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Stability of anisotropic perfect fluid spheres with electrical charge when the cosmological constant is included

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Part I

The Esculpi/Aloma mystery

1 Abstract

The main task of this thesis is to investigate the stability of anisotropic perfect fluid spheres with electrical charge when the cosmological constant is included. However, before we get so far we needed to read articles about the subject. We have read articles about the stability of anisotropic fluid spheres with and without charge, we have read articles about the stability of isotropic fluid spheres with and without cosmological constant, but one of the reasons we've taken on this thesis, is that we could not find any articles covering the stability of all of these things at the same time.

It was Chandrasekhar who was the first to develop this pulsation equation in his work from 1964 [1], for an isotropic, uncharged perfect fluid sphere. In 1979 Irving Glazer [2] developed the same pulsation equation for an isotropic charged fluid sphere. These pulsation equations have been showed to be trustworthy, so any pulsation equation we reach for the anisotropic charged perfect fluid sphere with the cosmological constant included should match these equations for the isotropic charged and uncharged case, without the cosmological constant.

While reading different articles [4] we stumbled over an article written by M.Esculpi and E.Aloma [3] regarding charged anisotropic fluid spheres. In this article most of the calculations are left out, which is not necessarily a problem in itself, but they state that they use the conservation of the energy-momentum tensor, they site the calculations of the required perturbed quantities, and simply gives the end result, which is a fairly long pulsation equation. The problem, however, is that in the few equations given there are several mistakes. They could very well be simple typos, but we wanted to make sure that the end result was trustworthy. Since no calculations were shown, we had to reproduce them and see whether we reach the same result.

After that we shall develop the pulsation equation for an anisotropic charged fluid sphere when the cosmological constant is included. This is an equation that, to our knowledge, has not been developed. The last section in this thesis will be dedicated to develop the Tolman-Oppenheimer-Volkoff equation to go with the pulsation equation.

2 The field equations

We shall start by looking at the article of M.Esculpi and E.Aloma.
The line element is given by:

$$ds^2 = e^{\nu}\ dt^2 - e^{\lambda}\ \ - r^2(d\theta^2 + sin^2 d\phi^2). \tag{1}$$

The Einstein field equations are further given by:

$$-e^{-\lambda}\left[\frac{1}{r^2} - \frac{\lambda'}{r}\right] + \frac{1}{r^2} = 8\pi T_0^0, \tag{2}$$

$$-e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] + \frac{1}{r^2} = 8\pi T_1^1, \tag{3}$$

$$-\frac{-\lambda}{2}\left[\nu'' + \frac{\nu'^2}{2} + \frac{(\nu' - \lambda')}{r} - \frac{\nu'\lambda'}{2}\right] = 8\pi T_2^2, \tag{4}$$

$$-\frac{e^{-\lambda}}{r}\dot{\lambda} = -8\pi T_0^1, \tag{5}$$

where ν and λ are unknown functions of the radial coordinate r, primes denote derivatives with respect to the radial coordinate r, and dots denote derivatives with respect to the time coordinate x^0.

3 The energy-momentum tensor

The gravitational energy part of the energy-momentum tensor reads

$$T_{\mu}^{\nu(g)} = (\rho + p_t)u_\mu u^\nu - g_\mu^\nu p_t + (p_r - p_t)\chi_\mu \chi^\nu, \tag{6}$$

where ρ is the matter energy density, p_r is the radial pressure in the direction of χ_μ, p_t is the pressure in the two-space orthogonal to χ_μ , u_μ is the velocity four-vector of the fluid , and χ_μ is a unit space vector in the radial direction orthogonal to u_μ.

The electromagnetic field tensor is of the form

$$T_{\mu}^{\nu(em)} = \frac{1}{4\pi}\left(\frac{1}{4}f_{\alpha\beta}f^{\alpha\beta} - g^{\mu\nu}g^{\alpha\beta}f_{\mu\alpha}f_{\nu\beta}\right), \tag{7}$$

where the electromagnetic field tensor $f_{\mu\nu}$ is given in terms of the electromagnetic potentials A_μ

$$f_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} \tag{8}$$

If we consider the potential vector

$$A_\mu = (A_0, 0, 0, 0), \tag{9}$$

it follows that

$$f_{01} = -f_{10} = A_{0;1} \tag{10}$$

The Maxwell equations reads:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}f^{\mu\nu}\right) = 4\pi\sigma u^\mu, \tag{11}$$

σ is the charge density, and σu^μ is the current vector.

Before we go anywhere we need to establish how the u_μ's and χ_μ's are defined:

$$u^\mu = \frac{dx^\mu}{ds} \tag{12}$$

Our line element yields

$$1 = e^\nu\left(\frac{dt}{ds}\right)^2 - e^\lambda\left(\frac{dr}{ds}\right)^2 - r^2\left(\left(\frac{d\theta}{ds}\right)^2 + sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right) \tag{13}$$

Since we are only interested in radial motions we will put $\frac{\mathsf{d}\theta}{\mathsf{ds}} = \frac{\mathsf{d}\phi}{\mathsf{ds}} = 0$ so we get:

$$1 = e^\nu(u^0)^2 - e^\lambda(u^1)^2 \tag{14}$$

Considering u^1 we obtain

$$u^1 = \frac{dr}{dx^0} \times \frac{dx^0}{ds} = u^0\dot{\xi}, \tag{15}$$

where we defined $\dot{\xi} = \frac{dr}{dx^0}$
Solving (5) for u^0 and then (6) for u^1 to first order we find:

$$u^0 = e^{-\frac{\nu}{2}}, \tag{16}$$

and

$$u^1 = \dot{\xi} e^{-\frac{\nu}{2}}. \tag{17}$$

Here Esculpi/Aloma has made the first printing mistake, in writing $u^1 = \dot{\xi} e^{-\frac{\lambda}{2}}$
Now, the χ^μ's are not so obvious, but from the written definition we have what we need: χ^μ should be orthogonal to u^μ and orthonormal, meaning

$$\chi^\mu u_\mu = 0, \tag{18}$$

and

$$\chi^\mu \chi_\mu = -1. \tag{19}$$

Which leads to the same results as Esculpi/Aloma, namely

$$\chi^0 = e^{\frac{\lambda}{2} - \nu} \dot{\xi} \tag{20}$$

$$\chi^1 = e^{-\frac{\lambda}{2}} \tag{21}$$

For the calculation of the electromagnetic field tensor, it helps to know that under spherical symmetry only f_{01} and f_{10} are non-zero, and furthermore:

$$f_{10} = -f_{01}. \tag{22}$$

For the static case, $\dot{\xi}$ vanishes, and we are left with $u^\mu = (u^0, 0, 0, 0)$ and $\chi^\mu = (0, \chi^1, 0, 0)$
From (6) we obtain

$$T_0^{0(em)} = \frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}, \tag{23}$$

$$T_1^{1(em)} = \frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}, \tag{24}$$

$$T_2^{2(em)} = -\frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}, \tag{25}$$

$$T_3^{3(em)} = -\frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}. \tag{26}$$

From the Maxwell equations, equation (11) we further have

$$\sqrt{-g} \frac{\partial}{\partial r} \left(\sqrt{-g} f^{01} \right) = \frac{e^{-\frac{\lambda+\nu}{2}}}{r^2 sin\theta} \frac{\partial}{\partial r} \left(e^{\frac{\lambda+\nu}{2}} r^2 sin\theta f^{01} \right) = 4\pi\sigma e^{-\frac{\nu}{2}}. \tag{27}$$

Solving for f^{01} by integration we get

$$e^{\frac{\lambda+\nu}{2}} r^2 f^{01} = \int 4\pi\sigma r^2 e^{\frac{\lambda}{2}} dr. \tag{28}$$

The right hand side of this equation is merely the total charge within a sphere, so we call it Q:

$$Q(r) = \int_0^r 4\pi\sigma e^{\frac{\lambda}{2}} \tau^2 d\tau. \tag{29}$$

We then obtain

$$\sigma = \frac{Q'(r)}{4\pi r^2} e^{\lambda+\nu}. \tag{30}$$

By lowering the indices, and reversing the order of them, we obtain

$$e^{-\frac{\lambda+\nu}{2}} f_{10} = \frac{1}{r^2} \int 4\pi r^2 \sigma e^{\frac{\lambda}{2}} dr = \frac{Q(r)}{r^2}, \tag{31}$$

which yields

$$(f_{10})^2 = \frac{Q^2(r)}{r^4} e^{\lambda+\nu}. \tag{32}$$

We now find the components of the energy-momentum tensor

$$T_\mu^\nu = T_\mu^{(m)\nu} + T_\mu^{(em)\nu}. \tag{33}$$

We obtain

$$T_0^0 = \rho + \eta, \tag{34}$$

$$T_1^1 = -p_r + \eta, \tag{35}$$

$$T_2^2 = -p_t - \eta, \tag{36}$$

$$T_3^3 = -p_t - \eta, \tag{37}$$

where we follow Glazers notation and define

$$\eta = T_0^{0(em)} = \frac{Q^2}{8\pi r^2} = \frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}, \tag{38}$$

The pulsation equation itself comes from the covariant divergence of the energy-momentum tensor, namely

$$T^\nu_{\mu;\nu} = 0, \tag{39}$$

is a necessary identity. By choosing $\mu = 1$ and using that the covariant derivative can be written as

$$T^\nu_{\mu;\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} T_\mu^\nu - \frac{T^{\nu\alpha}}{2} \frac{\partial g_{\nu\alpha}}{\partial x^\mu}. \tag{40}$$

we get the following relation

$$\frac{\partial T_1^0}{\partial x^0} + \frac{\partial T_1^1}{\partial r} + T_1^0 \left(\frac{\dot{\nu} - \dot{\lambda}}{2} \right) + \frac{\nu'}{2} \left(T_1^1 - T_0^0 \right) + \frac{2}{r} \left(T_1^1 - \frac{T_2^2 - T_3^3}{2} \right) = 0. \tag{41}$$

For the static case, we distinguish the quantities describing this equilibrium state by a subscript zero. All time derivatives vanish for the equilibrium quantities by definition. Using equations (34)-(37), equation (41) yields a relation between the equilibrium values

$$\frac{\nu_0'}{2}\left(\rho_0 + p_{r0}\right) = \left(\eta_0' - p_{r0}'\right) + \frac{4}{r}\eta_0 - \frac{2}{r}\left(p_{r0} - p_{t0}\right). \tag{42}$$

Subtracting (2) by (3) we get the identity

$$8\pi\left(T_0^0 - T_1^1\right) = 8\pi(\rho + p_r) = e^{-\lambda}\left(\frac{\nu' + \lambda'}{r}\right). \tag{43}$$

4 Perturbations

Perturbing the equilibrium state we write

$$\lambda = \lambda_0(r) + \delta\lambda(r, x^0), \tag{44}$$

$$\nu = \nu_0(r) + \delta\nu(r, x^0), \tag{45}$$

$$\rho = \rho_0(r) + \delta\rho(r, x^0), \tag{46}$$

$$p_r = p_{r0}(r) + \delta p_{r0}(r, x^0), \tag{47}$$

$$p_t = p_{t0}(r) + \delta p_{t0}(r, x^0), \tag{48}$$

$$\eta = \eta_0(r) + \delta\eta(r, x^0). \tag{49}$$

For small perturbations, to first order we have:

$$e^{\lambda_0+\delta\lambda} = e^{\lambda_0}(1 + \delta\lambda), \tag{50}$$

$$e^{\nu_0+\delta\nu} = e^{\nu_0}(1 + \delta\nu). \tag{51}$$

To first order, all T^{μ}_{μ}'s remain the same, except for these perturbations to the quantities.

$$T^0_0 = \rho_0 + \delta\rho + \eta_0 + \delta\eta, \tag{52}$$

$$T^1_1 = -p_{r0} - \delta p_r + \eta_0 + \delta\eta, \tag{53}$$

$$T^2_2 = -p_{t0} - \delta p_t - \eta_0 + \delta\eta, \tag{54}$$

$$T^3_3 = -p_{t0} - \delta p_t - \eta_0 + \delta\eta, \tag{55}$$

Unlike the static case, we now get another non-zero entry in our energy-momentum tensor. Using equations (6), (16), (17), (20), and (21) we obtain

$$T^0_1 = -(\rho_0 + p_{r0} + \delta\rho + \delta p_r)e^{(\lambda_0-\nu_0)}(1 + \delta\lambda)(1 - \delta\nu)\dot{\xi}, \tag{56}$$

which to first order is

$$T^0_1 = -(\rho_0 + p_{r0})e^{(\lambda_0-\nu_0)}\dot{\xi}. \tag{57}$$

Now, the conservation of the energy-momentum-tensor becomes littered with these perturbed quantities, and it is equation (41) that we will develop into the pulsation equation:

$$T^{\alpha}_{1;\alpha} = \frac{\partial T^0_1}{\partial x^0} + T^0_1(\dot{\nu_0} + \dot{\delta\nu} + \dot{\lambda_0} + \dot{\delta\lambda}) - \frac{\nu_0' + \delta\nu'}{2}(T^0_0 - T^1_1) + \frac{\partial T^1_1}{\partial r} + \frac{2}{r}\left(T^1_1 - \frac{T^2_2 + T^3_3}{2}\right) = 0. \tag{58}$$

Inserting equations (52),(53),(54),(55),(57) into equation (58) and keeping only the first order terms

$$-\frac{\partial}{\partial x^0}\left(e^{\lambda_0-\nu_0}\dot{\xi}(\rho_0+p_{r0})\right)-\frac{\nu_0'}{2}\left(\rho_0+p_{r0}+\delta\rho+\delta p_r\right)-\frac{\delta\nu'}{2}\left(\rho_0+p_{r0}\right) \quad (59)$$
$$+\frac{\partial}{\partial r}\left(\eta_0-p_{r0}+\delta\eta-\delta p_r\right)+\frac{2}{r}\left(2\eta_0+2\delta\eta+p_{t0}-p_{r0}+\delta p_t-\delta p_r\right)=0.$$

Using equation (41) and cancelling the static solution we are left with the equation

$$\begin{aligned} e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\ddot{\xi} &= \left(\delta\eta'-\delta p_r'\right)-\frac{\nu_0'}{2}\left(\delta\rho+\delta p_r\right)-\frac{\delta\nu'}{2}\left(\rho_0+p_{r0}\right) \\ &\quad+\frac{4}{r}\delta\eta+\frac{2}{r}\left(\delta p_t-\delta p_r\right) \end{aligned} \quad (60)$$

This is the equation that will yield the "pulsation equation" once we insert the expressions for these perturbed quantities. Here, Esculpi/Aloma simply cite "Chandrasekhar's work" and claims that $\delta\eta$ has been obtain as proposed by Glazer. The quantities are easily derived, and it is therefore weird that of the six proposed results, three of them are flawed, and the result for $\delta\eta$ is not usable in the form given. This leaves quite an impact on the pulsation equation as there are many terms that could have been cancelled, but was not.

Compared to the isotropic case, here we have one extra variable, since the pressure p now contains two kinds of pressure, the radial pressure p_r and the tangential pressure p_t. Hence we will also need an extra equation to solve the system. We will keep it simple, as Esculpi/Aloma did, and look at the case where the tangential pressure p_t is proportional to the radial pressure p_r, i.e.

$$p_t = Cp_r, \quad (61)$$

which immediately yields

$$\delta p_t = C\delta p_r. \quad (62)$$

We now need to show what the other perturbed quantities look like. We shall start by looking at the field equations. Subtracting (3) from (2) (both static) we find

$$8\pi(T_0^0-T_1^1)=\frac{e^{-\lambda_0}}{r}(\lambda_0'+\nu_0'), \quad (63)$$

which we will write as

$$8\pi r e^{\lambda_0}(\rho_0+p_{r0})=(\nu_0'+\lambda_0'). \quad (64)$$

Inserting equation (64) into (5), we can integrate

$$\dot{\delta\lambda}=-8\pi r e^{\lambda_0}\dot{\xi}(\rho_0+p_{r0}), \quad (65)$$

to find

$$\delta\lambda=-8\pi r e^{\lambda_0}\xi(\rho_0+p_{r0}). \quad (66)$$

looking at (64) and (66) we see that we necessarily have

$$\delta\lambda = -\xi(\nu_0' + \lambda_0'). \tag{67}$$

Esculpi/Aloma gives the expression for $\delta\lambda$ with an extra factor of r^{-1}.

Taking the first field equation into account, equation (2) reads

$$-e^{-\lambda_0}(1-\delta\lambda)\left[\frac{1}{r^2} - \frac{\lambda_0' + \delta\lambda}{r}\right] + \frac{1}{r^2} = 8\pi(\rho_0 + \eta_0 + \delta\rho + \delta\eta), \tag{68}$$

and cancelling the static solution

$$-e^{-\lambda_0}\left[\frac{1}{r^2} - \frac{\lambda_0'}{r}\right] + \frac{1}{r^2} = 8\pi(\rho_0 + \eta_0). \tag{69}$$

We find that the perturbations relate to each other

$$e^{-\lambda_0}\delta\lambda\left(\frac{1}{r^2} - \frac{\lambda_0'}{r}\right) + \frac{e^{-\lambda_0}}{r}(\delta\lambda)' = 8\pi(\delta\rho + \delta\eta). \tag{70}$$

Multiplying the equation with r^2 we simplify the left-hand side

$$e^{-\lambda_0}\delta\lambda(1-\lambda_0' r) + e^{-\lambda_0}r(\delta\lambda)' = (e^{-r\lambda_0}\delta\lambda)' = 8\pi r^2(\delta\rho + \delta\eta) \tag{71}$$

And using (66) we get

$$8\pi r^2(\delta\rho + \delta\eta) = (re^{-\lambda_0}[-8\pi re^{\lambda_0}\xi(\rho_0 + p_{r0})])'. \tag{72}$$

Solving equation (72) for $\delta\rho$ we further have

$$\delta\rho = -\frac{1}{r^2}\left(r^2\xi(\rho_0 + p_{r0})\right)' - \delta\eta. \tag{73}$$

Here, Esculpi/Aloma has lost the negative-sign in the first term on the right hand side.

Finding a useful expression for $\delta\eta$ is particularly long-winded, but more or less straight-forward. While Esculpi/Aloma do give an expression for it, the expression they give is useless in it's given form. From their given expression it is easy to express $\delta\eta$ in a way that is very useful since it will simplify the pulsation equation considerably. Irving Glazer did this in his paper, and Esculpi/Aloma sited his paper, so they surely shouldn't be strangers to this simplification.

Let's just do a quick relabelling to make things less confusing.

$$f_{01} = E_0 + \delta E, \tag{74}$$

where E_0 and δE are the equilibrium and perturbed quantities of the electromagnetic field tensor respectively. The energy-momentum of the field now becomes

$$T_0^{0(em)} = \frac{e^{-(\lambda+\nu)}}{8\pi}(f_{10})^2 = \frac{e^{-(\lambda_0+\nu_0)}(1-\delta\lambda-\delta\nu)}{8\pi}(E_0 + \delta E)^2. \tag{75}$$

From equation (38) we defined the quantity η to be just this $T_0^{0(em)}$, and so also in the perturbed state:

$$\eta = \eta_0 + \delta\eta = \frac{e^{-(\lambda_0+\nu_0)}}{8\pi} E_0^2 (1 - (\delta\lambda + \delta\nu) + 2\frac{\delta E}{E_0})), \tag{76}$$

to first order. Looking at the Maxwell equations (11) in this perturbed state we find

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} \left(\sqrt{-g} f^{10}\right) = 4\pi\sigma u^1, \tag{77}$$

which reads

$$\frac{e^{-\frac{(\lambda_0+\nu_0)}{2}}}{r^2 sin\theta} \frac{\partial}{\partial x^0} \left(e^{-\frac{\lambda_0+\nu_0}{2}} (1 - \frac{\delta\lambda + \delta\nu}{2}) r^2 sin\theta (E_0 + \delta E) \right) = 4\pi\sigma_0 \dot{\xi} e^{\frac{-\nu_0}{2}}. \tag{78}$$

Since we this time differentiate with respect to time both r^2 and $sin\theta$ will cancel. This equation can be immediately integrated to give

$$\left(-\frac{\delta\lambda + \delta\nu}{2} \right) E_0 + \delta E = 4\pi\sigma_0 \xi e^{\frac{2\lambda_0+\nu_0}{2}}. \tag{79}$$

Multiplying both sides with $2E_0$ we find

$$E_0^2 \left(-(\delta\lambda + \delta\nu) + \frac{2\delta E}{E_0} \right) = 8\pi E_0 \sigma_0 \xi e^{\frac{2\lambda_0+\nu_0}{2}} \tag{80}$$

Inserting in (63) we have

$$\eta_0 + \delta\eta = \frac{e^{-(\lambda_0+\nu_0)}}{8\pi} E_0^2 (1 + \frac{8\pi\sigma_0\xi e^{\frac{2\lambda_0+\nu_0}{2}}}{E_0}) = \frac{e^{-(\lambda_0+\nu_0)}}{8\pi} E_0^2 + \sigma_0 \xi E_0 e^{\frac{-\nu_0}{2}}. \tag{81}$$

Remembering equations (32) and (38), for the static case we find

$$\eta_0 = \frac{e^{-(\lambda_0+\nu_0)}}{8\pi} E_0^2, \tag{82}$$

By equations (81) and (82) we must have that

$$\delta\eta = \sigma_0 \xi E_0 e^{-\frac{\nu_0}{2}} \tag{83}$$

This is the expression given in Esculpi/Aloma, but it is not very useful, as it doesn't simplify anything. Further investigation however yields a very simple and powerful expression for $\delta\eta$. Equation (38) can be written

$$\eta = \frac{Q^2(r)}{8\pi r^4} = \eta_0 + \delta\eta = \frac{(Q_0(r) + \delta Q)^2}{8\pi r^4} \tag{84}$$

To first order we have

$$\delta\eta = \frac{Q_0(r)\delta Q}{4\pi r^4} \tag{85}$$

we further have

$$(\eta_0)' = \frac{2Q_0(r)Q_0'(r)}{8\pi r^4} - \frac{Q_0^2(r)}{2\pi r^5}. \tag{86}$$

We defined $Q_0(r)$ earlier, in equation (29), and from equation (30) we find $Q_0(r)'$ as

$$Q_0'(r) = 4\pi r^2 \sigma_0 e^{\frac{\lambda_0}{2}}. \tag{87}$$

Equation (86) now reads

$$(\eta_0)' = \frac{Q_0(r)\sigma_0 e^{\frac{\lambda_0}{2}}}{r^2} - \frac{Q_0^2(r)}{2\pi r^5}. \tag{88}$$

From equations (22) and (31) we have

$$(\eta_0)' = -\sigma_0 E_0 e^{\frac{-\nu_0}{2}} - \frac{Q_0^2(r)}{2\pi r^5}, \tag{89}$$

and by employing equation (83), (89) can be written as

$$(\eta_0)' = -\frac{\delta\eta}{\xi} - \frac{4}{r}\eta_0. \tag{90}$$

This last equation may also be written in the following way

$$\delta\eta + (\eta_0' + \frac{4}{r}\eta_0)\xi = 0. \tag{91}$$

If you take a look at the perturbed quantities given by Esculpi/Aloma, you see that in the expression for δp_r, equation (48) in their paper, these terms can be cancelled. In the pulsation equation that follows this zero-term appears four(!) times, and could have been avoided altogether. Thus the calculations became much more complicated than they need to be.

Now all we need is the expression for δp_r and we can squeeze out the pulsation equation. To produce the expression for δp_r we need to make use of a supplementary condition, the conservation of baryon number i.e.

$$(Nu^\alpha)_{;\alpha} = 0. \tag{92}$$

This means that the total number of particles in the system remain unchanged. We write

$$N = N_0(r) + \delta N(r, x^0). \tag{93}$$

The contraction of the covariant derivative of a contra-variant vector Nu^α is given by

$$(Nu^\alpha)_{;\alpha} = \frac{\partial(Nu^\nu)}{\partial x^\nu} + Nu^\mu \frac{\partial(ln\sqrt{-g})}{\partial x^\mu}. \tag{94}$$

We remember equations (16) and (17), that $u^\mu = (e^{-\frac{\nu_0}{2}}, \dot{\xi} e^{-\frac{\nu_0}{2}}, 0, 0)$. Hence (71) reads, to first order

$$\begin{aligned}
&(N_0)\frac{\partial u^0}{\partial x^0} + \frac{\partial \delta N}{\partial x^0} u^0 + (N_0)\frac{\partial u^1}{\partial x^1} + \frac{\partial N_0}{\partial x^1} u^1 \\
&+ N_0 \left(\frac{\dot{\delta\lambda} + \dot{\delta\nu}}{2} \right) u^0 + N_0 \left(\frac{\lambda_0' + \nu_0'}{2} + \frac{2}{r} \right) = 0.
\end{aligned} \tag{95}$$

Inserting our expressions for u^μ we find

$$\begin{aligned}
&- N_0 e^{-\frac{\nu_0}{2}} \frac{\dot{\delta\nu}}{2} + \dot{\delta N} e^{-\frac{\nu_0}{2}} + (N_0 e^{-\frac{\nu_0}{2}} \dot{\xi})' + N_0 e^{\frac{-\nu_0}{2}} \frac{\dot{\delta\nu}}{2} + N_0 e^{-\frac{\nu_0}{2}} \frac{\dot{\delta\lambda}}{2} \\
&+ N_0 e^{-\frac{\nu_0}{2}} \dot{\xi} \left(\frac{\lambda_0' + \nu_0'}{2} + \frac{2}{r} \right) = 0,
\end{aligned} \tag{96}$$

where the first and the fourth term cancel. We can actually integrate this equation immediately and solve for δN

$$\delta N e^{-\frac{\nu_0}{2}} + \left(N_0 e^{-\frac{\nu_0}{2}} \xi \right)' + N_0 e^{-\frac{\nu_0}{2}} \delta\lambda + N_0 e^{-\frac{\nu_0}{2}} \xi \left(\frac{\lambda_0' + \nu_0'}{2} + \frac{2}{r} \right) = 0. \tag{97}$$

We find that equation (97) can be simplified to give

$$\delta N + \frac{e^{\frac{\nu_0}{2}}}{r^2} (N_0 e^{-\frac{\nu_0}{2}} r^2 \xi)' + N_0 \left(\delta\lambda + \xi \left(\frac{\lambda_0' + \nu_0'}{2} \right) \right) = 0 \tag{98}$$

And by (67) the third term here is actually zero, which means we get the very nice expression

$$\delta N = -\frac{e^{\frac{\nu_0}{2}}}{r^2} (N_0 e^{-\frac{\nu_0}{2}} r^2 \xi)'. \tag{99}$$

Now if

$$N = N(\rho(r, x^0), p_r(r, x^0)), \tag{100}$$

is the equation of state, it follows that

$$\delta N = \frac{\partial N}{\partial \rho} \delta\rho + \frac{\partial N}{\partial p_r} \delta p_r. \tag{101}$$

By equations (73), (91), and (99), equation (101) reads

$$\frac{\partial N}{\partial p_r} \delta p_r = -\frac{e^{\frac{\nu_0}{2}}}{r^2} (N_0 e^{-\frac{\nu_0}{2}} r^2 \xi)' - \frac{\partial N}{\partial \rho} \left[-\frac{1}{r^2} \left(r^2 \xi (\rho_0 + p_{r0}) \right)' + (\eta_0' + \frac{4}{r} \eta_0) \xi \right]. \tag{102}$$

Expanding some terms we obtain

$$\frac{\partial N}{\partial p_r} \delta p_r = -N_0' \xi - N_0 \frac{e^{\frac{\nu_0}{2}}}{r^2} (e^{-\frac{\nu_0}{2}} r^2 \xi)' - \frac{\partial N}{\partial \rho} \left[-\xi (p_{r0}' - \frac{4}{r} \eta_0 - \eta_0' + \rho_0') + \frac{\rho_0 + p_{r0}}{r^2} (r^2 \xi)' \right]. \tag{103}$$

Using equation (42)we can rewrite equation (103) in the following way

$$\begin{aligned}\frac{\partial N}{\partial p_r}\delta p_r =& - N_0'\xi - N_0\frac{e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' \\ &- \frac{\partial N}{\partial \rho}\left[\xi\left(\frac{\nu_0'}{2}(\rho_0+p_{r0})+\frac{2}{r}(p_{r0}-p_{t0})\right)-\frac{\rho_0+p_{r0}}{r^2}(r^2\xi)'-\xi\rho_0'\right].\end{aligned} \tag{104}$$

We collect the terms in the bracket and obtain

$$\begin{aligned}&-\frac{\partial N}{\partial \rho}\left[\xi\left(\frac{\nu_0'}{2}(\rho_0+p_{r0})+\frac{2}{r}(p_{r0}-p_{t0})\right)-\frac{\rho_0+p_{r0}}{r^2}(r^2\xi)'-\xi\rho_0'\right]= \\ &-\frac{\partial N}{\partial \rho}\left[\xi\frac{2}{r}(p_{r0}-p_{t0})-\frac{(\rho_0+p_{r0})e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'-\xi\rho_0'\right].\end{aligned} \tag{105}$$

Using equation (105), equation (104) now reads

$$\frac{\partial N}{\partial p_r}\delta p_r = -N_0'\xi-(N_0-(\rho_0+p_{r0})\frac{\partial N}{\partial \rho})\frac{e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'-\frac{\partial N}{\partial \rho}\left(\frac{2}{r}(p_{r0}-p_{t0})-\rho_0'\right). \tag{106}$$

We further have

$$N_0'(r,x^0)=N_0'(\rho(r,x^0),p_r(r,x^0))=\frac{\partial N_0}{\partial \rho_0}\frac{\partial \rho_0}{\partial r}+\frac{\partial N_0}{\partial p_{r0}}\frac{\partial p_{r0}}{\partial r}. \tag{107}$$

Keeping our analysis to first order, we must only bring $\frac{\partial N}{\partial p_r}$ to zero'th order.

$$\frac{\partial N}{\partial p_r}\delta p_r=\frac{\partial N_0}{\partial p_{r0}}\delta p_r, \tag{108}$$

and similar for $\frac{\partial N}{\partial \rho}$.

$$\frac{\partial N}{\partial \rho}\delta \rho=\frac{\partial N_0}{\partial \rho_0}\delta \rho. \tag{109}$$

Using equations (107), (108), and (109), equation (106) becomes

$$\begin{aligned}\delta p_r =& -\xi\left(\frac{\partial N_0}{\partial \rho_0}\frac{\partial \rho_0}{\partial r}+\frac{\partial N_0}{\partial p_{r0}}\frac{\partial p_{r0}}{\partial r}\right)\frac{\partial p_{r0}}{\partial N_0} \\ &-\frac{\partial p_{r0}}{\partial N_0}\left(N_0-(\rho_0+p_{r0})\frac{\partial N_0}{\partial \rho_0}\right)\frac{e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' \\ &-\frac{\partial N_0}{\partial \rho_0}\frac{\partial p_{r0}}{\partial N_0}\left(\frac{2}{r}(p_{r0}-p_{t0})\right)+\xi\frac{\partial p_{r0}}{\partial N_0}\frac{\partial N_0}{\partial \rho_0}\frac{\partial \rho_0}{\partial r}.\end{aligned} \tag{110}$$

Here the first and the last term on the right hand side cancel, two factors of the second term cancel, and we can can recognize one of the factors as the ratio of specific heats, that is

$$\gamma=\frac{1}{p_{r0}\frac{\partial N_0}{\partial p_{r0}}}\left(N_0-(\rho_0+p_{r0})\frac{\partial N_0}{\partial \rho_0}\right). \tag{111}$$

There is also an equivalent definition of this γ,

$$\gamma = \frac{(p_{r0}+\rho_0)}{p_{r0}}\frac{\partial p_{r0}}{\partial \rho_0}, \tag{112}$$

which yields

$$\frac{\partial p_{r0}}{\partial \rho_0} = \frac{\gamma p_{r0}}{\rho_0 + p_{r0}}. \tag{113}$$

Employing equations (111) and (112), equation (110) can now be written as

$$\delta p_r = -\xi p'_{r0} - \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2 \xi)' + \frac{\gamma p_{r0}\xi}{\rho_0 + p_{r0}}\frac{2}{r}(p_{t0} - p_{r0}). \tag{114}$$

Here Esculpi/Aloma again lost a factor, i.e. $\frac{e^{\frac{\nu_0}{2}}}{r^2}$ in the second term and they also carried along the three terms from equation (91) that cancel.
We now need the perturbed expression for ν'. Equation (3) when perturbed yields, to first order

$$-e^{-\lambda_0}(1-\delta\lambda)\left[\frac{\nu'_0}{r} + \frac{1}{r^2}\right] - e^{-\lambda_0}\left[\frac{\delta\nu'}{r}\right] = 8\pi(-p_{r0} - \delta p_r + \eta_0 + \delta\eta). \tag{115}$$

The static solution to (3) is:

$$-e^{-\lambda_0}\left[\frac{\nu'_0}{r} + \frac{1}{r}\right] + \frac{1}{r} = 8\pi\left(-p_{r0} + \eta_0\right). \tag{116}$$

Using equation (116) to cancel the static solution from (115) we find

$$\frac{e^{-\lambda_0}}{r}(\delta\nu') = -e^{-\lambda_0}\frac{\delta\lambda}{r}\left[\nu'_0 + \frac{1}{r}\right] + 8\pi(\delta p_r - \delta\eta). \tag{117}$$

Employing equation (66) we find

$$\frac{e^{-\lambda_0}}{r}(\delta\nu') = 8\pi(\delta p_r - \delta\eta) + 8\pi(\rho_0 + p_{r0})\xi\left(\nu'_0 + \frac{1}{r}\right). \tag{118}$$

Remembering equation (64) we can write the left hand side in the following way:

$$(\delta\nu')(\rho_0 + p_{r0}) = \left[\delta p_r - \delta\eta - (\rho_0 + p_{r0})\xi\left(\nu'_0 + \frac{1}{r}\right)\right](\nu'_0 + \lambda'_0). \tag{119}$$

This equation matches equation (50) from Esculpi/Aloma.

5 The pulsation equation

We shall assume perturbation in the form of radial oscillation,

$$\xi = e^{i\omega x^0}, \tag{120}$$

which leads to

$$\ddot{\xi} = -\xi\omega^2. \tag{121}$$

Using equations (61), (62), (73), (91),(114), and (119), equation (60) takes the following complicated form

$$\begin{aligned}
e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\xi\omega^2 = & \left((\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi - \frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right)' \\
& + \left(\frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}\frac{2}{r}(C-1)p_{r0}\right)' \\
& + \frac{\nu_0'}{2}\left(-\frac{1}{r^2}\left(r^2\xi(\rho_0+p_{r0})\right)' + (\eta_0' + \frac{4}{r}\eta_0)\xi - \xi p_{r0}'\right) \\
& + \frac{\nu_0'}{2}\left(-\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' + \frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}\frac{2}{r}(C-1)p_{r0}\right) \\
& + \left[(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi - (\rho_0+p_{r0})\xi\left(\nu_0' + \frac{1}{r}\right)\right]\frac{(\nu_0'+\lambda_0')}{2} \\
& + \left[-\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' + \frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}\frac{2}{r}(C-1)p_{r0}\right]\frac{(\nu_0'+\lambda_0')}{2} \\
& + \frac{4}{r}(\eta_0' + \frac{4}{r}\eta_0)\xi \\
& - \frac{2}{r}(C-1)\left[-\xi p_{r0}' - \frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right] \\
& - \frac{4}{r^2}\frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}(C-1)^2 p_{r0}.
\end{aligned} \tag{122}$$

We shall first look only at the 'isotropic' terms, the terms that do not involve $(C-1)$, thereafter we will look at the terms that do include the term $(C-1)$.

Starting with the 'isotropic' terms, in (122) we find

$$\begin{aligned}
& -\left(-(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi + \frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right)' + \frac{4}{r}(\eta_0' + \frac{4}{r}\eta_0)\xi \\
& + \frac{\nu_0'}{2}\left(-\frac{1}{r^2}\left(r^2\xi(\rho_0+p_{r0})\right)' + (\eta_0' + \frac{4}{r}\eta_0)\xi - \xi p_{r0}' - \frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right) \\
& + \left[(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi - \frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' - (\rho_0+p_{r0})\xi\left(\nu_0' + \frac{1}{r}\right)\right]\frac{(\nu_0'+\lambda_0')}{2}.
\end{aligned} \tag{123}$$

Even this is quite complicated, but we will simplify it step by step. Looking only at the terms with γ we collect them as follows

$$\begin{aligned}&-\left(\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right)' - \left(\frac{\nu_0'}{2}\right)\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' \\ &-\left(\frac{\nu_0'+\lambda_0'}{2}\right)\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'.\end{aligned} \tag{124}$$

These can be collected as a single term, i.e.

$$-e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{(\nu_0+\frac{\lambda_0}{2})}\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]'. \tag{125}$$

Remembering equations (42) and (61), we can write

$$(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi = \left(\frac{\nu_0'}{2}(\rho_0+p_{r0}) - \frac{2}{r}(C-1)p_{r0}\right)\xi. \tag{126}$$

Inserting this equation (123) takes the following form

$$\begin{aligned}&\left[\left(\frac{\nu_0'}{2}(\rho_0+p_{r0}) - \frac{2}{r}(C-1)p_{r0}\right)\xi\right]' \\ &- e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{(\nu_0+\frac{\lambda_0}{2})}\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right] \\ &+ \frac{\nu_0'}{2}\left(\frac{\nu_0'}{2}(\rho_0+p_{r0}) - \frac{2}{r}(C-1)p_{r0}\right)\xi \\ &+ \frac{(\nu_0'+\lambda_0')}{2}\left[\frac{\nu_0'}{2}(\rho_0+p_{r0})\xi - \frac{2}{r}(C-1)p_{r0}\xi - (\rho_0+p_{r0})\xi\left(\nu_0'+\frac{1}{r}\right)\right]\xi \\ &- \frac{\nu_0'}{2}\frac{1}{r^2}\left(r^2\xi(\rho_0+p_{r0})\right)' + \frac{4}{r}\left(\eta_0'+\frac{4}{r}\eta_0\right)\xi.\end{aligned} \tag{127}$$

Considering the terms in equation (127) containing $(\rho_0+p_{r0})\xi$ we have

$$\begin{aligned}&\left[\frac{\nu_0'}{2}(\rho_0+p_{r0})\xi\right]' + \left(\frac{(\nu_0')^2}{4}(\rho_0+p_{r0})\xi\right) + \\ &\frac{\lambda_0'+\nu_0'}{2}\left[\frac{\nu_0'}{2} - \left(\nu_0'+\frac{1}{r}\right)\right](\rho_0+p_{r0})\xi - \frac{\nu_0'}{2}\frac{1}{r^2}(r^2\xi(\rho_0+p_{r0})'.\end{aligned} \tag{128}$$

Expanding the first and last terms, factoring the derivative of $\frac{\nu_0'}{2}$ from the first term, and the derivative of r^2 in the last term, we obtain

$$\begin{aligned}&\left[\frac{\nu_0''}{2}(\rho_0+p_{r0})\xi\right] + \frac{\nu_0'}{2}[(\rho_0+p_{r0})\xi]' + \left(\frac{(\nu_0')^2}{4}(\rho_0+p_{r0})\xi\right) + \\ &\frac{\lambda_0'+\nu_0'}{2}\left[\frac{\nu_0'}{2} - \left(\nu_0'+\frac{1}{r}\right)\right](\rho_0+p_{r0})\xi - \frac{\nu_0'}{2}\frac{2}{r}\left(\xi(\rho_0+p_{r0})\right) \\ &- \frac{\nu_0'}{2}\left((\rho_0+p_{r0}\xi)\right)'.\end{aligned} \tag{129}$$

This expression can be drastically simplified, since two terms cancel, and the remaining terms can be written in the following way:

$$\frac{(\rho_0 + p_{r0})}{2}\xi\left[\nu_0'' - \frac{\lambda_0'\nu_0'}{2} - \frac{3\nu_0' + \lambda_0'}{r}\right]. \tag{130}$$

We rewrite equation (4) as

$$\begin{aligned}
&\frac{1}{2}\left[\nu_0'' + \frac{(\nu_0')^2}{2} + \frac{(\nu_0' - \lambda_0')}{r} - \frac{\lambda_0'\nu_0'}{2}\right] - \frac{(\nu_0')^2}{8} - \frac{2\nu_0'}{r} = \\
&- 8\pi e^{\lambda_0}(p_{t0} + \eta_0) - \nu_0'\left(\frac{\nu_0'}{8} + \frac{2}{r}\right).
\end{aligned} \tag{131}$$

We now see that equation (130) can be written as

$$-8\pi(\rho_0 + p_{r0})\xi e^{\lambda_0}(Cp_{r0} + \eta_0) - (\rho_0 + p_{r0})\xi\frac{\nu_0'}{4}\left(\frac{\nu_0'}{2} + \frac{8}{r}\right). \tag{132}$$

The second term here, using (42) reads

$$\begin{aligned}
&\xi\left[\frac{1}{2}\left((\eta_0' - p_{r0}') + \frac{4}{r}\eta_0 + \frac{2}{r}(C-1)p_{r0}\right)\right] \times \\
&\left[\frac{2}{\rho_0 + p_{r0}}\left((\eta_0' - p_{r0}') + \frac{4}{r}\eta_0 + \frac{2}{r}(C-1)p_{r0}\right) + \frac{8}{r}\right].
\end{aligned} \tag{133}$$

Here we separate the isotropic and anisotropic terms to obtain

$$\begin{aligned}
&\frac{\xi}{\rho_0 + p_{r0}}\left[((\eta_0' - p_{r0}') + \frac{4}{r}\eta_0)^2\right] + \frac{4\xi}{r}(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}') \\
&+ \frac{\xi}{\rho_0 + p_{r0}}\frac{4}{r}(C-1)P_{r0}(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}') \\
&+ \frac{\xi}{\rho_0 + p_{r0}}\frac{4}{r^2}(C-1)^2P_{r0}^2 + \frac{8}{r^2}\xi(C-1)p_{r0}.
\end{aligned} \tag{134}$$

Expanding the first term in (34) we have

$$\begin{aligned}
&\frac{\xi}{\rho_0 + p_{r0}}\left[(\eta_0' - p_{r0}')^2 + \frac{8}{r}\eta_0(\eta_0' - p_{r0}') + \frac{16}{r^2}\eta_0^2\right] + \frac{4\xi}{r}(\eta_0' + \frac{4}{r}\eta_0) - \frac{4\xi}{r}p_{r0} \\
&\frac{4}{r}\frac{\xi p_{r0}(C-1)}{\rho_0 + p_{r0}}\left[(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}') + \frac{1}{r}(C-1)p_{r0}\right] + \frac{8\xi}{r^2}(C-1)p_{r0}^2.
\end{aligned} \tag{135}$$

There are still terms in equation (122) that we have not considered, i.e. the anisotropic terms which read

$$-\left(\frac{2}{r}(C-1)p_{r0}\xi\right)' - \left(\frac{\nu_0'}{2} + \frac{\nu_0' + \lambda_0'}{2}\right)\left(\frac{2}{r}(C-1)p_{r0}\xi\right). \tag{136}$$

This expression can be collected to a single derivative,

$$-e^{-(\nu_0+\frac{\lambda_0}{2})}\left(e^{(\nu_0+\frac{\lambda_0}{2})}\frac{2}{r}(C-1)p_{r0}\xi)\right)' . \tag{137}$$

Employing expressions (128)-(137), expression (123) reads

$$\begin{aligned}
&-8\pi(\rho_0+p_{r0})\xi e^{\lambda_0}(Cp_{r0}+\eta_0)-e^{-(\nu_0+\frac{\lambda_0}{2})}\left(e^{(\nu_0+\frac{\lambda_0}{2})}\frac{2}{r}(C-1)p_{r0}\xi)\right)' \\
&-e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{(\nu_0+\frac{\lambda_0}{2})}\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]' \\
&-\frac{\xi}{\rho_0+p_{r0}}\left[(\eta_0'-p_{r0}')^2+\frac{8}{r}\eta_0(\eta_0'-p_{r0}')+\frac{16}{r^2}\eta_0^2\right]-\frac{4\xi}{r}(\eta_0'+\frac{4\xi}{r}\eta_0)+\frac{4}{r}p_{r0} \\
&-\frac{4}{r}\frac{\xi p_{r0}(C-1)}{\rho_0+p_{r0}}\left[(\eta_0'+\frac{4}{r}\eta_0-p_{r0}')+\frac{1}{r}(C-1)p_{r0}\right]-\frac{8}{r^2}(C-1)p_{r0}^2 \\
&+\frac{4}{r}(\eta_0'+\frac{4}{r}\eta_0)\xi.
\end{aligned} \tag{138}$$

Here some terms cancel, but we are not finished yet. Considering (122) again, we write the anisotropic terms as

$$\begin{aligned}
&+\left(\frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}\frac{2}{r}(C-1)p_{r0}\right)' \\
&+\frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}\frac{2}{r}(C-1)p_{r0}\left(\frac{\nu_0'+\nu_0'+\lambda_0'}{2}\right) \\
&+\frac{2}{r}(C-1)\left[\xi p_{r0}'+\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'-\frac{2}{r}\frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}\frac{2}{r}(C-1)p_{r0}\right].
\end{aligned} \tag{139}$$

Here the two first terms can be collected as a single derivative. Hence (139) reads

$$\begin{aligned}
&+e^{-\frac{\lambda_0+2\nu_0}{2}}\left(e^{\frac{\lambda_0+2\nu_0}{2}}\frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}\frac{2}{r}(C-1)p_{r0}\right)' \\
&+\frac{2}{r}(C-1)\left[\xi p_{r0}'+\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'-\frac{2}{r}\frac{\gamma p_{r0}\xi}{\rho_0+p_{r0}}\frac{2}{r}(C-1)p_{r0}\right].
\end{aligned} \tag{140}$$

Now, if we use the results from (138) and (140) in (122) we get the pulsation equation.

$$
\begin{aligned}
& - 8\pi(\rho_0 + p_{r0})\xi e^{\lambda_0}(p_{r0} + \eta_0) - e^{-(\nu_0 + \frac{\lambda_0}{2})} \left(e^{(\nu_0 + \frac{\lambda_0}{2})} \frac{2}{r}(C-1)p_{r0}\xi) \right)' \\
& - e^{-(\nu_0 + \frac{\lambda_0}{2})} \left[e^{\frac{3\nu_0 + \lambda_0}{2}} \frac{\gamma p_{r0}}{r^2} (e^{-\frac{\nu_0}{2}} r^2 \xi)' \right]' \\
& - \frac{\xi}{\rho_0 + p_{r0}} \left[(\eta_0' - p_{r0}')^2 + \frac{8}{r}\eta_0(\eta_0' - p_{r0}') + \frac{16}{r^2}\eta_0^2 \right] + \frac{4}{r}p_{r0} \\
& - \frac{4}{r}\frac{\xi p_{r0}(C-1)}{\rho_0 + p_{r0}} \left[(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}') + \frac{1}{r}(C-1)p_{r0} \right] - \frac{8}{r^2}(C-1)p_{r0}^2 \\
& + e^{-\frac{\lambda_0 + 2\nu_0}{2}} \left(e^{\frac{\lambda_0 + 2\nu_0}{2}} \frac{\gamma p_{r0}\xi}{\rho_0 + p_{r0}} \frac{2}{r}(C-1)p_{r0} \right)' \\
& + \frac{2}{r}(C-1) \left[\xi p_{r0}' + \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2} (e^{-\frac{\nu_0}{2}} r^2 \xi)' - \frac{\gamma p_{r0}\xi}{\rho_0 + p_{r0}} \frac{2}{r}(C-1)p_{r0} \right] \\
& = e^{\lambda_0 - \nu_0} \left(\rho_0 + p_{r0}\right) \xi \omega^2.
\end{aligned}
\tag{141}
$$

Here we separate some terms to compare with the pulsation equations we find in other papers regarding analysis' of stability of isotropic, charged perfect fluids. Equation (136) reads

$$
\begin{aligned}
& \frac{4}{r}p_{r0}' - 8\pi(\rho_0 + p_{r0})\xi e^{\lambda_0}(C p_{r0} + \eta_0) \\
& - e^{-\nu_0 + \frac{\lambda_0}{2}} \left[e^{(\frac{3\nu_0 + \lambda_0}{2})} \frac{\gamma p_{r0}}{r^2} (e^{-\frac{\nu_0}{2}} r^2 \xi)' \right]' \\
& - \frac{\xi}{\rho_0 + p_{r0}} \left[(\eta_0' - p_{r0}')^2 + \frac{8}{r}\eta_0(\eta_0' - p_{r0}') + \frac{16}{r^2}\eta_0^2 \right] \\
& - e^{-(\nu_0 + \frac{\lambda_0}{2})} \left(e^{(\nu_0 + \frac{\lambda_0}{2})} \frac{2}{r}(C-1)p_{r0}\xi) \right)' \\
& - \frac{4}{r}\frac{\xi p_{r0}(C-1)}{\rho_0 + p_{r0}} \left[(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}') + \frac{1}{r}(C-1)p_{r0} \right] \\
& - \frac{8}{r^2}(C-1)p_{r0}^2 + e^{-\frac{\lambda_0 + 2\nu_0}{2}} \left(e^{\frac{\lambda_0 + 2\nu_0}{2}} \frac{\gamma p_{r0}^2\xi}{\rho_0 + p_{r0}} \frac{2}{r}(C-1) \right)' \\
& + \frac{2}{r}(C-1) \left[\xi p_{r0}' + \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2} (e^{-\frac{\nu_0}{2}} r^2 \xi)' - \frac{2}{r}\frac{\gamma p_{r0}^2\xi}{\rho_0 + p_{r0}}(C-1) \right] \\
& = e^{\lambda_0 - \nu_0} \left(\rho_0 + p_{r0}\right) \xi \omega^2.
\end{aligned}
\tag{142}
$$

This is the equation we can compare with the pulsation equation of other authors. If we want to compare this equation to the one we find in Glazer. By setting $C = 1$ we can immediately write

$$\begin{aligned} e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\xi\omega^2 =& \frac{4}{r}p'_{r0} - 8\pi(\rho_0+p_{r0})\xi e^{\lambda_0}(p_{r0}+\eta_0) \\ & - e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{\frac{3\nu_0+\lambda_0}{2}}\frac{\gamma p_{r0}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]' \\ & - \frac{\xi}{\rho_0+p_{r0}}\left[(\eta'_0-p'_{r0})^2+\frac{8}{r}\eta_0(\eta'_0-p'_{r0})+\frac{16}{r^2}\eta_0^2\right]. \end{aligned} \tag{143}$$

This is the very same equation that we find in Glazer's paper on isotropic charged fluids. This equation is NOT reproduced in the results given by Esculpi/Aloma.
Setting the charge, $\eta_0 = 0$ as well we are left with the equation

$$\begin{aligned} e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\xi\omega^2 =& \frac{4}{r}p'_{r0} - 8\pi(\rho_0+p_{r0})\xi e^{\lambda_0}(p_{r0}) \\ & - e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{\frac{3\nu_0+\lambda_0}{2}}\frac{\gamma p_{r0}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]', \end{aligned} \tag{144}$$

which is the exact equation given by Chandrasekhar. Esculpi/Aloma do indeed reproduce this equation for isotropic non-charged fluids, but there is demonstrably something wrong with the pulsation equation. We found a plethora of mistakes, which may or may not be printing errors. It is easy to understand that they may have dropped a term in the rather tedious work to produce the pulsation equation. Which ever way they did it, an article containing this amount of mistakes should not make it through the proof-reading, which makes us wonder who, if any, proof-read this article.

We have done all these calculations that Esculpi/Aloma surely must have done too, to show how the pulsation equation is supposed to look in the case where the tangential pressure is proportional to the radial pressure.

Part II

The most general pulsation equation

It is now natural to wonder how the equation will look in the most general case. In the most general case, the tangential pressure will be a function of the radial pressure. We will include the cosmological constant also, and find the pulsation equation.

We will now undertake the task to produce this equation. We will start from scratch. Things will look more or less identical to the above case, until we start investigating the tangential pressure.

6 The field equations

Now, what we need to do is generalize this equation even further. We shall allow the tangential pressure to be any function of the radial pressure.
We have the line element

$$ds^2 = e^{\nu}\ dt^2 - e^{\lambda}\ - r^2(d\theta^2 + sin^2 d\phi^2), \tag{145}$$

and the Einstein field equations are further given by:

$$-e^{-\lambda}\left[\frac{1}{r^2} - \frac{\lambda'}{r}\right] + \frac{1}{r^2} = 8\pi T_0^0, \tag{146}$$

$$-e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] + \frac{1}{r^2} = 8\pi T_1^1, \tag{147}$$

$$-\frac{-\lambda}{2}\left[\nu'' + \frac{\nu'^2}{2} + \frac{(\nu' - \lambda')}{r} - \frac{\nu'\lambda'}{2}\right] = 8\pi T_2^2, \tag{148}$$

$$-\frac{e^{-\lambda}}{r}\dot{\lambda} = -8\pi T_0^1. \tag{149}$$

ν and λ are unknown functions of the radial coordinate r, primes denote derivatives with respect to the radial coordinate r, and dots denote derivatives with respect to the time coordinate x^0.

7 The energy-momentum tensor

The gravitational energy part of the energy-momentum tensor reads

$$T_{\mu}^{\nu(g)} = (\rho + p_t)u_\mu u^\nu - g_\mu^\nu p_t + (p_r - p_t)\chi_\mu \chi^\nu + \delta_\nu^\mu \Lambda, \tag{150}$$

where ρ is the matter energy density, p_r is the radial pressure in the direction of χ_μ, p_t is the pressure in the two-space orthogonal to χ_μ , u_μ is the velocity four-vector of the fluid , and χ_μ is a unit space vector in the radial direction orthogonal to u_μ. Λ is the cosmological constant.
The electromagnetic field tensor is of the form

$$T_{\mu}^{\nu(em)} = \frac{1}{4\pi}\left(\frac{1}{4}f_{\alpha\beta}f^{\alpha\beta} - g^{\mu\nu}g^{\alpha\beta}f_{\mu\alpha}f_{\nu\beta}\right), \tag{151}$$

where the electromagnetic field tensor $f_{\mu\nu}$ is given in terms of the electromagnetic potentials A_μ

$$f_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} \tag{152}$$

If we consider the potential vector

$$A_\mu = (A_0, 0, 0, 0), \tag{153}$$

it follows that

$$f_{01} = -f_{10} = A_{0;1} \tag{154}$$

The Maxwell equations reads:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}f^{\mu\nu}\right) = 4\pi\sigma u^\mu, \tag{155}$$

where σ is the charge density, and σu^μ is the current vector.
The u_μ's and χ_μ's are defined:

$$u^\mu = \frac{dx^\mu}{ds} \tag{156}$$

Our line element yields

$$1 = e^\nu\left(\frac{dt}{ds}\right)^2 - e^\lambda\left(\frac{dr}{ds}\right)^2 - r^2\left(\left(\frac{d\theta}{ds}\right)^2 + sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right) \tag{157}$$

Considering radial oscillations only we can put $\frac{d\theta}{ds} = \frac{d\phi}{ds} = 0$ so we get:

$$1 = e^\nu(u^0)^2 - e^\lambda(u^1)^2 \tag{158}$$

Considering u^1 we obtain

$$u^1 = \frac{dr}{dt}\times\frac{dt}{ds} = u^0\dot{\xi}, \tag{159}$$

where we defined $\dot{\xi} = \frac{dr}{dt}$
Solving (149) for u^0 and then (150) for u^1 to first order we find:

$$u^0 = e^{-\frac{\nu}{2}}, \tag{160}$$

and

$$u^1 = \dot{\xi} e^{-\frac{\nu}{2}}. \tag{161}$$

We define χ^{μ}' as being orthonormal and orthogonal to u^{μ}, yielding

$$\chi^{\mu} u_{\mu} = 0, \tag{162}$$

and

$$\chi^{\mu} \chi_{\mu} = -1 \tag{163}$$

Solving for χ^0 and χ^1 we find

$$\chi^0 = e^{\frac{\lambda}{2} - \nu} \dot{\xi}, \tag{164}$$

and

$$\chi^1 = e^{-\frac{\lambda}{2}} \tag{165}$$

Under spherical symmetry, there are only two non-zero components of the electromagnetic field tensor, namely f_{01} and f_{10}, which furthermore are antisymmetric:

$$f_{10} = -f_{01} \tag{166}$$

For the static case $\dot{\xi}$ vanishes, and we are left with $u^{\mu} = (u^0, 0, 0, 0)$ and $\chi^{\mu} = (0, \chi^1, 0, 0)$

Our electromagnetic field tensor then reads:

$$T_0^{0(em)} = \frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}, \tag{167}$$

$$T_1^{1(em)} = \frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}, \tag{168}$$

$$T_2^{2(em)} = -\frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}, \tag{169}$$

$$T_3^{3(em)} = -\frac{(f_{01})^2}{8\pi} e^{-(\lambda+\nu)}. \tag{170}$$

The Maxwell equations further read

$$\sqrt{-g} \frac{\partial}{\partial r} \left(\sqrt{-g} f^{01} \right) = \frac{e^{-\frac{\lambda+\nu}{2}}}{r^2 sin\theta} \frac{\partial}{\partial r} \left(e^{\frac{\lambda+\nu}{2}} r^2 sin\theta f^{01} \right) = 4\pi\sigma e^{-\frac{\nu}{2}}, \tag{171}$$

Solving for f^{01} by integration we get

$$e^{\frac{\lambda+\nu}{2}} r^2 f^{01} = \int 4\pi\sigma r^2 e^{\frac{\lambda}{2}} dr. \tag{172}$$

The right hand side of this equation is merely the total charge within a sphere, so we call it Q:

$$Q(r) = \int_0^r 4\pi\sigma e^{\frac{\lambda}{2}}\tau^2 d\tau \tag{173}$$

We then have

$$\sigma = \frac{Q'(r)}{4\pi r^2}e^{\lambda+\nu} \tag{174}$$

By lowering the indices, and reversing the order of them, we obtain

$$e^{-\frac{\lambda+\nu}{2}}f_{10} = \frac{1}{r^2}\int 4\pi r^2\sigma e^{\frac{\lambda}{2}}dr = \frac{Q(r)}{r^2}, \tag{175}$$

which yields

$$(f_{10})^2 = \frac{Q^2(r)}{r^4}e^{\lambda+\nu} \tag{176}$$

We now find the components of the energy-momentum tensor

$$T_\mu^\nu = T_\mu^{(m)\nu} + T_\mu^{(em)\nu} \tag{177}$$

We obtain

$$T_0^0 = \rho + \eta + \Lambda, \tag{178}$$

$$T_1^1 = -p_r + \eta + \Lambda, \tag{179}$$

$$T_2^2 = -p_t - \eta + \Lambda, \tag{180}$$

$$T_3^3 = -p_t - \eta + \Lambda, \tag{181}$$

where we follow Glazers notation and define

$$\eta = \frac{Q^2}{8\pi r^2}. \tag{182}$$

The pulsation equation itself comes from the covariant divergence of the energy-momentum tensor, meaning

$$T_{\mu;\nu}^\nu = 0, \tag{183}$$

which is a necessary identity. By choosing $\mu = 1$ and using that the covariant derivative can be written as

$$T_{\mu;\nu}^\nu = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}T_\mu^\nu - \frac{T^{\nu\alpha}}{2}\frac{\partial g_{\nu\alpha}}{\partial x^\mu} \tag{184}$$

we get the following relation

$$\frac{\partial T_1^0}{\partial x^0} + \frac{\partial T_1^1}{\partial r} + T_1^0\left(\frac{\dot{\nu} - \dot{\lambda}}{2}\right) + \frac{\nu'}{2}\left(T_1^1 - T_0^0\right) + \frac{2}{r}\left(T_1^1 - \frac{T_2^2 - T_3^3}{2}\right) = 0 \tag{185}$$

For the static case, we distinguish the quantities describing this equilibtrium state by a subscript zero. All time derivatives vanish for the equilibrium quantites by definition. Using equations (178)-(181), equation (186) yields a relation between the quilibrium values

$$\frac{\nu_0'}{2}\left(\rho_0 + p_{r0}\right) = \left(\eta_0' - p_{r0}'\right) + \frac{4}{r}\eta_0 - \frac{2}{r}\left(p_{r0} - p_{t0}\right). \tag{186}$$

Subtracting equation (147) from equation (146) we get the identity

$$8\pi\left(T_0^0 - T_1^1\right) = 8\pi(\rho + p_r) = e^{-\lambda}\left(\frac{\nu' + \lambda'}{r}\right) \tag{187}$$

8 Perturbations

Perturbing the equilibrium state, we write

$$\lambda = \lambda_0(r) + \delta\lambda(r, x^0), \tag{188}$$

$$\nu = \nu_0(r) + \delta\nu(r, x^0), \tag{189}$$

$$\rho = \rho_0(r) + \delta\rho(r, x^0), \tag{190}$$

$$p_r = p_{r0}(r) + \delta p_{r0}(r, x^0), \tag{191}$$

$$p_t = p_{t0}(r) + \delta p_{t0}(r, x^0), \tag{192}$$

$$\eta = \eta_0(r) + \delta\eta(r, x^0). \tag{193}$$

For small perturbations, to first order we have:

$$e^{\lambda_0+\delta\lambda} = e^{\lambda_0}(1+\delta\lambda), \tag{194}$$

$$e^{\nu_0+\delta\nu} = e^{\nu_0}(1+\delta\nu). \tag{195}$$

To first order, all T^{μ}_{μ}'s remain the same, except for these perturbations to the quantities, but unlike the static case, we have another non-zero entry in our Energy-Momentum Tensor. Using equations (160), (161), (164), and (165) in equation (150) we obtain

$$T^0_1 = -(\rho_0 + p_{r0} + \delta\rho + \delta p_r)e^{(\lambda_0-\nu_0)}(1+\delta\lambda)(1-\delta\nu)\dot{\xi}, \tag{196}$$

which to first order is

$$T^0_1 = -(\rho_0 + p_{r0})e^{(\lambda_0-\nu_0)}\dot{\xi}. \tag{197}$$

Now, the conservation of the energy-momentum-tensor becomes littered with these perturbed quantities, and it is equation (185) that we will develop into the pulsation equation:

$$T^{\alpha}_{1;\alpha} = \frac{\partial T^0_1}{\partial x^0} + T^0_1(\dot{\nu_0} + \delta\dot{\nu} + \dot{\lambda_0} + \delta\dot{\lambda}) - \frac{\nu_0' + \delta\nu'}{2}(T^0_0 - T^1_1) + \frac{\partial T^1_1}{\partial r} + \frac{2}{r}\left(T^1_1 - \frac{T^2_2 + T^3_3}{2}\right) = 0. \tag{198}$$

Inserting equations (178)-(181) and equation (197) into equation (198) and keeping it only to first order

$$\begin{aligned} &-\frac{\partial}{\partial x^0}\left(e^{\lambda_0-\nu_0}\dot{\xi}(\rho_0+p_{r0})\right) - \frac{\nu_0'}{2}(\rho_0 + p_{r0} + \delta\rho + \delta p_r) - \frac{\delta\nu'}{2}(\rho_0 + p_{r0}) \\ &+\frac{\partial}{\partial r}(\eta_0 - p_{r0} + \delta\eta - \delta p_r + \Lambda) + \frac{2}{r}(2\eta_0 + 2\delta\eta + p_{t0} - p_{r0} + \delta p_t - \delta p_r) = 0. \end{aligned} \tag{199}$$

Using equation (185) and cancelling the static solution we are left with the equation

$$\begin{aligned}-e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\ddot{\xi}+ = \left(\delta p_r' - \delta\eta'\right) + \frac{\nu_0'}{2}\left(\delta\rho+\delta p_r\right) + \frac{\delta\nu'}{2}\left(\rho_0+p_{r0}\right)\\ - \frac{4}{r}\delta\eta - \frac{2}{r}\left(\delta p_t - \delta p_r\right).\end{aligned} \tag{200}$$

This is the equation that will yield the pulsation equation.

We shall allow the tangential pressure to be any function of the radial pressure, so

$$p_t = p_t(p_r), \tag{201}$$

and we shall define the difference between the pressures as

$$\pi = p_t - p_r = \pi(p_r), \tag{202}$$

with equilibrium state

$$\pi_0 = p_{t0} - p_{r0}. \tag{203}$$

It follows that

$$\frac{d\pi}{dr} = \frac{d\pi}{dp_r}p_r', \tag{204}$$

and

$$\delta p_t - \delta p_r = \delta\pi = \frac{d\pi}{dp_r}\delta p_r. \tag{205}$$

Now, equation (200) reads

$$\begin{aligned}-e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\ddot{\xi} = \left(\delta p_r' - \delta\eta'\right) + \frac{\nu_0'}{2}\left(\delta\rho+\delta p_r\right) + \frac{\delta\nu'}{2}\left(\rho_0+p_{r0}\right)\\ - \frac{4}{r}\delta\eta - \frac{2}{r}\delta\pi.\end{aligned} \tag{206}$$

We now need to find the expressions for the other perturbations. We shall start by looking at the field equations. Subtracting (147) from (146) (both static) we find

$$8\pi(T_0^0 - T_1^1) = \frac{e^{-\lambda_0}}{r}(\lambda_0' + \nu_0'), \tag{207}$$

which we will write as

$$8\pi r e^{\lambda_0}(\rho_0 + p_{r0}) = (\nu_0' + \lambda_0'). \tag{208}$$

Inserting into (149), we can directly integrate

$$\dot{\delta\lambda} = -8\pi r e^{\lambda_0}\dot{\xi}(\rho_0 + p_{r0}), \tag{209}$$

to find

$$\delta\lambda = -8\pi r e^{\lambda_0}\xi(\rho_0 + p_{r0}), \tag{210}$$

and looking at (204) and (206) we see that we necessarily have that

$$\delta\lambda = -\xi(\nu_0' + \lambda_0') \tag{211}$$

Taking the first field equation, equation (146), into account

$$-e^{-\lambda_0}(1-\delta\lambda)\left[\frac{1}{r^2} - \frac{\lambda_0' + \delta\lambda'}{r}\right] + \frac{1}{r^2} = 8\pi(\rho_0 + \eta_0 + \delta\rho + \delta\eta + \Lambda), \tag{212}$$

and cancelling the static solution

$$-e^{-\lambda_0}\left[\frac{1}{r^2} - \frac{\lambda_0'}{r}\right] + \frac{1}{r^2} = 8\pi(\rho_0 + \eta_0 + \Lambda), \tag{213}$$

we find that the perturbations relate to each other

$$e^{-\lambda_0}\delta\lambda\left(\frac{1}{r^2} - \frac{\lambda_0'}{r}\right) + \frac{e^{-\lambda_0}}{r}(\delta\lambda)' = 8\pi(\delta\rho + \delta\eta). \tag{214}$$

Multiplying the equation with r^2 we simplify and find

$$e^{-\lambda_0}\delta\lambda(1 - \lambda_0' r) + e^{-\lambda_0} r(\delta\lambda)' = (e^{-r\lambda_0}\delta\lambda)' = 8\pi r^2(\delta\rho_0 + \delta\eta_0). \tag{215}$$

From equation (204) we get

$$8\pi r^2(\delta\rho_0 + \delta\eta_0) = (re^{-\lambda_0}[-8\pi r e^{\lambda_0}\xi(\rho_0 + p_{r0})])'. \tag{216}$$

Solving for $\delta\rho$ we further have

$$\delta\rho = -\frac{1}{r^2}\left(r^2\xi(\rho_0 + p_{r0})\right)' - \delta\eta. \tag{217}$$

We will now start investigating the perturbations, let us start with $\delta\eta$, which is the most tedious perturbation to find a suitable expression. Let's just do a quick relabelling to make things less confusing.

$$f_{01} = E_0 + \delta E, \tag{218}$$

where E_0 and δE is the equilibrium, and perturbed quantity of the electromagnetic field tensor respectively. The energy-momentum of the field now becomes

$$T_0^{0(em)} = \frac{e^{-(\lambda+\nu)}}{8\pi}(f_{10})^2 = \frac{e^{-(\lambda_0+\nu_0)}(1-\delta\lambda-\delta\nu)}{8\pi}(E_0 + \delta E)^2. \tag{219}$$

You might recognize that this is also the definition we have for η, which now becomes

$$\eta = \eta_0 + \delta\eta = \frac{e^{-(\lambda_0+\nu_0)}}{8\pi}E_0^2(1 - (\delta\lambda + \delta\nu) + 2\frac{\delta E}{E_0})), \tag{220}$$

to first order. Looking at the Maxwell equations (11) in this perturbed state we find

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^0}\left(\sqrt{-g}f^{10}\right)=4\pi\sigma u^1, \tag{221}$$

which reads

$$\frac{e^{-\frac{(\lambda_0+\nu_0)}{2}}}{r^2 sin\theta}\frac{\partial}{\partial x^0}\left(e^{-\frac{\lambda_0+\nu_0}{2}}(1-\frac{\delta\lambda+\delta\nu}{2})r^2 sin\theta(E_0+\delta E)\right)=4\pi\sigma_0\dot{\xi}e^{\frac{-\nu_0}{2}}. \tag{222}$$

Since we now differentiate with respect to time both r^2 and $sin\theta$ will cancel. This equation can immediately be integrated to give

$$\left(-\frac{\delta\lambda+\delta\nu}{2}\right)E_0+\delta E=4\pi\sigma_0\xi e^{\frac{2\lambda_0+\nu_0}{2}}. \tag{223}$$

Multiplying both sides of the last equation with $2E_0$ we find

$$E_0^2\left(-(\delta\lambda+\delta\nu)+\frac{2\delta E}{E_0}\right)=8\pi E_0\sigma_0\xi e^{\frac{2\lambda_0+\nu_0}{2}}. \tag{224}$$

Inserting in (203) we have

$$\eta_0+\delta\eta=\frac{e^{-(\lambda_0+\nu_0)}}{8\pi}E_0^2(1+\frac{8\pi\sigma_0\xi e^{\frac{2\lambda_0+\nu_0}{2}}}{E_0})=\frac{e^{-(\lambda_0+\nu_0)}}{8\pi}E_0^2+\sigma_0\xi E_0 e^{\frac{-\nu_0}{2}}. \tag{225}$$

Remembering equations (176) and (172), for the static case we find

$$\eta_0=\frac{e^{-(\lambda_0+\nu_0)}}{8\pi}E_0^2, \tag{226}$$

From equations (121) and (122) we must have

$$\delta\eta=\sigma_0\xi E_0 e^{-\frac{\nu_0}{2}}. \tag{227}$$

Let's keep moving forward, equation (182) reads

$$\eta=\frac{Q^2(r)}{8\pi r^4}=\eta_0+\delta\eta=\frac{(Q_0(r)+\delta Q)^2}{8\pi r^4}. \tag{228}$$

To first order we have

$$\delta\eta=\frac{Q_0(r)\delta Q}{4\pi r^4}, \tag{229}$$

and we further have

$$(\eta_0)'=\frac{2Q_0(r)Q_0'(r)}{8\pi r^4}-\frac{Q_0^2(r)}{2\pi r^5}. \tag{230}$$

While we defined $Q_0(r)$ earlier in equation (175), and from equation (176) we find $Q_0(r)'$ as

$$Q_0'(r) = 4\pi r^2 \sigma_0 e^{\frac{\lambda_0}{2}}. \tag{231}$$

Now equation (230) reads

$$(\eta_0)' = \frac{Q_0(r)\sigma_0 e^{\frac{\lambda_0}{2}}}{r^2} - \frac{Q_0^2(r)}{2\pi r^5}. \tag{232}$$

And it follows from (175), where we are careful with the definition of E_0 and the antisymmetry of $f_{\mu\nu}$ so we don't bring the wrong sign, that

$$(\eta_0)' = -\sigma_0 E_0 e^{\frac{-\nu_0}{2}} - \frac{Q_0^2(r)}{2\pi r^5}, \tag{233}$$

and looking at (226) and (227) we can write this as

$$(\eta_0)' = -\frac{\delta\eta}{\xi} - \frac{4}{r}\eta_0. \tag{234}$$

This last equation may also be written in the following way

$$\delta\eta + (\eta_0' + \frac{4}{r}\eta_0)\xi = 0. \tag{235}$$

We need the expression for δp_r, so we can squeeze out the pulsation equation. To produce the expression for δp_r we need to make use of a supplementary condition, the conservation of baryon number i.e.

$$(Nu^\alpha)_{;\alpha} = 0. \tag{236}$$

This means that the total number of particles in the system remain unchanged. We write

$$N = N_0(r) + \delta N(r, x^0). \tag{237}$$

The contraction of the covariant derivative of a contra-variant vector Nu^α is given by

$$(Nu^\alpha)_{;\alpha} = \frac{\partial(Nu^\nu)}{\partial x^\nu} + Nu^\mu \frac{\partial(ln\sqrt{-g})}{\partial x^\mu}. \tag{238}$$

We remember $u^\mu = (e^{-\frac{\nu_0}{2}}, \dot{\xi} e^{-\frac{\nu_0}{2}}, 0, 0)$. Hence (238) becomes to first order

$$\begin{aligned}(N_0)\frac{\partial u^0}{\partial x^0} + \frac{\partial \delta N}{\partial x^0}u^0 + (N_0)\frac{\partial u^1}{\partial x^1} + \frac{\partial N_0}{\partial x^1}u^1 \\ + N_0\left(\frac{\dot{\delta\lambda} + \dot{\delta\nu}}{2}\right)u^0 + N_0\left(\frac{\lambda_0' + \nu_0'}{2} + \frac{2}{r}\right) = 0.\end{aligned} \tag{239}$$

Inserting our expressions for u^μ equation (239) reads

$$\begin{aligned}&- N_0 e^{-\frac{\nu_0}{2}}\frac{\dot{\delta\nu}}{2} + \dot{\delta N} e^{-\frac{\nu_0}{2}} + (N_0 e^{-\frac{\nu_0}{2}}\dot{\xi})' + N_0 e^{\frac{-\nu_0}{2}}\frac{\dot{\delta\nu}}{2} + N_0 e^{-\frac{\nu_0}{2}}\frac{\dot{\delta\lambda}}{2} \\ &+ N_0 e^{-\frac{\nu_0}{2}}\dot{\xi}\left(\frac{\lambda_0' + \nu_0'}{2} + \frac{2}{r}\right) = 0,\end{aligned} \tag{240}$$

where the first and the fourth term cancel, and we can actually integrate this equation immediately and solve it for δN

$$\delta N e^{-\frac{\nu_0}{2}} + \left(N_0 e^{-\frac{\nu_0}{2}}\xi\right)' + N_0 e^{-\frac{\nu_0}{2}}\delta\lambda + N_0 e^{-\frac{\nu_0}{2}}\xi\left(\frac{\lambda_0' + \nu_0'}{2} + \frac{2}{r}\right) = 0. \tag{241}$$

We find that equation (241) can be simplified to give

$$\delta N + \frac{e^{\frac{\nu_0}{2}}}{r^2}(N_0 e^{-\frac{\nu_0}{2}} r^2 \xi)' + N_0\left(\delta\lambda + \xi\left(\frac{\lambda_0' + \nu_0'}{2}\right)\right) = 0. \tag{242}$$

And by (211) the third term here is zero, which means we get the very nice expression

$$\delta N = -\frac{e^{\frac{\nu_0}{2}}}{r^2}(N_0 e^{-\frac{\nu_0}{2}} r^2 \xi)'. \tag{243}$$

If

$$N = N(\rho(r, x^0), p_r(r, x^0)), \tag{244}$$

is the equation of state, it follows that

$$\delta N = \frac{\partial N}{\partial \rho}\delta\rho + \frac{\partial N}{\partial p_r}\delta p_r. \tag{245}$$

By equations (217), (235), and (243) this becomes

$$\frac{\partial N}{\partial p_r}\delta p_r = -\frac{e^{\frac{\nu_0}{2}}}{r^2}(N_0 e^{-\frac{\nu_0}{2}} r^2\xi)' - \frac{\partial N}{\partial \rho}\left[-\frac{1}{r^2}\left(r^2\xi(\rho_0 + p_{r0})\right)' + (\eta_0' + \frac{4}{r}\eta_0)\xi\right]. \tag{246}$$

Expanding some terms we obtain

$$\frac{\partial N}{\partial p_r}\delta p_r = -N_0'\xi - N_0\frac{e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2\xi)' - \frac{\partial N}{\partial \rho}\left[-\xi(p_{r0}' - \frac{4}{r}\eta_0 - \eta_0' + \rho_0') + \frac{\rho_0 + p_{r0}}{r^2}(r^2\xi)'\right]. \tag{247}$$

Using equations (186) and (202) we rewrite equation (247) in the following way

$$\begin{aligned}\frac{\partial N}{\partial p_r}\delta p_r = &- N_0'\xi - N_0\frac{e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2\xi)' \\ &- \frac{\partial N}{\partial \rho}\left[\xi\left(\frac{\nu_0'}{2}(\rho_0 + p_{r0}) + \frac{2}{r}\pi_0\right) - \frac{\rho_0 + p_{r0}}{r^2}(r^2\xi)' - \xi\rho_0'\right].\end{aligned} \tag{248}$$

we collect the terms in the bracket and obtain

$$\begin{aligned}&-\frac{\partial N}{\partial \rho}\left[\xi\left(\frac{\nu_0'}{2}(\rho_0 + p_{r0}) + -\frac{2\pi_0}{r}\right) - \frac{\rho_0 + p_{r0}}{r^2}(r^2\xi)' - \xi\rho_0'\right] = \\ &+\frac{\partial N}{\partial \rho}\left[\xi\frac{2\pi_0}{r} + \frac{(\rho_0 + p_{r0})e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2\xi)' + \xi\rho_0'\right].\end{aligned} \tag{249}$$

Using equation (249), equation (248) now reads

$$\frac{\partial N}{\partial p_r}\delta p_r = -N_0'\xi - (N_0 - (\rho_0 + p_{r0})\frac{\partial N}{\partial \rho})\frac{e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' + \frac{\partial N}{\partial \rho}\left(\frac{2\pi_0}{r} + \rho_0'\right). \quad (250)$$

We further have

$$N_0'(r, x^0) = N_0'(\rho(r, x^0), p_r(r, x^0)) = \frac{\partial N_0}{\partial \rho_0}\frac{\partial \rho_0}{\partial r} + \frac{\partial N_0}{\partial p_{r0}}\frac{\partial p_{r0}}{\partial r}. \quad (251)$$

We only take $\frac{\partial N}{\partial p_r}$ to zero'th order. The product $\frac{\partial N}{\partial p_r}\delta p_r$ remains first order

$$\frac{\partial N}{\partial p_r} = \frac{\partial N_0}{\partial p_{r0}}, \quad (252)$$

and similar for $\frac{\partial N}{\partial \rho}$.

$$\frac{\partial N}{\partial \rho} = \frac{\partial N_0}{\partial \rho_0}. \quad (253)$$

Using equations (251), (252), and (253), equation (250) becomes

$$\begin{aligned} \delta p_r = &- \xi\left(\frac{\partial N_0}{\partial \rho_0}\frac{\partial \rho_0}{\partial r} + \frac{\partial N_0}{\partial p_{r0}}\frac{\partial p_{r0}}{\partial r}\right)\frac{\partial p_{r0}}{\partial N_0} \\ &- \frac{\partial p_{r0}}{\partial N_0}\left(N_0 - (\rho_0 + p_{r0})\frac{\partial N_0}{\partial \rho_0}\right)\frac{e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' \\ &+ \frac{\partial N_0}{\partial \rho_0}\frac{\partial p_{r0}}{\partial N_0}\left(\frac{2\pi_0}{r}\right) + \xi\frac{\partial p_{r0}}{\partial N_0}\frac{\partial N_0}{\partial \rho_0}\frac{\partial \rho_0}{\partial r}. \end{aligned} \quad (254)$$

Here the first and the last term on the right hand side cancel, two factors of the second term cancel, and we can can recognize one of the factors as the ratio of specific heats, that is

$$\gamma = \frac{1}{p_{r0}\frac{\partial N_0}{\partial p_{r0}}}\left(N_0 - (\rho_0 + p_{r0})\frac{\partial N_0}{\partial \rho_0}\right). \quad (255)$$

There is also an equivalent definition of this γ,

$$\gamma = \frac{(p_{r0} + \rho_0)}{p_{r0}}\frac{\partial p_{r0}}{\partial \rho_0}, \quad (256)$$

which we will write as

$$\frac{\partial p_{r0}}{\partial \rho_0} = \frac{\gamma p_{r0}}{\rho_0 + p_{r0}}. \quad (257)$$

Employing equations (255) and (256), equation (254) can be written as

$$\delta p_r = -\xi p_{r0}' - \frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' + \frac{2\pi_0\gamma p_{r0}\xi}{r(\rho_0 + p_{r0})}. \quad (258)$$

The only perturbed expression we need now is that of ν', so let's take a look at equation (147) when perturbed, to first order

$$-e^{-\lambda_0}(1-\delta\lambda)\left[\frac{\nu_0'}{r}+\frac{1}{r^2}\right]-e^{-\lambda_0}\left[\frac{\delta\nu'}{r}\right]=8\pi(-p_{r0}-\delta p_r+\eta_0+\delta\eta+\Lambda). \quad (259)$$

The static solution to equation (147) is:

$$-e^{-\lambda_0}\left[\frac{\nu_0'}{r}+\frac{1}{r}\right]+\frac{1}{r}=8\pi\left(-p_{r0}+\eta_0+\Lambda\right). \quad (260)$$

Using equation (260) to cancel the static solution from (259) we find

$$\frac{e^{-\lambda_0}}{r}(\delta\nu')=-e^{-\lambda_0}\frac{\delta\lambda}{r}\left[\nu_0'+\frac{1}{r}\right]+8\pi(\delta p_r-\delta\eta). \quad (261)$$

Employing equation (210) we find

$$\frac{e^{-\lambda_0}}{r}(\delta\nu')=8\pi(\delta p_r-\delta\eta)+8\pi(\rho_0+p_{r0})\xi\left(\nu_0'+\frac{1}{r}\right). \quad (262)$$

Remembering equation (208) we can write equation (262) the following way:

$$(\delta\nu')(\rho_0+p_{r0})=\left[\delta p_r-\delta\eta-(\rho_0+p_{r0})\xi\left(\nu_0'+\frac{1}{r}\right)\right](\nu_0'+\lambda_0') \quad (263)$$

In all the expressions for the perturbed quantities, the cosmological constant has turned out to be irrelevant.

9 The pulsation equation

We shall assume perturbation in the form of radial oscillation,

$$\xi = e^{i\omega x^0}, \tag{264}$$

which leads to

$$\ddot{\xi} = -\xi\omega^2. \tag{265}$$

Using equations (202), (217), (235),(257), and (262), equation (200) takes the following complicated form

$$\begin{aligned}
e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\xi\omega^2 = &\left((\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi - \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2 \xi)'\right)' \\
&+ \left(\frac{2\pi_0\gamma p_{r0}\xi}{r(\rho_0+p_{r0})}\right)' \\
&+ \frac{\nu_0'}{2}\left(-\frac{1}{r^2}\left(r^2\xi(\rho_0+p_{r0})\right)' + (\eta_0' + \frac{4}{r}\eta_0)\xi - \xi p_{r0}'\right) \\
&+ \frac{\nu_0'}{2}\left(-\frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2 \xi)' + \frac{2\pi_0\gamma p_{r0}\xi}{r(\rho_0+p_{r0})}\right) \\
&+ \left[(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi - (\rho_0+p_{r0})\xi\left(\nu_0' + \frac{1}{r}\right)\right]\frac{(\nu_0'+\lambda_0')}{2} \\
&+ \left[-\frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2 \xi)' + \frac{2\pi_0\gamma p_{r0}\xi}{r(\rho_0+p_{r0})}\right]\frac{(\nu_0'+\lambda_0')}{2} \\
&- \frac{2}{r}\frac{d\pi}{dp_r}\left[-\xi p_{r0}' - \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2 \xi)' + \frac{2\pi_0\gamma p_{r0}\xi}{r(\rho_0+p_{r0})}\right] \\
&+ \frac{4}{r}(\eta_0' + \frac{4}{r}\eta_0)\xi.
\end{aligned} \tag{266}$$

We shall look first only at the 'isotropic' terms, the terms that do not involve π, thereafter we will look at the terms that do involve π.

Starting with the 'isotropic' terms, in (266) we find

$$\begin{aligned}
&-\left(-(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi + \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2 \xi)'\right)' + \frac{4}{r}(\eta_0' + \frac{4}{r}\eta_0)\xi \\
&+ \frac{\nu_0'}{2}\left(-\frac{1}{r^2}\left(r^2\xi(\rho_0+p_{r0})\right)' + (\eta_0' + \frac{4}{r}\eta_0)\xi - \xi p_{r0}' - \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2 \xi)'\right) \\
&+ \left[(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi - \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2 \xi)' - (\rho_0+p_{r0})\xi\left(\nu_0' + \frac{1}{r}\right)\right]\frac{(\nu_0'+\lambda_0')}{2}.
\end{aligned} \tag{267}$$

Even this is quite complicated, but we will simplify it, step by step. Looking only at the terms with γ we collect them as follows

$$\begin{aligned} &-\left(\frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}} r^2\xi)'\right)' - \left(\frac{\nu_0'}{2}\right)\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' \\ &-\left(\frac{\nu_0'+\lambda_0'}{2}\right)\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)' \end{aligned} , \tag{268}$$

which can be collected as a single term, i.e.

$$-e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{(\nu_0+\frac{\lambda_0}{2})}\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]' . \tag{269}$$

Remembering equations (268) and (202), we can rewrite the following terms in expression (267)

$$(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}')\xi = \left(\frac{\nu_0'}{2}(\rho_0+p_{r0}) - \frac{2\pi_0}{r}\right)\xi. \tag{270}$$

Now expression (267) takes the following form

$$\begin{aligned} &\left[\left(\frac{\nu_0'}{2}(\rho_0+p_{r0}) - \frac{2\pi_0}{r}\right)\xi\right]' \\ &- e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{(\nu_0+\frac{\lambda_0}{2})}\frac{\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right] \\ &+ \frac{\nu_0'}{2}\left(\frac{\nu_0'}{2}(\rho_0+p_{r0}) - \frac{2\pi_0}{r}\right)\xi \\ &+ \frac{(\nu_0'+\lambda_0')}{2}\left[\frac{\nu_0'}{2}(\rho_0+p_{r0})\xi - \frac{2\pi_0}{r}\xi - (\rho_0+p_{r0})\xi\left(\nu_0'+\frac{1}{r}\right)\right]\xi \\ &- \frac{\nu_0'}{2}\frac{1}{r^2}\left(r^2\xi(\rho_0+p_{r0})\right)' \end{aligned} \tag{271}$$

Considering the terms containing $(\rho_0+p_{r0})\xi$ in expression (271) we have

$$\begin{aligned} &\left[\frac{\nu_0'}{2}(\rho_0+p_{r0})\xi\right]' + \left(\frac{(\nu_0')^2}{4}(\rho_0+p_{r0})\xi\right) + \\ &\frac{\lambda_0'+\nu_0'}{2}\left[\frac{\nu_0'}{2} - \left(\nu_0'+\frac{1}{r}\right)\right](\rho_0+p_{r0})\xi - \frac{\nu_0'}{2}\frac{1}{r^2}(r^2\xi(\rho_0+p_{r0})'. \end{aligned} \tag{272}$$

Expanding the first and last terms, we obtain

$$\begin{aligned} &\left[\frac{\nu_0''}{2}(\rho_0+p_{r0})\xi\right] + \frac{\nu_0'}{2}\left((\rho_0+p_{r0})\xi\right)' + \left(\frac{(\nu_0')^2}{4}(\rho_0+p_{r0})\xi\right) + \\ &\frac{\lambda_0'+\nu_0'}{2}\left[\frac{\nu_0'}{2} - \left(\nu_0'+\frac{1}{r}\right)\right](\rho_0+p_{r0})\xi - \frac{\nu_0'}{2}\frac{2}{r}\big(\xi(\rho_0+p_{r0})\big) \\ &- \frac{\nu_0'}{2}\big((\rho_0+p_{r0}\xi)\big)'. \end{aligned} \tag{273}$$

This expression becomes quite simple since two terms cancel, and the remaining terms can be written in the following way:

$$\frac{(\rho_0 + p_{r0})}{2}\xi\left[\nu_0'' - \frac{\lambda_0'\nu_0'}{2} - \frac{3\nu_0' + \lambda_0'}{r}\right]. \tag{274}$$

By rewriting equation (148) as

$$-8\pi e^{\lambda_0}(\Lambda - p_{t0} - \eta_0) = \frac{1}{2}\left[\nu'' + \frac{(\nu_0')^2}{2} + \frac{\nu_0' - \lambda_0'}{r} - \frac{\nu_0'\lambda_0'}{2}\right], \tag{275}$$

we now see that equation (274) can be written as

$$8\pi e_0^{\lambda}(\rho_0 + p_{r0})(p_{t0} + \eta_0 - \Lambda)\xi - (\rho_0 + p_{r0})\frac{\nu_0'}{2}\left(\nu_0' + \frac{8}{r}\right)\xi. \tag{276}$$

Employing equation (186), we can write the last term in expression (132) as

$$\begin{aligned}&\xi\left[\frac{1}{2}\left((\eta_0' - p_{r0}') + \frac{4}{r}\eta_0 + \frac{2\pi_0}{r}\right)\right] \times \\ &\left[\frac{2}{\rho_0 + p_{r0}}\left((\eta_0' - p_{r0}') + \frac{4}{r}\eta_0 + \frac{2\pi_0}{r}\right) + \frac{8}{r}\right].\end{aligned} \tag{277}$$

Here we separate the isotropic and anisotropic terms to obtain

$$\begin{aligned}&\frac{\xi}{\rho_0 + p_{r0}}\left[((\eta_0' - p_{r0}') + \frac{4}{r}\eta_0)^2\right] + \frac{4\xi}{r}(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}') \\ &+ \frac{4\pi_0\xi}{r(\rho_0 + p_{r0})}(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}') \\ &+ \frac{4\pi_0^2\xi}{r^2(\rho_0 + p_{r0})} + \frac{8\pi_0}{r^2}\xi.\end{aligned} \tag{278}$$

Expanding the square in the first term in (278) we have

$$\begin{aligned}&\frac{\xi}{\rho_0 + p_{r0}}\left[(\eta_0' - p_{r0}')^2 + \frac{8}{r}\eta_0(\eta_0' - p_{r0}') + \frac{16}{r^2}\eta_0^2\right] + \frac{4\xi}{r}(\eta_0' + \frac{4}{r}\eta_0) - \frac{4\xi}{r}p_{r0} \\ &\frac{4\pi_0\xi}{r(\rho_0 + p_{r0})}\left[(\eta_0' + \frac{4\pi_0}{r}\eta_0 - p_{r0}') + \frac{\pi_0}{r}\right] + \frac{8\pi_0\xi}{r^2}.\end{aligned} \tag{279}$$

There are still terms in expression (271) that we have not considered, i.e. the anisotropic terms which read

$$-\left(\frac{2\pi_0}{r}\xi\right)' - \left(\frac{\nu_0'}{2} + \frac{\nu_0' + \lambda_0'}{2}\right)\left(\frac{2\pi_0}{r}\xi\right). \tag{280}$$

These terms can be collected to a single derivative,

$$-e^{-(\nu_0 + \frac{\lambda_0}{2})}\left(e^{(\nu_0 + \frac{\lambda_0}{2})}\frac{2\pi_0}{r}\xi)\right)'. \tag{281}$$

Taking into account expressions (272)-(281), expression (267) reads

$$\begin{aligned}
&- 8\pi(\rho_0 + p_{r0})\xi e^{\lambda_0}(p_{t0} + \eta_0 - \Lambda) - e^{-(\nu_0+\frac{\lambda_0}{2})} \left(e^{(\nu_0+\frac{\lambda_0}{2})} \frac{2\pi_0}{r}\xi) \right)' \\
&- e^{-(\nu_0+\frac{\lambda_0}{2})} \left[e^{(\nu_0+\frac{\lambda_0}{2})} \frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2} (e^{-\frac{\nu_0}{2}} r^2 \xi)' \right]' \\
&- \frac{\xi}{\rho_0 + p_{r0}} \left[(\eta_0' - p_{r0}')^2 + \frac{8}{r}\eta_0(\eta_0' - p_{r0}') + \frac{16}{r^2}\eta_0^2 \right] - \frac{4\xi}{r}(\eta_0' + \frac{4\xi}{r}\eta_0) + \frac{4}{r}p_{r0} \\
&- \frac{4\pi_0\xi}{r(\rho_0 + p_{r0})} \left[(\eta_0' + \frac{4}{r}\eta_0 - p_{r0}') + \frac{\pi_0}{r} \right] - \frac{8\pi_0^2}{r^2} \\
&+ \frac{4}{r}(\eta_0' + \frac{4}{r}\eta_0)\xi.
\end{aligned} \tag{282}$$

Here some terms cancel, but we are not finished yet. Considering equation (266) again, we write the anisotropic terms as

$$\begin{aligned}
&+ \left(\frac{\gamma p_{r0}\xi}{\rho_0 + p_{r0}} \frac{2\pi_0}{r} \right)' \\
&+ \frac{\gamma p_{r0}\xi}{\rho_0 + p_{r0}} \frac{2\pi_0}{r} \left(\frac{\nu_0' + \nu_0' + \lambda_0'}{2} \right) \\
&+ \frac{2\pi_0}{r} \left[\frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2} (e^{-\frac{\nu_0}{2}} r^2 \xi)' - \frac{4\pi_0 \gamma p_{r0} \xi}{r^2(\rho_0 + p_{r0})} \right] + \frac{2\xi}{r}\pi_0'
\end{aligned} \quad . \tag{283}$$

Here the two first terms can be collected as a single derivative. Hence (283) reads

$$\begin{aligned}
&+ e^{-\frac{\lambda_0+2\nu_0}{2}} \left(e^{\frac{\lambda_0+2\nu_0}{2}} \frac{2\pi_0\gamma p_{r0}\xi}{r(\rho_0 + p_{r0})} \right)' \\
&+ \frac{2\pi_0}{r} \left[\frac{\gamma p_{r0} e^{\frac{\nu_0}{2}}}{r^2} (e^{-\frac{\nu_0}{2}} r^2 \xi)' - \frac{2}{r} \frac{\gamma p_{r0}\xi}{\rho_0 + p_{r0}} \frac{2\pi_0}{r} \right] + \frac{2\xi}{r}\pi_0'.
\end{aligned} \tag{284}$$

Now, if we use the results from (282) and (284) in (266) we get the pulsation equation.

$$
\begin{aligned}
&-8\pi(\rho_0+p_{r0})\xi e^{\lambda_0}(p_{t0}+\eta_0-\Lambda)-e^{-(\nu_0+\frac{\lambda_0}{2})}\Bigg(e^{(\nu_0+\frac{\lambda_0}{2})}\frac{2\pi_0}{r}\xi)\Bigg)'\\
&-e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{\frac{3\nu_0+\lambda_0}{2}}\frac{\gamma p_{r0}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]'\\
&-\frac{\xi}{\rho_0+p_{r0}}\left[(\eta_0'-p_{r0}')^2+\frac{8}{r}\eta_0(\eta_0'-p_{r0}')+\frac{16}{r^2}\eta_0^2\right]+\frac{4}{r}p_{r0}'\\
&-\frac{4\xi\pi_0}{r(\rho_0+p_{r0})}\left[(\eta_0'+\frac{4}{r}\eta_0-p_{r0}')+\frac{\pi_0}{r}\right]-\frac{8\pi_0^2}{r^2}\\
&+e^{-\frac{\lambda_0+2\nu_0}{2}}\left(e^{\frac{\lambda_0+2\nu_0}{2}}\frac{2\pi_0\gamma p_{r0}\xi}{r(\rho_0+p_{r0})}\right)'+\frac{2\pi_0'\xi}{r}\\
&+\frac{d\pi}{dp_r}\left[\frac{2\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^3}(e^{-\frac{\nu_0}{2}}r^2\xi)'-\frac{4\pi_0p_{r0}\gamma\xi}{r^2(\rho_0+p_{r0})}\right]\\
&=e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\xi\omega^2.
\end{aligned}
\tag{285}
$$

Here, we separate some terms to compare with the pulsation equations we find in other papers regarding analysis of stability of isotropic, charged perfect fluids. Equation (285) reads

$$
\begin{aligned}
&\frac{4}{r}p_{r0}'-8\pi(\rho_0+p_{r0})\xi e^{\lambda_0}(p_{t0}+\eta_0-\Lambda)\\
&-e^{-\nu_0+\frac{\lambda_0}{2}}\left[e^{(\frac{3\nu_0+\lambda_0}{2})}\frac{\gamma p_{r0}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]'\\
&-\frac{\xi}{\rho_0+p_{r0}}\left[(\eta_0'-p_{r0}')^2+\frac{8}{r}\eta_0(\eta_0'-p_{r0}')+\frac{16}{r^2}\eta_0^2\right]\\
&-e^{-(\nu_0+\frac{\lambda_0}{2})}\Bigg(e^{(\nu_0+\frac{\lambda_0}{2})}\frac{2\pi_0}{r}\xi)\Bigg)'\\
&-\frac{4\pi_0\xi}{r(\rho_0+p_{r0})}\left[(\eta_0'+\frac{4}{r}\eta_0-p_{r0}')+\frac{\pi_0}{r}\right]\\
&-\frac{8\pi_0}{r^2}p_{r0}+e^{-\frac{\lambda_0+2\nu_0}{2}}\Bigg(e^{\frac{\lambda_0+2\nu_0}{2}}\frac{2\gamma\pi_0p_{r0}\xi}{r(\rho_0+p_{r0})})\Bigg)'\\
&+\frac{2\pi_0'\xi}{r}+\frac{d\pi}{dp_r}\left[\frac{2\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^3}(e^{-\frac{\nu_0}{2}}r^2\xi)'-\frac{4\pi_0p_{r0}\gamma\xi}{r^2(\rho_0+p_{r0})}\right]\\
&=e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\xi\omega^2.
\end{aligned}
\tag{286}
$$

This is the equation we can compare with the pulsation equation of other authors. Let us compare this equation to the one we find in Glazer's Ph.D-Thesis [2]. By setting the cosmological constant $\Lambda = 0$, and $\pi_0 = 0$, equivalent to

$p_t = p_r$ we can immediately write

$$\begin{aligned} e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\xi\omega^2 =& \frac{4}{r}p'_{r0} - 8\pi(\rho_0+p_{r0})\xi e^{\lambda_0}(p_{r0}+\eta_0) \\ & - e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{\frac{3\nu_0+\lambda_0}{2}}\frac{\gamma p_{r0}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]' \\ & - \frac{\xi}{\rho_0+p_{r0}}\left[(\eta'_0-p'_{r0})^2+\frac{8}{r}\eta_0(\eta'_0-p'_{r0})+\frac{16}{r^2}\eta_0^2\right], \end{aligned} \tag{287}$$

which matches the equation found by Glazer for isotropic charged fluid spheres.

Setting the charge, $\eta_0 = 0$ as well we are left with the equation

$$\begin{aligned} e^{\lambda_0-\nu_0}\left(\rho_0+p_{r0}\right)\xi\omega^2 =& \frac{4}{r}p'_{r0} - 8\pi(\rho_0+p_{r0})\xi e^{\lambda_0}p_{r0} \\ & - e^{-(\nu_0+\frac{\lambda_0}{2})}\left[e^{\frac{3\nu_0+\lambda_0}{2}}\frac{\gamma p_{r0}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]', \end{aligned} \tag{288}$$

which is, as we expected, the same as Chandrasekhar's pulsation equation for an uncharged isotropic fluid sphere [1].

The cosmological constant seemed to be of little importance to how the pulsation equation looks, only appearing once.However, there is one more thing we want to investigate i.e. the Tolman-Oppenheimer-Volkoff equation.

Part III

The most general TOV-equation

The Tolman-Oppenheimer-Volkoff equation is developed by inserting the two first field equations, equations (146) and (147), into the the equation of static equilibrium, equation (186).

We write equation (146) as

$$-e^{-\lambda_0}[1-r\lambda'_0]+1 = 8\pi(\rho_0+\eta_0+\Lambda)r^2, \tag{289}$$

which can also be written on the following form

$$\frac{d}{dr}\left(re^{-\lambda_0}\right) = 1-8\pi r^2(\rho_0+\eta_0+\Lambda). \tag{290}$$

By integrating this equation from 0 to r we find

$$e^{-\lambda_0} = \frac{1}{r}\left[\int_0^r 1-8\pi r^2(\rho_0+\eta_0+\Lambda)dr\right]. \tag{291}$$

which we may write as

$$e^{-\lambda_0} = \left[1-\frac{8\pi r^2}{3}\Lambda-\frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right] \tag{292}$$

Now we look at equation (147), and we write

$$-e^{-\lambda_0}[\nu_0' r+1]+1=8\pi r^2(\eta_0-p_{r0}+\Lambda) \tag{293}$$

Inserting equation (292) we find

$$-\left[1-\frac{8\pi r^2}{3}\Lambda-\frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right][\nu_0' r+1]+1=8\pi r^2(\eta_0-p_{r0}+\Lambda), \tag{294}$$

and solving for ν_0' we obtain

$$\nu_0'=\frac{1-8\pi r^2(\eta_0-p_{r0}+\Lambda)}{r\left[1-\frac{8\pi r^2}{3}\Lambda-\frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]}-\frac{1}{r}. \tag{295}$$

Collecting the two terms, this becomes

$$\nu_0'=\frac{1-8\pi r^2(\eta_0-p_{r0}+\Lambda)-\left[1-\frac{8\pi r^2}{3}\Lambda-\frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]}{r\left[1-\frac{8\pi r^2}{3}\Lambda-\frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]}, \tag{296}$$

which takes the form

$$\nu_0'=\frac{8\pi r^2(p_{r0}-\eta_0)+\frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr-\frac{16\pi}{3}\Lambda r^2}{r\left[1-\frac{8\pi r^2}{3}\Lambda-\frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]}. \tag{297}$$

Equation (186), solved for p_{r0}' reads

$$p_{r0}'=-\frac{\nu_0'}{2}(\rho_0+p_{r0})+\eta_0'+\frac{4}{r}\eta_0+\frac{2}{r}\pi_0. \tag{298}$$

Inserting equation (297) we obtain

$$\begin{aligned}p_{r0}'=&\frac{4\pi r^2(\eta_0-p_{r0})-\frac{4\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr+\frac{8\pi}{3}\Lambda r^2}{r\left[1-\frac{8\pi r^2}{3}\Lambda-\frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]}\times(\rho_0+p_{r0})\\&+\eta_0'+\frac{4}{r}\eta_0+\frac{2}{r}\pi_0.\end{aligned} \tag{299}$$

If we were to insert this equation into the pulsation equation we obtained in equation (286) we would get the rather messy

$$
\begin{aligned}
&\frac{4}{r}\left[\frac{4\pi r^2(\eta_0 - p_{r0}) - \frac{4\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr + \frac{8\pi}{3}\Lambda r^2}{r\left[1 - \frac{8\pi r^2}{3}\Lambda - \frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]}\right] \times (\rho_0 + p_{r0})\xi \\
&+ \frac{4}{r}(\eta_0' + \frac{4}{r}\eta_0 + \frac{2}{r}\pi_0)\xi - 8\pi(\rho_0 + p_{r0})\xi e^{\lambda_0}(p_{t0} + \eta_0 - \Lambda) \\
&- e^{-\nu_0 + \frac{\lambda_0}{2}}\left[e^{(\frac{3\nu_0+\lambda_0}{2})}\frac{\gamma p_{r0}}{r^2}(e^{-\frac{\nu_0}{2}}r^2\xi)'\right]' \\
&- \frac{\xi}{\rho_0 + p_{r0}}\left(\frac{4\pi r^2(\eta_0 - p_{r0}) - \frac{4\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr + \frac{8\pi}{3}\Lambda r^2}{r\left[1 - \frac{8\pi r^2}{3}\Lambda - \frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]} + \frac{4}{r}\eta_0 + \frac{2}{r}\pi_0\right)^2 \\
&+ \frac{\xi}{\rho_0 + p_{r0}}\frac{8}{r}\eta_0\left[\frac{4\pi r^2(\eta_0 - p_{r0}) - \frac{4\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr + \frac{8\pi}{3}\Lambda r^2}{r\left[1 - \frac{8\pi r^2}{3}\Lambda - \frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]} \times (\rho_0 + p_{r0})\right. \\
&\left. + \frac{2}{r}\eta_0 + \frac{2}{r}\pi_0\right] - e^{-(\nu_0 + \frac{\lambda_0}{2})}\left(e^{(\nu_0 + \frac{\lambda_0}{2})}\frac{2\pi_0}{r}\xi\right)' \\
&+ \frac{4\pi_0\xi}{r(\rho_0 + p_{r0})}\left[\frac{4\pi r^2(\eta_0 - p_{r0}) - \frac{4\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr + \frac{8\pi}{3}\Lambda r^2}{r\left[1 - \frac{8\pi r^2}{3}\Lambda - \frac{8\pi}{r}\int_0^r(\rho_0+\eta_0)r^2dr\right]} \times (\rho_0 + p_{r0}) + \frac{1}{r}\pi_0\right] \\
&- \frac{8\pi_0}{r^2}p_{r0} + e^{-\frac{\lambda_0 + 2\nu_0}{2}}\left(e^{\frac{\lambda_0+2\nu_0}{2}}\frac{2\gamma\pi_0 p_{r0}\xi}{r(\rho_0 + p_{r0})})\right)' \\
&+ \frac{2\pi_0'\xi}{r} + \frac{d\pi}{dp_r}\left[\frac{2\gamma p_{r0}e^{\frac{\nu_0}{2}}}{r^3}(e^{-\frac{\nu_0}{2}}r^2\xi)' - \frac{4\pi_0 p_{r0}\gamma\xi}{r^2(\rho_0 + p_{r0})}\right] \\
&= e^{\lambda_0 - \nu_0}\,(\rho_0 + p_{r0})\,\xi\omega^2.
\end{aligned}
\tag{300}
$$

This equation is not a pretty sight. We would much rather write equation (286) as is, and keep the TOV-equation nearby.

10 Concluding remarks

We reproduced all the calculations that were left out in the Esculpi/Aloma article, and our pulsation equation turned out different from what Esculpi/Aloma showed. The article contained several peculiarities, including several mistakes. Also, they cite the work of Glazer in producing the expression for $\delta\eta$, yet the expression they give is useless as given, and in the pulsation equation they have not substituted this expression.

We also developed the pulsation equation for the most general case, where we have included anisotropy, where the tangential pressure is any function of the radial pressure, and electrical charge is included, as is the cosmological constant. At first glance, the cosmological constant is only apparent in a single term in the pulsation equation. To go with this, we showed what the TOV-equation looks like in the most general case. We now see that the cosmological constant affects the pulsation equation in more than just the one term, and we see how all these parameters, charge, anisotropy, gravitational energy and the cosmological constant all intertwine to affect the radial pressure-gradient, thereby affecting the stability of the sphere.

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