ON A UNIQUENESS PROPERTY OF SECOND CONVOLUTIONS

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1. Introduction and Main Result

Let M_∞ denote the space of all finite nontrivial complex Borel measures on the real line whose variation has a fast decay at $-\infty$:

$$(1)\qquad \int_{-\infty}^{0} e^{r|t|}\, d|\mu(t)| < \infty, \qquad \text{for every} \quad r > 0.$$

It follows from (1) that the Fourier-Stieltjes transform of every measure $\mu \in M_\infty$,

$$\hat{\mu}(z) := \int_{-\infty}^{\infty} e^{izt}\, d\mu(t),$$

converges uniformly on compact subsets of the upper half-plane $\mathsf{C}_+ := \{z \in \mathsf{C} : \operatorname{Im} z > 0\}$ to a function analytic in C_+. Let $l(\mu) := \inf \operatorname{supp} \mu$ denote the left boundary of the support of μ, and μ^{n*} the nth convolution power of μ.

The following uniqueness property of nth convolutions of measures from M_∞ was discovered in connection with some probabilistic results (see for example [1], [7], [8], [9], [10] and the literature therein): Let $n \geq 3$ be an integer, and let $\mu \in M_\infty$ be such that $l(\mu) = -\infty$. Then every half-line $(-\infty, a)$, $a \in \mathsf{R}$, is a uniqueness set for the nth convolution μ^{n*}, in the sense that the implication holds: Suppose $\nu \in M_\infty$ and

(2) there exists $a \in \mathsf{R}$ such that $\mu^{n*}|_{(-\infty,a)} = \nu^{n*}|_{(-\infty,a)}$. Then $\mu^{n*} = \nu^{n*}$.

It is also known that property (2) does not hold for $n = 2$. An easy way to check this is to take two measures $\xi_1, \xi_2 \in M_\infty$ such that $l(\xi_1 + \xi_2) = -\infty$ and $\xi_1 * \xi_2 = 0$ on some half-line $(-\infty, a)$. Then the measures $\mu = \xi_1 + \xi_2$ and $\nu = \xi_1 - \xi_2$ belong to M_∞, $l(\mu) = -\infty$ and we have

$$(\mu^{2*} - \nu^{2*})|_{(-\infty,a)} = 4\xi_1 * \xi_2|_{(-\infty,a)} = 0.$$

For example, let $\xi_j \in M_\infty$ be the measures with Fourier-Stieltjes transforms

$$\hat{\xi}_j(z) = e^{(-1)^j e^{-iz}}, \qquad j = 1, 2. \tag{3}$$

From $\hat{\xi}_1\hat{\xi}_2 = 1$, we see that $\xi_1 * \xi_2$ is the unit measure concentrated at the origin, so that $(\xi_1 + \xi_2)^{2*} - (\xi_1 - \xi_2)^{2*} = 4\xi_1 * \xi_2 = 0$ on $(-\infty, 0)$.

It turns out that there cannot be more than two different second convolutions which agree on a half-line. The aim of this note is to prove the following

THEOREM 1. *Assume a measure $\mu \in M_\infty$ satisfies $l(\mu) = -\infty$. Suppose there exists $a \in \mathsf{R}$ and measures $\nu, \phi \in M_\infty$ such that*

$$\mu^{2*}|_{(-\infty,a)} = \nu^{2*}|_{(-\infty,a)} = \phi^{2*}|_{(-\infty,a)}, \tag{4}$$

and $\nu^{2*} \neq \phi^{2*}$. Then either $\nu^{2*} = \mu^{2*}$ or $\phi^{2*} = \mu^{2*}$.

An immediate corollary is the following uniqueness property of the second convolutions:

COROLLARY 2. *For every $\mu \in M_\infty, l(\mu) = -\infty$, there is a real number $a_0 = a_0(\mu)$ such that μ^{2*} is uniquely determined by its values on $(-\infty, a)$, $a > a_0$, i.e. if $\nu \in M_\infty$ and there exists $a > a_0$ such that $\mu^{2*}|_{(-\infty,a)} = \nu^{2*}|_{(-\infty,a)}$, then $\mu^{2*} = \nu^{2*}$.*

We also mention a uniqueness result for squares of analytic functions:

COROLLARY 3. *Assume functions f, g and h are analytic in the punctured unit disk $0 < |z| < 1$, and that f has an essential singularity at the origin. Suppose that both functions $f^2 - g^2$ and $f^2 - h^2$ have a pole or a removable singularity at the origin and $g^2 \neq h^2$. Then either $g^2 = f^2$ or $h^2 = f^2$.*

This is just a particular case of Theorem 1 for measures concentrated on the set of integers, and follows from it by the change of variable $z = \exp(-it)$.

2. Remarks

1. Observe that condition (1) is crucial for the uniqueness property (2): The property (2) does not in general hold for measures whose Fourier–Stieltjes transform is not analytic in C_+, see [7], [8] and [1]. A comprehensive survey of results on this and similar uniqueness properties can be found in [9].

2. As it was observed in [7], the uniqueness property of nth convolutions (2) is closely connected with the Titchmarsh convolution theorem and its extensions. The classical Titchmarsh convolution theorem states that if ξ_1 and ξ_2 are finite Borel measures satisfying $l(\xi_j) > -\infty$, $j = 1, 2$, then $l(\xi_1 * \xi_2) = l(\xi_1) + l(\xi_2)$. This is not true for measures with unbounded support,

for there exist measures ξ_j, $j = 1, 2$, $l(\xi_1) = -\infty$, such that $l(\xi_1 * \xi_2) > -\infty$. Such measures can be taken from M_∞, see example (3). However, it was shown in [8] that the conclusion of Titchmarsh convolution theorem holds true whenever the variation of measures satisfies a condition at $-\infty$ more restrictive than (1):

$$\int_{-\infty}^{0} e^{r|t|\log|t|} d|\mu(t)| < \infty, \qquad \text{for every} \quad r > 0. \tag{5}$$

Second convolutions of such measures enjoy the uniqueness property above ([7], [8]). Moreover, examples similar to (3) show that restriction (5) cannot be weakened. Analogous results for unbounded measures were established in [2].

Observe that extensions of the Titchmarsh convolution theorem have also applications in the theory of invariant subspaces, see [2], [3] and [4].

3. The Titchmarsh convolution theorem has been extended to linearly dependent measures: the equality

$$l\,(\xi_1 * \cdots * \xi_n) = \sum_{j=1}^{n} l(\xi_j)$$

holds for linearly dependent measures $\xi_j \in M_\infty$, $j = 1, \ldots, n$, $n \geq 3$, in “general position”, for the precise statement see [5]. Our proof of Theorem 1 below is a fairly easy consequence of this result.

3. Proof of Theorem 1

The following lemma is a particular case of Theorem 4 in [5]:

Lemma 4. (i) *Suppose measures $\xi_1, \xi_2, \xi_3 \in M_\infty$ are linearly independent over* C. *Then*

$$l(\xi_1 * \xi_2 * \xi_3 * (\xi_1 + \xi_2 + \xi_3)) = l(\xi_1) + l(\xi_2) + l(\xi_3) + l(\xi_1 + \xi_2 + \xi_3). \tag{6}$$

(ii) *Suppose measures $\xi_1, \xi_2 \in M_\infty$ are linearly independent over* C *and* $|a_1| + |a_2| \neq 0$. *Then*

$$\begin{aligned} &l\,(\xi_1 * \xi_2 * (\xi_1 + \xi_2) * (a_1\xi_1 + a_2\xi_2)) \\ &\qquad = l(\xi_1) + l(\xi_2) + l(\xi_1 + \xi_2) + l(a_1\xi_1 + a_2\xi_2). \end{aligned}$$

For the convenience of the reader, we recall shortly the main ideas of the proof in [5]. To prove, say (6), by the Titchmarsh convolution theorem, it suffices to verify the implication

$$l\,(\xi_1 * \xi_2 * \xi_3 * (\xi_1 + \xi_2 + \xi_3)) > -\infty \;\Rightarrow\; l(\xi_j) > -\infty, \quad j = 1, 2, 3.$$

We may assume that $\xi_1 * \xi_2 * \xi_3 * (\xi_1 + \xi_2 + \xi_3) = 0$ on $(-\infty, 0)$, so that the product of the Fourier-Stieltjes transforms $\hat{\xi}_1\hat{\xi}_2\hat{\xi}_3(\hat{\xi}_1 + \hat{\xi}_2 + \hat{\xi}_3)$ belongs to the Hardy space $H^\infty(\mathsf{C}_+)$. Hence, the zero set of the product, and so the zero set of each factor satisfies the Blaschke condition. Now one can use the following argument: If functions f_j, $j = 1, \dots, n$, $n \geq 2$, are analytic in the unit disk, linearly independent and such that the zeros of each f_j and the sum $f_1 + \dots + f_n$ satisfy the Blaschke condition in the disk, then each f_j must have "slow" growth in the disk. A sharp statement follows from H. Cartan's second main theorem for analytic curves, see Theorem D in [5]. This argument proves that the growth of each $\hat{\xi}_j$ in C_+ must satisfy a certain restriction. Next, we have additional information that each function $\hat{\xi}_j$ is bounded in every horizontal strip in C_+. This allows one to improve the previous estimate to show that numbers b_j exist such that $\hat{\xi}_j(z)\exp(ib_j z) \in H^\infty(\mathsf{C}_+)$, $j = 1, 2, 3$. This means that $l(\xi_j) \geq -b_j > -\infty$, $j = 1, 2, 3$.

We shall also need a simple lemma:

LEMMA 5. *Suppose $\mu \in M_\infty$ is such that $l(\mu^{2*}) > -\infty$. Then $l(\mu) > -\infty$.*

Indeed, we may assume that $\mu^{2*} = 0$ on $(-\infty, 0)$, so that $(\hat{\mu})^2 \in H^\infty(\mathsf{C}_+)$. Since $\hat{\mu}$ is analytic in C_+, we obtain $\hat{\mu} \in H^\infty(\mathsf{C}_+)$. Consider now convolutions $\mu * p_n$, where p_n is any sequence of smooth functions concentrated on $[0, \infty]$ which converges weakly to the delta-function concentrated at the origin. We have $\hat{p}_n\hat{\mu} \in (H^\infty \cap H^1)(\mathsf{C}_+)$. A standard argument involving inverse Fourier transform along the line $\operatorname{Im} z = y$ as $y \to \infty$, proves that $l(\mu * p_n) \geq 0$. Taking the limit as $n \to \infty$, we conclude that $l(\mu) \geq 0$.

PROOF OF THEOREM 1. Suppose measures $\mu, \nu, \phi \in M_\infty, l(\mu) = -\infty$, satisfy (4) for some $a \in \mathsf{R}$, and $\nu^{2*} \neq \phi^{2*}$. Set $\xi_1 := (\mu + \nu)/2, \xi_2 := (\mu - \nu)/2$, and $\eta_1 := (\mu + \phi)/2, \eta_2 := (\mu - \phi)/2$. To prove the theorem, it suffices to show that one of the measures ξ_j, η_j, $j = 1, 2$, is trivial.

Let us assume that it is not so, and show that this leads to a contradiction. Since

$$(\mu^{2*} - \nu^{2*})|_{(-\infty,a)} = 4\xi_1 * \xi_2|_{(-\infty,a)} = 0,$$

$$(\mu^{2*} - \phi^{2*})|_{(-\infty,a)} = 4\eta_1 * \eta_2|_{(-\infty,a)} = 0,$$

we have

$$l(\xi_1 * \xi_2) > -\infty, \; l(\eta_1 * \eta_2) > -\infty. \tag{7}$$

Let us show that (7) implies $l(\mu) > -\infty$, which contradicts the assumption $l(\mu) = -\infty$.

We shall consider several cases. First, assume that ξ_1 and ξ_2 are linearly dependent. Then $\mu = \xi_1 + \xi_2 = (1+b)\xi_2$, for some $b \in \mathsf{C}$, $b \neq 0$, and so

$$\mu^{2*} = (1+b)^2 \xi_2^{2*} = \frac{(1+b)^2}{b} \xi_1 * \xi_2.$$

By (7), this gives $l(\mu^{2*}) > -\infty$. Lemma 5 yields $l(\mu) > -\infty$.

Assume now that ξ_1 and ξ_2 are linearly independent. From $\mu = \xi_1 + \xi_2 = \eta_1 + \eta_2$ we have $\eta_2 = \xi_1 + \xi_2 - \eta_1$. Now (7) gives

$$-\infty < l(\xi_1 * \xi_2 * \eta_1 * \eta_2) = l(\xi_1 * \xi_2 * \eta_1 * (\xi_1 + \xi_2 - \eta_1)).$$

If ξ_1, ξ_2 and η_1 are linearly independent, then by part (i) of Lemma 4, we obtain $l(\xi_j) > -\infty$, $j = 1, 2$, and so $l(\mu) > -\infty$. If ξ_1, ξ_2 and η_1 are linearly dependent, we have $\eta_1 = c_1\xi_1 + c_2\xi_2$, for some $c_1, c_2 \in \mathsf{C}$. Hence,

$$-\infty < l(\xi_1 * \xi_2 * \eta_1 * \eta_2) = l\,(\xi_1 * \xi_2 * (c_1\xi_1 + c_2\xi_2)) * ((1-c_1)\xi_1 + (1-c_2)\xi_2).$$

If either $c_j \neq 0$, $j = 1, 2$, or $1 - c_j \neq 0$, $j = 1, 2$, then part (ii) of Lemma 4 implies $l(\xi_j) > -\infty$, and so $l(\mu) > -\infty$. Otherwise, we may assume that $c_1 = 0$ and $1 - c_2 = 0$. This gives

$$-\infty < l(\xi_1 * \xi_2 * \eta_1 * \eta_2) = l(\xi_1^{2*} * \xi_2^{2*}).$$

From (7) and Lemma 5 we conclude that $l(\xi_j) > -\infty$, $j = 1, 2$, which shows that $l(\mu) > -\infty$.

REFERENCES

1. Blank, N. M., *Distributions whose convolutions coincide on the half axis*. (Russian) Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. No. 41 (1984), 17–25.
2. Borichev, A. A., *The generalized Fourier transform, the Titchmarsh theorem and almost analytic functions*. (Russian) Algebra i Analiz 1 (1989), no. 4, 17–53; translation in Leningrad Math. J. 1 (1990), 825–857.
3. Borichev, A. A., *Errata: "The generalized Fourier transform, the Titchmarsh theorem and almost analytic functions"*. Algebra i Analiz 2 (1990), no. 5, 236–237.
4. Domar, Y., *Extension of the Titchmarsh convolution theorem with applications in the theory of invariant subspaces*, Proc. London. Math. Soc. (3) 46 (1983), 288–300.
5. Gergün, S., Ostrovskii, I. V., Ulanovskii, A., *On the Titchmarsh convolution theorem*. Ark. Mat. 40 (2002), 55–71.
6. Hayman, W. K., *Subharmonic Functions. Vol. 2*, London Math. Soc. Monographs 20, Academic Press, London 1989.

7. Ostrovskii, I. V., *Support of a convolution of finite measures, and measures determined uniquely by the restriction to a half-line*. (Russian) Dokl. Akad. Nauk Ukrain. SSR Ser. A 1984, no. 3, 8–12.
8. Ostrovskii, I. V., *Generalization of the Titchmarsh convolution theorem and the complex-valued measures uniquely determined by their restrictions to a half-line*. pp. 256–283 in Stability Problems for Stochastic Models (Uzhgorod, 1984), Lecture Notes in Math. 1155, Springer, Berlin 1985.
9. Ostrovskii, I. V., Ulanovskii, A., *Classes of complex-valued Borel measures that can be uniquely determined by restrictions*. (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 170 (1989), Issled. Linein. Oper. Teorii Funktsii. 17, 233–253, 325; translation in J. Soviet Math. 63 (1993), 246–257.
10. Ramachandran, B., *On the results of Ibragimov, Titov and Blank on distribution functions on the real line—their convolution powers coinciding on a half-line*, J. Indian Statist. Assoc. 36 (1998), 75–81.