Whether to attack a terrorist's resource stock today or tomorrow

Kjell Hausken

Abstract

A terrorist defends an asset which grows from the first to the second period and is attacked. With large asset growth, the terrorist's first period outcome is low caused by a large attack. With no expenditure constraint, the terrorist's total outcome is positive. With equal attack and defense inefficiencies, when the terrorist as defender requires his first period outcome to be positive, the attacker eliminates the asset in the first period when asset growth multiplied with the square roots of the terrorist's and attacker's discount parameters exceed $4/\sqrt{3}$. This gives maximum conflict in the first period. Growth and the two discount parameters are strategic complements. The range for the attack inefficiency divided by defense inefficiency, which causes negative first period outcome for the terrorist, increases with asset growth. The attacker refrains from asset elimination in the first period due to strength (weakness) if the ratio is below (above) the range.

Keywords: Terrorism; Terror capacity; Threat; Conflict; Two periods; Dynamics; Growth; Discounting; Defense inefficiency; Attack inefficiency; Contest success function; Time substitution

---

1. Introduction

After Hezbollah's capture of two Israeli soldiers on July 12, 2006, Israel launched attacks with the stated objective of eliminating Hezbollah presence between Israel's border with Lebanon and the Litani River, 18–25 miles north of the border. Hezbollah, founded in 1982 as an umbrella organization of various radical Islamic Shi'ite groups, earned two ministers in the government and 14 seats in the 128-member Lebanese Parliament after the 2005 election. Israel believes Hezbollah earns financial, military, and political support from Syria and Iran.

In addition to po-litical activities, Hezbollah is estimated to have a few thousand militants and activists. It applies guerilla warfare, operates military capacity such as artillery from caves, tunnels and basements in southern Lebanon, and has intermingled within the infrastructure of around 300,000 Lebanese civilians who live south of the Litani River. Israel withdrew from Lebanon in 2000, and has during 2000–2006 assessed Hezbollah's military buildup during these six years.

This case matches the phenomenon analyzed in this article. Hezbollah as the defender possesses an asset perceived as a threat by Israel. Israel as the attacker assesses whether to attack the asset today or tomorrow, that is, in the first period, or the second period, or in both periods. The asset grows from the first to the second period. The attacker is not interested in the asset, but seeks to eliminate it. The defender chooses the optimal defense in both periods. Results are generated dependent on the growth rate, different discount parameters for the defender and attacker, the defense inefficiency, and the attack inefficiency. A conventional static analysis considers the tradeoffs each actor makes between how much to invest to defend or attack the asset at a given point in time, dependent on the opponent's investment. This article additionally analyzes the tradeoffs each actor makes concerning investments through time. Each actor's investment in each time period accounts for the same actor's investment in the other time period, and for the other actor's investments in both periods. The tradeoffs are crucially different for the defender and attacker in the two periods, and depend on the five parameters in sometimes unexpected ways.

Conflicts are over allocation of assets but also over avoidance of threats. A terrorist possessing an asset such as a resource stock constitutes a threat which other actors may not willingly accept. Furthermore, if the terrorist's asset has growth potential through time, the threat may increase. The assessment of other actors is whether to ignore or downgrade or eliminate the threat today, or whether to choose a similar or alternative strategy in the future. Although static conflict theory is well researched, dynamic conflict theory is analytically challenging (Hirshleifer, 1995, p. 31). This article contributes to the dynamics of conflict. A defender who may be a terrorist is equipped with an asset which constitutes a threat to others. An attacker chooses a strategy for how to attack the asset through time.

Contributions to the dynamics of conflict are rare, but a few examples exist. Applying intertemporal optimization without strategic interaction, Sethi (1979) analyzes a continuous dynamic pilfering thief. Reuveny and Maxwell (2001) consider two rival groups, each dependent on a single contested renewable resource. They develop differential time equations where groups continuously allocate their members between resource harvesting and resource appropriation. They seek to maximize their income. This gives nonlinear dynamic interaction between conflict, the two populations, and the resource. Hausken (2005a) generalizes the model to account for within-group collective action, different resource stocks and efficiencies of harvesting for the two groups, and variable decisiveness of fighting between the two groups. Maxwell and Reuveny (2005) further investigate continuous conflict over renewable natural resources. They find that Hirshleifer's (1991) "paradox of power" is self-correcting, and that if production causes damage to disputed resources, introducing conflict enhances social welfare.

For game theory more generally, Fudenberg and Maskin (1986) have shown that "any individually rational outcome can arise as a Nash equilibrium in infinitely repeated games with sufficiently little discounting."[1] Their result has often been used to show that cooperation rather than conflict can be sustained in long-term relationships. The prisoner's dilemma has often been used for illustration (Axelrod, 1984). For the battle of the sexes where one player values the future while the other is myopic, Hausken (2005b) shows that the first player is more inclined through conflictful behavior to risk a conflict in the present when the future is important. Similarly, Skaperdas (1996) show how increased importance of the future may harm cooperation. These mixed results make it quite appropriate to analyze whether a two period growth model suppresses or amplifies conflict, possibly differently in the two periods.

Whether growth of a terrorist's asset deters or encourages attack has to the author's knowledge not been considered in the literature. The literature on economic growth is by its nature dynamic, but has not been explicitly linked to conflict. The linkage has instead been made to income distribution, human capital, fertility, trade development, money, etc. One example of a focus within this literature, provided by Benhabib and Rustichini (1996), is how the level of wealth and the degree of inequality affects growth.

The attacker's decision in an intertemporal conflict model can also be conceptualized as how to make investment substitutions across time. A large attack in the first period is detrimental to asset growth and reduces the opportunities for the downgraded defender in the second period. Conversely, a too modest attack in the first period may also be suboptimal since the defender may enjoy a too advantaged position in the second period, and constitute a substantial threat to others. Although substitutions across targets have been analyzed (Hausken, 2006), intertemporal substitutions against the same target have hardly been analyzed. There are a few examples with a somewhat different focus. First, Enders and Sandler (2003) suggest that a terrorist may compile and accumulate resources during times when the defender's investments are high, awaiting times when the defender may relax his efforts and choose lower investments. Second, in preventing terrorism, Keohane and Zeckhauser (2003, pp. 201, 224) show that "the optimal control of terror stocks will rely on both ongoing abatement and periodic cleanup" of "a terrorist's 'stock of terror capacity'." Their work is influenced by Arrow et al.'s (1951) and Scarf's (1960) (S, s) model of inventory management. The optimal policy in the face of stochastic demand for a product is to replenish inventory up to a level given by S every time it falls to or below s. These contributions do not consider a strategic opponent.

Section 2 develops the two-period game. Section 3 solves the game. Section 4 assumes an expenditure constraint for the defender. Section 5 considers numerical examples and Section 6 concludes.

2. A two-period game

Consider a terrorist, or more generally any collective or individual actor, e.g. a firm, with an asset R which it seeks to defend. The asset can be thought of as a threat such as a resource stock of terrorist capacity, but may be anything of value to the terrorist such as its organization, its possessions, a physical or non-physical commodity, an information set, etc. The terrorist is under attack by another actor which seeks to destroy the asset, or a largest possible fraction of it. The other actor finds no value in the terrorist's asset, but rather considers it as a threat, does not want to utilize it or transform it for own use, and confines attention to eliminating

[1] Defining V^* as "the set of individually rational payoffs," the Folk Theorem states: "For any $(v_1, \ldots, v_n) \in V^*$, if players discount the future sufficiently little, there exists a Nash equilibrium of the infinitely repeated game where, for all i, player i's average payoff is v_i" (Fudenberg and Maskin, 1986, p. 537).

it. The other actor may be a government, another terrorist or firm in the same or in another industry, or more generally any collective or individual actor which lawfully or unlawfully seeks to destroy the asset. Whether or not this other actor has assets is not under consideration in this paper. In period i, $i = 1, 2$, the terrorist, referred to as the defender, incurs an effort T_i to defend his asset, hereafter referred to as an investment. The effort may be hiring competent personnel such as lawyers, engineers, and security guards, or implementation of procedures to protect the asset while ensuring that it is available and accessible as the defender requires, or investment in technology such as physical barriers, firewalls, and intrusion detection systems, encryption techniques, access control mechanisms, to ascertain that the defense is optimal. The defense investment expenditure is F_i, where $\partial F_i/\partial T_i > 0$. We consider the simple case $F_i = BT_i$, where B is the inefficiency of investment. Higher B means greater inefficiency, and $1/B$ is the efficiency.[2]

Independently and simultaneously in period i, the other actor, referred to as the attacker, incurs an effort t_i to destroy the asset. The attacker seeks to be more competent than the lawyers, engineers, and security guards employed by the defender, seeks to circumvent the defense procedures, and seeks to break through the firewalls, penetrate the intrusion detection systems, encryption techniques, and access control mechanisms, thus attempting to design the optimal attack. Analogously, the attack investment expenditure is f_i, $\partial f_i/\partial t_i > 0$. We assume $f_i = bt_i$, where b is the inefficiency of investment, and $1/b$ is the efficiency. Both the expenditures BT_i and bt_i can be capital and/or labor, and both the defender and attacker are assumed risk neutral.[3]

We assume that the contest between the defender and attacker takes the form that is common in the conflict and rent seeking literature (Hirshleifer, 1995; Skaperdas, 1996), with one difference. The defender gets a fraction H_i, and seeks to maximize this fraction. The attacker, in contrast, seeks to minimize the fraction H_i, and is not concerned about the remaining fraction $1 - H_i$, as in the rent seeking literature, $\partial H_i/\partial T_i > 0$, $\partial H_i/\partial t_i < 0$. The fraction H_i can be thought of as an actual value, or an expected value. We use the common ratio formula[4]

$$H_i = \frac{T_i^m}{T_i^m + t_i^m} \tag{2}$$

where m is a decisiveness parameter.[5] The defender's outcome in period i is U_i, which he seeks to maximize. The attacker's outcome in period i is u_i, which she seeks to minimize. Based on this reasoning, the outcomes in the first period are

$$U_1 = \frac{T_1^m}{T_1^m + t_1^m} R - BT_1, \qquad u_1 = \frac{T_1^m}{T_1^m + t_1^m} R + bt_1. \tag{3}$$

We subtract the cost expenditure BT_1 for the defender, and add the cost expenditure bt_1 for the attacker, since they seek maximization and minimization, respectively. After the contest in the first period the defender retains the smaller asset $RT_1^m/(T_1^m + t_1^m)$. Defenders are continuously

2 B can also be interpreted as the unit cost of defense investment, where T_i is generally continuous.

3 An alternative analysis may for example assume that the attacker is risk seeking while the defender is risk averse. Assuming risk neutrality simplifies the analysis. Much of the economic conflict literature related to production, appropriation, defense, and rent seeking assumes risk neutrality. See Skaperdas (1991) for an exception.

4 An alternative is the logit or difference form $H_i = e^{mt_i}/(e^{mt_i} + e^{mT_i})$.

5 $m = 0$ gives equal distribution, and $0 < m < 1$ gives a disproportional advantage of investing less than one's opponent. When $m = 1$, the investments have proportional impact, $m > 1$ gives a disproportional advantage of investing more than one's opponent, and $m = \infty$ implies "winner-takes-all."

worn down by attackers, especially if attackers are successful, and are often also worn down by the wear and tear of time. With nothing to counteract this wearing down, the assets of defenders would gradually become smaller and smaller, and eventually disappear. Usually, defenders of assets set in motion processes to ensure that assets grow over time. Such growth may be due to a blossoming economy, discovering and exploiting market opportunities, hard and skillful work, outside funding, or the benefit of being at the right place at the right time. If an asset is converted into money and placed in a bank, the asset earns an interest rate. If the asset is invested in the stock market, dividends may be paid out. Workers and consultants can be employed to grow assets, and contracts with various actors can be made to ensure further growth. To account for asset growth from the first period to the second period, we multiply with a growth parameter a. That is, the defender's asset $RT_1^m/(T_1^m+t_1^m)$ after the first period grows to $aRT_1^m/(T_1^m+t_1^m)$ at the start of the second period. Especially, if $a=1$, there is no growth from the first to the second period. The defender simply starts the second period with the same asset it retained after the first period. If $a>1$, there is positive growth. If $0<a<1$, the asset deteriorates which means negative growth. Finally, if $a=0$, the asset vanishes, the defender is driven out of business, and the second period is not worth playing.

In the second period the actors invest analogously as in the first period. Independently and simultaneously, the defender incurs an effort T_2 and the attacker incurs an effort t_2. Based on the reasoning above, the outcomes in the second period are

$$U_2=\frac{T_2^m}{T_2^m+t_2^m}a\frac{T_1^m}{T_1^m+t_1^m}R-BT_2, \qquad u_2=\frac{T_2^m}{T_2^m+t_2^m}a\frac{T_1^m}{T_1^m+t_1^m}R+bt_2. \tag{4}$$

Note that the term $aRT_1^m/(T_1^m+t_1^m)$ is present in both U_2 and u_2 since this is what is under attack in the second period.

For the two-period game as a whole, with discounting Δ for the defender and δ for the attacker for the second period, the total outcomes U and u for the defender and attacker are

$$U=U_1+\Delta U_2, \qquad u=u_1+\delta u_2. \tag{5}$$

If the attacker does not attack in any of the periods, and the defender keeps her resource in both periods, the outcomes are $U=(1+\Delta a)R$ to the defender and $u=(1+\delta a)R$ to the attacker. Conversely, if the defender gives up her asset without defense, the attacker invests arbitrarily little and both actors' outcomes are zero $U=u=0$.

3. Solving the two-period game

The game is solved with backward recursion, starting with the second period, assuming a subgame perfect equilibrium. We thereafter find the optimal solution in the first period, taking into account that the actors' choices in the second period must be in equilibrium. Differentiating (4), and solving the first order conditions $\partial U_2/\partial T_2=0$ and $\partial u_2/\partial t_2=0$ gives

$$T_2=\frac{amb^mB^{m-1}\frac{T_1^m}{T_1^m+t_1^m}R}{(B^m+b^m)^2}, \qquad t_2=\frac{BT_2}{b}. \tag{6}$$

Intuitively, the ratio T_2/t_2 of the investments in the second period is inverse proportional to the ratio B/b of the inefficiencies. For analytical tractability, we hereafter set $m=1$. Inserting (6) into (5) gives

$$U=\frac{T_1}{T_1+t_1}\left(1+\frac{\Delta ab^2}{(B+b)^2}\right)R-BT_1, \qquad u=\frac{T_1}{T_1+t_1}\left(1+\frac{\delta ab(2B+b)}{(B+b)^2}\right)R+bt_1. \tag{7}$$

When $a=0$, or $b=0$, or $\Delta=0$ and $\delta=0$, which means zero growth, or zero attack inefficiency, or discount parameters equal to zero for both actors, the total outcomes for the two-period game in (7) equal the outcomes U_1 and u_1 in (3) for the first period. The second term in each of the two brackets in (7), multiplied with the factor outside each bracket, express the additional outcomes the defender and attacker get in the second period. These two terms depend on the three parameters a, b, B. Additionally, for the defender it depends on Δ, and for the attacker it depends on δ. The asset R operates proportionally outside the brackets. Differentiating (7), and solving the first order conditions $\partial U/\partial T_1=0$ and $\partial u/\partial t_1=0$ gives

$$T_1=\frac{b[(B+b)^2+\delta ab(2B+b)][(B+b)^2+\Delta ab^2]^2}{(B+b)^2[B^3+Bb^2(3+\delta a)+B^2b(3+2\delta a)+b^3(1+\Delta a)]^2}R,$$

$$t_1=\frac{B[(B+b)^2+\delta ab(2B+b)]^2[(B+b)^2+\Delta ab^2]}{(B+b)^2[B^3+Bb^2(3+\delta a)+B^2b(3+2\delta a)+b^3(1+\Delta a)]^2}R. \tag{8}$$

The denominators in T_1 and t_1 are equivalent. The two brackets in the numerators in T_1 and t_1 are also equivalent, but oppositely squared.

Inserting $a=0$ or $\Delta=\delta=0$ into (8) and (3) when $m=1$ gives

$$T_1=\frac{bR}{(B+b)^2}, \qquad t_1=\frac{BT_1}{b}, \qquad U_1=\frac{b^2R}{(B+b)^2}, \qquad u_1=\frac{b(2B+b)R}{(B+b)^2}. \tag{9}$$

When $a=0$ or $\Delta=\delta=0$, the first period effectively operates as the second and last period. The ratio T_1/t_1 of the investments is then inverse proportional to the ratio B/b of the inefficiencies, as is generally the case in the second period shown in (6). Inserting $b=B$ into (9) gives

$$T_1=t_1=\frac{R}{4B}, \qquad U_1=U=\frac{R}{4}, \qquad u_1=u=\frac{3R}{4}. \tag{10}$$

Hence when equally matched $b=B$, and no growth $a=0$ or $\Delta=\delta=0$, each actor gets 50% of the asset. Both actors' investment expenditures are $\frac{1}{4}$ of the asset R. Hence the defender keeps $\frac{1}{4}$ of her asset. The attacker adds the expenditures $\frac{1}{4}$ to 50% of the asset value and gets the outcome $\frac{3}{4}$ which he seeks to minimize.

Inserting (8) into (6) for $m=1$ gives

$$T_2=\frac{ab^2[(B+b)^2+\Delta ab^2]}{(B+b)^2[B^3+Bb^2(3+\delta a)+B^2b(3+2\delta a)+b^3(1+\Delta a)]}R, \qquad t_2=\frac{BT_2}{b}. \tag{11}$$

Inserting the first and second period investments in (8) and (11) into (3)–(5) gives the outcomes[6]

$$U_1=\frac{b^2R((b+B)^2+ab^2\Delta)((b+B)^4-ab(-(b-B)(b+B)^2+abB(b+2B)\delta)\Delta)}{(b+B)^2(B^3+b^2B(3+a\delta)+bB^2(3+2a\delta)+b^3(1+a\Delta))^2},$$

$$u_1=\frac{bR((b+B)^2+ab^2\Delta)((b+2B)((b+B)^4+3abB(b+B)^2\delta+a^2b^2B(b+2B)\delta^2)+ab^3(b+B)^2\Delta)}{(b+B)^2(B^3+b^2B(3+a\delta)+bB^2(3+2a\delta)+b^3(1+a\Delta))^2},$$

[6] All calculations in this article are made using the Mathematica software package (http://www.wolfram.com).

$$
\begin{aligned}
U_2 &= bT_2 = \frac{ab^3R((b+B)^2+ab^2\Delta)}{(b+B)^2(B^3+b^2B(3+a\delta)+bB^2(3+2a\delta)+b^3(1+a\Delta))},\\
u_2 &= \frac{b(b+2B)}{B}t_2 = \left(1+\frac{2B}{b}\right)U_2 = \frac{ab^2(b+2B)R((b+B)^2+ab^2\Delta)}{(b+B)^2(B^3+b^2B(3+a\delta)+bB^2(3+2a\delta)+b^3(1+a\Delta))},\\
U &= \frac{b^2R((b+B)^2+ab^2\Delta)^3}{(b+B)^2(B^3+b^2B(3+a\delta)+bB^2(3+2a\delta)+b^3(1+a\Delta))^2},\\
u &= \frac{bR((b+B)^2+ab(b+2B)\delta)((b+B)^2+ab^2\Delta)((b+2B)((b+B)^2+2abB\delta)+ab^3\Delta)}{(b+B)^2(B^3+b^2B(3+a\delta)+bB^2(3+2a\delta)+b^3(1+a\Delta))^2}.
\end{aligned} \tag{12}
$$

Proposition 1.

(i) *With no first period outcome restraint for the defender, an interior solution with positive first period and second period investments T_1, t_1, T_2, t_2 for the defender and attacker always exists.*

(ii) $\frac{\partial T_2}{\partial a} > 0$, $\frac{\partial t_2}{\partial a} > 0$, $\frac{\partial T_1}{\partial \Delta} > 0$, $\frac{\partial T_2}{\partial \Delta} > 0$, $\frac{\partial t_2}{\partial \Delta} > 0$, $\frac{\partial t_1}{\partial \delta} > 0$, $\frac{\partial T_2}{\partial \delta} < 0$, $\frac{\partial t_2}{\partial \delta} < 0$, $\frac{\partial T_1}{\partial B} < 0$.

Proof. Follows from (8), (11), U_1 in (12), and Appendix A. □

Since the attacker seeks to eliminate the asset, she always attacks in the first period, causing a need for the defender to defend his asset unless he is constrained by his first period outcome which can be low or negative. The second period is the last period. The defender seeks to defend his asset, and the attacker seeks to eliminate it.

Both actors' second period investments increase with growth, and increase as the defender's discount parameter increases. A growing asset is important to defend, and more so if the second period is important for the defender. The attacker responds with increased investment in the second period. The defender also invests more to defend the asset in the first period if the second period is important for him as expressed with a large discount parameter.

The defender invests more in the first period if her discount parameter is large. If the second period is important for the defender, she invests more in the first period to downgrade the asset. Both actors' second period investments decrease as the attacker's discount parameter increases. Finally, the defender's first period investment decreases with the defense inefficiency, since the defender is thereby place in a disadvantaged position and is induced to invest less.

We define U_i/T_i as the rate of return on investment for the defender in period i. Outcome is interpreted as return for the defender. In the second period, the rate is b, as shown in (12). Intuitively, the defender's rate of return increases if the attack inefficiency increases, which is to the detriment of the attacker. When $a = 0$ or $\Delta = \delta = 0$, the first period effectively operates as the last period, giving $U_1/T_1 = b$.

All the six outcomes in (12) are positive except U_1 where the requirement is

$$
U_1 > 0 \quad \text{when} \quad a < \frac{2(b+B)^2}{\Delta b(B-b)+\sqrt{\Delta}b\sqrt{4B(b+2B)\delta+(b-B)^2\Delta}} = a^*. \tag{13}
$$

Proposition 2.

(i) *With no expenditure constraint for the defender, his total outcome and second period outcome are always positive, though may be less worth than the asset R he started out with.*

(ii) *The defender's first period outcome is positive when* $a < a^*$ *defined in* (13).

(iii) *With expenditure constraint* $U_1 > 0$ *for the defender, he loses his entire asset in the first period when* $a > a^*$*, after which the game ends. This constitutes maximum conflict in the first period.*

(iv) *The attacker always has positive outcomes which she seeks to minimize.*

(v) $\frac{\partial U_2}{\partial a} > 0$, $\frac{\partial u_2}{\partial a} > 0$, $\frac{\partial u}{\partial a} > 0$, $\frac{\partial U_2}{\partial \Delta} > 0$, $\frac{\partial u_2}{\partial \Delta} > 0$, $\frac{\partial U}{\partial \Delta} > 0$, $\frac{\partial u}{\partial \Delta} > 0$, $\frac{\partial U_2}{\partial \delta} < 0$, $\frac{\partial u_2}{\partial \delta} < 0$, $\frac{\partial U}{\partial \delta} < 0$, $\frac{\partial u}{\partial \delta} > 0$, $\frac{\partial U}{\partial B} < 0$, $\frac{\partial U_2}{\partial b} > 0$, $\frac{\partial u_2}{\partial b} > 0$, $\frac{\partial U}{\partial b} > 0$, $\frac{\partial u}{\partial b} > 0$.

Proof. Follows from (12), (13) and Appendix A. □

If asset growth is too large as expressed in (13), $a > a^*$, the defender's first period outcome is negative. This is detrimental for the defender if he is subject to the expenditure constraint such as $U_1 > 0$, or $U_1 > k$ where k is positive or negative, which he very well may be, since he then loses his entire asset through elimination by the attacker. The reason is that the defender chooses zero first period investment $T_1 = 0$ which guarantees zero outcome rather than negative outcome in the first period. The defender is in a preferred situation if subject to no expenditure constraint. Then his total outcome is always positive, regardless of growth and the other parameters. The reason is that the positive second period outcome more than compensates for the possibly negative first period outcome. This causes a positive total outcome which, though, may be less worth than the resource R the defender started out with.

Increasing growth and an increasing discount parameter for the defender gives an increasing second period outcome for the defender, and increasing second period and total outcomes for the attacker, who is disadvantaged by both growth and that the defender values the future. Also, the defender's total outcome increases as his discount parameter increases. In contrast, the defender's second period and total outcome decreases as the attacker's discount parameter increases. The attacker's second period outcome also decreases as the attacker's discount parameter increases, which is to the advantage of the defender. However, the attacker's total outcome increases as the attacker's discount parameter increases. The defender's total outcome decreases with the defense inefficiency, which is to his disadvantage. Both the defender's and attacker's second period and total outcome increase with the attack inefficiency since such increase is to the advantage of the defender and disadvantage of the attacker.

Proposition 3 *(Strategic complementarity).*

(i) *For the first period investments and outcomes* T_1, t_1, U_1, u_1*, and the total outcomes* U *and* u*, growth never operates alone but is always a strategic complement with either the defender's discount parameter, expressed as* Δa*, or a strategic complement with the attacker's discount parameter, expressed as* δa.

(ii) *For the second period investments and outcomes* T_2, t_2, U_2, u_2*, growth is a strategic complement, but also operates once proportionally alone.*

Proof. Follows from (8), (11), (12). □

Growth is the most essential parameter with largest flexibility. However, reduced (increased) growth can be compensated by higher (lower) discount parameters, with the constraint that discount parameters are between zero and one.

4. Expenditure constraint $U_1 > k$ for the defender

The defender may for a variety of reasons not have the capacity or willingness to accept low or negative outcome $U_1 < k$ in the first period, where k is positive or negative, despite the possibility of a positive total outcome $U > 0$. When the defender is a terrorist, operating outside the broadly accepted social norms of society, outside the legal structures and commonly used financial systems, and usually hidden from society at large, additional risk plays a role. The terrorist may have funders of various kinds, but not necessarily funders who accept low or negative outcomes at various points in time. Note that $U_1 < k$ can be a lenient or strict constraint. As expressed in (3), it means that the cost expenditure BT_1 the defender incurs in the first period cannot be too large compared with the fraction $RT_1^m/(T_1^m + t_1^m)$ it retains of the asset R it originally possessed. When $k = 0$, the former cannot exceed the latter. A defender's constraint may be quite strict, expressed with a positive k. With few own resources, and few sponsors and benefactors, a defender may only be able to sustain a cost expenditure that is a small part of the fraction of R it expects to retain. A good benchmark for analysis is to assume an expenditure constraint that equals what one expects to possess after a contest, which means $k = 0$. A bank, for example, may be unwilling to give a loan that exceeds the net worth of the customer. This section analyzes the implications of requiring $U_1 > k$ for the defender in the first period, where $k = 0$ is a convenient benchmark.

Inserting equal attack and defense inefficiencies $b = B$ into (8) and (12) gives

$$
\begin{aligned}
T_1 &= \frac{(4+3\delta a)(4+\Delta a)^2}{4B[8+(3\delta+\Delta)a]^2}R, \qquad t_1 = \frac{(4+3\delta a)^2(4+\Delta a)}{4B[8+(3\delta+\Delta)a]^2}R,\\
u_1 &= \frac{[48+9\delta a(4+\delta a)+4\Delta a](4+\Delta a)}{4[8+(3\delta+\Delta)a]^2}R,\\
U_1 &= \frac{(16-3\delta\Delta a^2)(4+\Delta a)}{4[8+(3\delta+\Delta)a]^2}R > 0 \quad \text{when} \quad a < 4/\sqrt{3\delta\Delta} \Leftrightarrow a\sqrt{\delta\Delta} < 4/\sqrt{3},\\
T_2 &= t_2 = \frac{a(4+\Delta a)}{4B[8+(3\delta+\Delta)a]}R, \qquad U_2 = bT_2 = \frac{a(4+\Delta a)}{4[8+(3\delta+\Delta)a]}R,\\
u_2 &= 3U_2 = \frac{3a(4+\Delta a)}{4[8+(3\delta+\Delta)a]}R,\\
U &= \frac{(4+\Delta a)^3}{4[8+(3\delta+\Delta)a]^2}R, \qquad u = \frac{[12+(6\delta+\Delta)a](4+3\delta a)(4+\Delta a)}{4[8+(3\delta+\Delta)a]^2}R. \qquad (14)
\end{aligned}
$$

Proposition 4. *Assume expenditure constraint $U_1 > k$ for the defender and equal attack and defense inefficiencies $b = B$.*

(i) *The attacker eliminates the entire asset in the first period when $k = 0$ and $a > 4/\sqrt{3\delta\Delta}$.*
(ii) *The attacker eliminates the entire asset in the first period when k is arbitrarily much negative and finite provided that growth a is sufficiently large.* $\text{Lim}_{a\to\infty} U_1 = -\infty$. *All other variables approach infinity as a approaches infinity.*
(iii) *When $k = 0$, growth and the two discount parameters are strategic complements, $a\sqrt{\delta\Delta} > 4/\sqrt{3}$, where the impact of each discount parameter is the square root of the impact of growth.*

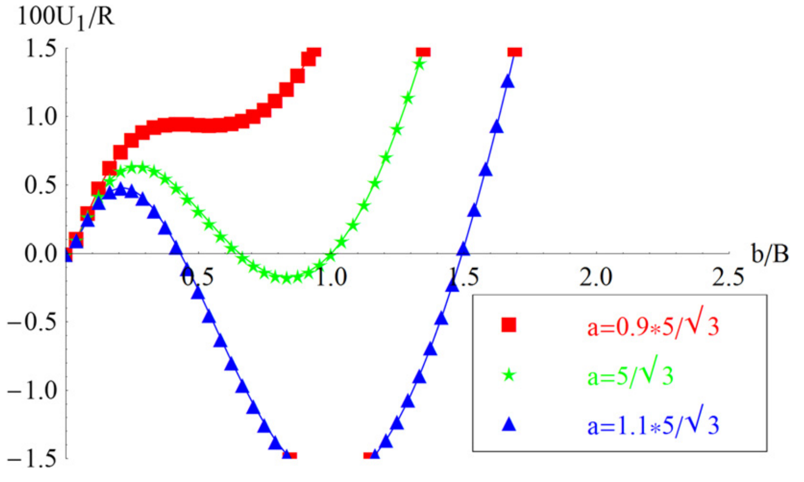

Fig. 1. The defender's first period outcome U_1 as a function of b/B, $\Delta = \delta = 4/5$.

Proof. (i) Follows from setting $U_1 > 0$ in (14). (ii) The variables T_1, t_1, U_1, u_1, U, u in (14) are of the third order in a in the numerator, and in the second order in a in the denominator. The variables T_2, t_2, U_2, u_2 in (14) are of the second order in a in the numerator, and in the first order in a in the denominator. (iii) Follows from (14). $\square$

If the attacker seeks to eliminate the entire asset in the first period, she is aided by large growth of the defender's asset, but also aided by both the defender's and attacker's discount parameters.

Analyzing the impact of different attack and defense inefficiencies b and B is complicated since setting $U_1 = 0$ in (12) and (13) gives a fourth order equation in both b and B. Fig. 1 plots the defender's first period outcome $100U_1/R$ as a function of b/B when $\Delta = \delta = 4/5$, for three growth values. Multiplication with 100 is for scaling purposes, and all variables are proportional to R. Inserting into Proposition 4 gives $a > 5/\sqrt{3}$ as a requirement for asset elimination when $b/B = 1$. Accordingly, the middle curve (star) sets $a = 5/\sqrt{3} \approx 2.89$ which causes $U_1 = 0$ when $b/B = 1$. When the attack inefficiency increases above one, to the disadvantage of the attacker, U_1 is positive and the asset is not eliminated in the first period. The attacker is too disadvantaged to do so. The attacker prefers not to eliminate the asset due to weakness. Conversely, when the attack inefficiency decreases below one, to the advantage of the attacker, U_1 is negative and the asset is not eliminated in the first period. But this holds only when $0.65 < b/B < 1$. When $b/B < 0.65$, U_1 is positive with no asset elimination in the first period. The reason is that the attacker is so advantaged by a low b/B that there is no reason to incur the extreme expenditure of eliminating the asset entirely in the first period. The attacker prefers not to eliminate the asset due to strength. The attacker is also advantaged by a low b/B in the second period, and the tradeoff across time involves optimal asset reduction in both periods.

The first curve (box) in Fig. 1 chooses lower growth $a = 0.9 * 5/\sqrt{3} \approx 2.60$ which causes $U_1 > 0$. There is less need for the attacker to eliminate the asset in the first period when the asset grows modestly into the second period. The third curve (triangle) in Fig. 1 chooses higher growth $a = 1.1 * 5/\sqrt{3} \approx 3.18$ which causes $U_1 > 0$ when $b/B > 1.49$. The substantial asset growth causes a threat leading to its first period elimination also within the range $1 < b/B < 1.49$, where the attacker is up to 49% more disadvantaged than the defender by her attack inefficiency. When $b/B > 1.49$, the attacker is too disadvantaged to eliminate the asset in the first period.

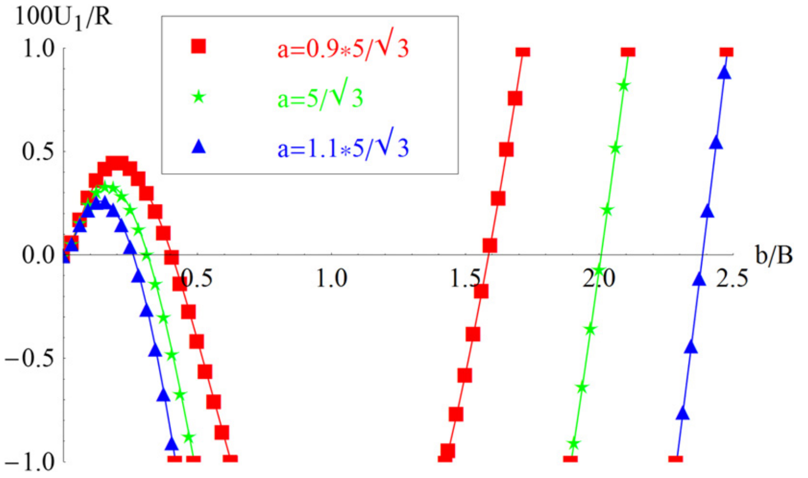

Fig. 2. The defender's first period outcome U_1 as a function of b/B, $\Delta = \delta = 1$.

At the lower end, when $b/B < 0.43$, the asset is not eliminated in the first period due to strength since it is expenditure efficient for the advantaged attacker to reduce the asset in both periods. Note that 0.43 for the third curve is lower than 0.65 for the second curve because of the higher asset growth for the third curve. When asset growth is large, the attacker needs to be advantaged by an even lower b/B in order to refrain from eliminating the asset in the first period due to strength.

Fig. 2 makes the same assumptions as Fig. 1, but increases both actors' discount parameters to $\Delta = \delta = 1$. Because of strategic complementarity, increasing the discount parameters has an impact similar to increasing growth. Consequently, the intermediate range of b/B within which the asset is eliminated in the first period increases.

Proposition 5. *U_1 increases concavely from zero when $b/B = 0$. When growth is large enough, expressed as $a > 4/\sqrt{3\delta\Delta}$ when $b/B = 1$, U_1 reaches a maximum, decreases, becomes negative for a range of b/B which includes $b/B = 1$, reaches a minimum, increases, becomes positive, and increases concavely toward R. Hence with large growth, the attacker chooses to eliminate the defender's asset in the first period within an intermediate range of b/B which includes $b/B = 1$. When b/B is above this range, the defender refrains from asset elimination in the first period due to weakness. When b/B is below this range, the defender refrains from asset elimination in the first period due to strength.*

Proof. See Appendix A. □

Proposition 5 throws light on Israel's strategy towards Hezbollah in the early years of the New Millennium. From 2000 until July 12, 2006 Israel refrained from attacking Hezbollah due to own strength. The growth a of Hezbollah's attack capacity was perceived not to be too high, b/B was low, and Hezbollah earned a positive first period outcome U_1. Israel perceived its own military superiority to be so significant that it tolerated Hezbollah's presence near its northern border. This situation is illustrated with the first curve (box) in Fig. 1 where b/B is low and U_1 is positive. Gradually, two mechanisms started to operate. The first was that Israel perceived Hezbollah's growth a to increase. Its support in southern Lebanon increased. Its military strength increased helped by perceived funding by other actors. And its threat became more overwhelming as perceived by Israel. The second mechanism was that Israel perceived Hezbollah's defense inefficiency B to decrease, which means lower unit cost of investing T_i. If b is constant, this causes b/B to increase. Referring to Fig. 1, these two mechanisms cause a downward vertical shift in U_1 because of increasing growth a, and a rightward horizontal shift in U_1 because of decreasing inefficiency B. This situation is illustrated with the second and third curves (star, triangle) in Fig. 1 where U_1 is negative. That is, the inevitable consequence is that Hezbollah's first period outcome U_1 gets decreased too much due to heavy attacks by Israel seeking to eliminate Hezbollah's attack capacity. In situations like this, which are quite common, the strategic decision by defenders such as Hezbollah is whether and how long it is willing or capable to sustain a low and possibly negative first period outcome U_1. A defender without such willingness or capability gets eliminated. A defender that perseveres and accepts a low or negative U_1 through time, may earn a reputation for standing up against an adversary which, combined with other advantages may cause a positive second period outcome and a positive total outcome in the long run.

5. Numerical examples

Assume $R = 100$, $B = 1$, and $\Delta = \delta = 4/5$ and consider four benchmarks. First, if the attacker does not attack in any of the periods, and the defender keeps her resource untouched in both periods, the outcomes are $U = u = 100 + 80a$ which increases from 100 to infinity when a increases from 0 to infinity. The defender seeks to maximize her outcome U and the attacker seeks to minimize his outcome u. Second, if the defender gives up her asset without defense, the attacker invests arbitrarily little and both actors' outcomes are zero $U = u = 0$. Third, and moving now to equilibrium situations, if the actors are equally matched $b = B = 1$ and no growth $a = 0$, (10) gives $T_1 = t_1 = U_1 = U = 25$ and $u_1 = u = 75$. Fourth, with infinite growth $a = \infty$, then $U_1 = -\infty$ and all other variables are infinite. Table 1 considers 15 numerical examples where we use the growth values $a = 5/\sqrt{3} \approx 2.89$, $a = 0.9 * 5/\sqrt{3} \approx 2.60$, and $a = 1.1 * 5/\sqrt{3} \approx 3.18$ from Fig. 1, and allow the attack inefficiency b to vary.

Table 1
The 10 variables dependent on a and b when $R = 100$, $B = 1$, $\Delta = \delta = 4/5$

a	b	T_1	t_1	T_2	t_2	U_1	u_1	U_2	u_2	U	u
2.60	0.2	10.72	82.93	4.13	20.69	0.73	28.04	0.83	9.09	1.39	35.32
2.89	0.2	10.50	84.11	4.45	22.24	0.60	27.92	0.89	9.79	1.31	35.74
3.18	0.2	10.28	85.26	4.74	23.72	0.48	27.81	0.95	10.44	1.24	36.16
4.65	0.2	9.36	90.64	6.044	30.22	0	27.49	1.21	13.30	0.97	38.13
0	1	25	25	0	0	25	75	0	0	25	75
1	1	29.39	39.18	10.71	10.71	13.47	82.04	10.71	32.14	22.04	107.76
2.60	1	35.52	59.82	24.20	24.20	1.74	97.08	24.20	72.60	21.10	155.16
2.89	1	36.60	63.40	26.42	26.42	0	100.	26.42	79.25	21.13	163.40
3.18	1	37.68	66.95	28.59	28.59	−1.67	102.96	28.59	85.77	21.21	171.57
5	1	44.44	88.89	41.67	41.67	−11.11	122.22	41.67	125	22.22	222.22
∞	b	∞	∞	∞	∞	$-\infty$	∞	∞	∞	∞	∞
2.60	2	47.02	34.80	33.18	16.59	10.45	127.07	66.36	132.72	63.53	233.24
2.89	2	49.67	37.40	36.59	18.29	7.38	131.85	73.18	146.36	65.92	248.93
3.18	2	52.28	40.01	39.98	19.99	4.37	136.67	79.95	159.91	68.33	264.59
3.60	2	56.16	43.84	44.95	22.48	0	143.85	89.90	179.80	71.92	287.69

Consider the defender's first period outcome when the actors are equally matched $b = B = 1$. When $a = 1$, both actors invest more in the first period than in the second period, and the first period outcomes are somewhat higher than the second period outcomes. If the defender requires a first period outcome of $U_1 \geqslant 13.47$, growth cannot exceed $a = 1$. If it does, she withdraws in the first period with outcome $T_1 = U_1 = T_2 = U_2 = U = 0$. As a exceeds 2.89, the defender's first period outcome is negative. It is $U_1 = -1.67$ when $a = 3.18$. If the defender can sustain that, her second period outcome and total outcome are $U_2 = 28.59$ and $U = 21.21$. The outcome is $U_1 = -11.11$ when $a = 5$. With infinite growth, she can sustain a negative infinite first period outcome which gets compensated with a positive infinite second period outcome and total outcome.

The four first rows in Table 1 gives a five-fold advantage to the attacker in terms of $b = 0.2$. This corresponds to a leftward shift in Fig. 1. The defender's first period outcomes are small, but they are all positive. The defender's second period outcomes are also small. The attacker perceives the defender as a small nuisance and refrains from eliminating the asset due to own strength expressed with the low b. Growth $a = 4.65$ is needed to reduce the defender's first period outcome to $U_1 = 0$ in equilibrium. A higher first period investment by the attacker is not worth wile, and a lower investment allows the asset to be slightly more than a nuisance in the second period. For this example the defender's total outcome decreases from $U = 1.39$ to $U = 0.97$ as growth increases from $a = 2.60$ to $a = 4.65$.

The four last rows in the table gives a two-fold disadvantage to the attacker in terms of $b = 2$. This corresponds to a rightward shift in Fig. 1. The defender's first period outcomes are larger and all positive. The defender's second period outcomes are even larger. The attacker perceives the defender as an overwhelming threat and refrains from eliminating the asset due to own weakness expressed with the high b. Growth $a = 3.60$ is needed to reduce the defender's first period outcome to $U_1 = 0$ in equilibrium. A higher first period investment by the attacker is too costly, and a lower investment allows the asset to be unacceptably large in the second period. For this example the defender's total outcome increases from $U = 63.53$ to $U = 71.92$ as growth increases from $a = 2.60$ to $a = 3.60$.

6. Conclusion

A defender who may be a terrorist possesses an asset perceived as a threat by an attacker who seeks to downgrade or eliminate it. A contest success function determines the relative fraction retained by the defender in each of two periods. The fraction may grow from the first to the second period. The attacker seeks to minimize the defender's fraction of the asset through time.

With no first period outcome restraint for the defender, an interior solution with positive first period and second period investments for the defender and attacker always exists. Since the attacker seeks to eliminate the asset, she always attacks in the first period, causing a need for the defender to defend the asset. Both actors' second period investments increase with asset growth, and increase as the defender's discount parameter increases. A growing asset is important to defend, and more so if the second period is important for the defender.

With no expenditure constraint for the defender, his second period outcome and total outcome are always positive, though may be less worth than his initial asset. The defender's first period outcome is positive when his asset growth is moderate since he constitutes no formidable threat. A defender with large asset growth is guaranteed a first period low or negative income when the ratio of the attack and defense inefficiencies is intermediate within a certain range. If the defender does not accept a low or negative first period outcome, his asset is eliminated and the game ends. This constitutes maximum conflict in the first period. The defender is in a preferred situation if subject to no expenditure constraint. Then his total outcome is always positive, regardless of growth and the other parameters. The reason is that the positive second period outcome more than compensates for the possibly negative first period outcome.

Increasing growth and an increasing discount parameter for the defender gives an increasing second period outcome for the defender, and increasing second period and total outcomes for the attacker, who is disadvantaged by both growth and that the defender values the future. The defender's total outcome increases as his discount parameter increases. In contrast, the defender's second period and total outcome decrease as the attacker's discount parameter increases. The attacker's second period outcome also decreases as the attacker's discount parameter increases, which is to the advantage of the defender. However, the attacker's total outcome increases as the attacker's discount parameter increases. The defender's total outcome decreases with the defense inefficiency, which is to his disadvantage. Both the defender's and attacker's second period and total outcome increase with the attack inefficiency since such increase is to the advantage of the defender and disadvantage of the attacker.

Growth and the two discount parameters are strategic complements. The growth parameter has the largest flexibility. Reduced (increased) growth can be compensated by higher (lower) discount parameters, though discount parameters are constrained between zero and one. For the first period investments and outcomes, and the total outcomes, growth never operates alone but is always a strategic complement with either the defender's discount parameter or the attacker's discount parameter. For the second period investments and outcomes, growth is a strategic complement, but also operates once proportionally alone. If the attacker seeks to eliminate the entire asset in the first period, she is aided by large growth of the defender's asset, but also aided by both the defender's and attacker's discount parameters.

Assume that the defender requires his first period outcome to be positive, which means that his cost expenditure of investment in the first period cannot exceed the fraction of the asset that he retains after the contest in the first period. Then, with equal attack and defense inefficiencies, the attacker eliminates the entire asset in the first period when the asset growth multiplied with the square roots of the defender's and attacker's discount parameters exceed $4/\sqrt{3}$. Hence when both discount parameters equal one, the attacker eliminates assets that grow more than 131%.

The range for the ratio of the attack and defense inefficiencies, which causes negative first period outcome for the defender, increases both upwards and downwards with increasing asset growth. The range is finite and bounded strictly above zero. Interestingly, the attacker refrains from asset elimination in the first period due to weakness if the ratio is above the range, or due to strength if the ratio is below the range. First, if the attack inefficiency is too large, the attacker refrains from asset elimination due to weakness. With infinitely large attack inefficiency, the defender keeps his entire asset in the first period and may become a substantial threat in the second period due to asset growth. Second, and conversely, if the attack inefficiency is too low, the attacker refrains from asset elimination due to strength. She may degrade the asset somewhat, but knows that her superiority is overwhelming and finds it optimal to downgrade the asset in both periods. She may attack moderately today, moderately tomorrow, but always has the option of eliminating the asset if it grows out of bounds.

The model is illustrated with Israel's strategy towards Hezbollah. From 2000 until July 12, 2006 Israel refrained from attacking Hezbollah due to own strength. Gradually, Hezbollah's asset growth increased, and its defense inefficiency decreased. This caused Israel to launch a large first period attack which reduced Hezbollah's first period outcome. Defenders without willingness or capability to sustain low or negative first period outcomes get eliminated. In contrast, defenders with such willingness or capability may persevere through asset growth and compensation in the second period.

Appendix A. Proof of propositions

Proof of Propositions 1 and 2.

$$\frac{\partial T_2}{\partial a} = \frac{b^2R((b+B)^5 + ab^2(2(b+B)^3 + abB(b+2B)\delta)\Delta + a^2b^5\Delta^2)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\frac{\partial t_2}{\partial a} = \frac{bBR((b+B)^5 + ab^2(2(b+B)^3 + abB(b+2B)\delta)\Delta + a^2b^5\Delta^2)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\frac{\partial T_1}{\partial \Delta} = \frac{2ab^3BR((b+B)^2 + ab(b+2B)\delta)^2((b+B)^2 + ab^2\Delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^3} > 0,$$

$$\frac{\partial T_2}{\partial \Delta} = \frac{a^2b^4BR((b+B)^2 + ab(b+2B)\delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\frac{\partial t_2}{\partial \Delta} = \frac{a^2b^3B^2R((b+B)^2 + ab(b+2B)\delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\frac{\partial t_1}{\partial \delta} = \frac{2aB(b+2B)R((b+B)^2 + ab(b+2B)\delta)(b(b+B)^2 + ab^3\Delta)^2}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^3} > 0,$$

$$\frac{\partial T_2}{\partial \delta} = -\frac{a^2b^3B(b+2B)R((b+B)^2 + ab^2\Delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} < 0,$$

$$\frac{\partial t_2}{\partial \delta} = -\frac{a^2b^2B^2(b+2B)R((b+B)^2 + ab^2\Delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} < 0,$$

$$\begin{aligned}\frac{\partial T_1}{\partial B} = &-\Big(2bR\big((b+B)^2 + ab^2\Delta\big)\big((b+B)^7 + ab(b+B)^4(2b+B)(b+3B)\delta \\ &+ a^2b^2(b+B)^2(b+2B)\big(b^2+3bB+B^2\big)\delta^2 + ab^2\big((b+B)^4(b+3B) \\ &+ ab(b+B)^2\big(2b^2+12bB+11B^2\big)\delta + a^2b^2(b+2B)\big(b^2+5bB+5B^2\big)\delta^2\big)\Delta \\ &+ a^3b^6B\delta\Delta^2\big)\Big)\Big/\Big((b+B)^3\big(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta)\big)^3 \\ &< 0,\end{aligned}$$

$$\frac{\partial U_2}{\partial a} = \frac{b^3R((b+B)^5 + ab^2(2(b+B)^3 + abB(b+2B)\delta)\Delta + a^2b^5\Delta^2)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\frac{\partial u_2}{\partial a} = \frac{b^2(b+2B)R((b+B)^5 + ab^2(2(b+B)^3 + abB(b+2B)\delta)\Delta + a^2b^5\Delta^2)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\begin{aligned}\frac{\partial u}{\partial a} = &\Big(b^3R\big((b+B)^4(b+2B)\delta\big((b+B)^2(b+3B) + 3abB(b+2B)\delta\big) \\ &+ \big(2B^2(b+B)^6 + 3ab(b+B)^4(b+2B)\big(b^2+2bB+2B^2\big)\delta \\ &+ 6a^2b^2B(b+B)^3(b+2B)^2\delta^2 + 2a^3b^3B^2(b+2B)^3\delta^3\big)\Delta \\ &+ 3a^2b^4(b+2B)\delta\big((b+B)^3 + abB(b+2B)\delta\big)\Delta^2 + a^3b^7(b+2B)\delta\Delta^3\big)\Big) \\ &\Big/\Big((b+B)^2\big(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta)\big)^3 \quad > 0,\end{aligned}$$

$$\frac{\partial U_2}{\partial \Delta} = \frac{a^2 b^5 BR((b+B)^2 + ab(b+2B)\delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\frac{\partial u_2}{\partial \Delta} = \frac{a^2 b^4 B(b+2B)R((b+B)^2 + ab(b+2B)\delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\frac{\partial U}{\partial \Delta} = \frac{ab^4 R((b+B)^2 + ab^2\Delta)^2(3B^3 + b^2B(5+3a\delta) + bB^2(7+6a\delta) + b^3(1+a\Delta))}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^3} > 0,$$

$$\frac{\partial u}{\partial \Delta} = \frac{2ab^3 B^2 R((b+B)^2 + ab(b+2B)\delta)^3}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^3} > 0,$$

$$\frac{\partial U_2}{\partial \delta} = -\frac{a^2 b^4 B(b+2B)R((b+B)^2 + ab^2\Delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} < 0,$$

$$\frac{\partial u_2}{\partial \delta} = -\frac{a^2 b^3 B(b+2B)^2 R((b+B)^2 + ab^2\Delta)}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} < 0,$$

$$\frac{\partial U}{\partial \delta} = -\frac{2aB(b+2B)R(b(b+B)^2 + ab^3\Delta)^3}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^3} < 0,$$

$$\frac{\partial u}{\partial \delta} = \frac{ab^3(b+2B)R((b+B)^2 + ab^2\Delta)^2(3B^3 + b^2B(5+3a\delta) + bB^2(7+6a\delta) + b^3(1+a\Delta))}{(b+B)^2(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^3} > 0,$$

$$\frac{\partial U}{\partial B} = -\Big(2R\big(b(b+B)^2 + ab^3\Delta\big)^2\big((b+B)^5 + ab^2(b+B)^2(b+3B)\delta$$

$$+ ab^2\big(2(b+B)^2(b+2B) + ab\big(b^2 + 6bB + 6B^2\big)\delta\big)\Delta + a^2b^5\Delta^2\big)$$

$$\Big/\Big((b+B)^3\big(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta)\big)^3 \quad < 0,$$

$$\frac{\partial U_2}{\partial b} = \frac{ab^2BR(3(b+B)^5 + ab(b+B)^3(b+4B)\delta + ab^2(5(b+B)^3 + ab(b^2 + 7bB + 8B^2)\delta)\Delta + 2a^2b^5\Delta^2)}{(b+B)^3(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta))^2} > 0,$$

$$\frac{\partial u_2}{\partial b} = \Big(abBR\big((b+B)^5(b+4B) + ab(b+B)^3(b+2B)^2\delta$$

$$+ ab^2\big((b+B)^3(b+8B) + ab(b+2B)^2(b+3B)\delta\big)\Delta + 2a^2b^5B\Delta^2\big)\Big)$$

$$\Big/\Big((b+B)^3\big(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta)\big)^2 \quad > 0,$$

$$\frac{\partial U}{\partial b} = \Big(2bBR\big((b+B)^2 + ab^2\Delta\big)^2\big((b+B)^5 + ab^2(b+B)^2(b+3B)\delta$$

$$+ ab^2\big(2(b+B)^2(b+2B) + ab\big(b^2 + 6bB + 6B^2\big)\delta\big)\Delta + a^2b^5\Delta^2\big)$$

$$\Big/\Big((b+B)^3\big(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta)\big)^3 \quad > 0,$$

$$\frac{\partial u}{\partial b} = \Big(2B^2R\big((b+B)^3\big((b+B)^6 + 2ab(b+B)^3\big(2b^2 + 6bB + 3B^2\big)\delta$$

$$+ 3a^2b^2(b+B)(b+2B)\big(b^2 + 4bB + 2B^2\big)\delta^2 + a^3b^3(b+2B)^3\delta^3\big)$$

$$+ ab^2\big((b+B)^6(b+3B) + 3ab(b+B)^4(b+2B)(2b+3B)\delta$$

$$+ 3a^2b^2(b+B)^3(b+2B)(b+6B)\delta^2 + a^3b^3(b+2B)^3(b+3B)\delta^3\big)\Delta$$

$$+ 3a^3b^6\delta\big((b+B)^3 + abB(b+2B)\delta\big)\Delta^2 + a^4b^9\delta\Delta^3\big)$$

$$\Big/\Big((b+B)^3\big(B^3 + b^2B(3+a\delta) + bB^2(3+2a\delta) + b^3(1+a\Delta)\big)^3 \quad > 0.$$

Proof of Proposition 5.

Inserting $s = b/B$ into the first equation in (12) gives

$$U_1 = \frac{s^2 R((1+s)^4 + as((s-1)(1+s)^2 - as(2+s)\delta)\Delta)(1 + s(2+s+as\Delta))}{(1+s)^2(1+s(3+2a\delta+s(3+s+a\delta+as\Delta)))^2}. \quad \text{(A1)}$$

Applying L'Hopital's rule eight times on the numerator and denominator gives

$$\lim_{b/B\to\infty} U_1 = R \quad \text{(A2)}$$

so the defender keeps his entire resource at the limit when the ratio of the attack and defense inefficiencies approaches infinity, independently of growth. Also, inserting $s = b/B = 0$ into (A1) gives

$$\lim_{b/B\to 0} U_1 = 0 \quad \text{(A3)}$$

so the defender loses his entire resource at the limit when the ratio of the attack and defense inefficiencies approaches zero, independently of growth. The calculations

$$\left(\frac{\partial U_1}{\delta(b/B)}\right)_{b/B=0} = 0, \qquad \left(\frac{\partial^2 U_1}{\partial(b/B)^2}\right)_{b/B=0} = 2R > 0 \quad \text{(A4)}$$

show that U_1 increases concavely in b/B when $b/B = 0$. Hence U_1 is positive when b/B is arbitrarily small but positive. However, we know from (14) where $b/B = 1$ that growth can be chosen large enough, $a > 4/\sqrt{3\delta\Delta}$, such that U_1 is negative, $U_1 < 0$. Finally, we know from (A2) that U_1 reaches R asymptotically and thus concavely as b/B approaches infinity, independently of growth. In other words, U_1 increases concavely from zero when $b/B = 0$. When growth is large enough, U_1 reaches a maximum, decreases, becomes negative for a range of b/B which includes $b/B = 1$, reaches a minimum, increases, becomes positive, and increases concavely toward R.

References

Arrow, K.J., Harris, T., Marschak, J., 1951. Optimal inventory policy. *Econometrica* 19, 250–272.

Axelrod, R., 1984. T*he Evolution of Cooperation.* Basic Books, New York.

Benhabib, J., Rustichini, A., 1996. Social conflict and growth. *J. Econ. Growth* 1 (1), 125–142.

Enders, W., Sandler, T., 2003. What do we know about the substitution effect in transnational terrorism? In: Silke, A., Ilardi, G. (Eds.), *Researching Terrorism: Trends, Achievements, Failures.* Frank Cass, Ilfords, UK. http://www-rcf.usc.edu/~sandler/substitution2ms.pdf.

Fudenberg, D., Maskin, E., 1986. The folk theorem in repeated games with discounting or with incomplete information. *Econometrica* 54 (3), 533–554.

Hausken, K., 2005a. Production and conflict models versus rent seeking models. *Public Choice* 123, 59–93.

Hausken, K., 2005b. The battle of the sexes when the future is important. *Econ. Letters* 87 (1), 89–93.

Hausken, K., 2006. Income, interdependence, and substitution effects affecting incentives for security investment. *J. Acc. Public Pol.* 25 (6), 629–665.

Hirshleifer, J., 1991. The paradox of power. *Econ. Politics* 3, 177–200.

Hirshleifer, J., 1995. Anarchy and its breakdown. *J. Polit. Economy* 103 (1), 26–52.

Keohane, N., Zeckhauser, R.J., 2003. The ecology of terror defense. *J. Risk Uncertainty* 26 (2/3), 201–229.

Maxwell, J.W., Reuveny, R., 2005. Continuing conflict. *J. Econ. Behav. Organ.* 58, 30–52.

Reuveny, R., Maxwell, J.W., 2001. Conflict and renewable resources. *J. Conflict Resolution* 45 (6), 719–742.

Scarf, H., 1960. The Optimality of (S, s) policies for the dynamic inventory problem. In: *Proceedings of the First Stanford Symposium on Mathematical Methods in Social Sciences.* Stanford Univ. Press, Stanford, CA. Sethi,

S., 1979. Optimal pilfering policies for dynamic continuous thieves. *Manage. Sci.* 25 (6), 535–542.

Skaperdas, S., 1991. Conflict and attitudes toward risk. *Amer. Econ. Rev.* 81, 116–120.

Skaperdas, S., 1996. Contest success functions. *Econ. Theory* 7, 283–290.