Protection vs. Separation in Parallel Non-Homogeneous Systems

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ABSTRACT

The article considers strategic defense and attack of a system consisting of N functionally identical parallel elements of n types distinguished by element's performance. The elements can be separated in order to reduce the damage caused by an outside attack. The defender distributes its resource between separation and protecting the elements from attacks. The attacker attacks optimally. The vulnerability of each element is determined by a contest between the defender and the attacker expressed as a contest success function. Ten propositions are developed with and without resource constraints to show when separation is efficient. With two different types of elements, the defender can never justify separation by maintaining its contest success against the attacker for one type of elements. To possibly justify separation, the contest success must be decreased against one type and increased against the other type. For the case of two types of elements a condition is developed for the highest separation cost the defender is willing to incur if it chooses not to protect any of the elements of type 1. Without resource constraints, separation is not efficient if the unit costs of protection are equal for all elements, and the unit costs of attack are the same for all elements. However, separation is efficient if lower unit costs of defense can be obtained through the separation process for sufficiently many of the elements, especially those elements with high performance, or if the number of elements of the same type is large.

Keywords: Risk, survivability, vulnerability, reliability, optimization, defense, attack, protection, separation, contest intensity, game theory, non-homogeneous system.

BASIC DEFINITIONS

Performance — quantitative measure of task performing intensity of element or system (capacity, productivity, processing speed, task completion time etc.)

Element — lowest-level part of system, which is characterized by its vulnerability and performance

Vulnerability — probability of destruction

Protection — technical or organizational measure aimed at reduction of destruction probability of system elements in the case of attack

Separation — action aimed at preventing simultaneous destruction of several elements in the case of single attack (can be performed by spatial dispersion, by encapsulating different elements into different protective casings, by using different power sources etc.)

1. INTRODUCTION

Determining risk reduction strategies applying reliability theory have usually assumed a static external threat [7, 14, 15, 17, 18, 19]. Bier and Abhichandani [4] and Bier *et al.* [5] assume that the defender minimizes the success probability and expected damage of an attack [7, 14, 15, 17, 18, 19]. Levitin [16] determines the expected damage for any distribution of the attacker's effort and any separation and distribution of the defender's effort. The September 11, 2001 attack illustrated that major threats today involve strategic attackers. There is a need to proceed beyond earlier research and assume that both the defender and attacker of a system of components are fully strategic optimizing agents [8, 10, 13, 20, 22, 27, 28].[1]

This article considers systems like power generators, water supply systems, telecommunications systems, or more generally any complex system characterized by some kind of cumulative performance. A system consists of N functionally identical[2] parallel elements of n types with different performance. The elements can be separated and protected. The defender distributes its resource between separation and protecting the elements from outside attacks. This gives N contests between the defender and attacker. Separation is, and should also be, a rightfully acknowledged and standard practice for reducing system risk in the face of intentional attacks and non-intentional impacts, see [17, 21, 26]. The separation is aimed at preventing the simultaneous destruction of several elements in the case of a single attack. It can be performed by spatial dispersion, by encapsulating different elements into different protective casings, by using different power sources, etc. The protection is a technical or organizational measure aimed at the reduction of the destruction probability of system elements in the case of attack.

The protection of elements of the same type is the same. The attacker attacks optimally. The vulnerability of each element is determined by a contest between the defender and the attacker. The contest is expressed as a contest success function modeled with the common ratio form.

Each element is assumed to either work at a nominal performance or be destroyed (electronic devises, mechanical equipment etc.). This is realistic since multi-state systems that can work with different performance rates can usually be decomposed into two-state elements (for example, steam generator's capacity can vary depending on availability of heating sections which can be made dichotomous).

The article studies the system from the reliability diagram structure point of view. The developed propositions should be considered as technical aspects that support the decision making process when the system defense is planned. The article analyzes how protection and separation in a parallel non-homogeneous system are influenced by strategic optimal behavior of a defender seeking the system to function, and an attacker seeking the system to malfunction.

In order to clearly understand this influence and simplify the initial study, we assume completely rational agents which constitutes a fine benchmark. Future research can introduce various mental level behavioral assumptions such as boundedly rational agents. Furthermore, the article assumes that both the defender and attacker have complete information about the structure of the game, the strategy sets (which specify the ranges for the free choice variables), and all parameters. The later also constitutes a benchmark. In practice, the attacker may not be fully informed about the characteristics of the system, and the agents may not be fully informed about each other's unit costs. For example, the attacker may have compiled partial intelligence, may be partially informed, and may even in direct physical attacks not always be able to validate the exact sizes of all parameters. Modeling incomplete information about parameters is left for future research.

We believe that more complete models considering incomplete information and taking into account psychological aspects can be developed further based on the presented benchmark Empirical validation of the results is also left for future research, which can be done applying the experimental approaches normally used in decision theory.

Section 2 presents the model and analyzes separation with resource constraints for the defender and attacker. Section 3 introduces utility maximization with resource constraints. Section 4 considers utility maximization without resource constraints and without separation. Section 5 incorporates separation. Section 6 assesses the separation efficiency when defender and attacker act optimally without resource constraints. Section 7 concludes.

2. ANALYZING SEPARATION WITH RESOURCE CONSTRAINTS

We consider a system consisting of N functionally identical parallel elements. The total cumulative performance of the parallel elements in the system is G. If no separation is imposed, a single successful attack destroys the entire system. For the special case that all elements are identical, the system can be separated into N independent identical elements with performance $g = G/N$ each. That is, for this special case each element can perform the same function as the system, but with N times smaller performance. A single successful attack on the separated system can destroy only one element, and cause system cumulative performance reduction from G to $G(N–1)/N$.

Generally, the system consists of n types of elements characterized by different performance. One example is a power generating system composed from n types of generators with different capacity. The number of elements of type i is N_i such that $\sum_{i=1}^{n} N_i = N$. The total system performance is equal to the sum of performances of the elements.

Each element of type i has performance g_i. The total performance of the elements of type i is $N_i g_i$. The total system performance is $G = \sum_{i=1}^{n} N_i g_i$. In order to reduce the expected damage caused by an attack the defender can separate the system into N elements and protect each one of the separated elements.

The total attacker's fixed resource is R. The total defender's fixed resource is r. The defender's resource is distributed between separation s and protection $r–s$. The defender's strategic decision variable is s. The defender's strategy is to determine whether the separation cost s justifies separation when the objective is to minimize the expected damage. That is, there exists an upper acceptable separation cost s which justifies separation for the defender. In order for the defender to justify the separation cost, he must furnish a contest success on each element that in an overall sense is sufficiently larger than the contest success of the attacker.

The attacker knows the performance of each element and seeks to maximize the expected damage.

Assumption 1: Both agents distribute their efforts equally across the functionally identical elements.

This assumption is relevant in situations where the attacker cannot direct the attack exactly against certain targets (for example, low precision missile attack against a group of separated targets) and the defender cannot protect only a subset of targets (for example in the case of anti-aircraft defense in the area where the targets are located). In such situations one should assume that both the attacker and the defender distribute their effort evenly among the elements.

Assumption 1 applies for the first four propositions in this section. Lacking information, the attacker distributes his effort evenly among the elements. This means that the attacker makes no strategic decision, and thus has no strategic decision variable. The attacker thus plays no game in the beginning of this section. Toward the end of this section, in Proposition 5, the attacker has complete information about the defender's effort distribution and is able to direct the attack exactly against certain targets. Then the attacker's strategy is to attack unprotected elements with an arbitrarily small but positive effort, and to distribute his remaining resource evenly across the protected elements.

The vulnerability of any element is determined by a contest between the defender and the attacker. The contest is expressed as a contest success function modeled with the common ratio form, see [9, 23, 25]. For the case when both the defender and the attacker evenly distribute their efforts over all N elements, which implies division with N, the vulnerability of any element is

$$v = \frac{(R/N)^m}{(R/N)^m + [(r-s)/N]^m} = \frac{R^m}{R^m + (r-s)^m}. \qquad (1)$$

where $m \geq 0$ is a parameter[3] that expresses the intensity of the contest, $\partial v/\partial R > 0$, $\partial v/\partial (r-s) < 0$. If the attacker exerts high effort, he is likely to win the contest which gives high vulnerability. If the defender exerts high effort, he is likely to win the contest which gives low vulnerability. Since the agents have limited budgets, and separation is costly for the defender, there are limits to how high efforts the agents exert.

Analogously to the formulation by [9], [12, page 30] and [24, page 102], the defender has a resource r transformable into two kinds of efforts. The first is separation effort s designed to separate the system into N elements. The second is protective effort r-s aimed at protecting the elements. The defender allocates the same resource r_i for protection of each one of identical elements of type i, and $N_i r_i$ to protect the group of elements of type i. Proportional protection cost is often realistic since more elements take up more space, which means that more protective casing or a larger bunker is required. Analogously, the attacker has a resource R transformable into an attack on the system. The attacker attacks each element of type i with resource R_i, and attacks the group of elements of type i with resource $N_i R_i$. After allocating the attack and the protection resources to elements of type i for $1 \leq i \leq n–1$, the resource remaining to attack and protect each element of type n is

$$r_n = \frac{r - \sum_{i=1}^{n-1} N_i r_i - s}{N_n}, \quad R_n = \frac{R - \sum_{i=1}^{n-1} N_i R_i}{N_n}. \qquad (2)$$

which follow by implication since r and R are fixed. The strategic choice variables r_i and R_i for $1 \le i \le n-1$, and r_n and R_n in (2), jointly provide the connection between effort and resource consumption for each agent. Though separation requires the attacker to spread its attack across a larger number of separated elements, the decreased amount of resources available for the defender to defend each element makes each element more vulnerable, despite the decreased attack effort against each element.

The vulnerability v_i of the group of elements of type i is

$$v_i = \frac{(N_i R_i)^m}{(N_i r_i)^m + (N_i R_i)^m} = \frac{R_i^{\,m}}{r_i^{\,m} + R_i^{\,m}} \tag{3}$$

where the parameter m is the same for all elements of any type. In contrast to equation (1), equation (3) holds when the defender and attacker distribute their efforts unevenly across the N elements. We interpret the resources as normalized resources. Assume that the defender transforms its resource r_i measured in some currency into protective effort x times as efficiently as the attacker transforms its resource R_i measured in the same currency into attack. In order for r_i and R_i to be matched as efforts against each other in the contest success function, the attacker must have x times as much of its resource in the given currency as does the defender. Analogously, assume that the defender transforms its resource r_i measured in some currency into protecting one of the N_i elements y times as efficiently as it transforms its resource r_j measured in the same currency into protecting one of the N_j elements, $i, j = 1,\ldots, n$, $i \neq j$. In order for r_i and r_j to operate in the same calculation, the defender must allocate y times as much of r to constitute r_j compared with constituting r_i. The same normalization applies for transforming r into separation s. This means that although r, r_i, s, R, and R_i, $i = 1,\ldots,n$, may be similar when measured in some currency, they may be different when accounting for the efficiency by which each agent transforms its resource into effort [9, 12, 24].[4]

For N identical elements, the probability that k out of N elements are destroyed by the attacker is $\binom{N}{k} v^k (1-v)^{N-k}$. The damage caused by destruction of k out of N elements is $kg = kG/N$. The expected damage is

$$D = \frac{G}{N}\sum_{k=0}^{N} k \binom{N}{k} v^k (1-v)^{N-k} = Gv \tag{4}$$

This means that the damage is proportional to the cumulative system performance, proportional to the element vulnerability, and independent of the number N of elements. The expected system performance is $G - D = G(1-v)$. If the vulnerability is 0, the damage is 0, and the expected performance is G. If the vulnerability is 1, the damage is G, and the expected performance is 0.

Proposition 1: Separation of homogeneous elements with even allocation of the defender's effort is not beneficial for the defender.

Proof: Separation only increases the expected damage, with no added benefit. The proof is a special case of the proof of Proposition 2, shown below.

Without separation the expected damage of the system is

$$D = vG = \frac{R^m}{r^m + R^m}\sum_{i=1}^{n} N_i g_i . \tag{5}$$

The probability that k_i out of N_i elements are destroyed by the attacker is $\binom{N_i}{k_i} v_i^{k_i} (1-v_i)^{N_i-k_i}$. The joint probability across the n sets of elements is the product where i runs from 1 to n. The damage caused by destruction of k_i out of N_i elements is $\sum_{i=1}^{n} k_i g_i$. The expected damage is

$$D=\sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2}\cdots\sum_{k_n=0}^{N_n}\sum_{i=1}^{n}k_i g_i\prod_{j=1}^{n}\binom{N_j}{k_j}v_j^{k_j}(1-v_j)^{N_j-k_j}=\sum_{i=1}^{n}v_i N_i g_i=\sum_{i=1}^{n}\frac{R_i^m}{R_i^m+r_i^m}N_i g_i \tag{6}$$

The logic of (6) is that we determine the vulnerability v_i of an element of type i, and multiply it with the performance $N_i g_i$ of all N_i elements of type i. The expected damage follows from determining and summing this expression across all n groups.

Proposition 2. *If the ratio between the defender's and attacker's resources allocated to protect and attack an element of type i is the same for any i and equals the total resource minus separation cost, divided by the attacker's total resource,* $r_i/R_i=(r-s)/R$, $i=1,\ldots,n$, *then the vulnerability is the same for all elements of all types, separation is not beneficial, and the expected damage is*

$$D=\frac{R^m}{(r-s)^m+R^m}\sum_{i=1}^{n}N_i g_i \tag{7}$$

Proof: Inserting $r_i/R_i=(r-s)/R$ into (3) gives $v_i=R^m/[(r-s)^m+R^m]$. Inserting into (6) gives (7). Separation is not beneficial since the expression for D in (7) with separation is always larger than the expression for D without separation in (5) for any $s>0$.

Since D in (7) is always larger than D in (5), separation is not beneficial when $r_i/R_i=(r-s)/R$, $i=1,\ldots,n$. Two examples of Proposition 2 are resource allocation proportional to the sizes of the groups of identical elements, i.e. $r_i=(r-s)N_i\Big/\sum_{i=1}^{n}N_i$ and $R_i=RN_i\Big/\sum_{i=1}^{n}N_i$, and resource allocation proportional to the cumulative performance of the groups of identical elements, i.e. $r_i=(r-s)\;N_i g_i\Big/\sum_{i=1}^{n}N_i g_i$ and $=R_i=\;RN_i g_i\Big/\sum_{i=1}^{n}N_i g_i$.

Clearly, separation is not always ineffective. For example, inserting $n=2$, $N_1=N_2=m=r=R=1$, $r_1=0.98$, $r_2=0.01$, $s=0.01$, $R_1=0.01$, $R_2=0.99$, $g_1=2$, $g_2=1$ gives the expected damage $D=1.0102$ with separation in (7). Without separation in (5) the expected damage is $D=1.5$. Hence with these assumptions the defender prefers to separate. The reason for this result is that the defender protects the high performing elements heavily, while the attacker attacks the low performing elements. Although such an example may sound unlikely, it may occur in practice when the defender and attacker are incompletely informed about each other.

Comparing (6) and (5), separation is efficient if

$$\frac{R^m}{r^m+R^m}\sum_{i=1}^{n}N_i g_i>\sum_{i=1}^{n}\frac{R_i^{\,m}}{r_i^{\,m}+R_i^{\,m}}N_i g_i\quad\Rightarrow\quad\sum_{i=1}^{n}\left(\frac{R^m}{r^m+R^m}-\frac{R_i^{\,m}}{r_i^{\,m}+R_i^{\,m}}\right)N_i g_i>0 \tag{8}$$

For $n=2$ types of elements with sizes N_1 and N_2, (8) can be written as

$$\left(\frac{R^m}{r^m+R^m}-\frac{R_1^{\,m}}{r_1^{\,m}+R_1^{\,m}}\right)N_1 g_1+\left(\frac{R^m}{r^m+R^m}-\frac{(R-N_1R_1)^m}{(r-N_1r_1-s)^m+(R-N_1R_1)^m}\right)N_2 g_2>0 \tag{9}$$

Solving with respect to s gives

$$s<r-N_1r_1-\left(\frac{1}{\left(\frac{R^m}{r^m+R^m}-\frac{R_1^{\,m}}{r_1^{\,m}+R_1^{\,m}}\right)\frac{N_1g_1}{N_2g_2}+\frac{R^m}{r^m+R^m}}-1\right)^{\frac{1}{m}}(R-N_1R_1)=\bar{s} \tag{10}$$

where $\bar{s}$ is the upper acceptable separation cost which justifies separation. In order for the defender to justify the separation cost, he must furnish a contest success on each element that in an overall sense is sufficiently larger than the contest success of the attacker. The defender's contest success may be lower on some elements than on other elements, but it must be sufficiently higher on at least some elements. The inequality (10) is somewhat hard to interpret. One simplification is to assume that the defender is somehow able to maintain its contest success against the attacker for all elements of type 1. This means that $r_1/R_1 = r/R$. However, inserting this gives $s < 0$, as expected, which means that separation is never efficient.

Proposition 3: With two different types of elements, the defender can never justify separation by maintaining its contest success against the attacker for one type of elements. To possibly justify separation, the contest success must be decreased against one type and increased against the other type.

Proof: Follows from inserting $r_1/R_1 = r/R$ into (10) which implies $s < 0$ which means that no separation cost can be incurred to defend elements of type 2.

Proposition 3 implies that the defender must choose lower contest success for one type of elements and higher contest success for the other type of elements. To make a clean study of this, assume that $r_1 = 0$ which means that the defender chooses not to protect any of the elements of type 1.

Proposition 4: Assume two types of elements and that the defender chooses not to protect any of the elements of type 1, $r_1 = 0$. The defender prefers to incur the cost s of separation when

$$s < r - \left(\frac{1 + \frac{N_1 g_1}{N_2 g_2}}{\left(\frac{R}{r}\right)^m - \frac{N_1 g_1}{N_2 g_2}} \right)^{\frac{1}{m}} (R - N_1 R_1) = \bar{s} \tag{11}$$

which is satisfied when N_1 is large or R_1 is large, regardless of r, R, m, or when (R/r^m) is moderately larger than $N_1 g_1/N_2 g_2$ for appropriate parameter values.

Proof: Equation (11) follows from inserting $r_1 = 0$ into (10). First, when $N_1 R_1$ is large, regardless of r, R, m, the term multiplied with the ratio in (11) is small which makes the RHS in (11) positive and somewhat smaller than r, which justifies s up to this value. Second, $(R/r)^m = N_1 g_1/N_2 g_2$ causes division with zero in the ratio which causes the RHS to be negative. Third, $\lim_{R\to\infty} \bar{s} = \left(1 - \left(1 + \frac{N_1 g_1}{N_2 g_2} \right)^{\frac{1}{m}} \right) r < 0$ is also negative. But, for example $R = r$ and $N_1 g_1/N_2 g_2 = 1/2$ gives $\bar{s} = r(1 - 3^{1/m}) + 3^{1/m} N_1 R_1$ which is positive for many parameter values. One example is $m = r = N_1 = R_1 = 1$ which gives $\bar{s} = 1$.

Proposition 4 shows that the defender can justify the separation cost when many elements N_1 are not defended, without the attacker knowing this so that R_1 is chosen high. This is possible only when the attacker is incompletely informed.

Assumption 2: The attacker has complete information about the defender's effort distribution and can choose any distribution of his effort across the elements. The defender has no information about the attacker's effort distribution and assumes that the attacker for any defender's strategy can respond with resource distribution that maximizes the damage to the system.

This assumption is realistic in a two period game where the defender chooses strategy in the first period, the attacker chooses strategy in the second period, and the attacker is able to direct the attack exactly against certain targets. If the attacker is completely informed about the defender not defending the N_1 elements, he benefits from investing an arbitrarily small but positive amount R_1 to attack each of the N_1 elements.

Proposition 5: Assume that the defender chooses not to protect any of the elements of type 1, $r_1 = 0$. If the attacker knows this, he invests an arbitrarily small but positive amount $R_1 = 0^+$ into attacking each of the N_1 elements. The consequence is that the defender cannot justify the separation cost.

Proof: Inserting $R_1 = 0$ into (11) gives

$$s < \left(1 - \left(\frac{\left(\frac{R}{r}\right)^m \left(1 + \frac{N_1 g_1}{N_2 g_2}\right)}{\left(\frac{R}{r}\right)^m - \frac{N_1 g_1}{N_2 g_2}}\right)^{\frac{1}{m}}\right) r = \overline{s} \tag{12}$$

The numerator in the ratio is always larger than the denominator and hence is negative.

3. UTILITY MAXIMIZATION WITH RESOURCE CONSTRAINTS

This section also assumes fixed resources r and R for the defender and attacker.

Assumption 3: The defender and attacker make their strategic choices simultaneously and independently. Both agents can choose any distribution of their effort across the elements.

Assumption 3 applies for the remainder of the article, and means that the defender and attacker play a simultaneous game with each other. This means that neither agent knows the other agent's strategy before choosing its own strategy. The defender does not know how the attacker distributes its resource across the n types. The attacker knows that the system exists and has N functionally identical parallel elements of n types, but does not know how the defender distributes its resource across the types and how much of the resource is distributed to separation. The defender seeks to minimize his expenses caused by investments into the defense and the expected damage, that is

$$D + r = \sum_{i=1}^{n} v_i N_i g_i + r = G - \sum_{i=1}^{n} (1 - v_i) N_i g_i + r\,. \tag{13}$$

Since $G = \sum_{i=1}^{n} N_i g_i$ is constant, minimizing $D + r$ is equivalent to maximizing the defender's utility

$$u = \sum_{i=1}^{n} (1 - v_i) N_i g_i - r \tag{14}$$

In contrast, the attacker seeks to maximize the expected damage. Hence we model the attacker's utility as

$$U = D - R = \sum_{i=1}^{n} v_i N_i g_i - R \tag{15}$$

The defender has n–1 strategic decision variables $r_1, \ldots, r_{n-1}$. The attacker has n–1 strategic decision variables $R_1, \ldots, R_{n-1}$. For the defender, r_n follows from (2) when r is fixed, and is thus not a strategic decision variable. Analogously, for the attacker, R_n follows from (2) when R is fixed, and is thus not a strategic decision variable. The two agents choose their strategic decision variables independently and simultaneously. In order to differentiate with respect to r_1 and R_1 for the first type of elements, we write the utilities in (14) and (15) as

$$u = \frac{r_1^{\,m}}{r_1^{\,m} + R_1^{\,m}} N_1 g_1 + \sum_{i=2}^{n-1} (1 - v_i) N_i g_i + \frac{\left(r - N_1 r_1 - \sum_{i=2}^{n-1} N_i r_i - s\right)^m}{\left(r - N_1 r_1 - \sum_{i=2}^{n-1} N_i r_i - s\right)^m + \left(R - N_1 R_1 \sum_{i=2}^{n-1} N_i R_i\right)^m} N_n g_n - r \tag{16}$$

$$U = \frac{R_1^{\ m}}{r_1^{\ m} + R_1^{\ m}} N_1 g_1 + \sum_{i=2}^{n-1} v_i N_i g_i + \frac{\left(R - N_1 R_1 - \sum_{i=2}^{n-1} N_i R_i\right)^m}{\left(r - N_1 r_1 - \sum_{i=2}^{n-1} N_i r_i - s\right)^m + \left(R - N_1 R_1 \sum_{i=2}^{n-1} N_i R_i\right)^m} N_n g_n - R \qquad (17)$$

The first order conditions for the set of elements of type 1 are

$$\frac{\partial u}{\partial r_1} = mN_1 \left(\frac{g_1 r_1^{m-1} R_1^m}{(r_1^m + R_1^m)^2} - \frac{g_n \left(\frac{r - N_1 r_1 - \sum_{i=2}^{n-1} N_i r_i - s}{N_n}\right)^{m-1} \left(\frac{R - N_1 R_1 \sum_{i=2}^{n-1} N_i R_i}{N_n}\right)^m}{\left[\left(\frac{r - N_1 r_1 - \sum_{i=2}^{n-1} N_i r_i - s}{N_n}\right)^m + \left(\frac{R - N_1 R_1 \sum_{i=2}^{n-1} N_i R_i}{N_n}\right)^m\right]^2} \right) = 0 \qquad (18)$$

and

$$\frac{\partial U}{\partial R_1} = mN_1 \left(\frac{g_1 R_1^{m-1} r_1^m}{(r_1^m + R_1^m)^2} - \frac{g_n \left(\frac{r - N_1 r_1 - \sum_{i=2}^{n-1} N_i r_i - s}{N_n}\right)^{m} \left(\frac{R - N_1 R_1 \sum_{i=2}^{n-1} N_i R_i}{N_n}\right)^{m-1}}{\left[\left(\frac{r - N_1 r_1 - \sum_{i=2}^{n-1} N_i r_i - s}{N_n}\right)^m + \left(\frac{R - N_1 R_1 \sum_{i=2}^{n-1} N_i R_i}{N_n}\right)^m\right]^2} \right) = 0 \qquad (19)$$

The second order conditions for the set of elements of type 1 are

$$\frac{\partial^2 u}{\partial r_1^2} = -mN_1 \left(\frac{g_1 r_1^{m-2} R_1^{2m} \left((1+m) \frac{r_1^m}{R_1^m} + (1-m) \right)}{(r_1^m + R_1^m)^3} \right.$$

$$+\frac{g_n N_1 N_n\left(r-N_1 r_1-\sum_{i=2}^{n-1} N_i r_i-s\right)^{m-2}\left(R-N_1 R_1-\sum_{i=2}^{n-1} N_i R_i\right)^{2m}\left((1+m)\frac{\left(r-N_1 r_1-\sum_{i=2}^{n-1} N_i r_i-s\right)^m}{\left(R-N_1 R_1-\sum_{i=2}^{n-1} N_i R_i\right)^m}-(1-m)\right)}{\left[\left(r-N_1 r_1-\sum_{i=2}^{n-1} N_i r_i-s\right)^m+\left(R-N_1 R_1-\sum_{i=2}^{n-1} N_i R_i\right)^m\right]^3}\Bigg) \quad (20)$$

and

$$\frac{\partial^2 U}{\partial R_1^2} = -mN_1\left(\frac{g_1 r_1^m R_1^{2m-2}\left((1+m)+(1-m)\frac{r_1^m}{R_1^m}\right)}{(r_1^m+R_1^m)^3}\right.$$
$$\left.+\frac{g_n N_1 N_n\left(r-N_1 r_1-\sum_{i=2}^{n-1} N_i r_i-s\right)^m\left(R-N_1 R_1-\sum_{i=2}^{n-1} N_i R_i\right)^{2m-2}\left((1+m)-(1-m)\frac{\left(r-N_1 r_1-\sum_{i=2}^{n-1} N_i r_i-s\right)^m}{\left(R-N_1 R_1-\sum_{i=2}^{n-1} N_i R_i\right)^m}\right)}{\left[\left(r-N_1 r_1-\sum_{i=2}^{n-1} N_i r_i-s\right)^m+\left(R-N_1 R_1-\sum_{i=2}^{n-1} N_i R_i\right)^m\right]^3}\right) \quad (21)$$

Observe the joint occurrence of the same kinds of terms in the two expressions in (18) and (19). Careful solution of the two equations gives

$$r_1=\frac{R_1(r-\sum_{i=2}^{n-1} N_i r_i - s)}{R-\sum_{i=2}^{n-1} N_i R_i}=\frac{R_1(N_1 r_1+N_n r_n)}{N_1 R_1+N_n R_n} \quad\Rightarrow\quad r_1=\frac{R_1}{R_n} r_n \quad (22)$$

Inserting (22) into the first order condition for the defender or attacker and solving gives

$$R_1=\frac{g_1(R-\sum_{i=2}^{n-1} N_i R_i)}{N_1 g_1+N_n g_n} \quad (23)$$

Rearranging terms gives

$$\frac{R_1}{g_1}=\frac{R-\sum_{i=1}^{n-1} N_i R_i}{N_n g_n}=\frac{N_n R_n}{N_n g_n} \quad\Rightarrow\quad R_n=\frac{g_n}{g_1}R_1 \quad (24)$$

Performing the same analysis as above for the second type of elements analogously gives $R_n = g_n R_2/g_2$. Hence we generally have

$$R_i = \frac{g_i}{g_1} R_1 \tag{25}$$

Inserting (25) into (23), simplifying, and generalizing from type 1 to type *i*, gives

$$R_i = \frac{g_i R}{\sum_{i=1}^{n} N_i g_i} \tag{26}$$

Inserting into (22) gives $r_1 = g_1 r_n/g_n$. Hence generally

$$r_i = \frac{g_i}{g_1} r_1 \tag{27}$$

Inserting into the definition of r and solving with respect to r_1 gives

$$r = \sum_{i=1}^{n} N_i r_i + s = \frac{r_1}{g_1} \sum_{i=1}^{n} N_i g_i + s \quad \Rightarrow \quad r_1 = \frac{g_1 (r-s)}{\sum_{i=1}^{n} N_i g_i} \tag{28}$$

Generalizing from type 1 to type *i* gives

$$r_i = \frac{g_i (r-s)}{\sum_{i=1}^{n} N_i g_i} = \frac{r-s}{R} R_i \tag{29}$$

Inserting (26) and (29) into the second order conditions in (20) and (21) gives

$$\frac{\partial^2 u}{\partial r_1^2} = -m N_1 \frac{(r-s)^{m-2} R^{2m} \left((1+m) \frac{(r-s)^m}{R^m} + (1-m) \right)}{((r-s)^m + R^m)^3} \left(\frac{1}{g_1 \left(\sum_{i=1}^{n} N_i g_i \right)^2} + \frac{g_n N_1 N_n}{\left(1 - \frac{\sum_{i=1}^{n-1} N_i g_i}{\sum_{i=1}^{n} N_i g_i} \right)^2} \right) \tag{30}$$

and

$$\frac{\partial^2 U}{\partial R_1^2} = -m N_1 \frac{(r-s)^{m} R^{2m-2} \left((1+m) + (1-m) \frac{(r-s)^m}{R^m} \right)}{((r-s)^m + R^m)^3} \left(\frac{1}{g_1 \left(\sum_{i=1}^{n} N_i g_i \right)^2} + \frac{g_n N_1 N_n}{\left(1 - \frac{\sum_{i=1}^{n-1} N_i g_i}{\sum_{i=1}^{n} N_i g_i} \right)^2} \right) \tag{31}$$

Generalizing from type 1 to type i gives

$$\frac{\partial^2 u}{\partial r_i^2} = -mN_i \frac{(r-s)^{m-2} R^{2m}\left((1+m)\frac{(r-s)^m}{R^m} + (1-m)\right)}{((r-s)^m + R^m)^3} \left(\frac{1}{g_i\left(\sum_{i=1}^{n} N_i g_i\right)^2} + \frac{g_n N_i N_n}{\left(1 - \frac{\sum_{i=1}^{n-1} N_i g_i}{\sum_{i=1}^{n} N_i g_i}\right)^2} \right) \tag{32}$$

and

$$\frac{\partial^2 U}{\partial R_i^2} = -mN_i \frac{(r-s)^{m} R^{2m-2}\left((1+m) + (1-m)\frac{(r-s)^m}{R^m}\right)}{((r-s)^m + R^m)^3} \left(\frac{1}{g_i\left(\sum_{i=1}^{n} N_i g_i\right)^2} + \frac{g_n N_i N_n}{\left(1 - \frac{\sum_{i=1}^{n-1} N_i g_i}{\sum_{i=1}^{n} N_i g_i}\right)^2} \right) \tag{33}$$

which are satisfied as negative when

$$\frac{m-1}{m+1} < \frac{(r-s)^m}{R^m} < \frac{m+1}{m-1} \tag{34}$$

Inserting (26) and (29) into (3), the vulnerability v_i of type i is

$$v_i = \frac{R^m}{(r-s)^m + R^m} \tag{35}$$

Proposition 6. With resource constraints, resource investments of both defender and attacker for a given type of elements are proportional to the performance of that type. The defender and attacker allocate their resources so that the vulnerability of each type is the same. The vulnerability increases in the contest intensity m.

Proof: Follows from equations (26), (28), (35).

Inserting (35) into (14) and (15), the utilities are

$$u = \frac{(r-s)^m}{(r-s)^m + R^m}\sum_{i=1}^{n} N_i g_i - r, \quad U = \frac{R^m}{(r-s)^m + R^m}\sum_{i=1}^{n} N_i g_i - R, \quad \frac{u+r}{U+R} = \frac{(r-s)^m}{R^m} \tag{36}$$

The attacker's utility in (36) reveals that the same expected damage as in (7) applies, which implies the following proposition.

Proposition 7. With resource constraints, separation is not beneficial.

Proof: Follows from comparing (36) and (5) which gives equal expected damage with and without separation when $s = 0$.

4. UTILITY MAXIMIZATION WITHOUT RESOURCE CONSTRAINTS AND WITHOUT SEPARATION

Assume that the defender and attacker have unlimited resources. The defender incurs an effort t, which is the defender's strategic choice variable, at unit cost a of protecting the system. The attacker incurs an effort T, which is the attacker's strategic choice variable, at unit cost A of attacking the system. The vulnerability v of the system is

$$v = \frac{T^m}{T^m + t^m} \tag{37}$$

The expected damage of the system is

$$D = vG = v\sum_{i=1}^{n} N_i g_i \tag{38}$$

where we use the same performance definition $G = \sum_{i=1}^{n} N_i g_i$ as in the previous section to allow for comparison. The defender seeks to maximize the utility

$$u = (1-v)\sum_{i=1}^{n} N_i g_i - at \tag{39}$$

The attacker's utility is

$$U = v\sum_{i=1}^{n} N_i g_i - AT \tag{40}$$

The first order conditions are

$$\frac{\partial u}{\partial t} = \frac{mt^{m-1}T^m}{(t^m + T^m)^2}\sum_{i=1}^{n} N_i g_i - a = 0, \quad \frac{\partial U}{\partial T} = \frac{mT^{m-1}t^m}{(t^m + T^m)^2}\sum_{i=1}^{n} N_i g_i - A = 0 \tag{41}$$

which are solved to yield

$$t = \frac{AT}{a} = \frac{ma^{m-1}A^m}{(a^m + A^m)^2}\sum_{i=1}^{n} N_i g_i, \quad T = \frac{mA^{m-1}a^m}{(a^m + A^m)^2}\sum_{i=1}^{n} N_i g_i, \quad v = \frac{a^m}{a^m + A^m} \tag{42}$$

Inserting $t = AT/a$ into the second order conditions gives

$$\frac{\partial^2 u}{\partial t^2} = -\frac{mt^{m-2}T^m\left((1+m)t^m + (1-m)T^m\right)}{(t^m + T^m)^3}\sum_{i=1}^{n} N_i g_i = -\frac{mt^{m-2}T^{2m}\left((1+m)\frac{A^m}{a^m} + (1-m)\right)}{(t^m + T^m)^3}\sum_{i=1}^{n} N_i g_i,$$

$$\frac{\partial^2 U}{\partial T^2} = -\frac{mT^{m-2}t^m\left((1+m)T^m + (1-m)t^m\right)}{(t^m + T^m)^3}\sum_{i=1}^{n} N_i g_i = -\frac{mT^{2m-2}t^m\left((1+m) + (1-m)\frac{A^m}{a^m}\right)}{(t^m + T^m)^3}\sum_{i=1}^{n} N_i g_i \tag{43}$$

which are satisfied as negative when

$$\frac{m-1}{m+1} < \frac{A^m}{a^m} < \frac{m+1}{m-1} \tag{44}$$

Inserting (42) into (39) and (40) gives the

$$u=\frac{A^m[A^m+a^m(1-m)]}{(a^m+A^m)^2}\sum_{i=1}^{n}N_ig_i,\quad U=\frac{a^m[a^m+A^m(1-m)]}{(a^m+A^m)^2}\sum_{i=1}^{n}N_ig_i \tag{45}$$

Proposition 8: Without resource constraints and without separation, the defender withdraws from the contest over the system when $m > 1 + (A/a)^m$. The attacker withdraws when $m > 1 + (a/A)^m$. If both these are satisfied, the agent with the highest unit cost of effort, a or A, withdraws. The agent that withdraws earns zero utility, while the other agent earns utility G and exerts arbitrarily small but positive effort. If $m > 2$ and $a = A$, the agents are tied at effort $t = T = G/2$ and utilities $u = U = 0$.

Proof: Follows from (45) when $u < 0$ or $U < 0$.

To illustrate that an agent withdraws from contests over elements that contribute to negative utility, which is only possible when the contest intensity m is large, we can rewrite (45) as

$$u=\frac{A^m\max\{0,A^m+a^m(1-m)\}}{(a^m+A^m)^2}\sum_{i=1}^{n}N_ig_i,\quad U=\frac{a^m\max\{0,a^m+A^m(1-m)\}}{(a^m+A^m)^2}\sum_{i=1}^{n}N_ig_i \tag{46}$$

Inserting $a = A = 1$ into (37), (42), (45) gives

$$t=T=\frac{m}{4}\sum_{i=1}^{n}N_ig_i,\quad v=\frac{1}{2},\quad u=U=\frac{(2-m)}{4}\sum_{i=1}^{n}N_ig_i \tag{47}$$

5. UTILITY MAXIMIZATION WITHOUT RESOURCE CONSTRAINTS AND WITH SEPARATION

Assume that the defender and attacker have unlimited resources, and separate as in section 3. The total system performance is $G=\sum_{i=1}^{n}N_ig_i$. The defender incurs an effort t_i (defender's strategic choice variable) at unit cost a_i of protecting each element of type i, and incurs s as separation cost. The linkage between this section and the previous one is that $t=\sum_{i=1}^{n}N_it_i$, which sums up the protection across all elements to get the protection of the system. The attacker incurs an effort T_i (attacker's strategic choice variable) at unit cost A_i of attacking each element of type i, where $T=\sum_{i=1}^{n}N_iT_i$. Different unit costs a_i and A_i for different types of elements reflect different vulnerabilities for different types of elements. Note that a_i and a, and A_i and A, are often comparable in size since these are unit costs, and not actual costs which are reflected by t_i and T_i for elements and t and T for the system. The vulnerability v_i of the group of elements of type i is

$$v_i=\frac{(N_iT_i)^m}{(N_iT_i)^m+(N_it_i)^m}=\frac{T_i^m}{T_i^m+t_i^m} \tag{48}$$

The expected damage has the same expression as in section 3, i.e.

$$D=\sum_{i=1}^{n}v_iN_ig_i \tag{49}$$

but v_i is now modeled differently. The defender seeks to maximize the utility

$$u=\sum_{i=1}^{n}((1-v_i)N_ig_i-a_iN_it_i)-s \tag{50}$$

The attacker's utility is

$$U = \sum_{i=1}^{n} (v_i N_i g_i - A_i N_i T_i) \tag{51}$$

The first order conditions for the set of elements of type i are

$$\frac{\partial u}{\partial t_i} = \frac{m N_i g_i t_i^{m-1} T_i^m}{(t_i^m + T_i^m)^2} - a_i N_i = 0, \quad \frac{\partial U}{\partial T_i} = \frac{m N_i g_i T_i^{m-1} t_i^m}{(t_i^m + T_i^m)^2} - A_i N_i = 0 \tag{52}$$

which are solved to yield

$$t_i = \frac{A_i T_i}{a_i} = \frac{m g_i a_i^{m-1} A_i^m}{(a_i^m + A_i^m)^2}, \quad T_i = \frac{m g_i A_i^{m-1} a_i^m}{(a_i^m + A_i^m)^2}, \quad v_i = \frac{a_i^m}{a_i^m + A_i^m} \tag{53}$$

Inserting $t_i = A_i T_i / a_i$ into the second order conditions gives

$$\frac{\partial^2 u}{\partial t_i^2} = -\frac{m N_i g_i t_i^{m-2} T_i^m \left((1+m) t_i^m + (1-m) T_i^m\right)}{(t_i^m + T_i^m)^3} = -\frac{m N_i g_i t_i^{m-2} T_i^{2m} \left((1+m) \frac{A_i^m}{a_i^m} + (1-m)\right)}{(t_i^m + T_i^m)^3},$$
$$\frac{\partial^2 U}{\partial T_i^2} = -\frac{m N_i g_i T_i^{m-2} t_i^m \left((1+m) T_i^m + (1-m) t_i^m\right)}{(t_i^m + T_i^m)^3} = -\frac{m N_i g_i T_i^{2m-2} t_i^m \left((1+m) + (1-m) \frac{A_i^m}{a_i^m}\right)}{(t_i^m + T_i^m)^3} \tag{54}$$

which are satisfied as negative when

$$\frac{m-1}{m+1} < \frac{A_i^m}{a_i^m} < \frac{m+1}{m-1} \tag{55}$$

Inserting (53) into (50) and (51) gives the utilities

$$u = \sum_{i=1}^{n} \frac{A_i^m [A_i^m + a_i^m (1-m)] N_i g_i}{(a_i^m + A_i^m)^2} - s, \quad U = \sum_{i=1}^{n} \frac{a_i^m [a_i^m + A_i^m (1-m)] N_i g_i}{(a_i^m + A_i^m)^2} \tag{56}$$

Proposition 9. Without resource constraints, the defender withdraws from the contest over element of type i when $m > 1 + (A_i/a_i)^m$. The attacker withdraws from the contest over element of type i when $m > 1 + (a_i/A_i)^m$. If both these are satisfied, the agent with the highest unit cost of effort, a_i or A_i, withdraws. The agent that withdraws earns zero utility for that element, while the other agent earns utility $N_i g_i$ for that element and exerts arbitrarily small but positive effort for that element. If $m>2$ and $a_i = A_i$, the agents are tied at effort $t=T=N_i g_i/2$ and earns zero utilities over element i. Additionally, when the separation cost s is so large that the utility u is negative, the defender withdraws from all contests and does not separate.

Proof: Follows from (56) when $u < 0$ or $U < 0$.

Observe the similarity between Propositions 8 and 9. Proposition 8 assumes no separation and is valid for the system as a whole. In contrast, Proposition 9 assumes separation, and is valid for each element, as well as for the system as a whole. To illustrate that an agent withdraws from contests over elements that contribute to negative utility, which is only possible when the contest intensity m is large, we can rewrite (56) as

$$u = \sum_{i=1}^{n} \frac{A_i^m N_i g_i \max\{0, A_i^m + a_i^m (1-m)\}}{(a_i^m + A_i^m)^2} - s, \quad U = \sum_{i=1}^{n} \frac{a_i^m N_i g_i \max\{0, a_i^m + A_i^m (1-m)\}}{(a_i^m + A_i^m)^2} \tag{57}$$

There are many ways in which the defender can divide the overall separation cost s into separation cost s_i for each element of type i, $s=\sum_{i=1}^{n} s_i$. One possibility is $s_i = sN_i\ g_i \Big/ \sum_{i=1}^{n} N_i g_i$, but the cost can also be divided influenced by the parameters a_i, A_i, m. If the defender decides to design such a cost division formula for separation, s_i can be incorporated inside the summation sign in the defender's utility in (56) to determine from which contests the defender prefers to withdraw.

Inserting $a_i = a$ and $A_i = A$ into (56) gives

$$u=\frac{A^m[A^m+a^m(1-m)]}{(a^m+A^m)^2}\sum_{i=1}^{n} N_i g_i - s,\quad U=\frac{a^m[a^m+A^m(1-m)]}{(a^m+A^m)^2}\sum_{i=1}^{n} N_i g_i \tag{58}$$

Inserting $a = A = 1$ gives

$$t_i=T_i=\frac{mg_i}{4},\quad v_i=\frac{1}{2},\quad u=\frac{(2-m)}{4}\sum_{i=1}^{n} N_i g_i - s,\quad U=\frac{(2-m)}{4}\sum_{i=1}^{n} N_i g_i \tag{59}$$

which can be compared with (47). Note in particular how g_i in the expression for $t_i = T_i$ in (59) corresponds to $\sum_{i=1}^{n} N_i g_i$ in the expression for $t = T$ in (47), and how the utilities are equal aside from the separation cost.

6. ASSESSING SEPARATION EFFICIENCY WHEN DEFENDER AND ATTACKER ACT OPTIMALLY WITHOUT RESOURCE CONSTRAINTS

Comparing the defender's utility with separation in (57) with the defender's utility without separation in (46), separation is efficient if

$$\sum_{i=1}^{n}\frac{A_i^m N_i g_i \max\{0,A_i^m+a_i^m(1-m)\}}{(a_i^m+A_i^m)^2}-s>\frac{A^m\max\{0,A^m+a^m(1-m)\}}{(a^m+A^m)^2}\sum_{i=1}^{n} N_i g_i \tag{60}$$

which can be rewritten as

$$s<\sum_{i=1}^{n}\left(\frac{\max\{0,1+(a_i/A_i)^m(1-m)\}}{((a_i/A_i)^m+1)^2}-\frac{\max\{0,1+(a/A)^m(1-m)\}}{((a/A)^m+1)^2}\right)N_i g_i \tag{61}$$

Proposition 10: Assume no resource constraints. (1) When $a_i = a$ and $A_i = A$ for all $i = 1, \ldots, n$, separation is not efficient. (2) Separation is efficient when a_i/A_i is smaller than a/A for sufficiently many of the n types of elements so that (61) is satisfied.

Proof: (1) Inserting $a_i = a$ and $A_i = A$ into (61) gives $s < 0$. (2) Equating the two ratios in (61) gives a second order equation with two roots

$$(a_i/A_i)^m=(a/A)^m \quad \text{or} \quad (a_i/A_i)^m=\frac{(a/A)^m+m+1}{(a/A)^m(m-1)-1}>0 \quad \text{when} \quad m>1+(A/a)^m \tag{62}$$

The second root does not apply since it is positive only when m is so large that the numerator in the second ratio in (61) is negative which unacceptably gives negative utility. Hence the first root applies. The denominator of the first ratio is smaller than the denominator of the second ratio when $a_i/A_i < a/A$. The parenthesis in (61) is multiplied with $N_i g_i$, which means that the inequality is more easily satisfied for those elements where $N_i g_i$ is large.

Proposition 10 implies that the defender is willing to undertake the separation if lower unit costs of defense can be obtained through the separation process for sufficiently many of the elements so that (61) is satisfied. Especially important are elements with high performance (g_i is large), or if there are many of them of the same type (N_i is large). This means that the defender's benefit of lower unit costs of defense for these important elements outweighs the separation cost.

Let us consider an example. First, inserting m = 1 into (61) gives

$$s < \sum_{i=1}^{n} \left(\frac{1}{(a_i / A_i + 1)^2} - \frac{1}{(a / A + 1)^2} \right) N_i g_i \tag{63}$$

Second, inserting n = 3 and a/A = 1 into (63) gives

$$s < \left(\frac{1}{(a_1 / A_1 + 1)^2} - \frac{1}{4} \right) N_1 g_1 + \left(\frac{1}{(a_2 / A_2 + 1)^2} - \frac{1}{4} \right) N_2 g_2 + \left(\frac{1}{(a_3 / A_3 + 1)^2} \quad \frac{1}{4} \right) N_3 g_3 \tag{64}$$

Third, inserting $a_1/A_1 = 1/2$, $a_2/A_2 = 2/3$, $a_2/A_2 = 4/3$ into (64) gives

$$s < \frac{7}{36} N_1 g_1 + \frac{11}{100} N_2 g_2 - \frac{13}{196} N_3 g_3 \tag{65}$$

The inequality is most easily satisfied when $N_1 g_1$ and $N_2 g_2$ are large, and $N_3 g_3$ is small.

7. CONCLUSION

The article considers strategic defense and attack of a system which consists of functionally identical parallel elements of n types with different performance. The elements can be separated and protected. The defender distributes its resource between separation and protecting the elements from outside attacks. Separation leaves fewer resources for protecting each separated element, but prevents simultaneous destruction of all the elements. The protection of elements of the same type is the same. The defender distributes its resource between separation and protecting the elements from outside attacks. The vulnerability of each element is determined by an attacker-defender contest success function.

Ten propositions are developed. The first seven propositions assume resource constraints. First, separation of homogeneous elements with even allocation of the defender's effort is not beneficial for the defender. Second, if the ratio between the defender's and attacker's resources allocated to protect and attack an element of type i equals the total resource minus separation cost, divided by the attacker's total resource, then the vulnerability is the same for all elements of all types, and separation is not beneficial. An example is presented which illustrates that separation can be efficient. The example is such that the defender protects the high performing elements heavily, while the attacker attacks the low performing elements. Third, with two different types of elements, the defender can never justify separation by maintaining its contest success against the attacker for one type of elements. To possibly justify separation, the contest success must be decreased against one type and increased against the other type. Fourth, for the case of two types of elements a condition is developed for the highest separation cost the defender is willing to incur if it chooses not to protect any of the elements of type 1. Fifth, if the attacker knows that the defender chooses not to protect any of the elements of type 1, the attacker invests arbitrarily little to attack type 1 elements. The consequence is that the defender cannot justify the separation cost.

Sixth, introducing utility maximization and assuming resource constraints, the resource investments of both defender and attacker for a given type of elements are proportional to the performance of that type. The defender and attacker allocate their resources so that the vulnerability of each type of elements is the same. The vulnerability increases in the contest intensity m. Seventh, in this case separation is not beneficial.

Eighth, without resource constraints, without separation and when the contest intensity is sufficiently high, the agent with the highest unit cost of effort withdraws from defending/attacking the system, earning zero utility, while the other agent earns maximum utility. Ninth, without resource constraints and with separation, the agent with the highest unit cost of effort for one specific element withdraws from defending/attacking that element. Additionally, when the separation cost is so large that the utility is negative, the defender withdraws from all contests and does not separate. Tenth, without resource constraints, separation is not efficient if the unit costs of defense are equal for all elements, and the unit costs of attack are the same for all elements. However, separation is efficient if lower unit costs of defense can be obtained through the separation process for sufficiently many of the elements, especially those elements with high performance, or if the number of elements of the same type is large.

NOTES

1. See [8] for an analysis where one agent defends each component in a system, [13] and [10] for interdependence between components, and [20], [27], [28], and [22] applying game theory for components in isolation. See [11] for defense and attack of series and parallel systems.
2. The term functionally identical (or functionally equivalent) is used to state that the elements perform the same task sharing the work. The elements can differ by their characteristics (performance, reliability, cost etc.) The examples are: generating system consisting of different generators (they can be even of different nature: coal steam generators, diesel turbines etc.); computer grid consisting of different types of processors sharing a computation task.
3. When $m = 0$, the agents' efforts have equal impact on the vulnerability regardless of their size which gives 50% vulnerability. $0 < m < 1$ gives a disproportional advantage of investing less than one's opponent. When $m = 1$, the investments have proportional impact on the vulnerability. $m > 1$ gives a disproportional advantage of investing more effort than one's opponent (economies of scale). Finally, $m = \infty$ gives a step function where "winner-takes-all".
4. See [9], [12] and [24] for examples of modeling different efficiencies of transforming resources into effort.

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