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Shield versus sword resource distribution in K-round duels

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Abstract The paper considers optimal resource distribution between offense and defense in a duel. In each round of the duel two actors exchange attacks distributing the offense resources equally across K rounds. The offense resources are expendable (e.g. missiles), whereas the defense resources are not expendable (e.g. bunkers). The outcomes of each round are determined by a contest success functions which depend on the offensive and defensive resources. The game ends when at least one target is destroyed or after K rounds. We show that when each actor maximizes its own survivability, then both actors allocate all their resources defensively. Conversely, when each actor minimizes the survivability of the other actor, then both actors allocate all their resources offensively. We then consider two cases of battle for a single target in which one of the actors minimizes the survivability of its counterpart whereas the counterpart maximizes its own survivability. It is shown that in these two cases the minmax survivabilities of the two actors are the same, and the sum of their resource fractions allocated to offense is equal to 1. However, their resource distributions are different. In the symmetric situation when the actors are equally resourceful and the two contest intensities are equal, then the actor that fights for the destruction of its counterpart allocates more resources to offense. We demonstrate a methodology of game analysis by illustrating how the resources, contest intensities and number of rounds in the duels impact the survivabilities and resource distributions.

Keywords: survivability, duel, defense, attack, protection, contest intensity, game theory

List of symbols

r, R	Actors' resources
ρ	Ratio r/R between actor 1's and actor 2's resources
x, X	Offense-defense resource distribution parameters
K	Number of consecutive attacks
s, S	Target survivability (probability of survival in all K attacks)
v, V	Success probability of each attack
p_i, P_i	Probability that the target is destroyed in the i-th attack
μ, m	Contest intensities in attacks against the actors
$\theta(q\|K)$	Conditional survivability of target 2 in q rounds of a K-round duel
$\Theta(q\|K)$	Conditional survivability of target 1 in q rounds of a K-round duel

1 Introduction

We consider two actors who fight offensively and defensively (exchange attacks) with each other over K rounds or until one target is destroyed. Each actor determines the optimal balance between the offense and defense which depends on their resources, the contest intensities, and the number of rounds of attack available. The optimal strategy depends on whether each actor maximizes its own survivability, minimizes the survivability of the other actor, or whether the actor maximizes its own survivability while its counterpart minimizes this survivability.

Through history it has been important to distinguish between the offense and defense. For example, Clausewitz (1984/1832, Sec 6.1.2) argued for classical warfare for the "superiority of defense over attack": "The defender enjoys optimum lines of communication and retreat, and can choose the place for battle." The attacker is advantaged by surprise, but gets exposed by leaving fortresses and depots behind through extended operations. The defense gets improved by trench warfare and the machine gun (World War I), and by castles and fortresses with cannon fire from secure locations.[1] Tanks and aviation (World War II) increased the attacker's advantage. Attackers are more advantaged in the cyber era where defenders have to defend everywhere, while attackers can attack at many locations and at many points in time (Anderson 2001). Protective shields are examples of defensive measures, while swords exemplify offensive measures. Law distinguishes between active and passive defense which depends on whether the person passively shields itself, or whether it actively exerts an action to prevent being harmed. In this paper we also allow the actor to be offensive.

In earlier research Azaiez and Bier (2007) consider the optimal resource allocation for security in reliability systems. They determine closed-form results for moderately general systems, assuming that the cost of an attack against any given component increases linearly in the amount of defensive investment in that component. Bier et al.

[1] Hausken (2004) suggests that the superiority of the defense over the offense may be even larger for production facilities and produced goods than for Clausewitz's mobile army.

(2005) and Bier and Abhichandani (2002) assume that the defender minimizes the success probability and expected damage of an attack. Bier et al. (2005) analyze the protection of series and parallel systems with components of different values. They specify optimal defenses against intentional threats to system reliability, focusing on the tradeoff between investment cost and security. The optimal defense allocation depends on the structure of the system, the cost-effectiveness of infrastructure protection investments, and the adversary's goals and constraints. Levitin (2007) con-siders the optimal element separation and protection in complex multi-state series-parallel system and suggests an algorithm for determining the expected damage caused by a strategic attacker. Patterson and Apostolakis (2007) introduced importance measures for ranking the system elements in complex systems exposed to terrorist actions. Michaud and Apostolakis (2006) analyzed such measures of damage caused by the terror as impact on people, impact on environment, impact on public image etc.

Bier et al. (2007) assume that a defender allocates defense to a collection of locations while an attacker chooses a location to attack. They show that the defender allocates resources in a centralized, rather than decentralized, manner, that the optimal allocation of resources can be non-monotonic in the value of the attacker's outside option. Furthermore, the defender prefers its defense to be public rather than secret. Also, the defender sometimes leaves a location undefended and sometimes prefers a higher vulnerability at a particular location even if a lower risk could be achieved at zero cost. Dighe et al. (2009) consider secrecy in defensive allocations as a strategy for achieving more cost-effective attacker deterrence. Zhuang and Bier (2007) consider defender resource allocation for countering terrorism and natural disasters.

Attack is sometimes the best defense, but not always. This paper seeks to determine when it is optimal to stay on the defensive and await the blow, and when it is optimal to attack offensively. Each actor has a fixed resource which can be divided into two fractions allocated to defense and offense.

Section 2 presents the model. Section 3 analyzes the model. Section 4 concludes.

2 The model

Two actors participate in a duel in which they repeatedly attack each other. The total number of consecutive attacks is K, unless one actor is destroyed in attack i and the game ends. The actors have limited resources r and R. Each actor distributes its resource among the defense (protection) and offense (attack). The distribution is determined by the parameters x and X respectively: resources xr and XR are allocated to offense and resources $(1 - x)r$ and $(1 - X)R$ are allocated to defense. x and X are the two free choice variables. We denote the actor choosing x as actor 1 and the actor choosing X as actor 2. We assume that for both actors the offense resources are expendable (missiles) whereas the defense resources are not expendable (bunkers), which means that the actors use the same protection during the series of K attacks. The offense resources are distributed evenly across the K attacks, i.e. xr/K and XR/K for each attack. Assuming equal offense resource distribution across the K attacks constitutes a benchmark and is made for two reasons. The first is logistical. If the actor is a person, its strength is limited. It cannot suddenly triple its strength in one round. Despite becoming exhausted, the person may hold up well over the rounds. If the actor is an air force with a fixed number of airplanes, these can be equipped with only a finite number of missiles for each attack, before returning to base and redeploying for the subsequent attack. The second reason is analytical tractability. Assuming unequal offense resource distribution requires applying a different method of analysis.[2] Each actor can observe the outcome of each attack and cease the attacks if the counterpart is destroyed.

In order to determine the vulnerability of an attacked target we use the common ratio form contest success function (Skaperdas 1996; Tullock 1980) $w = T^h/(T^h + t^h)$, where w is the probability of target destruction, T is the attacker's effort, t is the defender's effort, $\partial w/\partial T > 0$, $\partial w/\partial t < 0$, and $h \geq 0$ is a parameter for the contest intensity. When $h = 0$, t and T have equal impact on w regardless of their size which gives 50% vulnerability. When $0 < h < 1$, there is a disproportional advantage of exerting less effort than one's opponent. When $h = 1$, the efforts have proportional impact on the w. When $h > 1$, exerting more effort than one's opponent gives a disproportional advantage. Finally, $h = \infty$ gives a step function where "winner-takes-all".

In our case we have attacks against two actors in each of the K duels. The contest intensities in these attacks can be different. We denote the contests intensities in attacks against actors 1 and 2 as m and μ respectively. The success probability of actor 1 investing an offense xr/K in each attack against the defense $(1 - X)R$ by actor 2 is

$$v = \frac{(xr/K)^\mu}{(xr/K)^\mu + [(1 - X)R]^\mu} = \frac{1}{1 + [(1 - X)K/(x\rho)]^\mu}. \tag{1}$$

Analogously, the success probability of actor 2 investing an offense XR/K against the defense $(1 - x)r$ by actor 1 in each attack is

$$V = \frac{(XR/K)^m}{(XR/K)^m + [(1 - x)r]^m} = \frac{1}{1 + [(1 - x)\rho K/X]^m}, \tag{2}$$

where $\partial V/\partial X > 0$, $\partial V/\partial R > 0$, $\partial V/\partial x > 0$, $\partial V/\partial r < 0$, and $m \geq 0$ is the intensity of the contest when actor 2 is offensive. The contest intensities can depend on target locations and, therefore, are different for the two attacks.

In order to destroy the counterpart target in the j-th attack, each actor must survive in all $j - 1$ previous attacks. The target can be destroyed in the j-th attack only if it has not been destroyed in any of the $j - 1$ previous attacks. Therefore the probabilities that the counterpart target is destroyed in the j-th attack are

$$p_j = v(1 - V)^{j-1}(1 - v)^{j-1},\ P_j = V(1 - V)^{j-1}(1 - v)^{j-1}. \tag{3}$$

[2] Consider in this regard Azaiez and Bier's (2007) work. They analyze the protection of n components, where each component may play the role of a round. They proved that the optimal defensive strategy was to strengthen the most attractive component (the most fragile in a series configuration) until all resources are depleted or this component becomes as attractive as the next one, and then strengthen simultaneously both "fragile" components until all resources are depleted or they become as attractive as the next one, and so on.

The probabilities of targets 1 and 2 destruction in K consecutive attacks are $\sum_{j=1}^{K} P_j$ and $\sum_{j=1}^{K} p_j$ respectively.

Since the success probability of any one of K attacks is the same, we obtain the survivabilities S and s of targets 1 and 2 as

$$
\begin{aligned}
S &= 1-\sum_{j=1}^{K} P_j = 1 - V\sum_{j=1}^{K}(1-V)^{j-1}(1-v)^{j-1} \\
&= 1-\frac{1}{1+[(1-x)\rho K/X]^m}\sum_{j=0}^{K-1}\left(\frac{[(1-x)\rho K/X]^m}{1+[(1-x)\rho K/X]^m}\,\frac{[(1-X)K/(x\rho)]^\mu}{1+[(1-X)K/(x\rho)]^\mu}\right)^j \\
&= \frac{[(1-x)\rho K/X]^m + (1+[(1-X)K/(x\rho)]^\mu)^{1-K}\left(\dfrac{[(1-x)\rho K/X]^m\,[(1-X)K/(x\rho)]^\mu}{\left(1+[(1-x)\rho K/X]^m\right)}\right)^K}{1+[(1-x)\rho K/X]^m+[(1-X)K/(x\rho)]^\mu} \\
s &= 1-\sum_{j=1}^{K} p_j = 1 - v\sum_{j=1}^{K}(1-V)^{j-1}(1-v)^{j-1} \\
&= 1-\frac{1}{1+[(1-X)K/(x\rho)]^\mu}\sum_{j=0}^{K-1}\left(\frac{[(1-x)\rho K/X]^m}{1+[(1-x)\rho K/X]^m}\,\frac{[(1-X)K/(x\rho)]^\mu}{1+[(1-X)K/(x\rho)]^\mu}\right)^j \\
&= \frac{[(1-X)K/(x\rho)]^\mu + \left(1+[(1-x)\rho K/X]^m\right)^{1-K}\left(\dfrac{[(1-x)\rho K/X]^m\,[(1-X)K/(x\rho)]^\mu}{(1+[(1-X)K/(x\rho)]^\mu)}\right)^K}{1+[(1-x)\rho K/X]^m+[(1-X)K/(x\rho)]^\mu},
\end{aligned}
\tag{5}
$$

where the sum is a geometric series. Observe that x, V, S correspond to actor 1 and its object, whereas X, v, s correspond to actor 2 and its object.

The probability that a target survives at least $q \le K$ rounds in a K-round duel follows from replacing K with q in the upper limit of the summation signs in (5). The conditional survivabilities $\Theta(q|K)$ and $\theta(q\,|K)$ of targets 1 and 2 are

$$
\begin{aligned}
\Theta(q|K) &= 1-\sum_{j=1}^{q} P_j \\
&= \frac{[(1-x)\rho K/X]^m + (1+[(1-X)K/(x\rho)]^\mu)^{1-q}\left(\dfrac{[(1-x)\rho K/X]^m\,[(1-X)K/(x\rho)]^\mu}{\left(1+[(1-x)\rho K/X]^m\right)}\right)^q}{1+[(1-x)\rho K/X]^m+[(1-X)K/(x\rho)]^\mu} \\
\theta(q|K) &= 1-\sum_{j=1}^{q} p_j \\
&= \frac{[(1-X)K/(x\rho)]^\mu + \left(1+[(1-x)\rho K/X]^m\right)^{1-q}\left(\dfrac{[(1-x)\rho K/X]^m\,[(1-X)K/(x\rho)]^\mu}{(1+[(1-X)K/(x\rho)]^\mu)}\right)^q}{1+[(1-x)\rho K/X]^m+[(1-X)K/(x\rho)]^\mu},
\end{aligned}
\tag{6}
$$

3 Analyzing the model

We consider four cases. First, in Sect. 3.1 each actor maximizes its own survivability (self-interest situation). Second, in Sect. 3.2 each actor minimizes the survivability of the other actor (mutual aggression). In Appendix A actor 1 minimizes the survivability s while actor 2 maximizes its survivability s (battle for s). Conversely, in Appendix B actor 1 maximizes its survivability S while actor 2 minimizes actor 1's survivability S (battle for S). In the battle for s and in the battle for S both actors focus exclusively on the survival of one of them, ignoring the survival of the other actor. Finally, Sect. 3.3 compares Appendices A and B.

3.1 Maximizing s and S

Assume that each actor maximizes its own survivability. If actor 1 chooses $x = 0$ and actor 2 chooses $X = 0$, the target destruction probabilities in (3) are $v = V = 0$ and the survivabilities in (5) are $s = S = 1$. No actor has an incentive to deviate unilaterally from this maximum survivability which thus constitutes an optimal solution where both actors are pacifistic. Both actors refrain from attack and focus exclusively on defense.

3.2 Minimizing S and s

Assume that each actor minimizes the survivability of the other actor. If actor 1 chooses $x = 1$ and actor 2 chooses $X = 1$, the target destruction probabilities in (3) are $v = V = 1$ and the survivabilities in (5) are $s = S = 0$. No actor has an incentive to deviate unilaterally from this minimum survivability which thus constitutes an optimal solution where both actors are maximally offensive. Both actors refrain from defense and focus exclusively on attack. See Appendix A for calculations of battle for s, and Appendix B for calculations of battle for S.

3.3 Comparing battle for s and battle for S solutions

We define $X^*(\text{opt } s)$ as the optimal X in the battle for s, and $X^*(\text{opt } S)$ as the optimal X in the battle for S.

Proposition 1 *When both actors focus exclusively on the survival of one of the actors, maximizing it and minimizing it, respectively, ignoring the survivability of the other actor, then the survivabilities of the two actors are the same, though their allocations to offense and defense are different.*

Proof Follows from the equivalence of (A3) and (B3) causing (A4), and the difference between (A6) and (B4).

Proposition 1 is interpreted as follows: when optimizing s, actor 2 chooses X^* from (A6) and actor 1's best response is $x^* = 1 - X^*$, which gives (A4). When optimizing S, if actor 2 chooses X^* from (A6), then actor 1 responds with some $x^* \neq 1 - X^*$, which gives S larger than in (A4). Actor 2 can achieve lower S by choosing X^* from (B4), and actor 1's best response is $x^* = 1 - X^*$, which gives (A4) again. That two different games and two different strategies cause the same survivabilities has to do with $x^* = 1 - X^*$ and the symmetry of the two situations. Both actors have different focuses in the two situations, though in a manner causing the same survivabilities.

The linkage between (A6) and (B4) is as follows: if we replace m with μ, μ with m and ρ with $1/\rho$, the situation becomes symmetrically opposite to the actors and, therefore, $X^*(\text{opt } S)$ in (B4) is equal to $x^*(\text{opt } s) = 1 - X^*(\text{opt } s)$ in (A6). After such replacement in (B4), summing (B4) and (1 gives $X^*(\text{opt } s) + X^*(\text{opt } S) = 1$.

Inserting $m = \mu$ and $\rho = 1$ into (A6) and (B4) (which corresponds to total symmetry in the duel) gives

$$X^*(\text{opt s}) = \frac{K^{\mu}}{2K^{\mu}+1} - \frac{K^{1-\mu}(K^{\mu})^{2K}}{(K^{\mu}+1)^{2K} - (K^{\mu})^{2K}}, \tag{7}$$

$$X^*(\text{opt S}) = \frac{K^{\mu}+1}{2K^{\mu}+1} + \frac{K^{1-\mu}(K^{\mu})^{2K}}{(K^{\mu}+1)^{2K} - (K^{\mu})^{2K}} \tag{8}$$

which sum to one. Equations (7) and (8) imply

$$X^*(\text{opt}S) - X^*(\text{opt}s) = \frac{1}{2K^{\mu}+1} + \frac{2K^{1-\mu}(K^{\mu})^{2K}}{(K^{\mu}+1)^{2K} - (K^{\mu})^{2K}} > 0. \tag{9}$$

This means that in the symmetric situation the actor that fights for survival must always be less aggressive, by spending more resources on defense, than the actor that fights for destruction of its counterpart.[3]

Figure 1 plots $X^*(\text{opt } s)$, $X^*(\text{opt } S)$, s^*, and S^* as functions of ρ for $K = 5$ and different m and μ. Observe that $X^*(\text{opt}S)$ is larger than $X^*(\text{opt } s)$ which illustrates non-symmetric situations where the actor that fights for the destruction of its counterpart is more aggressive than the actor that fights for its own survival. When both contest intensities are small the egalitarianism causes intermediate $X^*(\text{opt } s)$ and $X^*(\text{opt } S)$ which increase gradually as actor 1 becomes more resourceful (ρ increases) leading actor 2 to focus more on offense. When $\mu = 2$, $m = 0.5$, the contest where actor 2 is defensive is most intensive, causing a U shaped X^* when actor 2 battles defensively for low S. Actor 2 chooses high X^* (low $1 - X^*$) due to strength when ρ is low, and due to weakness when ρ is high. Actor 2 chooses low X^* when battling offensively for high s, caused by the low m. Conversely, when $m = 2$, $\mu = 0.5$, the contest where actor 2 is offensive is most intensive, causing an inverse U shaped X^* when actor 2 battles offensively for high s. Actor 2 chooses low X^* due to strength when ρ is low, and due to weakness when ρ is high. Actor 2 chooses high X^* (low $1 - X^*$) when battling defensively for low S, caused by the low μ. Finally, when both contest

[3] Numerical tests suggest that $X^*(\text{opt } S) - X^*(\text{opt } s) > 0$ for general m, μ and ρ, but we have not been able to prove that.

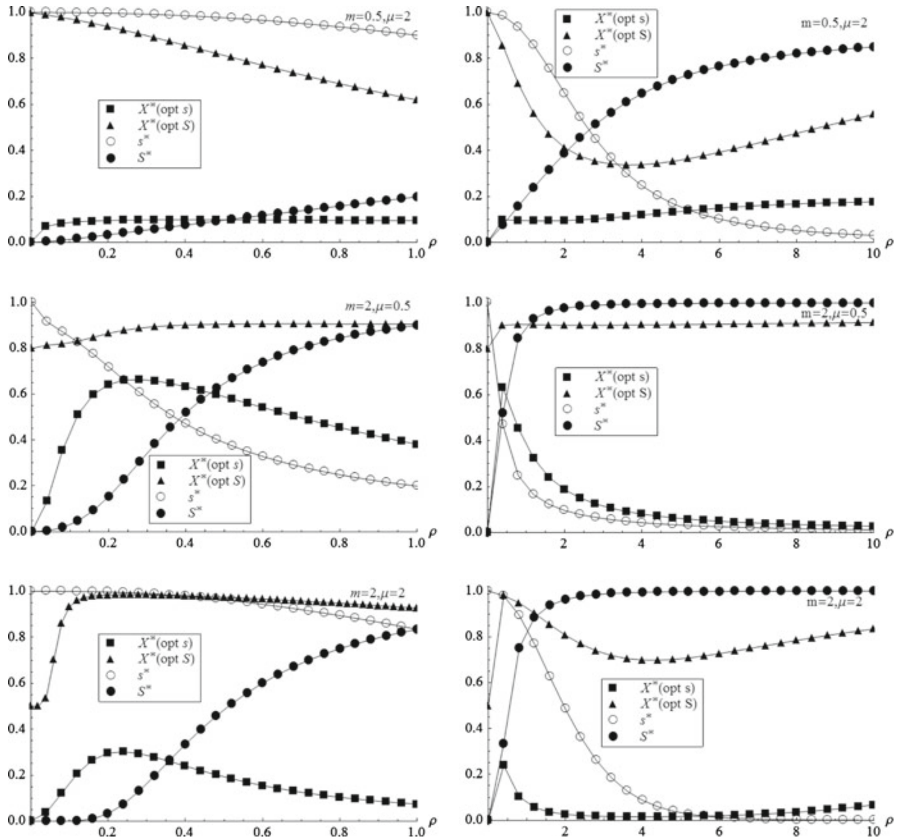

Fig. 1 X^*(opt s), X^*(opt S), s^*, and S^* as functions of ρ for $K = 5$ and different m and μ

intensities are high, X^*(opt s) is U shaped and X^*(opt S) is inverse U shaped, and the survivabilities change more abruptly dependent on which actor enjoys resource superiority.

Figure 2 plots X^*(opt s), X^*(opt S), s^*, and S^* as functions of K for different ρ, m and μ. When $\mu = 2$, $m = 0.5$, X^* (opt s) increases in K and X^*(opt S) decreases in K. When $\mu = 2$, $m = 0.5$, as K increases, actor 2 enjoys higher survivability s while actor 1 suffers lower survivability S. The reason is that a higher K makes actor 1 less successful in each attack when μ is large as expressed with a lower $(xr/K)^{\mu}$ in the numerator of (1) causing lower v. Conversely, when $\mu = 0.5$, $m = 2$, as K increases, actor 1 enjoys higher survivability S while actor 2 suffers lower survivability s, using the same reasoning with $(XR/K)^m$ in (2). When $m = 2$, $\mu = 2$, both actors enjoy higher survivabilities as K increases.

Figure 3 plots X^*(opt s), X^*(opt S), $\theta(q\,|K)$, and $\Theta(q\,|K)$ as functions of q for different ρ, m and μ when $K = 5$. In accordance with (A4) the conditional survivabilities decrease in q reflecting the risk of continued dueling. In some panels the

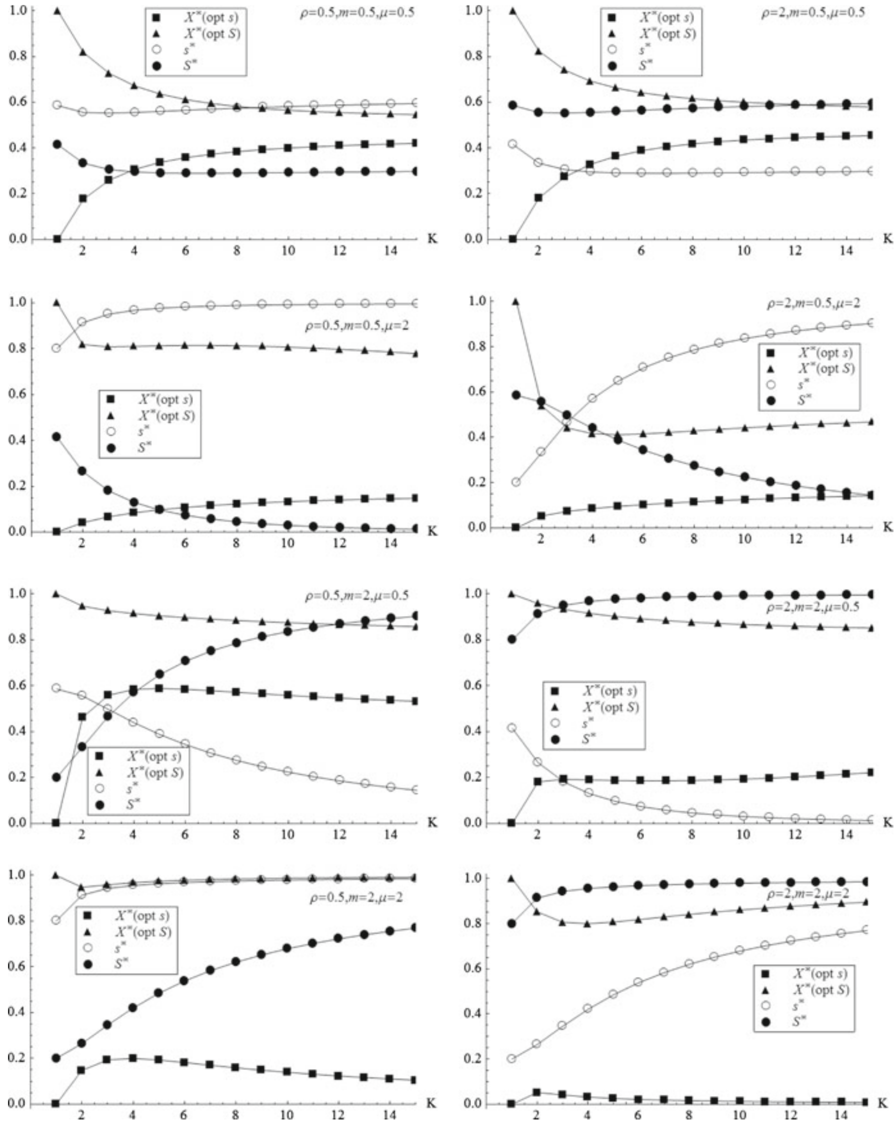

Fig. 2 $X^*(\text{opt } s)$, $X^*(\text{opt } S)$, s^*, and S^* as functions of K for different ρ, m and μ

decrease is strong and in others it is barely visible. For example, for $\rho = m = \mu = 2$ the function $\Theta(q\,|K)$ decreases from 0.99 for $q = 1$ t o0.963 for $q = 5$. For the more egalitarian contests $m = \mu = 0.5$, the conditional survivabilities $\theta(q\,|K)$, and $\Theta(q\,|K\,)$ are intermediate and their decreases in q are moderate. For these values of m and μ the inequality in (A5) where $q = 1$ becomes $1 \geq \rho$ which benefits actor 2 in the left panel and actor 1 in the right panel. When $m = 0.5$ and $\mu = 2$, the inequality in (A5) becomes $5^{1.5} \geq \rho^{2.5}$ which benefits actor 2 in both panels. Actor 2 benefits

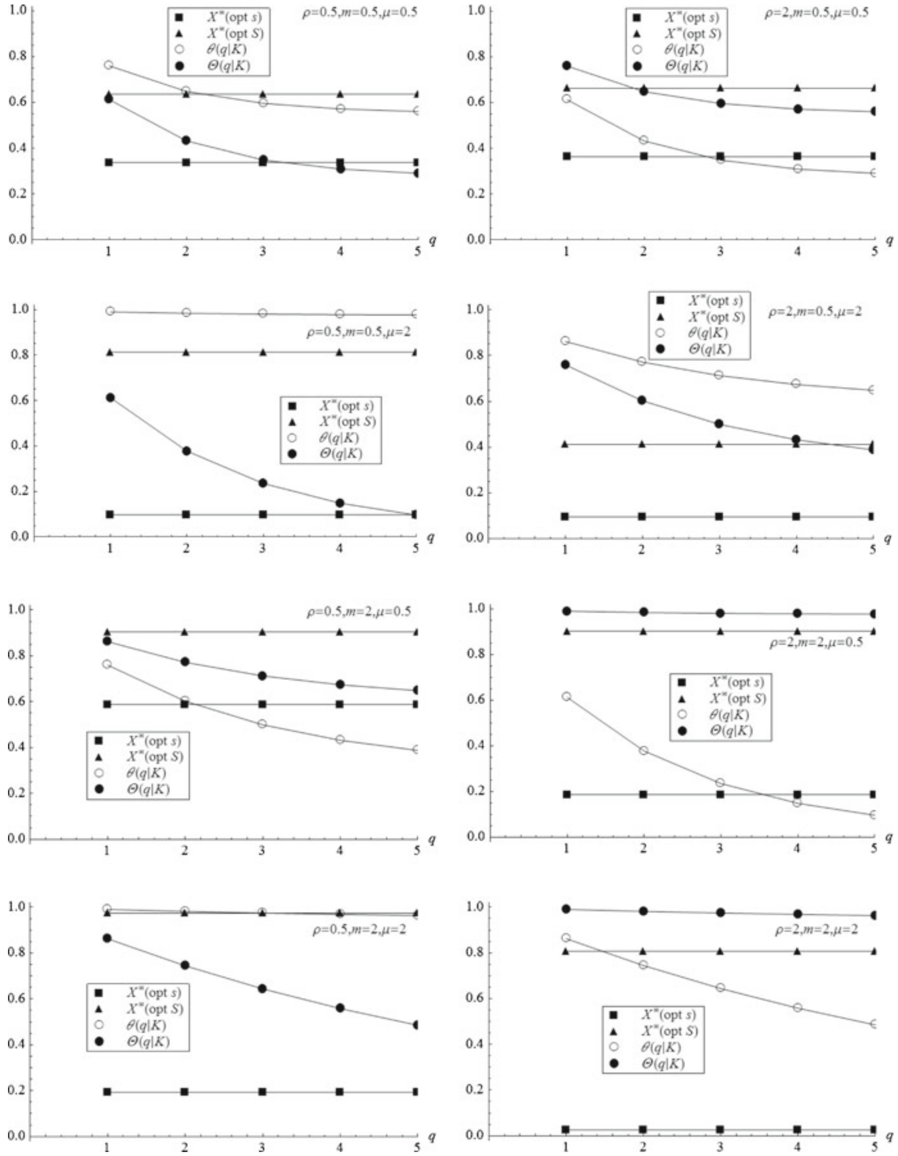

Fig. 3 $X^*(\text{opt } s)$, $X^*(\text{opt } S)$, $\theta(q|K)$, and $\Theta(q|K)$ as functions of q for different ρ, m, μ and $K = 5$

especially in the left panel, enjoying the low $\rho = 0.5$, while actor 1's $\Theta(q\ |K\ )$ decreases rapidly in q. When $m = 2$ and $\mu = 0.5$, the inequality in (A5) becomes $5^{-1.5} \geq \rho^{2.5}$ which benefits actor 1 in both panels. For the jointly intensive contests $m = \mu = 2$, the inequality in (A5) becomes $1 \geq \rho^4$ which benefits actor 2 in the left panel and actor 1 in the right panel. The survivability of the disadvantaged actor decreases rapidly in q because of the high contest intensities.

4 Conclusion

We consider K repeated duels in which two actors distribute their resources between offense and defense. The offense resources are expendable (e.g. missiles) and are distributed equally across K attacks, while the defense resources are not expendable (e.g. bunkers). The outcomes of the two duels in each round are determined by a contest success function which depends on the offensive and defensive resources. The game ends when at least one target is destroyed or after K rounds.

We show that when each actor maximizes its own survivability, then both actors allocate all their resources defensively. Conversely, when each actor minimizes the survivability of its counterpart, then both actors allocate all their resources offensively. We then consider two cases of battle for a single target in which one of the actors minimizes the survivability of its counterpart whereas the counterpart maximizes its own survivability. In these two cases the two actors' minmax survivabilities are the same, and the sum of their resource fractions allocated to offense equals 1. However, their allocations to offense and defense in the two cases are different. When the actors are equally resourceful and the two contest intensities are equal, then the actor that fights for the destruction of its counterpart always allocates more resources to offense. Analyzing the minmax solutions we demonstrate how the resources, contest intensities and number of attacks impact the survivabilities and allocations to offense and defense. It can be seen that the increase of the number of attacks in the duel K is favorable for the actor that has greater contest intensity when defending itself and lower contest intensity when attacking its counterpart.

Appendix A: Battle for s

When one actor chooses x to minimize s, and the other actor chooses X to maximize s, we get the two FOCs

$$
\begin{aligned}
\frac{\partial \mathrm{s}}{\partial \mathrm{x}} = {} & \Bigg(\Bigg(\bigg(\frac{K(1-x)\rho}{X}\bigg)^{m}+1\Bigg)^{-K}\Bigg(\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu}+1\Bigg)^{-K-1}\Bigg(K\Bigg(\bigg(\frac{K(1-x)\rho}{X}\bigg)^{m}\Bigg)^{K} \\
& \times\Bigg(\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu}\Bigg)^{K}\Bigg(\bigg(\frac{K(1-x)\rho}{X}\bigg)^{m}+\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu}+1\Bigg)\Bigg(mx\Bigg(\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu}+1\Bigg) \\
& -(x-1)\mu\Bigg(\bigg(\frac{K(1-x)\rho}{X}\bigg)^{m}+1\Bigg)\Bigg)+\Bigg(\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu}+1\Bigg)\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu} \\
& \times\Bigg((mx+(x-1)\mu)\bigg(\frac{K(1-x)\rho}{X}\bigg)^{m}+(x-1)\mu\Bigg)\Bigg(\Bigg(\bigg(\frac{K(1-x)\rho}{X}\bigg)^{m}\Bigg)^{K}\Bigg(\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu}\Bigg)^{K} \\
& -\Bigg(\bigg(\frac{K(1-x)\rho}{X}\bigg)^{m}+1\Bigg)^{K}\Bigg(\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu}+1\Bigg)^{K}\Bigg)\Bigg)\Bigg)\Bigg/\Bigg((x-1)x\Bigg(\bigg(\frac{K(1-x)\rho}{X}\bigg)^{m} \\
& +\bigg(\frac{K-KX}{x\rho}\bigg)^{\mu}+1\Bigg)^{2}\Bigg)=0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathrm{s}}{\partial \mathrm{X}} = &-\Bigg(\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)^{-K}\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)^{-K-1}\Bigg(K\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K \\
&\times\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K\left(\left(\frac{K(1-x)\rho}{X}\right)^m+\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)\Bigg(m(X-1)\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right) \\
&-X\mu\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)\Bigg)+\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)\left(\frac{K-KX}{x\rho}\right)^\mu\Bigg((m(X-1)+X\mu) \\
&\times\left(\frac{K(1-x)\rho}{X}\right)^m+X\mu\Bigg)\Bigg(\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K \\
&-\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)^K\Bigg)\Bigg)\Bigg)\Bigg/\Bigg((X-1)X\Bigg(\left(-\frac{K(x-1)\rho}{X}\right)^m \\
&+\left(\frac{K-KX}{x\rho}\right)^\mu+1\Bigg)^2\Bigg)=0 \qquad \text{(A1)}
\end{aligned}
$$

which can be written as

$$
\frac{\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)\left(\frac{K-KX}{x\rho}\right)^\mu\left(\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K-\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)^K\right)}{K\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K\left(\left(\frac{K(1-x)\rho}{X}\right)^m+\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)}
$$

$$
=\frac{mx\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)+(1-x)\mu\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)}{((1-x)\mu-mx)\left(\frac{K(1-x)\rho}{X}\right)^m+(1-x)\mu},
$$

$$
\frac{\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)\left(\frac{K-KX}{x\rho}\right)^\mu\left(\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K-\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)^K\right)}{K\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K\left(\left(\frac{K(1-x)\rho}{X}\right)^m+\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)}
$$

$$
=\frac{m(1-X)\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)+X\mu\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)}{(X\mu-m(1-X))\left(\frac{K(1-x)\rho}{X}\right)^m+X\mu} \qquad \text{(A2)}
$$

The two LHSs in (A2) are equal. Inspecting each term on the RHSs reveals that these are equal when $x = 1 - X$, which gives the optimal solution

$$
x^* = 1 - X^* \qquad \text{(A3)}
$$

which is inserted into (1), (2), (5), and (6) to yield

$$
\begin{aligned}
v^* &= \frac{1}{1+[K/\rho]^\mu},\ V^* = \frac{1}{1+[\rho K]^m}, \\
s^* &= \frac{[K/\rho]^\mu + (1+[\rho K]^m)\left(\frac{[\rho K]^m[K/\rho]^\mu}{(1+[\rho K]^m)(1+[K/\rho]^\mu)}\right)^K}{1+[\rho K]^m+[K/\rho]^\mu}, \\
S^* &= \frac{[\rho K]^m + (1+[K/\rho]^\mu)\left(\frac{[K/\rho]^\mu[\rho K]^m}{(1+[K/\rho]^\mu)(1+[\rho K]^m)}\right)^K}{1+[\rho K]^m+[K/\rho]^\mu}, \quad \frac{1-s^*}{1-S^*} = \frac{v^*}{V^*},
\end{aligned}
$$

$$\theta(q\,|K) = \frac{[K/\rho]^{\mu} + (1 + [\rho K]^m)\left(\frac{[\rho K]^m [K/\rho]^{\mu}}{(1+[\rho K]^m)(1+[K/\rho]^{\mu})}\right)^q}{1 + [\rho K]^m + [K/\rho]^{\mu}},$$

$$\Theta(q\,|K) = \frac{[\rho K]^m + (1 + [K/\rho]^{\mu})\left(\frac{[K/\rho]^{\mu} [\rho K]^m}{(1+[K/\rho]^{\mu})(1+[\rho K]^m)}\right)^q}{1 + [\rho K]^m + [K/\rho]^{\mu}} \tag{A4}$$

Since the ratio raised to q in (A4) is smaller than 1, it follows that $\partial\theta(q\,|K)/\partial q < 0$ and $\partial\Theta(q\,|K)/\partial q < 0$ when $q \geq 1$. Inserting $q = 1$ into (A4) gives

$$\theta(1\,|K) = \frac{[K/\rho]^{\mu}}{1 + [K/\rho]^{\mu}}, \quad \Theta(1\,|K) = \frac{[\rho K]^m}{1 + [\rho K]^m},$$
$$\theta(1\,|K) \geq \Theta(1\,|K) \Leftrightarrow K^{\mu-m} \geq \rho^{\mu+m} \tag{A5}$$

Solving (A2) when $x = 1 - X$ gives

$$X^* = \frac{m\gamma\left(\alpha^K\beta^K\,(K(\beta+\gamma)+\alpha\beta) - \alpha\beta\gamma^K\delta^K\right)}{\alpha\gamma\,(\beta(m+\mu)+\mu)\left(\alpha^K\beta^K - \gamma^K\delta^K\right) + K\alpha^K\beta^K\,(\alpha+\beta+1)\,(m\gamma-\delta\mu)}, \tag{A6}$$

where $\alpha = \left(\frac{K}{\rho}\right)^{\mu}$, $\beta = (K\rho)^m$, $\gamma = \alpha + 1$, $\delta = \beta + 1$.

Appendix B: Battle for *S*

When one actor chooses x to maximize S, and the other actor chooses X to minimize S, we get the two FOCs

$$\begin{aligned}
\frac{\partial \mathrm{S}}{\partial \mathrm{x}} = {} & \Bigg(\bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m + 1\bigg)^{-K-1} \bigg(\Big(\frac{K-KX}{x\rho}\Big)^{\mu} + 1\bigg)^{-K} \Bigg(K\bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m\bigg)^K \\
& \times \bigg(\Big(\frac{K-KX}{x\rho}\Big)^{\mu}\bigg)^K \bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m + \Big(\frac{K-KX}{x\rho}\Big)^{\mu} + 1\bigg)\bigg(mx\bigg(\Big(\frac{K-KX}{x\rho}\Big)^{\mu} + 1\bigg) \\
& -(x-1)\mu\bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m + 1\bigg)\bigg) - \Big(\frac{K(1-x)\rho}{X}\Big)^m \bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m + 1\bigg) \\
& \times \bigg((mx + (x-1)\mu)\Big(\frac{K-KX}{x\rho}\Big)^{\mu} + mx\bigg)\Bigg(\bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m\bigg)^K \bigg(\Big(\frac{K-KX}{x\rho}\Big)^{\mu}\bigg)^K \\
& - \bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m + 1\bigg)^K \bigg(\Big(\frac{K-KX}{x\rho}\Big)^{\mu} + 1\bigg)^K\Bigg)\Bigg)\Bigg) \Bigg/ \Bigg((x-1)x\bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m \\
& + \Big(\frac{K-KX}{x\rho}\Big)^{\mu} + 1\bigg)^2\Bigg) = 0 \\
\frac{\partial \mathrm{S}}{\partial \mathrm{X}} = {} & \Bigg(\bigg(\Big(\frac{K(1-x)\rho}{X}\Big)^m + 1\bigg)^{-K} \bigg(\Big(\frac{K-KX}{x\rho}\Big)^{\mu} + 1\bigg)^{-K} \\
& \times \Bigg(\frac{K\Big(\big(\frac{K(1-x)\rho}{X}\big)^m\Big)^K \Big(\big(\frac{K-KX}{x\rho}\big)^{\mu}\Big)^K \Big(\big(\frac{K(1-x)\rho}{X}\big)^m + \big(\frac{K-KX}{x\rho}\big)^{\mu} + 1\Big)\Big(X\mu\Big(\big(\frac{K(1-x)\rho}{X}\big)^m + 1\Big) - m(X-1)\Big(\big(\frac{K-KX}{x\rho}\big)^{\mu} + 1\Big)\Big)}{\big(\frac{K(1-x)\rho}{X}\big)^m + 1}
\end{aligned}$$

$$\times\left(\frac{K(1-x)\rho}{X}\right)^m\left((m(X-1)+X\mu)\left(\frac{K-KX}{x\rho}\right)^\mu+m(X-1)\right)$$
$$\times\left(\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K-\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)^K\right.$$
$$\left.\times\ \left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)^K\right)\Bigg)\Bigg/\left((X-1)X\left(\left(\frac{K(1-x)\rho}{X}\right)^m+\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)^2\right)=0 \tag{B1}$$

which can be written as

$$\frac{\left(\frac{K(1-x)\rho}{X}\right)^m\left(1+\left(\frac{K(1-x)\rho}{X}\right)^m\right)\left(\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K-\left(1+\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K\left(1+\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\right)}{K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(1+\left(\frac{K-KX}{x\rho}\right)^\mu+\left(\frac{K(1-x)\rho}{X}\right)^m\right)}$$
$$=\frac{mx\left(1+\left(\frac{K-KX}{x\rho}\right)^\mu\right)+(1-x)\mu\left(1+\left(\frac{K(1-x)\rho}{X}\right)^m\right)}{mx+(mx-(1-x)\mu)\left(\frac{K-KX}{x\rho}\right)^\mu},$$
$$\frac{\left(\frac{K(1-x)\rho}{X}\right)^m\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)\left(\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K-\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)^K\right)}{K\left(\left(\frac{K(1-x)\rho}{X}\right)^m\right)^K\left(\left(\frac{K-KX}{x\rho}\right)^\mu\right)^K\left(\left(\frac{K(1-x)\rho}{X}\right)^m+\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)}$$
$$=\frac{m(1-X)\left(\left(\frac{K-KX}{x\rho}\right)^\mu+1\right)+X\mu\left(\left(\frac{K(1-x)\rho}{X}\right)^m+1\right)}{(m(1-X)-X\mu)\left(\frac{K-KX}{x\rho}\right)^\mu+m(1-X)} \tag{B2}$$

The two LHSs in (B2) are equal. Inspecting each term on the RHSs reveals that these are equal when $x=1-X$, which gives the optimal solution

$$x^*=1-X^* \tag{B3}$$

and hence v^*, V^*, s^*, and S^* are the same as in (A4). However, X^* and x^* are not the same.

Solving (B2) when $x=1-X$ gives

$$X^*=-\frac{m\gamma\left(\alpha^K\beta^K\left(K(\alpha+\beta+1)-\beta\delta\right)+\beta\gamma^K\delta^{K+1}\right)}{\beta\delta\left(\alpha(m+\mu)+m\right)\left(\alpha^K\beta^K-\gamma^K\delta^K\right)-K\alpha^K\beta^K(\alpha+\beta+1)(m\gamma-\delta\mu)}, \tag{B4}$$

which is different from (A6).

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