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Gravitational waves from tachyonic preheating

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Contents

1 Units and conventions

Throughout this thesis, we will make use of the units, conventions and definitions stated here. They are for the most part adopted from [1].

We will use the units

$$\hbar = c = 1,$$

such that

$$[\text{lenght}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}.$$

For scalar quantities we adopt the following short notation for the derivatives.

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu},$$

and for a tensor of any order, e.g. $\xi_{\alpha\beta}$

$$\xi_{\alpha\beta,\mu} \equiv \frac{\partial \xi_{\alpha\beta}}{\partial x^\mu}, \quad \xi_{\alpha\beta,}{}^{\mu} = \frac{\partial \xi_{\alpha\beta}}{\partial x_\mu}. \tag{1.1}$$

For the Fourier transforms we follow these conventions

$$f(\boldsymbol{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} f(\boldsymbol{k}, \omega), \tag{1.2}$$

$$f(\boldsymbol{k}, \omega) = \int d^3x \; d\omega \; e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} f(\boldsymbol{x}, t), \tag{1.3}$$

where we for simplicity have used the same symbol for both configuration space and Fourier space.

For the metric tensor $g_{\mu\nu}$ we use the *mostly negative* signature $(1, -1, -1, -1)$.

2 Introduction

Most models of the universe, which are in agreement with current observations, include some form of extremely rapid expansion of space at a very early stage. This is called *cosmological inflation*, or just *inflation*.

Most models of inflation include one or more scalar fields. Before 2012 we did not know if scalar fields really do exist. It has been predicted by the standard model of particle physics in order to explain the masses of elementary particles. This is the so-called *Higgs field*, for which the corresponding particle, the *Higgs boson*, was detected at CERN in July 2012 [2][3].

Scalar fields have been the key ingredients in cosmological inflation models since the early 1980's, and the discovery of the Higgs particle provides new confidence both for cosmologist and particle physicists.

The simplest models of inflation involves only one scalar field called the *inflaton*. One obvious candidate could be the Higgs field itself as proposed in [4]. However, in this thesis, we will focus on a model of two coupled fields, called *hybrid inflation*.

When inflation ends, the universe is left in an empty and cold state, and (for a very short while) practically all of the energy in the universe we observe today is stored in the inflaton field(s) as potential energy. As the field rapidly decays, a huge amount of particles and radiation is created. This is the origin of the radiation and matter content of the universe today.

Most, if not all models of cosmic inflation predict generation of gravitational waves. Unlike electromagnetic waves, GW's propagate relatively undisturbed through space after they are created, and thus provides us with a way to probe the universe at very early times, right after inflation has ended. Now, the predicted GW spectrum due to inflation is model-dependent, and the observed spectrum should tell us if a specific model is possible or not. And in case neither model fits the data, we might even have to reconsider inflation altogether.

3 Einstein's Field Equations

The purpose of this section is to show how the energy momentum tensor comes about in Einstein's field equations, and how it relates to a physical source. Through this we obtain a general definition of the energy momentum tensor, which we will later use as a link between the presence of scalar fields and perturbations in the space time metric.

3.1 The Einstein-Hilbert Action and deduction of the vacuum field equations

The Einstein equations of empty space can be deduced from the principle of least action as explained in [5], by the variational principle,

$$\delta S_G = 0, \tag{3.1}$$

where S_G is the Einstein-Hilbert action related to gravitation,

$$S_G = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R. \tag{3.2}$$

where R is the Ricci tensor, $g \equiv det(g^{\mu\nu})$ and $\kappa = 8\pi G$. Note that we have left out the cosmological constant Λ. One reason is that it would be negligible during inflation, another reason is that it could be incorporated into the source term anyway. Varying S_G we get

$$\delta S_G = \frac{1}{2\kappa} \int d^4x \; (g^{\mu\nu}\sqrt{-g}\delta R_{\mu\nu} + R_{\mu\nu}\delta[g^{\mu\nu}\sqrt{-g}]). \tag{3.3}$$

By the arguments presented in [5], the first term takes the form of a total divergence, and thus, by Stoke's Theorem vanishes under the assumption that the metric and its derivatives are zero at the boundary of our integration region. Thus, what remains is the second term where

$$\delta[g^{\mu\nu}\sqrt{-g}] = \delta g^{\mu\nu}\sqrt{-g} + g^{\mu\nu}\delta\sqrt{-g}. \tag{3.4}$$

For any matrix we have [6, p.115]

$$det(M) = e^{Tr(lnM)}, \tag{3.5}$$

and so

$$\delta det(M) = det(M)Tr(M^{-1}\delta M). \tag{3.6}$$

We thus have

$$\delta g = g g^{\alpha\beta}\delta g_{\alpha\beta}, \tag{3.7}$$

and so

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}. \tag{3.8}$$

To express (3.8) in terms of $\delta g^{\alpha\beta}$ we observe that

$$g^{\mu\nu}g_{\nu\beta} = \delta^{\mu}{}_{\beta}. \tag{3.9}$$

But $\delta(\delta^{\mu}{}_{\beta}) = 0$, and so

$$0 = \delta(g^{\mu\nu}g_{\nu\beta}) \tag{3.10}$$
$$= \delta g^{\mu\nu}g_{\nu\beta} + g^{\mu\nu}\delta g_{\nu\beta} \tag{3.11}$$
$$= g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu} + g_{\alpha\mu}g^{\mu\nu}\delta_{\nu\beta}, \tag{3.12}$$

which gives

$$\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}. \tag{3.13}$$

Thus we obtain

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}. \tag{3.14}$$

Substituting (3.14) into (3.4) we get

$$\delta[g^{\mu\nu}\sqrt{-g}] = \delta g^{\mu\nu}\sqrt{-g} - \frac{1}{2}g^{\mu\nu}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} \tag{3.15}$$

Now, with the first term of (3.3) vanishing, and substituting (3.15) into the second term we obtain

$$\delta S_G = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}. \tag{3.16}$$

With $\delta S_G = 0$ for arbitrary $\delta g^{\mu\nu}$, we get

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0, \tag{3.17}$$

which are Einstein's field equations for a vacuum.

3.2 The definition of the energy momentum tensor through addition of a matter/energy term

We can now include matter and energy by adding the corresponding action S_M to S_G, and use the variational principle on the sum of them

$$\delta(S_G + S_M) = \delta S_G + \delta S_M = 0, \tag{3.18}$$

of which the first term is given by (3.16), and the second is the subject of this section.

The action integral for matter and energy is

$$S_M = \int d^4x \sqrt{-g}\,\mathcal{L}_M, \tag{3.19}$$

and so

$$\delta S_M = \int d^4x \delta[\sqrt{-g}\,\mathcal{L}_M]. \tag{3.20}$$

Since the Lagrangian in general depends on both the metric and its derivatives we have

$$\delta[\sqrt{-g}\,\mathcal{L}_M] = \frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\delta g^{\mu\nu}{}_{,\lambda}, \tag{3.21}$$

which can be written as

$$\delta[\sqrt{-g}\,\mathcal{L}_M] = \frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}}\delta g^{\mu\nu} - \left\{\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\right\}_{,\lambda}\delta g^{\mu\nu} + (\text{a total divergence term}). \tag{3.22}$$

Thus

$$\delta S_M = \int d^4x \left[\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}} - \left\{\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\right\}_{,\lambda}\right]\delta g^{\mu\nu} \tag{3.23}$$

The variation of the total action thus becomes

$$0 = \delta(S_G + S_M) = \int d^4x \left[\frac{\sqrt{-g}}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \left(\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}} - \left\{\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\right\}_{,\lambda}\right)\right]\delta g^{\mu\nu}, \tag{3.24}$$

which leads to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{2\kappa}{\sqrt{-g}}\left[\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}} - \left\{\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\right\}_{,\lambda}\right]. \tag{3.25}$$

Now, with the definition

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left[\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}} - \left\{\frac{\partial[\sqrt{-g}\,\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\right\}_{,\lambda}\right], \tag{3.26}$$

we have

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.27}$$

which is the gravitational field equations for the general theory of relativity. κ is a proportionality constant determined by comparison with Newton's law of gravitation in the Newtonian limit and is

$$\kappa = \frac{8\pi G}{c^4} = 8\pi G, \quad (\text{for } c = 1). \tag{3.28}$$

We will use the results from this section when we take a closer look at the ideas of inflation in the next section.

4 Inflation

Cosmological inflation refer to an epoch of the evolution of the universe in which space undergoes a rapid exponential expansion. It was originally hypothesized by Alan Guth in the early 1980's [7], and solves the *horizon* and *flatness* problem. The key points of the horizon problem is that the universe at large scales is observed to be incredible homogeneous and isotropic. However, by the conventional big bang hypothesis, it should not have been. This is because regions of the sky we observe today, which have a separation of more than about two degrees, could not have been in causal contact before decoupling. However, the nearly isotropic cosmic microwave background (CMB) suggests that the whole observable universe has at some time been in causal contact.

The flatness problem is that in order for the universe to be as flat as we observe it to be, it must have been extremely flat in the very early universe. This is a problem because the universe therefore seems to be extremely fine tuned, which does not seem natural.

We are not sure that our current understanding of physics will hold when the universe was very close to the initial singularity. In fact we expect current physical laws to not have validity for energies at the order of the Planck energy or above. At Planck scales quantum gravity effects come into play and and therefore theory including quantum effects might offer a solution. As Alan Guth suggests in [7], one could for now choose to accept the initial conditions, and wait for a better understanding of the physics at these extreme energies. However, inflation solves the problems nicely within the regime of physics that we already know. Not only that, it also offers an excellent explanation for the large scale structures of the universe, and the small scale invariant fluctuations of the cosmic microwave background (CMB). These have their origin in quantum fluctuations during inflation. Much more on this topic can be found in [8].

4.1 Condition for inflation

For an expanding homogeneous and isotropic universe it is convenient to use the *Friedmann–Lemaître–Robertson–Walker* (FLRW) metric. The line element expressed in terms of radial coordinates is

$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right], \tag{4.1}$$

where a(t) is the *scale factor.*

Solving Einsteins field equations for this metric, and with the energy momentum tensor as given in (4.9) , we obtain the two *Friedmann equations*

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad \text{(Energy equation)} \tag{4.2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad \text{(Acceleration equation),} \tag{4.3}$$

from which we can obtain the continuity (or fluid) equation

$$\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)(\rho + p) = 0 \qquad \text{(Fluid equation).} \tag{4.4}$$

We observe from (4.3) that in order to have an accelerated expansion, there must be a *negative pressure* in accordance with

$$\rho + 3p < 0 \implies p < -\frac{1}{3}\rho. \tag{4.5}$$

How we could have a universe in such a peculiar state will be the topic of the next section.

4.2 Scalar field inflation

One way to effectively satisfy (4.5) is to simply introduce a *cosmological constant* into Einstein's equations (as for dark energy). This will have the effect of giving us an equation of state in which $p = -\rho$, and thus we have inflation. However, there is one problem; how could the inflation ever end? A better solution is to introduce a *scalar field*, which energy content drives the inflation. The energy of the scalar field consists of a kinetic and a potential part. Inflation is sensitive to how the total energy is shared among these parts, and thus, the dynamics of the field could bring it into a state in which inflation could no longer be sustained (see figure 1). We will now look into more of the details of the physics behind.

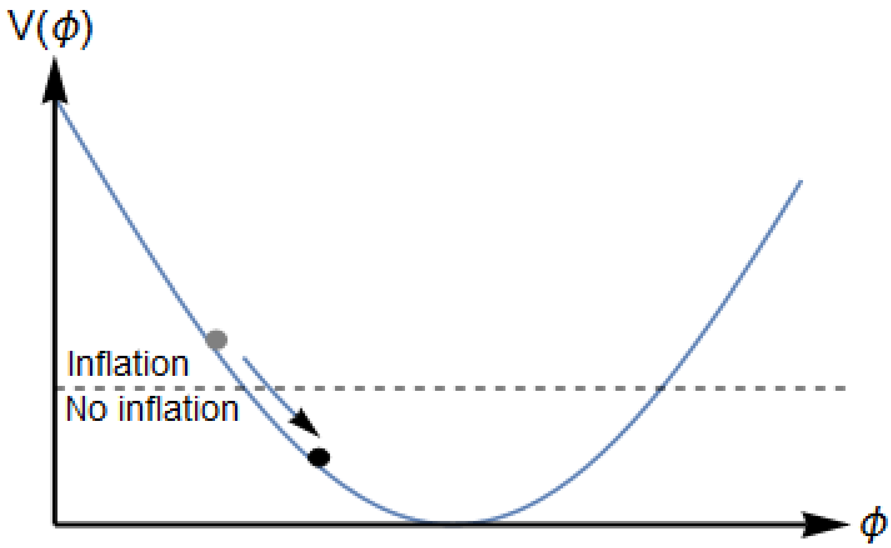

Figure 1: *Assume the field initially has a value ϕ with a potential $V(\phi)$ indicated by the grey dot where it causes inflation. The field loses potential to increasing kinetic energy and 'friction', and inflation stops when the potential goes below a certain value*

We assume inflation is driven by a scalar field ϕ, with the Lagrangian

$$\mathcal{L}_M = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{4.6}$$

where $V(\phi)$ is the (model dependent) field potential. Its energy momentum tensor obtained from (3.26) is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)\right). \tag{4.7}$$

Under the assumption of a homogeneous and isotropic universe, and in the comoving frame, we have no off-diagonal elements, and thus we can write

$$T_{\mu\nu} = diag\left(\frac{1}{2}\dot{\phi}^2 + V(\phi),\ \frac{1}{2}\dot{\phi}^2 - V(\phi),\ \frac{1}{2}\dot{\phi}^2 - V(\phi),\ \frac{1}{2}\dot{\phi}^2 - V(\phi)\right). \tag{4.8}$$

For a perfect fluid the energy momentum tensor can be expressed as [5]

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} \tag{4.9}$$

where ρ is the energy density, p is the pressure and u_μ is a covariant component of the fluids four-velocity. In the comoving frame we have that $u_\mu = (1, 0, 0, 0)$. In addition, for the assumption of a homogeneous and isotropic universe the metric is diagonal. Thus (4.9) becomes

$$T_{\mu\nu} = diag(\rho,\ p,\ p,\ p), \tag{4.10}$$

which, together with (4.8) yields

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{4.11}$$

The equation of state thus becomes

$$p = w\rho = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\rho. \tag{4.12}$$

Comparing this with (4.5), we observe that in order to have accelerated expansion the state of the scalar field has to satisfy

$$\frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} < -\frac{1}{3} \implies \dot{\phi}^2 < V(\phi). \tag{4.13}$$

Thus, in order to have accelerated expansion, the potential energy of the field must be at least twice the kinetic energy.

4.2.1 Slow roll

Now we use the definition of the Hubble parameter

$$H \equiv \frac{\dot{a}}{a} \tag{4.14}$$

and use the density and pressure given by (4.11) in (4.2) and (4.4) to obtain

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{4.15}$$

and

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi}, \tag{4.16}$$

where the latter is the scalar field equation of motion. In (4.15) we have neglected the $-k/a^2$-term, as it will quickly become negligible during the exponential increase in a. The dynamics of the scalar field inflation can now be described by simultaneously solving these two equations. However, for most potentials it is not possible to solve them analytically, and so it is usual to make some approximations that makes it possible. A very useful approximation is made through two assumptions. The first one being that the kinetic energy is so much smaller than the potential energy, that we can neglect it in (4.15). We thus obtain

$$H^2 = \frac{8\pi G}{3}V(\phi). \tag{4.17}$$

The second assumption is that the acceleration term in (4.16) is small compared to the kinetic term. Thus we can neglect it, and obtain

$$3H\dot{\phi} = -\frac{\partial V}{\partial \phi}. \tag{4.18}$$

This is the so-called slow-roll regime, in which computations regarding the cosmic microwave background and cosmological predictions becomes manageable analytically. We will not be considering such solutions further in this thesis.

4.3 Hybrid inflation

4.3.1 Spontaneous symmetry breaking

A *classical* field is stable if it is at rest either at a minimum or a maximum of its potential. However the latter is called a *metastable state* since even the slightest deviation from this point will cause it to accelerate towards the closest minimum. The real world however, is governed by the laws of quantum physics. Thus due to quantum fluctuations no field can be at rest at a maximum indefinitely. Sooner or later it will start accelerating away from its position. For a real field there are two ways it can go. This is shown in figure 2 with the field illustrated as a 'ball' located at the maximum of the potential

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v)^2, \tag{4.19}$$

where for the illustration we have set $\lambda = v = 1$. In this case the potential is symmetric about $\phi = 0$ where it has a global maximum. When quantum fluctuations sets the field off in either direction, the field would eventually end up at, or oscillate about, one of the two minima (depending the frictional terms in the fields equation of motion). In this case we say that the symmetry has been *spontaneously broken.*

Due to the random nature of the quantum fluctuations, we assume there is an equal chance for the field to fall in either direction. The expectation value of the field is therefore $\langle\phi\rangle = 0$. However, the expectation value for the deviation becomes $\sqrt{\langle\phi^2\rangle} = \pm v$. In equation (4.19) the lambda term is the so-called *self coupling constant*, and determines the bare mass μ of the field trough $\mu^2 \equiv \lambda v^2$. Using this definition and expanding (4.19) we then obtain

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\mu^4}{4\lambda}, \tag{4.20}$$

where we can now easier see the terms involved.

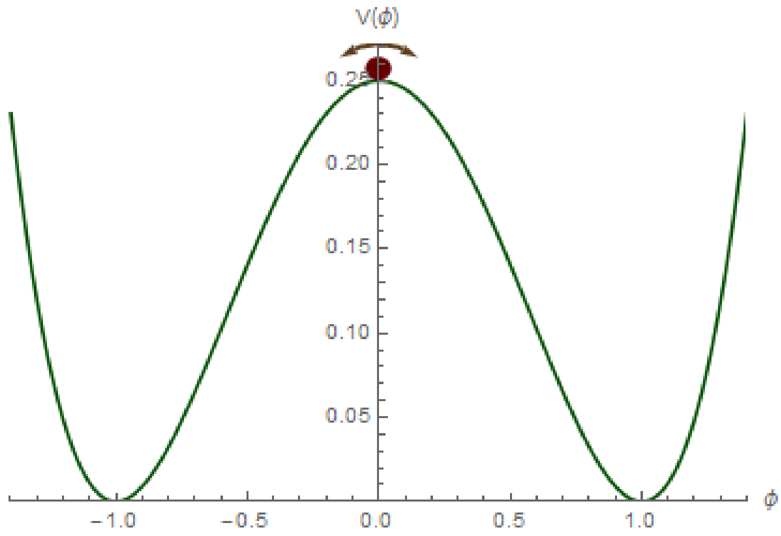

Figure 2:

Now, the *effective mass* m_ϕ of the field is defined through $m_\phi^2 \equiv \partial^2 V/\partial\phi^2$, which we note is also the curvature of the potential at a given value of ϕ. The differentiation yields

$$\partial^2 V/\partial\phi^2 = -\mu^2 + 3\lambda\phi^2, \tag{4.21}$$

and we observe that for $\phi = 0$ the effective mass squared becomes

$$m_\phi^2|_{\phi=0} = -\mu^2. \tag{4.22}$$

This seems to imply that we are dealing with imaginary masses which in in the theory of special relativity are associated to faster than light particles called *tachyons*. But unless we are willing to accept violations of causality, such particles do not exist. Instead, the imaginary mass term here represent an instability which came to be called a *tachyonic instability*. In inflationary cosmology, the violent particle production associated with decay from such an unstable state is called *tachyonic preheating*. Why preheating? Because *reheating* were assigned to another mechanism of particle production before the concept of preheating had been suggested. It now refer to processes taking place after preheating. We will not go into this subject here, but it can be found in [9][10].

Before we say more about tachyonic preheating, we need to say something about how the symmetry breaking field enters into a model of inflation

4.3.2 Hybrid inflation

Symmetry breaking fields have their place in inflation models because they provide a natural way for inflation to end. It also provides a very efficient mechanism for particle production at the end of inflation. In hybrid inflation models, the inflaton field is coupled to a symmetry breaking field which means that the potential energy density of each field depends on the value of both fields. As an example consider an inflaton field σ coupled to a symmetry breaking field ϕ (the potential is plotted in figure 3) such that their effective potential becomes

$$V(\phi,\sigma) = \frac{1}{2}m^2\sigma^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2 + \frac{1}{2}g^2\sigma^2\phi^2. \tag{4.23}$$

Here m is the bare mass of the inflaton field, and g is the coupling constant for the two fields. This is the standard hybrid inflation model proposed by Linde in the early 90's [11]. Expanding this out, and using the definition $\mu^2 \equiv \lambda v^2$ for the bare mass of the symmetry breaking field, we obtain

$$V(\phi,\sigma) = \frac{1}{2}m^2\sigma^2 + \frac{1}{2}(g^2\sigma^2 - \mu^2)\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\mu^4}{4\lambda}. \tag{4.24}$$

The effective masses corresponding to each field is

$$m_\sigma^2 = \frac{\partial^2 V}{\partial\sigma^2} = m^2 + g^2\phi^2 \tag{4.25}$$

$$m_\phi^2 = \frac{\partial^2 V}{\partial\phi^2} = g^2\sigma^2 - \mu^2 + 3\lambda\phi^2. \tag{4.26}$$

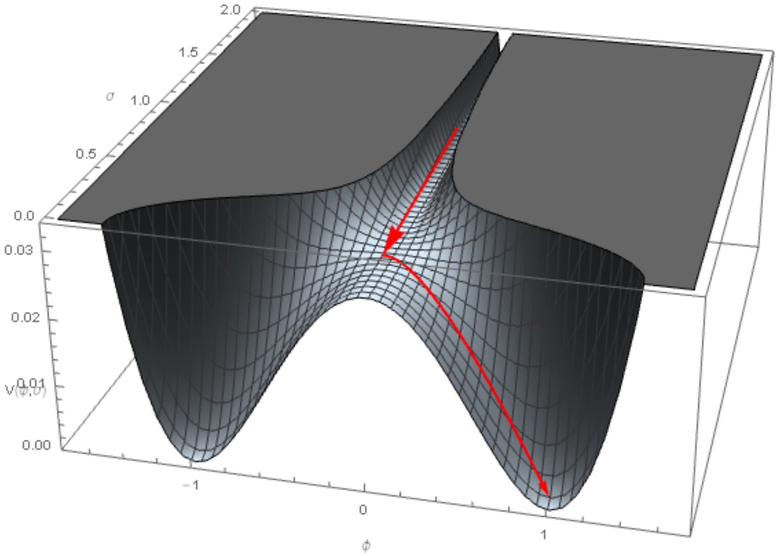

Figure 3: *The potential in equation (4.23) with $v = 1$,$m = 0.001$,$g = 1$ and $\lambda = 1/9$.* The red arrows indicate how the values of the fields develop with time.

Thus, for the symmetry breaking field, the effective mass depends on the value of the inflaton field. By assumption [12] the inflaton field start its slow roll from a very large value at $\sigma \gtrsim M_P$, where M_P is the Planck mass. Thus for $\mu << M_P$ the curvature m_ϕ^2 would be positive and large. Therefore ϕ would quickly become zero, and stay there during the slow roll. When σ goes below the value $\sigma_c \equiv \mu/g$, the symmetry breaking field acquires a negative mass squared, hence we have a tachyonic instability. Then if the so-called waterfall conditions [12]

$$\mu^3 << \lambda m M_P^2, \tag{4.27}$$

$$\mu^3 << \sqrt{\lambda} g m M_P^2, \tag{4.28}$$

are satisfied, the instability increases so quickly that it can be considered instantaneous. We can therefore to a very good approximation neglect the expansion of the universe during calculations under these conditions. The fields will then quickly decay into a huge amount of particles. This is the process of tachyonic preheating.

4.3.3 Tachyonic preheating

The essence of the simplest tachyonic preheating scenario is the following: At the end of inflation, the inflaton field σ comes to rest at zero where it has its minimum potential. At that point the effective mass squared m_ϕ^2 of the symmetry breaking field ϕ has acquired a negative value. As explained in section 4.3.1, the symmetry breaking field can not have the exact value of zero, where the field would be metastable. It has quantum fluctuations, for which the low momentum modes will grow exponentially as the field very quickly rolls down its potential. This is called *spinodal growth.* The maths in section 6.3 explains why only low momentum modes experience this growth. Through couplings to other fields they too will grow; some of which are the well known particles of the standard model, and probably also into exotic particles like dark matter. The mechanism of tachyonic preheating is highly non-homogeneous, and thus produces a lot of gradient energy, which in turn source gravitational waves.

5 Gravitational waves

Gravitational waves are propagating excitations in the geometry of space-time. Such excitations could have various origins. Perhaps the easiest to conceive is the excitations caused by two massive objects (e.g two black holes) spiralling into each other (which for the first time was observed by LIGO on September 14, 2015 [13]). Other possible sources from astronomical objects include slightly asymmetrical neutron stars and supernovas. Another class of sources are those associated with the very early universe, from the time of inflation. One such source could be vacuum fluctuations which are amplified to classical fluctuations during inflation. Here we will be concerned with GW's generated during *preheating* after inflation. In this section we will see how the energy density of gravitational waves is related to the energy momentum tensor of a general source. We will work in a linear approximation, which means that we will treat the metric excitations as small perturbations on a background metric.

5.1 Linear approximation

The gravitational interaction is weak, thus excitations of the metric are very small, even when produced by the most violent processes[14]. We can therefore as a good approximation write the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{5.1}$$

where $\bar{g}_{\mu\nu}$ is the background metric, and $h_{\mu\nu}$ is a small perturbing term. We assume the perturbations to be tiny, i.e. $|h_{\mu\nu}| << 1$, such that terms of order $h^2_{\mu\nu}$ or higher can be neglected. This is the linear approximation.

Since in our case we are neglecting effects due to the expanding universe, we assume a static Minkowski spacetime such that $\bar{g}_{\mu\nu} \to \eta_{\mu\nu} = diag(1, -1, -1, -1)$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{5.2}$$

For mechanisms of gravity wave production which extends over time, we should use the Friedmann–Lemaître–Robertson–Walker metric in flat space. This would result in the scale factor appearing inside the time integrals. In our case we consider them as being constant, and so they would play no role anyway. By this choice we avoid unnecessary complications, which presumably can be understood more easily after first understanding the simpler case considered here. A somewhat similar approach to ours where the expansion is taken into account can be found in [15].

Now recall Einstein's equations

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{5.3}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{5.4}$$

$R^{\mu\nu}$ is the Ricci tensor, and R is the Ricci scalar.

Both $R^{\mu\nu}$ and R are contractions of the Riemann tensor

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}{}_{\mu\nu,\beta} - \Gamma^{\alpha}{}_{\mu\beta,\nu} + \Gamma^{\alpha}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\beta}, \tag{5.5}$$

where $\Gamma^{\alpha}{}_{\beta\gamma}$ is the *metric connection*, also called *Christoffel symbol*. Its relation to the metric tensor is

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma}\big(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma}\big) \tag{5.6}$$

Now, in the linear approximation, since the derivatives of $\eta_{\mu\nu}$ vanish, and $|h_{\mu\nu}| \ll 1$, we can write

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} \eta^{\alpha\sigma}\big(h_{\sigma\beta,\gamma} + h_{\sigma\gamma,\beta} - h_{\beta\gamma,\sigma}\big) \tag{5.7}$$

With this in mind we go back to the Riemann tensor (5.5) and notice that, within the linear approximation, the two last terms can be neglected. Thus we have

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}{}_{\mu\nu,\beta} - \Gamma^{\alpha}{}_{\mu\beta,\nu}. \tag{5.8}$$

And substituting (5.7) into (5.8) we get

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}{}_{\mu\nu,\beta} - \Gamma^{\alpha}{}_{\mu\beta,\nu} = \frac{1}{2}\eta^{\alpha\sigma}\big(h_{\sigma\nu,\mu\beta} - h_{\mu\nu,\sigma\beta} - h_{\sigma\beta,\mu\nu} + h_{\mu\beta,\sigma\nu}\big) \tag{5.9}$$

Contraction yields the Ricci tensor to first order

$$R_{\mu\nu} = \frac{1}{2}\Big[h_{\mu}{}^{\alpha}{}_{,\alpha\nu} + h_{\nu}{}^{\alpha}{}_{,\alpha\mu} - h_{,\mu\nu} - \Box h_{\mu\nu}\Big], \tag{5.10}$$

where $h \equiv h^{\alpha}{}_{\alpha}$, and $\Box \equiv \partial_\alpha \partial^\alpha = \partial^2/\partial t^2 - \nabla^2$. Further contraction yields the Ricci scalar

$$R = h^{\alpha\beta}{}_{,\alpha\beta} - \Box h. \tag{5.11}$$

From the two last equations we construct the Einstein tensor

$$G_{\mu\nu} = \frac{1}{2}\Big[h_{\mu}{}^{\alpha}{}_{,\alpha\nu} + h_{\nu}{}^{\alpha}{}_{,\alpha\mu} - h_{,\mu\nu} - \Box h_{\mu\nu} - \eta_{\mu\nu}\big(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h\big)\Big], \tag{5.12}$$

which can be simplified by introducing

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h \tag{5.13}$$

which is called the *trace reversed* perturbation tensor because $\bar{h} = -h$. After some calculations we have

$$G_{\mu\nu} = \frac{1}{2}\left[\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha\nu} + \bar{h}_{\nu}{}^{\alpha}{}_{,\alpha\mu} - \eta_{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\alpha\beta} - \Box\bar{h}_{\mu\nu}\right]. \tag{5.14}$$

In the *harmonic gauge*

$$\bar{h}^{\alpha}{}_{\mu,\alpha} = 0, \tag{5.15}$$

only the last term survive to give us

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}. \tag{5.16}$$

Using this result in Einstein's equations (5.3) we have

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{5.17}$$

The minus sign in this equation has no role in our final results, and will be omitted. Thus we can write

$$\Box\bar{h}_{\mu\nu} = 16\pi G T_{\mu\nu}. \tag{5.18}$$

5.2 Transverse traceless gauge

In principle there could be ten independent solutions to such an equation as (5.18). However, by imposing the harmonic gauge condition in (5.15), we have already reduced the degrees of freedom by four. By properly choosing additional gauge conditions, it can be shown [5] that this will reduce to two independent solutions. The gauge conditions are

$$\bar{h}_{0\mu} = 0, \quad \bar{h}^{i}{}_{j,i} = 0, \quad \bar{h}^{i}{}_{i} = 0, \tag{5.19}$$

and together they are called the *transverse traceless gauge*. The first one requires that perturbations occur only in the space components of the metric (hence the use of latin indices for the remaining conditions). The second one is the harmonic condition from (5.15) which requires gravitational waves to be transverse to the direction of motion. The last condition require the metric perturbation to be traceless.

Due to the first and last condition in (5.19) we have

$$\bar{h}_{\mu\nu} = h_{\mu\nu}, \tag{5.20}$$

and the transverse traceless metric perturbation can be written as

$$h_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{12} & -h_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.21}$$

Using (5.20) and (5.21) in (5.18), we obtain

$$\Box h_{ij}^{TT} = 16\pi G T_{ij}^{TT}, \tag{5.22}$$

where T_{ij}^{TT} transverse-traceless spatial part of the source energy momentum tensor.

To avoid non-locality we do our calculations in momentum space [15]. The transverse traceless part of a tensor can be expressed by applying a projection operator (see section 5.4) to the original tensor,

$$h_{ij}^{TT} = \lambda_{ij,kl}h_{kl}. \tag{5.23}$$

In momentum space (5.22) becomes

$$\left(\frac{\partial^2}{\partial t^2} + k^2\right)h_{ij}^{TT}(\boldsymbol{k}, t) = 16\pi G T_{ij}^{TT}(\boldsymbol{k}, t), \tag{5.24}$$

where $k^2 \equiv |\boldsymbol{k}|$.

5.3 Green's function solution

For a cleaner look of our equations, we will hereafter omit the TT-superscripts in the metric perturbation such that

$$h_{ij}^{TT} \to h_{ij}. \tag{5.25}$$

We will now solve (5.24) by using a Green's function, such that

$$h_{ij}(\boldsymbol{k},t) = 16\pi G \int_0^t dt'\, \mathcal{G}(\boldsymbol{k}, t-t') T_{ij}^{TT}(\boldsymbol{k},t'), \tag{5.26}$$

where $\mathcal{G}(\boldsymbol{k}, t-t')$ is the Green's function. The Green's function equation corresponding to (5.24) is

$$\left(\frac{\partial^2}{\partial t^2} + k^2\right) \mathcal{G}(\boldsymbol{k}, t-t') = \delta(t-t'). \tag{5.27}$$

The Fourier expansion (in frequencies) of the Green's function is

$$\mathcal{G}(\boldsymbol{k}, t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \mathcal{G}(\boldsymbol{k},\omega) e^{-i\omega(t-t')}, \tag{5.28}$$

and we also know that

$$\delta(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}. \tag{5.29}$$

The Fourier expansion (in frequencies) of (5.27) thus becomes

$$-\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\omega^2 - k^2)\, \mathcal{G}(\boldsymbol{k},\omega) e^{-i\omega(t-t')} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}, \tag{5.30}$$

from which we obtain

$$\mathcal{G}(\boldsymbol{k},\omega) = -\frac{1}{\omega^2 - k^2}. \tag{5.31}$$

Plugging this result into (5.28) we obtain

$$\mathcal{G}(\boldsymbol{k}, t-t') = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2 - k^2} e^{-i\omega(t-t')}. \tag{5.32}$$

We notice that we have two simple poles, and proceed to solve it as a contour integral in the complex plane. We seek the retarded Green's function so that $\mathcal{G}(\boldsymbol{k}, t-t') = 0$ for $t < t'$. We now observe that in order for the integral to be defined when $t < t'$, we have to integrate in the upper half plane. Then in order for the result to become zero, we therefore shift the poles to below the real axis. Integration now in the upper half plane yields zero because there are no poles located inside the contour. Thus we write the Green's function as

$$\mathcal{G}(\boldsymbol{k}, t-t') = -\frac{1}{2\pi} \oint_C d\omega \frac{e^{-i\omega(t-t')}}{(\omega + k + i\epsilon)(\omega - k + i\epsilon)}, \quad \epsilon > 0, \tag{5.33}$$

where we have factored the denominator. To avoid clutter, we make the following definition before proceeding

$$f(\omega) \equiv \frac{e^{-i\omega(t-t')}}{(\omega + k + i\epsilon)(\omega - k + i\epsilon)}. \tag{5.34}$$

Now, for $t > t'$ we integrate in the lower half plane:

$$\begin{aligned} \mathcal{G}(\boldsymbol{k}, t-t') &= \lim_{\epsilon\to 0} -\frac{1}{2\pi} \oint_C d\omega f(\omega) \\ &= i \lim_{\epsilon \to 0} \big(Res[f(\omega); -k - i\epsilon] + Res[f(\omega); k - i\epsilon]\big) \end{aligned} \tag{5.35}$$

where the minus sign in (5.33) is cancelled due to integration in the clockwise direction. The two residues are

$$Res[f(\omega); -k - i\epsilon] = -\frac{e^{ik(t-t')} e^{-\epsilon(t-t')}}{2k}, \tag{5.36}$$

$$Res[f(\omega); k - i\epsilon] = \frac{e^{-ik(t-t')} e^{-\epsilon(t-t')}}{2k}. \tag{5.37}$$

Now we substitute these residues into (5.35), take the limit and divide by i. The Green's function then becomes

$$\mathcal{G}(\boldsymbol{k}, t-t') = \frac{\sin[k(t-t')]}{k}, \quad (t > t'). \tag{5.38}$$

Now we can bring this result back to where we started and substitute it into (5.26) to obtain

$$h_{ij}(\boldsymbol{k}, t) = 16\pi G \int_0^t dt' \frac{\sin[k(t-t')]}{k} T_{ij}^{TT}(\boldsymbol{k}, t'), \tag{5.39}$$

and its time derivative which we will have use for shortly is

$$\dot{h}_{ij}(\boldsymbol{k}, t) = 16\pi G \int_0^t dt' \cos[k(t-t')] T_{ij}^{TT}(\boldsymbol{k}, t'), \tag{5.40}$$

5.4 Projection operator

The transverse traceless part of a tensor can be represented by an operator acting on that tensor [15]. For our source tensor we can write

$$T_{ij}^{TT}(\boldsymbol{k}, t) = \lambda_{ij,kl} T_{kl}(\boldsymbol{k}, t), \tag{5.41}$$

where $\lambda_{ij,kl}$ is the projection operator given by

$$\lambda_{ij,kl}(\boldsymbol{k}) = P_{ik}(\boldsymbol{k}) P_{jl}(\boldsymbol{k}) - \frac{1}{2} P_{ij}(\boldsymbol{k}) P_{kl}(\boldsymbol{k}), \tag{5.42}$$

where

$$P_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{k^2}. \tag{5.43}$$

For later use, we include the following relations

$$\lambda_{ij,kl}(-\boldsymbol{k}) = \lambda_{ij,kl}(\boldsymbol{k}) \tag{5.44}$$

$$\lambda_{ij,kl}(\boldsymbol{k}) \lambda_{ij,mn}(\boldsymbol{k}) = \lambda_{kl,mn}(\boldsymbol{k}) \tag{5.45}$$

$$\lambda_{kl,mn}(\boldsymbol{k})(k_k,\, k_l,\, k_m,\, k_n) = (0,\, 0,\, 0,\, 0), \tag{5.46}$$

where k_j is the j'th component of vector $\boldsymbol{k}$. These relations were obtained by using the following observations:

$$P_{ij}(-\boldsymbol{k}) = P_{ij}(\boldsymbol{k}) \tag{5.47}$$

$$P_{ii}(\boldsymbol{k}) = 2 \tag{5.48}$$

$$P_{ij}(\boldsymbol{k}) P_{ik}(\boldsymbol{k}) = P_{jk}(\boldsymbol{k}) \tag{5.49}$$

$$P_{ij}(\boldsymbol{k}) k_j = 0. \tag{5.50}$$

5.5 Energy density

As before mentioned our aim is to find the energy density power spectrum of the gravitational waves which is sourced by the inhomogeneous decay of the symmetry braking field. It can be shown [14], that the energy density averaged over a volume V, much larger than the relevant wavelengths, is

$$\rho_{GW} = \frac{1}{32\pi G} \langle \dot{h}_{ij}(\boldsymbol{x}, t) \dot{h}_{ij}(\boldsymbol{x}, t) \rangle. \tag{5.51}$$

The Fourier expansion into k-space is,

$$\rho_{GW} = \frac{1}{32\pi G} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle \dot{h}_{ij}(\boldsymbol{k}, t) \dot{h}_{ij}(\boldsymbol{k}', t) \rangle e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}. \tag{5.52}$$

Now, substituting (5.40) for $\dot{h}$ and (5.41) for T^{TT} into the expression above, we obtain

$$\dot{h}_{ij}(\boldsymbol{k}, t) = 16\pi G \int_0^t d\tau \, \cos[k(t-\tau)] \lambda_{ij,kl}(\boldsymbol{k}) T_{ij}(\boldsymbol{k}, \tau). \tag{5.53}$$

We can now write

$$\begin{aligned}&\big\langle \dot{h}^{ij}(\boldsymbol{k},t)\dot{h}^{ij}(\boldsymbol{k}',t)\big\rangle = \\ &(16\pi G)^2 \int_0^t d\tau \int_0^t d\tau' \;\cos[k(t-\tau)]\cos[k'(t-\tau')]\lambda_{ij,kl}(\boldsymbol{k})\lambda_{ij,mn}(\boldsymbol{k}')\big\langle T_{kl}(\boldsymbol{k},\tau)T_{mn}(\boldsymbol{k}',\tau')\big\rangle.\end{aligned} \tag{5.54}$$

Later we will see that $\boldsymbol{k}' = -\boldsymbol{k}$ and therefore the two projectors simplifies to a single projector in accordance with (5.44). However, we keep it general for now.

Through (5.52) and (5.54) we can now calculate the energy density of gravitational waves from stochastic sources (if expansion can be neglected. Otherwise see [15]). The energy momentum tensor of a quantum field has quantum nature. The next section will for the most part deal with this tensor, its relation to the quantum field and how to calculate the expectation value in the integrand of (5.54).

6 The source

6.1 Energy momentum tensor

In general the energy momentum tensor for a scalar field is given by (4.7), but by the knowledge we gained in obtaining (5.22) it can be simplified [15]. First of all, we only need the spatial components. Secondly, the spatial part of the metric tensor is $g_{ij} = \delta_{ij} - h_{ij}$. The part involving the δ_{ij} is a pure trace, and will disappear under the transverse traceless condition. The remaining term involving h_{ij} will contribute through second order terms, which we will neglect. Thus the second term of (4.7) disappears and we are left with.

$$T_{ij} = \partial_i\phi\partial_j\phi. \tag{6.1}$$

However, what we need is an expression for the energy momentum tensor in momentum space. The Fourier expansion of the field and its derivatives are

$$\phi(\boldsymbol{x},t) = \int \frac{d^3p}{(2\pi)^3}\phi(\boldsymbol{p},t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}}, \quad \partial_j\phi(\boldsymbol{x},t) = i\int \frac{d^3p}{(2\pi)^3}p_j\phi(\boldsymbol{p},t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}}, \tag{6.2}$$

where p_j is the j'th component of $\boldsymbol{p}$. Our energy momentum tensor can then be written as

$$T_{ij}(\boldsymbol{x},t) = -\int \frac{d^3pd^3q}{(2\pi)^3(2\pi)^3}p_iq_j\phi(\boldsymbol{p},t)\phi(\boldsymbol{q},t)e^{i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}}. \tag{6.3}$$

The energy momentum tensor in momentum space is then obtained through a Fourier transform in the following way:

$$\begin{aligned}T_{ij}(\boldsymbol{k},t) &= -\int d^3x \int \frac{d^3pd^3q}{(2\pi)^3(2\pi)^3}p_iq_j\phi(\boldsymbol{p},t)\phi(\boldsymbol{q},t)e^{i(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{x}} \\ &= -\int \frac{d^3pd^3q}{(2\pi)^3(2\pi)^3}(2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})p_iq_j\phi(\boldsymbol{p},t)\phi(\boldsymbol{q},t) \\ &= -\int \frac{d^3p}{(2\pi)^3}p_i(k_j-p_j)\phi(\boldsymbol{p},t)\phi(\boldsymbol{k}-\boldsymbol{p},t).\end{aligned} \tag{6.4}$$

Using this result we can write

$$\begin{aligned}&\big\langle T_{kl}(\boldsymbol{k},\tau)T_{mn}(\boldsymbol{k}',\tau')\big\rangle = \\ &\quad\int \frac{d^3pd^3p'}{(2\pi)^3(2\pi)^3}p_k(k_l-p_l)p'_m(k'_n-p'_n)\big\langle\phi(\boldsymbol{p},\tau)\phi(\boldsymbol{k}-\boldsymbol{p},\tau)\phi(\boldsymbol{p}',\tau')\phi(\boldsymbol{k}'-\boldsymbol{p}',\tau')\big\rangle.\end{aligned} \tag{6.5}$$

The expectation value in the above expression is a so-called *4-point correlation function*. Since we are dealing with free fields, we are allowed, by Wick's theorem, to write the 4-point correlation function as a sum of products of 2-point correlation functions (also called 2-point *correlator*).

$$\begin{aligned}&\big\langle\phi(\boldsymbol{p},\tau)\phi(\boldsymbol{k}-\boldsymbol{p},\tau)\phi(\boldsymbol{p}',\tau')\phi(\boldsymbol{k}'-\boldsymbol{p}',\tau')\big\rangle = \\ &\qquad\big\langle\phi(\boldsymbol{p},\tau)\phi(\boldsymbol{k}-\boldsymbol{p},\tau)\big|0\big\rangle\big\langle\phi(\boldsymbol{p}',\tau')\phi(\boldsymbol{k}'-\boldsymbol{p}',\tau')\big\rangle+ \\ &\qquad\big\langle\phi(\boldsymbol{p},\tau)\phi(\boldsymbol{p}',\tau')\big\rangle\big\langle\phi(\boldsymbol{k}-\boldsymbol{p},\tau)\phi(\boldsymbol{k}'-\boldsymbol{p}',\tau')\big\rangle+ \\ &\qquad\big\langle\phi(\boldsymbol{p},\tau)\phi(\boldsymbol{k}'-\boldsymbol{p}',\tau')\big\rangle\big\langle\phi(\boldsymbol{k}-\boldsymbol{p},\tau)\phi(\boldsymbol{p}',\tau')\big\rangle.\end{aligned} \tag{6.6}$$

6.2 Field operators

From the expressions in (5.52), (5.54), (6.5) and (6.6) we have now established a link between the source field itself and the energy density of the gravitational waves it produces.

As before mentioned we assume the universe is being cooled by inflation to essentially $T = 0K$ before the tachyonic instability is triggered. The 2-point correlation functions from the last section can thus be expressed as *vacuum expectation values* $\langle 0|\hat{\phi}_{\boldsymbol{p}}\hat{\phi}_{\boldsymbol{q}}|0\rangle$, where the fields now are represented by field operators acting on the vacuum ground state. By vacuum ground state we mean the state in which $\hat{a}_{\boldsymbol{k}}|0\rangle = 0$, where $\hat{a}_{\boldsymbol{k}}$ is the annihilation operator of for a state of momentum $\boldsymbol{k}$.

The field operators in configuration space can be written as an expansion in momentum space by the following Fourier transforms (e.g. see [1])

$$\hat{\phi}(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^3}\hat{\phi}(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \hat{\pi}(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^3}\hat{\pi}(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{6.7}$$

where $\hat{\pi} \equiv \dot{\hat{\phi}}$.

Since the potential is changed at the time of the quench, we will have different expressions for the momentum modes before and after. We will now derive an expression for the field after the quench, which also depends on its potential before the quench. To do this we follow the reasoning in [16] and write the field modes just before the quench, near $t = 0$, as

$$\hat{\phi}(\boldsymbol{k},t) = \frac{1}{\sqrt{2\omega_k^+}}(\hat{a}_{\boldsymbol{k}}e^{-i\omega_k^+ t} + \hat{a}^\dagger_{-\boldsymbol{k}}e^{i\omega_k^+ t}), \quad \hat{\pi}(\boldsymbol{k},t) = \frac{-i\omega_k^+}{\sqrt{2\omega_k^+}}(\hat{a}_{\boldsymbol{k}}e^{-i\omega_k^+ t} - \hat{a}^\dagger_{-\boldsymbol{k}}e^{i\omega_k^+ t}), \tag{6.8}$$

where $\omega_k^+ = \sqrt{k^2+\mu^2}$, $k = |\boldsymbol{k}|$.

After the quench, $t > 0$, we can write

$$\hat{\phi}(\boldsymbol{k},t) = \hat{\alpha}_{\boldsymbol{k}}e^{-i\omega_k^- t} + \hat{\beta}_{\boldsymbol{k}}e^{i\omega_k^- t}, \quad \hat{\pi}(\boldsymbol{k},t) = -i\omega_k^-(\hat{\alpha}_{\boldsymbol{k}}e^{-i\omega_k^- t} - \hat{\beta}_{\boldsymbol{k}}e^{i\omega_k^- t}), \tag{6.9}$$

where $\omega_k^- = \sqrt{k^2-\mu^2}$, $k = |\boldsymbol{k}|$.

The expressions which are valid before the quench and those valid after the quench should agree at the time of the quench. We have chosen the quench to occur at $t = 0$ and thus we set them equal to each other at $t = 0$. We then obtain

$$\hat{\phi}(\boldsymbol{k},0) = \hat{\alpha}_{\boldsymbol{k}} + \hat{\beta}_{\boldsymbol{k}} = \frac{1}{\sqrt{2\omega_k^+}}(\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{\boldsymbol{k}}), \tag{6.10}$$

$$\hat{\pi}(\boldsymbol{k},0) = \hat{\alpha}_{\boldsymbol{k}} - \hat{\beta}_{\boldsymbol{k}} = \frac{\omega_k^+}{\omega_k^-\sqrt{2\omega_k^+}}(\hat{a}_{\boldsymbol{k}} - \hat{a}^\dagger_{\boldsymbol{k}}). \tag{6.11}$$

Now we solve for $\hat{\alpha}_{\boldsymbol{k}}$ and $\hat{\beta}_{\boldsymbol{k}}$ and get

$$\hat{\alpha}_{\boldsymbol{k}} = \frac{1}{2\sqrt{2\omega_k^+}}\left[\left(1+\frac{\omega_k^+}{\omega_k^-}\right)\hat{a}_{\boldsymbol{k}} + \left(1-\frac{\omega_k^+}{\omega_k^-}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\right], \tag{6.12}$$

$$\hat{\beta}_{\boldsymbol{k}} = \frac{1}{2\sqrt{2\omega_k^+}}\left[\left(1-\frac{\omega_k^+}{\omega_k^-}\right)\hat{a}_{\boldsymbol{k}} + \left(1+\frac{\omega_k^+}{\omega_k^-}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\right]. \tag{6.13}$$

The two ω's can be written together as

$$\omega_k^\pm = \sqrt{k^2 \pm \mu^2}. \tag{6.14}$$

Thus, for $t > 0$, we have an expression for our field operator $\hat{\phi}(\boldsymbol{k},t)$ and its time derivative $\hat{\pi}(\boldsymbol{k},t)$ through (6.9) and (6.12) - (6.14).

6.3 Correlation functions

In this section we derive an expression for a 2-point correlation function, as will be needed in in (6.6). From the previous section we have the field operator

$$\phi(\boldsymbol{k},t) = \hat{\alpha}_{\boldsymbol{k}} e^{-i\omega_k^- t} + \hat{\beta}_{\boldsymbol{k}} e^{i\omega_k^- t}, \quad t > 0. \tag{6.15}$$

However, since we are only going to consider the unstable modes where $|\boldsymbol{k}| < \mu$ we know that ω_k^- will always be imaginary, and thus we can write $\omega_k^- = i|\omega_k^-|$, and we define

$$\tilde{\omega}_k \equiv |\omega_k^-|. \tag{6.16}$$

Now we make the substitution $\omega_k^- \to i\tilde{\omega}_k$ in (6.12), (6.13) and (6.15), and obtain

$$\hat{\phi}(\boldsymbol{k},t) = \hat{\alpha}_{\boldsymbol{k}}\, e^{\tilde{\omega}_k t} + \hat{\beta}_{\boldsymbol{k}}\, e^{-\tilde{\omega}_k t}, \tag{6.17}$$

$$\hat{\alpha}_{\boldsymbol{k}} = \frac{1}{2\sqrt{2\omega_k^+}}\left[\left(1 - i\frac{\omega_k^+}{\tilde{\omega}_k}\right)\hat{a}_{\boldsymbol{k}} + \left(1 + i\frac{\omega_k^+}{\tilde{\omega}_k}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\right], \tag{6.18}$$

$$\hat{\beta}_{\boldsymbol{k}} = \frac{1}{2\sqrt{2\omega_k^+}}\left[\left(1 + i\frac{\omega_k^+}{\tilde{\omega}_k}\right)\hat{a}_{\boldsymbol{k}} + \left(1 - i\frac{\omega_k^+}{\tilde{\omega}_k}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\right]. \tag{6.19}$$

Further we define

$$W_k = 1 + i\frac{\omega_k^+}{\tilde{\omega}_k}, \quad \text{and thus:} \quad W_k^* = 1 - i\frac{\omega_k^+}{\tilde{\omega}_k}, \tag{6.20}$$

and substitute this into (6.18) and (6.19), which thus becomes

$$\hat{\alpha}_{\boldsymbol{k}} = \frac{1}{2\sqrt{2\omega_k^+}}\left[W_k^*\hat{a}_{\boldsymbol{k}} + W_k\hat{a}^\dagger_{-\boldsymbol{k}}\right], \qquad \hat{\beta}_{\boldsymbol{k}} = \frac{1}{2\sqrt{2\omega_k^+}}\left[W_k\hat{a}_{\boldsymbol{k}} + W_k^*\hat{a}^\dagger_{-\boldsymbol{k}}\right]. \tag{6.21}$$

We then substitute (6.21) into (6.17) and obtain

$$\hat{\phi}(\boldsymbol{k},t) = \frac{1}{2\sqrt{2\omega_k^+}}\left[\left(W_k^* e^{\tilde{\omega}_k t} + W_k e^{-\tilde{\omega}_k t}\right)\hat{a}_{\boldsymbol{k}} + \left(W_k e^{\tilde{\omega}_k t} + W_k^* e^{-\tilde{\omega}_k t}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\right]. \tag{6.22}$$

We then make another definition

$$Z(k,t) = W_k^* e^{\tilde{\omega}_k t} + W_k e^{-\tilde{\omega}_k t}, \quad \text{and thus:} \quad Z^*(k,t) = W_k e^{\tilde{\omega}_k t} + W_k^* e^{-\tilde{\omega}_k t}, \tag{6.23}$$

and substitute these into (6.22) to get

$$\hat{\phi}(\boldsymbol{k},t) = \frac{1}{2\sqrt{2\omega_k^+}}\left[Z(k,t)\hat{a}_{\boldsymbol{k}} + Z^*(k,t)\hat{a}^\dagger_{-\boldsymbol{k}}\right]. \tag{6.24}$$

We observe that

$$\hat{\phi}^\dagger(\boldsymbol{k},t) = \hat{\phi}(-\boldsymbol{k},t). \tag{6.25}$$

The two point correlation function can now be expressed as a vacuum expectation value, using the field operator given by (6.24)

$$\begin{aligned} &\langle 0|T\hat{\phi}(\boldsymbol{k},t)\hat{\phi}(\boldsymbol{k}',t')|0\rangle = \\ &\quad \frac{1}{8\sqrt{\omega_k^+\omega_{k'}^+}}\langle 0|T\left[Z(k,t)\hat{a}_{\boldsymbol{k}} + Z^*(k,t)\hat{a}^\dagger_{-\boldsymbol{k}}\right]\left[Z(k',t')\hat{a}_{\boldsymbol{k}'} + Z^*(k',t')\hat{a}^\dagger_{-\boldsymbol{k}'}\right]|0\rangle, \end{aligned} \tag{6.26}$$

where T is the time ordering operator. However, the only non zero term is the one containing $\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}'}$, and so we are left with

$$\begin{aligned} \langle 0|T\hat{\phi}(\boldsymbol{k},t)\hat{\phi}(\boldsymbol{k}',t')|0\rangle &= \frac{Z(k,t)Z^*(k',t')}{8\sqrt{\omega_k^+\omega_{k'}^+}}\langle 0|T\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}'}|0\rangle \\ &= \frac{Z(k,t)Z^*(k,t')}{8\omega_k^+}(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}'), \end{aligned} \tag{6.27}$$

where in the last equality we have used that $\langle 0|\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}'}|0\rangle = (2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')$, and that $Z^*(-k,t') = Z^*(k,t')$. We also note that time ordering has no effect since all time dependence can be brought outside the vacuum expectation value.

Now we can back-substitute for Z by (6.23) and (6.20) in (6.27), and rearrange to eventually obtain the vacuum expectation value

$$\langle 0|\hat{\phi}(\boldsymbol{k},t)\hat{\phi}(\boldsymbol{k}',t')|0\rangle = \frac{(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')}{4\omega_k^+}\Bigg[\left(1+\frac{\omega_k^{+2}}{\tilde{\omega}_k^2}\right)\cosh(\tilde{\omega}_k(t+t'))+ \left(1-\frac{\omega_k^{+2}}{\tilde{\omega}_k^2}\right)\cosh(\tilde{\omega}_k(t-t')) - i2\frac{\omega_k^+}{\tilde{\omega}_k}\sinh(\tilde{\omega}_k(t-t'))\Bigg]. \quad (6.28)$$

We can write this (as in [15]) as

$$\langle 0|\hat{\phi}(\boldsymbol{k},t)\hat{\phi}(\boldsymbol{k}',t')|0\rangle = (2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')F(k,t,t'), \quad (6.29)$$

where

$$F(k,t,t') = \frac{1}{4\omega_k^+}\Bigg[\left(1+\frac{\omega_k^{+2}}{\tilde{\omega}_k^2}\right)\cosh(\tilde{\omega}_k(t+t'))+ \left(1-\frac{\omega_k^{+2}}{\tilde{\omega}_k^2}\right)\cosh(\tilde{\omega}_k(t-t')) - i2\frac{\omega_k^+}{\tilde{\omega}_k}\sinh(\tilde{\omega}_k(t-t'))\Bigg]. \quad (6.30)$$

We can now use (6.29) in (6.6) to obtain

$$\begin{aligned}\langle 0 \,|\, \hat{\phi}(\boldsymbol{p},\tau)\hat{\phi}(\boldsymbol{k}-\boldsymbol{p},\tau)\hat{\phi}(\boldsymbol{p}',\tau')\hat{\phi}(\boldsymbol{k}'-\boldsymbol{p}',\tau')\,|\, 0\rangle =\\ F(p,\tau,\tau)F(p',\tau',\tau')(2\pi)^6\delta(\boldsymbol{k})\delta(\boldsymbol{k}')+\\ F(p,\tau,\tau')F(|\boldsymbol{k}-\boldsymbol{p}|,\tau,\tau')(2\pi)^6\delta(\boldsymbol{p}+\boldsymbol{p}')\delta(\boldsymbol{k}-\boldsymbol{p}+\boldsymbol{k}'-\boldsymbol{p}')+\\ F(p,\tau,\tau')F(|\boldsymbol{k}-\boldsymbol{p}|,\tau,\tau')(2\pi)^6\delta(\boldsymbol{p}+\boldsymbol{k}'-\boldsymbol{p}')\delta(\boldsymbol{k}-\boldsymbol{p}+\boldsymbol{p}').\end{aligned} \quad (6.31)$$

The first of these three terms will not contribute and is discarded. Substituting the remaining two terms into (6.5), and integrate over $\boldsymbol{p}'$ we obtain

$$\begin{aligned}&\langle T_{kl}(\boldsymbol{k},\tau)T_{mn}(\boldsymbol{k}',\tau')\rangle =\\ &2\int\frac{d^3p}{(2\pi)^3}\,p_k(k_l-p_l)[p_m(k_n-p_n)+p_n(k_m-p_m)]F(p,\tau,\tau')F(|\boldsymbol{k}-\boldsymbol{p}|,\tau,\tau')(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}').\end{aligned} \quad (6.32)$$

6.4 Transverse traceless energy momentum tensor

In (5.54) we had the following expression

$$\begin{aligned}&\langle \dot{h}^{ij}(\boldsymbol{k},t)\dot{h}^{ij}(\boldsymbol{k}',t)\rangle =\\ &(16\pi G)^2\int_0^t d\tau\int_0^t d\tau'\;\cos[k(t-\tau)]\cos[k'(t-\tau')]\lambda_{ij,kl}(\boldsymbol{k})\lambda_{ij,mn}(\boldsymbol{k}')\langle T_{kl}(\boldsymbol{k},\tau)T_{mn}(\boldsymbol{k}',\tau')\rangle.\end{aligned} \quad (6.33)$$

As mentioned in section 5.5 we now notice that the delta function in (6.32) implies that $\boldsymbol{k}' = -\boldsymbol{k}$. Together with the properties (5.44) and (5.45) the two projection operators simplifies in the following way

$$\lambda_{ij,kl}(\boldsymbol{k})\lambda_{ij,mn}(\boldsymbol{k}') \to \lambda_{ij,kl}(\boldsymbol{k})\lambda_{ij,mn}(-\boldsymbol{k}) \to \lambda_{ij,kl}(\boldsymbol{k})\lambda_{ij,mn}(\boldsymbol{k}) \to \lambda_{kl,mn}(\boldsymbol{k}). \quad (6.34)$$

We also have the property of (5.46), such that any components of $\boldsymbol{k}$ in (6.32) vanishes. What remains is then

$$\begin{aligned}\langle \dot{h}^{ij}(\boldsymbol{k},t)\dot{h}^{ij}(\boldsymbol{k}',t)\rangle = 2(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')(16\pi G)^2\int_0^t d\tau\int_0^t d\tau'\int\frac{d^3p}{(2\pi)^3}\\ \lambda_{kl,mn}(\boldsymbol{k})p_kp_lp_mp_n\cos[k(t-\tau)]\cos[k(t-\tau')]F(p,\tau,\tau')F(|\boldsymbol{k}-\boldsymbol{p}|,\tau,\tau').\end{aligned} \quad (6.35)$$

We will now see what effect the operator has on the vector components in (6.35). The only components are in the product $p_kp_lp_mp_n$, so we act on them by the projection operator which gives

$$\begin{aligned}&\lambda_{kl,mn}(\boldsymbol{k})p_kp_lp_mp_n =\\ &[\delta_{km}\delta_{ln}-\delta_{km}\hat{k}_l\hat{k}_n-\delta_{ln}\hat{k}_k\hat{k}_m-\frac{1}{2}(\delta_{kl}\delta_{mn}-\delta_{kl}\hat{k}_m\hat{k}_n-\delta_{mn}\hat{k}_k\hat{k}_l-\hat{k}_k\hat{k}_l\hat{k}_m\hat{k}_n)]p_kp_lp_mp_n =\\ &|\boldsymbol{p}|^4-|\boldsymbol{p}|^2(\boldsymbol{p}\cdot\hat{\boldsymbol{k}})^2-|\boldsymbol{p}|^2(\boldsymbol{p}\cdot\hat{\boldsymbol{k}})^2-\frac{1}{2}\big[|\boldsymbol{p}|^4-|\boldsymbol{p}|^2(\boldsymbol{p}\cdot\hat{\boldsymbol{k}})^2-|\boldsymbol{p}|^2(\boldsymbol{p}\cdot\hat{\boldsymbol{k}})^2-(\boldsymbol{p}\cdot\hat{\boldsymbol{k}})\big] =\\ &\frac{1}{2}[|\boldsymbol{p}|^2-(\boldsymbol{p}\cdot\hat{\boldsymbol{k}})^2]^2 = \frac{1}{2}\big[|\boldsymbol{p}|^2-|\boldsymbol{p}|^2cos^2\theta\big] = \frac{1}{2}p^4sin^4\theta,\end{aligned} \quad (6.36)$$

where $\hat{\boldsymbol{k}} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$ and θ is the angle between $\boldsymbol{p}$ and $\hat{\boldsymbol{k}}$.

Putting things together now, we combine (6.36), (6.35) and (5.52), integrate over $\boldsymbol{k}'$ and place the remaining momentum integrals to the left

$$\rho_{GW} = 16\pi G \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3}\, p^4 sin^4\theta \int_0^t d\tau \int_0^t d\tau' \ \cos[k(t-\tau)]\cos[k(t-\tau')]F(p,\tau,\tau')F(|\boldsymbol{k}-\boldsymbol{p}|,\tau,\tau'). \tag{6.37}$$

The quantity we seek is the derivative of the energy density with respect to $\ln k$. Integrating the $\boldsymbol{k}$ integral in spherical coordinates, and then taking the derivative with respect to lnk we obtain

$$\frac{\rho_{GW}}{d\ln k} = \frac{2Gk^3}{\pi^3} \int dpd\theta\, p^6 sin^5\theta \int_0^t d\tau \int_0^t d\tau' \ \ \cos[k(t-\tau)]\cos[k(t-\tau')]F(p,\tau,\tau')F(|\boldsymbol{k}-\boldsymbol{p}|,\tau,\tau'), \tag{6.38}$$

where we have also expressed the $\boldsymbol{p}$-integral in spherical coordinates, and integrated over the azimuthal angle to get an additional factor of 2π.

7 Performing the complete calculations

In this section we will go through the necessary calculations in (6.38), and show some of their intermediate results. The momentum integral can not be calculated analytically while the time integrals can. We therefore start with the time integrals.

7.1 Time integrals

We let Mathematica do the tedious calculations for us, and we obtain a rather large output. For readability we use the following definition in calculating our time integrals

$$q \equiv |\boldsymbol{k}-\boldsymbol{p}| = k^2 + p^2 - 2kp\cos\theta, \tag{7.1}$$

Mathematica is not so clever in making large results readable, so a bit of manual work went into obtaining a *more* readable result (though admittedly not so pleasing to the eyes still)

$$tint(\boldsymbol{k},\boldsymbol{p},t) \equiv \int_0^t d\tau \int_0^t d\tau' \ \cos[k(t-\tau)]\cos[k(t-\tau')]F(p,\tau,\tau')F(q,\tau,\tau') =$$

$$\begin{aligned}
&L_1 \cosh(t\tilde{\omega}_p) \\
&+L_2 \sinh(2t\tilde{\omega}_p)\sinh(2t\tilde{\omega}_q) \\
&+L_3 \cosh(2t\tilde{\omega}_p)\sinh(t\tilde{\omega}_q) \\
&+L_4 \cos(tk)\cosh(t\tilde{\omega}_p)\cosh(t\tilde{\omega}_q) \\
&+L_5 \cosh(t\tilde{\omega}_q)\sin(tk)\sinh(t\tilde{\omega}_p) \\
&+L_6 \cosh(t\tilde{\omega}_p)\sin(tk)\sinh(t\tilde{\omega}_q) \\
&+L_7 \cos(tk)\sinh(t\tilde{\omega}_p)\sinh(t\tilde{\omega}_q) \\
&+L_8 \cosh(t\tilde{\omega}_q)\sinh(t\tilde{\omega}_p) \\
&+L_9 \cosh(t\tilde{\omega}_p)\cosh(t\tilde{\omega}_q) \\
&+L_{10}\cos(tk) \\
&+L_{11}\sin(tk) \\
&+L_{12}
\end{aligned} \tag{7.2}$$

where L_1 through L_{12} are (7.3)

$$L_1 = \frac{\left(\omega_p^{+2} + \tilde{\omega}_p^2\right)\left(\omega_q^{+2} - \tilde{\omega}_q^2\right)\left(\tilde{\omega}_q^2 - \tilde{\omega}_p^2\right)}{8\omega_p^+\omega_q^+\tilde{\omega}_p^2\tilde{\omega}_q^2\left(\tilde{\omega}_p^4 + 2\left(k^2 - \tilde{\omega}_q^2\right)\tilde{\omega}_p^2 + \left(k^2 + \tilde{\omega}_q^2\right)^2\right)}$$

$$L_2 = -\frac{\left(\omega_p^{+2} + \tilde{\omega}_p^2\right)\left(\omega_q^{+2} + \tilde{\omega}_q^2\right)\left(\left(\tilde{\omega}_p^2 - \tilde{\omega}_q^2\right)^2 - k^4\right)}{8\omega_p^+\omega_q^+\tilde{\omega}_p\tilde{\omega}_q\left(\tilde{\omega}_p^4 + 2\left(k^2 - \tilde{\omega}_q^2\right)\tilde{\omega}_p^2 + \left(k^2 + \tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_3 = \frac{\left(\omega_q^{+2}+\tilde{\omega}_q^2\right)\left(\omega_p^{+2}\left(k^2+\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)^2+\tilde{\omega}_q^2\left(k^2-\tilde{\omega}_p^2+\tilde{\omega}_q^2\right)^2\right)}{4\omega_p^+\omega_q^+\left(\tilde{\omega}_q^5+2\left(k^2-\tilde{\omega}_p^2\right)\tilde{\omega}_q^3+\left(k^2+\tilde{\omega}_p^2\right)^2\tilde{\omega}_q\right)^2}$$

$$L_4 = -\frac{2\omega_p^{+2}\left(k^2+\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)^2+2\omega_q^{+2}\left(k^2-\tilde{\omega}_p^2+\tilde{\omega}_q^2\right)^2+\omega_p^+\omega_q^+\left(k^4-\left(\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)^2\right)}{4\omega_p^+\omega_q^+\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_5 = -\frac{k\left(8\omega_p^{+2}\left(k^2-\tilde{\omega}_p^2+\tilde{\omega}_q^2\right)\omega_q^{+2}+\omega_p^+\left(k^4+\left(\tilde{\omega}_q^2-\tilde{\omega}_p^2\right)\left(2k^2+3\tilde{\omega}_p^2+\tilde{\omega}_q^2\right)\right)\omega_q^+-4\tilde{\omega}_p^2\left(\left(k^2+\tilde{\omega}_p^2\right)^2-\tilde{\omega}_q^4\right)\right)}{8\omega_p^+\omega_q^+\tilde{\omega}_p\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_6 = -\frac{k\left(8\omega_p^{+2}\left(k^2+\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)\omega_q^{+2}+\omega_p^+\left(k^4+\left(\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)\left(2k^2+\tilde{\omega}_p^2+3\tilde{\omega}_q^2\right)\right)\omega_q^++4\tilde{\omega}_q^2\left(\tilde{\omega}_p^4-\left(k^2+\tilde{\omega}_q^2\right)^2\right)\right)}{8\omega_p^+\omega_q^+\tilde{\omega}_q\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_7 = \frac{4\omega_q^{+2}\left(\left(\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)^2-k^4\right)\tilde{\omega}_p^2+4\omega_p^{+2}\tilde{\omega}_q^2\left(\left(\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)^2-k^4\right)-\omega_p^+\omega_q^+\left(\tilde{\omega}_p^6+\left(2k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^4+\left(k^4-4\tilde{\omega}_q^2k^2-\tilde{\omega}_q^4\right)\tilde{\omega}_p^2+\tilde{\omega}_q^2\left(k^2+\tilde{\omega}_q^2\right)^2\right)}{8\omega_p^+\omega_q^+\tilde{\omega}_p\tilde{\omega}_q\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_8 = \frac{\left(\omega_p^{+2}+\tilde{\omega}_p^2\right)\left(\tilde{\omega}_q^8+\left(2k^2+\tilde{\omega}_p^2\right)\tilde{\omega}_q^6+\left(k^4-8\tilde{\omega}_p^2k^2-5\tilde{\omega}_p^4\right)\tilde{\omega}_q^4+3\tilde{\omega}_p^2\left(k^2+\tilde{\omega}_p^2\right)^2\tilde{\omega}_q^2+\omega_q^{+2}\left(\tilde{\omega}_p^6+\left(2k^2+\tilde{\omega}_q^2\right)\tilde{\omega}_p^4+\left(k^4-8\tilde{\omega}_q^2k^2-5\tilde{\omega}_q^4\right)\tilde{\omega}_p^2+3\tilde{\omega}_q^2\left(k^2+\tilde{\omega}_q^2\right)^2\right)\right)}{8\omega_p^+\omega_q^+\tilde{\omega}_p^2\tilde{\omega}_q^2\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_9 = \frac{\left(\omega_q^{+2}+\tilde{\omega}_q^2\right)\left(\left(\tilde{\omega}_p^2-\omega_p^{+2}\right)\left(\tilde{\omega}_q^2-\tilde{\omega}_p^2\right)\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)+2\left(\omega_p^{+2}+\tilde{\omega}_p^2\right)\left(\tilde{\omega}_p^6+\left(2k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^4+\left(k^4-4\tilde{\omega}_q^2k^2-\tilde{\omega}_q^4\right)\tilde{\omega}_p^2+\tilde{\omega}_q^2\left(k^2+\tilde{\omega}_q^2\right)^2\right)\right)}{8\omega_p^+\omega_q^+\tilde{\omega}_p^2\tilde{\omega}_q^2\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_{10} = -\frac{-2\left(\left(k^2-\tilde{\omega}_p^2+\tilde{\omega}_q^2\right)^2-4k^2\omega_p^{+2}\right)\omega_q^{+2}+\omega_p^+\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)\omega_q^++2\left(k^2\left(k^2+\tilde{\omega}_p^2+\tilde{\omega}_q^2\right)^2-\omega_p^{+2}\left(k^2+\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)^2\right)}{8\omega_p^+\omega_q^+\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_{11} = \frac{-2\left(\left(k^2-\tilde{\omega}_p^2+\tilde{\omega}_q^2\right)^2-4k^2\omega_p^{+2}\right)\omega_q^{+2}+\omega_p^+\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)\omega_q^++2\left(k^2\left(k^2+\tilde{\omega}_p^2+\tilde{\omega}_q^2\right)^2-\omega_p^{+2}\left(k^2+\tilde{\omega}_p^2-\tilde{\omega}_q^2\right)^2\right)}{8\omega_p^+\omega_q^+\left(\tilde{\omega}_p^4+2\left(k^2-\tilde{\omega}_q^2\right)\tilde{\omega}_p^2+\left(k^2+\tilde{\omega}_q^2\right)^2\right)^2}$$

$$L_{12} = \frac{\begin{aligned}&\left(\omega_q^{+2}-\tilde{\omega}_q^2\right)\tilde{\omega}_p^8+\left(\tilde{\omega}_q^4+\left(3\omega_p^{+2}-3\omega_q^+\omega_p^++\omega_q^{+2}\right)\tilde{\omega}_q^2+\left(2k^2-\omega_p^{+2}\right)\omega_q^{+2}\right)\tilde{\omega}_p^6\\&+\left(\tilde{\omega}_q^6+\left(8k^2-5\omega_p^{+2}-5\omega_q^{+2}+6\omega_p^+\omega_q^+\right)\tilde{\omega}_q^4+\left(3k^4-8\omega_q^{+2}k^2+2\omega_p^+\omega_q^+k^2+\omega_p^{+2}\left(6k^2+\omega_q^{+2}\right)\right)\tilde{\omega}_q^2+k^2\left(k^2-2\omega_p^{+2}\right)\omega_q^{+2}\right)\tilde{\omega}_p^4\\&+\left(\left(\tilde{\omega}_q^6-8k^2\tilde{\omega}_q^4+3k^4\tilde{\omega}_q^2+\omega_q^{+2}\left(-k^4+12\tilde{\omega}_q^2k^2+\tilde{\omega}_q^4\right)\right)\omega_p^{+2}+\omega_q^+\tilde{\omega}_q^2\left(k^2+\tilde{\omega}_q^2\right)\left(5k^2-3\tilde{\omega}_q^2\right)\omega_p^++\tilde{\omega}_q^2\left(2k^2+3\omega_q^{+2}-\tilde{\omega}_q^2\right)\left(k^2+\tilde{\omega}_q^2\right)^2\right)\tilde{\omega}_p^2\\&+\omega_p^{+2}\tilde{\omega}_q^2\left(k^2+\tilde{\omega}_q^2\right)^2\left(\tilde{\omega}_q^2-\omega_q^{+2}\right)\end{aligned}}{16\omega_p^+\omega_q^+\tilde{\omega}_p^2\left(k^2+\left(\tilde{\omega}_p-\tilde{\omega}_q\right)^2\right)^2\tilde{\omega}_q^2\left(k^2+\left(\tilde{\omega}_p+\tilde{\omega}_q\right)^2\right)^2}$$

We are not going to dwell more with this rather complicated intermediate result, but instead move on to calculations of the momentum integral in the next section.

7.2 Momentum integral

To calculate the momentum integral we have to do a few substitutions in (7.2)

$$\begin{aligned} q &\to |\boldsymbol{k}-\boldsymbol{p}| = k^2+p^2-2kp\cos\theta \\ \tilde{\omega}_p &\to \sqrt{\mu^2-p^2}, \quad p<\mu \\ \tilde{\omega}_q &\to \sqrt{\mu^2-(k^2+p^2-2kp\cos\theta)}, \quad |\boldsymbol{k}-\boldsymbol{p}|<\mu \\ \omega_p^+ &\to \sqrt{\mu^2+p^2} \\ \omega_q^+ &\to \sqrt{\mu^2+(k^2+p^2-2kp\cos\theta)} \end{aligned} \tag{7.4}$$

In order for the ω's in (7.4) to stay real, we restrict the values of k and p by introducing two Heaviside step functions (or better with UnitStep for Mathematica) in the integrand. Together with the definition in (7.2) substituted into (6.38) we obtain

$$\frac{\rho_{GW}}{d\ln k} = \frac{2Gk^3}{\pi^3}\int dpd\theta\, p^6\,(sin\theta)^5\, tint\,(\boldsymbol{k},\boldsymbol{p},t)\,\theta\,(\mu-p)\,\theta\left(\mu^2-\left(k^2+p^2-2kp\cos\theta\right)\right). \tag{7.5}$$

The integration is done numerically by Mathematica for a suitable number of values of k in the range $0<k/\mu<2$, and times t in the range $0\lesssim t\mu<4$ where we expect our free-field approximation to be valid.

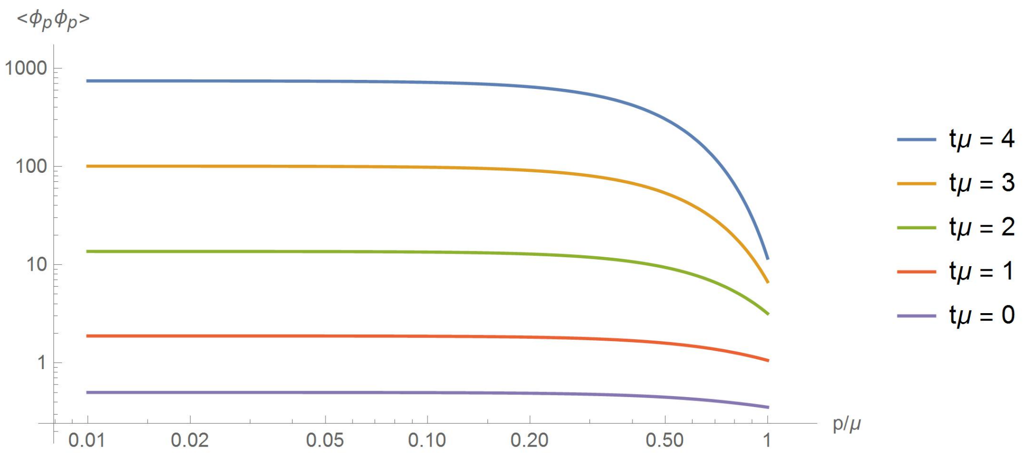

Figure 4: *2-point correlator as a function of modes at selected times.*

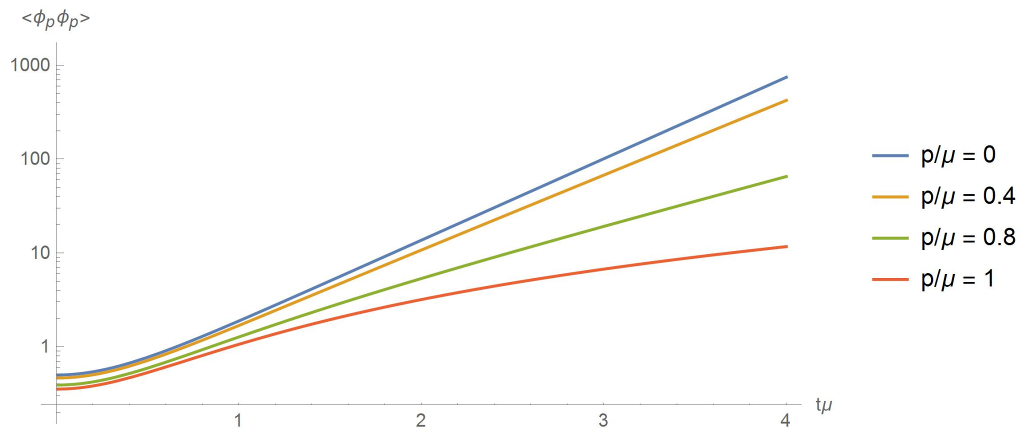

Figure 5: *2-point correlator as a function of time for some selected modes.*

8 Results

8.1 Equal time 2-point correlator

In our calculations we have used 2-point correlations functions as given by (6.28). In figure 4 we have plotted the equal time 2-point correlator as a function of p for some values of t. We see that we have exponential growth, most noticeable at $p = 0$.

In figure 5 we have plotted the equal time correlator as a function of t. We can clearly see how the modes quickly enters a stage of exponential growth. For $p = 0$ this continues through the whole time interval, while it flattens out with increasing p.

The energy density power spectrum we obtained numerically from (7.5) is presented as a plot in figure 6. We see that the gravitational waves has a peak at scales comparable to μ, from where the spectrum drops to zero at $k = 2$.

Now, dividing out the factor of k^3 in (7.5), we present the result in figure 7. To have a closer look at what happens for very small k, we focus on one time ($t = 2$), and in the interval $0.01 < k < 0.1$ in figure 8. We can now clearly see that it is close to constant near $k = 0$. Now, since this result is obtained from (7.5) divided by k^3, we conclude that for small k in our approximation we have

$$\frac{\rho_{GW}}{d \ln k} \propto k^3. \tag{8.1}$$

Validity of the approximation

Due to the free-field approximation the validity of this result will decrease with increasing time after the quench. This is because in the free field scenario we have neglected the ϕ^4 term of the potential. The question of what region of time this approximation is trustworthy is addressed in [16]. Following their procedure closely, we first assign a short notation for the correlator of the unstable modes, i.e. when $|k| < \mu$:

$$\varphi^2 \equiv \langle 0|\phi^2(\boldsymbol{x})|0\rangle_{unst}. \tag{8.2}$$

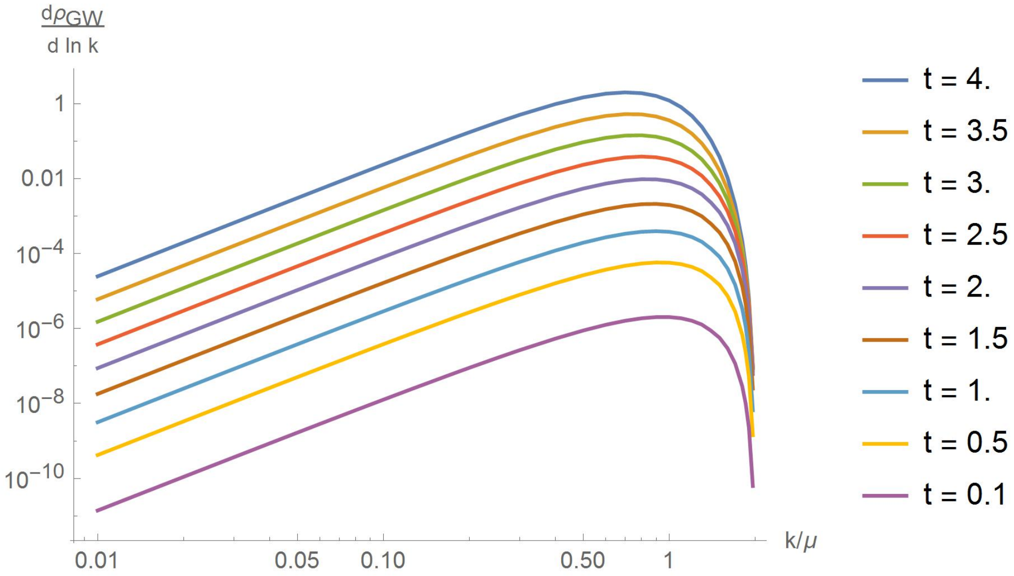

Figure 6: *Gravitational waves energy density power spectrum.*

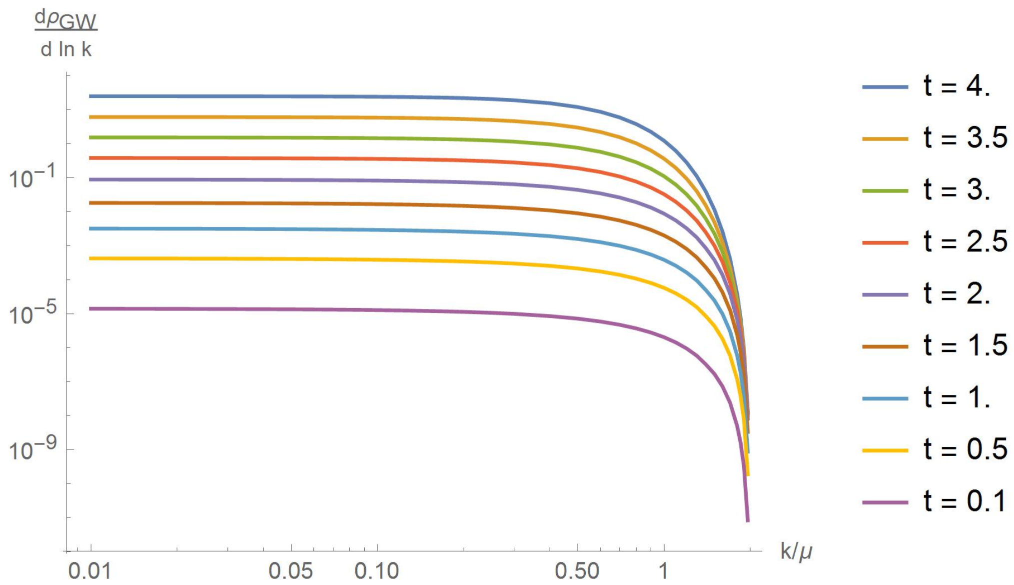

Figure 7: *The spectrum of figure 6 divided by* k^3.

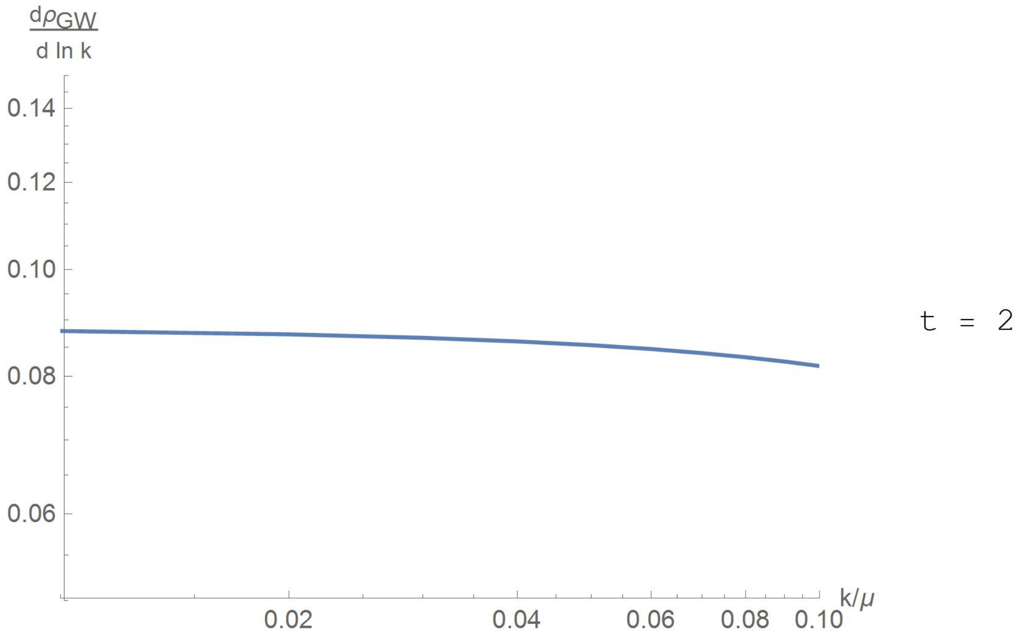

Figure 8: *Gravitational waves energy density power spectrum at $t = 2$, focused at lower momentum modes.*

We then say that we will no longer trust the result when time has passed a certain value t_{nl} (nl for 'non-linear'), for which the field has grown to a value coinciding with the inflection point of the true potential. The approximation we have made is

$$V(\phi) = \frac{\mu^4}{4\lambda} - \mu^2\phi^2 + \lambda\phi^4 \quad \rightarrow \quad \frac{\mu^4}{4\lambda} - \mu^2\phi^2, \tag{8.3}$$

where we have neglected the ϕ^4 term. The two potentials are illustrated in (figure. 10).

For small values of ϕ we have that $\phi^4 << \phi^2$, and our approximation is good. However, as ϕ grows, the approximation will eventually have too large errors to be useful. In line with [16] we choose to no longer trust the approximation past the inflection point where $\partial^2 V(\phi)/\partial\phi^2 = -\mu^2 + 3\lambda\phi^2$, i.e. when

$$\varphi^2 = \frac{\mu^2}{3\lambda} = \frac{v^2}{3}, \tag{8.4}$$

where v is the field value at the minimum of its potential. Thus we will not trust our calculations for field values beyond one third of its growth toward the minimum of its potential. In terms of time since the quench, we trust our results in the interval $0 < t < t_{nl}$. We can find φ^2 by integrating (6.28) over the $\boldsymbol{p}$'s:

$$\varphi^2 = \frac{1}{2\pi^2} \int_{|p|<\mu} dp\, p^2 F(p, t, t). \tag{8.5}$$

With $\lambda = 1/9$ we plot $\varphi^2 - \frac{\mu^2}{3\lambda}$ in figure 9, and obtain that $t_{nl} \approx 4.2$. Thus we do not trust calculations beyond $t = 4.2$.

9 Conclusion

In this thesis we have presented the basic ideas for cosmological inflation, and how the process of tachyonic preheating generates gravitational waves with a peak in the spectrum for modes comparable to the energy scale μ of the process. We have seen that for low momentum modes, the spectrum is proportional to k^3. A natural next step would be to leave the approximation and calculate the spectrum for the full potential. This requires a simulation approach as has been done in [15][17]. Another issue this thesis does not touch is in what frequency range the produced gravitational waves would be today, and what hope there would be for them to be detected by existing or planned detectors given different potentials, coupling constants and energy scales. Such issues are also discussed in [15][17].

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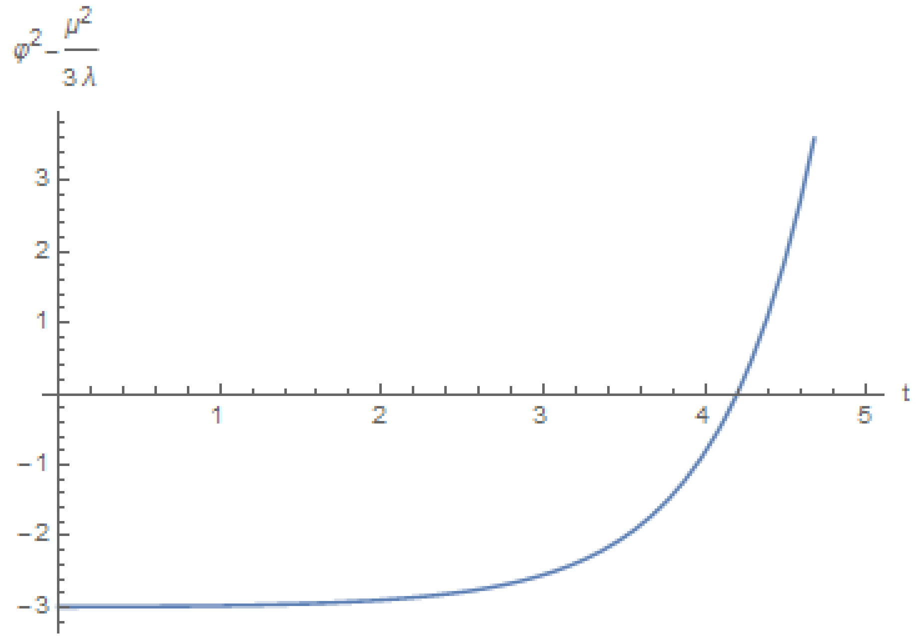

Figure 9: *We find t_{nl} where $\varphi^2 - \frac{\mu^2}{3\lambda} = 0$.*

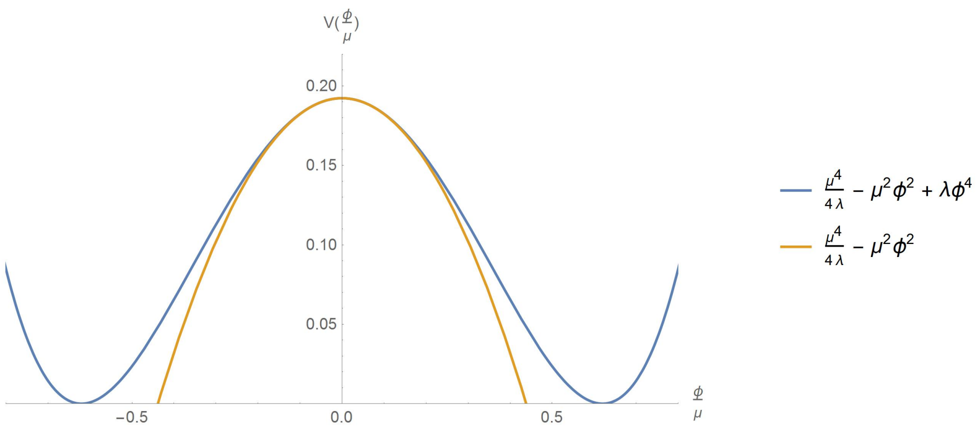

Figure 10: *The approximated potential deviates from the actual potential. At some point in time, as the system evolves, the deviation grows larger than what we can accept.*

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