Reputation, incomplete information, and differences in patience in repeated games with multiple equilibria

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Abstract

A game with multiple equilibria and incomplete information, which allows for reputation building, is repeated infinitely many times. Increasing differences in patience contribute to a greater likelihood of cooperation. As one player becomes sufficiently more patient than the other player, both players benefit, and both players' risk limits, and the conflict between the players, decrease.

Keywords: Conflict; Risk limits; Repeated game; Discounting; Reputation; Incomplete information

The literature has produced mixed results regarding whether players should choose conflict today to reap benefits tomorrow. First, the Folk theorem (Fudenberg and Maskin, 1986) is often taken to imply cooperation in long-term relationships, which is correct when the prisoner's dilemma is played repeatedly. Second, Hausken (2005) has for the battle of the sexes where player 1 values the future and player 2 is myopic, shown that player 1 prefers conflict in the present when the future is important. Similarly, Skaperdas and Syropoulos (1996) equip each agent with a resource which can be allocated into production versus arms. They show that increased importance of the future may harm cooperation. This article considers a broad class of games with multiple equilibria, and introduces incomplete information which allows the players to build reputations. The objective of the article is to understand how the players' payoffs, risk limits, and the conflict between them are influenced by different emphases on the future. The players are concerned about how the conflict between them evolves. The notion of risk limits was essential in Zeuthen's (1930) work. He originated the principle of risk dominance as a dominance relation based on comparing the various players' risk limits.[1]

Table 1

Two-person two-strategy game with two equilibria

	I	II
I	a_1,a_2	t_1,t_2
II	d_1,d_2	b_1,b_2

Consider the game in Table 1 where $a_1 \geq b_1 \geq t_1$, $b_2 \geq a_2 \geq t_2$, $a_1 \geq d_1$, $b_2 > d_2$ or $a_1 > d_1$, $b_2 \geq d_2$.[2] The two pure strategy equilibria are (a_1,a_2) and (b_1,b_2). Row player 1 prefers (a_1,a_2), and col umn player 2 prefers (b_1,b_2).[3]

In the finitely and infinitely repeated versions of the game in Table 1 the two Nash equilibria are subgame perfect.[4] In the infinitely repeated game the following two strategies constitute a subgame

[1] Subsequently Ellsberg (1961) discussed the principle related to his paradoxes. Thereafter Harsanyi (1977) analyzed Zeuthen's principle, and Harsanyi and Selten (1988:90) applied the notion of risk dominance as a criterion for equilibrium selection.

[2] Table 1 encompasses games 64–69 in Rapoport and Guyer's (1966:213) ordinal taxonomy. The most well-known of these are the Battle of the Sexes (game 68), Chicken (game 66), and game 69 with several names such as "Let George do it", "Apology", "Hero", "Sacrificed leader". The games 64, 65, 67 are hybrid asymmetric games.

[3] In the finitely and infinitely repeated versions of the game in Table 1 the two Nash equilibria are subgame perfect. In the infinitely repeated game the following two strategies constitute a subgame perfect equilibrium with payoff (a_1,a_2) in each period: Player 1: Choose strategy I when challenged, unless strategy 2 was chosen in the past, then always choose strategy II. Player 2: Choose strategy I unless player 1 failed to choose strategy I in the past, then always choose strategy II. The justification for the subgame perfect equilibrium with payoff (b_1,b_2) in each period is analogous. For these two subgame perfect equilibria one player acquires a reputation for recalcitrance, the other for acquiescence. One problem with these two equilibria is that the reputation is never tested. Table 1 is equivalent to the probably most well-known example of entry deterrence, viz. the chain store game on normal form, when $a_1=d_1 \geq b_1 \geq t_1$, $b_2 \geq a_2=d_2 \geq t_2$, where player 1 is the incumbent (fight = strategy I, acquiesce = strategy II) and player 2 the entrant (stay out = strategy I, enter = strategy II). Both games have the same two Nash equilibria, but the chain store game in its finitely repeated version has only one unique subgame perfect equilibrium (proved by backward induction); the entrant enters and the incumbent does not fight. Kreps and Wilson (1982) and Milgrom and Roberts (1982) were first to formalize reputation effects, where a small amount of incomplete information can be sufficient to overcome Selten's (1978) chain store paradox. As Kreps and Wilson (1982:255) point out, the second equilibrium (the entrant stays out and the incumbent chooses the strategy "fight if entry") i s "imperfect" and "not so plausible as the first. It depends on an expectation by the entrant of the <incumbent's> behavior that, faced with the fait accompli of entry, would be irrational behavior for the <incumbent>." For text book treatments see e.g. Fudenberg and Tirole (1991:369–374), Osborne and Rubinstein (1994:105–106,239–243), Rasmusen (1989:85–118, 2001:110,129), Wilson (1985:31–33).

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One way around this problem is to introduce incomplete information so that reputations can be built.[5] A literature on reputation bounds has emerged, expressing the average discounted payoffs the players can guarantee to themselves. The first systematic treatment was presented by Fudenberg and Levine (1989, 1992).[6] Player 1 prefers the equilibrium (a_1,a_2), and gets t_1 in the threat point, so we define $a_{1\infty}$ as player 1's lower bound. Analogously, player 2 prefers the equilibrium (b_1,b_2), and gets t_2 in the threat point, so we define $b_{2\infty}$ as player 2's lower bound. For players involved in reputation building Schmidt (1993) determines for infinitely repeated games with conflicting interests and simultaneous moves in each period the two lower bounds

$$a_{1\infty} = \left(1-\mu_2^0\delta_1^{k_1(\mu_1^*,\delta_2)}\right)t_1 + \mu_2^0\delta_1^{k_1(\mu_1^*,\delta_2)}a_1, \quad b_{2\infty} = \left(1-\mu_1^0\delta_2^{k_2(\mu_2^*,\delta_1)}\right)t_2 + \mu_1^0\delta_2^{k_2(\mu_2^*,\delta_1)}b_2, \qquad (1)$$

expressed as average discounted payoffs, where[7]

$$k_1(\mu_1^*,\delta_2) = \frac{([N_1]+1)\ln\mu_1^*}{\ln(1-\varepsilon_1)}, \quad k_2(\mu_2^*,\delta_1) = \frac{([N_2]+1)\ln\mu_2^*}{\ln(1-\varepsilon_2)}, \qquad (2)$$

$$N_1 = \frac{\ln(1-\delta_2) + \ln(a_2-t_2)-\ln(b_2-t_2)}{\ln\delta_2}, \quad N_2 = \frac{\ln(1-\delta_1) + \ln(b_1-t_1)-\ln(a_1-t_1)}{\ln\delta_1},$$

$$\varepsilon_1 = \frac{(1-\delta_2)^2(a_2-t_2)}{(b_2-t_2)} + \delta_2^{([N_1]+1)}(1-\delta_2), \quad \varepsilon_2 = \frac{(1-\delta_1)^2(b_1-t_1)}{(a_1-t_1)} + \delta_1^{([N_2]+1)}(1-\delta_1),$$

where $[N_i]$ is the integer value of N_i, and $\mu_i^0>0$, $\mu_i^*=1-\mu_i^0>0$ are the probabilities of the "normal" and "committed" (always challenges) types, respectively, of player i, $i=1,2$. Eq. (1) states that if the probability μ_2^0 of the normal type of player 2 is close to one, and if player 1 is very patient, then the lower bound $0_{1\infty}$ for player 1 is close to his commitment payoff a_1. The bound $a_{1\infty}$ for player 1 (w.l.o.g.) is valid and reputation building has impact only when player 1 is sufficiently more patient than player 2.

[5] To allow a role for reputation at least one player must have private information that persists over time, this player must be likely to take several actions in sequence, and the player must be unable to commit in advance to the sequence of actions she will take (Wilson, 1985; Kreps and Wilson, 1982).

[6] See Celentani et al. (1996), Cripps et al. (1996), Sorin (1999), and Watson (1996) for subsequent treatments.

[7] For the literature on reputation bounds treated systematically first by Fudenberg and Levine (1989, 1992), see e.g. Celentani et al. (1996), Cripps et al. (1996), Sorin (1999), Watson (1996). See also Fudenberg and Levine (1989, 1992), Eq. (1) corresponds in Schmidt's (1993) article to (30) in Theorem 3, Eq. (2) corresponds to (22), (17), (18), and Eq. (3) corresponds to (28). (There is a printing error in (23) which does not follow from (18) and (37)).
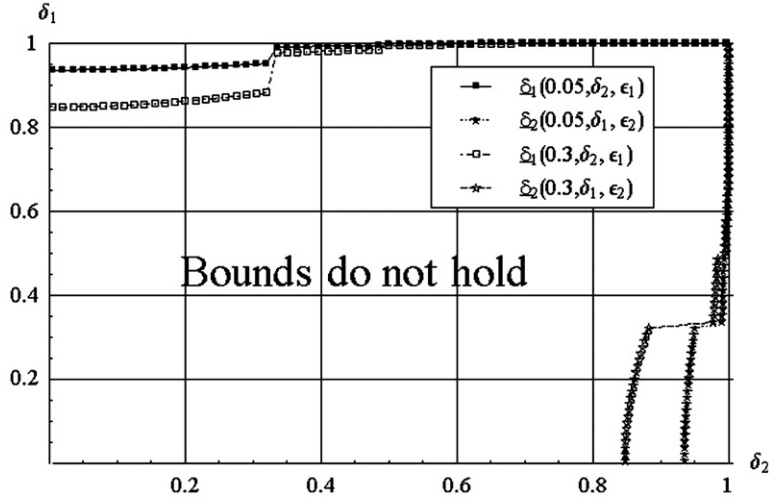
Fig. 1. $\underline{\delta}_1(\mu_1^*, \delta_2, \varepsilon_1)$ and $\underline{\delta}_2(\mu_2^*, \delta_1, \varepsilon_2)$ for $\mu_1^* = \mu_2^* = 0.05$ and $\mu_1^* = \mu_2^* = 0.3$.

That is (Schmidt, 1993:337), for any $\delta_2 < 1$, $\mu_1^* > 0$, $\varepsilon_1 > 0$, $\exists \delta_1 (\mu_1^*, \delta_2, \varepsilon_1) < 1$ s.t. for any $\delta_1 \geq \underline{\delta}_1$ $(\mu_1^*, \delta_2, \varepsilon_1)$ the average payoff of the normal type of player 1 is at least $a_{1\infty}$, where [8]

$$\underline{\delta}_1(\mu_1^*, \delta_2, \varepsilon_1) = \left(\frac{a_1 - t_1 - \varepsilon_1}{a_1 - t_1}\right)^{\frac{1}{k_1(\mu_1^*, \delta_2)}}, \quad \underline{\delta}_2(\mu_2^*, \delta_1, \varepsilon_2) = \left(\frac{b_2 - t_2 - \varepsilon_2}{b_2 - t_2}\right)^{\frac{1}{k_2(\mu_2^*, \delta_1)}}, \tag{3}$$

illustrated in Fig. 1 when $\mu_1^* = \mu_2^* = 0.05$ and $\mu_1^* = \mu_2^* = 0.3$, assuming $(a_1, a_2) = (4,3)$, $(b_1, b_2) = (3,4)$, $(t_1, t_2) = (2,2)$.[9] The area where the bounds do not hold is substantial. $[N_1]$ jumps discretely from 0 to 1 when $\delta_2 = 0.33$, from 1 to 2 when $\delta_2 = 0.5$, from 2 to 3 when $\delta_2 = 0.59$, from 3 to 4 when $\delta_2 = 0.65$, etc., which explain the discrete jumps in Fig. 1.

When the bound $a_{1\infty}(b_{2\infty})$ is valid for player 1 (2), the associated payoff to player 2 (1) compatible with (1) is

$$a_{2\infty} = \left(1 - \mu_2^0 \delta_1^{k_1(\mu_1^*, \delta_2)}\right) t_2 + \mu_2^0 \delta_1^{k_1(\mu_1^*, \delta_2)} a_2, \quad b_{1\infty} = \left(1 - \mu_1^0 \delta_2^{k_2(\mu_2^*, \delta_1)}\right) t_1 + \mu_1^0 \delta_2^{k_2(\mu_2^*, \delta_1)} b_1. \tag{4}$$

illustrated in Fig. 2 for $\delta_2 = 0.2$. That is, when $\mu_1^* = \mu_2^* = 0.3$ and $\delta_2 = 0.2$, player 1's discount factor must be above $\underline{\delta}_1(0.3, 0.2, \varepsilon_1) = 0.861519$ for the reputation bound $a_{1\infty}$ to be valid. The reputation bound $a_{1\infty}$ increases from $a_{1\infty} = 3.44$ when $\delta_1 = 0.861519$ to $a_{1\infty} = 3.90$ when $\delta_1 = 1$, which means that player 1 benefits from getting more patient. Player 2's associated payoff $a_{2\infty}$ is determined from Eq. (4), and increases from $a_{2\infty} = 2.72$ to $a_{2\infty} = 2.95$ in the same interval. In other words, as δ_1 becomes sufficiently

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[8] Eq. (1) corresponds in Schmidt's (1993) article to (30) in Theorem 3, Eq. (2) corresponds to (22), (17), (18), and Eq. (3) corresponds to (28). (There is a printing error in (23) which does not follow from (18) and (37)).
[9] Schmidt (1993:334) defines "conflicting interests" so that the commitment strategy of one player holds the other player down to his minmax payoff. This is satisfied when $d_2 > a_2$ or $d_1 > b_1$. See Cripps et al. (1996) for an analysis of games also without conflicting interests, which gives weaker bounds. I thank Larry Samuelson for a discussion about this point.
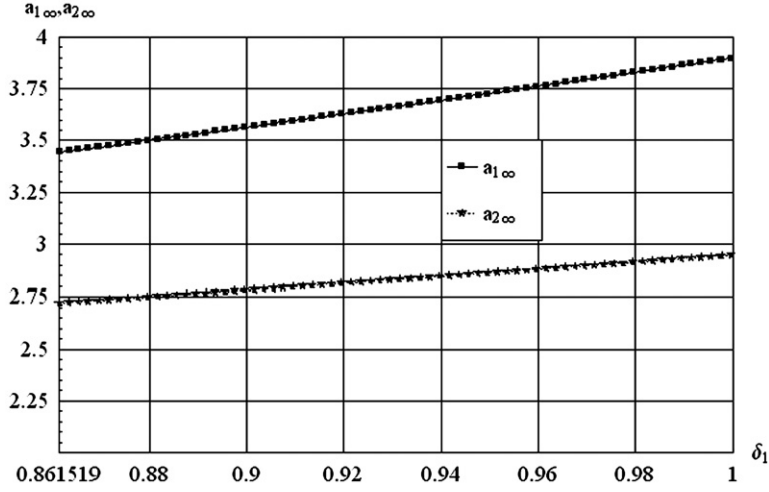
Fig. 2. The bound $a_{1\infty}$ and payoff $a_{2\infty}$ as functions of $\delta_1 \geq \underline{\delta}_1$ $(0.3,0.2,\varepsilon_1)=0.861519$.

larger than δ_2, player 1 can increase his lower bound $a_{1\infty}$ toward his most preferred payoff a_1, which also increases player 2's payoff toward $a_{2\infty}$. Let us formulate this as a property.

Property 1. *As one player becomes sufficiently more patient than the other player, both players benefit.*

Let us define player 1's risk limit as $r_1 = (a_1 - b_1)/(a_1 - t_1)$ in the static game in Table 1, which reaches its minimum at $r_1 = 0$ when player 1 is indifferent between the two equilibria, and reaches its maximum at $r_1 = 1$ when player 1 is indifferent between the threat point and the non-preferred equilibrium. In the infinitely repeated game player 1 can guarantee $a_{1\infty}$ rather than b_1 to himself. We thus define player 1's risk limit in the infinitely repeated game by replacing player 1's least preferred equilibrium payoff b_1 with $a_{1\infty}$. Player 1 strives toward the payoff a_1, and as the difference between a_1 and $a_{1\infty}$ decreases, player 1's risk limit decreases. We analogously define player 2's risk limit as $r_2 = (b_2 - a_2)/(b_2 - t_2)$ in the static game, replacing a_2 with $a_{2\infty}$ in the repeated game. Hence when player 1's lower bound $a_{1\infty}$ is valid, we define the risk limits and conflict measure

$$r_{11\infty} = \frac{a_1-a_{1\infty}}{a_1-t_1} = 1-\mu_2^0\delta_1^{k_1(\mu_1^*,\delta_2)}, \quad r_{21\infty} = \frac{b_2-a_{2\infty}}{b_2-t_2} = 1-\frac{(a_2-t_2)\mu_2^0\delta_1^{k_1(\mu_1^*,\delta_2)}}{b_2-t_2},$$
$$c_{r1\infty} = r_{11\infty}r_{21\infty}, \quad \delta_1 \geq \underline{\delta}_1(\mu_1^*,\delta_2,\varepsilon_1) \tag{5}$$

Defining conflict as the product of the players' risk limits has not been done earlier in the literature, but Axelrod (1970) has defined conflict in a static game such as Table 1 as

$$c = \frac{(a_1-b_1)(b_2-a_2)}{(a_1-t_1)(b_2-t_2)} \tag{6}$$

which happens to equal the product r_1r_2. In a two-dimensional utility diagram for the two players, Axelrod defines conflict as the relation of the small rectangle $(a_1-b_1)(b_2-a_2)$ of conflictful behavior to
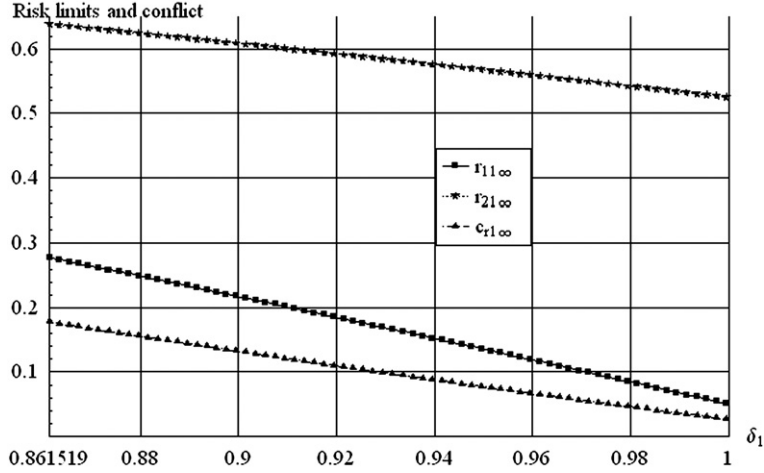
Fig. 3. The risk limits and conflict measure when $\delta_1 \geq \underline{\delta}_1$ $(0.3,0.2,\varepsilon_1)=0.861519$.

the large rectangle $(a_1 - t_1)(b_2 - t_2)$ of joint demand. Axelrod (1970:57) refers to the small rectangle as "the proportion of the joint demand area which is infeasible,"[10] and to the large rectangle as the area of joint demand spanned out by the threat point (t_1,t_2) and the outmost point determined by the best payoff (a_1,b_2) each player can possibly obtain under his most favorable circumstances.

When player 2's lower bound $b_{2\infty}$ is valid, we analogously define

$$r_{12\infty} = 1 - \frac{(b_1-t_1)\mu_1^0 \delta_2^{k_2(\mu_2^*,\delta_1)}}{a_1-t_1}, \quad r_{22\infty} = \frac{b_2-b_{2\infty}}{b_2-t_2} = 1 - \mu_1^0 \delta_2^{k_2(\mu_2^*,\delta_1)}, \tag{7}$$
$$c_{r2\infty} = r_{12\infty}r_{22\infty}, \quad \delta_2 \geq \underline{\delta}_2(\mu_2^*,\delta_1,\varepsilon_2).$$

Eq. (5) is illustrated in Fig. 3 for $\delta_2=0.2$. As player 1's patience increases beyond $\delta_1 \geq \underline{\delta}_1(0.3,0.2,\varepsilon_1)$, both risk limits and the conflict measure decrease. Increased difference between the players' emphasis on the future causes them to be more inclined to "cooperate" on the equilibrium preferred by the patient player. Conversely, as the players' emphasis on the future gets more similar, the conflict between them increases. Only the player who most successfully engages in costly reputation building in the present, which involves insisting on playing his preferred equilibrium to deter the other player from getting his preferred equilibrium, and which requires a high emphasis on the future, increases his chances to get his preferred equilibrium in the long run.

The benefits of "playing hard" are primarily in the future and can be gained only by choosing conflict today. Without explicit conflict in the present that higher payoff cannot be obtained. Players who place greater value on the future are more likely to choose conflict in order to reap those future benefits. Note the similarity between this reasoning and that of Skaperdas and Syropoulos (1996) and Garfinkel and Skaperdas (2000), where, despite a short-run incentive to settle a conflict, there can be long-term "compound rewards to cheating", or "long-term compounding rewards to going to war." If these compounded gains are large enough then conflictful and not "cooperative" behavior is the equilibrium.

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[10] It is the jointly infeasible expectation of an additional gain in case of a conflict.

Property 2. *As one player becomes sufficiently more patient than the other player, both players' risk limits, and the conflict between the players, decrease.*

The Properties 1 and 2 jointly mean that both players benefit from an increasing discrepancy in the players' discount factors, which causes lower risk limits and reduced conflict. That is, increasing differences in patience contribute to a greater likelihood of cooperation.

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