the Bianchi identities need to be fulfilled. Imposing these and computing the symmetric 2-tensor $\Box R_{ab}$, we note that this is of the following form:

$$\Box R = A\boldsymbol{\ell}\boldsymbol{\ell} + B(\boldsymbol{\ell}\tilde{\boldsymbol{m}}) + C\tilde{\boldsymbol{\ell}\boldsymbol{m}}. \tag{32}$$

If B or C is non-zero, then the previous computations imply that, by considering possibly four more derivatives, its Walker or Kundt. The requirements $B = C = 0$ impose additional conditions on the spin coefficients. Eventually, after possibly four more derivatives, this also implies its Walker or Kundt.

The Weyl tensor. Let us now consider the self-dual (or anti-self-dual by orientation reversion) Weyl tensor. This needs to be of type III, N, or O, see [7]. If it is of type III, then $(\mathsf{W}^+)^2$ as a bivector operator is of type N. Consider thus the case of type N. By discrete symmetries, we can thus assume that (in a shorthand notation)

$$W^+ = \phi(\boldsymbol{\ell} \wedge \boldsymbol{m})(\boldsymbol{\ell} \wedge \boldsymbol{m}). \tag{33}$$

We note that the discrete symmetry that acts on boost-weight space as $(b_1, b_2) \mapsto (-b_2, -b_1)$ leaves W^+ invariant. By computing the second covariant derivative, $\nabla_b \nabla_a W^+$, we pick out the following components (including their boost weights):

$$\begin{aligned}
&\boldsymbol{n}_a \boldsymbol{n}_b (\tilde{\boldsymbol{m}} \wedge \boldsymbol{m})(\tilde{\boldsymbol{m}} \wedge \boldsymbol{m}) : \quad \propto \kappa^2,\ (2, 0)\\
&\tilde{\boldsymbol{m}}_a \tilde{\boldsymbol{m}}_b (\tilde{\boldsymbol{m}} \wedge \boldsymbol{m})(\boldsymbol{\ell} \wedge \boldsymbol{n}) : \quad \propto \sigma^2,\ (0, -2)\\
&\boldsymbol{m}_a \boldsymbol{m}_b (\tilde{\boldsymbol{m}} \wedge \boldsymbol{m})(\tilde{\boldsymbol{m}} \wedge \boldsymbol{m}) : \ \propto \rho^2,\ (0, 2).
\end{aligned}$$

By the $\mathbf{N}^G$-property of W^+ and $\nabla^{(2)} W^+$, and using the remaining discrete symmetry, we thus obtain the following cases:

$$\kappa = \sigma = 0, \quad \text{or} \quad \kappa = \rho = 0.$$

Consider first $\kappa = \sigma = 0$. Computing $\nabla_d \nabla_c \nabla_b \nabla_a W^+$, in particular the component $\boldsymbol{m}_a \boldsymbol{m}_b \boldsymbol{m}_c \boldsymbol{m}_d (\tilde{\boldsymbol{m}} \wedge \boldsymbol{n})(\tilde{\boldsymbol{m}} \wedge \boldsymbol{n}) \propto \rho^4$ of boost weight $(2, 2)$. Again, utilizing the remaining discrete symmetry, this must be zero. Thus, $\kappa = \sigma = \rho = 0$.

Hence, we are left with $\kappa = \rho = 0$, while σ need not be zero. Assume thus that $\sigma \neq 0$. Using the second derivative once again, but this time the components

$$\begin{aligned}
&\tilde{\boldsymbol{m}}_a \boldsymbol{n}_b (\boldsymbol{m} \wedge \boldsymbol{n})(\boldsymbol{\ell} \wedge \boldsymbol{m}) \propto\ \tilde{\kappa}\sigma,\ (1, 1)\\
&\tilde{\boldsymbol{m}}_a \tilde{\boldsymbol{m}}_b (\boldsymbol{m} \wedge \boldsymbol{n})(\boldsymbol{\ell} \wedge \boldsymbol{m}) \propto \tilde{\rho}\sigma,\ (0, 0);
\end{aligned} \tag{34}$$

thus, $\tilde{\kappa} = \tilde{\rho} = 0$. From Law's equation (3.4*a*) in [4], it now implies that $\tilde{\sigma}\sigma = 0$; hence, $\tilde{\sigma} = 0$.

Thus we are in the situation where we obtain one of the following cases:

(1) $\kappa = \tilde{\kappa} = \rho = \tilde{\rho} = \tilde{\sigma} = 0, \sigma \neq 0$.
(2) $\kappa = \rho = \sigma = 0$.

It is important here that we keep track of the components of the lower derivatives.

Consider next the first case where $\sigma \neq 0$. Then using the fourth derivative, we obtain the component

$$\tilde{\boldsymbol{m}}_a \boldsymbol{m}_b \boldsymbol{\ell}_c \tilde{\boldsymbol{m}}_d (\tilde{\boldsymbol{m}} \wedge \boldsymbol{n})(\tilde{\boldsymbol{m}} \wedge \boldsymbol{m}) \propto \sigma^2 \tilde{\tau}^2$$

of boost weight (0,0); consequently, $\tilde{\tau} = 0$ and thus all the tilded variables $\tilde{\kappa} = \tilde{\rho} = \tilde{\sigma} = \tilde{\tau} = 0$, and this is thus a Walker space.

We are left to consider the second case where $\kappa = \rho = \sigma = 0$. If $\tau = 0$, we have a Walker space. Assume thus that $\tau \neq 0$. By computing the fourth derivative, we note that one of the components

$$\tilde{\boldsymbol{m}}_a \tilde{\boldsymbol{m}}_b \boldsymbol{\ell}_c \boldsymbol{\ell}_d (\boldsymbol{\ell} \wedge \boldsymbol{n})(\boldsymbol{\ell} \wedge \boldsymbol{n}) \propto \tau^4.$$

This component has boost weight (−2,−2) and has the same boost weight as W^+ under the exchange of tilded spin coefficients with non-tilded ones. After a lengthy computation, sometimes needing to go to eighth order, we obtain that $\tilde{\kappa} = \tilde{\sigma} = \tilde{\rho} = 0$ (analogously as above). Thus, implying that this is a Kundt space.

If W^+ but $W^- \neq 0$, then we can consider the discrete symmetry which interchanges tilded spin coefficients with non-tilded ones: $\tilde{x} \leftrightarrow x$, where x is the spin coefficients. Then an identical computation as above implies that the space is either Walker with an invariant null 2-plane, or Kundt. The theorem follows then from these considerations.

Although the argument involves eight derivatives, it is suspected that the number of derivatives needed is less than this. In particular, no examples of spaces which are VSI_k but not VSI_{k+1} are known for $k > 3$. The example equation (1) is VSI_3 but not VSI_4; however, this is a Walker metric which is a restricted class. This example, and an explanation of how this example can be extended to other similar examples, will be given later. However, a question still remains: are there examples of non-Walker metrics which are VSI_k but not VSI_{k+1} for $k > 3$?

3.4. Neutral VSI metrics

3.4.1. 4D neutral case: Kundt metrics. Using

$$\mathrm{d}s^2 = 2(\boldsymbol{\ell}\boldsymbol{n} - \boldsymbol{m}\tilde{\boldsymbol{m}}), \tag{35}$$

we will consider the pseudo-Riemannian Kundt case for which the transverse space is 2D. Requiring the **N**-property, this must be flat space (see [8, 15]). Therefore, we can write

$$-2\boldsymbol{m}\tilde{\boldsymbol{m}} = 2\,\mathrm{d}U\,\mathrm{d}U = -\mathrm{d}T^2 + \mathrm{d}X^2.$$

There are two classes of 4D Neutral Kundt VSI metrics; they can be written as [8, 15]

$$\mathrm{d}s^2 = 2\,\mathrm{d}u(\mathrm{d}v + H\,\mathrm{d}u + W_{\mu_1}\,\mathrm{d}x^{\mu_1}) + 2\,\mathrm{d}U\,\mathrm{d}V, \tag{36}$$

where

Null case:

$$\begin{aligned} W_{\mu_1}\,\mathrm{d}x^{\mu_1} &= vW_U^{(1)}(u,U)\,\mathrm{d}U + W_U^{(0)}(u,U,V)\,\mathrm{d}U + W_V^{(0)}(u,U,V)\,\mathrm{d}V,\\ H &= vH^{(1)}(u,U,V) + H^{(0)}(u,U,V). \end{aligned} \tag{37}$$

Spacelike/timelike case:

$$\begin{aligned} W_{\mu_1}\,\mathrm{d}x^{\mu_1} &= vW^{(1)}\,\mathrm{d}X + W_T^{(0)}(u,T,X)\,\mathrm{d}T + W_X^{(0)}(u,T,X)\,\mathrm{d}X,\\ H &= \frac{v^2}{8}(W^{(1)})^2 + vH^{(1)}(u,T,X) + H^{(0)}(u,T,X), \end{aligned} \tag{38}$$

and

$$W^{(1)} = -\frac{2\epsilon}{X}, \quad \text{where } \epsilon = 0, 1. \tag{39}$$

We note that these possess an invariant null line if $W^{(1)} = 0$, and a 2D invariant null plane if $W_V^{(0)} = 0$ for the null case[5].

3.4.2. 4D neutral signature: Walker metrics. This class of metrics provides us with a new set of VSI metrics which have not been considered before. This is due to the fact that these VSI metrics do not in general possess the **N**-property, but rather the weaker requirement of the $\mathbf{N}^G$-property.

Using the following Walker form:

$$\mathrm{d}s^2 = 2\,\mathrm{d}u(\mathrm{d}v + A\,\mathrm{d}u + C\,\mathrm{d}U) + 2\,\mathrm{d}U(\mathrm{d}V + B\,\mathrm{d}U), \tag{40}$$

the result can be summarized in the following theorem:

[5] In order for the spacelike/timelike case to possess an invariant null 2-plane, it needs to be a special case of the null case.

Theorem 3.3. *Consider the metric (40), where*

$$\begin{aligned} A &= vA_1(u,U) + VA_2(u,U) + A_0(u,U), \\ B &= VB_1(u,v,U) + B_0(u,v,U) \\ C &= C_1(u,v,U) + VC_2(u,U) + C_0(u,U). \end{aligned} \tag{41}$$

Then the following holds:

(1) The metric is a VSI_1 *space. If*

$$A_2\frac{\partial^2 B_1}{\partial v^2} \neq 0, \quad \text{or} \quad A_2\frac{\partial^3 C_1}{\partial v^3} \neq 0,$$

then it is not VSI_2.

(2) If

$$\begin{aligned} B_1 &= vB_{11}(u,U) + B_{10}(u,U) \\ C_1 &= v^2C_{12}(u,U) + vC_{11}(u,U) + C_{10}(u,U), \end{aligned} \tag{42}$$

then it is a VSI_3 *space. If in addition,*

$$A_2\frac{\partial^4 B_0}{\partial v^4} \neq 0,$$

then it is not VSI_4.

(3) If equation (42) holds and, in addition

$$B_0 = v^3B_{03}(u,U) + v^2B_{02}(u,U) + vB_{01}(u,U) + B_{00}(u,U), \tag{43}$$

then the space is VSI.

The proof is that this result is partly by direct computation of the curvature tensors and requiring $\mathbf{N}^G$-property. Let us indicate how the proof goes and in the process we elude to how these can be generalized.

Starting with the Walker form equation (40) we can compute the Riemann tensor. We note that the metric gives Riemann components in the lower triangular part of boost-weight space. Let us for short use notation such that the basis 1-forms are

$$\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \boldsymbol{\omega}^3, \boldsymbol{\omega}^4\} = \{\mathrm{d}u,\ \mathrm{d}v + A\,\mathrm{d}u + C\,\mathrm{d}U,\ \mathrm{d}U,\ \mathrm{d}V + B\,\mathrm{d}U\}. \tag{44}$$

Then a component of a tensor would have the boost weight as follows (indices downstairs):

$$(b_1, b_2) = (\#(1) - \#(2), \#(3) - \#(4));$$

i.e. the component R_{1223}, say, will have boost weight $(-1, 1)$.

For the Walker metric the components of interest in relation to the $\mathbf{N}^G$-property are

$$(2,-2): R_{1414} = -B_{,vv}, \tag{45}$$

$$(1,-1): R_{1214} = -\tfrac{1}{2}C_{,vv}, \quad R_{1434} = -B_{,vV} \tag{46}$$

$$(0,0): R_{1212} = -A_{,vv}, \quad R_{1234} = -\tfrac{1}{2}C_{,vV} \quad R_{3434} = -B_{,VV} \tag{47}$$

$$(-1,1): R_{1223} = A_{,vV}, \quad R_{2334} = \tfrac{1}{2}C_{,VV}, \tag{48}$$

$$(-2,2): R_{2323} = -A_{,VV}, \tag{49}$$

while $(R)_{(b_1,b_2)} = 0$ for $b_1 + b_2 > 0$. Thus, the Riemann tensor automatically satisfies the $\mathbf{S}_1^G$-property. In order for it to satisfy the $\mathbf{N}^G$-property, we can set the components $(2,-2)$, $(1,-1)$ and $(0,0)$ to zero. Solving these equations gives the functional dependences as given in (41). This is thus a VSI_0 space. Indeed, by direct computation we note that ∇(Riemann) satisfies the $\mathbf{N}^G$-property also, hence, it is in addition VSI_1.

Assume then that (41) is satisfied. Regarding the $\nabla^{(2)}$(Riemann) we note that this does not necessarily satisfy the $\mathbf{N}^G$-property (thus not VSI_2). One such non-vanishing scalar is $R_{abcd;ef}R^{abcd;ef}$. However, componentwise, the critical components are

$$R_{1424;33} = A_2(B_1)_{,vv}, \quad R_{2324;11} = \tfrac{1}{2}A_2(2(B_1)_{,vv} - (C_1)_{,vvv}).$$

These give rise to the conditions mentioned and equating these to zero gives the solutions (42). Satisfying equation (42) will now give the $\mathbf{N}^G$-property and thus VSI_2; indeed, VSI_3 by direct computation.

Thus assume (41) and (42) are satisfied. Computing $R_{abcd;efgh}R^{abcd;efgh}$, we obtain

$$R_{abcd;efgh}R^{abcd;efgh} = 576[(B_0)_{,vvvv}]^2 A_2^4.$$

Hence, it is not VSI_4 if $A_2(B_0)_{,vvvv} \neq 0$. Requiring that $(B_0)_{,vvvv} = 0$ gives the solution in (43) and by inspection, $\nabla^{(4)}$(Riemann) satisfies the $\mathbf{N}^G$-property. This is sufficient for the metric to be VSI.

We note that this proof also provides us with examples of metrics being VSI_3 but not VSI_4. For example, if (41) and (42) are satisfied, but A_2, $(B_0)_{,vvvv} \neq 0$, then it is VSI_3 but not VSI_4. The example given in the introduction, equation (1) is perhaps the simplest member of this family.

Similarly, metrics being VSI_1 but not VSI_2 can be found analogously; as a simple set of examples of metrics of this kind

$$\mathrm{d}s^2 = 2\,\mathrm{d}u(\mathrm{d}v + V\,\mathrm{d}u + bv^3\,\mathrm{d}U) + 2\,\mathrm{d}U(\mathrm{d}V + aVv^2\,\mathrm{d}U), \tag{50}$$

where a and b are constants, or functions depending on (u, U), not both being zero.

4. Discussion

In this paper we have studied pseudo-Riemannian metrics with the degenerate curvature structure in the sense that they are not characterized by their polynomial curvature invariants. In particular, we related these to the $\mathbf{S}_1^G$-property. Specifically, we have three main results:

(1) In a pseudo-Riemannian space of arbitrary dimension and signature, a space (tensor) not characterized by its polynomial invariants possesses the $\mathbf{S}_1^G$-property.
(2) In the special case where the invariants vanish, the space (tensor) must possess the $\mathbf{N}^G$-property.
(3) In 4D neutral signature, a VSI space is either Kundt or a Walker space.

Indeed, in the latter case we constructed a new family of Walker VSI spaces. This shows that in the pseudo-Riemannian case, these Walker metrics can provide new examples of metrics not being characterized by their invariants. Indeed, using the ideas given in this paper, examples of VSI Walker metrics can be given in any signature $(k, k+m)$ where $k \geqslant 2$. As an example, the following is a neutral VSI Walker metric (with a 3D invariant null space) in six dimensions:

$$\mathrm{d}s^2 = 2\,\mathrm{d}u(\mathrm{d}v + V\,\mathrm{d}u) + 2\,\mathrm{d}U(\mathrm{d}V + \mathcal{V}\,\mathrm{d}U) + 2\,\mathrm{d}\mathcal{U}(\mathrm{d}\mathcal{V} + v^7\,\mathrm{d}\mathcal{U}).$$

In future work, pseudo-Riemannian VSI metrics will be studied further and the ultimate aim is a full classification of VSI metrics in any dimension and signature.

References

[1] Pravda V, Pravdova A, Coley A and Milson R 2002 *Class. Quantum Grav.* **19** 6213 (arXiv:gr-qc/0209024)
Coley A, Milson R, Pravda V and Pravdova A 2004 *Class. Quantum Grav.* **21** 5519
Coley A, Fuster A, Hervik S and Pelavas N 2006 *Class. Quantum Grav.* **23** 7431

[2] Coley A, Hervik S and Pelavas N 2009 *Class. Quantum Grav.* **26** 025013 (arXiv:0901.0791)
Coley A, Hervik S, Papadopoulos G and Pelavas N 2009 *Class. Quantum Grav.* **26** 105016 (arXiv:0901.0394)
Coley A, Hervik S and Pelavas N 2006 *Class. Quantum Grav.* **23** 3053 (arXiv:gr-qc/0509113)

[3] Walker A G 1949 *Q. J. Math.* **20** 135–45
Walker A G 1950 *Q. J. Math.* **1** 69–79

[4] Law P R 2009 *J. Geom. Phys.* **59** 1087–126

[5] Hervik S 2011 *Class. Quantum Grav.* **28** 215009 (arXiv:1109.2551 [gr-qc])

[6] Coley A, Milson R, Pravda V and Pravdova A 2004 *Class. Quantum Grav.* **21** L35 (arXiv:gr-qc/0401008)
Pravda V, Pravdová A, Coley A and Milson R 2002 *Class. Quantum Grav.* **19** 6213 (arXiv:0710.1598)
Coley A 2008 *Class. Quantum Grav.* **25** 033001 (arXiv:0710.1598)

[7] Hervik S and Coley A 2011 *Int. J. Geom. Methods Mod. Phys.* **08** 1679 (arXiv:1008.3021 [gr-qc])

[8] Hervik S and Coley A 2011 *Class. Quantum Grav.* **28** 015008 (arXiv:1008.2838 [gr-qc])

[9] Hervik S and Coley A 2010 *Class. Quantum Grav.* **27** 095014 (arXiv:1002.0505)
Coley A and Hervik S 2009 *Class. Quantum Grav.* **27** 015002 (arXiv:0909.1160)

[10] Hervik S 2011 *Class. Quantum Grav.* **28** 157001 (arXiv:1107.3210 [gr-qc])

[11] Goodman R and Wallach N R 2009 *Symmetry, Representations and Invariants* (Berlin: Springer)

[12] Richardson R W and Slodowy P J 1990 *J. London Math. Soc. (2)* **42** 409–29

[13] Eberlein P and Jablonski M 2009 *Contemp. Math.* **491** 283

[14] Penrose R and Rindler W 1986 *Twistors and Space-Time* vols $1/2$ (Cambridge: Cambridge University Press)
Dunajski M 2009 *J. Phys. A: Math. Theor.* **42** 404004
Dunajski M 2002 *Proc. R. Soc. Lond.* A **458** 1205–22
Dunajski M and Tod P 2010 *Math. Proc. Camb. Phil. Soc.* **148** 485 (arxiv:0901.2261v3 [math.DG])
Galaev A S 2010 arXiv:1002.2064v1 [math.DG]

[15] Alcolado A, MacDougall A, Coley A and Hervik S 2012 *J. Geom. Phys.* **62** 594 (arXiv:1104.3799 [math-ph])

[16] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C A and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge: Cambridge University Press)

[17] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol 1 (New York: Interscience)