Note on the invariant classification of vacuum type D spacetimes

A. Coley

Department of Mathematics and Statistics

Dalhousie University, Halifax, NS B3H 3J5, Canada

aac@mathstat.dal.ca

and

S. Hervik

Faculty of Science and Technology

University of Stavanger, N-4036 Stavanger, Norway

sigbjorn.hervik@uis.no

Abstract

We illustrate the fact that the class of vacuum type D spacetimes which are $\mathcal{I}$-*non-degenerate* are invariantly classified by their scalar polynomial curvature invariants.

1 Introduction

In Lorentzian spacetimes, identical metrics are often given in different coordinate systems, which disguises their true equivalence. It is consequently of fundamental importance to have an invariant way to distinguish spacetime metrics. The invariant classification developed by Karlhede [1], based on the Newman-Penrose (NP) formalism [2] and the Cartan equivalence method, is widely used to characterize Lorentzian spacetimes in terms of their Cartan scalars in general relativity [3].

However, perhaps the easiest way of distinguishing metrics is through their *scalar polynomial curvature invariants*, which are scalars obtained by contraction from a polynomial in the Riemann tensor and its covariant derivatives, due to the fact that inequivalent invariants implies inequivalent metrics. In [4] the class of four-dimensional Lorentzian manifolds that can be completely characterized locally (in the sense defined below) by their scalar polynomial curvature invariants was determined. In particular, for any given Lorentzian spacetime, $(\mathcal{M}, g)$, let $\mathcal{I}$ denote the set of all scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives. If there does not exist a metric deformation of g having the same set of invariants as g, then we call the set of invariants *$\mathcal{I}$-non-degenerate*, and the spacetime metric g is called *$\mathcal{I}$-non-degenerate*. This means that for a metric which is $\mathcal{I}$-non-degenerate, the invariants locally characterize the spacetime uniquely. In [4] it was proven that a four-dimensional Lorentzian spacetime metric is either *$\mathcal{I}$-non-degenerate* or *degenerate Kundt*.

This important result implies that metrics not determined by their scalar polynomial curvature invariants (at least locally) must be of degenerate Kundt form. These Kundt metrics therefore correspond to degenerate metrics in the sense that many such spacetimes can have identical scalar polynomial invariants. The Kundt class is defined as those spacetimes admitting a null vector ℓ that is geodesic, expansion-free, shear-free and twist-free [3, 5]. It follows that there exists a *kinematic* frame in which the NP scalars $\kappa = \sigma = \rho = \epsilon$ all vanish.

In a *degenerate Kundt* spacetime there exists a common null frame in which the geodesic, expansion-free, shear-free and twist-free null vector ℓ is also the null vector in which all positive boost weight terms of the Riemann tensor and its covariant derivatives are zero [5]. Any metric in the degenerate Kundt class can be written in a canonical form [5, 6].

Clearly, by knowing which spacetimes can be characterized by their scalar curvature invariants alone, the computations of the invariants (i.e., simple polynomial scalar invariants) is much more straightforward and can be done algorithmically (i.e., the full complexity of the equivalence method is not necessary). On the other hand, the Cartan equivalence method also contains, at least in principle, the conditions under which the classification is complete (although in practice carrying out the classification for the more general spacetimes can be very difficult). Therefore, in a sense, the full machinery of the Cartan equivalence method is only necessary for the classification of the degenerate Kundt spacetimes [7].

2 Vacuum type D spacetimes

In this note we shall illustrate this by considering the class of vacuum type D spacetimes. These spacetimes were studied by Kinnersley [8], who divided the spacetimes into 4 subclasses (I-IV) (which subdivides further into 10 distinct classes of metrics). This simple class of spacetimes are of physical interest, and have been classified using the Karlhede method [9]. Recently, an earlier invariant classification of the vacuum type D spacetimes based on the NP formalism [10] has been extended by exploiting some NP identities using the GHP formalism [11]. Clearly the classification of vacuum type D spacetimes is still of interest. We shall show that we can classify these spacetimes, which are, in general, *$\mathcal{I}$-non-degenerate* [4], using simple scalar polynomial curvature invariants. As an illustration we shall present the results for the Kinnersley class I spacetimes (the other cases work in a similar way). Note that this method does not reproduce the determination of the upper bound (2) on the order of the Cartan scalars for these spacetimes [11, 9].

2.1 Kinnersley class I metric

The Kinnersley class I Petrov type D vacuum metric is given by [8]:

$$
\begin{aligned}
\underset{.}{\mathrm{s}}^2 &= -2S\underset{.}{\mathrm{u}}^2 + 2\underset{.}{\mathrm{u}}\underset{.}{\mathrm{r}} + 4D^{-1}lSy\underset{.}{\mathrm{u}}\underset{.}{\mathrm{x}} - 4D^{-1}lSx\underset{.}{\mathrm{u}}\underset{.}{\mathrm{y}} - 2D^{-1}ly\underset{.}{\mathrm{r}}\underset{.}{\mathrm{x}} + 2D^{-1}lx\underset{.}{\mathrm{r}}\underset{.}{\mathrm{y}} \\
&\quad + \left(-2D^{-2}l^2Sy^2 - D^{-2}zz^*\right)\underset{.}{\mathrm{x}}^2 + 4D^{-2}l^2Sxy\underset{.}{\mathrm{x}}\underset{.}{\mathrm{y}} + \left(-2D^{-2}l^2Sx^2 - D^{-2}zz^*\right)\underset{.}{\mathrm{y}}^2,
\end{aligned}
$$

where the variables (labelled 0–3) are (u, r, x, y), $z = r+il$ is a complex variable, l, C, $(2Cil+m)$ and $(-2Cil+m)$ are constants, and

$$
\begin{aligned}
S &\equiv 2Cl^2(zz^*)^{-1} - C + \frac{1}{2}(2Cil+m)r(zz^*)^{-1} + \frac{1}{2}(-2Cil+m)r(zz^*)^{-1} \\
D &\equiv \frac{1}{2}Cx^2 + \frac{1}{2}Cy^2 + 1.
\end{aligned}
$$

There are 4 algebraically independent (complex) Cartan invariants, which are [1]:

$$
\begin{aligned}
\Psi_2 &= -(2Cil+m)z^{-3} \\
\nabla\Psi_{20'} &= 3(2Cil+m)S^{\frac{1}{2}}z^{-4} = \nabla\Psi_{31'} \\
\nabla^2\Psi_{20'} &= -12(2Cil+m)Sz^{-5} = \nabla^2\Psi_{42'} \\
\nabla^2\Psi_{31'} &= -3C(2Cil+m)z^{-4}z^{*-1} + \frac{3}{2}(2Cil+m)^2z^{-5}z^{*-1} - 3(2Cil+m)Sz^{-4}z^{*-1} \\
&\quad -12(2Cil+m)Sz^{-5}.
\end{aligned}
$$

We calculate the 4 (complex) scalar polynomial invariants:

$$
\begin{aligned}
I &\equiv \frac{1}{2}\Psi_{abcd}\Psi^{abcd} = 3(2Cil+m)^2z^{-6}, \\
C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} &= 24(2Cil+m)^2z^{-6} + 24(-2Cil+m)^2z^{*-6}
\end{aligned}
$$

[1] The worksheets, based on the work of J. Åman, were kindly provided by J. Skea. We follow the notation of these worksheets and [9]. Note that $Re(\Psi_2) = (2Cil+m)z^{-3} + (-2Cil+m)z^{*-3}$ is an independent function.

$$\Psi^{(abcd;e)f'}\Psi_{(abcd;e)f'} = 180(2Cil+m)^2 S z^{-8}$$
$$C^{\alpha\beta\gamma\delta;\mu}C_{\alpha\beta\gamma\delta;\mu} = 720(2Cil+m)^2 S z^{-8} + 720(-2Cil+m)^2 S z^{*-8},$$

where Ψ_{abcd} is the Weyl spinor, $a, b, ..$ are spinor indices, and $\alpha, \beta, ...$ are frame indices.

We see that $(\Psi_2)^2 \sim I$, $(\nabla\Psi_{20'})^2 \sim \Psi^{(abcd;e)f'}\Psi_{(abcd;e)f'}$, and $\nabla^2\Psi_{20'} \sim \Psi^{(abcd;e)f'}\Psi_{(abcd;e)f'}/I$. These can be used to solve for the 4 real parameters (r, l, C, m) in terms of the scalar polynomial invariants. Then by inserting these into the expression for $\nabla^2\Psi_{31'}$ we can write also this Cartan invariant in terms of polynomial invariants. Therefore, all of the Cartan invariants can be expressed in terms of scalar polynomials.

3 Discussion

All of the other cases in the Kinnersley classes can be dealt with in a similar way [8]. The Schwarzschild vacuum type D spacetime belongs to the Kinnersley class I and, as discussed in [4], in canonical coordinates there are two functionally independent (as functions of the two parameters r and M) scalar polynomial invariants [2], $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = 48M^2r^{-6}$ and $C^{\alpha\beta\gamma\delta;\mu}C_{\alpha\beta\gamma\delta;\mu} = 720(r-2M)M^2r^{-9}$, and all of the algebraically independent Cartan scalars Ψ_2, $\nabla^2\Psi_{20'}$, $\nabla^2\Psi_{31'}$, and $\nabla^2\Psi_{42'}$ are related to these two polynomial curvature invariants (see also [12] for the invariant classification of the Schwarzschild spacetime). The Kerr solution belongs to Kinnersley class IIA; this spacetime has been invariantly characterized intrinsically (using, in addition to algebraic scalar invariants, differential Weyl concomitants) [13].

In general, the vacuum type D spacetimes in the Kinnersley classes are $\mathcal{I}$*-non-degenerate*; from the Bianchi identities the covariant derivative of the Weyl tensor is of algebraic type

[2] By setting $l = 0$, $C = -(1/2)(-m/M)^{2/3}$, and by rescaling the radial coordinate, $r \mapsto (-m/M)^{1/3}r$, we see that the invariants in the Schwarzschild metric are equivalent to those in the Kinnersley metric (where $l = 0$ is the Schwarzschild case).

I (for non-flat spacetimes) except in some very special cases (i.e., some special degenerate Kundt spacetimes are included in the Kinnersley classes [8]). There is a sense in which the type D^k degenerate Kundt spacetimes can also be completely characterized by their scalar polynomial curvature invariants [4, 5]; however, the vacuum type D^k spacetimes are trivial.

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References

[1] A. Karlhede, 1980, Gen. Rel. Grav. **12**, 693.

[2] E. Newman and R. Penrose, 1962, J. Math. Phys. **3**, 566.

[3] H. Stephani, D. Kramer, M. A. H. MacCallum, C. A. Hoenselaers, E. Herlt 2003 *Exact solutions of Einstein's field equations, second edition* (Cambridge University Press; Cambridge).

[4] A. Coley, S. Hervik and N. Pelavas, 2009, Class. Quant. Grav. **26**, 025013 [arXiv:0901.0791].

[5] A. Coley, S. Hervik, G. Papadopoulos and N. Pelavas, 2009, Class. Quant. Grav. **26**, 105016 [arXiv:0901.0394].

[6] A. Coley, R. Milson, V. Pravda and A. Pravdova, 2004, Class. Quant. Grav. **21**, L35 [gr-qc/0401008]; V. Pravda, A. Pravdová, A. Coley and R. Milson, 2002, Class. Quant. Grav. **19**, 6213 [arXiv:0710.1598]. A. Coley, 2008, Class. Quant. Grav. **25**, 033001.

[7] A. Coley, R. Milson, G. Papadopoulos, preprint.

[8] W. Kinnersley, 1969, J. Math. Phys. **10**, 1195.

[9] J. E. Aman and A. Karlhede, 1981, preprint **46** Institute Theoretical Physics, University of Stockholm.

[10] J. Collins *et al.*, 1991, Class. Quant. Grav. **7**, 2005 & **8**, L215.

[11] S. B. Edgar, A. Garcia-Parrado and J. M. Martin-Garcia, 2009, Class. Quant. Grav. **26**, 105022 [arXiv0812.1232].

[12] J. J. Ferrando and J. A. Saez, 1998, Class. Quant. Grav. **15**, 1323.

[13] J. J. Ferrando and J. A. Saez, 2009, Class. Quant. Grav. **26**, 075013.