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What is the optimal maturity for bonds in a given risk/non-risk portfolio?

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Abstract

The paper analyzes whether there is an optimal maturity for bonds given a set stock/bond portfolio. Mean-variance optimization has been used to combine a portfolio consisting of a given asset allocation of stocks and a dynamic proportion in bonds that gives the highest estimated return for the lowest risk, by changing the maturity of the bonds. The paper uses data for the Norwegian All-share index, and short-term and long-term Norwegian government bonds. The data was divided into three parts; total: 1994-2013, period 1: 1994-2003, and period 2: 2004-2013. The optimal portfolio weights differ from each period, but with similar trend, the portfolio consisted of the obligatory All-Share index and a combination between 3-month, 3-year and 5-year government bonds. Generally, the higher the weight proportion on the risky asset, the portfolio will be better off investing the remaining allocation on long-term government bonds. This was the case for all periods where the risky assets (stocks) had the portfolio majority. For portfolios with a lower risk proportion, it was more unclear, as they are very sensitive to changes in correlation amongst stocks and bonds.

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Preface

The Master Thesis represents the end of our Master of Science in Business and Administration, with specialization in Applied Finance, at the University of Stavanger. The main object of the thesis is to learn how to apply scientific methods to an applied issue.

The decision to write about the effect bonds duration has on the optimization of portfolio was originally an idea from Grieg Investors that sought more information on the subject. We seized the opportunity to learn more on the subject, as we believe that right investment of financial resources is vital for a country such as Norway. Norway is well known by economists for its pension fund: Government Pension Fund – Global (GPGF). The current investment strategy has a weight distribution with approximately 60 percent in the stock market, 35 percent in the bond market, and the remaining 5 percent in real estate. Diversification has been an important factor to optimize the GPGF to achieve steady return in turbulent times (Oljefondet 2014).

The financial market has gone through some turbulent periods, like the Dot.com bubble, “Black Monday” and the most recent financial crises of 2008, the latter is still affecting the economy. Nearly 6 years after the crises the S&P500 and the Dow Jones indices have gained what they lost (and some more), but the aftermath has caused significant fall in the interest rates, which is currently kept extremely low at 1.7 percent on 10-year treasury bonds (USA Today 2012). This makes the bond market less appealing for investors, and is therefore an interesting subject that should be taken more into account when investing, as there so far has been devoted little attention to the effect bonds maturity has on optimizing portfolios.

We would like to thank our supervisor Bernt Arne Ødegaard for important guidance, advises and constructive feedback. We would also like to thank Kjetil Svihus from Grieg Investor. And last but not least, would we like to thank each other for valuable support and overcoming obstacles together.

1 Introduction

Diversification is an important concept within finance, where several investment companies take great effort in seeking further information on how to better diversify the investment strategy. There have been few studies regarding the effect of bonds maturity on portfolios; therefore, we decided to seek more information on the subject. Is there an optimal maturity on bonds that can hedge the portfolio even further?

The paper studies the optimal weight distributions on bonds with various maturity in a given risk/non-risk portfolio. In other words, can varying the maturity on bonds hedge a portfolio combining stocks and bonds? Our hypothesis is that if interest rates decline, bond prices will fall while stock prices rise. If you then have the majority of the portfolio in risky assets, such as stocks, the rise in stocks will make the fall in bond prices negligible, and smooth out the interest rate risk. In this case, long-term maturity on the bonds may be advisable. On the other hand, with a portfolio with a high proportion of bonds, the stock proportion has a smaller impact on the portfolio if the interest rate falls. In this scenario, it might be advised to keep bonds with shorter maturity dates. This issue is important to reduce the risk in portfolios especially with the majority invested in bonds. Based on this, we have the following research question:

What is the optimal maturity for bonds in a given risk/non-risk portfolio?

We have chosen to write this subject, as we believe there is high degree of actuality, especially for a country such as Norway, where proper investment is essential for the high financial assets for future generations to come.

To find an optimal maturity for a portfolio we first need to distinguish what optimal really means in this sense. The paper will therefore use the Portfolio Theory developed by Harry Markowitz as framework to discuss optimality. The Markowitz mean-variance frontier is a selection of portfolios with an optimal asset allocation that give the highest estimated return for a given level of risk. The paper analyses the mean-variance portfolio when the portfolio consists of the All-share stock index and government bonds with various maturities.

We approached the research question by collecting empirical data on Norwegian government bonds and the Norwegian All-share stock index, by using historical data for the period 1994 through 2013. In the second and the third section of the paper we have focused on the theory and given a short background of the Norwegian financial market, as this is the foundation for the rest of the thesis. Thereafter, we created a portfolio problem, where we illustrate what you

gain/lose by combining a portfolio of different asset allocations in equities and bonds. We found historical return and standard deviation for the All-share index and Norwegian government bonds. Based on these estimates, together with the correlation and the covariance, we used mean-variance optimization to create efficient frontiers of portfolios containing the different asset allocations. The main object has been to find empirical evidence on whether or not you can benefit from diversification by investing in bonds with various maturity. During our last two sections the empirical findings with a discussion and eventually a conclusion has been presented.

2 Theory

In this chapter, we introduce various theoretical building blocks that is used for portfolio investing. The main theories are within modern portfolio theory.

2.1 Portfolio Management

A portfolio is a collection of different investment assets, such as real estate, stocks and bonds. There are two decisions each investor would have to make when constructing their portfolio; asset allocation and security selection. The first is the choice on which assets classes to hold, while the latter is on which specific securities to hold within the different assets class. In this paper we focus primarily on portfolio problems that contain investments in stocks and government bonds. We will therefore start by explain these concepts shortly.

2.1.1 Asset Allocation/portfolio construction

The process of construction a portfolio consists of deciding the composition of the portfolio of risky assets versus non-risk assets. An example of a method of portfolio construction is the “top-down”-method. It works by first deciding how much of the total portfolio goes in to each asset class, afterwards the investors choose particular securities to be held in each asset class. The Government Pension Fund (GPF) has a balanced mixture portfolio: a “top-down” portfolio, consisting of approximately 60 percent in stocks, 35 percent in bonds, and 5 percent in real estate, with a regular rebalance (Oljefondet 2014).

2.1.2 Fixed-income securities (Bonds)

A bond is a security sold by governments or corporations that obligates the borrower to make future cash flows to the buyer over a specific period. There is three ways the buyer can achieve a return on the bonds: regular coupon payments, price changes in bond and the reinvestment income that are reinvested until maturity (Bodie et al. 2011). There are two types of bonds:

Zero-coupon bonds do not make a coupon payment, the only payment is the face value done on maturity date. In order to entice a buyer for this zero-coupon is to sell the bond at a discount: the bond price is lower than the face value. Bond price and face value will converge as the maturity date is approaching.

$$\text{Zero coupon bond price} = \frac{F}{(1+r)^t}$$

Coupon bonds pay regular interest payments to the holder including the face value at maturity. Whereas zero-coupon bonds normally sold at a discount this is not necessarily the case for coupon bonds. A coupon bond may trade at a discount, at premium or at par. A premium bond is trading at higher value than the face value and a par bond is trading at face value.

$$\text{Bond price} = \frac{C}{(1+i)} + \frac{C}{(1+i)^n} \frac{M}{(1+i)^n}$$

By Investing in fixed-income securities one needs to account for interest rates, as changes in the rates can have significant effect on the bonds value, especially when investing in long-term bonds. Therefore, there is an inverse correlation between the yield to maturity (interest rates) and the value of the fixed-income securities as higher (lower) YTM leads to lower (higher) price in bond (Berk, DeMarzo 2011).

2.1.3 Maturity and duration

Maturity is the time period before a bond is paid out by the borrower to the buyer of the security.

Duration is a measurement of how many years, or periods, it takes the bond to be paid out by its entire cash flows. Duration measures how sensitive a bond is to changes in interest rate, as interest rate risk is the key risk for bond investors (Bodie et al. 2011). Theory implies that as the interest rate rise, bond prices will fall, and the higher the duration the greater the interest rate risk is. Higher duration on bonds are more sensitive to changes in interest rates. Bonds with longer maturities are more sensitive for changes than shorter maturity bonds.

For zero coupon bonds, the duration is equal to the bond's time to maturity. As there only is a payment at maturity, while for bonds with higher coupon rates the duration will be smaller since they make coupon payments upfront.

2.1.4 Term structure theory

Term structure theory or the expectations hypothesis describes the relationship between bond yields of different maturities. It reflects the expectations the market has to the future changes in interest rate. Investors seek higher return for longer maturity bonds, because of the risk in changes in future interest rates and the effect monetary policy might have on interest rates (Bodie, et al. 2011).

2.2 Modern portfolio theory

The modern portfolio theory describes how to use diversification to achieve maximum portfolio return for a chosen level of portfolio risk that would optimize the asset allocation. It is applied on the basis that the investor is rational and risk-averse, meaning the investors wants the highest possible return, for the lowest possible risk. A rational investors knows that by wanting a higher return you must accept a higher risk.

2.2.1 Markowitz

Markowitz was the first to develop a quantitative theory on Modern Portfolio Theory, such as the mean-variance optimization. The theory is based on the assumption of risk averse investors; investors needs to be compensated for increased risk. The efficient frontier is found by using the expected return, standard deviations, and covariance for a number of assets. The portfolio that is tangent with the efficient frontier and the capital allocation line maximizes reward-to-variability-ratio or Sharpe-ratio, as shown in the graph below. Therefore, when investing in a portfolio, you should place the investments above the global minimum-variance portfolio, otherwise your portfolio is not efficient as there is another portfolio that has a higher return for the same amount of risk (Bodie, et al. 2011).

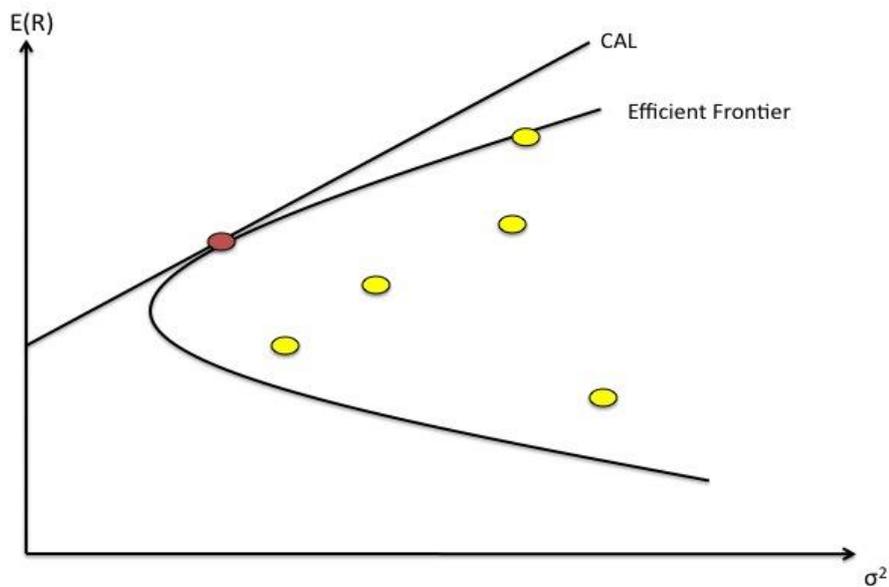


Figure 1 Efficient frontier: shows the gain of diversifying when investing in multiple stocks. Risk along the x-axis and estimated return along the y-axis

2.2.2 Diversification

Diversification is a strategy that lowers the unsystematic risk, by investing in a diversified portfolio of various assets. In other words to not put all the eggs in one basket. In order to successfully lower unsystematic it is vital that the assets are not perfectly correlated. The idea is that better performing stocks will neutralize the lower performing stocks, by smoothing out the unsystematic risk. The systematic risk cannot be avoided as this represents the macroeconomic factors as a whole such as business cycle, inflation, interest rates and exchange rates (Bodie et al. 2011).

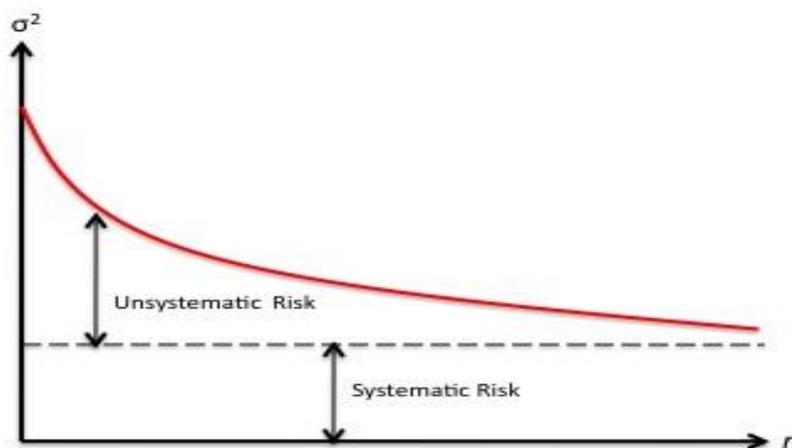


Figure 2 Risk: unsystematic risk decreases as the number of assets increase. Risk along the y-axis and number of stocks along the x-axis

In the case of a portfolio combination including combining equities and government bonds there might be good opportunities for diversification. Equities depend approximately of 80-90 percent on the unsystematic risk, while bonds highly depending on systematic risk and changes in monetary policy (Ang 2012).

2.2.2.1 **Correlation and Covariance analysis**

In statistics, covariance and correlation coefficient are important tools to measure how different variables change together.

2.2.2.1.1 Covariance

Covariance measures how stocks move relatively to each other. The greater value of the covariance the more the variables tend to move in the same direction. Whenever positive value, the variables are moving in the same direction, while for negative values the variables move in the opposite direction. When the assets returns have a negative covariance there is potential for diversification.

$$cov(X, Y) = \sum_{i=1}^N \frac{(x_i - \bar{x})(y_i - \bar{y})}{N}$$

2.2.2.1.2 Correlation

The correlation coefficient is a normalized version of the covariance, but shows the same tendency.

$$cor(X, Y) = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

The values range between -1 and +1, and is perfect correlated if equal to 1, and perfect negative correlation if the values equal -1. Diversification of a portfolio is possible as long as the relation between is not perfectly correlated.

2.2.3 **Sharpe ratio**

The Sharpe ratio gives us vital information when comparing different portfolios, as it gives the return adjusted for portfolio risk. Therefore, a portfolio with a low return can be better than a portfolio with higher return, if the lower return portfolio has a higher Sharpe-ratio.

$$Sharpe = \frac{r_p - r_f}{\sigma_p}$$

Sharpe-ratio is calculated using expected return less risk free rate, divided by portfolio standard deviation

2.2.4 Risk Tolerance

Individuals have different risk tolerance; investors willingness to reduce the expected returns by lower the exposure to risk, which is better known as risk aversion. Whether or not the investor is risk averse, risk neutral or risk seeking, is therefore an important factor that investors should adjust for when making investments. Efficient Frontiers illustrate different efficient portfolios with the highest return given the risk. Where you chose to be on the efficient frontier line depends on your risk tolerance, as the entire line above the global minimum-variance global portfolio is an efficient portfolio solution. The factor that eventually decides where to invest depends on the individual investor's utility, based on the expected return and risk for given portfolios. The higher the score, the higher expected return and standard deviation, and vice versa.

$$U = E(r) - 1/2 A\sigma^2$$

The utility is measured by the estimated return subtracted by half of the investors risk aversion times the variance of the returns

3 The Norwegian Financial Market

This section will include a general overview of the Norwegian financial market and extraordinary events, during the period in research. The intention is to clarify what might make the Norwegian market differ from other financial markets and as an explanation for the fluctuations.

3.1 1994-2003

The Norwegian Financial market went through turbulent times during the first period of this research. To get a better understanding of the given period, a short summary of the decade leading up to 1994 will be taken into consideration. The 1980s was a volatile period, with shocks like "Black Monday". The event started in the U.S financial market, but had severe influence on the Norwegian financial market that suffered from a recession until a small recovery in 1989. Then in 1990 new all-time high was reached, only to find the country in a new recession later that same year.

In 1994 through 1998 new all-time records were achieved, which would last until early 1998. There was a drastic decrease in the oil prices this year, and as the Norwegian economy is sensitive to changes in oil-prices, this had major impact on the country. The country would experience high volatility in the oil-index in the upcoming years that would lead to

fluctuations in the financial market. The Dot.com bubble in the early 2000s was another international event that would influence the economy during this period. The market had become more dependent on international markets and more transparent, which meant that trends in international markets, such as Western Europe and the U.S, would play a great impact on Norway's economy. This was noticeable in 2000-2003 when a new recession reached the country as a consequent of a recession in the U.S after a long period of increasing values.

From 2002, increased collaboration and alliances with other markets and financial institutions had a positive effect on the financial market. The market grew and the volume on Oslo Børs increased (OsloBørs 2014).

Interest rates in Norway fluctuated greatly caused by the different business-cycle during this period. The interest rates until 1998 were reflected by the recovery in the economy, with low interest rates. In 1998, the interest rate doubled, as a consequent of turbulent times in international markets and trading partners. The oil prices had also a significant drop. Thereafter a new period of increased oil prices, increased economic activity and lower interest rates dominated the market until 2002. From 2002 to 2004, the interest rate kept decreasing as a way of trying to stimulate the economy (regjeringen.no).

3.2 2004-2013

The past decade has been extremely turbulent and very suspenseful for the investors of the Oslo stock exchange. As many of the biggest companies noted on the Oslo Stock exchange are directly either oil production, or oil-service companies, this makes the oil price a leading indicator for the All-share index according to Gabrielsen and Holtet (2009).

From 2004, the world economy had major events that affected the oil price greatly, for instant: Hurricane Katrina in 2005 and the US invasion of Iraq. Both incidences had great positive impact on oil prices. The all-time high oil price was \$147 per barrel during the summer of 2008. Then the financial crisis in late 2008 caused the oil price to plummet, reaching a low of \$35 in 2009. In 2010 oil prices steadily increased, and has been relatively stable ever since. In January 2014, the Brent crude oil (North Sea Oil) price was \$110.

The Oslo Børs Index followed the oil prices and steadily increased from 2004 until 2008. After reaching an all-time high in summer of 2008, the financial crisis hit the market hard. In 2009, the index was back at 2004 levels. From 2010, the All-share index has seen a huge

growth, with a bit setback in 2011. In late 2013, the All-share again reached the same level as before the financial crisis and is now regularly climbing to all-time highs. The growth of the index was due to the recovery of the oil prices and monetary policy, such as lowered interest rates and the growth in the general world economy.

Interest in Norway has never been lower and economic profiles have contrary views whether now is the time for an interest increase. According to analysis from Norges Bank, the Norwegian key interest rate will be kept at the present rate towards mid-2015, and thereafter increase gradually (PPR 2014). Until then the government bonds yields will most likely remain low. Low interest rates has a direct effect on government bonds, therefore, the yield on bonds has dropped. The growth in the index and low yields on the bond indices during the last decade has caused an inverse relationship and uncorrelated returns between stock index and bond indices, as we will discuss later on.

3.2.1 The Norwegian Bond Market

The Norwegian bond market is a special case compared to other economies, as the corporate bonds suffer from illiquidity, while government papers are liquid. According to Rakkestad, Skjeltnor and Ødegaard (2012) the corporate issued bond market is improving, as both the trading activity and volume has increased from 1999 to 2011, but still not enough so that it can be labeled as liquid. According to Østhus and Ueland (2013) Norwegian government bonds are trading at an average of 250 days per year, meaning all trading days. This combined with a low spread implies that bonds are liquid assets. Our focus in this paper is on the government issued bonds, consisting of: 3-month, 6-month, 1-year, 3-years and 5-year government bonds, respectively. Therefore, this statement of corporate bonds does not affect our results when working under this scope. This could however play an important impact if making further research within the field optimizing portfolios using bonds with various maturities and duration.

4 Data analysis

The data analysis is the evaluation process of the collected data that is being preceded and analyzed. The data is collected by secondary sources, and then to be tested and measured to find the correlation between the variables, in this case for the bonds and All-share index. The data will be presented as correlation coefficients, covariance matrices, and descriptive data to illustrate portfolio problems, efficient frontiers and optimal risky portfolios.

4.1 Methodology

In the paper, we have focused our methodology on secondary data and quantitative analysis. The data is based on historical interpretation and observation over a period of 20 years, 1994-2013, respectively. We have used chronological ordering of time series data, which we have used to interpret our estimate for the future; as the past might influence the future. However, the process is stochastic as the future is impossible to foresee. The reliability of the numerical data is greatly enhanced as the data is collected from reliable sources as Oslo Børs and Norges Bank, which strengthen the reliability of the research.

4.2 Limitation

The empirical analysis in this chapter is based on financial variables. The study is quite broad; we have therefore set some limitations, by only focusing on the Norwegian financial market, with a given stock index and government bonds. As mentioned before, we have collected secondary data for All-share index and Norwegian government bonds with various maturities, from a 20 years period, from 1994 throughout 2013. Corporate bonds have not been taken into consideration, as there is limited availability. These limitations will create some biases; however, the data we have collected will hopefully be sufficient as empirical evidence to show a trend to answer for the research question. As the research question's objective is to illustrate the optimal maturity for bonds in a portfolio, to maximize the return and minimize the volatility, we have used government bonds with various maturities. We have also limited the research by not taken any additional investments into consideration, such as real estate.

Similar studies on portfolio theory uses excess return, but since we have focused on short term government bonds, which is close to risk free investment, we have chosen to not include this in this paper.

4.3 Portfolio Problem

The portfolio problem, as mention in the theory chapter, consists of deciding how much of the investment should be invested in risky assets and non-risk assets. In our analysis, the risky assets are the All-share index, and 3-year and 5-year government bond. The non-risk assets are the short-term government bonds, the 3-month and 6-month. The All-Share index is an estimate of the market weighted index, and for Norway's case this would indicate a close correlation with the oil price-index, as the energy sector index is the main natural resource and income for the country.

Portfolio problems is important to take into consideration before investors make their investment, as the choice of asset allocation will depend on the investor's risk willingness and how to maximize the utility, which we mentioned in the theory section.

The portfolio problems in the paper are illustrated to achieve better understanding of combining a stock index and government bond indices with various maturities. The intention is to show to what extent the various maturities of bonds might influence the portfolio's return and the standard deviation. And how you best benefit from diversification.

We have compared the All-Share index and government bonds with various maturities, 3-month, 6-month, 1-year, 3-year and 5-year, respectively. The data is based on historical data in a 20 years period, from 1994 through 2013. The historical data will thereafter be used to predict the expected return.

We have divided the research into three parts, the first takes the entire period of 20 years to account, thereafter the period is divided into two parts, the first for period 1994 through 2003 and the latter for period 2004 through 2013. We have chosen to do so to see if the two periods would imply similar results. These two periods have fluctuated greatly; with significant changes in the historical return for both the stock index and bond's return. Incidents like for instant the financial crises that hit the market in 2007/08, have played a major impact on the financial market. The trend prior to the crises, the trends during the period, and the period after the crises have played major impact worldwide. The turbulent financial times might create some biases in our results and the general trend, but by separating the periods we might see whether there are different empirical evidence and conclusions for investment that will be conveyed. After analyzing the historical data we also used current interest rates for the second period to estimate the same descriptive statistics, correlation and covariance, and last but not least the efficient frontier given certain weights.

4.3.1 Asset allocation

The asset allocation have been influenced by the GPF's "top-down"-method, and two additional portfolios with the majority in either risk or non-risk assets classes. We did this to study whether the optimal maturity changes as the asset allocation changes. This was done to illustrate the outcomes if different asset allocations.

1. Portfolio 1 consists of 20 percent in the stock index and 80 percent in bonds
2. Portfolio 2 consists of 80 percent in the stock index and 20 percent in bonds

3. Portfolio 3 consists of 60 percent in the stock index and 40 percent in bonds (GPFG)

The weight portfolio GPFG has chosen to use is often used as investment strategy for corporate pension plans or as individual's retirement plans. The portfolio consists of 60 % equities and 40 % (long-term) bonds is known as the "balanced" mixture, which according to Armstrong (1999) beat the market for ten years until 1997.

The two other asset allocations are used to illustrate the differences between a portfolio with the majority weight in either equities or securities. Individual investors have different investment aspects, both when it comes to risk tolerance and life situations. To fit investors individual needs we have therefore focused on these two approaches where 0.8/0.2 stock/bonds could be for people that are risk seeking, for example young people with steady income and no kids. While 0.2/0.8 stocks/bonds could be for investors that are risk averse and are expecting life changes, such as starting a family, or reaching their retirement age. The financial situation might change during a lifetime, and the need for rebalancing or reallocate the portfolio is often necessary.

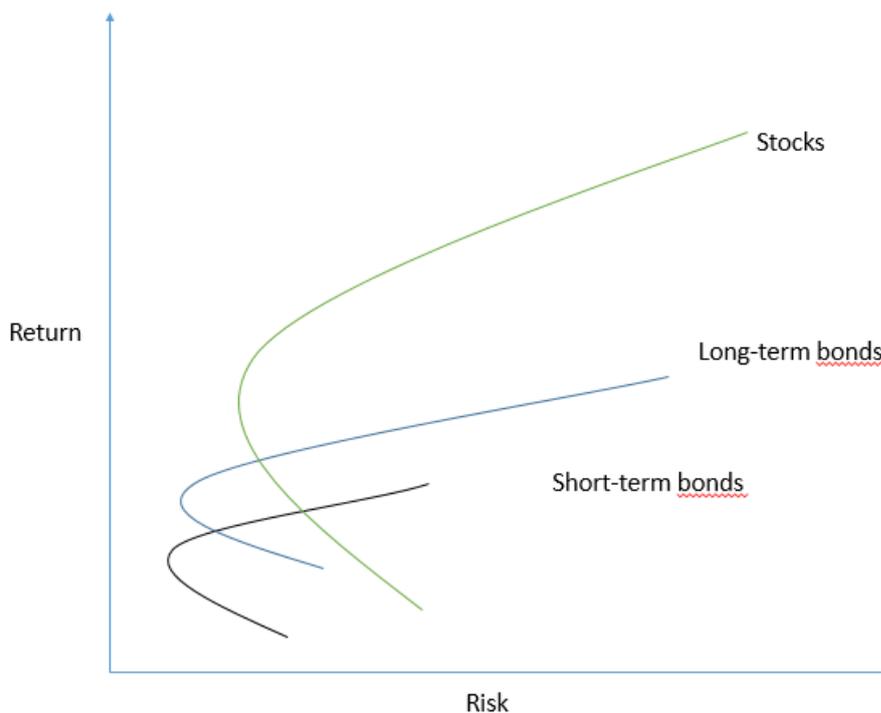


Figure 3 Projected efficient frontier for different asset classes. Stocks have the highest return and SD, long-term bonds have lower return and SD than stocks, while short-term bonds have the lowest return and SD

We expect that a portfolio consisting only of stock indices will carry a high estimated return with a high risk. By combining the portfolio with either long or short-term bonds will decrease both the risk and the estimated return. The lowest level of volatility and expected return would likely be a combination of stocks and short-term bonds.

5 Descriptive statistics

In this part of the paper descriptive statistics of our results will be presented. Descriptive statistics is a quantitative approach to collect and present information. The data will be presented in tables, matrixes and graphs, to illustrate the outcomes for the portfolio problems with historical returns and standard deviation (risk) as main objects.

The results are based on historical data on government bonds of various maturities and All-share (stock) index for a 20 years period, from 1994 through 2013. The period has been divided into three parts; the first is for the entire period, the second for the first 10 years, while the latter is for the last 10 years. The reason for this division is to allocate biases, and to find a common trend that can imply empirical evidence to a concrete conclusion. Thereafter, the same research has been done based on current interest rate on the second period.

First, the historical estimated returns will be presented. Second, portfolios consisted of different asset allocations combining equities and 3-month bonds, and combining equities and 5-year bonds, are presented with tables to illustrate the reward-to-variability. The last section of this chapter illustrated correlation coefficients to better understand the benefits of diversification.

5.1 Historical returns

The first graph illustrates the total return if the entire investment was in one asset:



Figure 4 Historical return on All-share and government bonds 1994-2013. January 2003 bonds return caught up to the All-share return. Since then All-share returns have been much higher than for bonds.

The graph implies that there are significant differences between the historical returns for the stock index and the government bonds with various maturities. The stock index have on average the highest return, however, the stocks also carries the highest volatility among the variables. The increased level of risk is compensated by the higher return. Government bonds with higher maturities carry higher volatility and higher return then the ones with shorter maturity. While the stock market is both rewarding and risky, the average return and standard deviation are relatively stable if investing in government bonds. In periods during stock market crashes, government bonds exceed the given return of the stock index.

The graph also implies a great change in ex-posted return for the two periods investigated, especially for the historical return on stocks that have exceed its previous return in several occasions. The results during these two periods might therefore lead to significant different outcomes. To get a broader understanding of the two periods and the graph above, the arithmetic average annual return was calculated. The results imply differences, as the historical return on government bonds for period 1994 through 2003 is almost the double as for period 2003 through 2013. The historical return for 3-month bonds were in the first period 5.52 percent while in the second period only 2.66 percent, this is a decrease of almost 3 percent during these few years. For the long-term government bonds, in this case 5-year, the increase is not as significant; nevertheless, for the first period the ex-posted return was 6.96 percent compared to 4.68 percent in the latter. This is a decrease of more than 2 percent. On the contrary, for the All-share index, the second period had an arithmetic average return of

14.65 percent, more than 4 percent higher than for the first period, with “only” 10.12 percent return.

The two graphs below illustrate the given portfolio problem by combining All-Share index and the government bonds with 3-month and 5-year maturities. The combination of the All-share index and bonds will show similar outcome for these two graphs as for Figure 4. The trend for the remaining government bonds, 6-month, 1-year and 3-year, will imply the same results, therefore, we have chosen to only illustrate one for short-term and one for long-term.

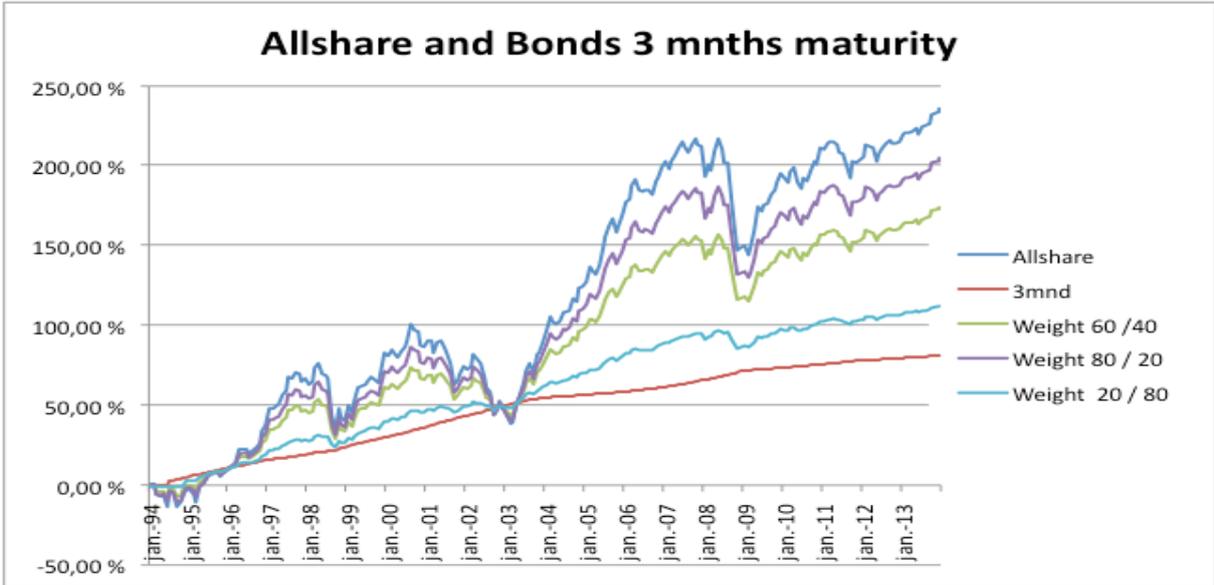


Figure 5 Historical return on combinations of All-share and 3-month bond with various weights

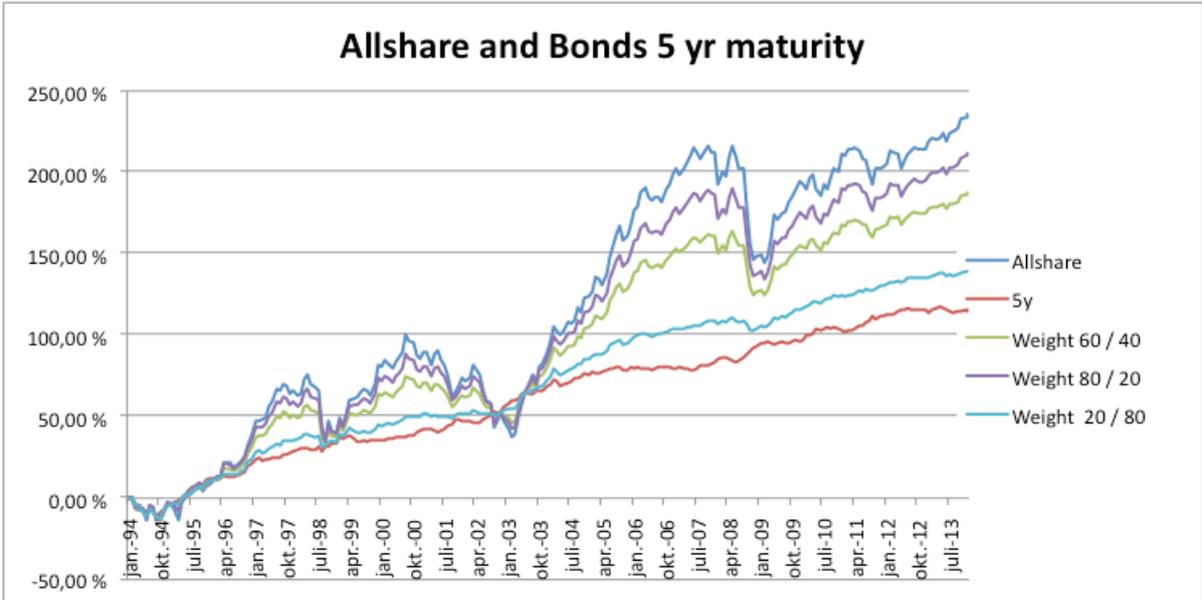


Figure 6 Historical return on combinations of All-share and 5-year bond with various weights

The graphs illustrate the historical returns for combined assets with different weight distributions. By including a bond index to a portfolio that currently consisting of only the All-share index, the volatility and return will decrease. The recession brought on by the Dot.com-bubble from 2000 until 2003 caused such a big drop in the All-share index that bonds had a positive effect on the combined portfolio in the start of 2003. By the end of 2003, the All-share had again surpassed the bonds short lead.

The table below shows the relation between reward-to-variability for given asset allocations.

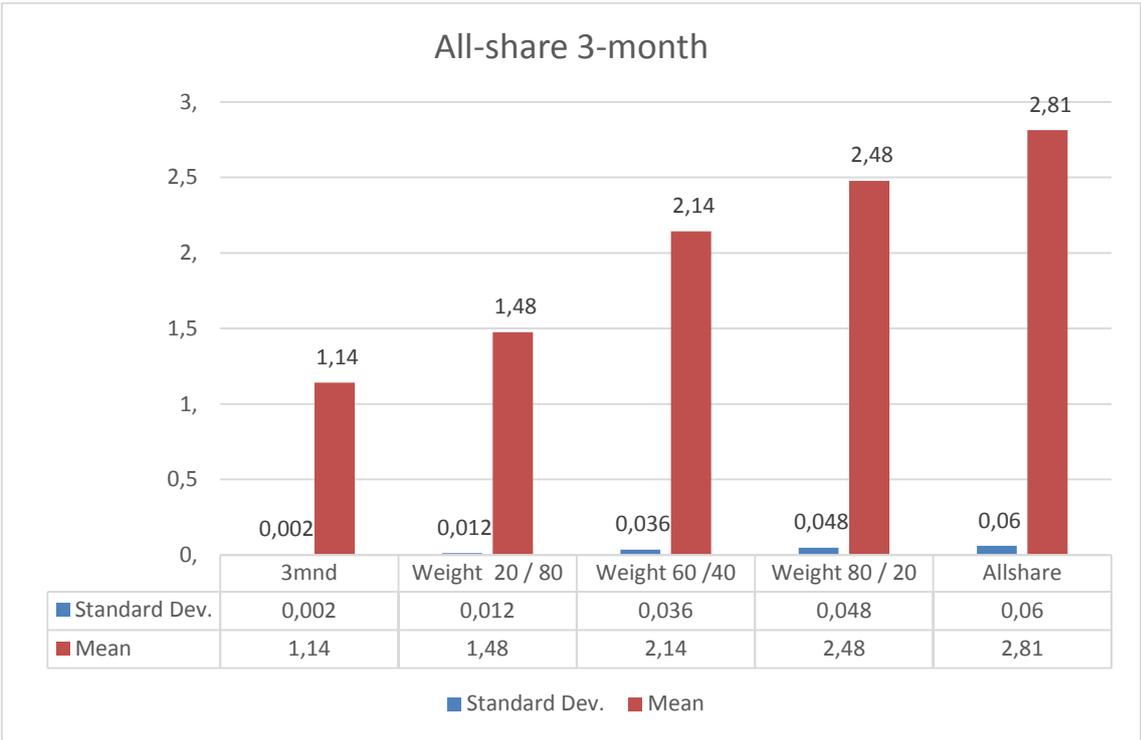


Figure 7 reward-to-variability for 3-month and All-share. Illustrates what you gain in return and increase in risk as the weight increases on the All-share

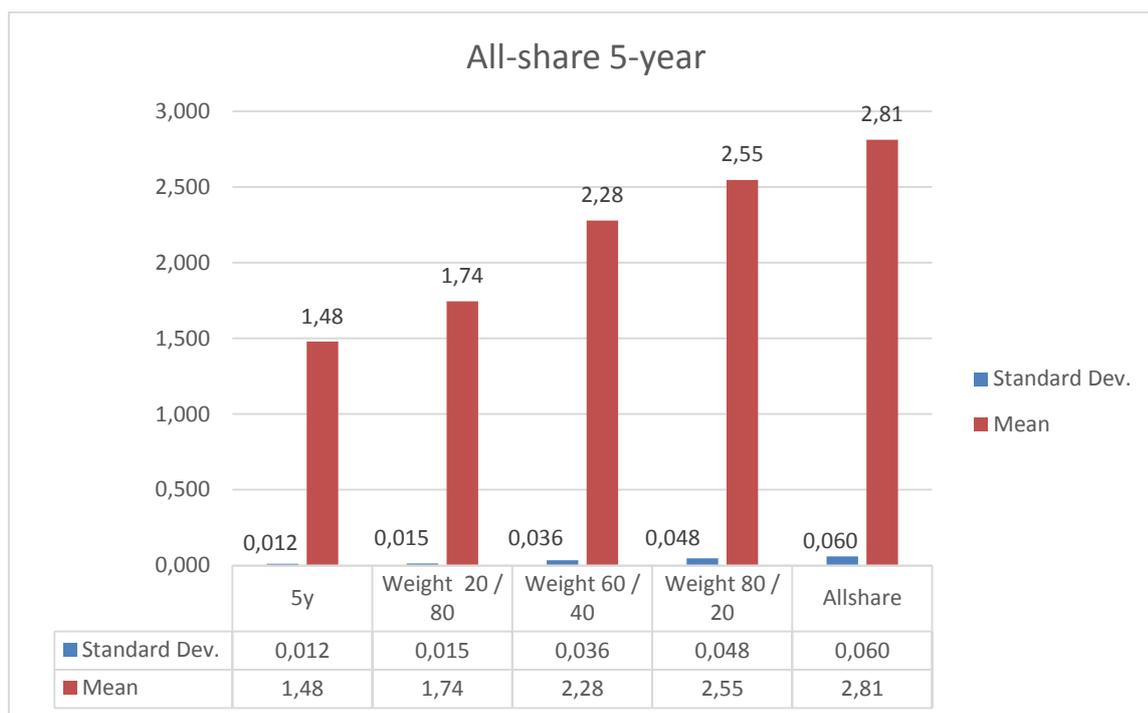


Figure 8 reward-to-variability for 5-year and All-share. Illustrates what you gain in return and increase in risk as the weight increases on the All-share

If investing in only two assets, with a higher weight portion in equities, the standard deviation would remain almost equal for the 3-month compared to the 5-year bonds, while the estimated return increases by 0.14 for the 60 stock and 40 bonds, and by 0.07 for 0.8 stock and 0.2 bonds. This implies that with the majority in equities you would benefit by investing the remained weight in long-term bonds. The results are mainly affected by the lower covariance between these assets. This is something we will discuss later on in the section of Efficient Frontier.

5.2 Correlation and Covariance analysis

The All-Share index has a negative correlation with the government issued bonds during the entire period 1994-2013 and during the last period, 2004-2013. We suspect the reason is what Gulko (2002) labeled as a “flight-to-safety”-effect, where the correlation amongst stocks and bonds becomes increasingly negative during market crashes. The financial crises had a severe influence on the Norwegian financial market during this time. The results for the first period 1994-2003, implies a small positive correlation between the stocks and the bonds. We suspect this is caused by the relatively similar returns in this period for both the stock index and the government bonds.

Correlation 1994 - 2013						
Index	Allshare	3-month	6-month	1-year	3-year	5-year
Allshare	1					
3-month	-0,1251	1				
6-month	-0,0624	0,9405	1			
1-year	-0,0515	0,7591	0,9081	1		
3-year	-0,0862	0,3897	0,5663	0,8213	1	
5-year	-0,0951	0,2792	0,4308	0,6809	0,9470	1

Figure 9 Correlation between All-share and Government Bonds. Note that 3-month has the lowest correlation to the All-share

Correlation 1994 - 2003						
Index	Allshare	3-month	6-month	1-year	3-year	5-year
Allshare	1					
3-month	0,0666	1				
6-month	0,2095	0,9205	1			
1-year	0,2534	0,7284	0,915	1		
3-year	0,2095	0,4576	0,6787	0,8855	1	
5-year	0,1376	0,3623	0,552	0,766	0,9538	1

Figure 10 Correlation between All-share and government bonds, period 1994-2003. Note that there is a positive correlation for all bonds.

Correlation 2004 - 2013						
Index	Allshare	3-month	6-month	1-year	3-year	5-year
Allshare	1					
3-month	-0,3507	1				
6-month	-0,3201	0,9299	1			
1-year	-0,3781	0,7392	0,8880	1		
3-year	-0,4194	0,3529	0,4869	0,7789	1	
5-year	-0,3578	0,2530	0,3609	0,6276	0,9398	1

Figure 11 Correlation between All-share and government bonds, period 2004-2013. Negative correlation for all bonds, 3-year has the lowest.

The negative correlation implies that the bonds and stocks move in opposite direction, therefore, a combination of the two assets will reduce the standard deviation and greatly benefit from diversification. A portfolio of two assets will benefit from diversification as long as the correlation between the two assets in the portfolio is less than 1. The correlation coefficients imply that the assets only have perfect correlation with themselves; therefore, investing in more than one of the assets will reduce the risk.

The correlation coefficients are higher among the long-terms bonds and among short-terms bonds. For example, the correlation between 3-year and 5-year is above 0.9 and the same goes for 3-month and 6-month bonds. These securities will have the same trend; so if one declines

(increases) the other will most likely decline (increases) as well. This correspond to the term structure theory stated by Bodie et al (2011).

Johnson (et al 2014) argues that the conventional wisdom of negative correlation for bonds-stocks has been caused of low inflation risk. With increased interferes by the government and/or central banks that we see now a days, as fiscal and monetary policies, the inflation risk may rise. Consequently, the correlation investors have been taken for granted, might become less influential for investments in the future. Based on this, Johnson (et al 2014) expects the correlation between stocks and bonds to remain negative in the short run. In the long run however, the correlation might be higher or even positive, as was seen during our first period 1994-2003 of research.

6 Efficient Frontier

In the following section, we use the historical data and estimations for the equities and bonds of various maturity to illustrate the outcomes for portfolio combinations given the chosen asset allocations.

6.1 Efficient frontier

In this chapter, we have used Markowitz modern portfolio theory on Efficient Frontier to construct portfolios combining the All-Share index and the government bonds. The first step was to create a covariance matrix and find the correlation coefficient between the variables, which was done in the previous section. The whole process has been developed on monthly, historical data from Oslo Børs in a period of twenty years, from 1994 through 2013. The next step was to find different efficient frontiers for the period, first without any interference, thereafter, different criteria where set. The weight distributions on the different assets, based on GPFG allocation and two other portfolios with the majority of the weight on either stock or bonds.

The mean variance and optimum weight distribution for the efficient frontier are listed in the tables below the graphs. The covariance, standard deviations and complete efficient frontier calculations are listed in the appendix. The idea was to see what impact the various maturities have had to the mean-variance portfolio.

6.2 Analysis of the efficient frontiers

In the analysis of the efficient frontier, different graphs will be displayed, given the asset allocations and time horizon. This would give an estimate of how much more risk you undertake as you alter your estimate returns. The first criterion was to divide the portfolios into three different weight distributions, as 0.2/0.8, 0.8/0.2 and 0.6/0.4. The first 0.2/0.8 implies that 20 percent of the portfolio is locked to the stock index and 80 percent is divided among the government bonds with various maturities. The second illustrates 80 percent in stocks and 20 percent divided among the bonds, while the last one has 60 percent in stocks and the remaining 40 percent in bonds, the latter influenced by the GPF. The intention was to optimize the portfolio by finding the right maturity for the bonds given the different asset allocations combining bonds and stocks. The second criterion was dividing the research into different periods to distinguish biases. The first section takes the whole period into consideration, 1994-2013; thereafter the period is divided into two equal periods, 1994-2003 and 2004-2013. Eventually, estimated returns based on the current interest rates are used to find the expected return.

6.2.1 Efficient frontier whole period without interference

The following graph shows the efficient frontier including all possible assets.

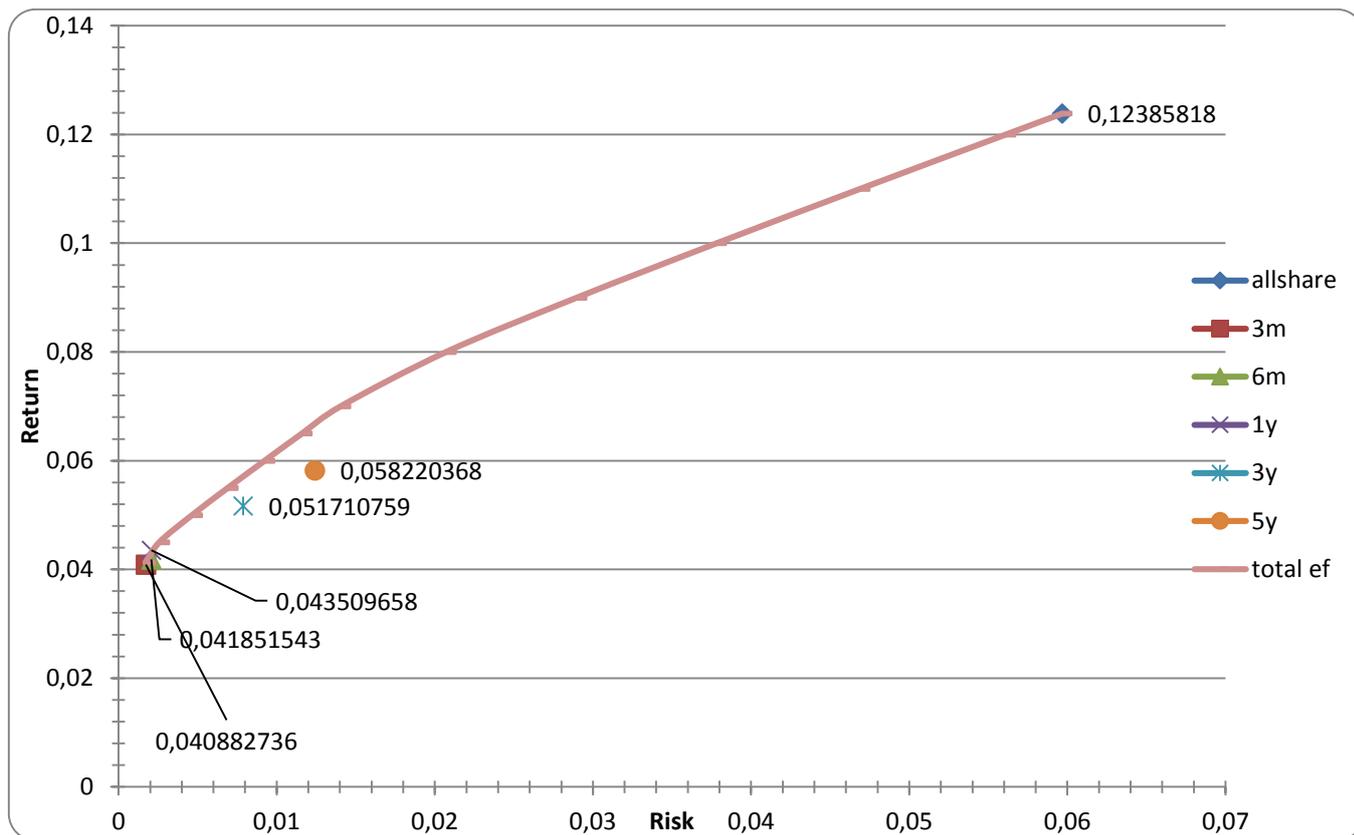


Figure 12 Efficient Frontier for all assets. Note that the standard deviation of an efficient portfolio gives a higher mean to the same standard deviation as any single asset; this illustrates the effect of diversification

The efficient frontier without interference implies that the long-term securities, 3-year and 5-year bonds, are not efficient investments, as there are other portfolios that have higher return for the same level of risk. In the research, the short-term indices are close to risk-free and low return, while the stock index carries the highest return and highest volatility. Therefore, efficient frontiers made with these assets should be somewhere on the line between the risk-free and the stock index.

6.2.2 Efficient frontier for portfolios with given risk-weight

The following part contains estimation of efficient frontiers with portfolios with a given risk-weight. The efficient frontier is illustrated and the combinations of asset allocations are presented in tables. Generally, efficient portfolios consist of assets with the lowest covariance, as these gives great potential opportunities to benefit from diversification. What is important is that the assets do not move down together.

6.2.2.1 Total

In the period as a whole, 1994 through 2013, the All-Share index and government bonds had an inverse relationship, with 3-month government bonds being the lowest of negative 0.125, which gives great potential for diversification.

The following diagrams show the different efficient frontiers when under different weights portfolios.

6.2.2.1.1 Given risk/non-risk weights 0.2/0.8

		Min Var			Highest slope				Highest mean	
Mean	5,75 %	5,86 %	6,00 %	6,20 %	6,35 %	6,50 %	6,60 %	7,00 %	7,13 %	
SD	1,18 %	1,18 %	1,19 %	1,21 %	1,23 %	1,26 %	1,29 %	1,42 %	1,48 %	
Slope	4,854	4,956	5,059	5,143	5,161	5,147	5,121	4,927	4,828	
All-share	0,20	0,20	0,20	0,20	0,20	0,20	0,20	0,20	0,20	
3-month	0,80	0,74	0,65	0,54	0,44	0,32	0,24	0,00	0,00	
6-month	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	
1-year	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	
3-year	0,00	0,00	0,00	0,00	0,03	0,13	0,19	0,21	0,00	
5-year	0,00	0,06	0,15	0,26	0,33	0,36	0,37	0,59	0,80	

Figure 13 Efficient portfolios with weights: 0.2/0.8

Figure 14 displays the efficient frontier with a given estimate of how much more risk you undertakes as you alter your estimate returns. The global minimum variance portfolio gives an estimated return of 5.9 percent with a standard deviation of 1.182 percent. The weights are 0.2 on stock index (given) and divided by 0.74 on 3-month bonds and 0.06 on 5-year bonds. By trading of more and more off the short-term bonds for the long-term bonds, the estimated return increases as well as the risk. Any portfolios above the minimum-variance portfolio are efficient portfolios. Where on the efficient frontier you chose to be is up to the individual investor's risk-tolerance.

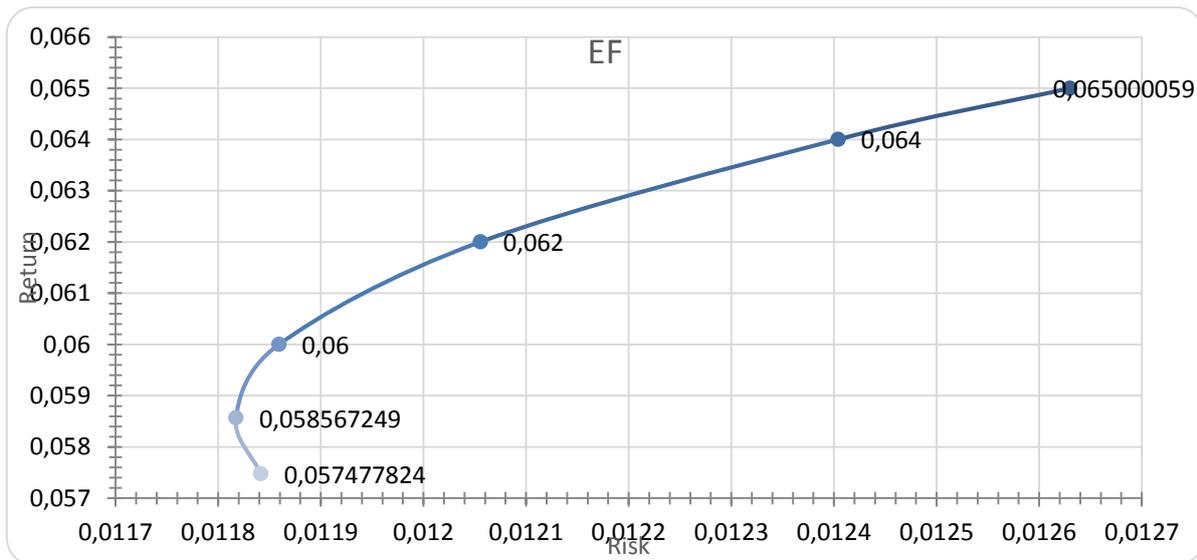


Figure 14 Efficient frontier for 0.2/0.8 stocks/bonds

6.2.2.1.2 Given risk/non-risk weights 0.8/0.2

					Min Var,Highest mean/slope
Mean	10,73%	10,80%	10,90%	11,00%	11,07%
SD	4,770%	4,766%	4,762%	4,758%	4,757%
Slope	2,249	2,266	2,289	2,312	2,328
All-share	0,80	0,80	0,80	0,80	0,80
3-month	0,20	0,16	0,10	0,04	0,00
6-month	0,00	0,00	0,00	0,00	0,00
1-year	0,00	0,00	0,00	0,00	0,00
3-year	0,00	0,00	0,00	0,00	0,00
5-year	0,00	0,04	0,10	0,16	0,20

Figure 15 Efficient portfolios with weights 0.8/0.2

In this instance, the majority of the portfolio is invested in equities, with the given portfolio weights of 0.8 in the stock index and 0.2 divided between the government bonds. As the diagram illustrate, the weight distribution with the highest mean also have the lowest standard deviation. The efficient frontier graph then stops at the global minimum variance, implying that the optimal portfolio would be a distributed with 0.8 in the All-Share index, 0.1 in 3-month bonds and 0.1 in 5-year bonds.

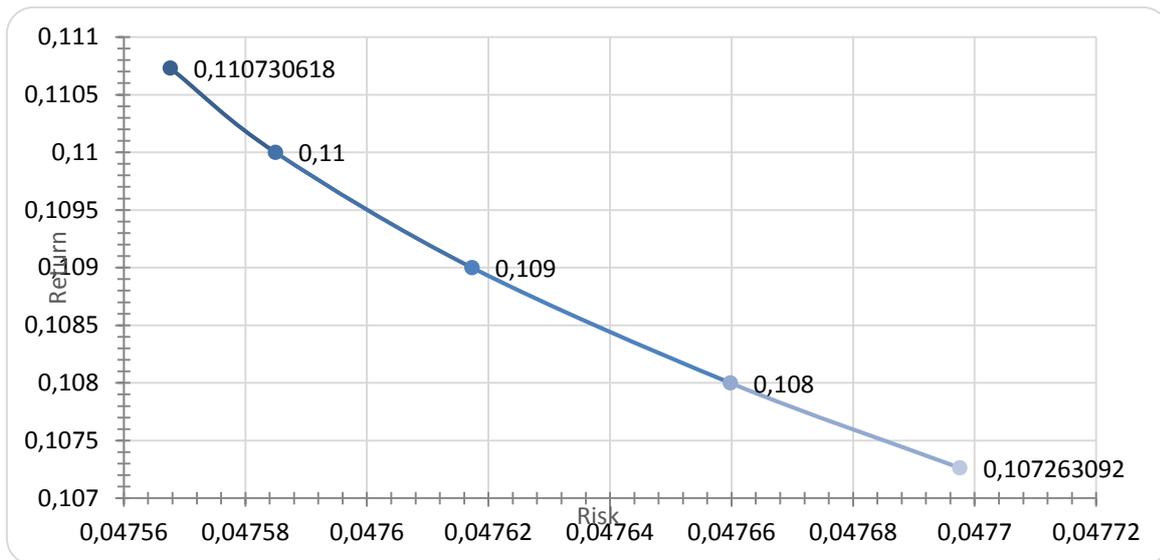


Figure 16 Efficient frontier for 0.8/0.2 stocks/bonds

6.2.2.1.3 Given risk/non-risk weights 0.6/0.4

	Short term		Min Var			Highest mean/highest slope
Mean	9,07%	9,25%	9,47%	9,60%	9,70%	9,76%
SD	3,572%	3,565%	3,562%	3,563%	3,565%	3,568%
Slope	2,538	2,595	2,658	2,694	2,721	2,736
All-share	0,60	0,60	0,60	0,60	0,60	0,60
3-month	0,40	0,29	0,17	0,09	0,03	0,00
6-month	0,00	0,00	0,00	0,00	0,00	0,00
1-year	0,00	0,00	0,00	0,00	0,00	0,00
3-year	0,00	0,00	0,00	0,00	0,00	0,00
5-year	0,00	0,11	0,23	0,31	0,37	0,40

Figure 17 Efficient portfolios with weights 0.6/0.4

The global minimum variance gives a return of 9.47 percent with a standard deviation of 3.562 percent. Efficient portfolio should consist of a maximum weight of 0.17 on 3-month and at least 0.23 on 5-year bonds. The weight distribution should gradually transfer from 3-month to 5-year.

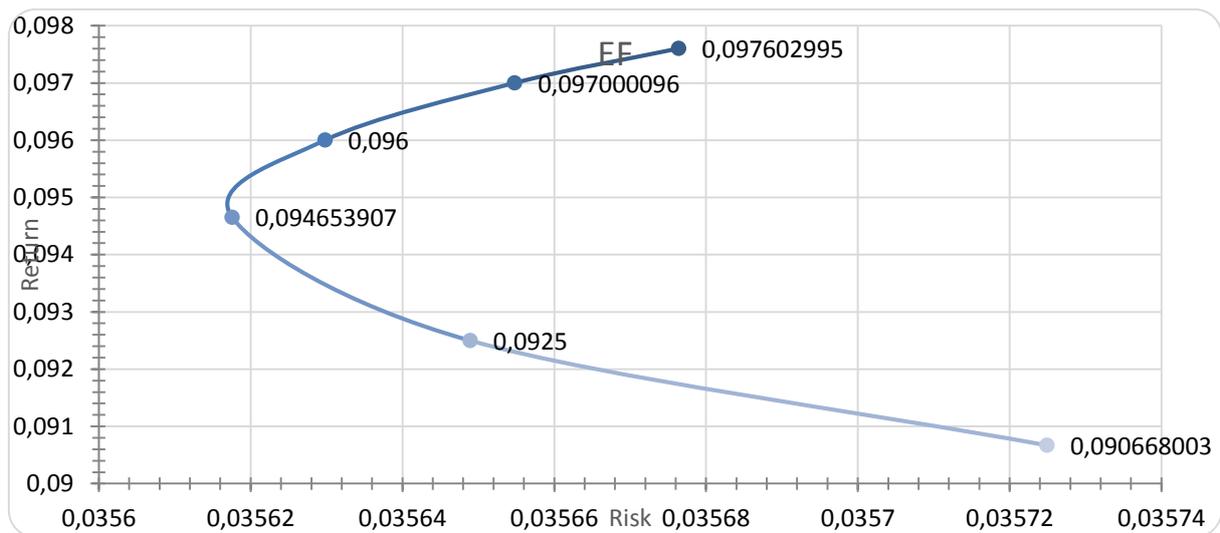


Figure 18 Efficient frontier with 0.6/0.4 in stocks/bonds

6.2.2.2 Period 1: 1994-2003

In the first period the All-share index and the government bonds have a positive correlation with 3-month government bonds being the lowest of 0.066

6.2.2.2.1 Given risk/non-risk weights 0.2/0.8

	Min Var	Highest slope					Highest mean
Mean	6,44 %	6,50 %	6,75 %	7,00 %	7,25 %	7,50 %	7,59 %
SD	1,16 %	1,17 %	1,24 %	1,34 %	1,47 %	1,62 %	1,68 %
Slope	5,55	5,56	5,46	5,22	4,93	4,63	4,51
All-share	0,20	0,20	0,20	0,20	0,20	0,20	0,20
3-month	0,80	0,76	0,58	0,41	0,17	0,00	0,00
6-month	0,00	0,00	0,00	0,00	0,00	0,00	0,00
1-year	0,00	0,00	0,00	0,00	0,01	0,00	0,00
3-year	0,00	0,00	0,00	0,00	0,19	0,19	0,00
5-year	0,00	0,04	0,22	0,39	0,44	0,61	0,80

Figure 19 Efficient portfolios with weights 0.2/0.8

The global minimum variance portfolio gives an estimated return of 6.44 percent with a standard deviation of 1.16 percent. The result implies that the entire investment should be divided between the stock index (given 0.2) and the remaining 0.80 should be invested in 3-month bonds, as the standard deviation for 3-month bonds is the lowest and the covariance between these two assets are the significant lowest. By trading of more and more off the short-term bond for the long-term bonds, the estimated return increases as well as the risk. To achieve an efficient portfolio your main investment should be on 3-month bonds, 3-year bonds and/or 5-year bonds, this because the covariance between these are smaller given the return. The final choice of investment would eventually depend on the investor's risk tolerance.

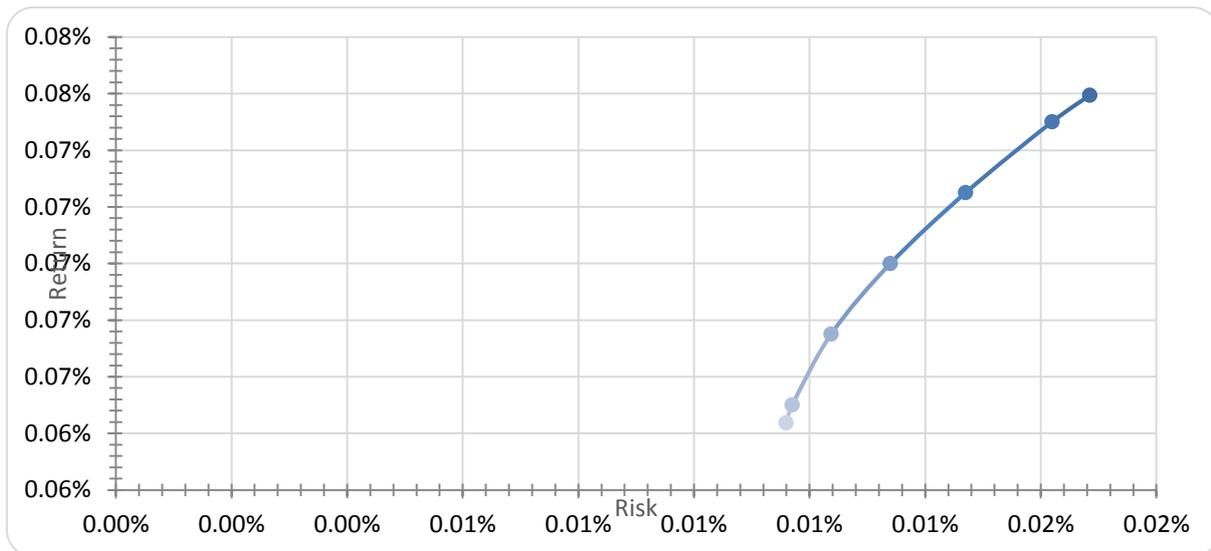


Figure 20 Efficient frontier with 0.2/0.8 in stocks/bonds

6.2.2.2.2 Given risk/non-risk weights 0.8/0.2

	Min Var				Highest slope
Mean	9,20 %	9,25 %	9,35 %	9,45 %	9,49 %
SD	4,59 %	4,60 %	4,62 %	4,63 %	4,64 %
Slope	2,00	2,01	2,03	2,04	2,05
All-share	0,80	0,80	0,80	0,80	0,80
3-month	0,20	0,16	0,09	0,03	0,00
6-month	0,00	0,00	0,00	0,00	0,00
1-year	0,00	0,00	0,00	0,00	0,00
3-year	0,00	0,00	0,00	0,00	0,00
5-year	0,00	0,04	0,11	0,17	0,20

Figure 21 Efficient portfolios with weights 0.8/0.2

The second graph illustrate a similar trend as the previous, there would be a small, almost linear increase in both the return and standard deviation by adding 5-year bonds to the portfolio. The global minimum-variance portfolio would be a combination of the stock index and the 3-month bond, which would give an estimated return of 9.2 percent with a standard deviation of 4.59 percent. The highest estimated return would be with a combination of stock and 5-year bonds, with an estimated return of 9.5 percent and standard deviation of 4.64 percent.

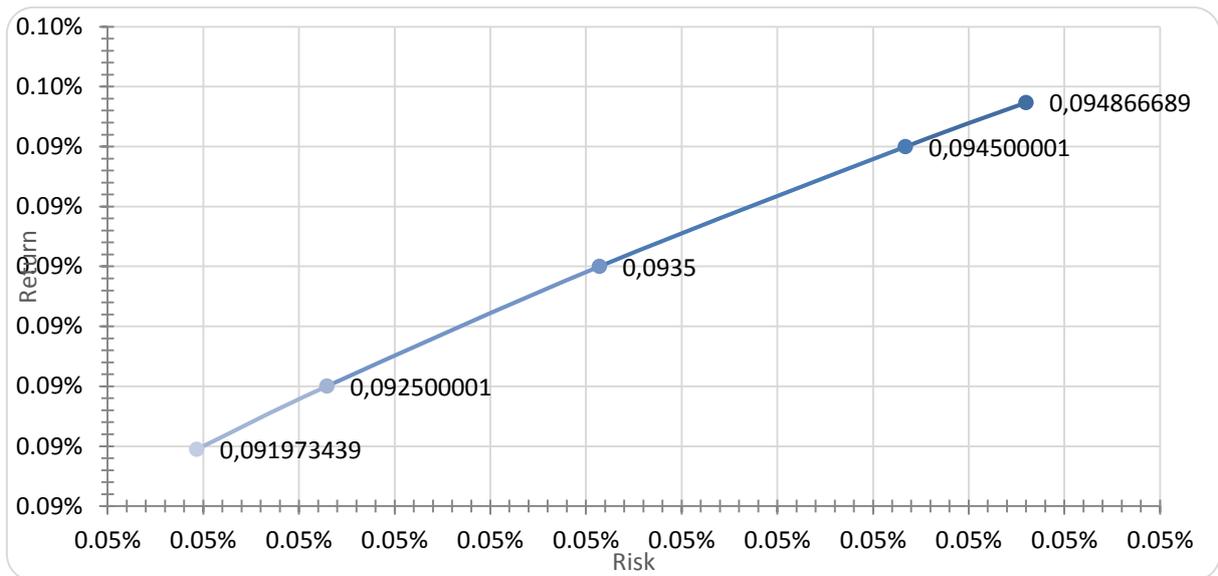


Figure 22 Efficient frontier with 0.8/0.2 in stocks/bonds

6.2.2.2.3 Given risk/non-risk weights 0.6/0.4

	Min Var				Highest slope		Highest mean
Mean	8,28 %	8,40 %	8,50 %	8,62 %	8,75 %	8,86 %	
SD	3,45 %	3,47 %	3,49 %	3,51 %	3,54 %	3,56 %	
Slope	2,40	2,42	2,44	2,46	2,47	2,49	
Allshare	0,60	0,60	0,60	0,60	0,60	0,60	
3mnd	0,40	0,32	0,20	0,16	0,03	0,00	
6mnd	0,00	0,00	0,00	0,00	0,03	0,00	
1y	0,00	0,00	0,02	0,00	0,02	0,00	
3y	0,00	0,00	0,07	0,00	0,01	0,00	
5y	0,00	0,08	0,10	0,24	0,32	0,40	

Figure 23 Efficient portfolios with weights 0.6/0.4

The last graph implies a very similar linear trend to the other two graphs in this period. The global minimum variance would be a combination of stocks and 3-month bonds that give a return of 8.28 percent with standard deviation of 3.45 percent, the highest mean would be a combination of stocks and 5-year bonds, with a return of 8.86 percent with standard deviation of 3.56 percent.

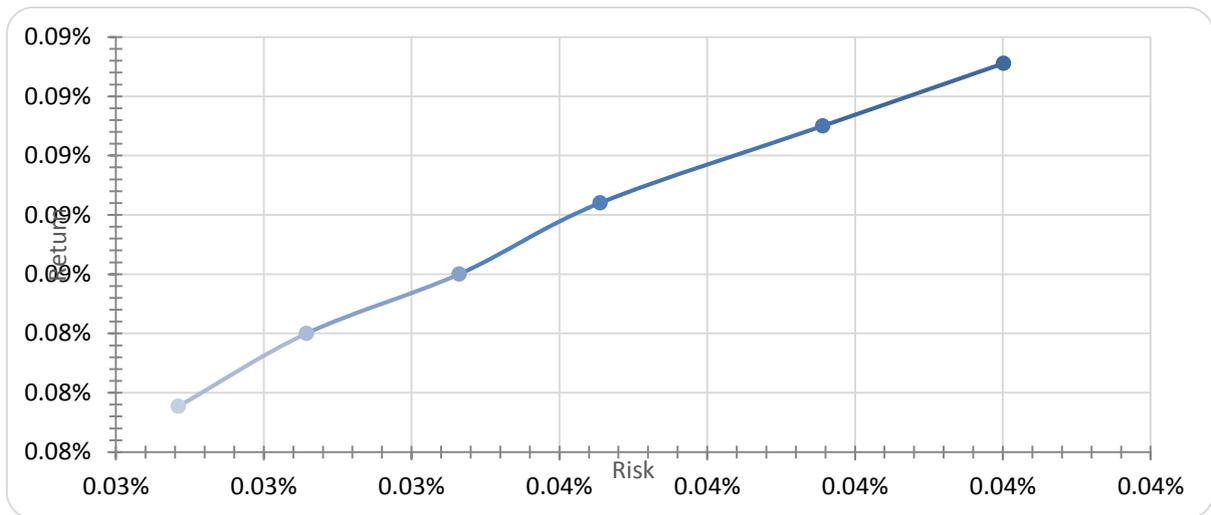


Figure 24 Efficient frontier with 0.6/0.4 in stocks/bonds

6.2.2.3 Period 2: 2004-2013

In the second period the relation is inverse with the lowest correlation coefficient between All-share index and 6-month bonds, while 5-year bonds have the lowest covariance of negative 0.0002

6.2.2.3.1 Given risk/non-risk weights 0.2/0.8

					Min Var			Highest slope
Mean	5,06 %	5,25 %	5,50 %	5,75 %	5,87 %	6,00 %	6,25 %	6,37 %
SD	1,21 %	1,17 %	1,14 %	1,12 %	1,12 %	1,12 %	1,15 %	1,17 %
Slope	4,20	4,48	4,83	5,13	5,25	5,35	5,45	5,46
All-share	0,20	0,20	0,20	0,20	0,20	0,20	0,20	0,20
3-month	0,80	0,64	0,43	0,23	0,12	0,02	0,00	0,00
6-month	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
1-year	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
3-year	0,00	0,16	0,37	0,57	0,68	0,78	0,52	0,37
5-year	0,00	0,00	0,00	0,00	0,00	0,00	0,28	0,43

Figure 25 Efficient portfolios with weights 0.2/0.8

The global minimum-variance portfolio gives an estimated return of 2.87 percent with a standard deviation of 1.12 percent. The minimum-variance portfolio consists of 0.20 in stock index (given), 0.12 on 3-month bonds and 0.68 on 3-year government bonds. The 3-month bonds are gradually traded off to 3-year and 5-year bonds as the efficient portfolio moves up.

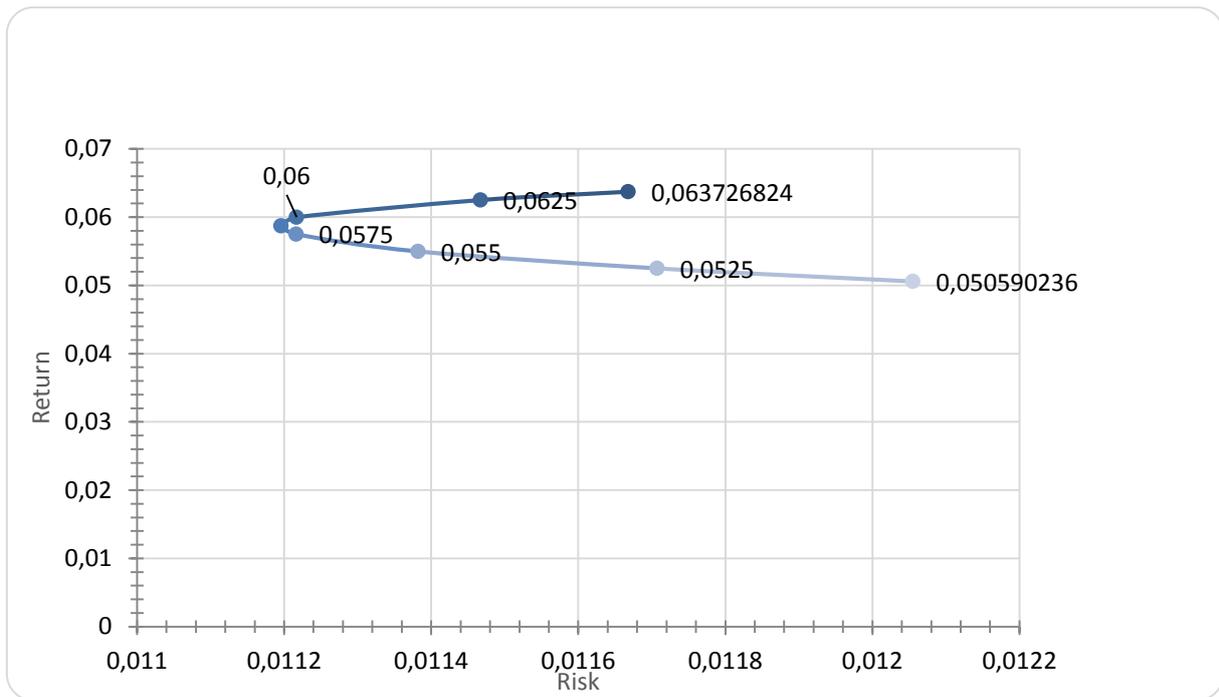


Figure 26 Efficient frontier with 0.2/0.8 in stocks/bonds

6.2.2.3.2 Given risk/non-risk weights 0.8/0.2

									Min Var, Highest mean/slope
Mean	12,26 %	12,35 %	12,38 %	12,45 %	12,50 %	12,55 %	12,60 %	12,65 %	12,66 %
SD	4,935 %	4,916 %	4,910 %	4,896 %	4,887 %	4,881 %	4,875 %	4,870 %	4,869 %
Slope	2,483	2,512	2,521	2,543	2,558	2,571	2,584	2,598	2,600
All-share	0,80	0,80	0,80	0,80	0,80	0,80	0,80	0,80	0,80
3-month	0,20	0,11	0,10	0,04	0,00	0,00	0,00	0,00	0,00
6-month	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
1-year	0,00	0,02	0,00	0,00	0,00	0,00	0,00	0,00	0,00
3-year	0,00	0,07	0,10	0,16	0,20	0,13	0,07	0,01	0,00
5-year	0,00	0,00	0,00	0,00	0,00	0,07	0,13	0,19	0,20

Figure 27 Efficient portfolios with weights 0.8/0.2

The global minimum variance portfolio gives an estimated return of 12.7 percent with a standard deviation of 4.869 percent. The portfolio consists of only the All-Share (0.80) and 5-year government bonds. The minimum variance portfolio also gives the highest mean, so this is the only efficient portfolio.

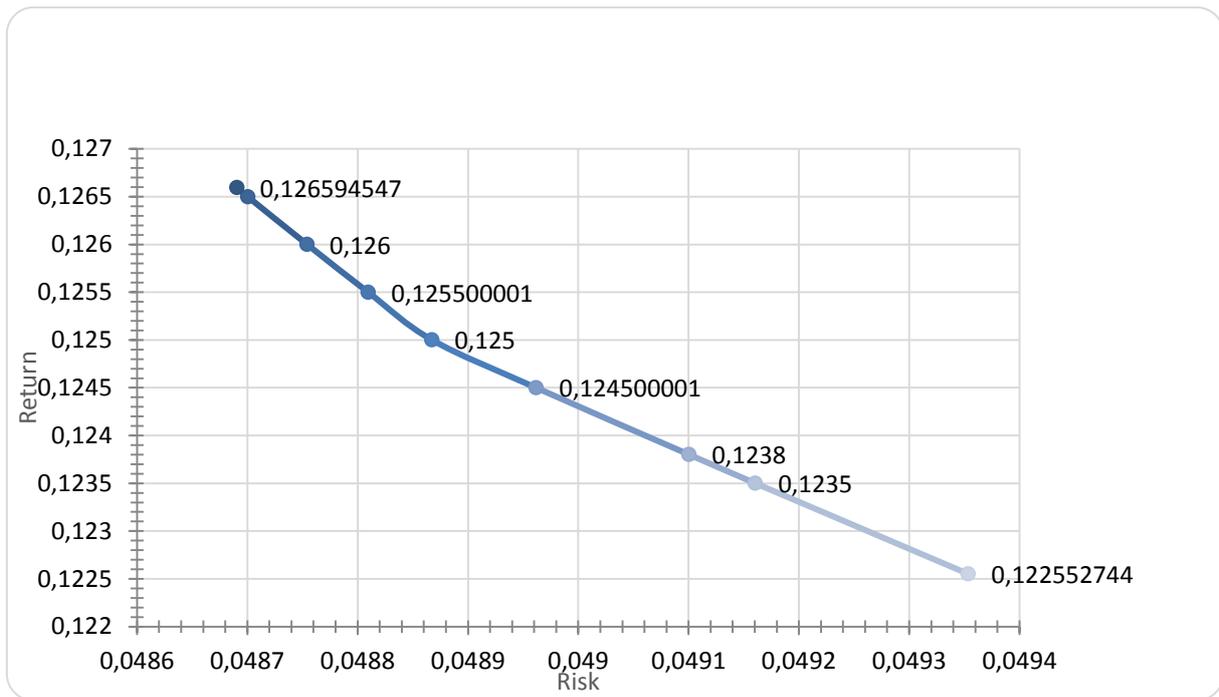


Figure 28 Efficient frontier with 0.8/0.2 in stocks/bonds

6.2.2.3.3 Given risk/non-risk weights 0.6/0.4

						Min Var, Highest mean/slope
Mean	9,86 %	10,00 %	10,12 %	10,25 %	10,50 %	10,66 %
SD	3,69 %	3,66 %	3,64 %	3,62 %	3,59 %	3,57 %
Slope	2,67	2,73	2,78	2,83	2,93	2,98
All-share	0,60	0,60	0,60	0,60	0,60	0,60
3-month	0,40	0,28	0,18	0,07	0,00	0,00
6-month	0,00	0,00	0,00	0,00	0,00	0,00
1-year	0,00	0,00	0,00	0,00	0,00	0,00
3-year	0,00	0,12	0,22	0,33	0,20	0,00
5-year	0,00	0,00	0,00	0,00	0,20	0,40

Figure 29 Efficient portfolios with weights 0.6/0.4

The global minimum variance portfolio gives an estimated return of 10.66 percent with a standard deviation of 3.57 percent; it also gives the highest possible return. This implies that this is the only efficient portfolio. Portfolio consists of 0.60 All-Share (given) and 0.40 5-year

government bonds.

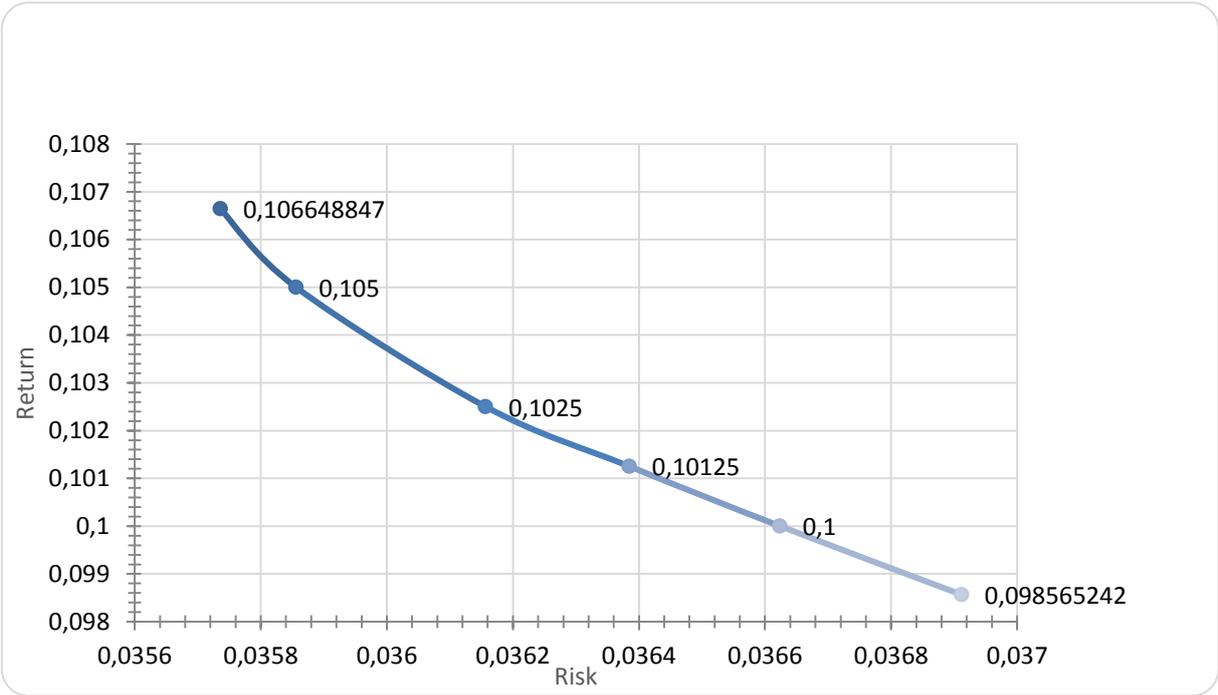


Figure 30 Efficient frontier with 0.6/0.4 in stocks/bonds

6.2.2.4 *Period 2: 2004-2013 with current interest*

In this part of the paper current interest rates have been used to reflect the market today. This would give an estimate of investments made in the current market

6.2.2.4.1 Given risk/non-risk weights 0.2/0.8

				Min Var			Highest slope		Highest mean
Mean	4,05 %	4,20 %	4,25 %	4,30 %	4,35 %	4,40 %	4,46 %	4,50 %	4,58 %
SD	1,17 %	1,13 %	1,12 %	1,12 %	1,12 %	1,13 %	1,14 %	1,16 %	1,18 %
Slope	3,472	3,725	3,788	3,839	3,871	3,887	3,893	3,890	3,868
All-share	0,20	0,20	0,20	0,20	0,20	0,20	0,20	0,20	0,20
3-month	0,00	0,00	0,02	0,12	0,03	0,04	0,06	0,07	0,10
6-month	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
1-year	0,74	0,36	0,21	0,00	0,00	0,00	0,00	0,00	0,00
3-year	0,06	0,44	0,56	0,68	0,73	0,59	0,43	0,31	0,10
5-year	0,00	0,00	0,00	0,00	0,04	0,17	0,31	0,42	0,61

Figure 31 Efficient portfolios with weights 0.2/0.8

The global minimum variance portfolio gives an estimated return of 4.3 percent with a standard deviation of 1.12 percent. The mean-variance portfolio consists of 0.20 weights in stock index (given), 0.12 on 3-month government bonds and 0.68 on 3-year government bonds. The weight on the 3-month bond is gradually traded off to more 3-year and 5-year as the efficient portfolio moves up.

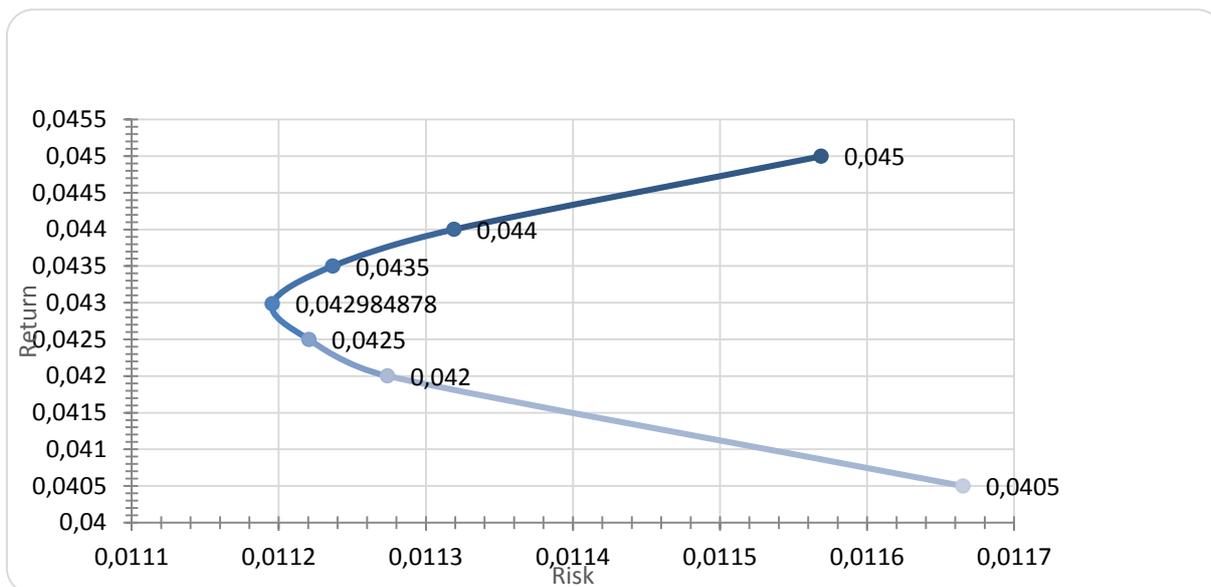


Figure 32 Efficient frontier with 0.2/0.8 in stocks/bonds

6.2.2.4.2 Given risk/non-risk 0.8/0.2

					Min Var, Highest mean/slope
Mean	12,00 %	12,05 %	12,10 %	12,15 %	12,16 %
SD	4,93 %	4,90 %	4,88 %	4,87 %	4,87 %
Slope	2,44	2,46	2,48	2,49	2,50
All-share	0,80	0,80	0,80	0,80	0,80
3-month	0,00	0,00	0,00	0,00	0,00
6-month	0,00	0,00	0,00	0,00	0,00
1-year	0,20	0,07	0,00	0,00	0,00
3-year	0,00	0,13	0,14	0,03	0,00
5-year	0,00	0,00	0,06	0,17	0,20

Figure 33 Efficient portfolios with weights 0.8/0.2

The global minimum variance portfolio gives an estimated return of 12.16 percent with a standard deviation of 4.869 percent. The portfolio consists of the All-Share (0.80) and 5-year (0.20) government bond. The mean variance portfolio gives also the highest mean, so this is the only efficient portfolio.

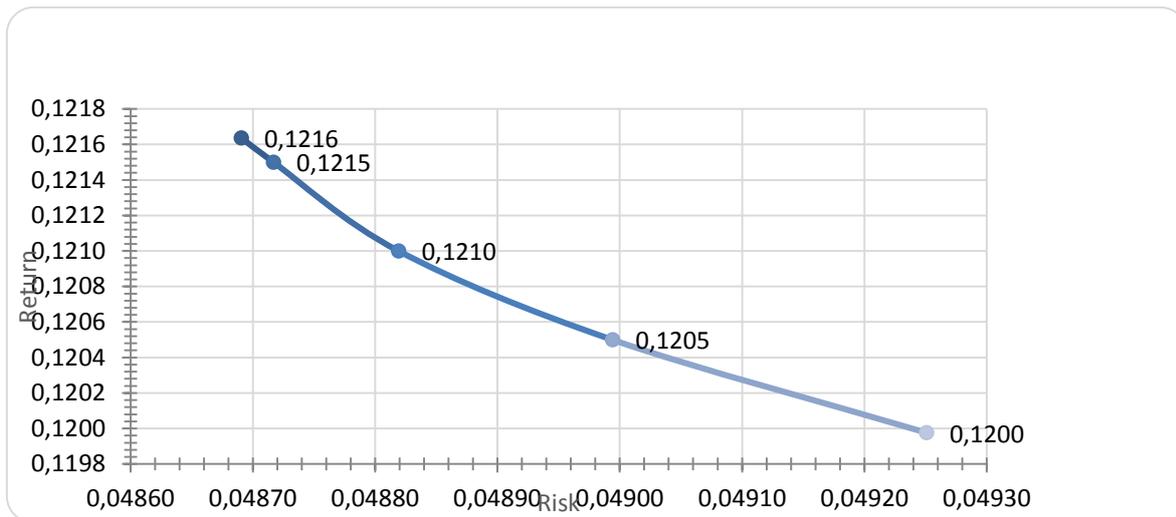


Figure 34 Efficient frontier with 0.8/0.2 in stocks/bonds

6.2.2.4.3 Given risk/non-risk weights 0.6/0.4

							Min Var, Highest mean/slope
Mean	9,34 %	9,40 %	9,45 %	9,50 %	9,60 %	9,65 %	9,67 %
SD	3,67 %	3,64 %	3,62 %	3,60 %	3,58 %	3,58 %	3,57 %
Slope	2,54	2,58	2,61	2,64	2,68	2,70	2,71
All-share	0,60	0,60	0,60	0,60	0,60	0,60	0,60
3-month	0,00	0,00	0,00	0,00	0,00	0,00	0,00
6-month	0,00	0,00	0,00	0,00	0,00	0,00	0,00
1-year	0,40	0,25	0,12	0,00	0,00	0,00	0,00
3-year	0,00	0,15	0,28	0,39	0,17	0,05	0,00
5-year	0,00	0,00	0,00	0,01	0,23	0,35	0,40

Figure 35 Efficient portfolios with weights 0.6/0.4

The global minimum variance portfolio gives an estimated return of 9.67 percent with a standard deviation of 3.57 percent; it also gives the highest possible return meaning this is the only efficient portfolio. Portfolio consists of 0.60 All-Share (given) and 0.40 5-year government bonds.

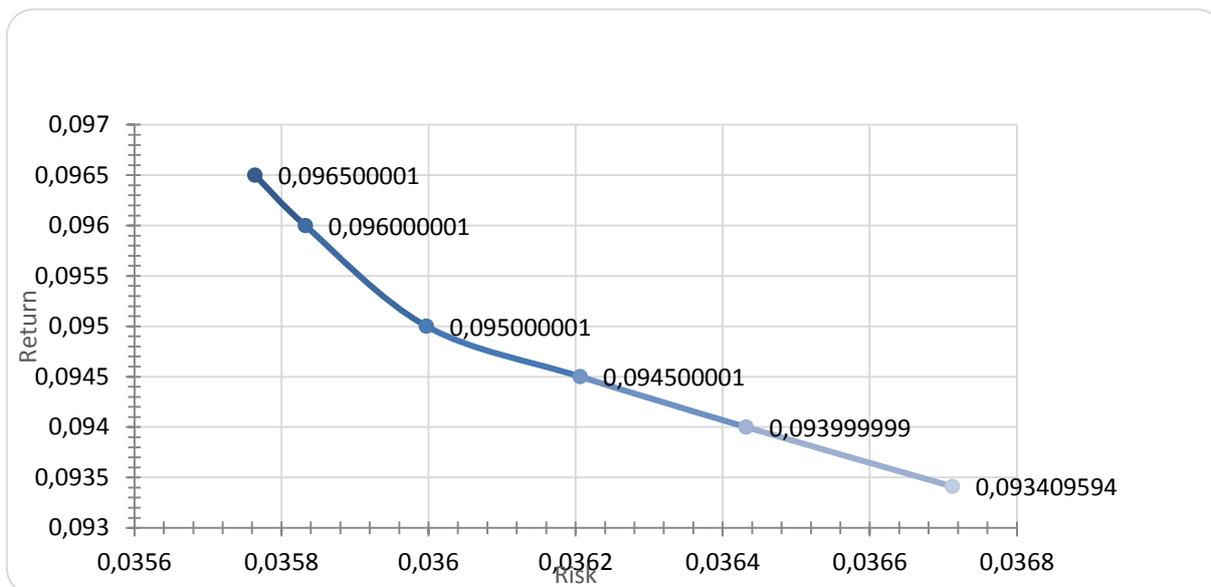


Figure 36 Efficient frontier with 0.6/0.4 in stocks/bonds

6.3 Comparison of different efficient frontiers

Figure 37 illustrates the combination of the stock index together with the government bonds with various maturities. This to better understand how correlation and covariance interfere with portfolio risk, as the covariance/correlation among the variables are lower and plays a more important part than does the standard deviation for the variables separately.

The line with the highest estimated return for the given risk is in this case a combination between the All-share index and long-term government bonds, 3-years and 5-years, respectively.

The choice of investment will depend on the investor's risk tolerance, on whether or not the investor is risk seeking or risk averse. A risk averse can achieve a greater utility by investing in a combination of only short-term bonds and the stock index as this combination gives the lowest risk possible. By utilizing only long-term bonds and the stock index, both the return and risk are marginally higher. After the standard deviation exceeds roughly 1 percent, the efficient frontier implies that the portfolio with short-term maturity bonds has a lower estimated return at the same level of risk than the portfolio with long-term maturity bonds.

The portfolio would not be efficient.

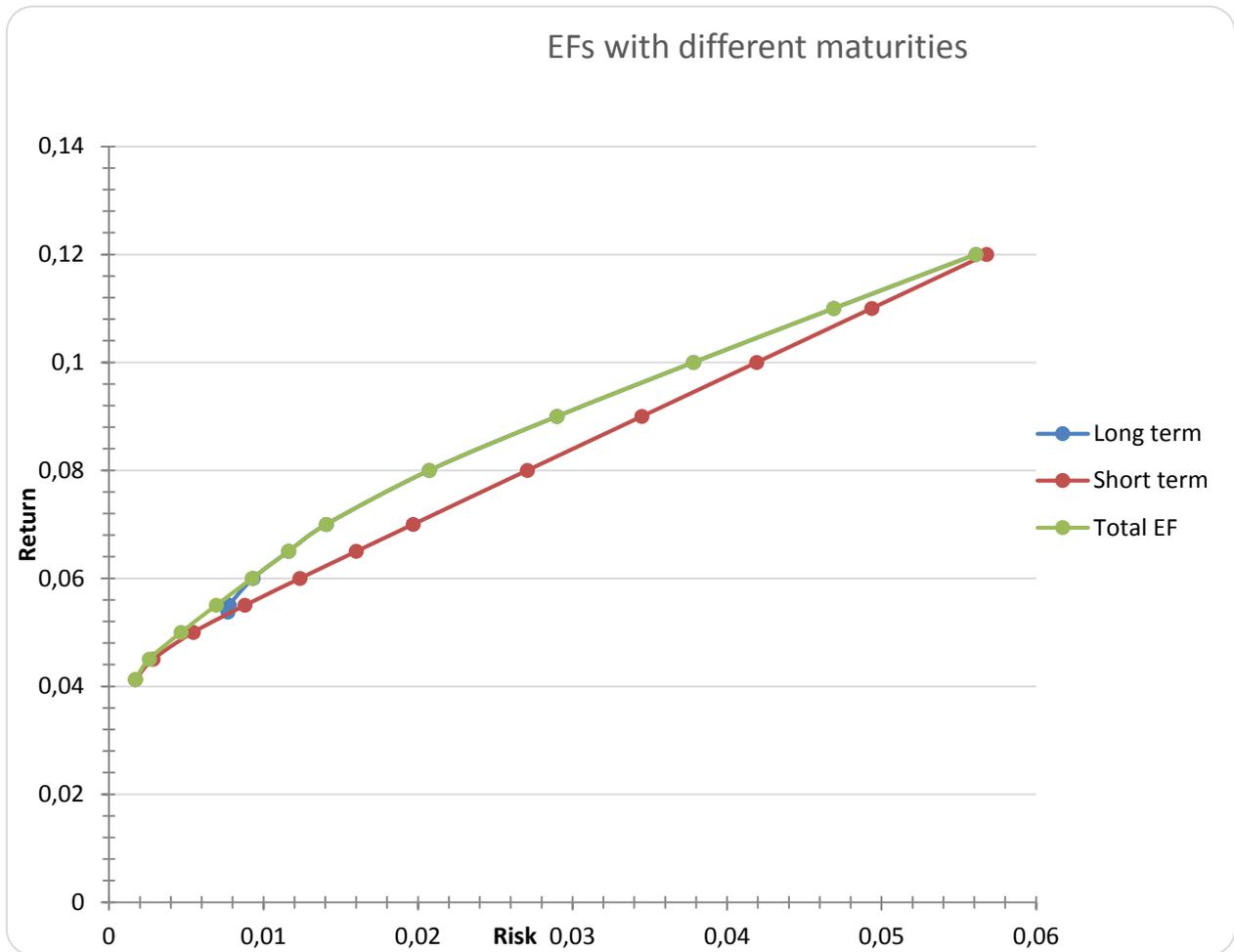


Figure 37 Illustration of Efficient frontiers using a combination of All-share and short and long-term government bond maturities

6.3.1 Discussions about and highest Sharpe-ratio for the different Efficient Frontiers

As mentioned in the introduction to analysis of efficient frontier, the lower the covariance and standard deviation the higher the potential for diversification. This is illustrated in the findings, as portfolios often only consist of All-share index, 3-month and 5-year bonds. In some cases the portfolio weights shift in favor of 3-year bonds rather than 5-year bonds, this is caused by a combination of 3-year bonds and the All-share giving a lower combined standard deviation, while losing only marginally return and covariance. In this instance, what you gain in lower standard deviation amongst the 3-year bond and the All-Share gives a greater benefit than the loss in covariance. The combination of short-term bonds and long-term bonds also corresponds with the term-structure theory that implies a lower correlation among bonds with longer spread in maturity.

As the proportion in equities increases so does the overall risk. As Ilmanen (2011) claims, the popular 0.6 stocks and 0.4 bonds portfolio is insufficiently diversified because 95 percent of the portfolio risk will come from the risk-proportion. For portfolio allocation of 0.8/0.2 stock/bond the stock component will account for an even higher percent of portfolio volatility. This coincides with our findings for portfolios of 0.6/0.4 and 0.8/0.2 stock/bond, as trading off long-term maturity for short-term maturity will have a small effect on the overall portfolio risk.

6.3.2 Optimal risky portfolio throughout the periods

The optimal risky portfolio is the portfolio combination with the highest Sharpe-ratio (Business Missouri). The optimal risky portfolio for the different portfolio weights gives different results depending on the measured periods.

6.3.2.1 0.2/0.8 stock/bond

The total period, 1994-2013, the optimal risky portfolio consisted of 0.2 in stocks, 0.4 in 3-month, 0.06 in 3-year and 0.34 5-year bonds.

In period 1, 1994-2003, we get the result we anticipated: if you have the majority of the portfolio on bonds with only a small proportion in stocks, you should invest the majority in short-term bonds. The optimal risky portfolio consisted of 0.20 stocks, 0.76 in 3-month bonds and 0.04 in 5-year bonds.

For the last period, 2004-2014, the optimal risky portfolio consists of only long-term bonds. We suspect this is mainly caused by the low correlation between long-term bonds and the All-share index during the financial crisis, and the high return in for long-term bonds relatively to short-term bonds. Research done by Gulko (2002) and Connolly et.al. (2005) suggest that during steep stock market declines, with increased equity volatility the correlation between stocks and bonds fall. Considering this, the correlation results for the last period might have been an anomaly, so we would not base a future investment decision on this allocation.

6.3.2.2 0.6/0.4 and 0.8/0.2

For portfolios with the majority of the asset allocation in stocks, 0.6/0.4 and 0.8/0.2 stock/bond, the optimal risky portfolio consisted of the obligatory stock, and only the 5-year bonds. This was the case for all periods, 1994-2013, 1994-2003, and 2004-2013. This coincides with our original hypothesis from, that the more you invest in stocks, the higher maturity you should invest in bonds.

So in short, our findings coincide with our hypothesis that the more of the portfolio proportion that is invested in stocks, the higher the bonds maturity should be to achieve the optimal risky portfolio. Predicting the future is impossible, but do not mind trying: based on our findings, which implied the same results during very different periods, our hypothesis might hold also in the future. This would suggest that for portfolios with the major weight portion in risky assets the remaining weights should consist of long-term bonds.

Even though the second period, 2004-2013, was afflicted with extraordinary events all periods gave the same results, so we believe this is something we can base future investments on.

7 Predicting for the future

Will the coming generations be influenced by the post-financial crises? The future is uncertain, and it is impossible to predict what might happen. Based on historical estimates people have taken a flight-to-safe approach after longer periods of economic instability, with higher investment in low-risk assets, such as government bonds (Gulko 2002). Norway's economy has been less affected by the financial crises than most of the country's business partners, such as the United States and the European Union. The volatile economy has influenced the US and the EU interest rates, which is currently historical low. The European Central Bank (ECB) recently published a new report to decrease the refinancing rate by ten basis points to 0.15 percent, and according to ECB President Draghi (2014) the rate will remain at this level for a sustained period of time, to keep inflation rates under control. Even though economic activity in Norway has been less affected directly by the financial crises, the country depends on their international business partners and other big financial markets, due to increased globalization. The key interest rate in Norway is also currently historical low. The predictions from several Head of economists, including the Head of Sparebank1 Holvik (2014), imply that the interest rate will remain low, or might even decline further. Holvik predict that there might not be needed increases in the rate level for the next three to five years.

Based on historical estimates there is a close relation with the level of interest rates and the government bonds, while an inverse relation with the stock indices. Our prediction of the future is, based on our results and statements from Draghi and Holvik that the interest will remain low for the years to come and good outcomes for the stock indices during this period. However, the interest rates will eventually increase. What will then happened? We predict that the increased interest rates will have a negative effect on the stock indices, like the

historical estimates, which might lead to a market decline. During previous increases in the interest rates, the stock indices have become less desirable caused by higher loans and lower expectations by investors. Investors are currently placing the majority of their capital in the stock market as low-risk assets can barely beat the inflation. In the future, it will be advisable for the general investor to shift some of their portfolio weight to more safe, low-risk assets, in order to hedge their investment from an inevitable interest rate increase. The vital thing is to learn from past mistakes.

“What we learn from history that we do not learn from history.” Georg Wilhelm Friedrich Hegel, German

philosopher, 1770-1831

8 Conclusion

We have used the historical data for Norway to illustrate the outcomes for portfolio combinations involving combining equities and bonds of various maturities. The data implies different outcomes for optimum, but there are some similarities. Our results suggest that a majority of the portfolio should consist of the obligatory All-share index, and/or a mix between the short-term 3-month bonds, and long-term 3-year and 5-year government bonds, respectively. This is regardless of the weight distribution.

The data for the two periods differ greatly, so there are some variations in the combination of stocks and bonds with various maturity portfolios. Hence, we fail to give a clear conclusion to what maturity investors should take advantage off, to reduce the unsystematic risk. However, keep in mind that these analyses are sensitive to how the correlation between the stock market and government bonds might be in the long run.

The portfolios where the global minimum variance and the highest return was at the same point, the combination consisted of the stock index and mainly long-term 5-year government bonds. This was the case for 0.80/0.20 stock/bond allocation for both the entire period 1994 – 2013 and in the second period 2004-2013, and for the second period the combination of 0.60/0.40 in stock/bond suggested the same results. Indicating that portfolios with higher weight in the stock index, the remaining investment should be in long-term bonds to get you better off. The combination will diversify your portfolio as you receive higher return while lowering the risk. This merges with the hypothesis we had in the introduction that portfolios

consisting of a high percentage in stocks (risky-assets) would benefit from bonds with long duration, as the interest rate risk is reduced. While for portfolios with a weight of 0.2 in risky assets, the results are not as obvious; here the individual investor's risk tolerance would decide the remaining weights on the non-risk assets (government bonds).

When studying the optimized Sharpe-ratio portfolio, we found that the case was clear for portfolios with at least 0.6/0.4 stock/bond allocations: higher maturity for bonds gives the highest Sharpe-ratio. These findings coincide with our original hypothesis that as the stock portion increases the bonds maturity should also increase. For 0.2/0.8 stock/bond allocations, it was more uncertain as the correlation varies during the periods and in turn, the optimal allocation is sensitive to changes in correlation among the stock and bond assets. The portfolio choice will eventually depend on the investor's life situation and risk tolerance.

Readers have to be aware of the limitations in the study. The research is based on historical data from the Norwegian financial market, on the monthly return on the Oslo Børs' All-share stock index and government bonds of various maturities. The research does not take height for corporate bonds, nor for different stock indices, which might influence our results. The Norwegian economy and financial market correlates significantly by the fluctuation in the energy sector, e.g. changes in the oil prices, our findings might therefore be limited to financial markets in similar situation.

For further research we would recommend to collect similar data from other financial markets that contain high liquid corporate bonds. Since corporate bonds have higher yields, we would assume that these could have an effect on the efficient portfolios. In addition, it could be interesting to use several stock indices, as the indices could interact differently with the government bonds and give different results.

9 References

9.1 Articles / Books

Berk, J., DeMarzo, P. (2011) *Corporate Finance (2nd ed.)*. Pearson Education Limited

Bodie, Z., Kane, A. & Marcus, AJ. (2011). *Investment and Portfolio Management (9th ed.)*. McGraw-Hill Irwin

Connoly, R., Sun, L., Stivers, C., (2003). *Stock Implied Volatility, Stock Turnover and the Stock-Bond Return Relation*. Working Paper FED Atlanta

Fabozzi, F., Fong, G. (1994). *Advanced Fixed Income Portfolio Management: The State of the Art*. McGraw Hill

Svetlozar, T., Stoyan, S., Fabozzi, F., (2008). *Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The ideal Risk, Uncertainty and Performance Measures*. John Wiley & Sons, Inc

Gulko, L. (2002). *Decoupling*. The Journal of Portfolio Management, 28(3), 59-66

9.2 Online articles / websites

Ang, A. (2009). *Fixed Income*. Retrieved June 2014.

From: <http://www0.gsb.columbia.edu/faculty/aang/book/Fixed%20Income%2007-26-2012.pdf>

Armstrong, F. (1999). *Long Term vs. Short Term Bonds*. Retrieved June 2014.

From: <http://investorsolutions.com/knowledge-center/investing-for-keeps-article-series/long-term-vs-short-term-bonds/>

Business Missouri. *Risky Portfolio*. Retrieved June 2014.

From: <http://business.missouri.edu/yanx/fin333/lectures/Riskyportfolio%20short.pdf>

Børsens Historie. Retrieved June 2014.

From: <http://www.oslobors.no/Oslo-Boers/Om-oss/Boersens-historie>

Duration. Retrieved June 2014.

From: <http://www.investinganswers.com/financial-dictionary/bonds/duration-1288>

En strategi for sysselsetting og verdiskapning. Retrieved June 2014.

From: <http://www.regjeringen.no/en/dep/fin/dok/nouer/2000/nou-2000-21/4.html?id=360188>

Euro Area Interest Rate: ECB Cuts rate to record low. Retrieved June 2014.

From <http://www.tradingeconomics.com/euro-area/interest-rate>

Fed Ties interest rates to 6.5% unemployment. Retrieved June 2014.

From <http://www.usatoday.com/story/money/business/2012/12/12/fed-buys-treasuries/1762459/>

Gabrielsen, H., Holtet, M. (2009). *Oljeprisens påvirkning på Oslo Børs*. Retrieved June 2014.
From:

<http://brage.bibsys.no/xmlui/bitstream/handle/11250/168393/Gabrielsen%20og%20Holtet%202009.pdf?sequence=1>

Holvik, E. (2014) *Må avlyse all snakk om renteøkning de neste årene*. Retrieved June 2014.
From: <http://www.dn.no/nyheter/okonomi/2014/06/15/-m-avlyse-alt-snakk-om-rentekning-de-neste-rene>.

Johnson, N., Vasant, N., Page, S., Pedersen, N., Sapra, S. (2013). *The Stock-Bond Correlation*. Retrieved June 2014.,

From:
http://media.pimco.com/Documents/PIMCO_Quantitative_Research_Stock_Bond_Correlation_Oct2013.pdf

Oljefondet. Retrived June 2014.

From: <http://www.nbim.no/>

Pengepolitisk Rapport 1.14. Retrieved June 2014.

From: http://www.norges-bank.no/Upload/Publikasjoner/PPR/PPR_1_14/PPR_1_14.pdf

Ødegaard, B.A. (2014) *Empirics of the Oslo Stock Exchange. Basic, descriptive results 1980-2013*. Retrieved June 2014.

From: http://finance.bi.no/~bernt/wps/empirics_ose_basics/index.html

Ødegaard, B.A. (2014) *Empirics of the Oslo Stock Exchange: Asset pricing results. 1980-2012*. Retrieved June 2014.

From: http://finance.bi.no/~bernt/wps/empirics_ose_asset_pricing/index.html

Qualitative versus quantitative research. Retrieved June 2014.

From: <http://www.wilderdom.com/research/QualitativeVersusQuantitativeResearch.html>

10 Appendix

10.1 Historical Returns of All-share index and Government bonds with various maturities

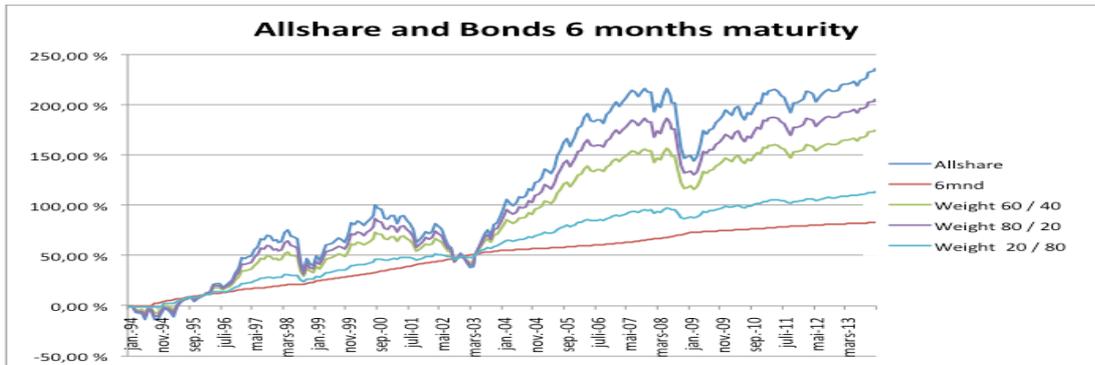


Figure 38 Historical Return on the Norwegian All-share stock index and 6-month government bonds with various asset allocations, period 1994-2013

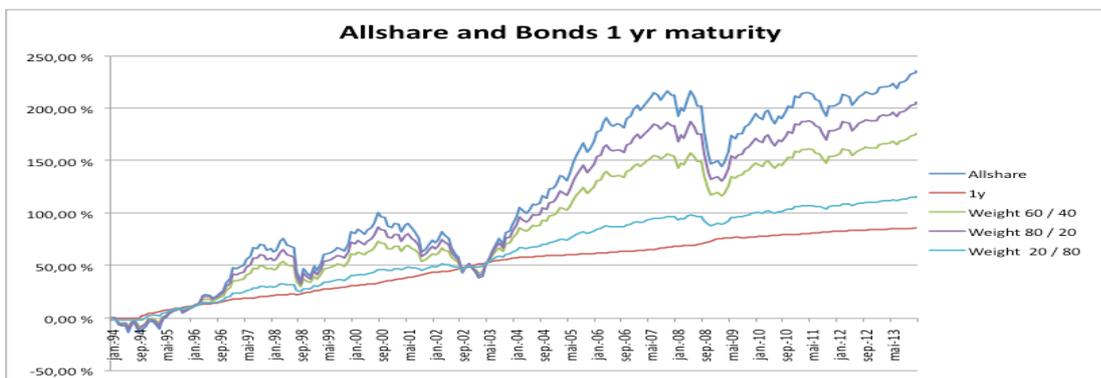


Figure 39 Historical Return on the Norwegian All-share stock index and 1-year government bonds with various asset allocations, period 1994-2013.

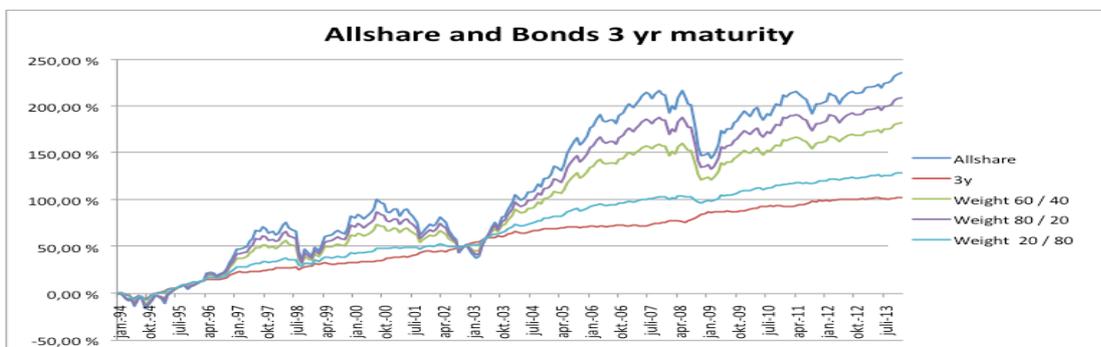


Figure 40 Historical Return on the Norwegian All-share stock index and 3-year government bonds with various asset allocations, period 1994-2013

10.2 Arithmetic annual return

Arithmetic annual return			
Index	1994-2013	1994-2003	2004-2013
Allshare	12,39 %	10,12 %	14,65 %
3-month	4,09 %	5,52 %	2,66 %
6-month	4,19 %	5,57 %	2,80 %
1-year	4,35 %	5,79 %	2,91 %
3-year	5,17 %	6,48 %	3,87 %
5-year	5,82 %	6,96 %	4,68 %

Figure 41 Arithmetic annual return of the All-share index and the government bonds from the monthly historical return

10.3 Covariance table the All-share stock index and Norwegian government treasuries

Covariance marix 1994 - 2013						
	Allshare	3mnd	6mnd	1y	3y	5y
Allshare	0,003561					
3mnd	-1,3E-05	2,96E-06				
6mnd	-7,7E-06	3,34E-06	4,26E-06			
1y	-9,3E-06	3,96E-06	5,68E-06	9,18E-06		
3y	-4,1E-05	5,28E-06	9,2E-06	1,96E-05	6,2E-05	
5y	-7E-05	5,96E-06	1,1E-05	2,56E-05	9,25E-05	0,000154

Figure 42 Covariance matrix of All-share and government bonds calculated from the historical return during period 1994-2013. There is an inverse covariance between the equities and bonds during this period, implying that the variables move in the opposite direction

Covariance marix 1994 - 2003						
	Allshare	3mnd	6mnd	1y	3y	5y
Allshare	0,003296					
3mnd	4,9E-06	1,64E-06				
6mnd	2,14E-05	2,1E-06	3,17E-06			
1y	4,42E-05	2,84E-06	4,95E-06	9,25E-06		
3y	0,000103	5,02E-06	1,04E-05	2,31E-05	7,34E-05	
5y	0,000107	6,29E-06	1,33E-05	3,16E-05	0,000111	0,000184

Figure 43 Covariance matrix of All-share and government bonds calculated from the historical return during period 1994-2003. There is a positive covariance between the equities and bonds during this period, implying that the variables move

in the same direction

Covariance matrix 2004 - 2013						
	Allshare	3mnd	6mnd	1y	3y	5y
Allshare	0,003819					
3mnd	-2,6E-05	1,45E-06				
6mnd	-3,2E-05	1,84E-06	2,69E-06			
1y	-5,8E-05	2,22E-06	3,63E-06	6,23E-06		
3y	-0,00018	2,95E-06	5,54E-06	1,35E-05	4,82E-05	
5y	-0,00024	3,37E-06	6,53E-06	1,73E-05	7,21E-05	0,000122

Figure 44 Covariance matrix of All-share and government bonds calculated from the historical return during period 2004-2013. There is an inverse covariance between the equities and bonds during this period, implying that the variables move in the opposite direction

10.3.1 Calculating the covariance matrix

Covariance between two returns:

$$cov(X_i, X_j) = E(X_i - \mu_i)(X_j - \mu_j)$$

$$\sigma_{ii} = E(X_i - \mu_i)^2$$

Where W is the portfolio weight. All weights are nonnegative, so the sum to one.

$$w_1 + w_2 + \dots + w_n = w'e = 1$$

Return of the portfolio r_p , is expressed by the weight and mean of the assets

$$r_p = w_1X_1 + w_2X_2 + \dots + w_nX_n = \sum_{i=1}^n w_i X_i = w'X$$

The vector of weights and the expected asset returns expresses expected portfolio return

$$Er_p = w_1\mu_1 + w_2\mu_2 + \dots + w_n\mu_n = \sum_{i=1}^n w_i\mu_i = w'\mu$$

The variance of portfolio returns is expressed by the mean of the weights and the covariance between them.

$$\sigma_{rp}^2 = E(r_p - Er_p)^2 = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}$$

The covariance of all asset returns can be arranged in a matrix and expressed as

$$\sigma_{rp}^2 = w'\Sigma w$$

Σ is a $n \times n$ matrix of the covariance

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \sigma_{2n} \\ \sigma_{n1} & \sigma_{n2} & \sigma_{nn} \end{pmatrix}$$

Optimization behind the first formulation mean-variance analysis:

$$\min w' \Sigma w$$

$$s. t. w' e = 1$$

$$w' \mu \geq R_*$$

$$w \geq 0$$

R is the lower bound on expected performance.

Optimization problem behind the second formulation mean-variance analysis

$$\text{Max } w' \mu$$

$$s. t. w' e = 1$$

$$w' \Sigma \leq R^*$$

$$w \geq 0$$

R is the upper bound on portfolio return.

(Svetlozar et al. 2008)

10.4 Standard deviation

$$\text{Cov}(r_D r_E) = \rho_{DE} \sigma_D \sigma_E$$

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2w_D w_E \sigma_D \sigma_E \rho_{DE}$$

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2}$$

$$\mu = \frac{1}{N} \sum_{i=1}^N x_i$$

Standard Deviation 1993 - 2013						
	Allshare	3-month	6-month	1-year	3-year	5-year
Weight 1	0,0598	0,0017	0,0021	0,003	0,0079	0,0124
60 / 40		0,0358	0,0358	0,0358	0,0357	0,0358
80 / 20		0,0478	0,0448	0,0478	0,0477	0,0477
20 / 80		0,0119	0,012	0,0121	0,013	0,0148
Covariance		0,	0,	0,	0,	-0,0001

Figure 45 Standard deviation for various asset allocations

Standard Deviation 1993 - 2003						
	Allshare	3-months	6-month	1-year	3-year	5-year
Weight 1	0,0577	0,0013	0,0018	0,0031	0,0086	0,0136
60 / 40		0,0346	0,0347	0,0349	0,0355	0,0357
80 / 20		0,0461	0,0462	0,0463	0,0465	0,0466
20 / 80		0,0116	0,0119	0,0124	0,0146	0,0169
Covariance		0,	0,	0,	0,0001	0,0001

Figure 46 Standard deviation for various asset allocation 1994-2003

Standard Deviation 2004 - 2013						
	Allshare	3-month	6-month	1-year	3-year	5-year
Weight 1	0,06206	0,0012	0,0016	0,0025	0,007	0,0111
60 / 40		0,0371	0,037	0,0369	0,0362	0,0359
80 / 20		0,0495	0,0495	0,0495	0,0491	0,0489
20 / 80		0,0121	0,0121	0,0118	0,0113	0,0124
Covariance		0,	0,	-0,0001	-0,0002	-0,0002

Figure 47 Standard deviation for various asset allocation 2004-2013