Published for SISSA by Springer

Received: *November 17, 2015*
Revised: *January 5, 2016*
Accepted: *January 15, 2016*
Published: *February 1, 2016*

# Submaximal conformal symmetry superalgebras for Lorentzian manifolds of low dimension

**Paul de Medeiros**[1]

*E-mail:* p.f.demedeiros@gmail.com

Abstract: We consider a class of smooth oriented Lorentzian manifolds in dimensions three and four which admit a nowhere vanishing conformal Killing vector and a closed two-form that is invariant under the Lie algebra of conformal Killing vectors. The invariant two-form is constrained in a particular way by the conformal geometry of the manifold. In three dimensions, the conformal Killing vector must be everywhere causal (or null if the invariant two-form vanishes identically). In four dimensions, the conformal Killing vector must be everywhere null and the invariant two-form vanishes identically if the geometry is everywhere of Petrov type N or O. To the conformal class of any such geometry, it is possible to assign a particular Lie superalgebra structure, called a conformal symmetry superalgebra. The even part of this superalgebra contains conformal Killing vectors and constant R-symmetries while the odd part contains (charged) twistor spinors. The largest possible dimension of a conformal symmetry superalgebra is realised only for geometries that are locally conformally flat. We determine precisely which non-trivial conformal classes of metrics admit a conformal symmetry superalgebra with the next largest possible dimension, and compute all the associated submaximal conformal symmetry superalgebras. In four dimensions, we also compute symmetry superalgebras for a class of Ricci-flat Lorentzian geometries not of Petrov type N or O which admit a null Killing vector.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, Differential and Algebraic Geometry, Space-Time Symmetries

ArXiv ePrint: 1406.5166

[1]Department of Mathematics and Natural Sciences, University of Stavanger, N-4036, Stavanger, Norway, from April 2016.



doi:10.1007/JHEP02(2016)008

JHEP02(2016)008

## Contents



JHEP02(2016)008

# 1 Introduction

The characterisation of non-trivial background geometries which support some amount of rigid (conformal) supersymmetry has attracted much attention in the recent literature [1–38]. The primary motivation being that it is often possible to obtain important exact results for quantum field theories defined on such backgrounds, with many novel holographic applications [4, 6, 12, 15, 18, 28, 33, 34, 36, 38]. Perhaps the most systematic strategy for generating admissible backgrounds is by taking a rigid limit of some local supergravity coupling, such that the dynamics of the gravity supermultiplet is effectively frozen out [1]. The resulting bosonic supergravity background supports rigid supersymmetry, with the supersymmetry parameter constrained by setting to zero the supersymmetry variation of the fermions in the gravity supermultiplet. For bosonic supersymmetric backgrounds of conformal supergravity [39–42], the supersymmetry parameter typically obeys a particular conformally invariant first order PDE, known as a 'twistor spinor equation', with respect to a certain superconnection whose precise form is dictated by the structure of the conformal gravity supermultiplet.

The Lie superalgebra which encodes the rigid (conformal) supersymmetry of a bosonic (conformal) supergravity background is known as the *(conformal) symmetry superalgebra* of the background ([16, 17]) [43–47]. The even part of this superalgebra contains (conformal) Killing vectors, which generate (conformal) isometries of the background, together with R-symmetries of the associated rigid supermultiplet. The odd part contains (twistor) spinors valued in certain R-symmetry representations which generate rigid (conformal) supersymmetries of the background. The virtue of the (conformal) symmetry superalgebra construction is that it often reveals special geometrical properties of the background based on the type and amount of rigid (conformal) supersymmetry it supports. For example, in dimensions eleven, ten and six, this approach was used recently in [48, 49] to prove that any bosonic supersymmetric supergravity background possessing more than half the maximal amount of supersymmetry is necessarily (locally) homogeneous.

The simplest class of conformal symmetry superalgebras contain odd elements which obey a 'geometric' twistor spinor equation, with respect to the Levi-Civita connection. Their generic structure was described in some detail in [16], where it was found that the inclusion of a non-trivial R-symmetry is crucial in solving the odd-odd-odd component of the Jacobi identity for the superalgebra. Indeed, this extra ingredient is what distinguishes the construction in [16] from several earlier ones [50–53]. More general conformal symmetry superalgebras are further complicated by the presence of some assortment of non-trivial background fields (other than the metric). The details of these background fields depend on the composition of the conformal gravity supermultiplet but one common feature is the presence of R-symmetry gauge fields. In section 5 of this paper, we shall explore a natural generalisation of the construction in [16] based on the gauging of R-symmetry. For Lorentzian geometries, we find that the resulting structure generically defines a Lie superalgebra only if the R-symmetry is one-dimensional and the background has dimension three or four. Indeed, these are precisely the cases where the bosonic sector of a conformal gravity supermultiplet contains only the metric and the R-symmetry gauge field.

It is a well-known and useful fact that geometric twistor spinors 'square' (in a sense which can be made precise) to conformal Killing vectors. More generally, for a conformal symmetry superalgebra, there is a similar (albeit somewhat more complicated) squaring map defined by the odd-odd bracket [16, 17]. Of course, if a pseudo-Riemannian spin manifold admits a conformal Killing vector, it need not admit a geometric twistor spinor.[1] However, as was shown in [6, 8, 15], at least for a certain class of Lorentzian geometries

[1] In Euclidean and Lorentzian signatures, up to local conformal equivalence, the classification of those geometries which do admit a nowhere vanishing geometric twistor spinor was established in [54–58].

which need not admit a geometric twistor spinor, the existence of a nowhere vanishing conformal Killing vector with a particular causal character is in fact locally equivalent to the existence of a nowhere vanishing twistor spinor that is defined with respect to a particular connection with non-trivial intrinsic torsion. The precise form of this intrinsic torsion is dictated by the local isotropy of the twistor spinor. Moreover, in dimensions three and four, with one-dimensional R-symmetry, this data recovers precisely the defining conditions for a bosonic supersymmetric conformal supergravity background.

If it is possible to define a quantum field theory on a background preserving a large amount of (conformal) supersymmetry, it is often the case that the theory is particularly well-behaved. Backgrounds which admit a conformal symmetry superalgebra with the largest possible dimension are necessarily locally conformally flat. In Lorentzian signature, any such conformal symmetry superalgebra has compact R-symmetry and is isomorphic to one of the well-known conformal superalgebras classified by Nahm in [59]. However, the general structure of conformal symmetry superalgebras with the next largest possible, or *submaximal*, dimension (for backgrounds that are not locally conformally flat) is much less clear. Our goal here will be to elucidate this structure for Lorentzian geometries in three and four dimensions which admit a conformal symmetry superalgebra with gauged one-dimensional R-symmetry. Our strategy will make use of some recent progress [60–62] which has determined the submaximal dimension of the Lie algebra of conformal Killing vectors for any Lorentzian manifold. We will also utilise some earlier results [63–65] on the classification of (conformal) Killing vectors for Lorentzian manifolds of low dimension. We then employ the results of [6, 15] to deconstruct a null (in four dimensions) or timelike (in three dimensions) conformal Killing vector in terms of the charged twistor spinors which form the odd part of the conformal symmetry superalgebra.

The organisation of this paper is as follows. We begin in section 2 by reviewing some essential features of the conformal geometry of Lorentzian manifolds. This will include the use of proper conformal scalars and gradients to represent the action of the conformal group in terms of isometries and homotheties for some representative geometry in a given conformal class. We will also provide a summary of the main results of [60–62] concerning sharp upper bounds on the submaximal dimension of the conformal algebra for Lorentzian geometries. In section 3, we review some basic properties of Clifford algebras, spinor modules and their invariant bilinear forms, together with the vital concepts of spinorial Lie derivative and twistor spinor. In the process, we take the opportunity to note some of our basis conventions to be used in the forthcoming analysis. In section 4, we briefly recap the definition of a real Lie superalgebra, focussing on the conceptualisation of certain axioms that will facilitate our description of conformal symmetry superalgebras. We will also define here a particular family of real Lie superalgebras that encompasses all the conformal symmetry superalgebras obtained in later sections. In section 5, we summarise the construction of conformal symmetry superalgebras in [16, 17] and propose a certain generalisation based on the gauging of R-symmetry. The conditions which are sufficient for the existence of such a Lie superalgebra are found to be satisfied identically only for abelian R-symmetry in dimensions three and four. We then focus on the classification of submaximal conformal symmetry superalgebras in these two cases.



JHEP02(2016)008

The classification in three dimensions is obtained in section 6. It begins in sections 6.1 and 6.2 with a synopsis of null triads and Majorana spinors in three dimensions. With respect to this framework, we then describe in detail the intimate connection [8, 15] between causal conformal Killing vectors (in section 6.3) and charged twistor spinors (in section 6.4) on Lorentzian three-manifolds. (Charged, that is, with respect to the gauged abelian R-symmetry.) At least locally, it follows that the existence of a nowhere vanishing charged (or uncharged) twistor spinor is characterised by the existence of a nowhere vanishing timelike (or null) conformal Killing vector. The classification of submaximal conformal symmetry superalgebras is contained in section 6.5, according to whether the geometry in question admits a nowhere vanishing conformal Killing vector that is either null or timelike. The null case is described in section 6.5.1 and the timelike case is described in section 6.5.2. Up to local conformal equivalence, we prove that there are precisely three types of Lorentzian three-manifold with a timelike conformal Killing vector which admit submaximal conformal symmetry superalgebras, where the dimension of the conformal algebra is four. Representative geometries are found to correspond to a certain class of locally stationary metrics with four Killing vectors (see section 8 for a summary).

The classification in four dimensions is obtained in section 7. It begins in sections 7.1, 7.2 and 7.3 with a synopsis of null tetrads, Majorana spinors and Petrov types in four dimensions. With respect to this framework, we then describe in detail the intimate connection [6] between null conformal Killing vectors (in section 7.4) and charged twistor spinors (in section 7.5) on Lorentzian four-manifolds. At least locally, it follows that the existence of a nowhere vanishing charged twistor spinor is characterised by the existence of a nowhere vanishing null conformal Killing vector (the twistor spinor is uncharged only if the geometry is of Petrov type N or O). The classification of submaximal conformal symmetry superalgebras is contained in section 7.6. From the results of [60, 75], it is immediately apparent that the submaximal conformal symmetry superalgebras here are associated with geometries of Petrov type N. Up to local conformal equivalence, we prove that there are precisely two types of Lorentzian four-manifold with a null conformal Killing vector which admit submaximal conformal symmetry superalgebras, where the dimension of the conformal algebra is seven. Representative geometries are found to correspond to a certain class of locally homogeneous plane wave metrics with six Killing vectors and a proper homothetic conformal Killing vector (see section 8 for a summary). Using the results of [65], in section 7.6.2, we proceed to compute the symmetry superalgebras for a class of 'physically admissible'[2] Ricci-flat Lorentzian four-manifolds with a null Killing vector that are of Petrov type II and D. We also compute the conformal symmetry superalgebra for the most symmetric geometry in this class, which is the unique representative of Petrov type D.

Section 8 contains a detailed summary of our main results.

## 2 Conformal Killing vectors

Let $M$ be a smooth oriented manifold equipped with a Lorentzian metric $g$ whose associated Levi-Civita connection will be denoted by $\nabla$. We take $M$ to have dimension $d > 2$.

[2] In the sense that their energy-momentum tensor does not violate the dominant energy condition.



JHEP02(2016)008

Let $\mathfrak{X}(M)$ denote the space of vector fields on $M$ (i.e. sections of the tangent bundle $TM$). Let $\|X\|^2 = g(X,X)$ denote the norm squared of any $X \in \mathfrak{X}(M)$ with respect to $g$. At a point in $M$, $X$ may be either *spacelike* (if $\|X\|^2 > 0$), *timelike* (if $\|X\|^2 < 0$) or *null* (if $\|X\|^2 = 0$). If $\|X\|^2 \leq 0$ then $X$ is said to be *causal*. At each point in $M$, clearly the sign of $\|X\|^2$ with respect to any positive multiple of $g$ is the same, so the aforementioned causal properties of a vector field depend only on the conformal class $[g]$ of $g$.

The Lie derivative $\mathcal{L}_X$ along any $X \in \mathfrak{X}(M)$ defines an endomorphism of the space of tensor fields on $M$. The Lie bracket of vector fields is defined by $[X,Y] = \mathcal{L}_X Y = \nabla_X Y - \nabla_Y X \in \mathfrak{X}(M)$, for all $X, Y \in \mathfrak{X}(M)$. This equips $\mathfrak{X}(M)$ with the structure of a Lie algebra. Furthermore

$$\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}\,, \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(M)$. Whence, the Lie derivative defines on the space of tensor fields a representation of the Lie algebra of vector fields.

The subspace of *conformal Killing vectors* in $\mathfrak{X}(M)$ is defined by

$$\mathfrak{C}(M,g) = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = -2\sigma_X g\}\,, \tag{2.2}$$

for some real function $\sigma_X$ on $M$. For any $X, Y \in \mathfrak{C}(M,g)$, using (2.1), it follows that $[X,Y] \in \mathfrak{C}(M,g)$ with

$$\sigma_{[X,Y]} = \nabla_X \sigma_Y - \nabla_Y \sigma_X\ . \tag{2.3}$$

Whence, restricting the Lie bracket to $\mathfrak{C}(M,g)$ defines a (finite-dimensional) Lie subalgebra of conformal Killing vectors on $(M,g)$.

Any $X \in \mathfrak{C}(M,g)$ with $\sigma_X$ constant is called *homothetic* and let $\mathfrak{H}(M,g)$ denote the subspace of homothetic conformal Killing vectors on $(M,g)$. Any $X \in \mathfrak{H}(M,g)$ with $\sigma_X \neq 0$ is said to be *proper*. Any $X \in \mathfrak{H}(M,g)$ with $\sigma_X = 0$ is called *isometric* and let $\mathfrak{K}(M,g)$ denote the subspace of isometric conformal Killing vectors (i.e. Killing vectors) on $(M,g)$. From (2.3), clearly $[\mathfrak{H}(M,g), \mathfrak{H}(M,g)] < \mathfrak{K}(M,g)$ so restricting to the subspace of Killing vectors on $(M,g)$ defines the ideal $\mathfrak{K}(M,g) \lhd \mathfrak{H}(M,g)$. Furthermore, given any $X, Y \in \mathfrak{H}(M,g)$ with $\sigma_X \neq 0$, then $Y - \frac{\sigma_Y}{\sigma_X} X \in \mathfrak{K}(M,g)$. Whence, either $\mathfrak{H}(M,g) = \mathfrak{K}(M,g)$ or $\dim(\mathfrak{H}(M,g)/\mathfrak{K}(M,g)) = 1$.

A real function $\phi$ on $M$ is called a *conformal scalar* if $\nabla_X \phi = p_\phi \sigma_X \phi$, for all $X \in \mathfrak{C}(M,g)$, in terms of some $p_\phi \in \mathbb{R}$ ($\phi$ is said to be *proper* if $p_\phi \neq 0$). A real one-form $\upsilon$ on $M$ is called a *conformal one-form* if $\mathcal{L}_X \upsilon = p_\upsilon \mathrm{d}\sigma_X$, for all $X \in \mathfrak{C}(M,g)$, in terms of some $p_\upsilon \in \mathbb{R}$ ($\upsilon$ is said to be *proper* if $p_\upsilon \neq 0$). If $\mathrm{d}\upsilon = 0$ then $\upsilon$ is called a *conformal gradient*. For example, if $\phi$ is a (proper) conformal scalar, then $\mathrm{d}(\ln \phi)$ is a (proper) conformal gradient. If $\upsilon$ is a proper conformal gradient then, at least locally, $\upsilon = p_\upsilon \mathrm{d}\varphi$ for some real function $\varphi$ such that, for each $X \in \mathfrak{C}(M,g)$, $\sigma_X - \nabla_X \varphi = s_X$ for some $s_X \in \mathbb{R}$.

Any metric $\tilde{g}$ in the same conformal class $[g]$ as $g$ is of the form $\tilde{g} = \mathrm{e}^{2\omega} g$, in terms of some real function $\omega$ on $M$. Each $X \in \mathfrak{C}(M,g)$ (with conformal factor $\sigma_X$) is also in $\mathfrak{C}(M,\tilde{g})$ but with conformal factor $\tilde{\sigma}_X = \sigma_X - \nabla_X \omega$. Thus, we may assign the Lie algebra $\mathfrak{C}(M,[g])$ of conformal Killing vectors on $(M,g)$ to the conformal class $[g]$. Of course, there may be a preferred metric in $[g]$ with respect to which the conformal Killing vectors in $\mathfrak{C}(M,[g])$ are most conveniently represented (e.g. via a homothetic or isometric action). For example, if

$(M,g)$ admits a proper conformal scalar $\phi$, then $\mathfrak{C}(M,[g]) = \mathfrak{K}(M, \mathrm{e}^{2\omega} g)$ for $\omega = \frac{1}{p_\phi} \ln \phi$. Alternatively, if $(M,g)$ admits a proper conformal gradient $\upsilon$, then $\mathfrak{C}(M,[g]) = \mathfrak{H}(M, \mathrm{e}^{2\omega} g)$ for $\omega = \varphi$ and $\tilde{\sigma}_X = s_X$. More generally, $\mathfrak{C}(M,[g])$ is said to be *conformally isometric* if it can be represented by $\mathfrak{K}(M, \mathrm{e}^{2\omega} g)$ or *conformally homothetic* if it can be represented by $\mathfrak{H}(M, \mathrm{e}^{2\omega} g)$ (with $\dim(\mathfrak{H}(M, \mathrm{e}^{2\omega} g)/\mathfrak{K}(M, \mathrm{e}^{2\omega} g)) = 1$), for some choice of $\omega$.

Important aspects of the conformal geometry of $(M,g)$ are characterised by its Weyl tensor $W$ and Cotton-York tensor $C$. If $d > 3$, then $W = 0$ only if $(M,g)$ is locally conformally flat. For any $X \in \mathfrak{C}(M,[g])$, $\mathcal{L}_X W = -2\sigma_X W$ which implies $\nabla_X \|W\|^2 = 4\sigma_X \|W\|^2$, where $\|W\|^2$ denotes the scalar norm-squared of $W$ with respect to $g$. Thus, if $d > 3$, any $(M,g)$ with $\|W\|^2$ nowhere vanishing is conformally isometric, with $\mathfrak{C}(M,[g]) = \mathfrak{K}(M, \|W\| g)$ (i.e. $\phi = \|W\|^2$ is a proper conformal scalar with $p_\phi = 4$). If $d = 3$, then $W$ vanishes identically and $C = 0$ only if $(M,g)$ is locally conformally flat. In this case, for any $X \in \mathfrak{C}(M,[g])$, $\mathcal{L}_X C = 0$ which implies $\nabla_X \|C\|^2 = 6\sigma_X \|C\|^2$, where $\|C\|^2$ denotes the scalar norm-squared of $C$ with respect to $g$. Thus, if $d = 3$, any $(M,g)$ with $\|C\|^2$ nowhere vanishing is conformally isometric, with $\mathfrak{C}(M,[g]) = \mathfrak{K}(M, \|C\|^{2/3} g)$ (i.e. $\phi = \|C\|^2$ is a proper conformal scalar with $p_\phi = 6$).

For any $X \in \mathfrak{C}(M,[g])$, $\nabla_X \|X\|^2 = -2\sigma_X \|X\|^2$ so if $\dim \mathfrak{C}(M,[g]) = 1$ then $\phi = \|X\|^{-2}$ defines a proper conformal scalar with $p_\phi = 2$ provided $X$ is nowhere null. In this case, $\mathfrak{C}(M,[g])$ is therefore always conformally isometric. Alternatively, if $(M,g)$ is conformally flat then $\mathfrak{C}(M,[g]) \cong \mathfrak{so}(d,2)$, which is neither conformally homothetic nor conformally isometric.

In fact, the dimension of $\mathfrak{C}(M,[g])$ can never exceed $\binom{d+2}{2}$ and equals it only if $(M,g)$ is locally conformally flat. An important problem in conformal geometry is to determine the next largest value of $\dim \mathfrak{C}(M,[g])$, or *submaximal dimension*, which can be realised for some $(M,g)$ that is not conformally flat. In Lorentzian signature, this problem was recently solved in [60–62]. Any $(M,g)$ that is not locally conformally flat must have $\dim \mathfrak{C}(M,[g]) \leq 4 + \binom{d-1}{2}$ for any $d > 3$ and $\dim \mathfrak{C}(M,[g]) \leq 4$ for $d = 3$. These upper bounds are sharp in that, for every $d > 2$, there are explicit examples for which they are saturated.

In $d = 4$, the conformal class of $(M,g)$ can be characterised locally as being of Petrov type I, II, D, III, N or O, depending on which components of the Weyl tensor vanish identically. Theorem 5.1.3 in [60] provides sharp upper bounds on $\dim \mathfrak{C}(M,[g])$ for each Petrov type. Type O means $W = 0$ so $(M,g)$ is locally conformally flat and $\dim \mathfrak{C}(M,[g]) = 15$. Type N must have $\dim \mathfrak{C}(M,[g]) \leq 7$, type D must have $\dim \mathfrak{C}(M,[g]) \leq 6$ while $\dim \mathfrak{C}(M,[g]) \leq 4$ for types I, II and III. It was shown in [64] that $\mathfrak{C}(M,[g])$ is conformally isometric only if $(M,g)$ admits a proper conformal scalar. Furthermore, from theorem 3 in [64], it follows that if $(M,g)$ does not admit a proper conformal scalar then it must be locally conformally equivalent to either Minkowski space (type O) or a plane wave (type N). Whence, any $(M,g)$ of type I, II, D or III must have $\mathfrak{C}(M,[g])$ conformally isometric.

## 3 Twistor spinors

Let us now assume that $M$ has vanishing second Stiefel-Whitney class so the bundle $\mathrm{SO}(M)$ of oriented pseudo-orthonormal frames lifts to $\mathrm{Spin}(M)$ by the assignment of a *spin structure*. For $d \leq 3$, this lift is always unobstructed.



JHEP02(2016)008

The *Clifford bundle* $\mathrm{C}\ell(TM)$ over $(M, g)$ is defined by the relation

$$\boldsymbol{XY} + \boldsymbol{XY} = 2g(X, Y)\mathbf{1}\,, \tag{3.1}$$

for all $X, Y \in \mathfrak{X}(M)$, where each multi-vector field $\Phi$ on $M$ is associated with a section $\boldsymbol{\Phi}$ of $\mathrm{C}\ell(TM)$. At each point $x \in M$, the exterior algebra of $T_xM \cong \mathbb{R}^{d-1,1}$ is isomorphic, as a vector space, to the Clifford algebra $\mathrm{C}\ell(T_xM)$ (the metric $g$ and its inverse provide a duality between multi-vector fields and differential forms on $M$). The canonical volume form $\varepsilon$ for the metric $g$ on $M$ defines a unique idempotent section $\mathbf{\Gamma}$ of $\mathrm{C}\ell(TM)$. If $d$ is odd, $\mathbf{\Gamma}$ is central in $\mathrm{C}\ell(TM)$. If $d$ is even, $\mathbf{\Gamma}\boldsymbol{X} = -\boldsymbol{X}\mathbf{\Gamma}$, for all $X \in \mathfrak{X}(M)$.

The Clifford algebra $\mathrm{C}\ell(T_xM)$ is $\mathbb{Z}_2$-graded such that elements with even and odd degrees are assigned grades 0 and 1 respectively. The grade 0 elements span an ungraded associative subalgebra $\mathrm{C}\ell^0(T_xM) < \mathrm{C}\ell(T_xM)$. The degree two elements span a Lie subalgebra $\mathfrak{so}(T_xM) < \mathrm{C}\ell^0(T_xM)$, where $\mathrm{C}\ell^0(T_xM)$ is understood as a Lie algebra whose brackets are defined by commutators.

At each point $x \in M$, the set of invertible elements in $\mathrm{C}\ell(T_xM)$ forms a multiplicative group $\mathrm{C}\ell^\times(T_xM)$. The vectors $\boldsymbol{X} \in \mathrm{C}\ell(T_xM)$ with $\|X\|^2 = \pm 1$ generate the subgroup $\mathrm{Pin}(T_xM) < \mathrm{C}\ell^\times(T_xM)$. The group $\mathrm{Spin}(T_xM) = \mathrm{Pin}(T_xM) \cap \mathrm{C}\ell^0(T_xM)$, which also follows by exponentiating $\mathfrak{so}(T_xM) < \mathrm{C}\ell^0(T_xM)$.

The *pinor* module is defined by the restriction to $\mathrm{Pin}(T_xM)$ of an irreducible representation of $\mathrm{C}\ell(T_xM)$. Every Clifford algebra is isomorphic, as an associative algebra with unit, to a matrix algebra and it is a simple matter to deduce their irreducible representations. The *spinor* module is defined by the restriction to $\mathrm{Spin}(T_xM)$ of an irreducible representation of $\mathrm{C}\ell^0(T_xM)$. Note that restricting to $\mathrm{Spin}(T_xM)$ an irreducible representation of $\mathrm{C}\ell(T_xM)$ need not define an irreducible spinor module. If $d$ is even, $\mathrm{C}\ell(T_xM)$ has a unique irreducible representation which descends to a reducible representation when restricted to $\mathrm{Spin}(T_xM)$, yielding a pair of inequivalent irreducible (*chiral*) spinor modules associated with the two eigenspaces of $\mathbf{\Gamma}$ on which $\mathbf{\Gamma} = \pm\mathbf{1}$. If $d$ is odd, $\mathrm{C}\ell(T_xM)$ has two inequivalent irreducible representations which are isomorphic to each other when restricted to $\mathrm{Spin}(T_xM)$. The isomorphism here is provided by the central element $\mathbf{\Gamma}$ and corresponds to Hodge duality in the exterior algebra. In either case, the spinor module defined at each point in $M$ defines a principle bundle $\mathrm{Spin}(M)$ and its associated vector bundle $\$(M)$ is called the *spinor bundle* over $M$.

Let $\mathfrak{S}(M)$ denote the space of spinor fields on $M$ (i.e. sections of $\$(M)$). If $d$ is even, $\mathfrak{S}(M) = \mathfrak{S}_+(M) \oplus \mathfrak{S}_-(M)$, where $\mathfrak{S}_\pm(M)$ denote the subspaces of chiral spinor fields (defined via projection operators $\mathbf{P}_\pm = \frac{1}{2}(\mathbf{1} \pm \mathbf{\Gamma})$) on which $\mathbf{\Gamma} = \pm\mathbf{1}$. The action of $\nabla$ induced on $\mathfrak{S}(M)$ is compatible with the Clifford action, i.e.

$$\nabla_X(\boldsymbol{Y}\psi) = (\nabla_X\boldsymbol{Y})\psi + \boldsymbol{Y}\nabla_X\psi\,, \tag{3.2}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\psi \in \mathfrak{S}(M)$. Furthermore,

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X)\psi = \nabla_{[X,Y]}\psi + \frac{1}{2}\boldsymbol{R}(X, Y)\psi\,, \tag{3.3}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\psi \in \mathfrak{S}(M)$, in terms of the Riemann tensor $R$ of $g$.

When required, we let $\{\partial_\mu \,|\, \mu = 0, 1, \ldots, d-1\}$ denote a local coordinate basis on $\mathfrak{X}(M)$. The volume form on $(M, g)$ is given by $\varepsilon = \pm\sqrt{|g|}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{d-1}$, in terms of the dual basis $\{\mathrm{d}x^\mu \,|\, \mu = 0, 1, \ldots, d-1\}$ of differential forms on $M$. With respect to this basis, the action of the Levi-Civita connection is defined by $\nabla_\mu \partial_\nu = \Gamma^\rho_{\mu\nu}\partial_\rho$ in terms of the Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \ . \tag{3.4}$$

Components of the Riemann tensor are given by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\sigma} \,, \tag{3.5}$$

and let $R_{\rho\sigma\mu\nu} = g_{\rho\alpha}R^\alpha{}_{\sigma\mu\nu}$. The Ricci tensor has components $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and the scalar curvature is $R = g^{\mu\nu}R_{\mu\nu}$.

Let $\{\mathbf{\Gamma}_{\mu_1 \ldots \mu_k} | k = 0, 1, \ldots, d\}$ denote a basis for sections of the Clifford bundle $\mathrm{C}\ell(TM)$, such that

$$\mathbf{\Gamma}_{\mu_1 \ldots \mu_k} = \mathbf{\Gamma}_{[\mu_1} \ldots \mathbf{\Gamma}_{\mu_k]} \equiv \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{|\sigma|} \mathbf{\Gamma}_{\mu_{\sigma(1)}} \ldots \mathbf{\Gamma}_{\mu_{\sigma(k)}} \,, \tag{3.6}$$

for degree $k > 0$ (i.e. unit weight skewsymmetrisation of $k$ distinct degree one basis elements) and the identity element $\mathbf{1}$ for $k = 0$. Let $\{e^\alpha_\mu\}$ denote the components of a pseudo-orthonormal frame on $(M, g)$. By definition, $g_{\mu\nu} = e^\alpha_\mu e^\beta_\nu \eta_{\alpha\beta}$, in terms of the canonical Minkowskian metric $\eta$ on $\mathbb{R}^{d-1,1}$. Components $\{\omega^{\alpha\beta}_\mu\}$ of the associated spin connection are defined by the 'no torsion' condition $\mathrm{d}e^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0$. In component form, the action of $\nabla$ on any $\psi \in \mathfrak{S}(M)$ is given by

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4}\omega^{\alpha\beta}_\mu \mathbf{\Gamma}_{\alpha\beta}\psi \ . \tag{3.7}$$

In terms of these basis conventions, the term $\boldsymbol{R}(X, Y) = \frac{1}{2} X^\mu Y^\nu R_{\mu\nu\rho\sigma}\mathbf{\Gamma}^{\rho\sigma}$ on the right hand side of (3.3).

There always exists on $\mathfrak{S}(M)$ a non-degenerate bilinear form $\langle -, - \rangle$ with the properties

$$\begin{aligned} \langle \psi, \varphi \rangle &= \sigma \langle \varphi, \psi \rangle \\ \langle \boldsymbol{X}\psi, \varphi \rangle &= \tau \langle \psi, \boldsymbol{X}\varphi \rangle \\ X\langle \psi, \varphi \rangle &= \langle \nabla_X \psi, \varphi \rangle + \langle \psi, \nabla_X \varphi \rangle \,, \end{aligned} \tag{3.8}$$

for all $\psi, \varphi \in \mathfrak{S}(M)$ and $X \in \mathfrak{X}(M)$, with respect to a pair of fixed signs $\sigma$ and $\tau$ (see [16, 67, 68] for more details). The possible choices for $\sigma$ and $\tau$ depend critically on both $d$ and the signature of $g$. The sign $\sigma = \pm 1$ indicates whether $\langle -, - \rangle$ is symmetric or skewsymmetric. The third line in (3.8) says that $\langle -, - \rangle$ is spin-invariant. For $d$ even, this implies $\langle \mathbf{\Gamma}\psi, \varphi \rangle = (-1)^{d/2}\langle \psi, \mathbf{\Gamma}\varphi \rangle$. Whence,

$$\begin{aligned} \langle \psi_\pm, \varphi_\mp \rangle &= 0 \quad \text{if } d = 0 \bmod 4 \\ \langle \psi_\pm, \varphi_\pm \rangle &= 0 \quad \text{if } d = 2 \bmod 4 \,, \end{aligned} \tag{3.9}$$

for all $\psi_\pm, \varphi_\pm \in \mathfrak{S}_\pm(M)$. To any pair $\psi, \varphi \in \mathfrak{S}(M)$, let us assign a vector field $\xi_{\psi,\varphi}$ defined such that

$$g(X, \xi_{\psi,\varphi}) = \langle \psi, \boldsymbol{X}\varphi \rangle \,, \tag{3.10}$$

for all $X \in \mathfrak{X}(M)$. From the first two properties in (3.8), it follows that $\xi_{\psi,\varphi} = \sigma\tau\xi_{\varphi,\psi}$, for all $\psi, \varphi \in \mathfrak{S}(M)$ and $X \in \mathfrak{X}(M)$.

Now let the dual $\overline{\psi}$ of any $\psi \in \mathfrak{S}(M)$ with respect to $\langle -, - \rangle$ be defined such that $\overline{\psi}\varphi = \langle \psi, \varphi \rangle$, for all $\varphi \in \mathfrak{S}(M)$. From any $\psi, \varphi \in \mathfrak{S}(M)$, one can define $\psi\overline{\varphi}$ as an endomorphism of $\mathfrak{S}(M)$. Whence, it can be expressed relative to the basis (3.6), with coefficients proportional to multi-vectors of the form $\overline{\varphi}\mathbf{\Gamma}^{\mu_1\cdots\mu_k}\psi$. Such expressions are known as *Fierz identities*, full details of which can be found in section 4 of [16].

The *spinorial Lie derivative* [69–72] along any $X \in \mathfrak{C}(M,[g])$ is defined by

$$\mathcal{L}_X = \nabla_X + \frac{1}{4}\mathbf{d}\boldsymbol{X}\,, \tag{3.11}$$

where $\mathbf{d}\boldsymbol{X} = (\nabla_\mu X_\nu)\mathbf{\Gamma}^{\mu\nu}$.[3] The spinorial Lie derivative (3.11) obeys

$$\begin{aligned}\mathcal{L}_X(\nabla_Y\psi) - \nabla_Y(\mathcal{L}_X\psi) &= \nabla_{[X,Y]}\psi + \frac{1}{4}\mathbf{d}\boldsymbol{\sigma_X} \wedge \boldsymbol{Y}\psi \\ \mathcal{L}_X(\boldsymbol{\nabla}\psi) - \boldsymbol{\nabla}(\mathcal{L}_X\psi) &= \sigma_X\boldsymbol{\nabla}\psi - \frac{1}{2}(d-1)(\boldsymbol{\nabla}\sigma_X)\psi\,,\end{aligned} \tag{3.12}$$

for all $X \in \mathfrak{C}(M,[g])$, $Y \in \mathfrak{X}(M)$ and $\psi \in \mathfrak{S}(M)$. Moreover, for all $X, Y \in \mathfrak{C}(M,[g])$ and any $w \in \mathbb{R}$, using (3.11) and (2.3), it follows that

$$(\mathcal{L}_X + w\sigma_X\mathbf{1})(\mathcal{L}_Y + w\sigma_Y\mathbf{1}) - (\mathcal{L}_Y + w\sigma_Y\mathbf{1})(\mathcal{L}_X + w\sigma_X\mathbf{1}) = \mathcal{L}_{[X,Y]} + w\sigma_{[X,Y]}\mathbf{1}\,. \tag{3.13}$$

Whence, the map $X \mapsto \mathcal{L}_X + w\sigma_X\mathbf{1}$ defines on $\mathfrak{S}(M)$ a representation of $\mathfrak{C}(M,[g])$.

With respect to a metric $\tilde{g} = \mathrm{e}^{2\omega}g$ in $[g]$, compatibility with the Clifford relation (3.1) requires that $\tilde{\boldsymbol{X}} = \mathrm{e}^{\omega}\boldsymbol{X}$, for all $X \in \mathfrak{X}(M)$. Moreover, given any $\psi \in \mathfrak{S}(M)$, $\tilde{\psi} = \mathrm{e}^{\omega/2}\psi$ defines the corresponding spinor field with respect to $\tilde{g}$. For $w = \frac{1}{2}$, the representation of $\mathfrak{C}(M,[g])$ on $\mathfrak{S}(M)$ defined by

$$\hat{\mathcal{L}}_X = \mathcal{L}_X + \frac{1}{2}\sigma_X\mathbf{1}\,, \tag{3.14}$$

for all $X \in \mathfrak{C}(M,[g])$, is known as the *Kosmann-Schwarzbach Lie derivative*. It is worthy of note because only for this particular value of $w$ does (3.14) define a conformally equivariant operator on $\mathfrak{S}(M)$, i.e. if $g \mapsto \mathrm{e}^{2\omega}g$ then

$$\hat{\mathcal{L}}_X \mapsto \mathrm{e}^{\omega/2}\hat{\mathcal{L}}_X\mathrm{e}^{-\omega/2}\,, \tag{3.15}$$

for all $X \in \mathfrak{C}(M,[g])$. The *Penrose operator*

$$\mathcal{P}_X = \nabla_X - \frac{1}{d}\boldsymbol{X}\boldsymbol{\nabla}\,, \tag{3.16}$$

acts on $\mathfrak{S}(M)$ along any $X \in \mathfrak{X}(M)$. It is also conformally equivariant on $\mathfrak{S}(M)$ and, using (3.12), obeys

$$\hat{\mathcal{L}}_X(\mathcal{P}_Y\psi) - \mathcal{P}_Y(\hat{\mathcal{L}}_X\psi) = \mathcal{P}_{[X,Y]}\psi\,, \tag{3.17}$$

for all $X \in \mathfrak{C}(M,[g])$, $Y \in \mathfrak{X}(M)$ and $\psi \in \mathfrak{S}(M)$.

[3] In a slight abuse of notation, we use the same symbol for a vector field and its dual one-form with respect to $g$.

The subspace of *conformal Killing (or twistor) spinors* in $\mathfrak{S}(M)$ is defined by

$$\mathfrak{Z}(M,[g]) = \left\{ \psi \in \mathfrak{S}(M) \;\middle|\; \nabla_X \psi = \frac{1}{d} \boldsymbol{X} \boldsymbol{\nabla} \psi \;,\; \forall\, X \in \mathfrak{X}(M) \right\} . \tag{3.18}$$

By construction, $\mathfrak{Z}(M,[g]) = \ker \mathcal{P}$ and conformal equivariance of the Penrose operator explains why the subspace (3.18) is assigned to the conformal class $[g]$ rather than to the particular metric $g$ on $M$. Furthermore, (3.17) shows that $\mathfrak{Z}(M,[g])$ is preserved by the action of the Kosmann-Schwarzbach Lie derivative (3.14). A key property of twistor spinors is that they 'square' to conformal Killing vectors, in the sense that the vector field defined by (3.10) is $\xi_{\psi,\varphi} \in \mathfrak{C}(M,[g])$, for any $\psi, \varphi \in \mathfrak{Z}(M,[g])$.

With respect to a particular metric in $[g]$, any $\psi \in \mathfrak{Z}(M,[g])$ with $\frac{1}{d} \boldsymbol{\nabla} \psi = \lambda \psi$, for some $\lambda \in \mathbb{C}$, is said to be *Killing* if $\lambda \neq 0$ or *parallel* if $\lambda = 0$. The non-zero constant $\lambda$ is called the *Killing constant* of a Killing spinor $\psi$.

Taking a derivative of the defining equation for any $\psi \in \mathfrak{Z}(M,[g])$ yields the following conditions

$$\nabla_X \boldsymbol{\nabla} \psi = \frac{d}{2} \boldsymbol{K}(X) \psi \quad , \qquad \boldsymbol{\nabla}^2 \psi = -\frac{d}{4(d-1)} R\, \psi \,, \tag{3.19}$$

and combining (3.19) with (3.3) implies the important integrability conditions

$$\boldsymbol{W}(X,Y) \psi = 0 \quad , \qquad \boldsymbol{C}(X,Y) \psi = \frac{1}{d} \boldsymbol{W}(X,Y) \boldsymbol{\nabla} \psi \,, \tag{3.20}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\boldsymbol{K}(X) = X^\mu K_{\mu\nu} \boldsymbol{\Gamma}^\nu$, $\boldsymbol{W}(X,Y) = \frac{1}{2} X^\mu Y^\nu W_{\mu\nu\rho\sigma} \boldsymbol{\Gamma}^{\rho\sigma}$ and $\boldsymbol{C}(X,Y) = X^\mu Y^\nu C_{\mu\nu\rho} \boldsymbol{\Gamma}^\rho$, in terms of the basis conventions described above. The Schouten tensor $K$ has components

$$K_{\mu\nu} = \frac{1}{d-2} \left( -R_{\mu\nu} + \frac{1}{2(d-1)} g_{\mu\nu} R \right) . \tag{3.21}$$

The Weyl tensor $W$ has components

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + g_{\mu\rho} K_{\nu\sigma} - g_{\nu\rho} K_{\mu\sigma} - g_{\mu\sigma} K_{\nu\rho} + g_{\nu\sigma} K_{\mu\rho} \,, \tag{3.22}$$

and we define $\| W \|^2 = W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$. The Cotton-York tensor $C$ has components

$$C_{\mu\nu\rho} = \nabla_\mu K_{\nu\rho} - \nabla_\nu K_{\mu\rho} \,, \tag{3.23}$$

and we define $\| C \|^2 = C_{\mu\nu\rho} C^{\mu\nu\rho}$.

There are a number of classification results concerning the existence of twistor spinors (with and without zeros) on $(M,g)$ in different dimensions and signatures. It can be shown that $\dim \mathfrak{Z}(M,[g]) \leq 2 \dim \mathfrak{S}(M)$ and, from (3.20), it follows that this bound is saturated only if $(M,g)$ is locally conformally flat. The classification in $d \geq 3$ of all local conformal equivalence classes of Lorentzian spin manifolds which admit a twistor spinor without zeros was established by Baum and Leitner [54–58]. Their results generalise the classification in $d = 4$ obtained earlier by Lewandowski in [73] which established that any Lorentzian spin manifold admitting a twistor spinor must be locally conformally equivalent to either a pp-wave, a Fefferman space or $\mathbb{R}^{3,1}$. In $d = 3$, any Lorentzian spin manifold admitting a twistor spinor must be locally conformally equivalent to either a pp-wave or $\mathbb{R}^{2,1}$. In $d > 4$, there are a few more distinct classes of Lorentzian manifolds which admit a non-vanishing twistor spinor (see [16] for more details). Any such geometry is locally conformally equivalent to either a Lorentzian Einstein-Sasaki manifold (if $d$ is odd) or the direct product of a Lorentzian Einstein-Sasaki manifold with a Riemannian manifold admitting Killing spinors.

## 4 Lie superalgebras

A real Lie superalgebra consists of a $\mathbb{Z}_2$-graded real vector space $\mathcal{S}$ (with even part $\mathcal{S}_{\bar{0}}$ and odd part $\mathcal{S}_{\bar{1}}$) that is equipped with the following additional structure.

A real bilinear map $[-,-] : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ which respects the $\mathbb{Z}_2$-grading such that

$$[\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{0}}] \subset \mathcal{S}_{\bar{0}} \ , \qquad [\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{1}}] \subset \mathcal{S}_{\bar{1}} \ , \qquad [\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] \subset \mathcal{S}_{\bar{0}} \ . \tag{4.1}$$

For all $u, v \in \mathcal{S}_{\bar{0}}$ and $\alpha, \beta \in \mathcal{S}_{\bar{1}}$,

$$[u, v] = -[v, u] \ , \qquad [u, \alpha] = -[\alpha, u] \ , \qquad [\alpha, \beta] = [\beta, \alpha] \ . \tag{4.2}$$

Since $[-,-]$ is symmetric bilinear on $\mathcal{S}_{\bar{1}}$, $[\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}]$ is defined by specifying $[\alpha, \alpha] \in \mathcal{S}_{\bar{0}}$ for all $\alpha \in \mathcal{S}_{\bar{1}}$ (i.e. via polarisation, any $[\alpha, \beta] = \frac{1}{2}([\alpha + \beta, \alpha + \beta] - [\alpha, \alpha] - [\beta, \beta])$).

Furthermore, $\mathcal{S}$ is subject to a Jacobi identity which constrains

$$\begin{aligned}
[[u, v], w] + [[v, w], u] + [[w, u], v] &= 0 \, , \\
[[u, v], \alpha] + [[v, \alpha], u] + [[\alpha, u], v] &= 0 \, , \\
[[u, \alpha], \beta] + [[\alpha, \beta], u] - [[\beta, u], \alpha] &= 0 \, , \\
[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] &= 0 \, ,
\end{aligned} \tag{4.3}$$

for all $u, v, w \in \mathcal{S}_{\bar{0}}$ and $\alpha, \beta, \gamma \in \mathcal{S}_{\bar{1}}$. The first three conditions in (4.3) have a simple conceptualisation. The first condition says that $\mathcal{S}_{\bar{0}}$ must be a real Lie algebra (i.e. it is the Jacobi identity for $\mathcal{S}_{\bar{0}}$). The second condition says that $\mathcal{S}_{\bar{1}}$ must be a real representation of $\mathcal{S}_{\bar{0}}$. The third condition says that the map $[-,-] : \mathcal{S}_{\bar{1}} \times \mathcal{S}_{\bar{1}} \to \mathcal{S}_{\bar{0}}$ must be $\mathcal{S}_{\bar{0}}$-equivariant. Notice that the fourth condition is symmetric trilinear on $\mathcal{S}_{\bar{1}}$ and therefore equivalent, via polarisation, to demanding $[[\alpha, \alpha], \alpha] = 0$, for all $\alpha \in \mathcal{S}_{\bar{1}}$.

If $[\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] = 0$ then clearly the third and fourth conditions in (4.3) are identically satisfied and a real Lie superalgebra $\mathcal{S}$ is defined by any real Lie algebra $\mathcal{S}_{\bar{0}}$ with real $\mathcal{S}_{\bar{0}}$-module $\mathcal{S}_{\bar{1}}$. We shall call $\mathcal{S}$ *proper* if $[\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] \neq 0$. If $[\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{1}}] = 0$ then the second and fourth conditions in (4.3) are identically satisfied. Consequently, $\mathcal{S}$ is then a proper real Lie superalgebra only if $[\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] \subset Z(\mathcal{S}_{\bar{0}})$ for a given real Lie algebra $\mathcal{S}_{\bar{0}}$ with non-trivial centre $Z(\mathcal{S}_{\bar{0}})$.

The proper real Lie superalgebras $\mathcal{S}$ we shall encounter in forthcoming sections all have $Z(\mathcal{S}_{\bar{0}})$ non-trivial, $\dim [\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] = 1$ and $\dim \mathcal{S}_{\bar{1}} \leq 2$. To facilitate their description, consider the following setup. Let $\mathfrak{h}$ be a real Lie algebra with ideal $\mathfrak{k} \lhd \mathfrak{h}$ such that $\dim(\mathfrak{h}/\mathfrak{k}) = 1$ and

$Z(\mathfrak{k})$ is non-trivial. Let $\mathfrak{r}$ be a one-dimensional real Lie algebra. Now define $\mathcal{S}_{\bar{0}} = \mathfrak{h} \oplus \mathfrak{r}$, as a real Lie algebra with $[\mathfrak{h}, \mathfrak{r}] = 0$, and let $\dim \mathcal{S}_{\bar{1}} = 2$. With respect to a choice of non-zero elements $z \in Z(\mathfrak{k})$, $h \in \mathfrak{h}/\mathfrak{k}$ and $r \in \mathfrak{r}$, and a choice of basis $\{\alpha, \beta\}$ on $\mathcal{S}_{\bar{1}}$, we define the $[\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{1}}]$ brackets

$$[k, \alpha] = 0 \; , \quad [k, \beta] = 0 \; , \quad [h, \alpha] = \frac{1}{2}\alpha \; , \quad [h, \beta] = \frac{1}{2}\beta \; , \quad [r, \alpha] = -\beta \; , \quad [r, \beta] = \alpha \; , \tag{4.4}$$

for all $k \in \mathfrak{k}$, and the $[\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}]$ brackets

$$[\alpha, \alpha] = [\beta, \beta] = z \; , \quad [\alpha, \beta] = 0 \; . \tag{4.5}$$

It follows that the Jacobi identity (4.3) is satisfied for this choice of brackets on $\mathcal{S}$ provided

$$[h, z] = z \; . \tag{4.6}$$

Taking (4.6) as part of the definition, the isomorphism class of the proper real Lie superalgebra $\mathcal{S}$ described above will be written $\mathcal{S}_2^{\triangleleft}(\mathfrak{h}|\mathfrak{r})$.

A more succinct form of (4.4) and (4.5) that will often be more convenient to use in the forthcoming discussion is given by

$$[k, \alpha_{\mathbb{C}}] = 0 \; , \quad [h, \alpha_{\mathbb{C}}] = \frac{1}{2}\alpha_{\mathbb{C}} \; , \quad [r, \alpha_{\mathbb{C}}] = i\alpha_{\mathbb{C}} \; , \quad [\alpha_{\mathbb{C}}, \alpha_{\mathbb{C}}] = 0 \; , \quad [\alpha_{\mathbb{C}}, \alpha_{\mathbb{C}}^*] = 2z \; , \tag{4.7}$$

for all $k \in \mathfrak{k}$, in terms of $\alpha_{\mathbb{C}} = \alpha + i\beta$ (and $\alpha_{\mathbb{C}}^* = \alpha - i\beta$). Of course, despite the appearance of the complex element $\alpha_{\mathbb{C}}$, this still encodes the same real Lie superalgebra $\mathcal{S}_2^{\triangleleft}(\mathfrak{h}|\mathfrak{r})$.

The Lie superalgebra $\mathcal{S}_2^{\triangleleft}(\mathfrak{h}|\mathfrak{r})$ contains several ideals and subalgebras that are worth naming, since they will also feature in the subsequent discussion. Let us define their isomorphism classes via the omission of certain combinations of the elements $h \in \mathfrak{h}/\mathfrak{k}$, $r \in \mathfrak{r}$ and $\beta \in \mathcal{S}_{\bar{1}}$ which figured in the construction above:

- $\mathcal{S}_2^{\circ}(\mathfrak{k}|\mathfrak{r}) \triangleleft \mathcal{S}_2^{\triangleleft}(\mathfrak{h}|\mathfrak{r})$ is defined by omitting $h \in \mathfrak{h}/\mathfrak{k}$.
- $\mathcal{S}_2^{\triangleleft}(\mathfrak{h}) \triangleleft \mathcal{S}_2^{\triangleleft}(\mathfrak{h}|\mathfrak{r})$ is defined by omitting $r \in \mathfrak{r}$.
- $\mathcal{S}_2^{\circ}(\mathfrak{k}) \triangleleft \mathcal{S}_2^{\triangleleft}(\mathfrak{h}|\mathfrak{r})$ is defined by omitting $h \in \mathfrak{h}/\mathfrak{k}$ and $r \in \mathfrak{r}$.
- $\mathcal{S}_1^{\triangleleft}(\mathfrak{h}) < \mathcal{S}_2^{\triangleleft}(\mathfrak{h})$ is defined by omitting $\beta \in \mathcal{S}_{\bar{1}}$.
- $\mathcal{S}_1^{\circ}(\mathfrak{k}) < \mathcal{S}_2^{\triangleleft}(\mathfrak{h})$ is defined by omitting $\beta \in \mathcal{S}_{\bar{1}}$ and $h \in \mathfrak{h}/\mathfrak{k}$.

Notice that the subscript in each of the proper real Lie superalgebras above denotes the dimension of its odd part. In the conformal symmetry superalgebras we shall encounter, $\mathfrak{r}$ will be represented by an abelian R-symmetry while $\mathfrak{h}$ (and $\mathfrak{k}$) will typically be represented by homothetic conformal Killing vectors (and Killing vectors).

| $d$ | 3 | 4 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Type | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{H}$ |
| $\mathcal{R}$ | $\mathfrak{so}(\mathscr{N})$ | $\mathfrak{u}(\mathscr{N} \neq 4)$ | $\mathfrak{su}(4)$ | $\mathfrak{sp}(1)$ | $\mathfrak{sp}(\mathscr{N})$ |
| $\mathcal{S}_{\max}$ | $\mathfrak{osp}(\mathscr{N}\|4)$ | $\mathfrak{su}(2,2\|\mathscr{N} \neq 4)$ | $\mathfrak{psu}(2,2\|4)$ | $\mathfrak{f}(4)$ | $\mathfrak{osp}(6,2\|\mathscr{N})$ |

**Table 1**. Generic conformal symmetry superalgebra data for Lorentzian $(M, g)$.

## 5 Conformal symmetry superalgebras and gauged R-symmetry

A certain class of pseudo-Riemannian spin manifolds which admit a twistor spinor may be equipped with a proper real Lie superalgebra structure that we refer to as a *conformal symmetry superalgebra* [16, 17]. A conformal symmetry superalgebra $\mathcal{S}$ contains conformal Killing vectors and constant R-symmetries in its even part $\mathcal{S}_{\bar{0}} = \mathcal{B}$ and twistor spinors valued in certain R-symmetry representations in its odd part $\mathcal{S}_{\bar{1}} = \mathcal{F}$.

Let $\mathcal{R}$ denote the real Lie algebra of R-symmetries. On any background $(M, g)$ that admits a conformal symmetry superalgebra $\mathcal{S}$, the even part $\mathcal{B} = \mathfrak{C}(M, [g]) \oplus \mathcal{R}$, as a real Lie algebra. The action of $\mathcal{B}$ on $\mathcal{F}$ which defines the $[\mathcal{B}, \mathcal{F}]$ bracket of $\mathcal{S}$ involves the action of $\mathfrak{C}(M, [g])$ on $\mathfrak{Z}(M, [g])$ defined by the Kosmann-Schwarzbach Lie derivative (3.14) and the action of $\mathcal{R}$ defined by the R-symmetry representation of $\mathcal{F}$. The $\mathfrak{C}(M, [g])$ part of the $[\mathcal{F}, \mathcal{F}]$ bracket of $\mathcal{S}$ involves pairing twistor spinors, using the spinorial bilinear form in (3.8) to make a conformal Killing vector (3.10), and projecting onto the $\mathcal{R}$-invariant part.

The type of spinor representation must be compatible with the type of R-symmetry representation in order to define $\mathcal{F}$ as a real $\mathcal{B}$-module. This puts restrictions on $\mathcal{R}$ according to the dimension $d$ of $M$ and the signature of $g$. In Lorentzian signature, the critical data is summarised in table 1. Entries in the 'Type' row in table 1 denote the ground field $\mathbb{K}$ over which the representation of $\mathcal{R}$ is defined. The dimension over $\mathbb{K}$ of this representation is denoted by $\mathscr{N}$. Entries in the '$\mathcal{S}_{\max}$' row of table 1 denote the conformal symmetry superalgebra $\mathcal{S} \cong \mathcal{S}_{\max}$ that is realised only when $(M, g)$ is locally conformally equivalent to Minkowski space.

If there exists a $\mathfrak{K}(M, g)$-invariant subspace $\mathcal{F}_\circ \subset \mathcal{F}$ such that the $\mathfrak{C}(M, [g])$ part of $[\mathcal{F}_\circ, \mathcal{F}_\circ]$ is in $\mathfrak{K}(M, g) < \mathfrak{C}(M, [g])$ then the background $(M, g)$ can be assigned the *symmetry superalgebra* $\mathcal{S}_\circ = \mathcal{B}_\circ \oplus \mathcal{F}_\circ$, where $\mathcal{B}_\circ = \mathfrak{K}(M, g) \oplus \mathcal{R}$.

Consider now the gauging of R-symmetry in the construction above. This amounts to promoting a given $\mathcal{R}$-module to a non-trivial vector bundle that is equipped with a connection $A$. Locally, elements in $\mathcal{F}$ now correspond to spinors on $(M, g)$ that are valued in sections of this vector bundle. By replacing all occurrences of the Levi-Civita connection $\nabla$ with the gauged connection $D = \nabla + A$ (i.e. in the Kosmann-Schwarzbach Lie derivative (3.14) and the Penrose operator (3.16)), the construction is made manifestly equivariant with respect to the gauged R-symmetry.

Following this prescription for a given conformal symmetry superalgebra $\mathcal{S}$ implies that each $\epsilon \in \mathcal{F}$ obeys a twistor spinor equation

$$D_X \epsilon = \frac{1}{d} \boldsymbol{X} \boldsymbol{D} \epsilon \,, \tag{5.1}$$

with respect to the gauged connection $D$, for all $X \in \mathfrak{X}(M)$. For any $\epsilon \in \mathcal{F}$, let $\Xi_\epsilon$ denote the component of $[\epsilon, \epsilon]$ in $\mathfrak{C}(M, [g])$. That $\Xi_\epsilon$ indeed remains a conformal Killing vector after gauging the R-symmetry follows directly from (5.1), using the fact that $\Xi_\epsilon$ is $\mathcal{R}$-invariant.

The action of $\mathfrak{C}(M, [g])$ on $\mathcal{F}$ is of the form

$$[X, \epsilon] = \hat{\mathcal{L}}_X \epsilon + (A_X + \rho_X) \cdot \epsilon \,, \tag{5.2}$$

for all $X \in \mathfrak{C}(M, [g])$ and $\epsilon \in \mathcal{F}$, in terms of some $\mathcal{R}$-valued function $\rho_X$ on $M$. Generically, (5.2) is not in $\mathcal{F}$ since $[X, \epsilon]$ does not obey the twistor spinor equation (5.1). However, this property does follow if

$$\iota_X F = D\rho_X \,, \tag{5.3}$$

for all $X \in \mathfrak{C}(M, [g])$, where $F = \mathrm{d}A + A \wedge A$ denotes the curvature of $D$.[4] For any $X, Y \in \mathfrak{C}(M, [g])$ which obey (5.3), $[X, Y] \in \mathfrak{C}(M, [g])$ also obeys (5.3) with

$$\rho_{[X,Y]} = F(X, Y) + D_X \rho_Y - D_Y \rho_X + [\rho_X, \rho_Y] \,. \tag{5.4}$$

The condition (5.3) ensures not only that $[\mathcal{B}, \mathcal{F}] \subset \mathcal{F}$ but also that the $[\mathcal{BBF}]$ and $[\mathcal{BFF}]$ components of the Jacobi identity (4.3) for $\mathcal{S}$ remain satisfied. Furthermore, $[\mathcal{F}, \mathcal{F}] \subset \mathcal{B}$ and the $[\mathcal{FFF}]$ component of the Jacobi identity for $\mathcal{S}$ remains satisfied as a consequence of manifest equivariance with respect to the gauged R-symmetry. Thus, $\mathcal{S}$ remains a Lie superalgebra after gauging the R-symmetry provided (5.3) is satisfied.

We can summarise this result more explicitly by introducing a local basis $\{\boldsymbol{e}_i\}$ for sections of the relevant R-symmetry vector bundles. At each point $x \in M$, $\{\boldsymbol{e}_i\}$ defines a basis for an $\mathcal{R}$-module $V$ of precisely the same type as in [16, 17]. In particular, we recall from table 1 that $V$ is orthogonal (type $\mathbb{R}$) in $d = 3$, Hermitian (type $\mathbb{C}$) in $d = 4$ and symplectic (type $\mathbb{H}$) in $d = 5, 6$. In the orthogonal (or symplectic) case, $V$ is equipped with a non-degenerate symmetric (or skewsymmetric) $\mathcal{R}$-invariant bilinear form that we will call $h$ (or $\omega$). With respect to a basis $\{\boldsymbol{e}^i\}$ on the dual module $V^*$ (defined such that $\boldsymbol{e}^i(\boldsymbol{e}_j) = \delta^i_j$), we identify $\boldsymbol{e}_i = h_{ij} \boldsymbol{e}^j$ in the orthogonal case (where $h_{ij} = h(\boldsymbol{e}_i, \boldsymbol{e}_j)$) and $\boldsymbol{e}_i = \omega_{ij} \boldsymbol{e}^j$ in the symplectic case (where $\omega_{ij} = \omega(\boldsymbol{e}_i, \boldsymbol{e}_j)$). In the Hermitian case, $\boldsymbol{e}_i$ and $\boldsymbol{e}^i$ are related by complex conjugation.

With respect to the basis above, a typical element $\epsilon \in \mathcal{F}$, subject to (5.1), is of the form

$$\epsilon = \begin{cases} \epsilon^i \boldsymbol{e}_i & \text{if } d = 3, 5 \\ \epsilon_+^i \boldsymbol{e}_i + \epsilon_{-i} \boldsymbol{e}^i & \text{if } d = 4 \\ \epsilon_+^i \boldsymbol{e}_i & \text{if } d = 6 \,, \end{cases} \tag{5.5}$$

[4] In fact, it is sufficient that $\Phi_X(Y) \cdot \epsilon = \frac{1}{d} \boldsymbol{Y} \boldsymbol{\Phi}_{\boldsymbol{X}} \cdot \epsilon$, for all $X \in \mathfrak{C}(M, [g])$, $Y \in \mathfrak{X}(M)$ and $\epsilon \in \mathcal{F}$, where $\Phi_X = \iota_X F - D\rho_X$. Generically this condition is weaker than (5.3) but is equivalent to it for all the cases of interest here.

where the components $\epsilon^i \in \mathfrak{S}(M)$ if $d = 3, 5$, $\epsilon^i_+ = (\epsilon_{-i})^* \in \mathfrak{S}_+(M)$ if $d = 4$ and $\epsilon^i_+ \in \mathfrak{S}_+(M)$ if $d = 6$. The element $\Xi_\epsilon \in \mathfrak{C}(M, [g])$ is defined such that

$$g(X, \Xi_\epsilon) = \begin{cases} h_{ij}\, \overline{\epsilon}^i \boldsymbol{X} \epsilon^j & \text{if } d = 3 \\ 2\, \overline{\epsilon}^i_+ \boldsymbol{X} \epsilon_{-i} & \text{if } d = 4 \\ \omega_{ij}\, \overline{\epsilon}^i \boldsymbol{X} \epsilon^j & \text{if } d = 5 \\ \omega_{ij}\, \overline{\epsilon}^i_+ \boldsymbol{X} \epsilon^j_+ & \text{if } d = 6\,, \end{cases} \tag{5.6}$$

for all $X \in \mathfrak{X}(M)$ and $\epsilon \in \mathcal{F}$. For any $\psi \in \mathfrak{S}(M)$, we recall that $\overline{\psi} = \langle \psi, - \rangle$ denotes its dual with respect to an admissible spinorial bilinear form $\langle -, - \rangle$ on $\mathfrak{S}(M)$ (defined in (3.8)). The associated element $\rho_{\Xi_\epsilon} \in C^\infty(M) \otimes \mathcal{R}$ is defined such that

$$(\rho_{\Xi_\epsilon} \cdot \psi)^i = \begin{cases} \frac{2}{3}(\overline{\epsilon}^i \boldsymbol{D} \epsilon^j - \overline{\epsilon}^j \boldsymbol{D} \epsilon^i) h_{kj} \psi^k & \text{if } d = 3 \\ (\overline{\epsilon}^i_+ \boldsymbol{D} \epsilon_{-j} - \overline{\epsilon}_{-j} \boldsymbol{D} \epsilon^i_+) \psi^j_+ - \frac{1}{4}(\overline{\epsilon}^j_+ \boldsymbol{D} \epsilon_{-j} - \overline{\epsilon}_{-j} \boldsymbol{D} \epsilon^j_+) \psi^i_+ & \text{if } d = 4 \\ \frac{3}{5}(\overline{\epsilon}^i \boldsymbol{D} \epsilon^j + \overline{\epsilon}^j \boldsymbol{D} \epsilon^i) \omega_{kj} \psi^k & \text{if } d = 5 \\ \frac{2}{3}(\overline{\epsilon}^i_+ \boldsymbol{D} \epsilon^j_+ + \overline{\epsilon}^j_+ \boldsymbol{D} \epsilon^i_+) \omega_{kj} \psi^k_+ & \text{if } d = 6\,, \end{cases} \tag{5.7}$$

for all $\epsilon, \psi \in \mathcal{F}$.

In terms of the data above, our result is that the assignment of a linear map $\rho : \mathfrak{C}(M, [g]) \to C^\infty(M) \otimes \mathcal{R}$ which obeys (5.3) is sufficient for the brackets

$$[X, \epsilon] = \hat{\mathcal{L}}_X \epsilon + (A_X + \rho_X) \cdot \epsilon\,, \qquad [\epsilon, \epsilon] = \Xi_\epsilon\,, \tag{5.8}$$

on the $\mathbb{Z}_2$-graded vector space $\mathfrak{C}(M, [g]) \oplus \mathcal{F}$ (for all $X \in \mathfrak{C}(M, [g])$ and $\epsilon \in \mathcal{F}$) to define a Lie superalgebra. Notice that the condition (5.3) does not involve elements in the image of $\rho$ that are in the kernel of $D$. Indeed, if we define

$$\mathfrak{R} = \{ R \in C^\infty(M) \otimes \mathcal{R} \,|\, DR = 0 \}\,, \tag{5.9}$$

as a real Lie algebra, then by appending to (5.8) the bracket

$$[R, \epsilon] = R \cdot \epsilon\,, \tag{5.10}$$

for all $R \in \mathfrak{R}$ and $\epsilon \in \mathcal{F}$, it follows that the $\mathbb{Z}_2$-graded vector space $\mathcal{S} = \mathcal{B} \oplus \mathcal{F}$, with $\mathcal{B} = \mathfrak{C}(M, [g]) \oplus \mathfrak{R}$, is also a Lie superalgebra. Elements in $\mathfrak{R}$ here generalise the constant R-symmetries in the ungauged construction of [16, 17].

Taking $X = \Xi_\epsilon$ in (5.3), for all $\epsilon \in \mathcal{F}$, typically constrains the form of $F$ because $\rho_{\Xi_\epsilon}$ is prescribed by (5.7). If $F = 0$, one can choose a gauge such that $\mathcal{S}$ recovers the form it took before gauging the R-symmetry. In Lorentzian signature, one finds that (5.3) is satisfied identically (with $F \neq 0$) for all conformal Killing vectors in $[\mathcal{F}, \mathcal{F}]$ only if $d = 3$ with $\mathcal{R} = \mathfrak{so}(2)$ or $d = 4$ with $\mathcal{R} = \mathfrak{u}(1)$. Notice that these are the only two cases in table 1



JHEP02(2016)008

where $\mathcal{R}$ is abelian. It follows that $F = \mathrm{d}A$ and $D\rho_X = \mathrm{d}\rho_X$ in (5.3), which is the local characterisation of

$$\mathcal{L}_X F = 0\,, \tag{5.11}$$

for all $X \in \mathfrak{C}(M,[g])$. Furthermore, it is precisely for these two cases that the data $(g, A)$ describes the full set of bosonic fields in an off-shell conformal gravity supermultiplet and (5.1) is the defining condition for bosonic supersymmetric vacua. The structure of field theories with rigid supersymmetry on such backgrounds has been explored recently in [6, 8, 15, 27].

## 6 $d = 3$

### 6.1 Null triads

Let $(\xi, \theta, \chi)$ denote a *null triad* of real vector fields on $(M, g)$, subject to the defining relations

$$\|\xi\|^2 = \|\theta\|^2 = g(\xi,\chi) = g(\theta,\chi) = 0\,, \qquad g(\xi,\theta) = \|\chi\|^2 = 1\,. \tag{6.1}$$

The relations (6.1) are preserved under the following transformations (which collectively generate $\mathrm{O}(2,1)$):

- $(\xi, \theta, \chi) \mapsto (\theta, \xi, \chi)$.
- $(\xi, \theta, \chi) \mapsto (\xi, \theta - \alpha\chi - \frac{1}{2}\alpha^2\xi, \chi + \alpha\xi)$, for any $\alpha \in \mathbb{R}$.
- $(\xi, \theta, \chi) \mapsto (\beta\xi, \beta^{-1}\theta, \pm\chi)$, for any $\beta \in \mathbb{R}\backslash\{0\}$.

It is convenient to use the null triad to express the metric and volume form on $M$ as

$$g = \xi \otimes \theta + \theta \otimes \xi + \chi \otimes \chi\,, \qquad \varepsilon = \xi \wedge \theta \wedge \chi\,. \tag{6.2}$$

### 6.2 Majorana spinors

The Clifford algebra $\mathrm{C}\ell(2,1) \cong \mathrm{Mat}_2(\mathbb{R}) \oplus \mathrm{Mat}_2(\mathbb{R})$ has two inequivalent irreducible representations, each isomorphic to $\mathbb{R}^2$, which are both identified with the unique irreducible representation of $\mathrm{C}\ell^0(2,1) \cong \mathrm{Mat}_2(\mathbb{R})$ after restricting to $\mathrm{Spin}(2,1)$. This restriction defines the Majorana spinor representation.

Relative to the basis conventions (3.6), Hodge duality in the exterior algebra implies

$$\mathbf{\Gamma}_{\mu\nu} = \varepsilon_{\mu\nu\rho}\mathbf{\Gamma}^\rho\,, \qquad \mathbf{\Gamma}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho}\mathbf{1}\,, \tag{6.3}$$

on Majorana spinors.

The bilinear form (3.8) on $\mathfrak{S}(M)$ is unique and skewsymmetric. For all $\psi, \varphi \in \mathfrak{S}(M)$, it follows that

$$\overline{\psi}\varphi = -\overline{\varphi}\psi\,, \qquad \overline{\psi}\mathbf{\Gamma}_\mu\varphi = \overline{\varphi}\mathbf{\Gamma}_\mu\psi\,, \tag{6.4}$$

and the associated Fierz identity is given by

$$\psi\overline{\varphi} = \frac{1}{2}((\overline{\varphi}\psi)\mathbf{1} + (\overline{\varphi}\mathbf{\Gamma}^\mu\psi)\mathbf{\Gamma}_\mu)\,. \tag{6.5}$$



JHEP02(2016)008

In terms of a unitary basis for $C\ell(2,1)$, all the quantities above are manifestly real.

For any $\epsilon \in \mathfrak{S}(M)$, let us define the real vector field

$$\xi_\epsilon^\mu = \bar{\epsilon}\boldsymbol{\Gamma}^\mu \epsilon \ . \tag{6.6}$$

We shall assume henceforth that $\xi_\epsilon$ is nowhere vanishing, which is so only if $\epsilon$ is nowhere vanishing. Furthermore, using (6.5), it follows that $\boldsymbol{\xi}_\epsilon \epsilon = 0$. Consequently, $\xi_\epsilon$ is null.

For any $\epsilon, \epsilon' \in \mathfrak{S}(M)$, let us also define

$$\zeta^\mu = \bar{\epsilon}\boldsymbol{\Gamma}^\mu \epsilon' \ , \qquad \kappa = \bar{\epsilon}\epsilon' \ . \tag{6.7}$$

The scalar $\kappa$ vanishes identically only if $\epsilon$ and $\epsilon'$ are linearly dependent. We shall assume henceforth that $\kappa \neq 0$. Using (6.5), it follows that

$$\boldsymbol{\xi}_\epsilon \epsilon' = -2\boldsymbol{\zeta}\epsilon = 2\kappa\epsilon \ . \tag{6.8}$$

Thus, $\xi_\epsilon$ and $\xi_{\epsilon'}$ are both null while $\zeta$ is spacelike, with

$$\|\zeta\|^2 = \kappa^2 = -\frac{1}{2} g(\xi_\epsilon, \xi_{\epsilon'}) \ , \qquad g(\xi_\epsilon, \zeta) = g(\xi_{\epsilon'}, \zeta) = 0 \ . \tag{6.9}$$

It is sometimes convenient to identify (6.6) and (6.7) in terms of the null triad introduced in section 6.1, such that

$$\xi_\epsilon = \xi \ , \qquad -\frac{1}{2}\kappa^{-2}\xi_{\epsilon'} = \theta \ , \qquad \kappa^{-1}\zeta = \chi \ . \tag{6.10}$$

In terms of this identification, $\epsilon' = \kappa\boldsymbol{\theta}\epsilon$. Since $(\epsilon, \epsilon')$ define a basis of Majorana spinors, it follows that any $\psi \in \mathfrak{S}(M)$ can be written

$$\psi = \alpha\epsilon + \beta\boldsymbol{\theta}\epsilon \, , \tag{6.11}$$

where $\alpha = \bar{\epsilon}\boldsymbol{\theta}\psi$ and $\beta = \bar{\epsilon}\psi$.

### 6.3 Causal conformal Killing vectors

The action of the Levi-Civita connection $\nabla$ on the null triad one-forms is constrained by the relations (6.1) such that

$$\begin{aligned} \nabla_\mu \xi_\nu &= p_\mu \xi_\nu - q_\mu \chi_\nu \\ \nabla_\mu \theta_\nu &= -p_\mu \theta_\nu + r_\mu \chi_\nu \\ \nabla_\mu \chi_\nu &= -r_\mu \xi_\nu + q_\mu \theta_\nu \, , \end{aligned} \tag{6.12}$$

in terms of three real one-forms $(p, q, r)$.

Given any $X \in \mathfrak{X}(M)$ with $\|X\|^2 = 0$, using the transformations below (6.1), one can always define a null triad such that $X = \xi$. From the first line in (6.12), it follows that $\xi$ is a conformal Killing vector only if

$$p_\theta = q_\xi = 0 \ , \qquad p_\xi = -2q_\chi \ , \qquad p_\chi = q_\theta \ . \tag{6.13}$$

The conformal factor (2.2) for $\xi \in \mathfrak{C}(M,[g])$ is $\sigma_\xi = q_\chi$. The condition $q_\xi = 0$ in (6.13) is equivalent to $\xi \wedge \mathrm{d}\xi = 0$. If $\xi$ is a Killing vector then (6.13) are satisfied with $\sigma_\xi = q_\chi = 0 = p_\xi$. If $\xi$ is $\nabla$-parallel then $p = q = 0$.

Identifying $\xi = \xi_\epsilon$ as in (6.10) implies

$$\nabla_\mu \epsilon = \frac{1}{2}(p_\mu \epsilon - q_\mu \boldsymbol{\theta} \epsilon) \ . \tag{6.14}$$

It follows that (6.13) are in fact necessary and sufficient for $\epsilon \in \mathfrak{Z}(M,[g])$. Since $\epsilon \in \mathfrak{Z}(M,[g])$ implies $\xi_\epsilon = \xi \in \mathfrak{C}(M,[g])$, clearly (6.13) are necessary for any $\epsilon \in \mathfrak{Z}(M,[g])$. The point is that, at least locally, the existence of a nowhere vanishing null conformal Killing vector is actually equivalent to the existence of a nowhere vanishing Majorana twistor spinor [8]. Furthermore, if $\xi_\epsilon$ is a Killing vector then $\epsilon$ is necessarily a Killing spinor (with Killing constant $-\frac{1}{2}q_\theta$). If $\xi_\epsilon$ is $\nabla$-parallel then $\epsilon$ is necessarily also $\nabla$-parallel.

Given any $X \in \mathfrak{X}(M)$ with $\|X\|^2 \neq 0$, using the transformations below (6.1), one can always define a null triad such that $X = \xi + \frac{1}{2}\|X\|^2\,\theta$. If $\|X\|^2 < 0$, let $\|X\|^2 = -4\kappa^2$ so that $X = \xi - 2\kappa^2\theta$. From the first two lines in (6.12), it follows that $X$ is a conformal Killing vector only if

$$\begin{aligned} &p_\theta = 0\,, \qquad q_\xi + 2\kappa^2 r_\xi = 2(\kappa^2 p_\chi - \partial_\chi \kappa^2)\,, \qquad p_\xi - 2\partial_\theta \kappa^2 = -2(q_\chi + 2\kappa^2 r_\chi)\,, \\ &p_\chi = q_\theta + 2\kappa^2 r_\theta\,, \qquad \kappa^2 p_\xi = \partial_\xi \kappa^2\ . \end{aligned} \tag{6.15}$$

The conformal factor $\sigma_X = -\partial_X(\ln\kappa)$. If $X$ is a Killing vector then (6.15) are satisfied with $p_\xi = 2\partial_\theta\kappa^2$.

Let us now identify the non-zero scalar $\kappa$ above with its namesake in (6.7). Identifying $\xi = \xi_\epsilon$ and $\theta = -\frac{1}{2}\kappa^{-2}\xi_{\epsilon'}$ as in (6.10) then implies $X = \xi - 2\kappa^2\theta = \xi_\epsilon + \xi_{\epsilon'}$. Let us name this timelike vector field $\Xi = \xi_\epsilon + \xi_{\epsilon'}$, with $\|\Xi\|^2 = -4\kappa^2$. With respect to the aforementioned identifications, it follows that

$$\begin{aligned} \nabla_\mu \epsilon &= \frac{1}{2}(p_\mu \epsilon - \kappa^{-1} q_\mu \epsilon') \\ \nabla_\mu \epsilon' &= -\kappa r_\mu \epsilon - \frac{1}{2}(p_\mu + \partial_\mu(\ln\kappa))\epsilon'\ . \end{aligned} \tag{6.16}$$

If $\Xi$ is a conformal Killing vector then

$$\mathrm{d}(\kappa^{-2}\Xi) = -\kappa^{-3}\rho_\Xi * \Xi\,, \tag{6.17}$$

where

$$\rho_\Xi = 2(\partial_\chi\kappa - \kappa p_\chi)\ . \tag{6.18}$$

Thus $\Xi \wedge \mathrm{d}\Xi = 4\kappa\rho_\Xi\,\xi\wedge\theta\wedge\chi$ is zero only if $\rho_\Xi = 0$.

### 6.4 Charged twistor spinors

Consider now the implications of the existence of a charged twistor spinor, which can be thought of locally as a pair of Majorana spinors $(\epsilon, \epsilon')$ both obeying (5.1) with respect to the action of the $\mathcal{R} = \mathfrak{so}(2)$ gauged connection

$$\begin{aligned} D_\mu \epsilon &= \nabla_\mu \epsilon + A_\mu \epsilon' \\ D_\mu \epsilon' &= \nabla_\mu \epsilon' - A_\mu \epsilon\ . \end{aligned} \tag{6.19}$$

An equivalent version of (6.19) follows by taking the real and imaginary parts of $D_\mu\epsilon_{\mathbb{C}} = \nabla_\mu\epsilon_{\mathbb{C}} - iA_\mu\epsilon_{\mathbb{C}}$, where $\epsilon_{\mathbb{C}} = \epsilon + i\epsilon'$. Because $(\epsilon,\epsilon')$ transform as a 2-vector under $\mathfrak{so}(2)$, $\epsilon_{\mathbb{C}} = \epsilon + i\epsilon'$ has unit charge under $\mathfrak{u}(1) \cong \mathfrak{so}(2)$.

The first consequence of (6.19) is that the timelike vector field $\Xi = \xi_\epsilon + \xi_{\epsilon'} \in \mathfrak{C}(M,[g])$. Thus, using (6.16), one finds that the five conditions in (6.15) are necessary in order for $(\epsilon,\epsilon')$ to define a charged twistor spinor with respect to $D$. The remaining three conditions which come from the twistor spinor equation for $(\epsilon,\epsilon')$ are precisely sufficient to fix all three components of $A$, such that

$$A_\mu = -\kappa r_\theta \xi_\mu + \frac{1}{2}\kappa^{-1} q_\xi \theta_\mu + (\partial_\theta \kappa - \kappa r_\chi)\chi_\mu \,, \tag{6.20}$$

in terms of the identifications in (6.10).

Thus we conclude that if $(M,g)$ admits a nowhere vanishing everywhere timelike conformal Killing vector $X$, then there must exist a nowhere vanishing charged twistor spinor pair $(\epsilon,\epsilon')$ on $(M,g)$, with $X = \Xi$, that is defined with respect to the gauged connection $A$ in (6.20). This characterisation of charged twistor spinors on a smooth orientable Lorentzian three-manifold was first obtained in [15].

Using (6.20), the twistor spinor equation for $(\epsilon,\epsilon')$ implies that the real function

$$\rho_\Xi = \frac{2}{3}(\bar\epsilon \boldsymbol{D}\epsilon' - \bar\epsilon' \boldsymbol{D}\epsilon) \,, \tag{6.21}$$

is identical to its namesake in (6.18). Moreover, it follows that

$$\iota_\Xi F = \mathrm{d}\rho_\Xi \,, \tag{6.22}$$

so (5.3) is satisfied identically for $X = \Xi$.

Differentiating the defining condition (5.1) with (6.19) gives

$$\frac{2}{3} D_\mu \boldsymbol{D}\epsilon_{\mathbb{C}} = K_{\mu\nu}\boldsymbol{\Gamma}^\nu \epsilon_{\mathbb{C}} - i(F_{\mu\nu}\boldsymbol{\Gamma}^\nu - \tilde F_\mu \mathbf{1})\epsilon_{\mathbb{C}} \,, \tag{6.23}$$

where $\tilde F_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F^{\nu\rho}$. Combining (6.23) with (3.3) implies the integrability condition

$$iC_{\mu\nu\rho}\boldsymbol{\Gamma}^\rho \epsilon_{\mathbb{C}} = (2\nabla_{[\mu}\tilde F_{\nu]}\mathbf{1} + \boldsymbol{\nabla} F_{\mu\nu})\epsilon_{\mathbb{C}} + \kappa^{-1}(F_{\mu\nu}\mathbf{1} - \tilde F_{[\mu}\boldsymbol{\Gamma}_{\nu]})(\rho_\Xi \mathbf{1} + 2\boldsymbol{\nabla}\kappa)\epsilon_{\mathbb{C}} \,, \tag{6.24}$$

using the identity

$$\frac{4}{3}\boldsymbol{D}\epsilon_{\mathbb{C}} = \kappa^{-1}(\rho_\Xi \mathbf{1} + 2\boldsymbol{\nabla}\kappa)\epsilon_{\mathbb{C}} \,. \tag{6.25}$$

It is convenient to define $\tilde C_{\mu\nu} = \frac{1}{2}\varepsilon_\mu{}^{\rho\sigma}C_{\rho\sigma\nu}$ in $d=3$. From the definition of the Cotton-York tensor (3.23), it follows that $\tilde C_{\mu\nu} = \tilde C_{\nu\mu}$, $\tilde C_{\mu\nu}g^{\mu\nu} = 0$ and $\nabla^\mu \tilde C_{\mu\nu} = 0$ identically. An equivalent, but somewhat more wieldy, form of (6.24) is given by

$$i\tilde C_{\mu\nu}\boldsymbol{\Gamma}^\nu \epsilon_{\mathbb{C}} = \kappa^{-1}\Big( (\rho_\Xi \tilde F_\mu - \nabla^\nu(\kappa F_{\mu\nu}))\mathbf{1} + \kappa \boldsymbol{\nabla}\tilde F_\mu + 2\tilde F_\mu \boldsymbol{\nabla}\kappa + 2\tilde F^\nu (\partial_{[\mu}\kappa)\boldsymbol{\Gamma}_{\nu]} - \frac{1}{2}\rho_\Xi \tilde F^\nu \boldsymbol{\Gamma}_{\mu\nu}\Big)\epsilon_{\mathbb{C}} \,. \tag{6.26}$$

From (6.26), it follows that

$$\frac{1}{2}\tilde C_{\mu\nu}\Xi^\nu = \rho_\Xi \tilde F_\mu - \nabla^\nu(\kappa F_{\mu\nu}) \,. \tag{6.27}$$

### 6.5 Conformal symmetry superalgebras

Let $(M, g)$ be any smooth orientable Lorentzian three-manifold equipped with a nowhere vanishing causal conformal Killing vector $X$ and a $\mathfrak{C}(M,[g])$-invariant closed two-form $F$. As we have explained, any null $X \in \mathfrak{C}(M,[g])$ defines a nowhere vanishing twistor spinor $\epsilon$ (with $X = \xi = \xi_\epsilon$) while any timelike $X \in \mathfrak{C}(M,[g])$ defines a nowhere vanishing charged twistor spinor pair $(\epsilon, \epsilon')$ (with $X = \Xi = \xi_\epsilon + \xi_{\epsilon'}$) that is charged with respect to the connection $A$ in (6.20) whose curvature is $F = \mathrm{d}A$. From section 5, we recall that this data is sufficient to assign to $(M,[g])$ a conformal symmetry superalgebra $\mathcal{S}$ with trivial $\mathcal{R}$ for null $X$ or gauged $\mathcal{R} = \mathfrak{so}(2)$ for timelike $X$.

#### 6.5.1 Null case

If $(M, g)$ admits a nowhere vanishing conformal Killing vector $\xi$ that is everywhere null then it is locally conformally equivalent to a pp-wave or $\mathbb{R}^{2,1}$. We shall assume henceforth that $(M, g)$ is not locally conformally flat. The general form of the three-dimensional pp-wave metric is

$$g_{\mathrm{pp}} = 2\mathrm{d}u\mathrm{d}v + H(u,x)\mathrm{d}u^2 + \mathrm{d}x^2\,, \tag{6.28}$$

in terms of Brinkmann coordinates $(u, v, x)$ on $M$, where $H$ is an arbitrary real function of $(u, x)$. In these coordinates, $\xi = \partial_v$ is a null Killing vector of $(M, g_{\mathrm{pp}})$. It is often convenient to adopt the notation $' = \partial_u$.

The only non-trivial component of the Riemann tensor of $g_{\mathrm{pp}}$ is $R_{uxux} = -\frac{1}{2}\partial_x^2 H$. Whence, $(M, g_{\mathrm{pp}})$ is flat only if $H$ is a linear function of $x$. The only non-trivial component of the Ricci tensor of $g_{\mathrm{pp}}$ is $R_{uu} = R_{uxux}$ and the scalar curvature $R = 0$. The only non-trivial component of the Cotton-York tensor of $g_{\mathrm{pp}}$ is $C_{uxu} = -\frac{1}{2}\partial_x^3 H$ whose scalar norm-squared $\|C\|^2 = 0$. Thus, $(M, g_{\mathrm{pp}})$ is conformally flat only if $H$ is a quadratic function of $x$. Since we are concerned with geometries that are not conformally flat, we shall assume henceforth that $(M, g_{\mathrm{pp}})$ has $\partial_x^3 H \neq 0$.

Any $X \in \mathfrak{C}(M,[g_{\mathrm{pp}}])$ must be of the form

$$X = \gamma_X \partial_u + (\alpha_X - \beta_X' x - \frac{\gamma_X''}{4}x^2 + 2c_X v)\partial_v + (\beta_X + (\frac{\gamma_X'}{2} + c_X)x)\partial_x\,, \tag{6.29}$$

in terms of three real functions $(\alpha_X, \beta_X, \gamma_X)$ of $u$ and a real number $c_X$ which obey

$$2(\alpha_X - \beta_X' x - \frac{\gamma_X''}{4}x^2)' + (\gamma_X H)' - 2c_X H + (\beta_X + (\frac{\gamma_X'}{2} + c_X)x)\partial_x H = 0\,. \tag{6.30}$$

The conformal factor is

$$\sigma_X = -\frac{\gamma_X'}{2} - c_X\,. \tag{6.31}$$

The expression (6.29) shows that the Lie bracket of the generic null Killing vector $\xi$ with any $X \in \mathfrak{C}(M,[g_{\mathrm{pp}}])$ is given by

$$[\xi, X] = 2c_X \xi\,. \tag{6.32}$$

Whence, the real line spanned by $\xi$ forms a one-dimensional ideal of $\mathfrak{C}(M,[g_{\mathrm{pp}}])$. Clearly $\xi$ is in the centre $Z(\mathfrak{C}(M,[g_{\mathrm{pp}}]))$ of $\mathfrak{C}(M,[g_{\mathrm{pp}}])$ only if every $X \in \mathfrak{C}(M,[g_{\mathrm{pp}}])$ has $c_X = 0$.

If $\xi \notin Z(\mathfrak{C}(M,[g_{\rm pp}]))$ then at least one $X \in \mathfrak{C}(M,[g_{\rm pp}])$ must have $c_X \neq 0$ and every other $Y \in \mathfrak{C}(M,[g_{\rm pp}])$ can be taken to have $c_Y = 0$ (i.e. if $Y \in \mathfrak{C}(M,[g_{\rm pp}])$ has $c_Y \neq 0$ then $\tilde{Y} = Y - \frac{c_Y}{c_X} X \in \mathfrak{C}(M,[g_{\rm pp}])$ has $c_{\tilde{Y}} = 0$).

The condition (6.30) indicates that the existence of an extra conformal Killing vector (in addition to $\xi$) puts constraints on the function $H$. If $\dim \mathfrak{C}(M,[g_{\rm pp}]) = 2$, one can fix $\gamma_X = 1$ in (6.29), (6.30) and (6.31) with respect to a conformally equivalent metric via

$$u \mapsto \int {\rm d}u\, \Omega^2(u)\,, \quad v \mapsto v - \frac{1}{2} x^2 \Omega^{-1} \partial_u \Omega\,, \quad x \mapsto \Omega x\,, \quad H \mapsto \Omega^{-2}(H - x^2 \Omega \partial_u^2 \Omega^{-1})\,, \tag{6.33}$$

if it is possible to identify $\Omega^2 = \gamma_X^{-1}$. In this case, $X$ is homothetic with respect to $\gamma_X^{-1} g_{\rm pp}$, with conformal factor $-c_X$. Whence, from (6.32), $X$ is a Killing vector only if $[\xi, X] = 0$, in which case (6.30) fixes

$$H = -2\alpha_X - \beta_X^2 + 2\beta_X' x + f(-x + \int {\rm d}u\, \beta_X)\,, \tag{6.34}$$

in terms of any real function $f$ of one variable whose third derivative is not zero. A similar, but more complicated, expression for $H$ emerges when $c_X \neq 0$.

Since $\|C\|^2$ vanishes identically on $(M, g_{\rm pp})$, it cannot be used to define a conformal scalar. However, the fact that the Lie derivative of $C$ along any conformal Killing vector $X$ is zero implies that $\partial_X C_{uxu} = -(2\partial_u X^u + \partial_x X^x) C_{uxu}$ on $(M, g_{\rm pp})$. Using (6.31), this gives $\partial_X C_{uxu} = (5\sigma_X + 4c_X) C_{uxu}$ for any $X \in \mathfrak{C}(M,[g_{\rm pp}])$. Consequently, if $\xi \in Z(\mathfrak{C}(M,[g_{\rm pp}]))$, then $\phi = \partial_x^3 H$ is a proper conformal scalar with $p_\phi = 5$ and $\mathfrak{C}(M,[g_{\rm pp}])$ is conformally isometric. If $\xi \notin Z(\mathfrak{C}(M,[g_{\rm pp}]))$ then $(M, g_{\rm pp})$ admits a proper conformal gradient with $\varphi = \frac{1}{5} \ln \partial_x^3 H$ and $s_X = -\frac{4}{5} c_X$, in which case $(M, g_{\rm pp})$ is conformally homothetic. In both cases, the isometric/homothetic action of $\mathfrak{C}(M,[g_{\rm pp}])$ is with respect to the metric $(\partial_x^3 H)^{2/5} g_{\rm pp}$ on $M$ (which need not be locally isometric to a pp-wave).

If $2 \leq \dim \mathfrak{C}(M,[g_{\rm pp}]) \leq 4$, we have seen above that the structure of $\mathfrak{C}(M,[g_{\rm pp}])$ depends critically on whether or not $\xi \in Z(\mathfrak{C}(M,[g_{\rm pp}]))$. If $\xi \in Z(\mathfrak{C}(M,[g_{\rm pp}]))$ then $\mathfrak{C}(M,[g_{\rm pp}]) = \mathfrak{K}(M,g)$ (i.e. conformally isometric). This requires the existence of a Lorentzian three-manifold $(M,g)$ with $2 \leq \dim \mathfrak{K}(M,g) \leq 4$ which admits a non-zero null $\xi \in Z(\mathfrak{K}(M,g))$. The classification, up to local isometry, of all three-dimensional Lorentzian geometries with Killing vectors is due to Kručkovič [63] (see also section 5 in [61]). Any Lorentzian three-manifold $(M,g)$ in [63] that is not conformally flat and admits a non-zero null $\xi \in Z(\mathfrak{K}(M,g))$ must have $\dim \mathfrak{K}(M,g) < 3$. If $\dim \mathfrak{K}(M,g) = 2$ then one can choose local coordinates $(u,v,x)$ on $M$ such that $(\partial_u, \xi = \partial_v) \in \mathfrak{K}(M,g) \cong \mathbb{R}^2$ and, for some positive function $\Omega$ of $x$, $g = \Omega(x) g_{\rm pp}$ in terms of the pp-wave metric $g_{\rm pp}$ in (6.28) of the form

$$2{\rm d}u{\rm d}v + H(x){\rm d}u^2 + {\rm d}x^2\,, \tag{6.35}$$

with $\partial_x^3 H \neq 0$.

If $\xi \notin Z(\mathfrak{C}(M,[g_{\rm pp}]))$ then $\mathfrak{C}(M,[g_{\rm pp}]) = \mathfrak{H}(M,g)$ (i.e. conformally homothetic) and $\xi \in Z(\mathfrak{K}(M,g))$. Just as in the previous case, $\mathfrak{K}(M,g) \cong \mathbb{R}^2$ is the only option if $\dim \mathfrak{C}(M,[g_{\rm pp}]) > 2$ (we have already covered the $\dim \mathfrak{C}(M,[g_{\rm pp}]) = 2$ case above). Clearly



JHEP02(2016)008

| $H$ | $\vartheta$ | $[\vartheta, \partial_u]$ | $\mathfrak{C}(M,[g_{\rm pp}])$ |
| --- | --- | --- | --- |
| $b + c\,\mathrm{e}^x$ | $u\partial_u - (v + bu)\partial_v - 2\partial_x$ | $-\partial_u + b\partial_v$ | $\mathfrak{b}[\mathrm{VI}_0]$ |
| $b - 2c\ln x$ | $-u\partial_u - (v + cu)\partial_v - x\partial_x$ | $\partial_u + c\partial_v$ | $\mathfrak{b}[\mathrm{IV}]$ |
| $b + c\,x^{-2\left(\frac{a-1}{a+1}\right)}$ | $-au\partial_u - (v - \frac{1}{2}(a-1)bu)\partial_v - \frac{1}{2}(a+1)x\partial_x$ | $a\partial_u - \frac{1}{2}(a-1)b\partial_v$ | $\mathfrak{b}[\mathrm{VI}]$ |

**Table 2**. Data for geometries with $\dim\mathfrak{C}(M,[g_{\rm pp}]) = 3$.

$\dim\mathfrak{C}(M,[g_{\rm pp}]) = 4$ is impossible since $\dim\mathfrak{C}(M,[g_{\rm pp}]) = \dim\mathfrak{H}(M,g) = \dim\mathfrak{K}(M,g)+1$. The only remaining option is $\dim\mathfrak{C}(M,[g_{\rm pp}]) = 3$. Since the homothetic action of $\mathfrak{C}(M,[g_{\rm pp}])$ is with respect to $g = (\partial_x^3 H)^{2/5} g_{\rm pp}$, where $g_{\rm pp}$ is of the form (6.35), the proper homothetic $X \in \mathfrak{H}(M,g)/\mathfrak{K}(M,g)$ obeys $\mathcal{L}_X g = \frac{8}{5}c_X g$ and $[\xi, X] = 2c_X\xi$ with $c_X \neq 0$. The non-zero value of $c_X$ is irrelevant and it is convenient to work in terms of $\vartheta := -\frac{1}{2c_X}X \in \mathfrak{H}(M,g)/\mathfrak{K}(M,g)$ which has $c_\vartheta = -\frac{1}{2}$. Solving $\mathcal{L}_\vartheta g = -\frac{4}{5}g$ and $[\vartheta,\xi] = \xi$ yields three inequivalent classes of solutions for $H$ and $\vartheta$ which are displayed in table 2. The parameters $a \in \mathbb{R}\backslash\{0, \pm 1, \frac{1}{3}\}$, $b \in \mathbb{R}$ and $c \in \mathbb{R}\backslash\{0\}$. For the functions $H$ in table 2, $\vartheta$ is homothetic not only with respect to $g = (\partial_x^3 H)^{2/5} g_{\rm pp}$ but also with respect to $g_{\rm pp}$ (albeit with a different constant conformal factor). For $H = b + c\,\mathrm{e}^x$, $\vartheta$ is a Killing vector with respect to $g_{\rm pp}$ and $(M, g_{\rm pp})$ is homogeneous. In the other two cases, $\vartheta$ is a Killing vector with respect to $x^{-2}g_{\rm pp}$.

For each geometry in table 2, it follows by direct calculation that the associated conformal symmetry superalgebra $\mathcal{S}$ is defined by the following common non-zero brackets

$$[\vartheta,\xi] = \xi\ , \quad [\vartheta,\epsilon] = \frac{1}{2}\epsilon\ , \quad [\epsilon,\epsilon] = \xi\ , \tag{6.36}$$

in terms of a (suitably normalised) parallel spinor $\epsilon \in \mathcal{F}$ on $(M, g_{\rm pp})$, together with the bracket $[\vartheta, \partial_u]$ displayed in the third column of table 2. Consequently $\mathcal{S} = \mathcal{S}_1^{\lhd}(\mathfrak{C}(M,[g_{\rm pp}]))$, in terms of the notation defined in section 4 (identifying $\vartheta = h$ and $\xi = z$). Entries in the rightmost column of table 2 denote the isomorphism class of $\mathcal{B} = \mathfrak{C}(M,[g_{\rm pp}])$ in terms of a three-dimensional indecomposable real Lie algebra in the Bianchi classification [77]. Up to isomorphism, there exists a basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ such that the Lie algebra

- $\mathfrak{b}[\mathrm{VI}_0]$ has non-zero Lie brackets $[\boldsymbol{e}_1, \boldsymbol{e}_3] = \boldsymbol{e}_2$, $[\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_1$.
- $\mathfrak{b}[\mathrm{IV}]$ has non-zero Lie brackets $[\boldsymbol{e}_1, \boldsymbol{e}_3] = \boldsymbol{e}_1$, $[\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_2 - \boldsymbol{e}_1$.
- $\mathfrak{b}[\mathrm{VI}]$ has non-zero Lie brackets $[\boldsymbol{e}_1, \boldsymbol{e}_3] = \boldsymbol{e}_2 + a\boldsymbol{e}_1$, $[\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_1 + a\boldsymbol{e}_2$.

For $\mathfrak{b}[\mathrm{VI}]$, each positive real number $a \neq 1$ corresponds to a distinct isomorphism class. The class $\mathfrak{b}[\mathrm{VI}_0]$ corresponds to the Poincaré algebra of $\mathbb{R}^{1,1}$.

### 6.5.2 Timelike case

Let us now assume that $(M,g)$ admits a nowhere vanishing conformal Killing vector $\Xi$ that is everywhere timelike. As we have explained, this data defines a nowhere vanishing charged twistor spinor pair $(\epsilon,\epsilon')$ such that $\Xi = \xi_\epsilon + \xi_{\epsilon'}$. We shall also insist that the closed

| $\phi$ | $\tilde{C}_{\mu\nu}\tilde{C}^{\mu\nu}$ | $\tilde{C}_{\mu\nu}\tilde{C}^{\nu}{}_{\rho}\tilde{C}^{\rho\mu}$ | $\tilde{F}_{\mu}\tilde{F}^{\mu}$ | $\tilde{C}_{\mu\nu}\tilde{F}^{\mu}\tilde{F}^{\nu}$ | $\tilde{F}^{\mu}\tilde{C}_{\mu\nu}\tilde{C}^{\nu\rho}\tilde{F}_{\rho}$ |
|---|---|---|---|---|---|
| $p_{\phi}$ | 6 | 9 | 4 | 7 | 10 |

**Table 3**. Some proper conformal scalars.

two-form $F = \mathrm{d}A$ defined by (6.20) is $\mathfrak{C}(M,[g])$-invariant so that we may assign to $(M,[g])$ a conformal symmetry superalgebra $\mathcal{S}$.

If $F = 0$ then locally $A = \mathrm{d}\lambda$ and the charged twistor spinor pair $(\epsilon, \epsilon')$ is equivalent to a pair of ordinary twistor spinors $(\cos\lambda\,\epsilon + \sin\lambda\,\epsilon', -\sin\lambda\,\epsilon + \cos\lambda\,\epsilon')$. Since $\epsilon$ and $\epsilon'$ are linearly independent, clearly $F = 0$ implies $(M,g)$ must be locally conformally flat.[5] In that case, $\mathfrak{C}(M,[g]) \cong \mathfrak{so}(3,2) \cong \mathfrak{sp}(4,\mathbb{R})$ and $\mathcal{S} \cong \mathfrak{osp}(2|4)$ (i.e. the $\mathscr{N} = 2$, $d = 3$ case in table 1). We shall assume henceforth that $(M,g)$ is not locally conformally flat so both the Cotton-York tensor $C$ and the two-form $F$ are not identically zero.

For any $X \in \mathfrak{C}(M,[g])$, $\mathcal{L}_X C = 0$ and $\mathcal{L}_X F = 0$ while $\mathcal{L}_X \tilde{C} = \sigma_X \tilde{C}$ and $\mathcal{L}_X \tilde{F} = \sigma_X \tilde{F}$ since $\mathcal{L}_X g = -2\sigma_X g$. Any non-zero scalar built from $\tilde{C}$, $\tilde{F}$ and $g$ therefore defines a conformal scalar $\phi$ (with weight $p_\phi$). Five different options are displayed in table 3. If just one $\phi$ in table 3 is nowhere vanishing then $\mathfrak{C}(M,[g])$ is conformally isometric. Conversely, let us examine what happens if all $\phi$ in table 3 are identically zero.

In this case, $\tilde{F}$ defines a non-zero vector field that is everywhere null with respect to $g$. As such, at least locally, there must exist a null triad $(\tilde{F}, G, H)$ such that

$$g_{\mu\nu} = 2\tilde{F}_{(\mu}G_{\nu)} + H_\mu H_\nu \; . \tag{6.37}$$

Using $\tilde{C}_{\mu\nu}\tilde{F}^\mu\tilde{F}^\nu = 0$, it follows that $\tilde{F}^\mu\tilde{C}_{\mu\nu}\tilde{C}^{\nu\rho}\tilde{F}_\rho = 0$ implies $\tilde{C}_{\mu\nu}\tilde{F}^\mu H^\nu = 0$. Moreover, $\tilde{C}_{\mu\nu}\tilde{C}^{\mu\nu} = 0$ and $\tilde{C}_{\mu\nu}g^{\mu\nu} = 0$ then imply $\tilde{C}_{\mu\nu}\tilde{F}^\nu = 0$ and $\tilde{C}_{\mu\nu}H^\mu H^\nu = 0$. Therefore

$$\tilde{C}_{\mu\nu} = \alpha\tilde{F}_\mu\tilde{F}_\nu + 2\beta\tilde{F}_{(\mu}H_{\nu)} \,, \tag{6.38}$$

in terms of a pair of real functions $\alpha$ and $\beta$ which are not both zero. The form of the Cotton-York tensor in (6.38) further implies that $\tilde{C}_{\mu\nu}\tilde{C}^{\nu}{}_{\rho}\tilde{C}^{\rho\mu}$ vanishes identically. Now taking $\mathcal{L}_X$ of (6.37) and using $\mathcal{L}_X g_{\mu\nu} = -2\sigma_X g_{\mu\nu}$ and $\mathcal{L}_X \tilde{F}_\mu = \sigma_X \tilde{F}_\mu$ implies that

$$\mathcal{L}_X G_\mu = -3\sigma_X G_\mu + \gamma_X H_\mu \; , \qquad \mathcal{L}_X H_\mu = -\gamma_X \tilde{F}_\mu - \sigma_X H_\mu \,, \tag{6.39}$$

in terms of some real function $\gamma_X$, for all $X \in \mathfrak{C}(M,[g])$. Taking $\mathcal{L}_X$ of (6.38) and using $\mathcal{L}_X \tilde{C}_{\mu\nu} = \sigma_X \tilde{C}_{\mu\nu}$ together with the expressions above yields

$$\partial_X \alpha = -\sigma_X \alpha + 2\gamma_X \beta \; , \qquad \partial_X \beta = \sigma_X \beta \,, \tag{6.40}$$

[5] Conversely, if $(M,g)$ is locally conformally flat, the integrability condition (6.26) does not imply $F = 0$. In this case, in terms of the conformally equivalent metric $\hat{g} = \kappa^{-2}g$, (6.26) just states that the one-form $\hat{*}F$ defines a Killing vector with respect to $\hat{g}$ and obeys $\mathrm{d}\hat{*}F = -\rho_{\Xi}F$. However, since $\mathfrak{C}(M,[g]) \cong \mathfrak{so}(3,2)$, it is the condition (5.11) that fixes $F = 0$. Of course, if $(M,g)$ is locally conformally flat and admits a non-zero $\mathfrak{K}(M,g)$-invariant $F$ which solves (6.26), one can define an associated symmetry superalgebra by restricting to $\mathfrak{K}(M,g) < \mathfrak{so}(3,2)$, as was done for a number of examples in [15].

| Class | $f$ | $a$ | $k$ | $l$ | $\mathfrak{K}(M,g)$ |
|---|---|---|---|---|---|
| IV.4 | 1 | 1 | $\partial_x - y\partial_t$ | $y\partial_x - x\partial_y + \frac{1}{2}(x^2-y^2)\partial_t$ | $\mathfrak{c}$[X] |
| IV.5 | $\mathrm{e}^x$ | $\neq \pm 1$ | $\partial_x - y\partial_y$ | $y\partial_x + \frac{1}{2}(\mathrm{e}^{-2x} - y^2)\partial_y - a\mathrm{e}^{-x}\partial_t$ | $\mathbb{R}\oplus\mathfrak{b}$[VIII] |
| IV.6 | $\sin x$ | - | $\cos y\,\partial_x - \frac{\sin y}{\sin x}(\cos x\,\partial_y + a\partial_t)$ | $-\sin y\,\partial_x - \frac{\cos y}{\sin x}(\cos x\,\partial_y + a\partial_t)$ | $\mathbb{R}\oplus\mathfrak{b}$[IX] |

**Table 4**. Data for geometries with $\dim \mathfrak{C}(M,[g]) = 4$.

for all $X \in \mathfrak{C}(M,[g])$. Whence, if $\beta \neq 0$ then $\phi = \beta$ defines a proper conformal scalar with $p_\phi = 1$. If $\beta = 0$ then $\phi = \alpha$ defines a proper conformal scalar with $p_\phi = -1$. Thus we have proved that $\mathfrak{C}(M,[g])$ is always conformally isometric in the timelike case.

This fact means that we may choose a representative metric $g$ in $[g]$ for any admissible geometry such that $\mathfrak{C}(M,[g]) = \mathfrak{K}(M,g)$. By definition, the geometry $(M,g)$ is therefore *stationary* because it is equipped with a timelike Killing vector $\Xi$. It is convenient to express the stationary metric

$$g = -4\kappa^2 \varpi \otimes \varpi + h\,, \tag{6.41}$$

in terms of the one-form $\varpi$ defined by $\varpi(X) = -\frac{1}{4\kappa^2} g(\Xi, X)$, for all $X \in \mathfrak{X}(M)$. By construction, $\iota_\Xi \varpi = 1$ and $\iota_\Xi h = 0$ since $\|\Xi\|^2 = -4\kappa^2$. It follows that $\mathcal{L}_\Xi \varpi = 0$ while $\partial_\Xi \kappa = -\sigma_\Xi \kappa = 0$ and $\mathcal{L}_\Xi h = -2\sigma_\Xi h = 0$ because $\sigma_\Xi = 0$. If $\mathrm{d}\varpi = 0$ then $(M,g)$ is *static*, which occurs only if $\Xi \wedge \mathrm{d}\Xi = 0$. Thus, from the observation below (6.18), $(M,g)$ is static only if $\rho_\Xi = 0$.

Any admissible geometry with $\dim \mathfrak{C}(M,[g]) = 4$ must be locally conformally equivalent to one of three Lorentzian geometries of class IV in [63] that is not conformally flat. All three of these class IV geometries are stationary and, when expressed in the form (6.41), have

$$2\kappa\varpi = \mathrm{d}t + \omega(x)\mathrm{d}y\,, \qquad h = \mathrm{d}x^2 + f(x)^2\mathrm{d}y^2\,, \tag{6.42}$$

in terms of local coordinates $(t,x,y)$ on $M$, for particular pairs of functions $\omega$ and $f$ that are related such that $\partial_x \omega = af$, for some non-zero real number $a$ (the precise data is displayed in table 4). This identification fixes $\Xi = -2\kappa\partial_t$ with $\kappa$ constant. The associated null triad one-forms are

$$\xi = \kappa(\mathrm{d}t + (\omega + f)\mathrm{d}y)\,, \qquad \theta = -\frac{1}{2\kappa}(\mathrm{d}t + (\omega - f)\mathrm{d}y)\,, \qquad \chi = \mathrm{d}x\,. \tag{6.43}$$

The volume form (6.2) is $\varepsilon = -f\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y$. The action of $\nabla$ on (6.43) can be defined via the triple of one-forms $(p,q,r)$ in (6.12), which are given by

$$\begin{aligned} p_\xi = p_\theta = q_\chi = r_\chi = 0\,, \qquad & q_\theta = -r_\xi = \frac{1}{2} f^{-1}\partial_x f\,, \\ p_\chi = \frac{1}{2}a\,, \qquad & q_\xi = \kappa^2(a + f^{-1}\partial_x f)\,, \qquad r_\theta = \frac{1}{4\kappa^2}(a - f^{-1}\partial_x f)\,. \end{aligned} \tag{6.44}$$

Comparison with (6.15) confirms that $\Xi$ is indeed a Killing vector. Moreover, from (6.18), $\rho_\Xi = -2\kappa p_\chi = -\kappa a$. Whence, none of the three class IV geometries is static.



JHEP02(2016)008

| Class | $\rho_k$ | $\rho_l$ | $\mathcal{S}$ |
|---|---|---|---|
| IV.4 | $-\frac{1}{2}y$ | $-\frac{1}{4}(x^2+y^2)$ | $\mathcal{S}_2^\circ(\mathfrak{c}[\mathrm{X}]\vert\mathcal{R})$ |
| IV.5 | $\frac{1}{2}(1-a^2)y\mathrm{e}^x$ | $\frac{1}{4}(1-a^2)(y^2\mathrm{e}^x+\mathrm{e}^{-x})$ | $\mathcal{S}_2^\circ(\mathbb{R}\vert\mathcal{R})\oplus\mathfrak{b}[\mathrm{VIII}]$ |
| IV.6 | $-\frac{1}{2}(1+a^2)\sin x\sin y$ | $-\frac{1}{2}(1+a^2)\sin x\cos y$ | $\mathcal{S}_2^\circ(\mathbb{R}\vert\mathcal{R})\oplus\mathfrak{b}[\mathrm{IX}]$ |

**Table 5**. Conformal superalgebra data for geometries with $\dim\mathfrak{C}(M,[g])=4$.

Substituting (6.43) and (6.44) into (6.20) yields

$$A=-\frac{1}{2}a(\mathrm{d}t+\omega\mathrm{d}y)+\frac{1}{2}\partial_x f\mathrm{d}y\ . \tag{6.45}$$

Whence, $F=\mathrm{d}A=\frac{1}{2}(\partial_x^2 f-a^2 f)\mathrm{d}x\wedge\mathrm{d}y$. Furthermore, substituting (6.43) and (6.44) into (6.16), and using (6.45), implies the gauged connection in (6.19) is given by

$$D_\mu=\partial_\mu-\frac{1}{4}a\boldsymbol{\Gamma}_\mu\ . \tag{6.46}$$

The identification $\Xi=\xi_\epsilon+\xi_{\epsilon'}=-2\kappa\partial_t$ and $\kappa=\bar{\epsilon}\epsilon'$ implies $\epsilon'=\boldsymbol{\Gamma}_t\epsilon$. Moreover, $\epsilon_{\mathbb{C}}=\epsilon+i\boldsymbol{\Gamma}_t\epsilon$ is a charged twistor spinor with respect to (6.46) only if $\epsilon$ is constant.

For each of the three Lorentzian geometries of class IV defined by table 4, there are four Killing vectors $(\partial_t,\partial_y,k,l)\in\mathfrak{K}(M,g)$, with $\partial_t\in Z(\mathfrak{K}(M,g))$. Entries in the rightmost column of table 4 denote the isomorphism class of $\mathfrak{K}(M,g)$ in terms of a four-dimensional real Lie algebra, within the classification scheme of [77, 78]. Up to isomorphism, there exists a basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4\}$ such that the Lie algebra

- $\mathfrak{c}[\mathrm{X}]$ has non-zero Lie brackets $[\boldsymbol{e}_2,\boldsymbol{e}_3]=\boldsymbol{e}_1$, $[\boldsymbol{e}_2,\boldsymbol{e}_4]=-\boldsymbol{e}_3$, $[\boldsymbol{e}_3,\boldsymbol{e}_4]=\boldsymbol{e}_2$.
- $\mathbb{R}\oplus\mathfrak{b}[\mathrm{VIII}]$ has non-zero Lie brackets $[\boldsymbol{e}_1,\boldsymbol{e}_2]=-\boldsymbol{e}_3$, $[\boldsymbol{e}_1,\boldsymbol{e}_3]=-\boldsymbol{e}_2$, $[\boldsymbol{e}_2,\boldsymbol{e}_3]=\boldsymbol{e}_1$.
- $\mathbb{R}\oplus\mathfrak{b}[\mathrm{IX}]$ has non-zero Lie brackets $[\boldsymbol{e}_1,\boldsymbol{e}_2]=\boldsymbol{e}_3$, $[\boldsymbol{e}_1,\boldsymbol{e}_3]=-\boldsymbol{e}_2$, $[\boldsymbol{e}_2,\boldsymbol{e}_3]=\boldsymbol{e}_1$.

The classes $\mathfrak{b}[\mathrm{VIII}]$ and $\mathfrak{b}[\mathrm{IX}]$ correspond respectively to the simple real Lie algebras $\mathfrak{so}(2,1)$ and $\mathfrak{so}(3)$.

In each case, using (6.45), it is straightforward to check that the condition $\mathcal{L}_X F=0$ from (5.11) is satisfied identically, for all $X\in\mathfrak{K}(M,g)$. Consequently, the locally equivalent condition $\iota_X F=\mathrm{d}\rho_X$ from (5.3) defines each function $\rho_X$ in (5.2), up to the addition of an arbitrary constant.[6] We have already noted that $\rho_\Xi=-\kappa a$ while, up to an arbitrary constant, $\rho_{\partial_y}=\frac{1}{2}(a\omega-\partial_x f)$. For the remaining Killing vectors $k$ and $l$, the non-constant parts of $\rho_k$ and $\rho_l$ are displayed in table 5. Using this data, it is straightforward to compute the associated submaximal conformal symmetry superalgebras $\mathcal{S}$, which are displayed in the rightmost column of table 5, in terms of the notation defined in section 4. In each case, $\mathcal{R}\cong\mathfrak{u}(1)$ and we can take $[\mathfrak{K}(M,g),\mathcal{F}]=0$ in $\mathcal{S}$. For some non-zero $R\in\mathcal{R}$ and all $\epsilon_{\mathbb{C}}\in\mathcal{F}$, we can take $[R,\epsilon_{\mathbb{C}}]=i\epsilon_{\mathbb{C}}$ and $[\epsilon_{\mathbb{C}},\epsilon_{\mathbb{C}}^*]=\Xi$.

[6] Any such constant term in (5.2) can be set to zero in the $[\mathcal{B},\mathcal{F}]$ bracket for the conformal symmetry superalgebra $\mathcal{S}$ via an appropriate compensating constant R-symmetry contribution.

## 7 $d = 4$

### 7.1 Null tetrads

Let $(\xi, \theta, \chi)$ denote a *null tetrad* of vector fields on $(M, g)$, subject to the defining relations

$$\|\xi\|^2 = \|\theta\|^2 = 0 = g(\xi, \chi) = g(\theta, \chi) = g(\chi, \chi) \ , \qquad g(\xi, \theta) = g(\chi, \chi^*) = 1 \ . \tag{7.1}$$

The elements $\xi$ and $\theta$ are real while $\chi$ is complex. The relations (7.1) are preserved under the following transformations (which collectively generate $\mathrm{O}(3,1)$):

- $(\xi, \theta, \chi) \mapsto (\theta, \xi, \chi)$.
- $(\xi, \theta, \chi) \mapsto (\xi, \theta - \alpha^*\chi - \alpha\chi^* - |\alpha|^2\xi, \chi + \alpha\xi)$, for any $\alpha \in \mathbb{C}$.
- $(\xi, \theta, \chi) \mapsto (\xi, \theta, \mathrm{e}^{i\beta}\chi)$, for any $\beta \in \mathbb{R}$.
- $(\xi, \theta, \chi) \mapsto (\gamma\xi, \gamma^{-1}\theta, \chi)$, for any $\gamma \in \mathbb{R}\backslash\{0\}$.

It is convenient to express the metric and orientation tensor on $M$ as

$$g = \xi \otimes \theta + \theta \otimes \xi + \chi \otimes \chi^* + \chi^* \otimes \chi \ , \qquad \varepsilon = i\,\xi \wedge \theta \wedge \chi \wedge \chi^* \ , \tag{7.2}$$

in terms of the null tetrad.

### 7.2 Majorana spinors

The Clifford algebra $\mathrm{C}\ell(3,1) \cong \mathrm{Mat}_4(\mathbb{R})$ has a unique irreducible Majorana spinor representation that is isomorphic to $\mathbb{R}^4$. On the other hand, its complexification (the Dirac spinor representation) decomposes into a pair of inequivalent irreducible chiral spinor representations, each isomorphic to $\mathbb{C}^2$, associated with the two eigenspaces of $\mathbf{\Gamma}$ on which $\mathbf{\Gamma} = \pm\mathbf{1}$. The action of a subalgebra $\mathrm{Mat}_2(\mathbb{C}) < \mathrm{Mat}_4(\mathbb{R})$ on $\mathbb{C}^2$ which commutes with the complex structure $i\mathbf{\Gamma}$ defines the action of $\mathrm{C}\ell(3,1)$ on each chiral projection (the two chiral projections transform in complex conjugate representations).

Relative to the Clifford algebra basis (3.6), taking $\mathbf{\Gamma} = \frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\mathbf{\Gamma}_{\mu\nu\rho\sigma}$, it follows that

$$\mathbf{\Gamma}_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\mathbf{\Gamma}^{\rho\sigma}\mathbf{\Gamma} \ , \qquad \mathbf{\Gamma}_{\mu\nu\rho} = -i\,\varepsilon_{\mu\nu\rho\sigma}\mathbf{\Gamma}^{\sigma}\mathbf{\Gamma} \ , \qquad \mathbf{\Gamma}_{\mu\nu\rho\sigma} = i\,\varepsilon_{\mu\nu\rho\sigma}\mathbf{\Gamma} \ . \tag{7.3}$$

There exists on $\mathfrak{S}(M)$ a skewsymmetric bilinear form (3.8), with respect to which

$$\overline{\psi}_\pm\varphi_\pm = -\overline{\varphi}_\pm\psi_\pm \ , \quad \overline{\psi}_\pm\mathbf{\Gamma}_\mu\varphi_\mp = \overline{\varphi}_\mp\mathbf{\Gamma}_\mu\psi_\pm \ , \quad \overline{\psi}_\pm\mathbf{\Gamma}_{\mu\nu}\varphi_\pm = \overline{\varphi}_\pm\mathbf{\Gamma}_{\mu\nu}\psi_\pm = \mp\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\,\overline{\varphi}_\pm\mathbf{\Gamma}^{\rho\sigma}\psi_\pm \ , \tag{7.4}$$

for all $\psi_\pm, \varphi_\pm \in \mathfrak{S}_\pm(M)$. The associated Fierz identities are given by

$$\begin{aligned} \psi_\pm\,\overline{\varphi}_\pm &= \frac{1}{2}\left((\overline{\varphi}_\pm\psi_\pm) - \frac{1}{4}(\overline{\varphi}_\pm\mathbf{\Gamma}^{\mu\nu}\psi_\pm)\mathbf{\Gamma}_{\mu\nu}\right)\mathbf{P}_\pm \\ \psi_\pm\,\overline{\varphi}_\mp &= \frac{1}{2}(\overline{\varphi}_\mp\mathbf{\Gamma}^{\mu}\psi_\pm)\mathbf{\Gamma}_\mu\mathbf{P}_\mp \ . \end{aligned} \tag{7.5}$$

In terms of a unitary basis for $C\ell(3,1)$, it follows that under complex conjugation

$$(\overline{\psi}_+\varphi_+)^* = \overline{\psi}_-\varphi_- \ , \quad (\overline{\psi}_+\mathbf{\Gamma}_\mu\varphi_-)^* = \overline{\psi}_-\mathbf{\Gamma}_\mu\varphi_+ \ , \quad (\overline{\psi}_+\mathbf{\Gamma}_{\mu\nu}\varphi_+)^* = \overline{\psi}_-\mathbf{\Gamma}_{\mu\nu}\varphi_- \ . \tag{7.6}$$

For a given $\epsilon \in \mathfrak{S}(M)$, we define

$$\xi^\mu_\epsilon = \overline{\epsilon}\Gamma^\mu\epsilon = 2\,\overline{\epsilon}_-\Gamma^\mu\epsilon_+ \ , \qquad \zeta_{\mu\nu} = \overline{\epsilon}_+\Gamma_{\mu\nu}\epsilon_+ \ . \tag{7.7}$$

From (7.6), it follows that the vector field $\xi_\epsilon$ is real while the two-form $\zeta$ is complex and obeys $\zeta_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\zeta^{\rho\sigma}$. The vector field $\xi_\epsilon$ is nowhere vanishing only if $\epsilon$ is nowhere vanishing, which we shall assume henceforth. Furthermore, using (7.5), one obtains

$$\|\xi_\epsilon\|^2 = 0 \ , \qquad \xi^\mu_\epsilon\zeta_{\mu\nu} = 0 \ , \qquad \zeta_{\mu\rho}\zeta^{\nu\rho} = 0 \ , \qquad \zeta^*_{\mu\rho}\zeta^{\nu\rho} = \frac{1}{2}g_{\mu\rho}\xi^\rho_\epsilon\xi^\nu_\epsilon \ . \tag{7.8}$$

It is convenient to identify (7.7) in terms of the null tetrad introduced in section 7.1, such that $\xi_\epsilon = \xi$. The second identity in (7.8) then implies $\xi \wedge \zeta = 0$, whence $\zeta = \xi \wedge \tau$, in terms of the complex one-form $\tau = \iota_\theta\zeta$. The remaining identities in (7.8) fix $\tau = \frac{1}{\sqrt{2}}\chi^*$, i.e. $\zeta = \frac{1}{\sqrt{2}}\xi \wedge \chi^*$ in terms of the null tetrad one-forms.

Using (7.5), it is possible to express any $\psi \in \mathfrak{S}(M)$ in terms of the null tetrad and the reference spinor $\epsilon$ that defines $\xi$. In particular,

$$\begin{aligned}\psi_+ &= \alpha\epsilon_+ + \beta\boldsymbol{\theta}\epsilon_- \\ \psi_- &= \alpha^*\epsilon_- + \beta^*\boldsymbol{\theta}\epsilon_+ \ ,\end{aligned} \tag{7.9}$$

where $\alpha = 2\overline{\epsilon}_-\boldsymbol{\theta}\psi_+$ and $\beta = 2\overline{\epsilon}_+\psi_+$. Moreover, it is easily verified that $\boldsymbol{\chi}\epsilon_- = 0$ and $\boldsymbol{\chi}^*\epsilon_- = \sqrt{2}\epsilon_+$.

### 7.3 Petrov types

The Weyl tensor $W$ of $g$ may also be expressed in terms of the null tetrad and its non-trivial components are characterised by five complex functions

$$\begin{aligned}&\Psi_0 = W(\xi,\chi,\xi,\chi) \ , && \Psi_1 = W(\xi,\theta,\xi,\chi) \ , && \Psi_2 = W(\xi,\chi,\theta,\chi^*) \ , \\ &\Psi_3 = W(\xi,\theta,\theta,\chi^*) \ , && \Psi_4 = W(\theta,\chi^*,\theta,\chi^*) \ ,\end{aligned} \tag{7.10}$$

called *Weyl scalars*. The conformal class of $g$ at each point in $M$ may be classified as being one of the following six *Petrov types*

- *Type I.* $\Psi_0 = 0$.
- *Type II.* $\Psi_0 = \Psi_1 = 0$.
- *Type D.* $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$.
- *Type III.* $\Psi_0 = \Psi_1 = \Psi_2 = 0$.
- *Type N.* $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$.
- *Type O.* $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$.

We shall only be concerned with geometries which have the same Petrov type at each point, and thus refer to the Petrov type of $(M,g)$. If $(M,g)$ is type I then it is called *algebraically general*, otherwise it is called *algebraically special*. If $(M,g)$ is type O then it is locally conformally flat (i.e. $W = 0$).

### 7.4 Null conformal Killing vectors

The relations (7.1) constrain the action of the Levi-Civita connection $\nabla$ on the null tetrad one-forms such that

$$\begin{aligned}
\nabla_\mu \xi_\nu &= 2\mathrm{Re}(p)_\mu \xi_\nu - q_\mu \chi_\nu - q^*_\mu \chi^*_\nu \\
\nabla_\mu \theta_\nu &= -2\mathrm{Re}(p)_\mu \theta_\nu + r_\mu \chi_\nu + r^*_\mu \chi^*_\nu \\
\nabla_\mu \chi_\nu &= -r^*_\mu \xi_\nu + q^*_\mu \theta_\nu - 2i\mathrm{Im}(p)_\mu \chi_\nu \,,
\end{aligned} \tag{7.11}$$

in terms of data that it is convenient to assemble into three complex one-forms $(p, q, r)$ on $M$.

Given any null vector field $X$ on $(M, g)$, it is always possible to define a null tetrad with respect to which $X = \xi$. From the first line in (7.11), it follows that $\xi$ is a conformal Killing vector only if

$$\mathrm{Re}(p)_\theta = 0 \,, \quad q_\xi = 0 = q_{\chi^*} \,, \quad 2\mathrm{Re}(p)_{\chi^*} = q_\theta \,, \quad 2\mathrm{Re}(p)_\xi + q_\chi + q^*_{\chi^*} = 0 \,. \tag{7.12}$$

Furthermore, $\xi$ is a Killing vector only if (7.12) are satisfied with $\mathrm{Re}(p)_\xi = 0$ (whence, $q_\chi$ must be pure imaginary). In that case,

$$\mathrm{d}\xi = 2(q_\theta \chi + q^*_\theta \chi^*) \wedge \xi + 2q_\chi\, \chi \wedge \chi^* \,. \tag{7.13}$$

Any Killing vector $X$ on $(M, g)$ obeys

$$\nabla_\mu \nabla_\nu X_\rho = -R_{\nu\rho\mu\sigma} X^\sigma \,, \tag{7.14}$$

in terms of the components $R_{\mu\nu\rho\sigma}$ of the Riemann tensor of $g$. For a null Killing vector $X = \xi$, (7.14) and (7.13) imply

$$R_{\mu\nu}\xi^\mu \xi^\nu = \nabla_\mu \xi_\nu \nabla^\mu \xi^\nu = 2|q_\chi|^2 \geq 0 \,, \tag{7.15}$$

where $R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\nu\sigma}$ is the Ricci tensor. The geometry $(M, g)$ is *Einstein* if it obeys $R_{\mu\nu} = \Lambda g_{\mu\nu}$, for some $\Lambda \in \mathbb{R}$ (if $\Lambda = 0$ then $(M, g)$ is *Ricci-flat*). Whence, if $(M, g)$ is Einstein and admits a null Killing vector $\xi$ then $q_\chi = 0$ and $\mathrm{d}\xi = k \wedge \xi$ (where $k = 2(q_\theta \chi + q^*_\theta \chi^*)$), which implies $\xi \wedge \mathrm{d}\xi = 0$. Indeed $\xi \wedge \mathrm{d}\xi = 0$ only if $q_\chi = 0$ so any $(M, g)$ with a null Killing vector $\xi$ for which $\xi \wedge \mathrm{d}\xi \neq 0$ cannot be Einstein.

The three-form $\xi \wedge \mathrm{d}\xi$ is called the *twist* of $\xi$ and $\xi$ is said to be *twisting* if $\xi \wedge \mathrm{d}\xi \neq 0$ or *non-twisting* if $\xi \wedge \mathrm{d}\xi = 0$. Restricting to the subspace $\xi^\perp = \{X \in \mathfrak{K}(M, g) \,|\, g(X, \xi) = 0\}$ defines a Lie subalgebra of $\mathfrak{K}(M, g)$ only if $\xi$ is non-twisting, thus ensuring that the associated hypersurface in $(M, g)$ is integrable. Furthermore, if $\xi$ is non-twisting then substituting $\mathrm{d}\xi = k \wedge \xi$ into (7.14) with $X = \xi$ leads to several more useful properties. Contracting the resulting expression with $\xi$ on different indices and with the inverse metric implies

$$\nabla_\xi k^\mu = f \xi^\mu \,, \quad R_{\mu\rho\nu\sigma}\xi^\rho \xi^\sigma = \frac{1}{4}\|k\|^2 \xi_\mu \xi_\nu \,, \quad R_{\mu\nu}\xi^\nu = -\frac{1}{2}\left( \nabla_\nu k^\nu + \frac{1}{2}\|k\|^2 - f \right) \xi_\mu \,, \tag{7.16}$$

where $f = \theta_\mu \nabla_\xi k^\mu$ and $\|k\|^2 > 0$ if $k \neq 0$. Substituting (7.16) into the definition of the Weyl tensor (3.22) then implies

$$W_{\mu\nu\rho\sigma}\xi^\rho\xi^\sigma = -\frac{1}{2}\left(\frac{1}{3}R + \nabla_\rho k^\rho - f\right)\xi_\mu\xi_\nu \ . \tag{7.17}$$

Thus it follows that any $(M, g)$ with a non-twisting null Killing vector has $\Psi_0 = \Psi_1 = 0$ and is therefore algebraically special. Moreover, if any such geometry is Ricci-flat with $\Psi_2 = 0$ then $k = 0$ so $\xi$ must be parallel and $(M, g)$ is necessarily of Petrov type N or O. Whence, any Ricci-flat $(M, g)$ with a null Killing vector must be of Petrov type II, D, N or O.

Let us conclude this section by illustrating some of the properties above via the introduction of local coordinates $(u, v, x, y)$ on $(M, g)$. If $(M, g)$ admits a nowhere vanishing null Killing vector $\xi$ then we can take $\xi = \partial_v$ tangent to a family of null geodesics, with geodesic distance parameterised by the affine coordinate $v$. A convenient local form of the metric in these adapted coordinates is

$$g = 2G(\mathrm{d}u + \alpha)(\mathrm{d}v + \beta + \frac{1}{2}H(\mathrm{d}u + \alpha)) + E^2(\mathrm{d}x^2 + \mathrm{d}y^2) \,, \tag{7.18}$$

in terms of three real functions $G$, $H$ and $E$ and two real one-forms $\alpha = \alpha_x \mathrm{d}x + \alpha_y \mathrm{d}y$ and $\beta = \beta_x \mathrm{d}x + \beta_y \mathrm{d}y$. All of these components are functions only of $(u, x, y)$. The null tetrad one-forms are identified such that

$$\xi = G(\mathrm{d}u + \alpha) \ , \qquad \theta = \mathrm{d}v + \beta + \frac{1}{2}G^{-1}H\,\xi \ , \qquad \chi = \frac{1}{\sqrt{2}}E(\mathrm{d}x + i\mathrm{d}y) \ . \tag{7.19}$$

If $\xi$ is non-twisting then integrability of its associated hypersurface implies $\xi = f\mathrm{d}u$, for some function $f$ of $(u, x, y)$. Up to a redefinition of $G$, this allows us to fix $\alpha = 0$ in (7.18). Furthermore, if $\xi$ is parallel, then $\mathrm{d}\xi = 0$ so $f$ must be a function only of $u$ and we can fix $f = 1$, $\beta = 0$ and $E = 1$. Indeed, the existence of a parallel null vector on $(M, g)$ characterises a four-dimensional pp-wave, with local metric as in (7.30). In $d = 4$, the pp-wave metric (7.30) is of Petrov type N or O. It is type O only if the real function $H$ of coordinates $(u, x, y)$ obeys $\partial_x^2 H = \partial_y^2 H$ and $\partial_x\partial_y H = 0$. It is a plane wave only if $H$ is a quadratic function of $(x, y)$.

Up to local isometry, there exists a classification of 'physically admissible' Ricci-flat $(M, g)$ with a null Killing vector. Chapter 24.4-5 in [65] contains a detailed summary of the local metrics and their Killing vectors. From table 24.2 in [65], if $(M, g)$ is a Ricci-flat type N pp-wave then $1 \leq \dim \mathfrak{K}(M, g) \leq 6$ (within this class, $\dim \mathfrak{K}(M, g)$ never equals 4 and equals 5 or 6 only for plane waves). The remaining solutions of type II and D are summarised in table 24.1 of [65]. The local metrics in this subclass correspond to (7.18) with $\alpha = \beta = 0$, $G = -x$, $E^2 = x^{-1/2}$ and $\partial_x(x\partial_x H) + x\partial_y^2 H = 0$. These geometries all have $1 \leq \dim \mathfrak{K}(M, g) \leq 4$. Contained within this subclass are the type II 'van Stockum' solutions [66] (for $\partial_u H = 0$) and a static type D solution with $\dim \mathfrak{K}(M, g) = 4$ (for $H$ constant). We shall return to compute symmetry superalgebras for these geometries in section 7.6.

### 7.5 Charged twistor spinors

Let us now examine the implications of the existence of a charged twistor spinor $\epsilon$, which can be thought of locally as a Majorana spinor obeying (5.1) with respect to the action of the $\mathcal{R} = \mathfrak{u}(1)$ gauged connection

$$D_\mu \epsilon = \nabla_\mu \epsilon + i A_\mu \boldsymbol{\Gamma} \epsilon \,, \tag{7.20}$$

i.e. $D_\mu \epsilon_\pm = \nabla_\mu \epsilon_\pm \pm i A_\mu \epsilon_\pm$.

The first consequence is that $\xi_\epsilon$ defined in (7.7) is a conformal Killing vector. Let us now deconstruct the defining condition $D_\mu \epsilon = \frac{1}{4} \boldsymbol{\Gamma}_\mu \boldsymbol{D} \epsilon$ for charged twistor spinor $\epsilon$ in terms of the null tetrad. Identifying $\xi_\epsilon = \xi$ then using (7.9) and (7.11) implies

$$\begin{aligned} \nabla_\mu \epsilon_+ &= p_\mu \epsilon_+ + \frac{1}{\sqrt{2}} q_\mu \boldsymbol{\theta} \epsilon_- \\ \nabla_\mu \epsilon_- &= p_\mu^* \epsilon_- + \frac{1}{\sqrt{2}} q_\mu^* \boldsymbol{\theta} \epsilon_+ \,. \end{aligned} \tag{7.21}$$

It follows that the defining condition for charged twistor spinor $\epsilon$ is equivalent to the following conditions on $p$, $q$ and $A$:

$$p_\theta + i A_\theta = 0 = p_\chi + i A_\chi \,, \quad q_\xi = 0 = q_{\chi^*} \,, \quad p_{\chi^*} + i A_{\chi^*} = q_\theta \,, \quad p_\xi + i A_\xi = -q_\chi \,. \tag{7.22}$$

Since $\xi_\epsilon \in \mathfrak{C}(M, [g])$ for any charged twistor spinor $\epsilon$, it is straightforward to identify a subset of conditions in (7.22) with precisely the conditions in (7.12) required for $\xi$ to be a conformal Killing vector. The conditions in (7.12) describe eight real constraints on $\mathrm{Re}(p)$ and $q$ which are contained in the twelve real constraints on $p$, $q$ and $A$ in (7.22). The four remaining constraints in (7.22) are precisely sufficient to fix all four components of $A$, such that

$$A_\mu = \frac{i}{2} (p_\mu + q_\chi \theta_\mu - q_\theta \chi_\mu - p_\mu^* - q_{\chi^*}^* \theta_\mu + q_\theta^* \chi_\mu^*) \,. \tag{7.23}$$

Thus we conclude that if $(M, g)$ admits a nowhere vanishing null conformal Killing vector $X$, then there must exist a nowhere vanishing charged twistor spinor $\epsilon$ on $(M, g)$, with $X = \xi_\epsilon$, that is defined with respect to the gauged connection $A$ in (7.23). This characterisation of charged twistor spinors on a smooth orientable Lorentzian four-manifold was first obtained in [6] (see also [8]).

For any charged twistor spinor $\epsilon$, the twist of $\xi_\epsilon = \xi$ can be written

$$\xi \wedge \mathrm{d}\xi = \frac{4}{3} \rho_\xi * \xi \,. \tag{7.24}$$

in terms of the real function

$$\rho_\xi = -\frac{3i}{4} \bar{\epsilon} \boldsymbol{\Gamma} \boldsymbol{D} \epsilon = -\frac{3i}{4} (q_\chi - q_{\chi^*}^*) \,. \tag{7.25}$$

It follows that

$$\iota_\xi F = \mathrm{d}\rho_\xi \,, \tag{7.26}$$

using $F = \mathrm{d}A$ from (7.23). Whence, the condition (5.3) is satisfied identically for $X = \xi$. Furthermore, as noted below (7.12), if $\xi$ is a Killing vector then $q_\chi$ is pure imaginary. In that case, $\xi$ is non-twisting only if $q_\chi = 0$ which, from (7.25), occurs only if $\rho_\xi = 0$.

Differentiating the defining condition for $\epsilon$ gives

$$D_\mu \boldsymbol{D}\epsilon_\pm = 2K_{\mu\nu}\boldsymbol{\Gamma}^\nu \epsilon_\pm \pm \frac{4i}{3} F_{\mu\nu}\boldsymbol{\Gamma}^\nu \epsilon_\pm - \frac{1}{3}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\boldsymbol{\Gamma}^\nu \epsilon_\pm \ . \tag{7.27}$$

Combining (7.27) with (3.3) implies the integrability condition

$$\frac{1}{4} W_{\mu\nu\rho\sigma}\boldsymbol{\Gamma}^{\rho\sigma}\epsilon_\pm = \mp \frac{i}{3} F^\rho{}_{[\mu}\boldsymbol{\Gamma}_{\nu]\rho}\epsilon_\pm \mp \frac{i}{3} F_{\mu\nu}\epsilon_\pm - \frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\epsilon_\pm \ . \tag{7.28}$$

Identifying $\xi_\epsilon = \xi$, in terms of the null tetrad, one finds that (7.28) is equivalent to the following conditions on the Weyl scalars (7.10):

$$\Psi_0 = 0 \ , \quad \Psi_1 = -\frac{i}{3} F(\xi, \chi) \ , \quad \Psi_2 = -\frac{i}{3}\left( F(\xi,\theta) + F(\chi,\chi^*) \right) \ , \quad \Psi_3 = -iF(\theta,\chi^*) \ . \tag{7.29}$$

The Petrov type of $(M, g)$ therefore determines which components of $F$ must vanish identically. For the algebraically special geometries, we have

- Type II $\iff F(\xi,\chi) = 0$.
- Type D $\iff F(\xi,\chi) = F(\theta,\chi) = 0$ and $\Psi_4 = 0$.
- Type III $\iff F(\xi,\chi) = 0$, $F(\xi,\theta) = 0 = F(\chi,\chi^*)$.
- Type N $\iff F = 0$ and $W \neq 0$.
- Type O $\iff F = 0$ and $W = 0$.

These equivalences were first obtained in [19]. Notice that if $F = 0$ then, locally, $A = \mathrm{d}\lambda$ and the charged twistor spinor $\epsilon_\pm$ is equivalent to an ordinary twistor spinor $\mathrm{e}^{\pm i\lambda}\epsilon_\pm$.

Up to local conformal equivalence, the classification of Lorentzian four-manifolds which admit a nowhere vanishing twistor spinor $\epsilon$ is due to Lewandowski [73]. Any such $(M, g)$ must be of type N or O. The admissible type N geometries are distinguished by the twist of the conformal Killing vector $\xi_\epsilon$. If the twist of $\xi_\epsilon$ vanishes then $(M, g)$ is in the conformal class of a pp-wave with $\epsilon$ parallel. If the twist of $\xi_\epsilon$ does not vanish then $(M, g)$ is in the conformal class of a Fefferman space [74].

Thus, at least locally, any algebraically special $(M, g)$ which admits a charged twistor spinor $\epsilon$ with $F \neq 0$ is of Petrov type II, D or III with null conformal Killing vector $\xi_\epsilon$.

### 7.6 Conformal symmetry superalgebras

Let $(M, g)$ be any smooth orientable Lorentzian four-manifold equipped with a nowhere vanishing null conformal Killing vector $X$ and a $\mathfrak{C}(M,[g])$-invariant closed two-form $F$. As we have explained, $X$ defines a nowhere vanishing charged twistor spinor $\epsilon$ (with $X = \xi_\epsilon$) that is charged with respect to the connection $A$ in (7.23) whose curvature is $F = \mathrm{d}A$. From section 5, we recall that this data is precisely what is needed to assign to $(M,[g])$ a conformal symmetry superalgebra $\mathcal{S}$ with gauged $\mathcal{R} = \mathfrak{u}(1)$.

### 7.6.1 Type N and O cases

From the discussion in section 7.5, it follows that if $F = 0$ then $(M, g)$ must have Petrov type N (if $W \neq 0$) or type O (if $W = 0$). If $(M, g)$ is of type O (i.e. locally conformally flat) then $\mathfrak{C}(M, [g]) \cong \mathfrak{so}(4, 2) \cong \mathfrak{su}(2, 2)$ and $\mathcal{S} \cong \mathfrak{su}(2, 2|1)$ (i.e. the $\mathscr{N} = 1$, $d = 4$ case in table 1). If $(M, g)$ is of type N, with $\mathcal{F}$ non-trivial, then it is locally conformally equivalent to a pp-wave or a Fefferman space.

The generic pp-wave metric in $d = 4$ is of the form

$$g_{\mathrm{pp}} = 2\mathrm{d}u\mathrm{d}v + H(u, x, y)\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \,, \tag{7.30}$$

in terms of Brinkmann coordinates $(u, v, x, y)$ on $M$, where $H$ is an arbitrary real function of $(u, x, y)$. In these coordinates, $\xi = \partial_v$ is a null Killing vector of $(M, g_{\mathrm{pp}})$. It is easily verified that $(M, g_{\mathrm{pp}})$ is of type N unless $\partial_x^2 H = \partial_y^2 H$ and $\partial_x \partial_y H = 0$. We shall assume henceforth that $(M, g_{\mathrm{pp}})$ is of type N. Using that

$$\nabla_u \epsilon = \partial_u \epsilon - \frac{1}{4}((\partial_x H)\mathbf{\Gamma}_x + (\partial_y H)\mathbf{\Gamma}_y)\boldsymbol{\xi}\epsilon \,, \quad \nabla_v \epsilon = \partial_v \epsilon \,, \quad \nabla_x \epsilon = \partial_x \epsilon \,, \quad \nabla_y \epsilon = \partial_y \epsilon \,, \tag{7.31}$$

for any $\epsilon \in \mathfrak{S}(M)$, it follows that any twistor spinor $\epsilon \in \mathfrak{Z}(M, [g_{\mathrm{pp}}])$ is actually $\nabla$-parallel. Moreover, using (7.31), it follows that $\nabla_\mu \epsilon = 0$ only if $\partial_\mu \epsilon = 0$ with $\boldsymbol{\xi}\epsilon = 0$. Consequently, $\mathfrak{Z}(M, [g_{\mathrm{pp}}]) \cong \ker \boldsymbol{\xi}$ and $\dim \mathfrak{Z}(M, [g_{\mathrm{pp}}]) = \frac{1}{2}\dim \mathfrak{S}(M) = 2$ in $d = 4$. It follows that the $[\mathcal{F}, \mathcal{F}]$ bracket for the conformal symmetry superalgebra $\mathcal{S} = \mathcal{B} \oplus \mathcal{F}$ ascribed to a pp-wave, with $\mathcal{F} \cong \mathfrak{Z}(M, [g_{\mathrm{pp}}])$, can be taken to be

$$[\epsilon, \epsilon] = \xi \,, \tag{7.32}$$

for all $\epsilon \in \ker \boldsymbol{\xi}$ (see section 7.1 of [16] for the proof).

If the function $H$ in (7.30) is quadratic in $(x, y)$, we recover a special class of pp-waves called *plane waves* (see [76] for a comprehensive review). Let

$$g_{\mathrm{pw}} = 2\mathrm{d}u\mathrm{d}v + (\alpha(u)(x^2 - y^2) + 2\beta(u)xy + \gamma(u)(x^2 + y^2))\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \,, \tag{7.33}$$

denote the generic plane wave metric in $d = 4$, in terms of three real functions $\alpha$, $\beta$ and $\gamma$ of $u$. Having assumed that $(M, g_{\mathrm{pw}})$ is of type N, $\alpha$ and $\beta$ must not both be zero.

In addition to $\xi = \partial_v$, $\mathfrak{K}(M, g_{\mathrm{pw}})$ contains Killing vectors of the form

$$k(f) = f_x \partial_x + f_y \partial_y - (x\partial_u f_x + y\partial_u f_y)\partial_v \,, \tag{7.34}$$

where

$$\partial_u^2 f_x = (\alpha + \gamma)f_x + \beta f_y \,, \qquad \partial_u^2 f_y = \beta f_x - (\alpha - \gamma)f_y \,, \tag{7.35}$$

and each component $f_x$ and $f_y$ is a function only of $u$. It follows that

$$[\xi, k(f)] = 0 \,, \qquad [k(f), k(\tilde{f})] = \varpi(f, \tilde{f})\xi \,, \tag{7.36}$$

for any $f$ and $\tilde{f}$ obeying (7.35), where

$$\varpi(f, \tilde{f}) = -f_x \partial_u \tilde{f}_x - f_y \partial_u \tilde{f}_y + \tilde{f}_x \partial_u f_x + \tilde{f}_y \partial_u f_y \,, \tag{7.37}$$

| Class | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|
| I | $a$ | $b$ | $c$ |
| II | $u^{-2}a$ | $u^{-2}b$ | $u^{-2}c$ |
| III | $a\cos(2u)+b\sin(2u)$ | $b\cos(2u)-a\sin(2u)$ | $c$ |
| IV | $u^{-2}(a\cos(2\ln u)+b\sin(2\ln u))$ | $u^{-2}(b\cos(2\ln u)-a\sin(2\ln u))$ | $u^{-2}c$ |

**Table 6**. Data for plane wave metrics $g_{\rm pw}$ (7.33) in $d=4$ with $\dim\mathfrak{H}(M,g_{\rm pw})=7$. Up to local conformal equivalence, the classes I $\cong$ II and III $\cong$ IV.

is constant. Whence, the Lie algebra spanned by $\xi$ and all linearly independent $k(f)$ as in (7.34) (with $f$ solving (7.35)) is isomorphic to the five-dimensional Heisenberg Lie algebra $\mathfrak{heis}_2$. Furthermore, it follows that $\mathfrak{Z}(M,[g_{\rm pw}])$ is invariant under this $\mathfrak{heis}_2$. That is,

$$[\xi,\epsilon]=\mathcal{L}_\xi\epsilon=0\;,\qquad [k(f),\epsilon]=\mathcal{L}_{k(f)}\epsilon=0\,, \tag{7.38}$$

for all $\epsilon\in\mathfrak{Z}(M,[g_{\rm pw}])$ and $k(f)\in\mathfrak{K}(M,g_{\rm pw})$, in terms of the spinorial Lie derivative (3.11).

A generic plane wave also admits a proper homothetic conformal Killing vector proportional to $2v\partial_v+x\partial_x+y\partial_y$. In fact, $\vartheta=-\frac{1}{2}(2v\partial_v+x\partial_x+y\partial_y)$ obeys

$$[\vartheta,\xi]=\xi\;,\qquad [\vartheta,k(f)]=\frac{1}{2}k(f)\;,\qquad [\vartheta,\epsilon]=\hat{\mathcal{L}}_\vartheta\epsilon=\frac{1}{2}\epsilon\,, \tag{7.39}$$

for all $k(f)\in\mathfrak{K}(M,g_{\rm pw})$ and $\epsilon\in\mathfrak{Z}(M,[g_{\rm pw}])$, in terms of the Kosmann-Schwarzbach Lie derivative (3.14).

In $d=4$, any $(M,g)$ that is not of type O must have $\dim\mathfrak{C}(M,[g])\leq 7$ and, from theorem 5.1.3 in [60], $\dim\mathfrak{C}(M,[g])=7$ requires $(M,g)$ to be of type N. More precisely, it is known [75] that $\dim\mathfrak{C}(M,[g])=7$ only if $(M,g)$ is locally conformally equivalent to a homogeneous plane wave with an extra Killing vector. Up to local isometry, there are precisely four classes of such geometries defined by the particular $(\alpha,\beta,\gamma)$ displayed in table 6. In each case, $(M,g_{\rm pw})$ is not conformally flat provided the real numbers $a$ and $b$ are not both zero and is Ricci-flat only if the real number $c$ is zero. Class I, with constant $(\alpha,\beta,\gamma)$, defines a symmetric space. The coordinate transformation

$$(u,v,x,y)\mapsto(\mathrm{e}^u,v-\frac{1}{4}(x^2+y^2),\mathrm{e}^{u/2}x,\mathrm{e}^{u/2}y)\,, \tag{7.40}$$

identifies the classes I $\cong$ II and III $\cong$ IV, up to local conformal isometry. Details of the extra Killing vector $l\in\mathfrak{K}(M,g_{\rm pw})$ and its related brackets for each of these four classes is displayed in table 7. Entries in the two rightmost columns of table 7 apply to any $k(f)=k(f_x,f_y)$ as in (7.34) and $\epsilon\in\mathfrak{Z}(M,[g_{\rm pw}])$. In each case,

$$[\vartheta,l]=0\;. \tag{7.41}$$

Thus, at least locally, we have just two distinct conformal classes of type N metrics with $\dim\mathfrak{C}(M,[g])=7$. It is convenient to take the class I and III entries in table 6 to define



JHEP02(2016)008

| Class | $l$ | $[l,\xi]$ | $[l,k(f_x,f_y)]$ | $[l,\epsilon]=\hat{\mathcal{L}}_l\epsilon$ |
|---|---|---|---|---|
| I | $\partial_u$ | 0 | $k(\partial_u f_x, \partial_u f_y)$ | 0 |
| II | $u\partial_u - v\partial_v$ | $\xi$ | $k(u\partial_u f_x, u\partial_u f_y)$ | $\frac{1}{2}\epsilon$ |
| III | $\partial_u - x\partial_y + y\partial_x$ | 0 | $k(\partial_u f_x - f_y, \partial_u f_y + f_x)$ | $\frac{i}{2}\mathbf{\Gamma}\epsilon$ |
| IV | $u\partial_u - v\partial_v - x\partial_y + y\partial_x$ | $\xi$ | $k(u\partial_u f_x - f_y, u\partial_u f_y + f_x)$ | $\frac{1}{2}(\mathbf{1}+i\mathbf{\Gamma})\epsilon$ |

**Table 7**. The extra $l \in \mathfrak{K}(M,g_{\mathrm{pw}})$ and its brackets for the four classes in table 6.

a representative metric for each of these classes. In both cases, the homogeneous plane wave $(M,g_{\mathrm{pw}})$ has $\mathfrak{C}(M,[g_{\mathrm{pw}}]) = \mathfrak{H}(M,g_{\mathrm{pw}})$ and the null Killing vector $\xi \in Z(\mathfrak{K}(M,g_{\mathrm{pw}}))$. The submaximal conformal symmetry superalgebra $\mathcal{S} = \mathcal{B} \oplus \mathcal{F}$ associated with each class has $\mathcal{B} = \mathfrak{H}(M,g_{\mathrm{pw}}) \oplus \mathcal{R}$ and $\mathcal{F} = \mathfrak{Z}(M,[g_{\mathrm{pw}}]) \cong \ker \boldsymbol{\xi}$, where $\mathcal{R} = \mathfrak{u}(1)$. The explicit brackets for $\mathcal{S}$ are prescribed by (7.32), (7.36), (7.38), (7.39), (7.41) and table 7. For some non-zero $R \in \mathcal{R}$, we can take $[R,\epsilon] = i\mathbf{\Gamma}\epsilon$, for all $\epsilon \in \mathcal{F}$. It follows that the class I and III representative plane wave geometries both yield submaximal conformal symmetry superalgebras $\mathcal{S} \cong \mathcal{S}_2^{\triangleleft}(\mathfrak{H}(M,g_{\mathrm{pw}})|\mathcal{R})$, in the notation of section 4 (identifying $\vartheta = h$, $\xi = z$ and $R = r$).

### 7.6.2 Physically admissible type II and D cases

From the discussion in section 7.4, we recall that if $(M,g)$ is equipped with a null Killing vector $\xi$ (and $F \neq 0$) then it must be of Petrov type II or D with $\xi$ non-twisting in order to be Ricci-flat. Let us now take advantage of the classification in [65] of 'physically admissible' Ricci-flat geometries which admit a null Killing vector and compute their associated symmetry superalgebras. We shall also compute the conformal symmetry superalgebra for the unique admissible Ricci-flat geometry of Petrov type D. In general, the symmetry superalgebra $\mathcal{S}_\circ$ and conformal symmetry superalgebra $\mathcal{S}$ associated with any admissible geometry $(M,g)$ are isomorphic only if $\mathfrak{K}(M,g) = \mathfrak{C}(M,[g])$ and $\mathcal{F}_\circ = \mathcal{F}$. It is worth recalling from theorem 3 in [64] that any $(M,g)$ not of Petrov type N or O has $\mathfrak{C}(M,[g])$ conformally isometric. Of course, if $(M,g)$ is Ricci-flat with $\mathfrak{C}(M,[g]) \neq \mathfrak{K}(M,g)$, the particular geometry $(M,\tilde{g})$ for which $\mathfrak{C}(M,[g]) = \mathfrak{K}(M,\tilde{g})$ need not be Ricci-flat.

In terms of the local coordinates introduced at the end of section 7.4, each geometry $(M,g)$ within this class has a null Killing vector $\xi = \partial_v$ with respect to a metric $g$ of the form (7.18) with $\alpha = \beta = 0$, $G = -x$ and $E = x^{-1/4}$. The only non-trivial component of the Ricci tensor of $g$ is $R_{uu}$, which vanishes only if $\partial_x(x\partial_x H) + x\partial_y^2 H = 0$. By explicit calculation of the Weyl tensor of $g$, one finds that the Weyl scalars $\Psi_0 = \Psi_1 = \Psi_3 = 0$ while $\Psi_2 = \frac{1}{8}x^{-3/2}$. The remaining Weyl scalar $\Psi_4$ vanishes only if

$$\partial_x(x^{3/2}\partial_x H) = x^{3/2}\partial_y^2 H \ , \qquad \left(x\partial_x + \frac{3}{4}\right)\partial_y H = 0 \ . \tag{7.42}$$

Thus $(M,g)$ is indeed generically of type II, and is of type D only if $H$ obeys (7.42). If $(M,g)$ is of type D and Ricci-flat then $H$ is necessarily constant.



JHEP02(2016)008

Identification of the null tetrad one-forms as in (7.19) implies that the volume form (7.2) is $\varepsilon = x^{1/2}\,\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}x \wedge \mathrm{d}y$. The action of $\nabla$ on (7.19) can be defined via the the triple of complex one-forms $(p, q, r)$ in (7.11), which are given by

$$p = \frac{1}{4}x^{-1}\left(\mathrm{d}x - \frac{i}{2}\mathrm{d}y\right) ,$$
$$q = -\frac{1}{2\sqrt{2}}x^{1/4}\mathrm{d}u\ , \quad r = -\frac{1}{2\sqrt{2}}x^{-3/4}\left(\mathrm{d}v + \left(\frac{1}{2}H + x(\partial_x - i\partial_y)H\right)\mathrm{d}u\right) . \qquad (7.43)$$

Comparison with (7.12) and (7.25) confirms that $\xi$ is indeed a non-twisting Killing vector with $\rho_\xi = 0$.

Substituting (7.19) and (7.43) into (7.23) yields

$$A = \frac{3}{8}x^{-1}\mathrm{d}y\ . \qquad (7.44)$$

Whence, $F = -\frac{3}{8}x^{-2}\mathrm{d}x\wedge\mathrm{d}y$. Furthermore, substituting (7.19) and (7.43) into (7.21), and using (7.44), implies the gauged connection in (7.20) is given by

$$D_\mu = \partial_\mu + \frac{1}{4}x^{-1/2}\mathbf{\Gamma}_\mu\mathbf{\Gamma}_x\ . \qquad (7.45)$$

The identification $\xi = \xi_\epsilon = \partial_v$ implies $\mathbf{\Gamma}_v\epsilon_\pm = 0$ while, from (7.3), $i\mathbf{\Gamma}_{xy}\epsilon_\pm = \pm x^{-1/2}\epsilon_\pm$. Consequently, using $\boldsymbol{\chi}\epsilon_+ = \sqrt{2}\epsilon_-$ and $\boldsymbol{\chi}^*\epsilon_+ = 0$, it follows that $\mathbf{\Gamma}_x\epsilon_\pm = x^{-1/4}\epsilon_\mp$. Moreover, $\epsilon$ is a charged twistor spinor with respect to (7.45) only if $\epsilon$ is constant.

All the non-generic cases, where $\xi = \partial_v$ is not the only Killing vector in $\mathfrak{K}(M,g)$, are summarised in table 24.1 of [65]. The pertinent data is displayed in table 8 (the parameters $a, b \in \mathbb{R}\backslash\{0\}$ and real functions $f, h$ are constrained such that $\partial_x(x\partial_x H) + x\partial_y^2 H = 0$). Entries in the fourth column of table 8 denote the isomorphism class of $\mathfrak{K}(M,g)$. In addition to $\mathfrak{b}[\mathrm{VI}_0]$ that we encountered at the end of section 6.5.1, $\mathfrak{b}[\mathrm{II}]$ is isomorphic to the three-dimensional Heisenberg Lie algebra $\mathfrak{heis}_1$ while $\mathfrak{a}$ denotes the two-dimensional nonabelian Lie algebra.

Every $X \in \mathfrak{K}(M,g)$ is of the form

$$X = \alpha(u\partial_u - v\partial_v) + \beta\partial_u + \gamma\partial_y + h(u)\partial_v\ , \qquad (7.46)$$

for some choice of real numbers $\alpha$, $\beta$ and $\gamma$ and a real function $h$ of $u$. Thus, from (7.44), it follows that $\iota_X F = \frac{3}{8}\gamma x^{-2}\mathrm{d}x$. Whence, the condition (5.3) is indeed satisfied for any $X \in \mathfrak{K}(M,g)$ of the form (7.46), with $\rho_X = -\frac{3}{8}\gamma x^{-1}$, up to the addition of an arbitrary constant. Using this data, it is straightforward to compute the associated symmetry superalgebras $\mathcal{S}_\circ$ which are displayed in the rightmost column of table 8, in terms of the notation defined in section 4. In each case, $\mathcal{R} = \mathfrak{u}(1)$ and the even-odd bracket in (5.2) is of the form $[X, \epsilon] = \frac{1}{2}\alpha\epsilon$, for all $\epsilon \in \mathcal{F}_\circ$ and $X \in \mathfrak{K}(M,g)$ of the form (7.46). For some non-zero $R \in \mathcal{R}$ and all $\epsilon \in \mathcal{F}_\circ$, we can take $[R, \epsilon] = i\mathbf{\Gamma}\epsilon$ and $[\epsilon, \epsilon] = \xi$.

The type D geometry $(M, g)$ from table 8 has $\mathfrak{C}(M, [g]) = \mathfrak{H}(M, g)$. If $\vartheta \in \mathfrak{H}(M,g)/\mathfrak{K}(M,g)$ is normalised such that $[\vartheta, \xi] = \xi$ then

$$\vartheta = -v\partial_v - 2x\partial_x - 2y\partial_y\ . \qquad (7.47)$$

| Type | $H(u,x,y)$ | $\{X\}$ | $\mathfrak{K}(M,g)$ | $\mathcal{S}_\circ$ |
|---|---|---|---|---|
| II | $f(u,x)$ | $\{\partial_y, \partial_v\}$ | $\mathbb{R}^2$ | $\mathbb{R} \oplus \mathcal{S}_2^\circ(\mathbb{R}\vert\mathcal{R})$ |
| II | $f(ay-bu,x)$ | $\{b\partial_y + a\partial_u, \partial_v\}$ | $\mathbb{R}^2$ | $\mathbb{R} \oplus \mathcal{S}_2^\circ(\mathbb{R}\vert\mathcal{R})$ |
| II | $f(u,x) - y\partial_u h(u)$ | $\{\partial_y + h(u)\partial_v, \partial_v\}$ | $\mathbb{R}^2$ | $\mathbb{R} \oplus \mathcal{S}_2^\circ(\mathbb{R}\vert\mathcal{R})$ |
| II | $u^{-2} f(y - a\ln(u), x)$ | $\{a\partial_y + u\partial_u - v\partial_v, \partial_v\}$ | $\mathfrak{a}$ | $\mathcal{S}_2^\triangleleft(\mathfrak{a}\vert\mathcal{R})$ |
| II | $f(x)\mathrm{e}^{-2ay}$ | $\{\partial_y + a(u\partial_u - v\partial_v), \partial_u, \partial_v\}$ | $\mathfrak{b}[\mathrm{VI}_0]$ | $\mathcal{S}_2^\triangleleft(\mathfrak{b}[\mathrm{VI}_0]\vert\mathcal{R})$ |
| II | $f(x) + ay$ | $\{\partial_y - au\partial_v, \partial_u, \partial_v\}$ | $\mathfrak{b}[\mathrm{II}]$ | $\mathcal{S}_2^\circ(\mathfrak{b}[\mathrm{II}]\vert\mathcal{R})$ |
| II | $f(x)$ | $\{\partial_y, \partial_u, \partial_v\}$ | $\mathbb{R}^3$ | $\mathbb{R}^2 \oplus \mathcal{S}_2^\circ(\mathbb{R}\vert\mathcal{R})$ |
| D | $0$ | $\{\partial_y, u\partial_u - v\partial_v, \partial_u, \partial_v\}$ | $\mathbb{R} \oplus \mathfrak{b}[\mathrm{VI}_0]$ | $\mathbb{R} \oplus \mathcal{S}_2^\triangleleft(\mathfrak{b}[\mathrm{VI}_0]\vert\mathcal{R})$ |

**Table 8**. Data for Ricci-flat geometries with a null Killing vector and $F \neq 0$.

The five-dimensional Lie algebra $\mathfrak{H}(M,g)$ is thus defined by $\mathfrak{K}(M,g)$ together with the following non-trivial brackets

$$[\vartheta, \partial_v] = \partial_v \ , \qquad [\vartheta, \partial_y] = 2\partial_y \ . \tag{7.48}$$

From (7.47) and (7.44), it follows that $\vartheta$ obeys condition (5.3) with $\rho_\vartheta = \frac{3}{4}x^{-1}y$. In addition to the $[\mathfrak{K}(M,g), \mathcal{F}_\circ]$ brackets given above, (5.2) also prescribes

$$[\vartheta, \epsilon] = \frac{1}{2}\epsilon \,, \tag{7.49}$$

for all $\epsilon \in \mathcal{F}_\circ = \mathcal{F}$. Whence, the symmetry superalgebra $\mathcal{S}_\circ$ together with (7.48) and (7.49) define the conformal symmetry superalgebra $\mathcal{S} \cong \mathcal{S}_2^\triangleleft(\mathfrak{H}(M,g)|\mathcal{R})$ (in the notation of section 4) for the conformal class of the unique Ricci-flat type D geometry in table 8.

## 8 Summary of main results

The Lie algebra $\mathfrak{C}(M,[g])$ of conformal Killing vectors of any $d$-dimensional Lorentzian manifold $(M,g)$ that is not locally conformally flat has $\dim \mathfrak{C}(M,[g]) \leq 4 + \binom{d-1}{2}$ if $d > 3$ and $\dim \mathfrak{C}(M,[g]) \leq 4$ if $d = 3$ [60–62]. Non-trivial conformal classes of geometries for which these upper bounds are saturated will be referred to as being submaximal.

In this paper, we considered conformal symmetry superalgebras which can be ascribed to the conformal class of certain Lorentzian geometries, based on gauging the R-symmetry in the construction of [16, 17]. In particular, in $d = 3, 4$ with one-dimensional R-symmetry $\mathcal{R}$, up to local conformal isometry, we obtained a classification of submaximal Lorentzian geometries admitting a conformal symmetry superalgebra of the form $\mathcal{S} = \mathcal{B} \oplus \mathcal{F}$, with even part $\mathcal{B} = \mathfrak{C}(M,[g]) \oplus \mathcal{R}$ and odd part $\mathcal{F}$ containing (charged) twistor spinors.

In $d = 3$, we found in section 6.5.2 that any submaximal geometry with $\dim \mathfrak{C}(M,[g]) = 4$ which admits a conformal symmetry superalgebra $\mathcal{S}$ is locally conformally equivalent to one of the three types of stationary geometries $(M,g)$ displayed in table 9, in terms of

| Class | $g$ | $\mathfrak{K}(M,g)$ | $\mathcal{S}$ |
|---|---|---|---|
| IV.4 | $-(\mathrm{d}t + x\,\mathrm{d}y)^2 + \mathrm{d}x^2 + \mathrm{d}y^2$ | $\mathfrak{c}[\mathrm{X}]$ | $\mathcal{S}_2^{\circ}(\mathfrak{c}[\mathrm{X}]\|\mathcal{R})$ |
| IV.5 | $-(\mathrm{d}t + a\,\mathrm{e}^x\,\mathrm{d}y)^2 + \mathrm{d}x^2 + \mathrm{e}^{2x}\,\mathrm{d}y^2$ | $\mathbb{R}\oplus\mathfrak{b}[\mathrm{VIII}]$ | $\mathcal{S}_2^{\circ}(\mathbb{R}\|\mathcal{R})\oplus\mathfrak{b}[\mathrm{VIII}]$ |
| IV.6 | $-(\mathrm{d}t - a\cos x\,\mathrm{d}y)^2 + \mathrm{d}x^2 + (\sin x)^2\,\mathrm{d}y^2$ | $\mathbb{R}\oplus\mathfrak{b}[\mathrm{IX}]$ | $\mathcal{S}_2^{\circ}(\mathbb{R}\|\mathcal{R})\oplus\mathfrak{b}[\mathrm{IX}]$ |

**Table 9**. Data for submaximal classes with $\dim\mathfrak{C}(M,[g]) = 4$ in $d = 3$.

| Class | $\alpha$ | $\beta$ | $\gamma$ | $l$ |
|---|---|---|---|---|
| I | $a$ | $b$ | $c$ | $\partial_u$ |
| III | $a\cos(2u) + b\sin(2u)$ | $b\cos(2u) - a\sin(2u)$ | $c$ | $\partial_u - x\partial_y + y\partial_x$ |

**Table 10**. Data for submaximal classes with $\dim\mathfrak{C}(M,[g]) = 7$ in $d = 4$.

local coordinates $(t,x,y)$ on $M$. (The non-zero real number $a \neq \pm 1$ for class IV.5.) In each case, $\mathfrak{C}(M,[g]) = \mathfrak{K}(M,g)$ is spanned by four Killing vectors $(\partial_t, \partial_y, k, l)$ (details of $k$ and $l$ are given in table 4) and $\mathcal{R} \cong \mathfrak{u}(1)$. Charged twistor spinors in $\mathcal{F}$ are of the form $\epsilon_{\mathbb{C}} = (1+i\mathbf{\Gamma}_t)\epsilon$, for any constant Majorana spinor $\epsilon$. The brackets defining $\mathcal{S}$ can be written

$$[\mathfrak{K}(M,g), \epsilon_{\mathbb{C}}] = 0\ , \qquad [R, \epsilon_{\mathbb{C}}] = i\epsilon_{\mathbb{C}}\ , \qquad [\epsilon_{\mathbb{C}}, \epsilon_{\mathbb{C}}^*] = \partial_t\ , \tag{8.1}$$

for all $\epsilon_{\mathbb{C}} \in \mathcal{F}$, in terms of a non-zero $R \in \mathcal{R}$. The notation for $\mathcal{S}$ in table 9 encodes this information in the manner defined in section 4. The notation for the Lie algebras $\mathfrak{K}(M,g)$ in table 9 is defined at the end of section 6.5.2.

In $d = 4$, we found in section 7.6.1 that any submaximal geometry with $\dim\mathfrak{C}(M,[g]) = 7$ which admits a conformal symmetry superalgebra $\mathcal{S}$ is locally conformally equivalent to one of two types of homogeneous plane wave geometries $(M, g_{\mathrm{pw}})$, where

$$g_{\mathrm{pw}} = 2\mathrm{d}u\mathrm{d}v + (\alpha(u)(x^2-y^2) + 2\beta(u)xy + \gamma(u)(x^2+y^2))\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2\ , \tag{8.2}$$

in terms of local coordinates $(u,v,x,y)$ on $M$, with functions $\alpha$, $\beta$ and $\gamma$ as in table 10. In both cases, the real numbers $a$ and $b$ are not both zero. In each case, $\mathfrak{C}(M,[g_{\mathrm{pw}}]) = \mathfrak{H}(M,g_{\mathrm{pw}})$ is spanned by six Killing vectors $(\partial_v, k(f), l)$ and a proper homothetic conformal Killing vector $\vartheta = -\frac{1}{2}(2v\partial_v + x\partial_x + y\partial_y)$. The symbol $k(f)$ represents four linearly independent Killing vectors, and is defined in (7.34). Collectively, $(\partial_v, k(f))$ span a five-dimensional Heisenberg Lie algebra $\mathfrak{heis}_2$. The extra Killing vector $l$ is shown in table 10 and the Lie algebra of Killing vectors $\mathfrak{K}(M,g_{\mathrm{pw}}) \cong \mathbb{R}l \ltimes \mathfrak{heis}_2$. The R-symmetry $\mathcal{R} = \mathfrak{u}(1)$ and twistor spinors in $\mathcal{F}$ correspond to constant Majorana spinors in the kernel of $\mathbf{\Gamma}_v$. The brackets defining $\mathcal{S}$ can be written

$$[\mathfrak{K}(M,g), \epsilon] = 0\ , \qquad [\vartheta, \epsilon] = \frac{1}{2}\epsilon\ , \qquad [R, \epsilon] = i\mathbf{\Gamma}\epsilon\ , \qquad [\epsilon, \epsilon] = \partial_v\ , \tag{8.3}$$

for all $\epsilon \in \mathcal{F}$, in terms of a non-zero $R \in \mathcal{R}$. In both cases, it follows that $\mathcal{S} \cong \mathcal{S}_2^{\lhd}(\mathfrak{H}(M,g_{\mathrm{pw}})|\mathcal{R})$, in the notation of section 4.

## References

[1] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [arXiv:1105.0689] [INSPIRE].

[2] B. Jia and E. Sharpe, *Rigidly supersymmetric gauge theories on curved superspace*, *JHEP* **04** (2012) 139 [arXiv:1109.5421] [INSPIRE].

[3] H. Samtleben and D. Tsimpis, *Rigid supersymmetric theories in* 4*D Riemannian space*, *JHEP* **05** (2012) 132 [arXiv:1203.3420] [INSPIRE].

[4] C. Klare, A. Tomasiello and A. Zaffaroni, *Supersymmetry on curved spaces and holography*, *JHEP* **08** (2012) 061 [arXiv:1205.1062] [INSPIRE].

[5] T.T. Dumitrescu, G. Festuccia and N. Seiberg, *Exploring curved superspace*, *JHEP* **08** (2012) 141 [arXiv:1205.1115] [INSPIRE].

[6] D. Cassani, C. Klare, D. Martelli, A. Tomasiello and A. Zaffaroni, *Supersymmetry in Lorentzian curved spaces and holography*, *Commun. Math. Phys.* **327** (2014) 577 [arXiv:1207.2181] [INSPIRE].

[7] J.T. Liu, L.A. Pando Zayas and D. Reichmann, *Rigid supersymmetric backgrounds of minimal off-shell supergravity*, *JHEP* **10** (2012) 034 [arXiv:1207.2785] [INSPIRE].

[8] P. de Medeiros, *Rigid supersymmetry, conformal coupling and twistor spinors*, *JHEP* **09** (2014) 032 [arXiv:1209.4043] [INSPIRE].

[9] T.T. Dumitrescu and G. Festuccia, *Exploring curved superspace (II)*, *JHEP* **01** (2013) 072 [arXiv:1209.5408] [INSPIRE].

[10] A. Kehagias and J.G. Russo, *Global supersymmetry on curved spaces in various dimensions*, *Nucl. Phys.* **B 873** (2013) 116 [arXiv:1211.1367] [INSPIRE].

[11] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *Supersymmetric field theories on three-manifolds*, *JHEP* **05** (2013) 017 [arXiv:1212.3388] [INSPIRE].

[12] D. Martelli, A. Passias and J. Sparks, *The supersymmetric NUTs and bolts of holography*, *Nucl. Phys.* **B 876** (2013) 810 [arXiv:1212.4618] [INSPIRE].

[13] H. Samtleben, E. Sezgin and D. Tsimpis, *Rigid* 6*sD supersymmetry and localization*, *JHEP* **03** (2013) 137 [arXiv:1212.4706] [INSPIRE].

[14] S.M. Kuzenko, *Symmetries of curved superspace*, *JHEP* **03** (2013) 024 [arXiv:1212.6179] [INSPIRE].

[15] K. Hristov, A. Tomasiello and A. Zaffaroni, *Supersymmetry on three-dimensional lorentzian curved spaces and black hole holography*, *JHEP* **05** (2013) 057 [arXiv:1302.5228] [INSPIRE].

[16] P. de Medeiros and S. Hollands, *Conformal symmetry superalgebras*, *Class. Quant. Grav.* **30** (2013) 175016 [arXiv:1302.7269] [INSPIRE].

[17] P. de Medeiros and S. Hollands, *Superconformal quantum field theory in curved spacetime*, *Class. Quant. Grav.* **30** (2013) 175015 [arXiv:1305.0499] [INSPIRE].

JHEP02(2016)008

[18] D. Martelli and A. Passias, *The gravity dual of supersymmetric gauge theories on a two-parameter deformed three-sphere*, Nucl. Phys. **B 877** (2013) 51 [arXiv:1306.3893] [INSPIRE].

[19] D. Cassani and D. Martelli, *Supersymmetry on curved spaces and superconformal anomalies*, JHEP **10** (2013) 025 [arXiv:1307.6567] [INSPIRE].

[20] L.F. Alday, D. Martelli, P. Richmond and J. Sparks, *Localization on three-manifolds*, JHEP **10** (2013) 095 [arXiv:1307.6848] [INSPIRE].

[21] S.M. Kuzenko, *Super-Weyl anomalies in $N = 2$ supergravity and (non)local effective actions*, JHEP **10** (2013) 151 [arXiv:1307.7586] [INSPIRE].

[22] C. Klare and A. Zaffaroni, *Extended supersymmetry on curved spaces*, JHEP **10** (2013) 218 [arXiv:1308.1102] [INSPIRE].

[23] Y. Pan, *Rigid supersymmetry on 5-dimensional Riemannian manifolds and contact geometry*, JHEP **05** (2014) 041 [arXiv:1308.1567] [INSPIRE].

[24] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *The geometry of supersymmetric partition functions*, JHEP **01** (2014) 124 [arXiv:1309.5876] [INSPIRE].

[25] C. Closset and I. Shamir, *The $\mathcal{N} = 1$ chiral multiplet on $T^2 \times S^2$ and supersymmetric localization*, JHEP **03** (2014) 040 [arXiv:1311.2430] [INSPIRE].

[26] N.S. Deger, A. Kaya, H. Samtleben and E. Sezgin, *Supersymmetric warped AdS in extended topologically massive supergravity*, Nucl. Phys. **B 884** (2014) 106 [arXiv:1311.4583] [INSPIRE].

[27] S.M. Kuzenko, U. Lindström, M. Roček, I. Sachs and G. Tartaglino-Mazzucchelli, *Three-dimensional $\mathcal{N} = 2$ supergravity theories: From superspace to components*, Phys. Rev. **D 89** (2014) 085028 [arXiv:1312.4267] [INSPIRE].

[28] D. Cassani and D. Martelli, *The gravity dual of supersymmetric gauge theories on a squashed $S^1 \times S^3$*, JHEP **08** (2014) 044 [arXiv:1402.2278] [INSPIRE].

[29] L. Di Pietro, M. Dine and Z. Komargodski, *(Non-)decoupled supersymmetric field theories*, JHEP **04** (2014) 073 [arXiv:1402.3385] [INSPIRE].

[30] S.M. Kuzenko and G. Tartaglino-Mazzucchelli, *$N = 4$ supersymmetric Yang-Mills theories in $AdS_3$*, JHEP **05** (2014) 018 [arXiv:1402.3961] [INSPIRE].

[31] T. Anous, D.Z. Freedman and A. Maloney, *De Sitter supersymmetry revisited*, JHEP **07** (2014) 119 [arXiv:1403.5038] [INSPIRE].

[32] Y. Imamura and H. Matsuno, *Supersymmetric backgrounds from 5D $N = 1$ supergravity*, JHEP **07** (2014) 055 [arXiv:1404.0210] [INSPIRE].

[33] D. Farquet, J. Lorenzen, D. Martelli and J. Sparks, *Gravity duals of supersymmetric gauge theories on three-manifolds*, arXiv:1404.0268 [INSPIRE].

[34] L.F. Alday, M. Fluder, P. Richmond and J. Sparks, *Gravity dual of supersymmetric gauge theories on a squashed five-sphere*, Phys. Rev. Lett. **113** (2014) 141601 [arXiv:1404.1925] [INSPIRE].

[35] B. Assel, D. Cassani and D. Martelli, *Localization on Hopf surfaces*, JHEP **08** (2014) 123 [arXiv:1405.5144] [INSPIRE].

[36] L.F. Alday, M. Fluder, C.M. Gregory, P. Richmond and J. Sparks, *Supersymmetric gauge theories on squashed five-spheres and their gravity duals*, JHEP **09** (2014) 067 [arXiv:1405.7194] [INSPIRE].

[37] S.M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, *Symmetries of curved superspace in five dimensions*, JHEP **10** (2014) 175 [arXiv:1406.0727] [INSPIRE].

[38] D. Farquet and J. Sparks, *Wilson loops on three-manifolds and their M2-brane duals*, JHEP **12** (2014) 173 [arXiv:1406.2493] [INSPIRE].

[39] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, *Properties of conformal supergravity*, Phys. Rev. **D 17** (1978) 3179 [INSPIRE].

[40] E.S. Fradkin and A.A. Tseytlin, *Conformal supergravity*, Phys. Rept. **119** (1985) 233 [INSPIRE].

[41] P. van Nieuwenhuizen, *$D = 3$ conformal supergravity and Chern-Simons terms*, Phys. Rev. **D 32** (1985) 872 [INSPIRE].

[42] M. Roček and P. van Nieuwenhuizen, *$N \geq 2$ supersymmetric Chern-Simons terms as $D = 3$ extended conformal supergravity*, Class. Quant. Grav. **3** (1986) 43 [INSPIRE].

[43] J.M. Figueroa-O'Farrill, P. Meessen and S. Philip, *Supersymmetry and homogeneity of M-theory backgrounds*, Class. Quant. Grav. **22** (2005) 207 [hep-th/0409170] [INSPIRE].

[44] J.M. Figueroa-O'Farrill, E. Hackett-Jones and G. Moutsopoulos, *The Killing superalgebra of ten-dimensional supergravity backgrounds*, Class. Quant. Grav. **24** (2007) 3291 [hep-th/0703192] [INSPIRE].

[45] J. Figueroa-O'Farrill, *A Geometric construction of the exceptional Lie algebras $F_4$ and $E_8$*, Commun. Math. Phys. **283** (2008) 663 [arXiv:0706.2829] [INSPIRE].

[46] J. Figueroa-O'Farrill, E. Hackett-Jones, G. Moutsopoulos and J. Simon, *On the maximal superalgebras of supersymmetric backgrounds*, Class. Quant. Grav. **26** (2009) 035016 [arXiv:0809.5034] [INSPIRE].

[47] J.M. Figueroa-O'Farrill, *The homogeneity conjecture for supergravity backgrounds*, J. Phys. Conf. Ser. **175** (2009) 012002 [arXiv:0812.1258] [INSPIRE].

[48] J. Figueroa-O'Farrill and N. Hustler, *The homogeneity theorem for supergravity backgrounds*, JHEP **10** (2012) 014 [arXiv:1208.0553] [INSPIRE].

[49] J. Figueroa-O'Farrill and N. Hustler, *The homogeneity theorem for supergravity backgrounds II: the six-dimensional theories*, JHEP **04** (2014) 131 [arXiv:1312.7509] [INSPIRE].

[50] K. Habermann, *The twistor equation on Riemannian manifolds*, J. Geom. Phys. **7** (1990) 469.

[51] C. Duval and P.A. Horvathy, *On Schrödinger superalgebras*, J. Math. Phys. **35** (1994) 2516 [hep-th/0508079] [INSPIRE].

[52] F. Klinker, *Supersymmetric Killing Structures*, Commun. Math. Phys. **255** (2005) 419.

[53] H. Rajaniemi, *Conformal Killing spinors in supergravity and related aspects of spin geometry*, Ph.D. thesis, University of Edinburgh, Edinburgh, U.K. (2006).

[54] H. Baum, *Conformal Killing spinors and special geometric structures in Lorentzian geometry: a survey*, math/0202008 [INSPIRE].

[55] H. Baum and F. Leitner, *The twistor equation in Lorentzian spin geometry*, math/0305063.



JHEP02(2016)008

[56] H. Baum, *Holonomy groups of Lorentzian manifolds — A status report*, in *Global differential geometry*, C. Bär, J. Lohkamp and M. Schwarz et al., Springer Proceedings in Mathematics, Springer, Germany (2012).

[57] F. Leitner, *Conformal Killing forms with normalization condition*, *Rend. Circ. Mat. Palermo, suppl. Ser II* **75** (2005) 279.

[58] H. Baum, *Conformal Killing spinors and the holonomy problem in Lorentzian geometry - a survey of new results*, in *Symmetries and overdetermined systems of partial differential equations*, M. Eastwood and W. Miller eds., IMA Volumes in Mathematics, Springer, Germany (2008).

[59] W. Nahm, *Supersymmetries and their Representations*, *Nucl. Phys.* **B 135** (1978) 149 [INSPIRE].

[60] B. Kruglikov and D. The, *The gap phenomenon in parabolic geometries*, arXiv:1303.1307.

[61] B. Kruglikov and V. Matveev, *Submaximal metric projective and metric affine structures*, arXiv:1304.4426.

[62] B. Doubrov and D. The, *Maximally degenerate Weyl tensors in Riemannian and Lorentzian signatures*, arXiv:1305.3499. .

[63] G. Kručkovič, *Classification of three-dimensional Riemannian spaces according to groups of motions*, *Usp. Matem. Nauk (N.S.)* **9** (1954) 3.

[64] L. Defrise-Carter, *Conformal groups and conformally equivalent isometry groups*, *Commun. Math. Phys.* **40** (1975) 273.

[65] H. Stephani et al., *Exact solutions of Einstein's field equations*, 2nd edition, Cambridge University Press, Cambridge U.K. (2009).

[66] W.J. van Stockum, *The gravitational field of a distribution of particles rotating about an axis of symmetry*, *Proc. Roy. Soc. Edinburgh* **A 57** (1938) 135, see section 20.4 and 21.2.

[67] D.V. Alekseevsky and V. Cortés, *Classification of $N$-(super)-extended Poincaré algebras and bilinear invariants of the spinor representation of* $\mathrm{Spin}(p,q)$, math/9511215.

[68] D.V. Alekseevsky, V. Cortes, C. Devchand and A. Van Proeyen, *Polyvector super-Poincaré algebras*, *Commun. Math. Phys.* **253** (2004) 385 [hep-th/0311107] [INSPIRE].

[69] A. Lichnerowicz, *Spineurs harmoniques*, *C. R. Acad. Sci. Paris* **257** (1963) 7.

[70] Y. Kosmann, *Dérivées de Lie des spineurs*, *Ann. Mat. Pura Appl.* **91** (1972) 317.

[71] J.P. Bourguignon and P. Gauduchon, *Spinors, Dirac operators and metric variations* (in French), *Commun. Math. Phys.* **144** (1992) 581 [INSPIRE].

[72] K. Habermann, *The graded algebra and the lie derivative of spinor fields related to the twistor equation*, *J. Geom. Phys.* **18** (1996) 131.

[73] J. Lewandowski, *Twistor equation in a curved space-time*, *Class. Quant. Grav.* **8** (1991) L11 [INSPIRE].

[74] C. Fefferman, *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains*, *Ann. Math.* **103** (1976) 395.

[75] A.J. Keane and B.O.J. Tupper, *Conformal symmetry classes for pp-wave spacetimes*, *Class. Quant. Grav.* **21** (2004) 2037 [arXiv:1308.1683] [INSPIRE].




JHEP02(2016)008

[76] M. Blau, *Plane waves and Penrose limits — Gravity and string theory group*, http://www.blau.itp.unibe.ch/lecturesPP.pdf.

[77] L. Bianchi, *Sugli spazii a tre dimensioni che ammettono un gruppo continuo di movimenti*, *Soc. Ital. Sci. Mem. di Mat.* **11** (1898) 267.

[78] J. Patera, R. Sharp, P. Winternitz and H. Zassenhaus, *Invariants of real low dimension Lie algebras*, *J. Math. Phys.* **17** (1976) 986.

[79] N.L. Matiadou and A. Fellouris, *Classification of the five-dimensional Lie superalgebras over the real numbers*, *Int. J. Theor. Phys.* **46** (2007) 451 [inSPIRE].