HYPERKÄHLER FOURFOLDS AND KUMMER SURFACES

ATANAS ILIEV, GRZEGORZ KAPUSTKA, MICHAŁ KAPUSTKA, AND KRISTIAN RANESTAD

Abstract. We show that a Hilbert scheme of conics on a Fano fourfold double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along a divisor of bidegree $(2,2)$ admits a $\mathbb{P}^1$-fibration with base being a hyper-Kähler fourfold. We investigate the geometry of such fourfolds relating them with degenerated EPW cubes, with elements in the Brauer groups of $K3$ surfaces of degree 2, and with Verra threefolds studied in [Ver04]. These hyper-Kähler fourfolds admit natural involutions and complete the classification of geometric realizations of anti-symplectic involutions on hyper-Kähler 4-folds of type $K3^{[2]}$.

As a consequence we present also three constructions of quartic Kummer surfaces in $\mathbb{P}^3$: as Lagrangian and symmetric degeneracy loci and as the base of a fibration of conics in certain threefold quadric bundles over $\mathbb{P}^1$.

Contents

By a hyper-Kähler manifold or equivalently by an irreducible holomorphic symplectic (or IHS) $2n$-fold we mean a $2n$-dimensional simply connected compact Kähler manifold

2000 *Mathematics Subject Classification.* 14J10,14J40.
Key words and phrases. Kummer surfaces, Irreducible symplectic manifolds, hyperkähler varieties, Lagrangian fibrations, Lagrangian degeneracy loci, Brauer groups.

with trivial canonical bundle that admits a unique (up to a constant) non-degenerate holomorphic 2-form (called the symplectic form) and is not a product of two manifolds [Bea83]. In this paper we are studying the geometry of some families of IHS fourfolds that are deformation equivalent to the Hilbert scheme of two points on a $K3$ surface (of type $K3^{[2]}$).

Recall from [BD85] that Hilbert schemes of lines on smooth cubic hypersurfaces in $\mathbb{P}^5$ are IHS fourfolds of type $K3^{[2]}$ characterized by the fact that they admit a polarization of Beauville degree $q = 6$ (i.e degree $3 * 36$). In [O'G06] O'Grady described the complete family of polarized IHS fourfolds of $K3^{[2]}$ type with Beauville degree $q = 2$. He found out that such manifolds are double covers of sextic hypersurfaces defined as Lagrangian degeneracy loci. Next [IM11] described constructions of IHS fourfolds with $q = 2$ as bases of $\mathbb{P}^1$ fibrations on Hilbert schemes of conics on Fano fourfolds of degree 10.

The aim of this article is to investigate a special 19-dimensional family $\mathcal{U}$ of IHS fourfolds of type $K3^{[2]}$ admitting a polarization of Beauville degree $q = 4$ (i.e degree 48). In fact, the family $\mathcal{U}$ represents a component of the hyperelliptic locus in the moduli space of all IHS fourfolds of type $K3^{[2]}$ admitting a polarization of Beauville degree $q = 4$. The elements of the family $\mathcal{U}$ are obtained as double covers of some special Lagrangian degeneracy loci on a cone over $\mathbb{P}^2 \times \mathbb{P}^2$. The same family $\mathcal{U}$ is obtained by considering for a general Fano fourfold Y being the double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ branched along a bi-degree $(2,2)$ divisor (we call such Y Verra fourfolds) the Hilbert scheme $F(Y)$ of conics on Y. We show that a general fivefold $F(Y)$ admits a natural $\mathbb{P}^1$ fibration such that its base is an IHS fourfold in $\mathcal{U}$. Finally, we show also that the generic element from $\mathcal{U}$ is a moduli space of twisted sheaves on a $K3$ surface.

The IHS fourfolds from $\mathcal{U}$ appear naturally in the following context: Recall that van Geemen classified two torsion elements in the Brauer group $Br(S)$ of a general $K3$ surface S that admits a polarization of degree 2, [vG05]. He showed that there are three types of elements in $Br(S)_2 \simeq (\mathbb{Z}_2)^{21}$ and that they give rise to three type of varieties Y_{α_i} for $i = 1,2,3$ respectively:

- a smooth complete intersection of three quadrics in $\mathbb{P}^5$, or
- a cubic fourfold containing a plane, or
- a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along a hypersurface of bi-degree $(2,2)$;

such that a twist of the polarized Hodge structure defined by α_i is Hodge isometric to a primitive sublattice of the middle cohomology of Y_{α_i} for $i = 1,2,3$. There are direct geometric constructions relating (S,α_i) with the variety Y_{α_i}. In the first case Mukai [Muk87] showed that a moduli space of bundles on Y_{α_1} is isomorphic to S. In [Bho86] it is shown that Y_{α_1} is isomorphic to the moduli space of certain orthogonal bundles on S; giving the relation in the other direction. Note, however, that the twist is not apparent in these construction. One may ask whether the $K3$ surface Y_{α_1} of degree 8 is isomorphic to a moduli space of twisted sheaves on $K3$ surfaces of degree 2 with the twist α_1 [MSTVA14, §1].

In the second case for (S,α_2) a geometric relation was described in [MS12]. It was shown that a moduli space of twisted sheaves on (S,α_2) is birational to the IHS fourfold being the Hilbert scheme of lines on a cubic fourfold containing a plane. Our construction completes the picture by showing that the moduli space of twisted sheaves on (S,α_3) is isomorphic to an IHS fourfold from $\mathcal{U}$ i.e. is constructed from the Hilbert scheme of conics on the corresponding fourfold Y_{α_3}.

0.1. **Construction via Lagrangian Degeneracy loci.** Section 2 is devoted to the construction of elements of $\mathcal{U}$ as double covers of appropriate Lagrangian degeneracy loci inside a cone $C(\mathbb{P}^2\times\mathbb{P}^2)\subset\mathbb{P}^9$ over the Segre embedding of $\mathbb{P}^2\times\mathbb{P}^2$. This construction is analogous to the construction of EPW sextics [O'G06], [EPW01]. It is also naturally related to special EPW cubes [IKKR16]. Let us be more precise: Let U_1, U_2 be 3-dimensional vector spaces with fixed volume forms. Consider the cone over the Segre embedding of $\mathbb{P}(\wedge^2 U_1)\times\mathbb{P}(U_2)$

$$C_{U_1} := C(\mathbb{P}(\wedge^2 U_1)\times\mathbb{P}(U_2))$$

interpreted as a subset

$$C_{U_1} = \mathrm{G}(3, U_1\oplus U_2)\cap\mathbb{P}(\wedge^3 U_1\oplus(\wedge^2 U_1\otimes U_2))\subset\mathbb{P}(\wedge^3(U_1\oplus U_2)).$$

Note that we use the notation $\mathbb{P}(B)$ to denote the space of 1-dimensional subspaces of B. Consider the vector subspace $(\wedge^2 U_1\otimes U_2)\oplus(U_1\otimes\wedge^2 U_2)\subset\wedge^3(U_1\oplus U_2)$ equipped with a symplectic form corresponding to wedge product. Each point $[U]$ of the cone C_{U_1} corresponds to a three-space $U\subset U_1\oplus U_2$ such that $\dim(U\cap U_1)\geq 2$. To U we associate the Lagrangian subspace

$$\begin{aligned}\bar{T}_U := (\wedge^2 U\wedge(U_1\oplus U_2))/\wedge^3 U_1 \subset &(\wedge^3 U_1\oplus(\wedge^2 U_1\otimes U_2)\oplus(U_1\otimes\wedge^2 U_2))/\wedge^3 U_1\\ &\cong(\wedge^2 U_1\otimes U_2)\oplus(U_1\otimes\wedge^2 U_2).\end{aligned}$$

Let $\bar{A}\subset(\wedge^2 U_1\otimes U_2)\oplus(U_1\otimes\wedge^2 U_2)$ be a general Lagrangian subspace. To this subspace $\bar{A}$ we can associate degeneracy loci for each $k>0$:

$$D_k^{\bar{A}} = \{[U]\in C_{U_1} | \dim(\bar{T}_U\cap\bar{A})\geq k\}.$$

The variety $D_1^{\bar{A}}$ is a special quartic section of C_{U_1} that we call an EPW quartic section (abusing the name of the first degeneracy locus in $\mathrm{G}(3,6)$ considered in [DK15]). We shall prove that for a generic choice of $\bar{A}$ the fourfold $D_1^{\bar{A}}$ is singular exactly along the surface $D_2^{\bar{A}}\subset\mathbb{P}(\wedge^3(U_1\oplus U_2))$ which has degree 72. The main result of the above construction is the following:

Theorem 0.1. *For a generic choice of $\bar{A}\in\mathrm{LG}(9,(\wedge^2 U_1\otimes U_2)\oplus(U_1\otimes\wedge^2 U_2))$ there exists a natural double cover $X_{\bar{A}}\to D_1^{\bar{A}}$ branched along $D_2^{\bar{A}}$ such that $X_{\bar{A}}$ is an IHS fourfold of $K3^{[2]}$ type that admits a polarization of Beauville degree $q=4$.*

The proof is presented in Section 2. The subset $\mathcal{U}$ of the moduli space of polarized IHS fourfolds deformation equivalent to $K3^{[2]}$ and with polarization of Beauville degree 4 that parametrizes manifolds constructed in Theorem 0.1 is of dimension 19.

0.2. **Relation to EPW cubes.** The construction of EPW quartic sections is more natural when seen in the context of EPW cubes. Recall that in [IKKR16] we constructed a 20-dimensional family (locally complete) of polarized IHS sixfolds deformation equivalent to the Hilbert scheme of three points on a $K3$-surface (i.e. of type $K3^{[3]}$) and admitting a polarization of Beauville degree $q=4$. The elements of this family are natural double covers of special codimension 3 subvarieties of the Grassmannian $\mathrm{G}(3,6)$ that we called EPW cubes. The EPW quartic sections can be seen as subvarieties of special EPW cubes. Recall that for a Lagrangian subspace $A\subset\wedge^3 V_6$ we define

$$D_A^2 = \{[U]\in\mathbb{P}(\wedge^3 V_6) | \dim(A\cap((\wedge^2 U)\wedge V_6))\geq 2\}.$$

When A is general D_A^2 is called an EPW cube. If now $A\subset\wedge^3 V_6$ is a general Lagrangian subspace that contains $\wedge^3 U_1$, for some $U_1\subset V_6$ of dimension 3 then D_A^2 is a special EPW cube. Now for every decomposition $V_6 = U_1\oplus U_2$ we have a natural identification

$C_{U_1} = C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)) = T_{[U_1]} \cap \mathrm{G}(3, V_6)$, where $T_{[U_1]}$ is the projective tangent space to $\mathrm{G}(3, V_6)$ in $[U_1]$. Under this identification we have

$$D^1_{\bar{A}} = D^2_A \cap C_{U_1},$$

with $\bar{A} = A/(\wedge^3 U_1) \subset (\wedge^3 U_1)^\perp/(\wedge^3 U_1)$.

0.3. **Construction via Hilbert scheme.** Our second construction of IHS fourfolds from the family $\mathcal{U}$ is the subject of Section 3. It uses Hilbert schemes of conics on so-called Verra Fano fourfolds. Let U_1 and U_2 be 3-dimensional vector spaces. We call a *Verra fourfold* [Ver04], [Ili97] an element of the 19-dimensional family of Fano fourfolds which is the intersection Y of the cone $C(\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2 U_2)) \subset \mathbb{P}(\mathbb{C}\oplus(U_1\otimes\wedge^2 U_2))$ with a quadric hypersurface Q. Equivalently Y is the double cover of $\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2 U_2) = \mathbb{P}^2\times\mathbb{P}^2$ branched along a divisor Z of bi-degree $(2,2)$. The threefold Z will be called the *Verra threefold* associated to the Verra fourfold Y. Note that Z can be identified with the section of Y by the hyperplane polar to the vertex of the cone $C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2))$ via the quadric Q. Verra threefolds were introduced by A. Verra in [Ver04] as counterexamples to the Torelli problem for Prym varieties of unbranched double coverings of plane sextics.

The linear system of quadrics containing $C(\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2 U_2)) \subset \mathbb{P}(\mathbb{C}\oplus(U_1\otimes\wedge^2 U_2))$ is then naturally isomorphic to $\mathbb{P}(U_1\otimes\wedge^2 U_2)$, via a volume form on $U_1\otimes\wedge^2 U_2 \cong U_1\otimes U_2^\vee$. The linear system of quadrics containing $Y \subset \mathbb{P}^9$ is therefore naturally isomorphic to $\mathbb{P}(\mathbb{C}\oplus(U_1\otimes\wedge^2 U_2))$ and its dual is naturally isomorphic to $\mathbb{P}(\mathbb{C}\oplus(\wedge^2 U_1\otimes U_2))$. The fourfold Y admits two natural projections π_1 and π_2 onto $\mathbb{P}(U_1)$ and $\mathbb{P}(\wedge^2 U_2)$ respectively. We denote by $F(Y)$ the Hilbert scheme of plane conic curves on Y of type $(1,1)$ i.e. conics that projects to lines by both π_1 and π_2.

Let $[C] \in F(Y)$ be a $(1,1)$-conic on Y, then C spans a plane $\mathrm{P}_C \subset \mathbb{P}(\mathbb{C}\oplus(U_1\otimes\wedge^2 U_2))$, and the locus H_C of quadrics containing $Y \cup \mathrm{P}_C$ is a hyperplane, i.e. a point $[H_C] \in \mathbb{P}(\mathbb{C}\oplus(\wedge^2 U_1\otimes U_2))$ in the dual space. In this way we define a map

$$\psi_Q : F(Y) \to \mathbb{P}(\wedge^3 U_1 \oplus (\wedge^2 U_1\otimes U_2)); \qquad [C] \mapsto [H_C]$$

We identify the image of this map in the following way. Note that the quadric hypersurface Q, such that $Y = Q \cap C(\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2 U_2))$, induces a quadric $Q' \subset \mathbb{P}(U_1\otimes\wedge^2 U_2)$ defining the branch locus Z of the double cover $Y \to \mathbb{P}(U_1)\times\mathbb{P}(\wedge^2 U_2)$ via $Z = Q' \cap (\mathbb{P}(U_1)\otimes\mathbb{P}(\wedge^2 U_2))$. The quadric Q' is defined by a symmetric linear map $q' \colon (U_1\otimes\wedge^2 U_2) \to (\wedge^2 U_1\otimes U_2)$. The graph of such a symmetric map q' is a Lagrangian subspace that we denote $\bar{A}_Q \subset (\wedge^2 U_1\otimes U_2)\oplus(U_1\otimes\wedge^2 U_2)$. We shall prove that the image $\psi_Q(F(Y))$ coincides with the first degeneracy locus

$$D_1^{\bar{A}_Q} \subset C(\mathbb{P}(\wedge^2 U_1)\times\mathbb{P}(U_2)).$$

Furthermore by studying fibers of the map we obtain a factorization $\psi_Q = \rho\circ\phi$ with ϕ a $\mathbb{P}^1$ fibration and ρ a 2:1 map branched exactly in $D_2^{\bar{A}_Q}$. Combining this with Theorem 0.1 we obtain:

Theorem 0.2. *The Hilbert scheme of conics on a general Verra fourfold $Y = Q \cap C(\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2 U_2))$ admits a $\mathbb{P}^1$-fibration (a smooth map whose all fibers are isomorphic to $\mathbb{P}^1$) over the IHS fourfold $X_{\bar{A}_Q} \in \mathcal{U}$. Moreover, a general IHS fourfold $X \in \mathcal{U}$ appears in this way.*

As a consequence of the proof of Theorem 0.2 we observe furthermore that in the above notation the surface $D_2^{\bar{A}_Q}$ is on one hand isomorphic to the fixed locus of an antisymplectic involution on the IHS fourfold $X_{\bar{A}_Q}$ and on the other it admits an étale double cover by the Hilbert scheme of conics on the Verra threefold Z (see Proposition 3.6).

0.4. **Moduli space of twisted sheaves.** In Section 4 we show a further alternative construction of the elements of $\mathcal{U}$: as moduli spaces of twisted sheaves [Yos06] on $K3$ surfaces. More precisely we prove:

Theorem 0.3. *A general fourfold $X \in \mathcal{U}$ is isomorphic to the moduli space of stable twisted sheaves on a polarized $K3$ surface of degree 2 with a two-torsion Brauer element.*

0.5. **Properties.** Our main motivation to study the family $\mathcal{U}$ is to understand the relation between the three geometric constructions considered. As a result we present relations of different points of view: Hodge-theoretic, moduli-theoretic, geometric, and arithmetic. In particular we prove, that the generic element of $\mathcal{U}$ has Picard group of rank 2 does not admit any polarization of Beauville degree 2 and is not isomorphic to a moduli space of sheaves on a $K3$ surface. Moreover, each element of the family $\mathcal{U}$ admits two Lagrangian fibrations and is a $8:1$ ramified cover of $\mathbb{P}^2 \times \mathbb{P}^2$.

In section 2 we also discuss our construction in the context of the classification of automorphisms of IHS fourfolds of type $K3^{[2]}$. In particular, we shall see that $\mathcal{U}$ is the unique 19-dimensional irreducible family of IHS fourfolds of type $K3^{[2]}$ that is not in the closure of the family of double EPW sextics, such that each element admits an anti-symplectic involution [OW13]. In particular, the family $\mathcal{U}$ can be seen as a component of the hyperelliptic locus of the moduli space of polarized IHS fourfolds of type $K3^{[2]}$ with $q = 4$. Indeed, for a general IHS fourfold of type $K3^{[2]}$ with polarization of Beauville degree $q = 4$ the map defined by the polarization is birational. The following remains a challenge:

Problem 0.4. *Describe the generic polarized IHS fourfold of type $K3^{[2]}$ of Beauville degree $q = 4$.*

The description as double covers of Lagrangian degeneracy loci can also be applied to study degenerations of the family $\mathcal{U}$ and permit to complete the classification of geometric realizations of automorphisms of IHS of type $K3^{[2]}$ given in [MW15]. Note that as a direct consequence from [MW15, §5.1] we obtain the following:

Corollary 0.5. *Any IHS fourfold X of type $K3^{[2]}$ that admits non-symplectic automorphism of prime order $p \neq 3, 23$ is either in the closure of the family of double EPW sextics or in the closure of the family $\mathcal{U}$, or X is isomorphic to a moduli space of stable objects on a $K3$ surface and the automorphism is induced from an automorphism of the $K3$ surface.*

Finally in section 5 we study the invariants of the two dimensional fixed loci of the involution on the elements from the family $\mathcal{U}$. Recall that Beauville studied the invariants of the fixed loci of antisymplectic involutions on IHS fourfolds in general. In the case of 19-dimensional families of involutions on IHS fourfolds with $b_2 = 23$ it follows from [Bea11, Theorem 2] that the invariants of the fixed locus F are $K_F^2 = 288$ and $\chi(\mathcal{O}_F) = 37$. Using Proposition 3.6 we are able to deduce the invariants of a Hilbert schemes of conics on a Verra threefold Z. The computation of all invariants is included in Proposition 5.1.

0.6. **Relation to Kummer surfaces.** In section 1, we describe a "Baby case" of our constructions by presenting two constructions of the Kummer surfaces first as Lagrangian degeneracy loci (as in [EPW01, Theorem 9.2]) and next as a quotient of the base of a fibration on the Hilbert scheme of $(1,1)$-conics on a quadric section of a cone $C(\mathbb{P}^1 \times \mathbb{P}^2) \subset \mathbb{P}^6$ over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The relation to the description of the EPW quartic section is explained in Section 3.1. In particular, we shall see that the EPW quartic section admits two fibrations by Kummer surfaces. The descriptions of EPW quartic sections via Lagrangian degeneracy loci and Hilbert scheme fibration restrict to the obtained descriptions of Kummer surfaces.

Furthermore, in Section 1 we provide in addition a third construction for a general Kummer surface: as a component of the discriminant locus of the system of quadrics containing the Verra fourfold, or equivalently as the associated symmetric degeneracy locus.

Acknowledgements. We thank G. Mongardi for several helpful discussions, in particular for suggesting us Theorem 0.3. We thank A. Kresch, A. Kuznetsov and L. Manivel for helpful comments. A. Iliev was supported by SNU grant 0450-20130016, G. Kapustka by Iuventus plus 0301/IP3/2015/73 "Teoria reprezentacji oraz wlasności rozmaitości siecznych", M. Kapustka by NCN grant 2013/10/E/ST1/00688 and K. Ranestad by RCN grant 239015.

0.7. **Notation.** Let us explain here some of the notation used in the paper. Let V be a complex 6-dimensional complex vector space, and fix an isomorphism $vol : \wedge^6 V \to \mathbb{C}$. It induces a natural skew-symmetric form

$$\eta : \wedge^3 V \times \wedge^3 V \to \mathbb{C}, \quad (\omega, \omega') \mapsto vol(\omega \wedge \omega'). \tag{0.1}$$

We denote by $\mathrm{LG}_\eta(10, \wedge^3 V)$ the variety of 10-dimensional isotropic (i.e. Lagrangian) subspaces of $\wedge^3 W$ with respect to η. For any 3-dimensional subspace $U \subset V$, the 10-dimensional subspace

$$T_U := \wedge^2 U \wedge V \subset \wedge^3 V$$

belongs to $\mathrm{LG}_\eta(10, \wedge^3 V)$, and $\mathbb{P}(T_U)$ is the projective tangent space to $\mathrm{G}(3,V) \subset \mathbb{P}(\wedge^3 V)$ at $[U]$. Therefore, the family $\{T_U \mid [U] \in \mathrm{G}(3,V)\}$ forms a symplectic vector bundle of rank 10 over $\mathrm{G}(3,V)$.

For any $[A] \in \mathrm{LG}_\eta(10, \wedge^3 V)$ and $k \in \mathbb{N}$, we consider the following Lagrangian degeneracy locus, with natural scheme structure [PR97],

$$D_k^A = \{[U] \in \mathrm{G}(3,V) \mid \dim(A \cap T_U) \geq k\} \subset \mathrm{G}(3,V).$$

The variety D_2^A is an *EPW cube*. In the present paper we study special EPW cubes corresponding to the choice of Lagrangian space $A \in \Sigma$, where

$$\Sigma = \{[A] \in \mathrm{LG}_\eta(10, \wedge^3 V) \mid \mathbb{P}(A) \cap \mathrm{G}(3,V) \neq \varnothing\}$$

as in [O'G13] and [IKKR16]. From the same references we recall the notation for the following additional subsets of $\mathrm{LG}_\eta(10, \wedge^3 V)$:

$$\Delta = \{[A] \in \mathrm{LG}_\eta(10, \wedge^3 V) \mid \exists v \in V : \dim A \cap (v \wedge (\wedge^2 V)) \geq 3\},$$

$$\Gamma = \{A \in \mathrm{LG}_\eta(10, \wedge^3 V) \mid \exists [U] \in \mathrm{G}(3,V) : \dim A \cap T_U \geq 4\}.$$

For $[U_1] \in \mathbb{P}(A) \cap \mathrm{G}(3,V)$ the Lagrangian space $A \subset \wedge^3 V$ is contained in $(\wedge^3 U_1)^\perp$, and thus defines a Lagrangian space $\bar{A}_{U_1} \subset (\wedge^3 U_1)^\perp / (\wedge^3 U_1)$. Clearly

$$T_U \subset (\wedge^3 U_1)^\perp \subset \wedge^3 V$$

for any $[U] \in \mathrm{G}(3,V) \cap \mathbb{P}(T_{U_1})$ so we define

$$D_k^{\bar{A}_{U_1}} = \{[U] \in \mathrm{G}(3,V) \cap \mathbb{P}(T_{U_1}) \mid \dim(\bar{A}_{U_1} \cap (T_U / (\wedge^3 U_1))) \geq k\} = \mathrm{G}(3,V) \cap \mathbb{P}(T_{U_1}) \cap D_{k+1}^A.$$

The variety $D_1^{\bar{A}_{U_1}}$ is an EPW quartic section.

Denote after O'Grady [O'G13]:

$$\tilde{\Sigma} := \{([U],[A]) \in \mathrm{G}(3,V) \times \mathrm{LG}(10,\wedge^3 V) \mid \wedge^3 U \subset A\},$$
$$\tilde{\Sigma}(d) := \{([U],[A]) \in \tilde{\Sigma} \mid \dim(A \cap (\wedge^2 U \wedge V)) \geq d+1\},$$
$$\Theta_A := \{[U] \in \mathrm{G}(3,V) \mid \wedge^3 U \subset A\}.$$
$$\Sigma_+ = \{[A] \in \Sigma \mid \quad Card(\Theta_A) > 1\},$$

If $\pi\colon \mathrm{G}(3,V) \times \mathrm{LG}(10,\wedge^3 V) \to \mathrm{LG}(10,\wedge^3 V)$ is the projection, then we set $\Sigma[d] := \pi(\tilde{\Sigma}(d))$.

1. Kummer surfaces–the first case

In this section we present a special construction of the Kummer quartic surface as a first Lagrangian degeneracy locus and at the same time as a symmetric degeneracy locus, as well as the base of a fibration on the Hilbert scheme of conics on a Fano threefold. This shows, in particular, that the Kummer quartic can be seen as the "baby case" of the EPW sextic construction. In the section 3.1 we shall see that the Kummer quartic is a building block in the construction of our 19-dimensional family $\mathcal{U}$.

1.1. Kummer surfaces as Lagrangian degeneracy loci.

Denote by $V = V_2 \oplus V_4$ the complex 6-dimensional vector space decomposed in the direct sum of a 2-dimensional space V_2 and a 4-dimensional space V_4. Set an isomorphism $vol : \wedge^6 V = \wedge^2 V_2 \otimes \wedge^4 V_4 \to \mathbb{C}$ by fixing isomorphisms $vol_i : \wedge^i V_i \to \mathbb{C}$. The isomorphism induces a natural skew symmetric form

$$\eta : \wedge^3 V \times \wedge^3 V \to \mathbb{C}, \quad (\omega, \omega') \mapsto vol(\omega \wedge \omega'), \tag{1.1}$$

which restricts to a nondegenerate skew symmetric form $\eta_{2,4}$ on the 12-dimensional subspace

$$V_{2,4} = V_2 \otimes \wedge^2 V_4 \subset \wedge^3 V.$$

For each $v \in V_4$ the 6-dimensional subspace

$$F_v := V_2 \otimes V_4 \wedge v \subset V_{2,4}$$

is Lagrangian with respect to $\eta_{2,4}$. Let $A \subset V_{2,4}$ be a general Lagrangian 6-space, and let

$$D_i^A = \{[v] \in \mathbb{P}(V_4) | \operatorname{rank} A \cap (V_2 \otimes V_4 \wedge v) \geq i\}.$$

Lemma 1.1. *D_1^A is a Kummer quartic surface singular in D_2^A; a set of* 16 *points.*

Proof. Let $\mathrm{LG}(6, V_{2,4})$ denote the Lagrangian Grassmannian parameterizing the Lagrangian subspaces of $V_{2,4}$, and let $\mathcal{F}$ be the universal rank 6 quotient bundle on $\mathrm{LG}(6, V_{2,4})$. The map

$$\phi : \mathbb{P}(V_4) \to \mathrm{LG}(6, V_{2,4}); \quad [v] \mapsto [F_v]$$

is an embedding, and the pullback $\phi^*(\mathcal{F})$ is a rank 6 bundle $\mathcal{F}_{\mathbb{P}(V_4)}$ on $\mathbb{P}(V_4)$. By construction F_v is a direct sum of two copies of a plane $\mathbb{P}(V_4 \wedge v) \subset \mathbb{P}(\wedge^2 V_4)$, so $\mathcal{F}_{\mathbb{P}(V_4)}$ is a direct sum of two copies of a bundle F_0 on $\mathbb{P}(V_4)$ with total Chern class $c(F_0) = 1 + 2h + 2h^2$, where h is the class of hyperplane in $\mathbb{P}(V_4)$. Therefore $\mathcal{F}_{\mathbb{P}(V_4)}$ has total Chern class

$$c(\mathcal{F}_{\mathbb{P}(V_4)}) = 1 + 4h + 8h^2 + 8h^3 + 4h^4.$$

The class in $\mathbb{P}(V_4)$ of the degeneracy D_i^A is now the degeneracy of the natural map $\phi^*(A) \to \mathcal{F}_{\mathbb{P}(V_4)}$. The first bundle $\phi^*(A)$ is trivial, so, by the formulas of Pragacz and Ratajski [PR97, Theorem 2.1], these degeneracy classes are given by the Chern classes of $\mathcal{F}_{\mathbb{P}(V_4)}$:

$$[D_1^A] = c_1(\mathcal{F}_{\mathbb{P}(V_4)}) = 4h, \quad [D_2^A] = (c_1c_2 - 2c_3)(\mathcal{F}_{\mathbb{P}(V_4)}) = 16h^3.$$

□

Remark 1.2. Similarly, for any 3-dimensional subspace $U \subset V_4$, the subspace

$$V_2 \otimes \wedge^2 U \subset V_{2,4}$$

is Lagrangian with respect to $\eta_{2,4}$. The degeneracy loci

$$\hat{D}_i^A = \{[U] \in \mathbb{P}(V_4^\vee) | \operatorname{rank} A \cap (V_2 \otimes \wedge^2 U) \geq i\}, \ (i = 1, 2)$$

are then again a Kummer surface $\hat{D}_1^A$ and 16 points $\hat{D}_2^A$ forming the singular locus of $\hat{D}_1^A$.

The Lagrangian degeneracy loci D_i^A, may also be interpreted as symmetric degeneracy loci:

1.2. **Kummer surfaces as symmetric degeneracy loci.** Fix a decomposition $V_4 = \langle v_0 \rangle \oplus V_3$ and the Lagrangian subspace $F_{v_0} = V_2 \otimes V_4 \wedge v_0 \cong V_2 \otimes V_3$, and let $B \subset V_{2,4}$ be a Lagrangian subspace such that $F_{v_0} \cap B = 0$. Then B is naturally isomorphic to $F_{v_0}^\vee \cong V_2^\vee \otimes \wedge^2 V_3$. The Lagrangian space A is then the graph in $V_{2,4} = F_{v_0} \oplus B \cong F_{v_0} \oplus F_{v_0}^\vee$ of a linear symmetric map $F_{v_0} \to F_{v_0}^\vee$. Composing with the natural isomorphism $V_2 \otimes V_3 \to F_{v_0}$ and its transpose $F_{v_0}^\vee \to V_2^\vee \otimes \wedge^2 V_3$, we obtain a linear map

$$q_A : V_2 \otimes V_3 \to V_2^\vee \otimes \wedge^2 V_3$$

inducing a symmetric bilinear form that, by abuse of notation, we shall denote by the same name

$$q_A : (V_2 \otimes V_3) \times (V_2 \otimes V_3) \to \mathbb{C}.$$

Denote by $Q_A = \{[\alpha] | \ q_A(\alpha, \alpha) = 0\} \subset \mathbb{P}(V_2 \otimes V_3)$ the quadric defined by q_A. Abusing notation again Q_A will also be the quadric polynomial defined by $Q_A(\alpha) \coloneqq q_A(\alpha, \alpha)$ defining the quadric Q_A. Similarly, for every $v \in V_3$ the map $(v_2 \otimes v_3) \mapsto v_2 \otimes v_3 \wedge v$ extends linearly to a symmetric bilinear map

$$q_v : (V_2 \otimes V_3) \times (V_2 \otimes V_3) \to \mathbb{C}.$$

Denote by $Q_v = \{[\alpha] | \ q_v(\alpha, \alpha) = 0\} \subset \mathbb{P}(V_2 \otimes V_3)$ the quadric defined by q_v and again also the quadratic polynomial defining the quadric. Notice that Q_v vanishes on the Segre 3-fold

$$\Sigma_{2,3} = \{[v_2' \otimes v_3] \in \mathbb{P}(V_2 \otimes V_3) | \ v_2 \in V_2, v_3 \in V_3\},$$

and in fact $[v] \mapsto Q_v$ defines an isomorphism

$$\mathbb{P}(V_3) \to \mathbb{P}(H^0(\mathcal{I}_{\Sigma_{2,3}}(2))).$$

Let $S_A = \Sigma_{2,3} \cap Q_A$. Then there is similarly a natural isomorphism

$$V_4 \cong H^0(\mathcal{I}_{S_A}(2)); \quad v + \lambda v_0 \mapsto q_v + \lambda q_A.$$

Let

$$\mathcal{D}_i = \{[v] \in \mathbb{P}(V_4) | \operatorname{corank} q_v \geq i\}$$

be the i-th degeneracy locus in $\mathbb{P}(V_4)$ of the linear system of quadrics $\{Q_v | [v] \in \mathbb{P}(V_4)\}$. Since the quadrics in the ideal of $\Sigma_{2,3}$ have rank 4, i.e. corank 2, we get that $\mathcal{D}_1$ contains the plane $\mathbb{P}(H^0(\mathcal{I}_{\Sigma_{2,3}}(2)))$ with multiplicity 2, and $\mathcal{D}_2$ contains this plane with multiplicity 1. The relation between the Lagrangian loci D_i^A and the symmetric loci $\mathcal{D}_i$ is described in the following:

Lemma 1.3. $D_i^A \cup \mathbb{P}(H^0(\mathcal{I}_{\Sigma_{2,3}}(2))) = \mathcal{D}_i$

Proof. It suffices to show that if $\beta = q_A(\alpha)$ and $(\alpha \wedge v_0 + \beta) \in F_{v+\lambda v_0} \cap A$, then

$$(q_v + \lambda q_A)(\alpha) = 0.$$

To show this we may assume that

$$\alpha = v_2 \otimes v_3 + v_2' \otimes v_3' \in V_2 \otimes V_3$$

and let

$$q_A(\alpha) = \beta = v_2 \otimes \beta_1 + v_2' \otimes \beta_2$$

with $\beta_i \in \wedge^2 V_3$. Then

$$v_0 \wedge \alpha + \beta \in A \in F_{v+\lambda v_0} \cap A \quad \text{iff} \quad (v_0 \wedge \alpha + \beta) \wedge (v + \lambda v_0) = 0.$$

The right hand side is

$$v_0 \wedge \alpha \wedge v + \beta \wedge (v + \lambda v_0) =$$
$$v_2 \otimes (v_3 \wedge v \wedge v_0 + \beta_1 \wedge v + \lambda\beta_1 \wedge v_0) + v_2' \otimes (v_3' \wedge v \wedge v_0 + \beta_2 \wedge v + \lambda\beta_2 \wedge v_0) = 0$$

and is equivalent to

$$\beta_1 \wedge v = \beta_2 \wedge v = 0 \quad \text{and} \quad \lambda\beta_1 = -v_3 \wedge v, \lambda\beta_2 = -v_3' \wedge v.$$

But then

$$\begin{aligned}(q_v + \lambda q_A)(\alpha) &= v_2 \otimes v_3 \wedge v + v_2' \otimes v_3' \wedge v + \lambda v_2 \otimes \beta_1 + \lambda v_2' \otimes \beta_2 \\ &= v_2 \otimes v_3 \wedge v + v_2' \otimes v_3' \wedge v - v_2 \otimes v_3 \wedge v - v_2' \otimes v_3' \wedge v = 0\end{aligned}$$

so the implication and the lemma follows. □

Remark 1.4. The intersection $S_A = \Sigma_{2,3} \cap Q_A$ is a del Pezzo surface of degree 2. The plane $\mathbb{P}(H^0(\mathcal{I}_{\Sigma_{2,3}}(2))) \subset \mathbb{P}(H^0(\mathcal{I}_{S_A}(2)))$ intersects the Kummer surface $D_i^A \subset \mathbb{P}(H^0(\mathcal{I}_{S_A}(2)))$ in a plane quartic curve. One may show, that for general A, this curve is smooth. Considering the similar symmetric degeneracy locus of quadrics for a hyperplane section $S_A \cap H$ and a double hyperplane section $S_A \cap H \cap H'$, one may show that the corresponding plane quartics are a singular quartic and a double conic, respectively.

That the symmetric degeneracy locus $\mathcal{D}_1 \subset \mathbb{P}(H^0(\mathcal{I}_{S_A}(2)))$ has a component that is a Kummer surface can be seen considering conics on S_A. The surface $\mathcal{D}_1$ is clearly a sextic, being the discriminant of a space of quadrics in $\mathbb{P}^5$. Since the quadrics in the ideal of the Segre cubic scroll all have rank 4, the plane $\mathbb{P}(H^0(\mathcal{I}_{\Sigma_{2,3}}(2)))$ is a component of multiplicity 2 in $\mathcal{D}_1$, so the residual component is a quartic surface. We show that 16 pairs of conic curves on S_A correspond to 16 planes in $\mathbb{P}(H^0(\mathcal{I}_{S_A}(2)))$ that each contain 6 rank 4-quadrics that contain S_A, but not $\Sigma_{2,3}$. Furthermore there are 16 rank 4-quadrics on the quartic surface in $\mathcal{D}_1$ outside the plane $\mathbb{P}(H^0(\mathcal{I}_{\Sigma_{2,3}}(2)))$, so the quartic is a Kummer surface.

Let $\pi_1 : S_A \to \mathbb{P}^1$ and $\pi_2 : S_A \to \mathbb{P}^2$ be the two projections to the factors of $\Sigma_{2,3}$. Then, for a general quadric Q_A every line in the intersection $S_A = \Sigma_{2,3} \cap Q_A$ is contracted by the map π_1.

Proposition 1.5. *Assume that S_A is smooth and that every line in S_A is contracted by π_1. Then*

(1) *S_A contains 12 lines, that form the components of 6 singular conics.*

(2) S_A *contains* 32 *smooth conic sections that are not fibers of* π_1. *They form* 16 *pairs that each intersect in a scheme of length* 2.
(3) S_A *contains* 32 *pencils of twisted cubic curves, that are pairwise complementary in hyperplane sections.*

Proof. The fibers of the projection $\pi_1 : S_A \to \mathbb{P}^1$ are plane conics, so S_A is birational to a ruled surface. Let H be the class of a hyperplane section on A and F the class of a fiber, then the canonical divisor is, by adjunction on $\Sigma_{2,3}$,

$$K_{S_A} = -2H + F.$$

So $K_{S_A}^2 = 2$ and S_A is isomorphic to a rational ruled surface blown up in 6 points, and therefore has 6 singular conics, i.e. 12 lines that intersect in 6 pairs and (1) follows.

Consider next the projection $\pi_2 : S_A \to \mathbb{P}^2$. It is $2:1$ and given by divisors in the class $H - F$. The general curve in this class is an elliptic quartic curve which is mapped $2:1$ onto a line with 4 branch points. In particular, the branch curve in $\mathbb{P}^2$ is a quartic curve with 28 bitangent lines. The preimage in S_A of each of these lines is a pair of rational curves intersecting in 2 points lying over the two branch points. Now, every line L in S_A is mapped to a line by π_2, and $\pi_2^{-1}(\pi_2(L))$ is the union of L and a twisted cubic curve C_L with $C_L^2 = -1$. Since there are 12 lines on S_A, there must be 16 bitangents to the branch curve whose preimage in S_A does not contain a line. Since the preimages have degree 4 on S_A, they must decompose into two smooth conics that intersect in a scheme of length 2. On the other hand, any conic that is not in a fiber of π_1 must be section of π_1 and is therefore mapped to a line by π_2, so (2) follows.

Notice that each of these conic sections have self intersection -1 and intersect 15 other conic sections among the 32 in one point.

Consider any conic section C that is a section of π_1, and its complement C' in the preimage of its image by π_2. Then C intersect 6 lines in S_A, one from each singular fiber of π_1, while C' intersect the remaining 6. Let L be on of the lines intersecting C, then the divisor class $C+L$ contains a pencil of twisted cubic curves without basepoints on S_A. If L' is the line in S_A that intersect L, then $C' + L'$ contains a pencil of twisted cubic curves without basepoints and $C + L + C' + L' = H$. Now, if C'' is a conic section in S_A that do not intersect C, and L'' is a line that intersect C'' but neither of C and L, then $(C + L) \cdot (C'' + L'') = 0$ and the two divisor classes $C + L$ and $C'' + L''$ coincide. Since $(C' + L') \cdot (C'' + L'') = 3$, we also have $C' \cdot C'' = C' \cdot L'' = C'' \cdot L' = 1$. Let L'' be one of the 5 lines in S_A besides L' that do not intersect C, then $C' \cup L' \cup L''$ spans a hyperplane, so the divisor class $H - C' - L' - L''$ contains a unique curve C''', a conic section that must be a section of π_1. We may conclude that that in the pencil $|C + L|$ of twisted cubic curves there are 6 singular fibers. We conclude that each conic section C that is a section of π_1 is a component of a fiber in 6 pencils of twisted cubic curves, and that each such pencil has 6 singular fibers. Adding up we find 16 pairs of base point free pencils of twisted cubic curves on S_A and (3) follows. □

Notice that the linear span of each twisted cubic curve is contained in unique quadric that contains S_A, a quadric of rank at most 4 that does not belong to the ideal of $\Sigma_{2,3}$. A hyperplane section of this quadric that contains the twisted cubic, will contain a twisted cubic of the complementary pencil, so the quadric must have rank 4. On the other hand any rank 4 quadric in the ideal of S_A that does not contain $\Sigma_{2,3}$, will define on S_A two base point free pencils of twisted cubic curves. We may therefore conclude:

Corollary 1.6. *In the ideal of* S_A *there are exactly* 16 *quadrics of rank* 4 *that do not contain* $\Sigma_{2,3}$. *Each of them define a pair of base point free pencils of twisted cubic* *curves on S_A. Furthermore, let C and C' be a pair of conics in S_A that intersect in a scheme of length 2 and let P and P' be the planes spanned by these conics. Then the net of quadrics that contain S_A and P contains also P', and the net contains exactly 6 rank 4-quadrics that do not contain $\Sigma_{2,3}$.*

Proof. It remains only to remark that each quadric in the net that contain S_A and P contain both C' and the line of intersection $P \cap P'$, so also P'. □

The dual surface $K^\vee$ to a Kummer quartic surface K is also a Kummer quartic, with each plane tangent along a conic through 6 nodes on K corresponding to a node on $K^\vee$, so we conclude:

Corollary 1.7. *Let $D_1^A \subset \mathbb{P}(H^0(\mathcal{I}_{S_A}(2)))$ be the Kummer surface, such that $\mathcal{D}_1 = D_1^A \cup \mathbb{P}(H^0(\mathcal{I}_{\Sigma_{2,3}}(2))$. Then the dual Kummer surface*

$$(D_1^A)^\vee \subset \mathbb{P}(H^0(\mathcal{I}_{S_A}(2))^\vee)$$

is singular in each point $[H^0(\mathcal{I}_{S_A \cup \langle C\rangle}(2))] \in \mathbb{P}(H^0(\mathcal{I}_{S_A}(2))^\vee)$, where $C \subset S_A$ is any of the 32 conics whose spanning plane $\langle C\rangle$ is not contained in $\Sigma_{2,3}$. These conics occur in pairs that define the same point, thus accounting for the 16 nodes of $(D_1^A)^\vee$.

1.3. **Kummer surfaces from a Hilbert scheme of conics.** We relate the Lagrangian and symmetric descriptions of Kummer surfaces to the Hilbert scheme of conics in a certain Fano threefold.

First we note a general lemma that identifies the discriminant locus of a family of quadrics with base locus a quadric section of a cone with the discriminant of the family of quadrics defining the branch locus of the induced double cover.

Lemma 1.8. *Let $X \subset \mathbb{P}^n$ be a manifold defined by quadrics and let $CX \subset \mathbb{P}^{n+1}$ be a cone over X with vertex $p \in \mathbb{P}^{n+1}$. Let Q be a general quadric form in $\mathbb{P}^{n+1}$. Let $Y_Q = CX \cap \{Q = 0\}$ and let $Y_r \subset X$ be the branch locus of the $2:1$ map induced by the projection from p of Y_Q onto X. Let $D_{CX} \subset \mathbb{P}(H^0(\mathbb{P}^{n+1}, I_{CX}(2)))$ and $D_{Y_Q} \subset \mathbb{P}(H^0(\mathbb{P}^{n+1}, I_{Y_Q}(2)))$ be the discriminants. The projective space $\mathbb{P}(H^0(\mathbb{P}^{n+1}, I_{CX}(2)))$ is a hyperplane in $\mathbb{P}(H^0(\mathbb{P}^{n+1}, I_{Y_Q}(2)))$, so we consider the inclusions*

$$D_{CX} \subset D_{Y_Q} \subset \mathbb{P}(H^0(\mathbb{P}^{n+1}, I_{Y_Q}(2))).$$

Similarly, we consider the inclusions in $\mathbb{P}(H^0(\mathbb{P}^n, I_X(2)))$ and $\mathbb{P}(H^0(\mathbb{P}^n, I_{Y_r}(2)))$

$$D_X \subset D_{Y_r} \subset \mathbb{P}(H^0(\mathbb{P}^n, I_{Y_r}(2))).$$

Then there exists a linear isomorphism $\mathbb{P}(H^0(\mathbb{P}^n, I_{Y_r}(2))) \to \mathbb{P}(H^0(\mathbb{P}^{n+1}, I_{Y_Q}(2)))$ mapping $D_{Y_r} \setminus D_X$ isomorphically to $D_{Y_Q} \setminus D_{CX}$.

Proof. Observe that in an appropriate choice of coordinates in $\mathbb{P}^{n+1}$ we have

$$Q(z, x_0, \ldots, x_n) = z^2 - Q'(x_0, \ldots, x_n)$$

and p is the point $(0, \ldots, 0, 1)$. It is the clear that in this setup Y_r is defined in $\mathbb{P}^n$ with coordinates $x_0, \ldots x_n$ as $X \cap \{(x_0 : \cdots : x_n) | Q'(x_0 : \cdots : x_n) = 0\}$. Note that $H^0(I_{CX}(2)) = H^0(I_X(2))$. Consider the map:

$$\phi : H^0(\mathbb{P}^n, I_{Y_r}(2)) \to H^0(\mathbb{P}^{n+1}, I_{Y_Q}(2))$$

such that $\phi|_{H^0(I_X(2))} = id$ and $\phi(Q') = Q$. Clearly ϕ is an isomorphism that doesn't change the corank of the quadrics that do not belong to $I_X(2)$, while it increases the corank by one for each quadric in $I_X(2)$. The complement $D_{Y_r} \setminus D_X$ is therefore isomorphic to $D_{Y_Q} \setminus D_{CX}$. □

Consider the 6-space $\mathbb{P}(\mathbb{C}\oplus(V_2\otimes V_3))(=\mathbb{P}^6)$, a general quadric hypersurface Q_A in this space and the 3-fold obtained as the intersection

$$T_A = C(\mathbb{P}(V_2)\times\mathbb{P}(V_3))\cap Q_A \subset \mathbb{P}(\mathbb{C}\oplus(V_2\otimes V_3)).$$

Denote by p the vertex of $C(\mathbb{P}(V_2)\times\mathbb{P}(V_3))$, and let $H_{A,p}$ be the polar of p with respect to the quadric Q_A, and let $Q_{A,p}=Q_A\cap H_{A,p}$ and $S_A=T_A\cap H_{A,p}$. Following Lemma 1.8, the restriction map $H^0(\mathbb{P}(\mathbb{C}\oplus(V_2\otimes V_3),\mathcal{I}_{T_A}(2))\to H^0(H_{A,p},\mathcal{I}_{S_A}(2))$ is an isomorphism not just between the vector spaces, but also between the components of the discriminants residual to the planes $\mathbb{P}(H^0(\mathcal{I}_{C(\mathbb{P}(V_2)\times\mathbb{P}(V_3))}(2)))$ and $\mathbb{P}(H^0(\mathcal{I}_{\Sigma_{2,3}}(2)))$, respectively. The discriminant in $\mathbb{P}(H^0(\mathcal{I}_{T_A}(2)))$ is the union of the plane $\mathbb{P}(H^0(\mathcal{I}_{C(\mathbb{P}(V_2)\times\mathbb{P}(V_3))}(2)))$ and a surface that we therefore may identify with the Kummer surface D_1^A. Dual to D_1^A is the Kummer surface $(D_1^A)^\vee\subset\mathbb{P}(H^0(\mathcal{I}_{T_A}(2))^\vee$.

The 3-fold T_A has natural projections, $\pi_1: T_A\to\mathbb{P}(V_2)$ and $\pi_2: T_A\to\mathbb{P}(V_3)$. A conic in T_A that is mapped birationally to $\mathbb{P}(V_2)$ and birationally onto a line in $\mathbb{P}(V_3)$ is called a $(1,1)$-conic. We denote by $F(T_A)$ the Hilbert scheme of $(1,1)$-conics in T_A.

Proposition 1.9. *$F(T_A)$ admits a morphism*

$$\psi_{Q_A}: F(T_A)\to(D_1^A)^\vee\subset\mathbb{P}(H^0(\mathcal{I}_{T_A}(2))^\vee$$

whose general fiber is a pair of $\mathbb{P}^1$'s.

Proof. The proof requires several lemmas. First we define ψ_{Q_A}. For any $(1,1)$-conic $C\subset T_A$ we let P_C be the plane spanned by C. Then the subspace

$$H_C := H^0(\mathcal{I}_{T_A\cup P_C}(2))\subset H^0(\mathcal{I}_{T_A}(2))$$

has codimension one, and hence defines a point in

$$[H_C]\in\mathbb{P}(H^0(\mathcal{I}_{T_A}(2))^\vee.$$

We shall show that $(D_1^A)^\vee$ is the image of the map

$$\psi_{Q_A}: F(T_A)\to\mathbb{P}(H^0(\mathcal{I}_{T_A}(2))^\vee,\qquad [C]\mapsto[H_C].$$

First, however, we show that the general fiber of ψ_{Q_A} is a pair of $\mathbb{P}^1$'s.

Lemma 1.10. *Assume that Q_A is general, so that T_A is smooth. Let $[C]\in F(T_A)$, then the subscheme defined by the net of quadrics H_C is a complete intersection, the union of T_A and a quadric threefold Q_C of rank at most 4. For general C, the quadric Q_C has rank 4 with singular point $p_C\notin T_A$, and the intersection $Q_C\cap T_A$ is a Del Pezzo quartic surface inside T_A. The two pencils of planes in Q_C, define two pencils of $(1,1)$-conics on T_A.*

Proof. We first show that the quadrics in H_C define a complete intersection. Note that since C is a $(1,1)$-conic, the plane P_C is not contained in the cone $C(\mathbb{P}(V_2)\times\mathbb{P}(V_3))$, so the net of quadrics H_C cannot contain the cone. Therefore, the net of quadrics H_C contains a pencil of quadrics that contain this cone. The base locus of this pencil is the union of the cone and a $\mathbb{P}^4_C$ that intersects the cone in a quadric 3-fold Q_{CC}. If the net of quadrics H_C contains the $\mathbb{P}^4_C$, then Q_{CC} is a component of T_A, against the genericity of T_A. Therefore every component in the base locus of H_C has codimension 3 and H_C defines a complete intersection.

This base locus is therefore the union of T_A and a quadric 3-fold Q_C in $\mathbb{P}^4_C$. Since Q_C contains the plane P_C, it has rank at most 4, with equality for general C. The intersection $T_A\cap Q_C = Q_{CC}\cap Q_C$ is a Del Pezzo surface, which is smooth for a general C. In particular, the singular point p_C of the quadric Q_C cannot lie on this surface.

The two pencils of planes in Q_C, intersect T_A in two pencils of conics, both of type $(1,1)$. The fiber of the map $\psi_Q^{-1}(H_C)$ is therefore two disjoint $\mathbb{P}^1$'s. $\square$

Corollary 1.11. *The Hilbert scheme of* $(1,1)$*-conics* $F(T_A)$ *is a threefold.*

Proof. The general net of quadrics $H \subset H^0(\mathcal{I}_{T_A}(2))$ defines a reducible complete intersection $T_A \cup Q$, where Q is a quadric threefold. The quadric Q is singular for a codimension one, i.e. 2-dimensional family of nets H, in which case the pencil of planes in Q intersect T_A in $(1,1)$-conics. $\square$

To identify the image of ψ_{Q_A} with the Kummer surface $(D_1^A)^\vee$, we show that the net of quadrics $H_C \subset H^0(\mathcal{I}_{T_A}(2)$ defines a plane $\mathbb{P}(H_C)$ that is tangent to the discriminant D_1^A, so that the point $[H_C] \in (D_1^A)^\vee$.

First we show that when C is a $(1,1)$-conic on T_A, then the net of quadrics H_C contains a quadric Q_c that is singular in the base locus of H_C.

Lemma 1.12. *Let* $[C] \in F(T_A)$*, and let* Q_C *be the quadric* 3*-fold of rank at most* 4 *in* $\mathbb{P}^4_C$*, such that the base locus of* H_C *is* $T_A \cup Q_C$*. Let* $p_C \in Q_C$ *be the singular point. Then there is at least one quadric* $Q_c \subset \mathbb{P}(\mathbb{C} \oplus V_2 \otimes V_3)$ *that belongs to* H_C *and is singular at* p_C*.*

Proof. The net of quadrics H_C defines a complete intersection 3-fold $T_A \cup Q_C$ of degree 8, and $Q_C \subset \mathbb{P}^4_C$. There is a pencil of hyperplanes in $\mathbb{P}(\mathbb{C} \oplus V_2 \otimes V_3)$ that contain $\mathbb{P}^4_C$. Every quadric in H_C contains $p_C \in Q_C$, and has a tangent space at p_C that contains $\mathbb{P}^4_C$, so one of these quadrics, say Q_c is singular at the point p_C. $\square$

The next lemma implies that the plane $\mathbb{P}(H_C) \subset \mathbb{P}(H^0(\mathcal{I}_{T_A}(2))$ is tangent to the discriminant surface D_1^A.

Lemma 1.13. *Let* W *be a linear space of quadrics in a projective space* P *and let* $Z \subset P$ *be the base locus of the quadrics in* W*. Let* $D \subset W$ *be the discriminant. If* $[Q] \in W$ *is a singular quadric with singular point at* $p \in Z$*, then the discriminant* D *is singular at* $[Q]$*.*

Proof. The tangent space to D in W at a quadric $[Q]$ that is singular at $p \in P$ is the hyperplane in W of quadrics that vanish at p. So if p is in the base locus Z, then the hypersurface D is singular at $[Q]$. $\square$

Let $C \subset T_A$ be a general $(1,1)$-conic, let H_C be the net of quadrics vanishing on $T_A \cup P_C$, and let $T_A \cup Q_C$ be the base locus of H_C. Let $p_C \in \mathbb{P}(\mathbb{C} \oplus V_2 \otimes V_3)$ be the singular point in the quadric 3-fold Q_C of rank 4. Then, by Lemma 1.10, $p_C \notin T_A$ and, by Lemma 1.12, p_C is a singular point of a quadric Q_c in H_C. Therefore, by Lemma 1.13, $\mathbb{P}(H_C) \cap D_1^A$ is singular at $[Q_c]$, so $\mathbb{P}(H_C)$ is the tangent plane to D_1^A at $[Q_c]$.

In particular ψ_{Q_A} maps to $(D_1^A)^\vee$. Since $F(T_A)$ is a threefold and the fibers are curves, the map is onto.

$\square$

1.4. From the Hilbert scheme of conics to a Lagrangian degeneracy locus.

Finally we relate the base of the fibration on the Hilbert scheme $F(T_A)$ directly to the Lagrangian degeneracy locus defined in 1.1. Let us consider the space

$$T_A = C(\mathbb{P}(V_2) \times \mathbb{P}(V_3)) \cap Q_A \subset \mathbb{P}(\mathbb{C} \oplus (V_2 \otimes V_3)).$$

Choose a coordinate system in $\mathbb{C} \oplus (V_2 \otimes V_3)$ in such a way that $Q_A(z,x) = z^2 - Q'_A(x)$, i.e. such that $z = 0$ is the hyperplane polar to the vertex of the cone with respect to the quadric Q_A. Note that we then have $T_A \cap \{z = 0\} = S_A$. The quadric Q'_A

corresponds to a symmetric map $q'_A : V_2 \otimes V_3 \to (V_2 \otimes V_3)^\vee$. Let now $V_4 := \mathbb{C}v_0 \oplus V_3$. Thus $\wedge^2 V_4 = v_0 \wedge V_3 \oplus \wedge^2 V_3$ and:

$$V_2 \otimes \wedge^2 V_4 = (V_2 \otimes v_0 \wedge V_3) \oplus (V_2 \otimes \wedge^2 V_3).$$

We shall from now on interpret $V_2 \otimes \wedge^2 V_4$ as a subspace in $\wedge^3(V_2 \oplus V_4)$. Then, up to choices of volume forms vol_2 and vol_4 in V_2 and V_4 respectively, we have a natural skew-symmetric form $\eta_{2,4}$ on $V_2 \otimes \wedge^2 V_4$ induced by the wedge product. The decomposition $V_2 \otimes \wedge^2 V_4 = (V_2 \otimes v_0 \wedge V_3) \oplus (V_2 \otimes \wedge^2 V_3)$ is then a decomposition into a sum of two Lagrangian spaces with respect to $\eta_{2,4}$. Furthermore the graph $A \subset (V_2 \otimes V_3) \oplus (V_2 \otimes \wedge^2 V_3)$ of q'_A is also Lagrangian. Hence to A we can associate a Kummer surface:

$$\hat{D}_1^A := \{[U] \in \mathbb{P}(V_4^\vee)|\ \dim(A \cap (V_2 \otimes \wedge^2 U)) \geq 1\},$$

singular in

$$\hat{D}_2^A := \{[U] \in \mathbb{P}(V_4^\vee)|\ \dim(A \cap (V_2 \otimes \wedge^2 U)) \geq 2\}.$$

On the other hand the system of quadrics containing T_A is naturally isomorphic to $V_4 = \mathbb{C} \oplus V_3$ via

$$V_4 = \mathbb{C} \oplus V_3 \ni (z, v) \mapsto (z \cdot Q_A + Q_v) \in H^0(\mathcal{I}_{T_A}(2)),$$

with Q_v defined by $Q_v(z, x) = x \wedge x \wedge v$ where $(z, x) \in \mathbb{C} \oplus (V_2 \oplus V_3)$.

We, consider the map $\psi_{Q_A} : F(T_A) \to \mathbb{P}(H^0(\mathcal{I}_{T_A}(2))^\vee) = \mathbb{P}(V_4^\vee)$ associating to a conic C the system H_C of quadrics vanishing on T_A and the plane P_C spanned by C and prove:

Proposition 1.14. *The map ψ_{Q_A} factors as $\rho_{Q_A} \circ \phi_{Q_A}$, where $\phi_{Q_A} : F(T_A) \to X_A$ is a $\mathbb{P}^1$-fibration and $\rho_{Q_A} : X_A \to \hat{D}_1^A$ is $2:1$ onto its image. Furthermore X_A is an abelian surface, and ρ_{Q_A} is a double cover of its Kummer surface.*

In Proposition 1.9 we showed that $(D_1^A)^\vee$ is the image of ψ_{Q_A}. We shall now see that the image of ψ_{Q_A} is in fact also described as $\hat{D}_A^1$.

Lemma 1.15. *For any $(1,1)$-conic $C \subset T_A$ i.e. $[C] \in F(T_A)$ we have $\psi_{Q_A}([C]) \in \hat{D}_1^A$, furthermore if $C \subset S_A = T_A \cap \{z = 0\}$ then $\psi_{Q_A}([C]) \in \hat{D}_2^A$.*

Proof. Fix the notation above. We start by describing the map ψ_{Q_A} in coordinates. Consider three general points $(z_1, \beta_1), (z_2, \beta_2), (z_3, \beta_3) \in C \subset T_A \subset C(\mathbb{P}(V_2) \times \mathbb{P}(V_3))$. By assumption, $\beta_i \in V_2 \otimes V_3$ are decomposable tensors that can be written as $\beta_1 = u_1 \otimes (v_0 \wedge v_1), \beta_2 = u_2 \otimes (v_0 \wedge v_2), \beta_3 = (u_1 + u_2) \otimes (v_0 \wedge (v_1 + v_2))$ (recall that we interpret elements of V_3 as two-forms $v_0 \wedge *$) for appropriate choice of basis (u_1, u_2) of V_2 and (v_1, v_2, v_3) of V_3 satisfying $vol_4(v_0 \wedge v_1 \wedge v_2 \wedge v_3) = 1$, $vol_2(u_1 \wedge u_2) = 1$. We keep this basis until the end of the proof. Clearly the component of $\psi_{Q_A}(C) \in \mathbb{P}(V_4^\vee) = \mathbb{P}(\mathbb{C} \oplus \wedge^2 V_3)$ corresponding to the part $\wedge^2 V_3$ is then $v_1 \wedge v_2$. We need to determine the remaining part of $\psi_{Q_A}(C)$. Let $\alpha_i = q'_A(\beta_i) \in V_2 \otimes \wedge^2 V_3$ which is equivalent to $\alpha_i + \beta_i \in A \subset (V_2 \otimes V_3) \oplus (V_2 \otimes \wedge^2 V_3)$ and implies also $Q'_A(\beta_i) = \alpha_i \wedge \beta_i$. Since A is Lagrangian we have $(\alpha_i + \beta_i) \wedge (\alpha_j + \beta_j) = 0$ for all i, j which implies $\alpha_i \wedge \beta_j = \alpha_j \wedge \beta_i =: c_{i,j}$ for $i \neq j$. Now

$$\begin{aligned} Q_A(\lambda_1(z_i, \beta_i) + \lambda_2(z_j, \beta_j)) &= (z_i + \lambda z_j)^2 - Q'_A(\lambda_1 \beta_i + \lambda_2 \beta_j) = \\ &= (\lambda_1 z_i + \lambda_2 z_j)^2 - (\lambda_1 \alpha_i + \lambda_2 \alpha_j) \wedge (\lambda_1 \beta_i + \lambda_2 \beta_j) \\ &= \lambda_1^2 Q_A((z_i, \beta_i)) + \lambda_2^2 Q_A((z_j, \beta_j)) + 2\lambda_1 \lambda_2 (z_i z_j - c_{i,j}). \end{aligned}$$

But $Q_A((z_i, \beta_i)) = 0$, since $(z_i, \beta_i) \in C \subset T_A$. So we deduce that

$$Q_A(\lambda_1(z_i, \beta_i) + \lambda_2(z_j, \beta_j)) = 2\lambda_1 \lambda_2 (z_i z_j - c_{i,j})$$

Now

$$(t_0 Q_A + t_1 Q_{(v_1\wedge v_2)^*})(\lambda_1(z_i,\beta_i)+\lambda_2(z_j,\beta_j)) = 2t_0\lambda_1\lambda_2(z_iz_j - c_{i,j}) + 2t_1\lambda_1\lambda_2$$

it follows that the $\psi_{Q_A}(C) = [(z_iz_j - c_{i,j}, v_1\wedge v_2)] \in \mathbb{P}(\mathbb{C}\oplus\wedge^2 V_3))$ which means, in particular, that:

$$z_1z_2 - c_{1,2} = z_1z_3 - c_{1,3} = z_2z_3 - c_{2,3} =: c_C.$$

If now $U_C =< [(c_C, v_1\wedge v_2)] >^{\perp} \subset \mathbb{C}\oplus V_3$ we have

$$\wedge^2 U_C = \{\gamma \in (v_0\wedge V_3)\oplus\wedge^2 V_3 | \gamma\wedge v_1\wedge v_2 = \gamma\wedge v_1\wedge(c_C v_3 + v_0) = \gamma\wedge v_2\wedge(c_C v_3+v_0) = 0\}$$

We deduce

$$V_2\otimes\wedge^2 U_C = \{\omega\in V_2\otimes\wedge^2 V_4 | \omega\wedge v_1\wedge v_2 = \omega\wedge v_1\wedge(c_C v_3+v_0) = \omega\wedge v_2\wedge(c_C v_3+v_0) = 0\}$$

We shall prove that $A\cap(V_2\otimes\wedge^2 U_C)\neq 0$. We know $\sum_{i=1}^3 \lambda_i(\alpha_i+\beta_i)\in A$ for $\lambda_i\in\mathbb{C}$. It is therefore enough to prove that the following system of equations has a nonzero solution $(\lambda_1,\lambda_2,\lambda_3)$:

$$E_1(\lambda_1,\lambda_2,\lambda_3) := (\sum_{i=1}^3 \lambda_i(\alpha_i+\beta_i))\wedge v_1\wedge v_2 = 0$$

$$E_2(\lambda_1,\lambda_2,\lambda_3) := (\sum_{i=1}^3 \lambda_i(\alpha_i+\beta_i))\wedge v_1\wedge(v_0+c_C v_3) = 0$$

$$E_3(\lambda_1,\lambda_2,\lambda_3) := (\sum_{i=1}^3 \lambda_i(\alpha_i+\beta_i))\wedge v_2\wedge(v_0+c_C v_3) = 0$$

Observe now that $E_1(\lambda_1,\lambda_2,\lambda_3)\equiv 0$ since both $\alpha_i\wedge v_1\wedge v_2 = 0$ and $\beta_i\wedge v_1\wedge v_2 = 0$. Furthermore, we have:

$$E_2(\lambda_1,\lambda_2,\lambda_3)\wedge v_1 = E_2(\lambda_1,\lambda_2,\lambda_3)\wedge v_2 = E_2(\lambda_1,\lambda_2,\lambda_3)\wedge v_3 = E_2(\lambda_1,\lambda_2,\lambda_3)\wedge v_0 = 0,$$

as well as

$$E_3(\lambda_1,\lambda_2,\lambda_3)\wedge v_1 = E_3(\lambda_1,\lambda_2,\lambda_3)\wedge v_2 = E_3(\lambda_1,\lambda_2,\lambda_3)\wedge v_3 = E_3(\lambda_1,\lambda_2,\lambda_3)\wedge v_0 = 0.$$

Finally, the three equations

$$E_2(\lambda_1,\lambda_2,\lambda_3)\wedge u_1 = z_1^2\lambda_1 + z_1z_2\lambda_2 + z_1z_3\lambda_3 = 0,$$

$$E_3(\lambda_1,\lambda_2,\lambda_3)\wedge u_2 = z_1z_2\lambda_1 + z_2^2\lambda_2 + z_2z_3\lambda_3 = 0,$$

$$(E_2(\lambda_1,\lambda_2,\lambda_3)+E_3(\lambda_1,\lambda_2,\lambda_3))\wedge(u_1+u_2) = z_1z_3\lambda_1 + z_2z_3\lambda_2 + z_3^2\lambda_3 = 0,$$

are proportional, so the above equations reduce to the following linear equations in the λ_i: $E_2(\lambda_1,\lambda_2,\lambda_3)\wedge u_2 = 0$ and one of the above proportional equations. It follows that the system is a rank 2 system of equations, it hence admits a nontrivial solution implying $\dim(A\cap\wedge^2 U_C)\geq 1$, which proves:

$$\psi_{Q_A}([C])\in\hat{D}_1^A.$$

If now C is contained in the branch locus $S_A = T_A\cap z = 0$, then the system of equations is of rank 1 since the three above proportional equations vanish, so $\psi_{Q_A}([C])\in\hat{D}_2^A$. $\square$

We can now pass to the proof of Proposition 1.14.

Proof of Proposition 1.14. Note that there is a natural involution on $F(T_A)$ induced by the involution $(z,\beta) \mapsto (-z,\beta)$. A conic is fixed by this involution if and only if it is contained in $\{z = 0\}$. Note also that from the explicit formula it follows that the involution acts on the fibers of ψ_{Q_A}. From Lemma 1.10 we know that such a fiber is either a disjoint union of two $\mathbb{P}^1$s or a single $\mathbb{P}^1$. On the other hand, we know that a nontrivial involution on $\mathbb{P}^1$ has two fixed points. By Proposition 1.5, there are exactly 16 pairs of $(1,1)$-conics on $S_A = T_A \cap \{z = 0\}$. Each pair intersect in two points, so we deduce that the involution exchanges the $\mathbb{P}^1$s in the general fiber and acts on the $\mathbb{P}^1$s in 16 fibers whose images are the 16 singular points of the Kummer surface $\hat{D}_1^A$. Hence the Stein factorization of ψ_{Q_A} gives the desired decomposition. Moreover, $X_A \to \hat{D}_1^A$ is a double cover branched in the 16 singular points of $\hat{D}_1^A$ and is therefore an Abelian surface. □

Note that, as observed above, combining Proposition 1.14 with Proposition 1.9 we obtain.

Corollary 1.16. *The Kummer surface $\hat{D}_1^A$ is projective dual to the Kummer surface D_1^A.*

Remark 1.17. Note that Corollary 1.16 provides further analogies between our description of Kummer surfaces and that of EPW sextics. Indeed a choice of Lagrangian A provides two constructions leading to birational and projectively dual varieties $\hat{D}_1^A$ and D_1^A which are both Kummer surfaces. In the context of EPW sextics a choice of Lagrangian space also gives rise to two birational and projectively dual EPW sextics.

2. First construction - singular EPW cubes

In this section we present the first construction of the family $\mathcal{U}$. Let us first discuss a natural context where the elements from $\mathcal{U}$ appear. We shall investigate IHS fourfolds deformation equivalent to the Hilbert scheme of two points on a $K3$ surface (of $K3^{[2]}$-type) which admit an antisymplectic involution ι (i.e. that changes the sign of the symplectic form).

Involutions of $K3$ surfaces were first studied from a lattice-theoretic point of view by Nikulin [Nik80]. For higher dimensions a classification of invariant lattices of non-symplectic automorphisms of prime order was given in [BCS16] and [BCMS16]. The problem of finding a geometric realization of non-symplectic automorphisms on IHS fourfolds was addressed in [OW13] and [MW15].

It follows from [Bea11, 3.4, Theorem 2] and [O'G06] that there exists exactly one irreducible 20-dimensional family of IHS fourfolds of $K3^{[2]}$ type which admit anti-symplectic involutions. By [O'G06], the invariant polarisation in this family has Beauville degree $q = 2$ and the quotient of such an involution for a generic element is a special sextic hypersurface in $\mathbb{P}^5$ called an EPW sextic. In [OW13] the authors classified all the possible invariant lattices $H^2(X,\mathbb{Z})^\iota$ of 19 parameter families of IHS fourfolds of $K3^{[2]}$ type. They found that any such lattice is hyperbolic and 2-elementary. In [OW13, thm. 2.3] they distinguished five families of IHS fourfolds of $K3^{[2]}$-type with anti-symplectic involutions. In fact there are only four isomorphism classes of invariant sublattices $H^2(X,\mathbb{Z})^\iota \subset H^2(X,\mathbb{Z})$. They are U, $U(2)$, and $\langle 2\rangle \oplus \langle -2\rangle$ such that the generator g with $q(g) = -2$ has divisibility either 2 or 1 (we call them the cases 1,2,3,4 respectively). Moreover, they found that there is a unique 19-dimensional irreducible family that admits the invariant lattice from each of the cases $1,2,4$ respectively and two families in the case 3. The families in the cases $1,3,4$ admit polarisations of Beauville degree $q = 2$, it is not hard to see [MW15, Rem. 5.7] that they can be described as families of resolutions of special singular double EPW sextics.

Our aim is to study the geometry of the missing family of IHS fourfolds with involutions from the case 2 above i.e. with invariant lattice $U(2)$. Note that each element of this family admits a natural polarization of Beauville degree 4 and as proved in [Add16, Proposition. 4] the generic element of this family is not isomorphic to a moduli space of sheaves on a $K3$-surface. Note finally that the family with invariant lattice U also admits a polarisation of degree $q = 4$ that is invariant with respect to the non-symplectic involution.

From [OW13] there is only one possible invariant lattice of rank two

$$H^2(X,\mathbb{Z})^\iota := \{x \in H^2(X,\mathbb{Z}) | \iota^*(x) = x\}.$$

that does not admit a polarization of Beauville degree $q = 2$, namely:

$$U(2) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Let X be an IHS fourfold with an involution and an invariant lattice $U(2)$. Then, by [Bea11], the invariant lattice has signature $(1,1)$ for some n. In particular X is projective. Let h_1 and h_2 be the generators of the lattice with $q(h_1) = q(h_2) = 0$. We are interested in the invariant polarization $h = h_1 + h_2$ of Beauville degree $q(h_1 + h_2) = 4$. If (X, H) is a polarized IHS fourfold of type $(\mathrm{K3})^{[2]}$ with $q(H) = 4$, we infer $H^4 = 3 \cdot (4)^2 = 48$, and from the Riemann–Roch theorem we find $h^0(\mathcal{O}_X(H)) = 10$ [Nie03, Theorem 5.2]. Thus, our polarization h gives a map $f \colon X \to \mathbb{P}^9$ that factors through the involution ι. Hence, we expect that f is $2:1$ to a degree 24 fourfold. Our aim is to describe the image of this map. We shall first show that this image can be realized as a subset of a degenerated EPW cube and next prove that in fact X is an element of $\mathcal{U}$.

2.1. **Degenerate EPW cubes.** In this section we consider double EPW cubes constructed from a general Lagrangian subspace $A \in \Sigma$, in particular with $\mathbb{P}(A) \cap \mathrm{G}(3,V) = [U_1]$. Let $\mathcal{T}$ be the Lagrangian subbundle $\mathcal{T} \subset \mathcal{O}_{\mathrm{G}(3,V)} \otimes \wedge^3 V$ whose fiber over $[U] \in \mathrm{G}(3,V)$ is $\mathcal{T}_{[U]} = T_U = \wedge^2 U \wedge V$. The degeneracy locus $D_2^A = \{[U] \in \mathrm{G}(3,V) | \dim(A \cap T_U) \geq 2\}$ is called an EPW cube. Our 19-dimensional family of IHS fourfolds will be constructed from the subvariety $D_2^A \cap \mathbb{P}(T_{U_1})$, when $\mathbb{P}(A) \cap \mathrm{G}(3,V) = [U_1]$.

The following description of a projective tangent space $\mathbb{P}(T_U)$ to $\mathrm{G}(3,V)$ at $[U]$ is classical [Don77].

Lemma 2.1. *Let $[U] \in \mathrm{G}(3,V)$ and $\mathbb{P}(T_U) = \mathbb{P}(\wedge^2 U \wedge V) \subset \mathbb{P}(\wedge^3 V)$ be the embedded projective tangent space to $\mathrm{G}(3,V) \subset \mathbb{P}(\wedge^3 V)$ in $[U]$. Then the intersection*

$$C_U := \mathbb{P}(T_U) \cap \mathrm{G}(3,V)$$

is a cone in the 9-dimensional linear space $\mathbb{P}(T_U)$ with vertex $[U]$ over the Segre embedding of $\mathbb{P}(\wedge^2 U) \times \mathbb{P}(V/U)$.

Proof. See [Don77, Lemma 3.5]. The tangent space $\mathbb{P}(T_U)$ is spanned by the spaces U' that intersect U in codimension 1. These spaces are naturally parameterized by pairs (M_2, N_1), where $M_2 \subset U$ is 2-dimensional, $N_1 \subset V/U$ is 1-dimensional and $U' \cap U = M_2$, $U'/M_2 = N_1$. □

Note that for each $[U] \in C_{U_1}$ we have $[U_1] \in \mathbb{P}(T_U) \cap \mathbb{P}(A)$. It follows that $C_{U_1} \subset D_1^A$. Observe, that since A is Lagrangian and $\wedge^3 U_1 \subset A$ then $A \subset (\wedge^3 U_1)^\perp$. Similarly $T_U \subset (\wedge^3 U_1)^\perp$ for each $[U] \in C_{U_1}$. There is moreover an exact sequence:

$$(2.1)\qquad 0\to \mathcal{O}_{C_{U_1}}\to \mathcal{T}|_{C_{U_1}}\to \bar{\mathcal{T}}\to 0,$$

with $\bar{\mathcal{T}}$ a subbundle of the trivial bundle $\mathcal{O}_{C_{U_1}}\otimes(\wedge^3V)/(\wedge^3U_1)$ with fibers $\bar{T}_U = T_U/(\wedge^3U_1)$ over $[U]\in C_{U_1}$. Consider the space

$$(\wedge^3U_1)^\perp/(\wedge^3U_1)\subset(\wedge^3V)/(\wedge^3U_1)$$

equipped with the symplectic form $\bar{\eta}$ induced by η. Clearly both $\bar{A}=A/(\wedge^3U_1)$ and the fibers $\bar{T}_U=T_U/(\wedge^3U_1)$ of $\bar{\mathcal{T}}$ are contained in $(\wedge^3U_1)^\perp/(\wedge^3U_1)$ and are Lagrangian with respect to the symplectic form $\bar{\eta}$. The natural map

$$\iota: C_{U_1}\to \mathrm{LG}_{\bar{\eta}}(9,(\wedge^3U_1)^\perp/(\wedge^3U_1));\quad [U]\mapsto[\bar{T}_U]$$

is an embedding, since it is the restriction of the embedding

$$\mathrm{G}(3,V)\to \mathrm{LG}_\eta(10,\wedge^3V),\ \ [U]\mapsto[T_U]$$

to C_{U_1}. Denote the corresponding Lagrangian degeneracy loci by

$$D_k^{\bar{A}}=\{[U]\in C_{U_1}|\dim(\bar{T}_U\cap\bar{A})\geq k\}.$$

These degeneracy loci are simply the restrictions to $\iota(C_{U_1})$ of the universal degeneracy loci $\mathbb{D}_k^{\bar{A}}$ on $\mathrm{LG}_{\bar{\eta}}(9,(\wedge^3U_1)^\perp/(\wedge^3U_1))$ [PR97].

Lemma 2.2. *Let $[A]\in(\Sigma-(\Sigma_+\cup\Sigma[1]))\subset \mathrm{LG}_\eta(10,\wedge^3W)$ such that $[U_1]\in\mathbb{P}(A)\cap \mathrm{G}(3,V)$. Then $C_{U_1}\subset D_1^A$, and $D_i^{\bar{A}}=C_{U_1}\cap D_{i+1}^A$, when $i=1,2$. Furthermore, $D_1^{\bar{A}}$ is an intersection of $C_{U_1}\subset\mathbb{P}^9$ with a quartic hypersurface Q_A, and $D_2^{\bar{A}}$ is a surface of degree 72 contained in the singular locus of $D_1^{\bar{A}}$.*

Proof. First, we simply note that $\bar{A}\cap\bar{T}_U=(A\cap T_U)/(\wedge^3U_1)$, so we obtain $C_{U_1}\cap D_{i+1}^A = D_i^{\bar{A}}$. To compute invariants, recall that the $\mathbb{P}^9$-bundle $\mathbb{P}(\mathcal{T})$ is the projective tangent bundle on $\mathrm{G}(3,V)$, so $\mathcal{T}^\vee$ fits into an exact sequence

$$0\to\Omega_{\mathrm{G}(3,V)}(1)\to\mathcal{T}^\vee\to\mathcal{O}_{\mathrm{G}(3,V)}(1)\to 0.$$

Therefore $\mathcal{T}^\vee$ has total Chern class

$$\begin{aligned}c(\mathcal{T}^\vee)&=c(\mathcal{O}_{\mathrm{G}(3,V)}(1))/(\Omega_{\mathrm{G}(3,V)}(1))\\&=1+4\sigma_1+8\sigma_1^2+(8\sigma_1^2+6\sigma_1\sigma_2-6\sigma_3)+(24\sigma_1^2\sigma_2-24\sigma_1\sigma_3)\\&\quad+(30\sigma_1\sigma_2^2-30\sigma_2\sigma_3)+(10\sigma_2^3+24\sigma_1\sigma_2\sigma_3-24\sigma_3^2)+18\sigma_2^2\sigma_3+12\sigma_2\sigma_3^2+4\sigma_3^3\end{aligned}$$

where $\sigma_i=c_i(Q_G)$ and Q_G is the universal quotient bundle on $\mathrm{G}(3,V)$. Furthermore, by the exact sequence 2.1, $c_i(\overline{\mathcal{T}}^\vee)\cap C_{U_1}=c_i(\mathcal{T}^\vee)\cap C_{U_1}$ for all i. Applying the Pragacz Ratajski formulas [PR97, Theorem 2.1] for the classes of the Lagrangian degeneracy loci $D_i^{\bar{A}}$ we get

$$[D_1^{\bar{A}}]=c_1(\overline{\mathcal{T}}^\vee)\cap[C_{U_1}]=c_1(\mathcal{T}^\vee)\cap[C_{U_1}]=4\sigma_1\cap[C_{U_1}],$$

so $D_1^{\bar{A}}$ is an intersection of $C_{U_1}\subset\mathbb{P}^9$ with a quartic hypersurface Q_A. Furthermore

$$\begin{aligned}[D_2^{\bar{A}}]&=(c_2c_1-2c_3)(\overline{\mathcal{T}}^\vee)\cap[C_{U_1}]=(c_2c_1-2c_3)(\mathcal{T}^\vee)\cap[C_{U_1}]\\&=(c_2c_1-2c_3)(\mathcal{T}^\vee)\cap[C_{U_1}]=(16\sigma_1^3-12\sigma_1\sigma_2+12\sigma_3)\cap[C_{U_1}].\end{aligned}$$

The class of $[C_{U_1}]$ in $\mathrm{G}(3,V)$ is $(\sigma_2^2-\sigma_1\sigma_3)\cap[\mathrm{G}(3,V)]$, so

$$\deg D_2^{\bar{A}}=\int_{[\mathrm{G}(3,V)]}\sigma_1^2\cdot(\sigma_2^2-\sigma_1\sigma_3)\cdot(16\sigma_1^3-12\sigma_1\sigma_2+12\sigma_3)=\int_{[\mathrm{G}(3,V)]}36\sigma_1^2\sigma_2^3\sigma_3=72.$$

The last statement is a standard result on degeneracy loci. □

To proceed with the construction we need to know precisely the singular locus of the Lagrangian degeneracy locus $D_1^{\bar{A}}$.

Lemma 2.3. *Let* $[A] \in (\Sigma - (\Sigma_+ \cup \Sigma[1])$ *and let* $[U_1]$ *be the unique point in* $\Theta(A)$*. Then the Lagrangian locus* $C_{U_1} \cap D_2^A = D_1^{\bar{A}}$ *is smooth outside* $S_{\bar{A}} = D_2^{\bar{A}} = C_{U_1} \cap D_3^A$*. Moreover, the tangent cone to* $D_1^{\bar{A}}$ *in points of* $S_{\bar{A}}$ *is a cone over a smooth conic curve.*

Proof. The proof will be analogous to that of [IKKR16, Lemma 2.9]. Let $[U_1]$ be the unique point in $\Theta(A)$. Observe that, by assumption, $[U_1] \notin D_1^{\bar{A}}$. Fix $[U_0] \in C_{U_1} \cap D_1^{\bar{A}}$ and choose a 3-space U_∞ such that $U_\infty \cap U_1 = 0$ and $U_\infty \cap U_0 = 0$. Let

$$\mathfrak{U} = \{[U] \in \mathrm{G}(3,V) | U \cap U_\infty = 0\}.$$

It is an open neighbourhood of $[U_0]$ in $\mathrm{G}(3,V)$.

For $[U] \in \mathfrak{U}$ the Lagrangian space T_U defines a symmetric linear map $T_{U_0} \to T_{U_0}^\vee$ that we denote by q_U and a corresponding quadratic form on T_{U_0} that we denote by Q_U. We shall describe Q_U in local coordinates. Let $(u_1, u_2, u_3), (u_4, u_5, u_6)$ be a basis for U_0 resp. U_∞.

Observe that for any $[U] \in \mathrm{G}(3,V)$,

$$T_U \cap T_{U_\infty} = 0 \leftrightarrow U \cap U_\infty = 0$$

and that any such subspace U is the graph of a linear map $\beta_U : U_0 \to U_\infty$. In particular, there is an isomorphism:

$$\rho : \mathfrak{U} \to Hom(U_0, U_\infty); \quad [U] \mapsto \beta_U$$

whose inverse is the map

$$\alpha \mapsto [U_\alpha] := [(u_1 + \alpha(u_1)) \wedge (u_2 + \alpha(u_2)) \wedge (u_3 + \alpha(u_3))].$$

In the given basis for U_0 and U_∞ we let $B_U = (b_{i,j})_{i,j \in \{1\dots 3\}}$ be the matrix of the linear map β_U. In the dual basis, we let (m_0, M), with $M = (m_{i,j})_{i,j \in \{1\dots 3\}}$, be the coordinates in

$$T_{U_0}^\vee = (\wedge^3 U_0 \oplus \wedge^2 U_0 \otimes U_\infty)^\vee = (\wedge^3 U_0 \oplus Hom(U_0, U_\infty))^\vee$$

In these coordinates, the map

$$\iota : \mathfrak{U} \ni [U] \mapsto Q_U \in Sym^2 T_{U_0}^\vee$$

is defined by

$$Q_U(m_0, M) = \sum_{i,j \in \{1\dots 3\}} b_{i,j} M^{i,j} + m_0 \sum_{i,j \in \{1\dots 3\}} B_U^{i,j} m_{i,j} + m_0^2 \det B_U, \tag{2.2}$$

where $M^{i,j}$, $B_U^{i,j}$ are the entries of the matrices adjoint to M and B_U. To see this, write the map $\wedge^3 U_0 \oplus \wedge^2 U_0 \otimes U_\infty \to \wedge^3 U_\infty \oplus \wedge^2 U_\infty \otimes U_0$ whose graph is $\wedge^3 U \oplus \wedge^2 U \otimes U_\infty$ in coordinates, where U is the graph of the map $U_0 \to U_\infty$ given by the matrix B_U.

Let now q_A be the symmetric map $T_{U_0} \to T_{U_\infty} = T_{U_0}^\vee$ whose graph is A and Q_A the corresponding quadratic form. In this way

$$D_l^A \cap \mathfrak{U} = \{[U] \in \mathfrak{U} | \dim T_U \cap A) \geq l\} = \{[U] \in \mathfrak{U} | \operatorname{rk}(q_U - q_A) \leq 10 - l\},$$

hence D_l^A is locally defined by the vanishing of the $(11-l) \times (11-l)$ minors of the 10×10 matrix with entries being polynomials in $b_{i,j}$.

We now consider the restriction of the map $\iota : [U] \mapsto Q_U$ to $C_{U_1} \cap \mathfrak{U}$.

The map $f : U_0 \to U_\infty$, whose graph is U_1, has rank 1, since $[U_0] \in C_{U_1} \cap D_1^{\bar{A}}$ and $C_{U_1} \cap \mathfrak{U} = \{g \in Hom(U_0, U_\infty) | \mathrm{rk}(g - f) \le 1\}$. After possible changes of basis for U_0 and U_∞, we may assume that $f \in Hom(U_0, U_\infty)$ is given by a matrix with one nonzero entry in the upper left corner. The restriction of the map ι is then given by

$$Q_U(m_0, M) = \sum_{i,j \in \{1\ldots3\}} b_{i,j} M^{i,j} + m_0 \sum_{i,j \in \{2,3\}} B_U^{i,j} m_{i,j}. \tag{2.3}$$

We now observe that all quadrics Q_U with $[U] \in C_{U_1}$ are singular in the point $[U_1]$ with coordinates $m_0 = m_{1,1} \neq 0$ and $m_{i,j} = 0; (i,j) \neq (1,1)$. Passing to the quotient $T_{U_0}/\wedge^3 U_1$ and denoting the induced quadrics on the quotient space by $\bar{Q}_U$ and $\bar{Q}_A = Q_{\bar{A}}$ and the corresponding symmetric linear maps by $\bar{q}_U$ and $q_{\bar{A}}$ respectively. We have

$$\bar{Q}_U(M) = \sum b_{i,j} M^{i,j}. \tag{2.4}$$

We can now follow the proof of [IKKR16, Lemma 2.9] for the first degeneracy locus $D_1^{\bar{A}}$ around $[U_0]$. In $\mathfrak{U}$ the locus

$$D_1^{\bar{A}} \cap \mathfrak{U} = \{[U] \in \mathfrak{U} \cap C_{U_1} | \dim(T_U/\wedge^3 U_1) \cap \bar{A}) \ge l\} = \{[U] \in \mathfrak{U} \cap C_{U_1} | \mathrm{rk}(\bar{q}_U - q_{\bar{A}}) \le 9 - l\},$$

i.e. $D_1^{\bar{A}}$ is defined by the determinant of a 9×9 symmetric matrix with entries being regular functions on $\mathfrak{U} \cap C_{U_1}$. We may assume that $q_{\bar{A}}$ is given by a diagonal matrix with 0's and 1's on the diagonal, and let $K := \ker q_{\bar{A}} = \bar{A} \cap T_{U_0}/\wedge^3 U_1$. By 2.3, we know that the differential of the map $\iota|_{C_{U_1}}$ in $[U_0]$ maps onto the linear system of quadrics generating the ideal of the image $\hat{C}$ of the projection of the cone $C_{U_0} \subset \mathbb{P}(T_{U_0})$ from the point $[U_1]$. In other words, the linear forms of the matrix of polynomials

$$\bar{\iota} : \mathfrak{U} \cap C_{U_1} \ni [U] \mapsto \bar{Q}_U \in Sym^2(T_{U_0}/\wedge^3 U_1)^\vee$$

for a chosen coordinate chart of C_{U_1} in $[U_0]$ define the linear system of quadrics containing $\hat{C}$. We then observe that if $\mathbb{P}(A) \cap \mathrm{G}(3,6) = [U_1]$ then $\mathbb{P}(A/\wedge^3 U_1) \cap \hat{C} = \varnothing$ hence $K \cap \hat{C} = 0$ and remark that $\hat{C}$ satisfies the assertion of [IKKR16, Lemma 2.8]. More precisely, we have:

Lemma 2.4. *If $P \subset \mathbb{P}(T_{U_0}/\wedge^3 U_1) \smallsetminus \hat{C}$ is a linear subspace of dimension at most 1, then the restriction map $\mathbf{r}_P : H^0(\mathbb{P}(T_U), \mathcal{I}_{\hat{C}}(2)) \to H^0(P, \mathcal{O}_P(2))$ is surjective.*

Proof. Note that $\hat{C}$ is defined in $\mathbb{P}^8 = \mathbb{P}(T_{U_0}/\wedge^3 U_1)$ by 5 quadrics obtained as 2×2 minors of a 3×3 matrix of linear forms that do not involve the upper left entry. Since $\hat{C}$ is defined by quadrics the lemma is proven for $\dim P = 0$. If $\dim P = 1$ it is enough to observe that $\hat{C}$ can be seen as a cone over a section of the Grassmannian $\mathrm{G}(2,5)$ by two hyperplanes. Let $G = \mathrm{G}(2,5) \subset \mathbb{P}^9$ and consider the rational map $\delta : \mathbb{P}^9 \to \mathbb{P}^4$ defined by the quadrics that generate the ideal of G. Observe that the closures of the fibers of δ are $\mathbb{P}^5$ spanned by 4-dimensional quadrics in G. It follows that the image $\delta(l)$ of any line l , with $l \cap G = \varnothing$ is a smooth conic. If now $CG \subset \mathbb{P}^{10}$ is a cone over G, then the map defined by quadrics containing CG factorizes as the composition $\delta \circ \pi_p$ of the projection from the vertex p of the cone CG and δ. It follows that $\delta(\pi_p(l))$ is a conic for any line $l \subset \mathbb{P}^{10}$ such that $l \cap CG = \varnothing$. This means that the restriction map from the system of quadrics containing CG to quadrics on the line l is surjective if $l \cap CG = \varnothing$. Now, since $\hat{C}$ appears as a section of CG we conclude that the system of quadrics containing $\hat{C}$ contains the system of restrictions of quadrics containing CG. The latter restricts surjectively onto quadrics on the line P since $P \cap CG = \varnothing$, which proves the lemma. □

Let us now denote the components of $\Phi := \det(\bar{q}_U - q_{\bar{A}})$ of degree i by Φ_i. If now $[U_0] \in D_1^{\bar{A}} \smallsetminus D_2^{\bar{A}}$ then $\dim K = 1$, then $\Phi_0 = 0$ and Φ_1 is the linear entry of $(\bar{q}_U - q_{\bar{A}})$ corresponding to the restrictions to $Sym^2 K$. It follows, by Lemma 2.4, that $\Phi_1 \neq 0$ hence $D_1^{\bar{A}}$ is smooth in $[U_0]$. If now $[U_0] \in D_2^{\bar{A}}$, then $\dim K = 2$ so $\Phi_0 = \Phi_1 = 0$ and then Φ_2 is the determinant of the restriction of $\bar{q}_U - q_{\bar{A}}$ to $Sym^2 K$. Again, by Lemma 2.4, we get that Φ_2 is a rank 3 quadric which concludes the proof.

□

Lemma 2.5. *The variety $D_1^{\bar{A}}$ is integral.*

Proof. By Lemma 2.3, $D_1^{\bar{A}}$ is a divisor in C_{U_1} that is smooth outside the codimension two locus $D_2^{\bar{A}}$; in particular it is reduced. By Lemma 2.2, the locus $D_1^{\bar{A}}$ is the intersection of C_{U_1} with a quartic hypersurface, so if it was reducible, it would have singularities in codimension one which would contradict Lemma 2.3. Therefore $D_1^{\bar{A}}$ is integral. □

From Lemmas 2.3 and 2.5 we conclude that $D_1^{\bar{A}}$ is an irreducible 4-fold with quadratic singularities along the surface $D_2^{\bar{A}}$. We proceed to construct a natural resolution of singularities. For this define the incidences

$$\tilde{D}_1^{\bar{A}} = \{([U],[\omega]) \in C_{U_1} \times \mathrm{G}(1,\bar{A}) \mid \quad \overline{T}_U \supset \langle \omega \rangle\},$$

and

$$\tilde{\mathbb{D}}_1^{\bar{A}} = \{([L],[\omega]) \in \mathrm{LG}_{\bar{\eta}}(9,(\wedge^3 U_1)^{\perp}/(\wedge^3 U_1)) \times \mathrm{G}(1,\bar{A}) \mid \quad L \supset \langle \omega \rangle\},$$

which fit in the following diagram:

$$\begin{array}{ccc}
C_{U_1} & \xrightarrow{\iota} & \mathrm{LG}_{\bar{\eta}}(9,(\wedge^3 U_1)^{\perp}/(\wedge^3 U_1)) \\
\cup| & & \cup| \\
D_1^{\bar{A}} & \xrightarrow[\iota|_{D_1^{\bar{A}}}]{} & \mathbb{D}_1^{\bar{A}} \\
\uparrow{\scriptstyle \bar{\alpha}} & & \uparrow{\scriptstyle \phi} \\
\tilde{D}_1^{\bar{A}} & \xrightarrow[\tilde{\iota}]{} & \tilde{\mathbb{D}}_1^{\bar{A}}
\end{array}$$

where $\bar{\alpha}$ and ϕ are the projections on the first factor.

Lemma 2.6. *The variety $\tilde{D}_1^{\bar{A}}$ as well as the exceptional divisor E of $\bar{\alpha}$ are both smooth. In particular $\bar{\alpha}$ is a resolution of singularities of $D_1^{\bar{A}}$.*

Proof. Once we have proved Lemma 2.3, and observed that $D_3^{\bar{A}} = \varnothing$ the proof is completely analogous to [IKKR16, Lemma 3.3]. □

We can now perform the construction of a smooth double cover of $D_1^{\bar{A}}$ branched in $D_2^{\bar{A}}$. Note that the exceptional divisor in $\tilde{D}_1^{\bar{A}}$ is an even divisor. To see this, denote by H a Plücker hyperplane section on $\mathrm{LG}_{\bar{\eta}}(9,(\wedge^3 U_1)^{\perp}/(\wedge^3 U_1))$, denote by h a Plücker hyperplane section on $\mathrm{G}(3,V)$ restricted to C_{U_1}, and denote by R a Plücker hyperplane section on $\mathrm{G}(1,\bar{A})$. Then, by [IKKR16, Lemma 2.4] and the fact that $\iota^*(H) = c_1(\mathcal{T}^\vee) \cap C_{U_1}) = 4h$, the divisor E can be expressed as

$$E = \iota^*(H - 2R) = 4h - 2\iota^*(R).$$

Hence, E is divisible by 2 and there exists a unique double cover $\tilde{f} : \tilde{X}_{\bar{A}} \to \tilde{D}_1^{\bar{A}}$ branched along E. Clearly the preimage of $\tilde{f}^{-1}(E)$ is contracted by a birational morphism ψ

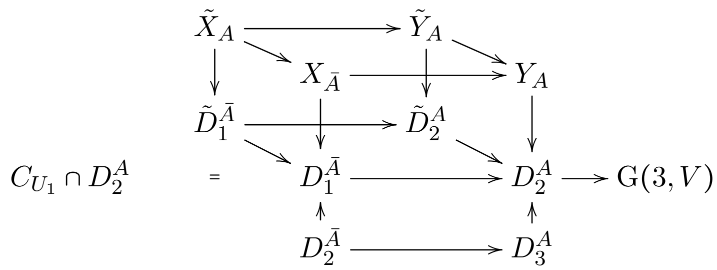
defined by some multiple of the system $\tilde{f}^*\bar{\alpha}^*H$ on $\tilde{X}_{\bar{A}}$. The proof that the image $X_{\bar{A}} = \psi(\tilde{X}_{\bar{A}})$ of this morphism is smooth is similar to the proof of [IKKR16, Proposition 3.1]. It amounts to observing that the restriction of ϕ to the strict transform on $\tilde{X}_{\bar{A}}$ of a generic surface linear section of $D_1^{\bar{A}}$ is the contraction of (-1)-curves on a smooth surface. Denote by

$$p_X\colon X_{\bar{A}} \to D_1^{\bar{A}} \tag{2.5}$$

the induced double cover, ramified over $D_2^{\bar{A}}$.

Let $[A] \in (\Sigma - (\Sigma_+ \cup \Sigma[1] \cup \Gamma)$ and let $[U_1] = \mathbb{P}(A) \cap \mathrm{G}(3,V)$. In [IKKR16, Section 3] a 6-fold double cover $Y_A \to D_2^A$ ramified along D_3^A is constructed over the second degeneracy locus $D_2^A \subset \mathrm{G}(3,V)$ of the Lagrangian subspace A. Note that our construction of the double cover p_X is just the restriction of that construction to $D_2^A \cap C_{U_1}$. Indeed, we proceed as shown in the diagram 2.6. The intersection $D_2^A \cap C_{U_1}$ coincides with $D_1^{\bar{A}}$ and $D_3^A \cap C_{U_1} = D_2^{\bar{A}}$. Also the resolution of singularities $\tilde{D}_2^A \to D_2^A$ restricts to the resolution of singularities $\tilde{D}_1^{\bar{A}} \to D_1^{\bar{A}}$. The double cover $\tilde{Y}_A \to \tilde{D}_2^A$ restricts to a double cover of $\tilde{D}_1^{\bar{A}}$ branched along E. It hence follows by uniqueness of double cover that the strict transform of $\tilde{D}_1^{\bar{A}}$ under the double cover $\tilde{Y}_A \to \tilde{D}_2^A$ is isomorphic to $\tilde{X}_A$.

(2.6)



Therefore $X_{\bar{A}}$ coincides with the strict transform of $D_1^{\bar{A}}$ under the double covering $Y_A \to D_2^A$. Finally p_X is then the restriction of the double cover $Y_A \to D_2^A$ to $X_{\bar{A}}$:

Proposition 2.7. *Let $[A] \in (\Sigma - (\Sigma_+ \cup \Sigma[1] \cup \Gamma)$, let $[U_1] = \mathbb{P}(A) \cap \mathrm{G}(3,V)$ and $p_Y : Y_A \to D_2^A \subset \mathrm{G}(3,V)$ be 6-fold double cover ramified over D_3^A. Then $p_X : X_{\bar{A}} \to D_1^{\bar{A}}$ coincides with the restriction of the double cover p_Y to the preimage $p_Y^{-1}(D_2^A \cap C_{U_1})$.*

We construct in this way a 19-dimensional family, parametrized by

$$\Sigma - (\Sigma_+ \cup \Sigma[1]),$$

of hyperkähler fourfolds admitting polarizations of degree 48 that define antisymplectic involutions.

2.2. **The construction.** We need to prove that $X_{\bar{A}}$ are hyper-Kähler manifolds.

Proposition 2.8. *Let $[A] \in (\Sigma - (\Sigma_+ \cup \Sigma[1] \cup \Gamma))$, let $[U_1] = \mathbb{P}(A) \cap \mathrm{G}(3,V)$ and let $p_X : X_{\bar{A}} \to D_1^{\bar{A}}$ be the double cover of (2.5). Then $X_{\bar{A}}$ is a smooth manifold with trivial first Chern class.*

Proof. The smoothness of $X_{\bar{A}}$ was noted above, so it remains to compute the canonical class. For this we start with $D_1^{\bar{A}}$, a quartic hypersurface section of the cone C_{U_1}, see Lemma 2.2, with quadratic singularities along the surface $D_2^{\bar{A}}$. Let $\tilde{C}_{U_1} \to C_{U_1}$ be the blowup of the cone C_{U_1} in the vertex. Then $\tilde{C}_{U_1}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2 \times \mathbb{P}^2$. The pullback h to $\tilde{C}_{U_1}$ of a hyperplane divisor on $\mathbb{P}^2 \times \mathbb{P}^2$, coincides with the pullback of a hyperplane divisor on C_{U_1}. The pullback of a canonical divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ is $-3h$, while the relative canonical divisor over $\mathbb{P}^2 \times \mathbb{P}^2$ is $-h$, so the canonical divisor on $\tilde{C}_{U_1}$ is $-4h$. By adjunction the fourfold $D_1^{\bar{A}}$ has trivial canonical sheaf. Since the singularities along the surface $D_2^{\bar{A}}$ are quadratic, the canonical divisor of the smooth fourfold $\tilde{D}_1^{\bar{A}}$ is half the class of the exceptional divisor. The double cover $\tilde{X}_{\bar{A}}$, therefore has canonical divisor equal to the ramification divisor $\tilde{E}$. On the smooth fourfold $X_{\bar{A}}$, this divisor is blown down, so $X_{\bar{A}}$ has trivial first Chern class. □

Theorem 2.9. *There exists a* 19*-dimensional family of polarized IHS fourfolds* (X, H) *such that* $|H|$ *defines a* $2:1$ *morphism to* $\mathbb{P}^9$ *and the image is the intersection of a cone over a Segre product* $\mathbb{P}^2 \times \mathbb{P}^2$ *with a special quartic* $Q_{\bar{A}}$*, with the branch locus being the surface* $S_{\bar{A}}$ *defined in Lemma 2.2. Moreover, each fourfold in this family admits two Lagrangian fibrations and a polarization with* $q = 4$*.*

Let us be more precise. Let $A \in \mathrm{LG}_\eta(10, \wedge^3 V)$ such that $\mathbb{P}(A)$ intersects transversally $\mathrm{G}(3, V)$ in one point (i.e. $[A] \in (\Sigma - (\Sigma_+ \cup \Sigma[1]))$. In this case without, loss of generality we let $X_{\bar{A}} \subset Y_A$ be the fourfold defined in (2.5) and Proposition 2.8.

In order to prove that $X_{\bar{A}}$ is IHS we need to find a degeneration of $X_{\bar{A}}$ that is birational to the Hilbert scheme of two points on a $K3$ surface. For this, we first consider, for $v \in V$, the 10-dimensional Lagrangian subspace

$$F_{[v]} := \langle v \rangle \wedge (\wedge^2 V) \subset \wedge^3 V.$$

Recall that

$$\Delta = \{[A] \in \mathrm{LG}_\eta(10, \wedge^3 V) | \quad \exists v \in V \colon \dim A \cap F_{[v]} \geq 3\}.$$

We shall use an $[A] \in \Sigma \cap \Delta$ to find the suitable degeneration.

By dimension count we infer the following, using the notation of 0.7:

Lemma 2.10. *The set* $(\Sigma \cap \Delta) - (\Sigma_+ \cup \Sigma[1] \cup \Gamma)$ *is nonempty of dimension* 18.

Proof. By a direct count, we first compute that $\dim(\Sigma \cap \Delta) = 53$. Let $v \in V$ and let $\mathbf{P} \subset \mathbb{P}(F_{[v]})$ be a plane. The set of triples $F_{[v]}$, $\mathbf{P}$ and $[U] \in \mathrm{G}(3, V) \cap \mathbf{P}^\perp$ depend on $5 + (3 * 7) + 6 = 32$ parameters. The set of Lagrangian subspaces A such that $\mathbb{P}(A) \supset \langle \mathbf{P}, [U] \rangle$ is isomorphic to a $\mathrm{LG}(6, 12)$ so its dimension is 21. It follows that $(\Sigma \cap \Delta)$ contains a component corresponding to general pairs $(\mathbf{P}, [U])$. We shall compute dimensions of the intersections of this component with Σ_+, $\Sigma[1]$ and Γ separately:

(1) For a general $[A] \in \Sigma_+ \cap \Delta$, the linear space $\mathbb{P}(A)$ contains a pair $(\mathbf{P}, [U])$ and a point $[U'] \in \mathrm{G}(3, V) \cap \langle \mathbf{P}, [U] \rangle^\perp$. Since $\mathbf{P}$ and $[U]$ are general, we have $\mathrm{G}(3, V) \cap \langle \mathbf{P}, [U] \rangle = [U]$ and

$$\dim(\mathrm{G}(3, V) \cap \langle \mathbf{P}, [U] \rangle^\perp) = 5.$$

It follows that $[U'] \notin \langle \mathbf{P}, [U] \rangle$ and the space of choices of U' is 5-dimensional. A dimension count yields $32 + 5 + dim(\mathrm{LG}(5, 10)) = 52$

(2) For a general $[A] \in \Sigma[1] \cap \Delta$, the linear space $\mathbb{P}(A)$ contains a pair $(\mathbf{P}, [U])$ and a line $l \subset \mathbb{P}(T_U)$ through $[U]$. Since A is Lagrangian we get that $l \subset \mathbb{P}(T_U) \cap \langle \mathbf{P}, [U] \rangle^\perp$, and the number of parameters for A given $(\mathbf{P}, [U])$ and the line l is $\dim \mathrm{LG}(5, 10) = 15$

When $\mathbf{P}$ and $[U]$ are general the number of parameters for l is

$$\dim(\mathrm{G}([U], 1, \mathbb{P}(T_U) \cap \langle \mathbf{P}, [U] \rangle^\perp)) = 5.$$

So summing up we get that $\Sigma[1] \cap \Delta$ has dimension $32 + 5 + \dim(\mathrm{LG}(5, 10)) = 52$.

(3) For a general $[A] \in \Gamma \cap \Sigma \cap \Delta$, the linear space $\mathbb{P}(A)$ contains a pair $(\mathbf{P}, [U])$, and intersects $T_{[U']}$ for some $[U'] \in \mathrm{G}(3, V)$ such that $\dim(\mathbb{P}(T_{U'}) \cap \mathbb{P}(A)) = 3$. Let $\dim(\mathbb{P}(T_{U'}) \cap \langle \mathbf{P}, [U] \rangle^\perp) = 5 + d_1$ and therefore $\dim(\mathbb{P}(T_{U'}) \cap \langle \mathbf{P}, [U] \rangle) = d_1 - 1$. The set of 4-dimensional subspaces $W_4 \subset \wedge^3 V$ such that $\mathbb{P}(W_4) \subset \mathbb{P}(T_{[U']}) \cap \langle \mathbf{P}, [U] \rangle^\perp$ and meets $\langle \mathbf{P}, [U] \rangle$ in dimension d_1 is a Schubert cycle of dimension 8 for $d_1 = 0$ and 9 for $d_1 = 1$. On the other hand the dimension of the set of Lagrangian spaces A such that $\mathbb{P}(A)$ contains $\langle \mathbb{P}(W_4), \mathbf{P}, [U] \rangle$ is

$$\dim(\mathrm{LG}(2 + d_1, 4 + 2d_1)) = \frac{(2 + d_1)(3 + d_1)}{2}.$$

To complete the dimension count we compute the dimension of the set of subspaces U' corresponding to $d_1 = 0$ and $d_1 = 1$. For $d_1 = 0$ the set of subspaces U' is an open set in $\mathrm{G}(3, V)$, so the dimension is 9, so the set of Lagrangian subspaces A in this case has dimension $32 + 9 + 3 + 8 = 52$. Whereas, for $d_1 = 1$ the set of subspaces U' such that $\mathbb{P}(T_{[U']}) \cap \langle \mathbf{P}, [U] \rangle \neq \emptyset$ has dimension 5, so the set of Lagrangian subspaces A in this case has dimension $32 + 5 + 6 + 9 = 52$.

□

Definition 2.11. Let $v_0 \in V$. We call $\mathrm{LG}_\eta(10, \wedge^3 V)^{v_0}$ the set of Lagrangian subspaces $A \subset \wedge^3 V$ that satisfy the following conditions:

(1) There exists a codimension 1 subspace $V_0 \subset V$ such that $\wedge^3 V_0 \cap A = 0$.
(2) $v_0 \in U$ for at most one $[U] \in \Theta_A$.
(3) If $v_0 \in U$ and $[U] \in \Theta_A$, then $A \cap (\wedge^2 U \wedge V) = \wedge^3 U$

Recall that for $A \in \mathrm{LG}(\wedge^3 V)^{v_0}$ O'Grady defined a surface $S_A(v_0)$ as follows [O'G13]: By the first two conditions $V = \langle v_0 \rangle \oplus V_0$. Consider the isomorphism

$$\lambda : \wedge^2 V_0 \to F_{v_0} = v_0 \wedge (\wedge^2 V); \qquad \alpha \mapsto v_0 \wedge \alpha.$$

Let $K_A^0 = \lambda^{-1}(A \cap F_{v_0}) \subset \wedge^2 V_0$. Given a volume form on V_0, there is an isomorphism $\wedge^3 V_0 \cong \wedge^2 V_0^\vee$, and hence the annihilator $\mathrm{Ann} K_A^0 \subset \wedge^3 V_0$ defines a linear section $F_A^0 = \mathbb{P}(\mathrm{Ann} K_A^0) \cap Gr(3, V_0) \subset \mathbb{P}(\wedge^3 V_0)$. Now, K_A^0 is 3-dimensional, so the linear section F_A^0 has codimension 3 in $Gr(3, V_0)$ and is a Fano 3-fold. The first assumption in 2.11 implies that A is the graph of a linear map

$$q_A : \wedge^2 V_0 \to \wedge^3 V_0 \subset \wedge^3 V$$

such that $q_A(\alpha) = \beta \leftrightarrow (v_0 \wedge \alpha + \beta) \in A$. Since A, $F_{[v_0]}$ and $\wedge^3 V_0$ are Lagrangian, the map q_A is symmetric, while $\ker q_A = K_A^0$, so q_A induces an isomorphism

$$\wedge^2 V_0 / K_A^0 \to \mathrm{Ann} K_A^0 \subset \wedge^3 V_0$$

whose inverse defines a quadratic form

$$Q_A^* : \beta \mapsto \mathrm{vol}(\alpha \wedge \beta), \quad \text{where } q_A(\alpha) = \beta$$

on $\mathrm{Ann} K_A^0$. The surface $S_A(v_0)$ is the intersection $F_A^0 \cap \{Q_A^* = 0\}$.

O'Grady proves that if Θ_A is finite, then $S_A(v_0)$ is reduced and irreducible with explicitly described singular locus. Moreover, if it has du Val singularities, then the minimal resolution $\overline{S_A(v_0)} \to S_A(v_0)$ is a $K3$ surface [O'G13, Corollaries 4.7 and 4.8].

Lemma 2.12. *Let $[A] \in (\Sigma \cap \Delta) - (\Sigma_+ \cup \Sigma[1] \cup \Gamma)$ be generic, then there exists a unique $[v] \in \mathbb{P}(V)$ such that $\dim(A \cap F_{[v]}) \geq 3$. Moreover, $[A] \in \mathrm{LG}(\wedge^3 V)^v$ and the surface $S_A(v)$ is a $K3$ surface with one node.*

Proof. Consider $F_{[v]}$ for a general $v \in V$ and a general projective plane $\mathbf{P} \subset \mathbb{P}(F_{[v]})$ and let $[U]$ be a general point in $\mathrm{G}(3,V) \cap \mathbf{P}^{\perp}$. By Lemma 2.10, the general Lagrangian space A such that $\mathbb{P}(A)$ contains $[U] \cup \mathbf{P}$ is an element of $(\Sigma \cap \Delta) - (\Sigma_+ \cup \Sigma[1] \cup \Gamma)$. Clearly A and v then satisfy the assumption of 2.11 i.e. $A \in \mathrm{LG}_{\eta}(10, \wedge^3 V)^v$.

We need to prove that $S_A(v)$ is a $K3$ surface with one node. We build on the proof of [O'G13, Proposition 4.6]. The Fano threefold $F_A = \mathbf{P}^{\perp} \cap \mathrm{G}(3,V_0)$ is smooth and the surface $S_A(v)$ is a quadric section of F_A that is smooth outside one point. It follows that the singularity is an ordinary double point. □

We denote by $\overline{S_A(v)} \to S_A(v)$ the minimal resolution of singularities on $S_A(v)$. Consider the 6-fold Lagrangian degeneracy locus $Y_A \subset \mathrm{G}(3,V)$ called an EPW cube in [IKKR16], defined as

$$Y_A = \{[U] \in \mathrm{G}(3,V) | \dim A \cap T_U \geq 2\}.$$

When $[A] \in (\Sigma \cap \Delta) - (\Sigma_+ \cup \Sigma[1])$ we shall define a rational map

$$\psi \colon S_A(v)^{[3]} \dashrightarrow Y_A,$$

as in [IKKR16, §4].

First we consider the natural isomorphism:

$$V^{\vee} = V_0^{\vee} \oplus \langle v_0^* \rangle \to H^0(\mathcal{I}_{S_A(v)}(2)); \qquad v^* + cv_0^* \mapsto q_{v^*} + cq_A^*,$$

where Q_{v^*} is the restriction to $\mathrm{Ann}K_A^0$ of the quadratic form on $\wedge^3 V_0$ defined by

$$Q_{v^*}(\omega) = \mathrm{vol}(\omega(v^*) \wedge \omega).$$

Let $[\beta_1]$ and $[\beta_2]$ be two points in $S_A(v)$, such that the line $\langle [\beta_1], [\beta_2] \rangle$ is not contained in $S_A(v)$, then $H^0(\mathcal{I}_{S_A(v) \cup \langle \beta_1, \beta_2 \rangle}(2))$ is a hyperplane in $H^0(\mathcal{I}_{S_A(v)}(2)) \cong V^{\vee}$. Therefore

$$\phi \colon S_A(v)^{[2]} \dashrightarrow \mathbb{P}(V); \quad ([\beta_1], [\beta_2]) \mapsto [H^0(\mathcal{I}_{S_A(v) \cup \langle \beta_1, \beta_2 \rangle}(2))]$$

defines a rational map. The rational map $\psi \colon S_A(v)^{[3]} \dashrightarrow \mathrm{G}(3,V)$ is now defined by

$$\psi(\beta_1, \beta_2, \beta_3) = [\langle \phi(\beta_1,\beta_2), \phi(\beta_1,\beta_3), \phi(\beta_2,\beta_3) \rangle] \in \mathrm{G}(3,V)$$

For general A, both ϕ and ψ are morphisms that are $2:1$ onto their image [IKKR16, Proposition 4.1].

We consider a restriction of the map ψ to show

Proposition 2.13. *Let $[A] \in (\Sigma \cap \Delta) - (\Sigma_+ \cup \Sigma[1] \cup \Gamma)$. Let $[U] = \mathrm{G}(3,V) \cap \mathbb{P}(A)$, and let $\bar{A} = A / \wedge^3 U$. Then $X_{\bar{A}}$ is birational to $\overline{S_A(v)}^{[2]}$.*

Proof. Let U be the unique element in Θ_A. Consider the decomposition $V = v_0 \oplus V_0$. By [O'G13, Corollary 4.7], the $K3$-surface $S := S_A(v_0)$ in $\mathbb{P}(\wedge^3 V_0)$ is singular in $p := [\wedge^3 U']$ where U' is the projection of U onto V_0. Moreover, by Lemma 2.12, the point p is a node in S. Let $\kappa : \overline{S} \to S$ be the blow up giving the resolution of the node p. Consider the following rational map ξ defined on pairs of distinct points on $\overline{S}$.

$$\xi : \overline{S}^{[2]} \to \mathrm{G}(3, H^0(\mathcal{I}_S(2)); \quad \xi([p_1,p_2]) = H^0(\mathcal{I}_{S \cup \langle l_{p_1}, l_{p_2} \rangle}(2)) \subset \mathrm{G}(3, H^0(\mathcal{I}_S(2)),$$

where l_{p_i} is the line spanned by p and $\kappa(p_i)$ when $p \neq \kappa(p_i)$ and the line in the tangent cone of p corresponding to p_i when $\kappa(p_i) = p$. Moreover, $\langle l_{p_1}, l_{p_2} \rangle$ is the plane spanned by l_{p_1} and l_{p_2}.

Lemma 2.14. *The map ξ is generically 2:1, well defined and unbranched outside a set of codimension 2.*

Proof. The proof is analogous to the proofs of Proposition 4.1 and Proposition 4.5 of [IKKR16]. We need only to observe that S is the intersection of a smooth Fano threefold F with a quadric Q and hence every twisted cubic passing through p is tangent to the quadric Q in p. This implies that the involution on $S_A(v)^{[3]}$ given by ψ above restricts to an involution on the locus of triples on $\overline{S}_A(v)$ containing p, and ξ can be considered as the restriction of ψ to this locus. □

The next step is to prove that the image of ξ is contained in the cone $C_U = \mathrm{G}(3,V) \cap \mathbb{P}(T_U) = \{[L] \in \mathrm{G}(3,V) : \dim(L \cap U) \geq 2\}$. By [IKKR16, Lemma 4.2], we have

$$\xi([p_1,p_2]) = \psi([p,p_1,p_2]) = \langle \phi([p,p_1]), \phi([p,p_2]), \phi([p_1,p_2]) \rangle$$

in the above notation. To prove that $\xi([p_1,p_2]) \in C_U$ it is enough to prove that $\phi([p,p_i]) \in U$. Let $i = 1$. We follow the proof of [IKKR16, Proposition 4.1]. Indeed let

$$\wedge^3 U = u_1 \wedge u_2 \wedge u_3 = v_0 \wedge \alpha + v_1 \wedge v_2 \wedge v_3$$

with $v_1, v_2, v_3 \in V_0$ and $\alpha \in \wedge^2\langle v_1,v_2,v_3\rangle$, then, by [O'G13, Corollary 4.7], the singular point of the $K3$ surface S is $p = v_1 \wedge v_2 \wedge v_3$. Without loss of generality we may then assume that $p_1 = v_1 \wedge v_4 \wedge v_5$. Then, by [IKKR16, Equation 4.1], we have $\phi([p,p_1]) = [\mathrm{vol}(\alpha \wedge v_1 \wedge v_4 \wedge v_5)v_0 + v_1]$. To check that it is an element of U we compute

$$\begin{aligned}&(\mathrm{vol}(\alpha \wedge v_1 \wedge v_4 \wedge v_5)v_0 + v_1) \wedge (v_0 \wedge \alpha + v_1 \wedge v_2 \wedge v_3)\\ &= v_1 \wedge v_0 \wedge \alpha + vol(\alpha \wedge v_1 \wedge v_4 \wedge v_5)v_0 \wedge v_1 \wedge v_2 \wedge v_3.\end{aligned}$$

The latter is an element of $\wedge^4\langle v_0,v_1,v_2,v_3\rangle$ and the wedge product with $v_4 \wedge v_5$,

$$\begin{aligned}&(v_1 \wedge v_0 \wedge \alpha + \mathrm{vol}(\alpha \wedge v_1 \wedge v_4 \wedge v_5)v_0 \wedge v_1 \wedge v_2 \wedge v_3) \wedge v_4 \wedge v_5\\ &= (-v_0 \wedge \alpha \wedge v_1 \wedge v_4 \wedge v_5 + \mathrm{vol}(\alpha \wedge v_1 \wedge v_4 \wedge v_5)(v_0 \wedge \ldots \wedge v_5)\\ &= (-\mathrm{vol}(\alpha \wedge v_1 \wedge v_4 \wedge v_5) + \mathrm{vol}(\alpha \wedge v_1 \wedge v_4 \wedge v_5))(v_0 \wedge \ldots \wedge v_5) = 0,\end{aligned}$$

so $(\mathrm{vol}(\alpha \wedge v_1 \wedge v_4 \wedge v_5)v_0 + v_1) \in U$. With the same argument for $i = 2$ we conclude that $\xi([p_1,p_2]) \in C_U$, in particular $\xi([p_1,p_2]) \in D^1_{\bar{A}} \subset C_U$.

Therefore $X_{\bar{A}} \to D^1_{\bar{A}}$ and $\xi : S^{[2]}_A \to D^1_{\bar{A}}$ are two double covers which are well defined and unbranched outside a set of codimension 2. It follows that $X_{\bar{A}}$ is birational to $S^{[2]}_A$ as in [IKKR16, §5] and further still following [IKKR16, §5] we get $X_{\bar{A}}$ is IHS and deformation equivalent to a $K3^{[2]}$ for general $\bar{A}$. □

Remark 2.15. The intersection lattice of $\bar{S}^{[2]}_A$, where $\bar{S}_A$ is the minimal resolution of the nodal S_A, is the diagonal matrix with entries $10, -2, -2$. After a change of base to (h_1, h_2, θ) we obtain:

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

We find that the map ξ is given by $h_1 + h_2$. Since there is a divisor with self-intersection -10 and divisibility 2 perpendicular to $h_1 + h_2$, it follows that ξ contracts a $\mathbb{P}^2$ to a point (see [HY15, §5.1] or [Mon15, §2]). We can identify this $\mathbb{P}^2$ as the set of pairs of points on S_A such that the line spanned by these points is contained in the threefold section of $\mathrm{G}(2,5)$ containing S_A.

Remark 2.16. We can find another 18-dimensional subfamily of $\mathcal{U}$ such that the elements are birational to the Hilbert scheme of two point on a $K3$ surface. Let us take a $K3$ surface S that is a hyperplane section of a Verra threefold $Z \subset \mathbb{P}^8$. The intersection lattice of $S^{[2]}$ is

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

After an integral linear change of coordinates the matrix takes the form:

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

with basis l_1, l_2, η'. Then $l_1 + l_2$ gives a $2:1$ map to an EPW quartic section containing the vertex of the cone and singular at it. We can show that this map contracts two planes $\mathbb{P}^2$ to this vertex point.

3. The second construction- the Hilbert scheme of conics on the Verra 4-fold

We describe the second construction of elements from $\mathcal{U}$ that is parallel to the construction of Kummer surfaces given in section 1.3. Let U_1 and U_2 be three dimensional complex vector spaces, fix moreover a volume form on each space $U_1, U_2^\vee$ such that $\wedge^2 U_1 = U_1^\vee$ and $\wedge^2 U_2^\vee = U_2$, and let $\eta : \wedge^3 U_1 \otimes \wedge^3 U_2^\vee \to \mathbb{C}$ be the product volume form. Let $Y \subset \mathbb{P}^9$ be the intersection of the cone $C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)) \subset \mathbb{P}(\mathbb{C} \oplus (U_1 \otimes \wedge^2 U_2))$ with a quadric hypersurface. Such a fourfold is a smooth Fano fourfold when Q is chosen generically: we call it a Verra fourfold. We have a 19-dimensional family of Verra fourfolds.

Note that a Verra fourfold is naturally a double cover of $\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)$. Its ramification locus Z is the intersection of Y with the hyperplane polar to the vertex of the cone via the quadric Q. In terms of coordinates, this means that if coordinates are chosen in such a way that Q is defined by a quadric $\{z^2 - Q' = 0\}$ then $Z = Y \cap \{z = 0\}$. We call Z the Verra threefold associated to Y. We shall sometimes also identify Z with the branch locus $\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2) \cap \{Q' = 0\}$.

Notice the following properties of Verra fourfolds.

Lemma 3.1. *If $Y \subset C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2))$ is a smooth Verra fourfold then:*

(1) *Y does not pass through the vertex of the cone $C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2))$;*
(2) *Y contains no quadric threefold;*
(3) *the preimage of each quadric surface $\mathbb{P}(L^\vee) \times \mathbb{P}(M^\vee) \subset \mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)$ by the double cover $Y \to \mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)$ is irreducible.*

Proof. Clearly Y being a smooth complete intersection of $C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2))$ with a quadric cannot pass through the singular point of the cone. For (2), if Y contained a quadric threefold, then this threefold would be contained in $C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2))$ and hence would be a cone over a quadric surface in $\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)$. This leads to a contradiction with (1). Finally assume that the preimage of some quadric surface $\mathbb{P}(L^\vee) \times \mathbb{P}(M^\vee) \subset \mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)$ is reducible. Then it must decompose as the union of two quadric surfaces and the branch locus of the projection onto $\mathbb{P}(L^\vee) \times \mathbb{P}(M^\vee)$ is then a double conic. It follows that the branch locus Z of the projection $Y \to \mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)$ meets the $\mathbb{P}^3$ spanned by $\mathbb{P}(L^\vee) \times \mathbb{P}(M^\vee)$ in a double conic. By Zak's Tangency theorem [Zak93], this implies that Z is singular and in consequence Y is also singular. □

The linear system of quadrics containing $C(\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2U_2))\subset\mathbb{P}(\mathbb{C}\oplus(U_1\otimes\wedge^2U_2))$ is naturally isomorphic to $\mathbb{P}(U_1\otimes\wedge^2U_2)$. In fact let $w\in U_1\otimes\wedge^2U_2=U_1\otimes(U_2)^\vee$ and $(w_0,w')\in\mathbb{C}\oplus(U_1\otimes\wedge^2U_2)=\mathbb{C}\oplus(U_1\otimes(U_2)^\vee)$, then

$$Q_w(w_0,w')=\eta(w\wedge w'\wedge w')$$

is a quadratic form on $\mathbb{C}\oplus(U_1\otimes\wedge^2U_2)$, and the map

$$U_1\otimes\wedge^2U_2\to H^0(\mathcal{I}_{C(\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2U_2))}(2));\quad w\mapsto Q_w$$

is an isomorphism. Thus

$$I_{Y,2}:=H^0(\mathcal{I}_Y(2))\cong\mathbb{C}\oplus(U_1\otimes\wedge^2U_2),$$

and the linear system of quadrics containing $Y\subset\mathbb{P}^9$ is naturally isomorphic to $\mathbb{P}(\mathbb{C}\oplus(U_1\otimes\wedge^2U_2))$ and is dual to $\mathbb{P}(\mathbb{C}\oplus(\wedge^2U_1\otimes U_2))$. By abuse of notation, we denote also by Q_w the quadric hypersurface corresponding to $[w]\in\mathbb{P}(U_1)\otimes\mathbb{P}(\wedge^2U_2)$.

Consider the two natural projections π_i of Y onto $\mathbb{P}(U_1)$ and $\mathbb{P}(\wedge^2U_2)$ for $i=1,2$ respectively. We denote by $F(Y)$ the Hilbert scheme of conics on Y of type $(1,1)$ i.e. conics that project to lines by both π_1 and π_2.

Let us now relate the Hilbert scheme $F(Y)$ corresponding to the quadric Q with an EPW quartic section. Let C be a conic on Y, then C spans a plane $\mathrm{P}_C\subset\mathbb{P}(\mathbb{C}\oplus(U_1\otimes\wedge^2U_2))$. Consider the locus H_C of quadrics containing $Y\cup\mathrm{P}_C$. Clearly H_C is a hyperplane in the space of quadrics containing Y i.e. naturally a point $H_C\in\mathbb{P}(\wedge^3U_1\oplus(\wedge^2U_1\otimes U_2))$. In this way we defined a morphism

$$\psi_Q:F(Y)\to\mathbb{P}(\wedge^3U_1\oplus(\wedge^2U_1\otimes U_2)).$$

Proposition 3.2. *The image $\psi_Q(F(Y))$ is isomorphic to an EPW quartic section.*

Proof. We first introduce the EPW quartic section that we claim is $\psi_Q(F(Y))$.

For that, choose a coordinate chart (z,β) on $\mathbb{C}\oplus(U_1\otimes\wedge^2U_2)$ in which $Q(z,\beta)=z^2-Q'(\beta)$. Note, that in this case $Q'\cap(\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2U_2))$ is the branch locus of the projection map of Y from the vertex of the cone.

Now, the vector space $((\wedge^2U_1)\otimes U_2)\oplus(U_1\otimes(\wedge^2U_2))$ is equipped with the symplectic form $\bar\eta(\alpha,\beta)=vol(\alpha\wedge\beta)$. Observe that $((\wedge^2U_1)\otimes U_2)\oplus(U_1\otimes(\wedge^2U_2))$ is then a decomposition into a sum of Lagrangian spaces with respect to $\bar\eta$. In particular $\bar\eta$ defines the canonical isomorphism $((\wedge^2U_1)\otimes U_2)^\vee\simeq(U_1\otimes(\wedge^2U_2))$. Now Q' defines a symmetric map $q':(U_1\otimes(\wedge^2U_2))\to(U_1\otimes(\wedge^2U_2))^\vee=((\wedge^2U_1)\otimes U_2)$, the graph of this map in $((\wedge^2U_1)\otimes U_2)\oplus(U_1\otimes(\wedge^2U_2))$ is a Lagrangian space that we call $\bar A_{Q'}$.

Since we know that the subset of the Hilbert scheme of conics in Y parameterizing smooth conics is dense in the whole Hilbert scheme of conics the following lemma completes the proof of the proposition.

Lemma 3.3. *Let $\mathbf{P}$ be a plane in $\mathbb{P}(\mathbb{C}\oplus(U_1\otimes\wedge^2U_2))$ meeting Y in a smooth conic curve C of type $(1,1)$, then the hyperplane $H_\mathbf{P}$ of quadrics containing $Y\cup\mathbf{P}$ is an element of the EPW quartic section $\bar D_1^{\bar A_{Q'}}$. Furthermore, if C is contained in the branch locus Z then $[H_\mathbf{P}]\in\bar D_2^{\bar A_{Q'}}$.*

Proof. Let us consider the cone

$$C(\mathbb{P}(U_1)\times\mathbb{P}(\wedge^2U_2))\subset\mathbb{P}(\mathbb{C}\oplus(U_1\otimes(\wedge^2U_2)))=\mathbb{P}((\wedge^3U_2)\oplus(U_1\otimes(\wedge^2U_2)))$$

as

$$\mathbb{P}((U_1\oplus U_2)\wedge(\wedge^2U_2))\cap\mathrm{G}(3,U_1\oplus U_2)\subset\mathbb{P}(\wedge^3(U_1\oplus U_2)).$$

Let

$$\mathbf{P} = \langle (z_1, \beta_1), (z_2, \beta_2), (z_3, \beta_3) \rangle,$$

such that

$$(z_i, \beta_i) \in C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)) \subset \mathbb{P}((U_1 \oplus U_2) \wedge (\wedge^2 U_2)) \cap \mathrm{G}(3, U_1 \oplus U_2) \subset \mathbb{P}(\wedge^3 (U_1 \oplus U_2)).$$

Since $\mathbf{P}$ meets $\mathrm{G}(3, U_1 \oplus U_2)$ in a conic curve, there exists then a basis $u_1, u_2, u_3, v_1, v_2, v_3$ of $U_1 \oplus U_2$ such that we have

$$\beta_1 = u_1 \wedge v_1 \wedge v_2, \beta_2 = u_2 \wedge v_1 \wedge v_3, \beta_3 = (u_1 + u_2) \wedge v_1 \wedge (v_2 + v_3).$$

In such basis the coordinate of $H_{\mathbf{P}} \in \mathbb{P}(\mathbb{C} \oplus (\wedge^2 U_1) \otimes U_2)$ corresponding to $\wedge^2 U_1 \wedge U_2$ is $u_1 \wedge u_2 \wedge v_1$. Moreover, by the definition of $\bar{A}_{Q'}$, for each $\beta \in U_1 \wedge \wedge^2 U_2$ there exists an $\alpha \in (\wedge^2 U_1) \wedge U_2$ such that $Q'(\beta) = \alpha \wedge \beta$ or equivalently $\alpha + \beta \in \bar{A}_{Q'}$. Let us denote by α_i the elements corresponding to β_i under the above. Since $(\alpha_i + \beta_i) \wedge (\alpha_j + \beta_j) = 0$ for all i, j we get $\alpha_i \wedge \beta_j = \alpha_j \wedge \beta_i =: c_{i,j}$ for $i \neq j$. Now

$$Q(\lambda_1 (z_i, \beta_i) + \lambda_2 (z_j, \beta_j)) = (z_i + \lambda z_j)^2 - Q'(\lambda_1 \beta_i + \lambda_2 \beta_j) =$$

$$= (z_i + \lambda z_j)^2 - (\lambda_1 \alpha_i + \lambda_2 \alpha_j) \wedge (\lambda_1 \beta_i + \lambda_2 \beta_j) = \lambda_1^2 q((z_i, \beta_i)) + \lambda_2^2 q((z_j, \beta_j)) + 2\lambda_1 \lambda_2 (z_i z_2 - c_{i,j}).$$

But $Q((z_i, \beta_i)) = 0$ by assumption, so

$$Q(\lambda_1 (z_i, \beta_i) + \lambda_2 (z_j, \beta_j)) = 2\lambda_1 \lambda_2 (z_i z_2 - c_{i,j}).$$

Now

$$(t_0 Q + t_1 Q_{(u_1 \wedge u_2 \wedge v_1)^*})(\lambda_1 (z_i, \beta_i) + \lambda_2 (z_j, \beta_j)) = 2t_0 \lambda_1 \lambda_2 (z_i z_j - c_{i,j}) + 2t_1 \lambda_1 \lambda_2.$$

It follows that the $H_{\mathbf{P}} = [(z_i z_j - c_{i,j}, u_1 \wedge u_2 \wedge v_1)] \in \mathbb{P}(\mathbb{C} \oplus U_1 \wedge (\wedge^2 U_2))$ which means, in particular, that:

$$z_1 z_2 - c_{1,2} = z_1 z_3 - c_{1,3} = z_2 z_3 - c_{2,3} = c_{\mathbf{P}}.$$

$H_{\mathbf{P}}$ is also an element of the cone $C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2)) \subset \mathbb{P}(\mathbb{C} \oplus \wedge^2 U_1 \wedge U_2)$

The corresponding $\bar{T}_{H_{\mathbf{P}}}$ is described by

$$\{\omega \in ((\wedge^2 U_1) \otimes U_2) \oplus ((\wedge^2 U_2) \otimes U_1) | \omega \wedge u_1 \wedge u_2 = \omega \wedge u_1 \wedge (v_1 + c_{\mathbf{P}} u_3) = \omega \wedge u_2 \wedge (v_1 + c_{\mathbf{P}} u_3) = 0\}.$$

We shall prove that $\bar{A}_{Q'} \cap \bar{T}_{H_{\mathbf{P}}} \neq 0$. We know that $\sum_{i=1}^3 \lambda_i (\alpha_i + \beta_i) \in \bar{A}_{Q'}$ for $\lambda_i \in \mathbb{C}$, it is therefore enough to prove that the following system of equations has a nonzero solution $(\lambda_1, \lambda_2, \lambda_3)$:

$$E_1(\lambda_1, \lambda_2, \lambda_3) := (\sum_{i=1}^3 \lambda_i (\alpha_i + \beta_i)) \wedge u_1 \wedge u_2 = 0$$

$$E_2(\lambda_1, \lambda_2, \lambda_3) := (\sum_{i=1}^3 \lambda_i (\alpha_i + \beta_i)) \wedge u_1 \wedge (v_1 + c_{\mathbf{P}} u_3) = 0$$

$$E_3(\lambda_1, \lambda_2, \lambda_3) := (\sum_{i=1}^3 \lambda_i (\alpha_i + \beta_i)) \wedge u_2 \wedge (v_1 + c_{\mathbf{P}} u_3) = 0$$

Observe now that $E_1(\lambda_1, \lambda_2, \lambda_3) \equiv 0$ since both $\alpha_i \wedge u_1 \wedge u_2 = 0$ and $\beta_i \wedge u_1 \wedge u_2 = 0$. Furthermore, we have:

$$E_2(\lambda_1, \lambda_2, \lambda_3) \wedge u_1 = E_2(\lambda_1, \lambda_2, \lambda_3) \wedge u_2 = E_2(\lambda_1, \lambda_2, \lambda_3) \wedge u_3 = E_2(\lambda_1, \lambda_2, \lambda_3) \wedge v_1 = 0,$$

as well as

$$E_3(\lambda_1, \lambda_2, \lambda_3) \wedge u_1 = E_3(\lambda_1, \lambda_2, \lambda_3) \wedge u_2 = E_3(\lambda_1, \lambda_2, \lambda_3) \wedge u_3 = E_3(\lambda_1, \lambda_2, \lambda_3) \wedge v_1 = 0.$$

Finally

$$E_2(\lambda_1,\lambda_2,\lambda_3)\wedge v_2 = z_1^2\lambda_1 + z_1z_2\lambda_2 + z_1z_3\lambda_3, \tag{3.1}$$

$$E_3(\lambda_1,\lambda_2,\lambda_3)\wedge v_3 = z_1z_2\lambda_1 + z_2^2\lambda_2 + z_2z_3\lambda_3, \tag{3.2}$$

$$(E_2(\lambda_1,\lambda_2,\lambda_3)+E_3(\lambda_1,\lambda_2,\lambda_3))\wedge (v_2+v_3) = z_1z_3\lambda_1 + z_2z_3\lambda_2 + z_3^2\lambda_3, \tag{3.3}$$

are proportional, so the above equations reduce to two linear equations in the λ_i:

$$E_2(\lambda_1,\lambda_2,\lambda_3)\wedge v_3 = 0$$

and one of the above 3 proportional equations. It follows that the linear system has rank 2 and therefore admits a nontrivial solution implying $\dim(\bar{A}_{Q'}\cap \bar{T}_{H_{\mathbf{P}}})\geq 1$, which proves the first part of the lemma.

It remains to prove that the image of the Hilbert scheme of $(1,1)$-conics contained in the ramification locus $Y\cap\{z=0\}$ of the projection maps to $\bar{D}_2^{\bar{A}_{Q'}}$. Clearly points on such conics satisfy $z_i=0$ and the three proportional equations above are then trivial, hence the system has two-dimensional solution i.e. $\dim(\bar{A}_{Q'}\cap \bar{T}_{H_{\mathbf{P}}})\geq 2$. □

□

We shall now describe the Stein factorization of the morphism

$$\psi_Q : F(Y)\to \bar{D}_1^{\bar{A}_{Q'}}.$$

Consider the diagram:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{F}) & \xrightarrow{\pi} & \mathbb{P}(\wedge^2 U_1)\times\mathbb{P}(U_2) \\ f\big\downarrow & & \\ \mathbb{P}(\mathbb{C}\oplus(\wedge^2U_1\otimes U_2)) & & \end{array}$$

where

$$\mathcal{F} = \pi_1^\vee((\mathcal{O}_{\mathbb{P}(\wedge^2U_1)}(1))\otimes\pi_2^\vee(\mathcal{O}_{\mathbb{P}(U_2)}(1)))\oplus\mathbb{C}) \tag{3.4}$$

is a vector bundle on $\mathbb{P}(\wedge^2U_1)\times\mathbb{P}(U_2)$ such that π is the projection. Moreover, f is given by $\mathcal{O}_{\mathbb{P}(F)}(1)$ such that the image of f is the cone over $\mathbb{P}(\wedge^2U_1)\times\mathbb{P}(U_2)$ and f is the blow-up of the vertex with exceptional divisor E.

Consider the rank 5 bundle $\mathcal{G}$ over $\mathbb{P}(\wedge^2U_1)\times\mathbb{P}(U_2)$, such that for $(L,M)\in\mathbb{P}(\wedge^2U_1)\times\mathbb{P}(U_2)$ the fiber $\mathcal{G}_{(L,M)}$ is

$$\mathbb{C}\oplus(L^\vee\otimes M^\vee)\subset\mathbb{C}\oplus(U_1\otimes\wedge^2U_2).$$

There is a natural restriction map $I_{Y,2}\to Sym^2\mathcal{G}^\vee_{(L,M)}$. When Y contains no quadric threefold, this map has rank 2, and the image is a pencil of quadric threefolds that defines a complete intersection that we denote by $D_{(L,M)}$. Thus for each $(L,M)\in\mathbb{P}(\wedge^2U_1)\times\mathbb{P}(U_2)$ there is a natural surjective restriction map

$$\pi_{(L,M)} : I_{Y,2}\to I_{D_{(L,M)},2} := H^0(\mathbb{P}(\mathcal{G}_{(L,M)}),\mathcal{I}_{D_{(L,M)}}(2))\subset Sym^2\mathcal{G}^\vee_{(L,M)}.$$

For each element $\mathfrak{Q}\in I_{D_{(L,M)},2}$, let $H_{\mathfrak{Q}}\subset I_{Y,2}$ be the hyperplane of quadrics whose image in $I_{D_{(L,M)},2}$ is proportional to $\mathfrak{Q}$. We define degeneracy loci

$$D_r^Q = \{([H_{\mathfrak{Q}}],(L,M))|\mathfrak{Q}\in I_{D_{(L,M)},2},\ \mathrm{rank}(\mathfrak{Q})\leq 5-r\}\subset\mathbb{P}(\mathbb{C}\oplus(\wedge^2U_1\otimes U_2))\times\mathbb{P}(\wedge^2U_1)\times\mathbb{P}(U_2).$$

Notice that $D_r^Q\subset\mathbb{P}(\mathcal{F})$. Consider the projections

$$f|_{D_r^Q} : D_r^Q\to\mathbb{P}(\mathbb{C}\oplus(\wedge^2U_1\otimes U_2));\quad ([H_{\mathfrak{Q}}],(L,M))\mapsto[H_{\mathfrak{Q}}]\quad r=1,2.$$

We claim

Lemma 3.4. $f(D_1^Q) = \psi_Q(F(Y))$.

Proof. A $(1,1)$-conic C is mapped to a unique pair of lines $\mathbb{P}(L^\vee) \subset \mathbb{P}(U_1)$ and $\mathbb{P}(M^\vee) \subset \mathbb{P}(\wedge^2 U_2)$, and is therefore contained in a unique complete intersection $D_{(L,M)}$, so if $\mathfrak{Q} \in I_{D_{(L,M)},2}$ is the quadric threefold that contains the plane P_C spanned by C, then $H_C \subset I_{Y,2}$ is the hyperplane of quadrics that contain P_C, i.e. $\psi[C] = f([H_\mathfrak{Q}],(L,M))$. On the other hand, if $\mathfrak{Q} \in I_{D_{(L,M)},2}$ is singular, then, by Lemma 3.1(4), it has rank 4 or 3, and the planes in $\mathfrak{Q}$ intersect $D_{(L,M)}$ in conics that are $(1,1)$-conics on Y. □

Next, we claim that f restricted to D_1^Q has an inverse $f^{-1} : f(D_1^Q) \to \mathbb{P}(\mathcal{F})$. Indeed, the quadrics in the ideal of Y define a rational map:

$$\mathbb{P}(\mathbb{C} \oplus (U_1 \otimes \wedge^2 U_2)) \dashrightarrow \mathbb{P}(\mathbb{C} \oplus (\wedge^2 U_1 \otimes U_2)).$$

The preimage of a point $p \in C\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2)$, outside the vertex, is the union

$$Y \cup \mathfrak{Q}_p,$$

where $\mathfrak{Q}_p \in I_{D_{(L,M)},2}$, and, by abuse of notation, at the same time $\mathfrak{Q}_p$ is a quadric threefold in $\mathbb{P}(C \oplus (L^\vee \otimes M^\vee))$. Therefore, the quadrics in the hyperplane $H_\mathfrak{Q}$ with $\mathfrak{Q} \in I_{D_{(L,M)},2}$ define the pair (L,M) and hence also $\mathfrak{Q}$, so f has an inverse.

We choose coordinates such that Y is the intersection of $C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2))$ with a quadric $\{z^2 - Q' = 0\}$, where $\{Q' = 0\}$ is a cone with vertex at the vertex of $C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2))$, and z is nonzero at the vertex.

The pencil $I_{D_{(L,M)},2}$ contains in general 5 rank 4 quadrics. One is the rank 4 quadric $C(\mathbb{P}(L^\vee) \times \mathbb{P}(M^\vee))$. The planes in this quadric intersect $D_{(L,M)}$ in conics that are contracted, by the projection to either $\mathbb{P}(U_1)$ or $\mathbb{P}(\wedge^2 U_2)$, so these are not $(1,1)$-conics. A plane in any of the other singular quadrics in $I_{D_{(L,M)},2}$, intersects $D_{(L,M)}$ in a $(1,1)$-conic. When $\mathfrak{Q} \in I_{D_{(L,M)},2}$ has rank 4, the fiber $\psi_Q^{-1}([H_\mathfrak{Q}])$ is therefore two $\mathbb{P}^1$'s of conics defined on $D_{(L,M)}$ by the two pencils of planes in $\mathfrak{Q}$. The two pencils coincide precisely when $\mathfrak{Q}$ has rank 3.

The double cover $Y \to \mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)$ is branched along the Verra threefold $Z = Y \cap \{z = 0\}$. It defines an involution on Y, that for each (L,M) restricts to an involution on $D_{(L,M)}$ and on each threefold quadric $\mathfrak{Q}$, where $\mathfrak{Q} \in I_{D_{(L,M)},2}$. In particular, when $\mathfrak{Q}$ has rank 4, the two pencils of planes in the quadric are interchanged by this involution.

Finally, when $\mathfrak{Q} \in I_{D_{(L,M)},2}$ has rank 3, then $D_{(L,M)}$ is singular in two points on the vertex of $\mathfrak{Q}$. So the double cover

$$D_{(L,M)} \to \mathbb{P}(L^\vee) \times \mathbb{P}(M^\vee) \subset \mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)$$

is branched along a curve with two singular points, i.e. a pair of conics $C \cup C'$, corresponding to the fixed points of the involution on the pencil of planes in $\mathfrak{Q}$. The pair of conics $C \cup C'$ lies in the hyperplane $\{z = 0\}$, i.e. in the Verra threefold $Z = Y \cap \{z = 0\}$. Conversely, a $(1,1)$-conic C in Z is mapped to a pair of lines $\mathbb{P}(L^\vee)$ and $\mathbb{P}(M^\vee)$ and is a component of the ramification locus of the double cover $D_{(L,M)} \to \mathbb{P}(L^\vee) \times \mathbb{P}(M^\vee)$. The other component C' is also a $(1,1)$-conic contained in Z and C and C' intersect in a scheme of length 2. The complete intersection $D_{(L,M)}$ is singular along this scheme, which is the intersection of the vertex of a rank 3 quadric $\mathfrak{Q} \in I_{D_{(L,M)},2}$ and $D_{(L,M)}$.

Thus, we have identified the set of pairs $([H_\mathfrak{Q}],(L,M)) \in D_2^Q$ where $\mathfrak{Q}$ has rank 3 with the set of pairs of $(1,1)$-conics $C \cup C'$ in Z that intersect in a scheme of length 2.

By Lemma 3.3, we infer $f(D_2^Q) = D_2^{\bar{A}_{Q'}}$. Summing up, we have precisely described the Stein factorization of the map ψ_Q.

Proposition 3.5. *The Stein factorization of ψ_Q is*

$$\psi_Q = \phi \circ \rho$$

with $\phi : F(Y) \to X_Q$ a $\mathbb{P}^1$ fibration and $\rho : X_Q \to D_1^{\bar{A}_{Q'}}$ a 2:1 cover branched precisely in $D_2^{\bar{A}_{Q'}}$.

Moreover, we have proven the following relation between the singular locus of an EPW quartic and its associated Verra threefold.

Proposition 3.6. *Let $Y = Q \cap C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2))$ be a general Verra fourfold and let $Z = (\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2)) \cap Q' = Y \cap \{z = 0\}$ be its associated Verra threefold. Then the map*

$$\psi_Q|_{F(Z)} : F(Z) \to C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2))$$

is an étale $2:1$ map to the set $D_2^{\bar{A}_{Q'}} \subset C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2))$. Thus the singular set of a general EPW quartic section admits an étale double cover being the Hilbert scheme of conics on the corresponding Verra threefold Z.

Finally, Theorem 0.2 appears also as a direct consequence of the above arguments.

Proof of Theorem 0.2. Let $X_{\bar{A}} \in \mathcal{U}$, then $X_{\bar{A}}$ is a double cover of $D_1^{\bar{A}}$ for some Lagrangian $\bar{A} \subset ((\wedge^2 U_1) \otimes U_2) \oplus (U_1 \otimes (\wedge^2 U_2))$. Let $Q_{\bar{A}} \subset \mathbb{P}((U_1 \otimes (\wedge^2 U_2))$ be the corresponding quadric and $Z_{\bar{A}}$ be the corresponding Verra threefold and $Y_{\bar{A}}$ the corresponding Verra fourfold. Then both $X_{Q_{\bar{A}}}$ and $X_{\bar{A}}$ appear as double covers of $D_1^{\bar{A}}$ branched in $D_2^{\bar{A}}$ hence are isomorphic. It follows that $X_{\bar{A}}$ is the base of a $\mathbb{P}^1$ fibration on $F(Y_{\bar{A}})$. For the converse we just need to recall that there is a 1:1 correspondence between general Lagrangian subspaces $\bar{A}$ and general quadrics $Q_{\bar{A}}$. □

Remark 3.7. Observe that if V_6 a 6-dimensional vector space and $[A] \in \mathrm{LG}(10, \wedge^3 V_6)$ such that $\mathbb{P}(A) \cap \mathrm{G}(3, V_6) = \{[U_1]\}$ then to A we associate a unique EPW quartic section $D^1_{\bar{A}}$ and also a unique Verra fourfold V_A. The Verra fourfold appears as follows. First, for a fixed choice of $[U_2] \in \mathrm{G}(3, V_6)$ such that $U_2 \cap U_1 = \{0\}$ consider

$$q_{A,U_2} : T_{[U_2]}/ < [U_2] > \to (T_{[U_1]}/ < [U_1] >) = (T_{[U_2]}/ < [U_2] >)^\vee$$

the symmetric map whose graph is $A/ < [U_1] >$ and let Q_{A,U_2} be the corresponding quadric. Let $C_{U_2} = T_{[U_2]} \cap \mathrm{G}(3, U)$ and $P_{U_2} = \mathbb{P}(\wedge^2 U_2) \times \mathbb{P}(U_1)$ be the corresponding Segre embedding

$$\mathbb{P}^2 \times \mathbb{P}^2 \subset T_{[U_2]}/ < [U_2] > \simeq (T_{[U_1]}/ < [U_1] >)^\vee.$$

Define $Z_{A,U_2} = P_{U_2} \cap Q_{A,U_2}$ the Verra threefold associated to A and U_2 and V_{A,U_2} the corresponding Verra fourfold. We claim that in fact Z_{A,U_2} (and in consequence V_{A,U_2}) is independent from the choice of U_2. Indeed, if we choose a different $[U_2'] \in \mathrm{G}(3, V_6)$ then we have a canonical isomorphism $T_{[U_2']}/ < [U_2'] > \simeq (T_{[U_1]}/ < [U_1] >)^\vee \simeq T_{[U_2]}/ < [U_2] >$ induced by the symplectic form and under this identification we have $Q_{A,U_2} - Q_{A,U_2'} \in H^0(I_{P_{U_2}}(2)) = H^0(I_{P_{U_2'}}(2))$.

3.1. **Two Lagrangian fibrations.** Observe that a general double EPW quartic section X admits two fibrations. Indeed, consider the composition of maps $X_{\bar{A}} \to D_1^{\bar{A}} \subset C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2)) = C(\mathbb{P}^2 \times \mathbb{P}^2)$, with $D_1^{\bar{A}}$ the EPW quartic section defined by the Lagrangian subspace $\bar{A} \subset (\wedge^3 U_1)^{\perp}/(\wedge^3 U_1)$. The projections to the factors of $\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2)$ induces two fibrations π_1 and π_2. Since $X_{\bar{A}}$ is IHS the fibers are abelian surfaces. Let us study these fibrations in more details. We shall consider the fibration of the EPW quartic section $D_1^{\bar{A}} \subset C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2))$.

Proposition 3.8. *The general fibers of the two natural fibrations* $\pi_1 : D_1^{\bar{A}} \to \mathbb{P}(\wedge^2 U_1)$ *and* $\pi_2 : D_1^{\bar{A}} \to \mathbb{P}(U_2)$ *of the EPW quartic section* $D_1^{\bar{A}}$ *are Kummer quartic surfaces.*

Proof. We consider the fibers of the second projection π_2, the fibers of π_1 are treated similarly. Let $v \in \mathbb{P}(U_2)$ be generic. Denote by $\mathbb{P}(V_2) \subset \mathbb{P}(\wedge^2 U_2)$ the line dual to v. This induces a subset $C(\mathbb{P}(U_1) \times \mathbb{P}(V_2)) \cap Q_{\bar{A}}$ of the corresponding Verra fourfold $C(\mathbb{P}(U_1) \times \mathbb{P}(\wedge^2 U_2)) \cap Q_{\bar{A}}$.

We can identify the fiber $\pi_2^{-1}(v)$ as the image by $\psi_{Q_{\bar{A}}}$ of the conics contained in $C(\mathbb{P}(U_1) \times \mathbb{P}(V_2)) \cap Q_{\bar{A}}$. It follows from Proposition 1.9 that this fiber is a Kummer surface. □

Remark 3.9. Note that from the adjunction formula π_1 and π_2 induces two Lagrangian fibrations on $X_{\bar{A}}$. The Kummer surfaces above can be seen as quotient of the Abelian surfaces in the fibers.

Remark 3.10. Note that also the original description of the EPW quartic section as a Lagrangian degeneracy locus induces naturally a description of the Kummer quartic fibers as Lagrangian degeneracy loci in $\mathbb{P}^3$. That description is consistent with Lemma 1.1 in the following sense. We analyze both fibrations separately:

(1) The fibers of $\pi_2 : D_1^{\bar{A}} \to \mathbb{P}(U_2)$. We know that

$$D_1^{\bar{A}} \subset C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2))$$

hence a fiber D_{u_2} of the projection $\pi_2 : D_1^{\bar{A}} \to \mathbb{P}(U_2)$ of a point $[u_2] \in \mathbb{P}(U_2)$ is the intersection of

$$P_{[u_2]} = \mathbb{P}((\wedge^3 U_1) \oplus ((\wedge^2 U_1) \otimes u_2))) \cap \mathrm{G}(3, U_1 \oplus U_2) = C(\mathbb{P}(\wedge^2 U_1)),$$

with the Lagrangian degeneracy locus $D_1^{\bar{A}}$:

$$D_{u_2} = P_{[u_2]} \cap \bar{D}_1^{\bar{A}} = \{[U] \in P_{[u_2]} \cap \mathrm{G}(3, \wedge^2 U_1 \oplus U_2) | \dim(\bar{T}_U \cap \bar{A}) \geq 1\}.$$

Let

$$K_4 = U_1 \oplus \langle u_2 \rangle.$$

then

$$\wedge^3 K_4 = \wedge^3 U_1 \oplus ((\wedge^2 U_1) \otimes u_2))) \subset \wedge^3 V.$$

Thus, for all $[U] \in P_{u_2}$ we have $\wedge^3 U \subset \wedge^3 K_4$ and

$$T_U = ((\wedge^2 U) \wedge V) \supset (\wedge^3 K_4)$$

Since T_U is Lagrangian with respect to the wedge product form on $\wedge^3 V$, we have $T_U \subset (\wedge^3 K_4)^{\perp}$.

Consider the 12-dimensional quotient space $(\wedge^3 K_4)^{\perp}/(\wedge^3 K_4)$, with the nondegenerate 2 form induced by the wedge product form. Then

$$T_U/(\wedge^3 K_4) \subset (\wedge^3 K_4)^{\perp}/(\wedge^3 K_4)$$

is a Lagrangian subspace. The Lagrangian subspace $A \subset \wedge^3 V$ contains $\wedge^3 U_1$ so has a Lagrangian quotient $\bar{A} \subset (\wedge^3 U_1)^\perp/(\wedge^3 U_1)$. It follows that the image $\bar{A}_{K_4} := Im\varphi$ of the natural projection

$$\varphi : \bar{A} \cap (\wedge^3 K_4)^\perp/(\wedge^3 U_1) \to (\wedge^3 K_4)^\perp/(\wedge^3 K_4)$$

is an isotropic subspace.

By the genericity of $\bar{A}$, we have $\bar{A} \cap ((\wedge^3 K_4)/(\wedge^3 U_1)) = 0$ (for every $u_2 \in U_2$), so $\dim(\bar{A}_{K_4}) = 6$ and $\bar{A}_{K_4}$ is Lagrangian (for every $u_2 \in U_2$).

Finally for $[U] \in P_{u_2}$ i.e. $U \subset K_4$,

$$[U] \in D_{u_2} \iff \dim(\bar{T}_U \cap \bar{A}) \geq 1 \iff \dim((T_U/(\wedge^3 K_4)) \cap \bar{A}_{K_4}) \geq 1,$$

i.e. the fiber D_{u_2} is a Lagrangian degeneracy locus associated to the family of Lagrangian subspaces

$$\{T_U/(\wedge^3 K_4) | U \subset K_4\}$$

and the fixed space $\bar{A}_{K_4}$ as Lagrangian subspaces of $(\wedge^3 K_4)^\perp/(\wedge^3 K_4)$.

With a choice of decomposition $V = K_4 \oplus K_2$ we may identify

$$(\wedge^3 K_4)^\perp/(\wedge^3 K_4) = (\wedge^2 K_4) \otimes K_2 \subset \wedge^3 V,$$

and identify the 6-dimensional subspace $\bar{A}_{K_4}$ with a Lagrangian subspace in $(\wedge^2 K_4) \otimes K_2$, finally we identify:

$$T_U = (\wedge^2 U) \otimes K_2 \subset (\wedge^2 K_4) \otimes K_2.$$

In this context $D_{u_2} \subset \mathbb{P}(K_4^\vee)$ is the first degeneracy locus

$$\{[U] \in \mathbb{P}(K_4^\vee) | \dim((\wedge^2 U) \otimes K_2) \cap \bar{A}_{K_4}) \geq 1\}.$$

This degeneracy locus was described in Section 1 as a Kummer quartic singular in 16 points given by:

$$\{[U] \in \mathbb{P}(K_4^\vee) | \dim((\wedge^2 U) \otimes K_2) \cap \bar{A}_{K_4}) \geq 2\}.$$

(2) Consider next a fiber of the first projection $\pi_1 : D_1^{\bar{A}} \to \mathbb{P}(\wedge^2 U_1)$ from the Lagrangian degeneracy locus

$$D_1^{\bar{A}} \subset C(\mathbb{P}(\wedge^2 U_1) \times \mathbb{P}(U_2)).$$

Let $M_2 \subset U_1$ be a 2- dimensional subspace and denote by D_{M_2} the fiber $\pi_1^{-1}([\wedge^2 M_2])$.

Let $U \supset M_2$ be 3-dimensional subspace of V, then

$$\wedge^3 U \in \wedge^2 M_2 \wedge V \subset \wedge^3 V.$$

Thus we may identify the sets $\{[U] \in \mathrm{G}(3,V) | U \supset M_2\} = P_{[M_2]}$ where

$$P_{[M_2]} := \mathbb{P}(\wedge^2 M_2 \wedge V) \subset \mathbb{P}(\wedge^3 V)$$

The fiber $\pi_1^{-1}([\wedge^2 M_2])$ is then

$$D_{M_2} = D_1^{\bar{A}} \cap P_{[M_2]}.$$

Notice that for each $[U] \in P_{M_2}$,

$$\wedge^2 M_2 \wedge V \subset T_U = ((\wedge^2 U) \wedge V).$$

In particular

$$\wedge^2 M_2 \wedge V \subset T_{U_1} = \wedge^3 U_1 \oplus ((\wedge^2 M_2) \otimes U_2) \subset \wedge^3 V.$$

Since T_U is Lagrangian with respect to the wedge product form on $\wedge^3 V$, we have $T_U \subset (\wedge^2 M_2 \wedge V)^\perp$. Consider the 12-dimensional quotient space

$$(\wedge^2 M_2 \wedge V)^\perp/(\wedge^2 M_2 \wedge V),$$

with the nondegenerate 2-form induced by the wedge product. Then

$$T_U/(\wedge^2 M_2 \wedge V) \subset (\wedge^2 M_2 \wedge V)^\perp/(\wedge^2 M_2 \wedge V)$$

is a Lagrangian subspace. The Lagrangian subspace $A \subset \wedge^3 V$ contains $\wedge^3 U_1$, so has a Lagrangian quotient $\bar{A} \subset \wedge^3 U_1^\perp / \wedge^3 U_1$. So the subspace $\bar{A} \cap ((\wedge^2 M_2 \wedge V)^\perp/(\wedge^3 U_1))$ of $\bar{A}$ is isotropic. The projection

$$\phi : \bar{A} \cap ((\wedge^2 M_2 \wedge V)^\perp/(\wedge^3 U_1)) \to (\wedge^2 M_2 \wedge V)^\perp/(\wedge^2 M_2 \wedge V)$$

therefore has an image

$$Im\,\phi := \bar{A}_{M_2} \subset (\wedge^2 M_2 \wedge V)^\perp/(\wedge^2 M_2 \wedge V).$$

which is isotropic. By the genericity of $\bar{A}$ we have $\bar{A} \cap ((\wedge^2 M_2 \wedge V)/(\wedge^3 U_1)) = 0$ (for every $M_2 \subset U_1$), so $\dim \bar{A}_{M_2} = 6$, and $\bar{A}_{M_2}$ is in fact a Lagrangian subspace for every $M_2 \subset U_1$.

Finally for $[U] \in P_{M_2}$ i.e. $U \supset M_2$,

$$[U] \in D_{M_2} \iff \dim(\bar{T}_U \cap \bar{A}) \geq 1 \iff \dim((T_U/((\wedge^2 M_2 \wedge V)) \cap \bar{A}_{M_2}) \geq 1,$$

i.e. the fiber $D_{M_2} \subset P_{[M_2]}$ is the first Lagrangian degeneracy locus associated to the family of Lagrangian subspaces

$$\{T_U/(\wedge^2 M_2 \wedge V) | U \supset M_2\}$$

and the fixed space $\bar{A}_{M_2}$ as Lagrangian subspaces of $(\wedge^2 M_2 \wedge V)^\perp/(\wedge^2 M_2 \wedge V)$, i.e.

$$D_{M_2} = \{[U] \in P_{[M_2]} | \dim(\wedge^2 U \wedge V/((\wedge^2 M_2) \wedge V)) \cap \bar{A}_{M_2}) \geq 1\}.$$

If we set $V_2 = M_2$ and decompose $V = V_2 \oplus V_4$, then

$$(\wedge^2 M_2 \wedge V)^\perp/(\wedge^2 M_2 \wedge V) \cong (\wedge^2 V_2 \wedge V_4)^\perp / \wedge^2 V_2 \wedge V_4 \cong V_2 \otimes \wedge^2 V_4.$$

On the one hand we can identify the space $\{U \supset M_2\}$ with $\{\langle v \rangle = U \cap V_4\}$. If A' is the Lagrangian subspace corresponding to $\bar{A}_{M_2}$ via these isomorphisms, then the fiber $\pi_1^{-1}([\wedge^2 M_2])$ is isomorphic to

$$D_{M_2} \cong \{[v] \in \mathbb{P}(V_4) | \mathrm{rank} A' \cap (V_2 \otimes V_4 \wedge v) \cap A') \geq 1\}.$$

This degeneracy locus is the Lagrangian degeneracy locus $D_1^{A'}$ of Lemma 1.1 and is a Kummer quartic surface singular in 16 points.

Corollary 3.11. *The general fibers of the fibrations $\mathbb{P}(\wedge^2 U_1) \leftarrow X_{\bar{A}} \to \mathbb{P}(U_2)$ are abelian surfaces. The projections factor through the double cover $X_{\bar{A}} \to D_1^{\bar{A}}$, which for each fiber is the double cover of a Kummer quartic surface branched in its 16 singular points.*

4. The third construction-moduli space of twisted sheaves

It was observed by G. Mongardi, that the generic element from the family $\mathcal{U}$ can be constructed as the moduli space of twisted sheaves on a $K3$ surface of degree 2. We know that the generic element from the family $X \in \mathcal{U}$ admits two Lagrangian fibrations $\pi_i : X \to \mathbb{P}^2$. In particular, we obtain two sextic curves as discriminant curves of the fibrations on the bases. The double cover of $\mathbb{P}^2$ branched along a curve of degree 6 is a $K3$ surface of degree 2. For a given X we can associate naturally two such surfaces. Those will be naturally the base of our moduli space of stable twisted sheaves.

Recall that the moduli space of stable twisted sheaves were described by Yoshioka in [Yos06]. In order to construct such a moduli space $M_v(S, \alpha) = M(v)$ we need to fix a $K3$ surface S with an element α in the Brauer group $Br(S)$ and a Mukai vector v. Recall that for a $K3$ surface we have

$$Br(S) = \mathrm{Hom}(T_S, \mathbb{Q}/\mathbb{Z}),$$

where

$$T_S = NS(S)^\perp = \{v \in H^2(S,\mathbb{Z}) \colon \ \forall\ m \in NS(S) \quad v \cdot m = 0\}$$

is the transcendental lattice of S. For a cyclic element $\alpha \in Br(S)$ of order n denote by

$$T_{\langle\alpha\rangle} = ker(\alpha \colon T_S \to \mathbb{Q}/\mathbb{Z}) \subset T_S$$

the sublattice of index n.

Now let S be a general $K3$ surface of degree 2 such that $NS(S) = \mathbb{Z}h$. In [vG05] van Geemen classified the order 2 elements in $Br(S)$ by classifying the possible index two sublattices of $T_S = \langle -2\rangle \oplus 2U \oplus 2E_8(-1)$ and found three possibilities. Recall that an element of order n in the group $Br(S)$ can be represented as a Brauer-Severi variety being a rank n bundle on S. As suggested to us by A. Kresch it is convenient to look at the three geometric realizations of the order two elements in $Br(S)$ in the following way:

- a $(2,2,2)$-complete intersection is the base locus of a net of quadric 4-folds, then the space of planes in the quadric 4-folds is generically $\mathbb{P}^3 \cup \mathbb{P}^3$ and over the sextic discriminant curve is just $\mathbb{P}^3$; Therefore it is a $\mathbb{P}^3$-bundle over the double cover of $\mathbb{P}^2$ branched along the discriminant sextic (an element of $Br(S)_2$ is also an element from $Br(S)_4$ so gives a rank 4 bundle).
- for the cubic fourfold containing a plane, the projection from the plane yields a quadric surface fibration over $\mathbb{P}^2$, the discriminant locus is a sextic curve, the space of lines in the quadrics give a $\mathbb{P}^1$-bundle over the double cover branched along the sextic.
- the double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ branched along a $(2,2)$ hypersurface is a quadric surface bundle (by the projection to the first factor), the discriminant locus is a sextic and the spaces of lines give a $\mathbb{P}^1$-bundle over the corresponding double cover.

We are interested in the last case, discussed in detail in [vG05, §9.8]. Then

$$T_{\langle\alpha_3\rangle} = \langle -2\rangle \oplus U \oplus U(2) \oplus 2E_8(-1)$$

is Hodge isometric to a primitive sublattice of the middle cohomology of the Verra fourfold. Note that $T_{\langle\alpha_3\rangle}$ admits two embeddings as an index 2 sublattice of T_S. Note also that Hassett and Varilly-Alvarado [HVA13] showed that the Brauer elements α_3 that we consider can obstruct the Hasse principle.

Proposition 4.1. *Let $X \in \mathcal{U}$ be general then X is isomorphic to the moduli space of stable twisted sheaves on a $K3$ surface of degree 2 with twist $\alpha_3 \in Br(S)$.*

Proof. Since the Picard group of X has rank two and X admits two Lagrangian fibrations it follows that the movable cone of X is isomorphic to the nef cone. Thus it is enough to prove that X is birational to the moduli space of twisted sheaves.

We argue similarly as [Add16] using the global Torelli theorem [Ver13]. Let us use the notation from [Huy15, Proposition 4.1]. We have to show that there is an embedding $H^2(X,\mathbb{Z}) \hookrightarrow \tilde{H}(S,\alpha_3)$ (into the Hodge structure of the twisted $K3$ surface see [Huy15, Definition 2.5]) that is compatible with the Hodge structure. Given the embedding, we find a vector $v \in \tilde{H}(S,\alpha_3)$ in the orthogonal complement of the image of $H^2(X,\mathbb{Z})$ having $(v,v) = 2$. For such v let $M(v) = M_v(S,\alpha_3)$ be the moduli space of stable twisted sheaves on S. We know from [Yos06, Theorem 3.19] that there is a distinguished embedding

$$H^2(M(v),\mathbb{Z}) \simeq v^\perp \hookrightarrow \tilde{H}(S,\alpha_3).$$

We deduce an isomorphism $H^2(X,\mathbb{Z}) \simeq H^2(M(v),\mathbb{Z})$ and conclude by the global Torelli theorem for deformations of $K3^{[2]}$ [Mar11, Corollary 9.8] that $M(v)$ and X are birational.

We denote the hyperbolic plane by U, i.e. $\mathbb{Z}^2$ with the intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Denote by

$$\tilde{\Lambda} = I \oplus J \oplus M \oplus N \oplus 2E_8(-1),$$

where I, J, M, N are copies of the hyperbolic lattice U. We can assume (by choosing an appropriated marking) that $H^2(X,\mathbb{Z}) = \Lambda \oplus \langle -2\rangle = \eta^\perp \subset \tilde{\Lambda}$, where $\Lambda \simeq 3U \oplus E_8(-1)$ and η is some element with $\eta^2 = 2$ contained in I. We know that the Hodge structure on $H^2(X,\mathbb{Z}) \otimes \mathbb{C}$ is determined by the choice of

$$x := \langle H^{2,0}(X)\rangle$$

such that the algebraic part $H^{1,1}(X)$ of $x^\perp$ contains the lattice $U(2) \subset \tilde{\Lambda}$. We are thus in the context of [Huy15, Lemma 2.6]. From the improved Eichler's criterion [BHPVdV04, thm I.2.9] it follows that there is a unique (up to $O(\tilde{\Lambda})$) embedding of a lattice of type $U(2) \oplus \langle \eta \rangle$ into $\tilde{\Lambda}$. In particular, we can assume that $U(2) \subset M \oplus N$ such that if u_1, u_2 and v_1, v_2 are standard generators of M and N respectively, then the image of the embedding is defined by $e = u_1 + v_1, f_2 = u_2 + v_2$. We find that the lattice generated by $\langle e, f\rangle$, where $f = u_2$ (or $f = v_2$) spans a hyperbolic plane U. We obtain a new special decomposition

$$\tilde{\Lambda} = U \oplus \Lambda \tag{4.1}$$

(this decomposition is different from the one in the definition of $\tilde{\Lambda}$). Since x is orthogonal to e, it admits a decomposition $x = \lambda e + \sigma$ with respect to (4.1) with $\sigma \in \Lambda \otimes \mathbb{C}$ and $\lambda \in \mathbb{C}$.

By the surjectivity of the period map we can find a $K3$ surface S that realizes $\sigma \in \Lambda \otimes \mathbb{C}$ (we have two such $K3$ surfaces). We claim that S admits a polarization of degree 2. Indeed, observe that $\eta \in I$, so we have $\eta \in \Lambda$. Moreover,

$$0 = (\eta.x) = (\eta.(\lambda e + \sigma)) = (\eta.\sigma) = (\eta.\bar{\sigma})$$

and $\eta^2 = 2$. It follows that η induces a polarisation of degree 2 on S; the claim follows.

Let us identify the twist. As in [Huy15, Lemma 2.6] we decompose with respect to (4.1) the element $f_2 = \gamma + 2f + ke$ with $\gamma \in \Lambda$. We compute that $\gamma = v_2 - u_2 \in \Lambda$ and denote $B := \frac{1}{2}\gamma = \frac{1}{2}(v_2 - u_2)$. We define the Brauer class $\alpha'_3 \in Br(S)$ as the image of B under the exponential map

$$\Lambda \otimes \mathbb{Q} \simeq H^2(S,\mathbb{Q}) \to H^2(S,\mathcal{O}_S^*)_{tors} \simeq Br(S).$$

Finally, if we identify $\mathbb{Z}e \subset U \subset \Lambda \oplus U$ (with respect to (4.1)) with $H^4(S,\mathbb{Z})$ we obtain an isometry with $\tilde{H}(S,\alpha'_3)$ and the Hodge structure determined by x on $\tilde{\Lambda}$ (and a second isometry for $f = v_2$).

In order to identify the element α'_3 with the element α_3 described above, we use [vG05, §2.1]. Indeed, we associate to B a map $b\colon T_S \to \mathbb{Q}/\mathbb{Z}$, where $T_S \subset \Lambda$ is the perpendicular lattice to η (in particular $T_S \simeq \langle -2\rangle \oplus 2U \oplus 2E_8(-1) = \langle -2\rangle \oplus \Lambda'$ and $B \in \Lambda'$), such that $b(t) = [t.B]$ (in particular $a_{\alpha'_3} = 0$ and $d = 1$ in the notation of [vG05, Proposition 9.2]) i. e. we are in the case of [vG05, Proposition 9.8]. □

Remark 4.2. By analogy with the generic cubic fourfold containing a plane we expect that the generic Verra fourfold is not rational. We hope that the IHS fourfold from $\mathcal{U}$ related to a Verra fourfold can be used to attack this problem [MS12].

Remark 4.3. We saw in the proof above that X admits two structures of moduli spaces of stable twisted sheaves on $K3$ surfaces of degree 2. The elements of these moduli spaces are torsion sheaves that are supported on curves on the linear system of degree 2 on S, so define two Lagrangian fibrations.

We are now ready to give a proof of our main Theorem.

Proof of Theorem 0.1. It follows from Theorem 2.9 and Proposition 2.13 that the Lagrangian degeneracy locus $D_1^{\bar{A}}$ admits a double cover branched along $D_2^{\bar{A}}$ being an IHS fourfold of type $K3^{[2]}$ such that the double cover $X_{\bar{A}} \to D_1^{\bar{A}}$ is given by an anti-symplectic involution. Moreover, $X_{\bar{A}}$ moves in a 19-dimensional family. It follows from [OW13] that the invariant lattice of the involution is one of the lattices U, $U(2)$, $\langle -2\rangle \oplus \langle 2\rangle$. On the other hand $X_{\bar{A}}$ admits a polarisation of Beauville degree 4 and, from Section 3.1, two Lagrangian fibrations: thus the invariant lattice is $U(2)$.

The fact that the Hilbert scheme of conics on Y admits a $\mathbb{P}^1$ fibration with base a fourfold from $\mathcal{U}$, follows from Proposition 3.6. The isomorphism with the moduli space of twisted sheaves follows from Proposition 4.1. □

5. The Fano surface of the Verra threefold Z

Let $Z = (\mathbb{P}_1^2 \times \mathbb{P}_2^2) \cap Q$ be a general Verra threefold, and let $F = F(Z)$ be the Fano surface of conics of bidegree $(1,1)$ on Z, i.e.

$$F = \{[C] : C \subset Z \text{ is a conic}, C \cdot h_1 = C \cdot h_2 = 1\}$$

where the h_i is the pullback to Z of a line in $\mathbb{P}_i^2$. On F there is a natural regular involution

$$i : F \to F,\ [C] \mapsto [C'] = i([C])$$

described as follows: Since any $[C] \in F$ is of bidegree $(1,1)$ then C can degenerate only to a pair $C_o = L + M$ of intersecting lines, one of bidegree $(1,0)$ and the other of bidegree $(0,1)$. Indeed if $C_o = 2L$ is a double line, then the bidegree $\deg(C_o) = (2,0)$ or $(0,2)$, a contradiction. Let $p_i : Z \to \mathbb{P}_i^2, i = 1,2$, be the two projections. Then, for any $[C] \in F$ the projections

$$L_i = p_i(C) \subset \mathbb{P}_i^2, i = 1,2$$

are lines, and the conic C lies on the smooth quadric surface

$$S_2 = L_1 \times L_2 \subset \mathbb{P}_1^2 \times \mathbb{P}_2^2.$$

Since Z is a quadratic section $\mathbb{P}_1^2 \times \mathbb{P}_2^2 \cap Q$ and $C \subset Z$, then

$$S_2 \cap Z = S_2 \cap Q = C + C',$$

where also C' is a $(1,1)$-conic on Z. It is bisecant to C. The involution on F is defined by

$$i : [C] \mapsto [C'].$$

Clearly $[C] = i([C'])$, and C' is the unique conic on Z bisecant to C. The Fano surface $F = F(Z)$ of the general Z is smooth, the involution $i : F \to F$ is regular and has no fixed points; in particular the *quotient Fano surface*

$$F_0 = F_0(Z) = F/i$$

of Z is smooth [Ili97], [DIM11]. Note that, by Proposition 3.6, the quotient Fano surface F_0 is isomorphic to the singular locus of the EPW quartic section associated to the Verra fourfold Y being the double cover of $\mathbb{P}^2_1 \times \mathbb{P}^2_2$ branched in Z. In particular F_0 is isomorphic to the fixed locus of an antisymplectic involution on an IHS fourfold of $K3$ type from the family $\mathcal{U}$.

5.1. **The two conic bundle structures on Z and invariants of the Fano surface.** [Ver04], [Ili97]

Let $Z = (\mathbb{P}^2_1 \times \mathbb{P}^2_2) \cap Q$ be general. For a point $x \in \mathbb{P}^2_i$ denote by $C_x = p_i^{-1}(x)$ the fiber of p_i over x. If $x \in \mathbb{P}^2_1$ (resp. $x \in \mathbb{P}^2_2$) then $C_x \subset Z$ is a conic of bidegree $(0,2)$ (resp. of bidegree $(2,0)$). For the general Z any degenerate fiber C_x of any of the two projections p_i has rank two, i.e. $C_x = p_i^{-1}(x) = L'_x + L''_x$ is a pair of lines, intersecting at a point

$$f_i(x) = L'_x \cap L''_x = \text{ Sing } C_x,$$

and the discriminant curves

$$\Delta_i = \{x \in \mathbb{P}^2_i : C_x = L'_x + L''_x\} \subset \mathbb{P}^2_i$$

are smooth plane sextics, see [Ver04]. The maps

$$f_i : \Delta_i \to \mathbb{P}^8,\ x \mapsto f_i(x) = \text{ Sing } C_x, i = 1,2$$

are called the the Steiner maps of the conic fibrations p_i.

Let

$$\tilde{\Delta}_i = \{([L], x) : L \subset C_x, x \in \Delta_i\}$$

be the curve of components of degenerate fibers $p_i^{-1}(x) = C_x = L'_x + L''_x$ of p_i, $i = 1,2$. Let

$$\pi_i : \tilde{\Delta}_i \to \Delta_i,\ ([L], x) \mapsto x$$

$i = 1,2$ be the induced étale double covering, and let $\varepsilon_i \in Pic_2(\Delta_i)$ be the 2-torsion sheaf defining π_i. Then for $i = 1,2$ the two coverings π_i (resp. the two pairs $(\Delta_i, \varepsilon_i)$) define two Prym varieties

$$Pr_i = \text{Prym}(\Delta_i, \varepsilon_i)$$

which are both principally polarized abelian varieties (p.p.a.v.) of dimension $9 = g(\Delta_i) - 1$. Let also

$$J(Z) = H^1(\Omega^2_Z)^* / H_3(Z, \mathbb{Z})$$

be the principally polarized (p.p.) intermediate jacobian of Z. By the results of [Ver04], Pr_1 and Pr_2 are both isomorphic to each other and to $J(Z)$ as p.p.a.v..

Proposition 5.1. *Let $Z = W \cap Q \subset \mathbb{P}^8$ be a general Verra threefold. Then:*

(A) The Fano surface $F = F(Z)$ has invariants $K^2 = 576$, $c_2 = 312$, $p_g = 82$, $q = 9$.

(B) The quotient Fano surface $F_0 = F(Z)/i$ has invariants $K^2 = 288$, $c_2 = 156$, $p_g = 36$, $q = 0$.

Proof. The irregularity $q(F) = 9$ follows from the Abel-Jacobi isomorphism $Alb(F) \cong J(Z)$, from where $q(F) = h^{1,0}(F) = h^{2,1}(Z) = 9$.

We have seen that F_0 is isomorphic to the fixed locus of the an IHS fourfold from $\mathcal{U}$. Starting from this, we compute the invariants of $F(Z)$ and F_0. The facts that $K^2_{F_0} = 288$ and $\chi(\mathcal{O}_{F_0}) = 37$ follows from [Bea11] since Y moves in a 19 dimensional family. By Noether's formula, we infer $c_2(F_0) = 156$.

Now, by [Bea82], the class of F in $J(Y)$ is $[F] = 2\Theta^7/7!$, where Θ is the theta divisor of the principal polarization on $J(Y)$. Moreover from [Voi90, Corollary 3.17] the Abel-Jacobi map is surjective. Thus, by [Voi90, Corollary 3.18], we deduce that the invariant part $H^0(K_{F(Z)})^+$ of the involution i on $H^0(K_{F(Z)})$ has dimension 36. It follows that $p_g(F_0) = 36$.

Since $F(Z) \to F_0$ is a 2-sheeted unbranched covering, then

$$K^2_{F(Z)} = 2 \cdot K^2_{F_0} = 576,$$

$$c_2(F(Z)) = 2 \cdot c_2(F_0) = 2 \cdot 156 = 156,$$

and $\chi(O_F) = 2 \cdot \chi(O_F) = 37 \cdot 2 = 74$. Therefore, since $p_g(F_0) - q(F_0) + 1 = \chi(O_{F_0})$ then

$$q(F_0) = p_g(F_0) - \chi(O_{F_0}) + 1 = 36 - 37 + 1 = 0.$$

Thus we find $q(F_0) = 0$. □

Remark 5.2. Note that the Chow group $CH^0(F(Z))$ was studied in [Voi90, Proposition 1.1]. In particular, we find a description of the class $2([C] + [C']) \in CH^0(F(Z))$ where C and C' are two involutive conics.

Remark 5.3. Note that the Hilbert scheme of $(1,1)$ conics on Z was already studied in [Ver04, §6] and [DIM11]. Verra considers a natural birational map $u\colon F(Z) \to D^6_Z$, where D^6_Z is the intersection of the closure of the locus of rank 6 quadrics containing Z but not $\mathbb{P}^2 \times \mathbb{P}^2$ with the locus of quadrics containing $\mathbb{P}^2 \times \mathbb{P}^2$.

Note that in [DIM11] it is proved that each nodal prime Fano threefold X_{10} of degree 10 is birational to a Verra threefold Z_X. From [DIM11, Proposition 6.6., §5.4] the Fano surface of conics on X_{10} and $(1,1)$ conics Z_X are two birational Beauville special subvarieties S^{odd} and S^{even} respectively.

By [Bea82], the class of Z in $J(Z)$ is $[Z] = 2\Theta^7/7!$, where Θ is the theta divisor of the principal polarization on $J(Z)$. Also K_Z is numerically equivalent (on Z) to $2\Theta|_Z$. Therefore we recompute

$$K^2_Z = (2\Theta|_Z)^2 = (2\Theta)^2.2\Theta^7/7! = 8\Theta^9/7! = 8.9!/7! = 8.8.9 = 576$$

since the Abelian 9-fold $J(Z)$ is principally polarized by Θ, this yields $\Theta^9/9! = 1$.

Remark 5.4. We also have the following relation $K_{F_0} = 2H + e$ where H is the hyperplane section on $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$ and e is the torsion divisor defining the cover i.

References

[Add16] Nicolas Addington. On two rationality conjectures for cubic fourfolds. *Math. Res. Lett.*, 23(1):1–13, 2016.

[BCMS16] S. Boissière, Ch. Camere, G. Mongardi, and A. Sarti. Isometries of ideal lattices and hyperkhler manifolds. *International Math. Research Notices*, 2016.

[BCS16] S. Boissière, Ch. Camere, and A. Sarti. Classification of automorphisms on a deformation family of hyperkä hler fourfolds by p-elementary lattices. *Kyoto Journal of Mathematics*, 2016.

[BD85] Arnaud Beauville and Ron Donagi. La variété des droites d'une hypersurface cubique de dimension 4. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(14):703–706, 1985.

[Bea82] A. Beauville. Sous-varietes speciales des varietes de prym. *Compositio Math.* ., 265(45):357–383, 1982.

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.

[Bea11] Arnaud Beauville. Antisymplectic involutions of holomorphic symplectic manifolds. *J. Topol.*, 4(2):300–304, 2011.

[Bho86] Usha N. Bhosle. Nets of quadrics and vector bundles on a double plane. *Math. Z.*, 192(1):29–43, 1986.

[BHPVdV04] Wolf Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004.

[DIM11] O. Debarre, A. Iliev, and L. Manivel. On nodal prime fano threefolds of degree 10. *Sci. China Math.*, 54:1591–1609., 2011.

[DK15] O. Debarre and A. Kuznetsov. Gushel–mukai varieties: classification and birationalities. *arXiv:1510.05448*, 2015.

[Don77] Ron Y. Donagi. On the geometry of Grassmannians. *Duke Math. J.*, 44(4):795–837, 1977.

[EPW01] David Eisenbud, Sorin Popescu, and Charles Walter. Lagrangian subbundles and codimension 3 subcanonical subschemes. *Duke Math. J.*, 107(3):427–467, 2001.

[Huy15] Daniel Huybrechts. The k3 category of a cubic fourfold, 2015.

[HVA13] Brendan Hassett and Anthony Várilly-Alvarado. Failure of the Hasse principle on general $K3$ surfaces. *J. Inst. Math. Jussieu*, 12(4):853–877, 2013.

[HY15] Brendan Hasset and Tschinkel Yuri. Extremal rays and automorphisms of holomorphic symplectic varieties. *arXiv:1506.08153*, 2015.

[IKKR16] Atanas Iliev, Grzegorz Kapustka, Michal Kapustka, and Kristian Ranestad. Epw cubes. *J. Reine Angew. Math.*, 2016. published online DOI 10.1515/crelle-2016-0044.

[Ili97] Atanas Iliev. The theta divisor of bidegree $(2,2)$ threefold in $\mathbf{P}^2 \times \mathbf{P}^2$. *Pacific J. Math.*, 180(1):57–88, 1997.

[IM11] Atanas Iliev and Laurent Manivel. Fano manifolds of degree ten and EPW sextics. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(3):393–426, 2011.

[Mar11] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 257–322. Springer, Heidelberg, 2011.

[Mon15] Giovanni Mongardi. A note on the Kähler and Mori cones of hyperkähler manifolds. *Asian J. Math.*, 19(4):583–591, 2015.

[MS12] Emanuele Macrì and Paolo Stellari. Fano varieties of cubic fourfolds containing a plane. *Math. Ann.*, 354(3):1147–1176, 2012.

[MSTVA14] Kelly McKinnie, Justin Sawon, Sho Tanimoto, and Anthony Várilly-Alvarado. Brauer groups on k3 surfaces and arithmetic applications, 2014.

[Muk87] Shigeru Mukai. Moduli of vector bundles on $K3$ surfaces and symplectic manifolds. *Sūgaku*, 39(3):216–235, 1987. Sugaku Expositions **1** (1988), no. 2, 139–174.

[MW15] G. Mongardi and M. Wandel. Induced automorphisms on irreducible symplectic manifolds. *J. Lond. Math. Soc. (2)*, 92(1):123–143, 2015.

[Nie03] Marc A. Nieper. Hirzebruch-Riemann-Roch formulae on irreducible symplectic Kähler manifolds. *J. Algebraic Geom.*, 12(4):715–739, 2003.

[Nik80] V.V. Nikulin. Finite groups of automorphisms of kählerian surfaces of type k3. *Moscow Math. Soc.*, 38:71–137., 1980.

[O'G06] Kieran G. O'Grady. Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics. *Duke Math. J.*, 134(1):99–137, 2006.

[O'G13] Kieran G. O'Grady. Double covers of EPW-sextics. *Michigan Math. J.*, 62:143–184, 2013.

[OW13] Hisanori Ohashi and Malte Wandel. Non-natural non-symplectic involutions on symplectic manifolds of $k3^{[2]}$-type. *arXiv:1305.6353 [math.AG]*, 2013.

[PR97] P. Pragacz and J. Ratajski. Formulas for Lagrangian and orthogonal degeneracy loci; Q-polynomial approach. *Compositio Math.*, 107(1):11–87, 1997.

[Ver04] Alessandro Verra. The Prym map has degree two on plane sextics. In *The Fano Conference*, pages 735–759. Univ. Torino, Turin, 2004.

[Ver13] Misha Verbitsky. Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Math. J.*, 162(15):2929–2986, 2013. Appendix A by Eyal Markman.

[vG05] Bert van Geemen. Some remarks on Brauer groups of $K3$ surfaces. *Adv. Math.*, 197(1):222–247, 2005.

[Voi90] Claire Voisin. Relations dans l'anneau de Chow de la surface des coniques des variétés de Fano, et applications. *Amer. J. Math.*, 112(6):877–898, 1990.

[Yos06] Kōta Yoshioka. Moduli spaces of twisted sheaves on a projective variety. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 1–30. Math. Soc. Japan, Tokyo, 2006.

[Zak93] F. L. Zak. *Tangents and secants of algebraic varieties*, volume 127 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993. Translated from the Russian manuscript by the author.

Seoul National University, Department of Mathematics, Gwanak Campus, Bldg. 27, Seoul 151-747, Korea
E-mail address: ailiev@snu.ac.kr

Institut für Mathematik Mathematisch-naturwissenschaftliche Fakultät Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich

Institute of Mathematics of the Jagiellonian University, ul Łojasiewicza 6 Kraków, Poland
E-mail address: grzegorz.kapustka@uj.edu.pl

University of Stavanger, Department of Mathematics and Natural Sciences, NO-4036 Stavanger, Norway

Institute of Mathematics of the Jagiellonian University, ul Łojasiewicza 6 Kraków, Poland
E-mail address: michal.kapustka@uis.no

University of Oslo, Department of Mathematics, PO Box 1053, Blindern, N-0316 Oslo, Norway
E-mail address: ranestad@math.uio.no