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Preface

The objective with this thesis is to assess recent advances in decision theory under uncertainty and to find out whether these can be used as an extension to the decision analysis normally performed in the industry.

Oil companies put a lot of effort into reducing the uncertainties in their decision making and invest in extensive exploration and front end study activities prior to project sanction. After a project is sanctioned the companies select reliable contractors and monitor performance closely to reduce uncertainties and ensure predictable project execution. Decision analysis may be performed to support decision making at several of these key stages of a project development where identified uncertainties are described by the use of probability distributions and Monte Carlo simulations. The analysis results are then normally represented by an uncertainty range and an expected value of the observable quantities. A qualitative judgement process or management review process is then performed to define a margin that addresses the uncertainty in the results.

Recent advances in decision theory do however have a potential to extend the quantitative modelling approach beyond current decision analysis practise. In this thesis, recent advances in decision theory models have been assessed and an attempt is made to capture the uncertainties and define a single equivalent sure amount for various types of decision problems. The single equivalent sure amount value defined by these extended decision analysis models can potentially support rational decision making and serve as a potential improvement to the management review process of decisions associated with uncertainty.

There are numerous papers to be found that describe normative and descriptive theoretical models for decision analysis, but very few of these are able to translate their theory into practical applications. The paper from Borgonovo & Marinacci (2015) that describe decisions under ambiguity do however stand out as an exception to the rest. This excellent paper describes the theoretical basis and illustrates this theory with numerical analysis of relevant decision problems.

In-depth interviews of a selected group of decision makers are included in the qualitative research to assess whether the theoretical models of decision analysis associated with uncertainty are known and used by the industry. Scenarios were also included in the in-depth interviews in order to see how consistent decision makers are when subject to decision problems associated with various forms of uncertainty.

Decision problems have been analysed that includes uncertainties relating to business risk, cost risk, production risk and accident risk. The theory and methods for decision analysis under risk and ambiguity are intended to give useful information that support rational decision making and improve the basis for decision support.

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11.06.2018

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1. Introduction

We all have to make a lot of important decisions during our lifetime. These decisions are typically associated with uncertainties and can among other things be related to choice of friends, lifetime partner, education, career or where to settle down and when to retire. Some of these uncertainties are known phenomena that can be reasonably well defined by random variations. Other types of uncertainties are unknown and are not that easy to define by random variations based on the information available at the time of the decision. These unknown or un-measureable uncertainties can for example be related to the choice to move to a new city where you do not know anyone or the choice of an education where you do not have any prior knowledge. In business life, oil companies are also faced with these types of measureable and un-measureable uncertainties in their decision making. The measureable uncertainties are normally defined by the use of probability distributions that describe the random variation of the relevant phenomena. The un-measureable uncertainties are more difficult to describe by the use of probability distribution as these refers to unknown phenomena's as a result of lack of information. Consequently if more relevant information is made available, the un-measureable uncertainty can be reduced. Oil companies do therefore invest in several activities in order to reduce the un-measureable uncertainties in their decision making. Exploration and appraisal wells and seismic surveys are performed to collect as much information as possible about the location and size of a reservoir. Front end studies are performed to define the solution and cost of a field development project prior to project sanction. Information of potential contractor's previous experience and performance is gathered and evaluated by the oil companies prior to contract awards to ensure predictable project execution within time and cost.

Sometimes in our lives, we take wrong decisions due to ignorance or simply bad judgement. We then have to live with the consequences which could eventually be broken relationships, lost opportunities or other types of losses. In the business world, we also sometimes see that oil companies make decisions that result in project delivery failures or lost production. These decisions could also be a result of ignorance of the uncertainties or bad judgement of the potential outcomes. This could for example be in relation to the choice of novel platform concepts or the choice of contractors with limited or no experience with implementation of the function-based regulations specific for the Norwegian Continental Shelf (NCS) as defined by the NORSOK[1)] standards. We have in the past even seen projects consisting of novel platform concepts that are to be delivered by contractors with limited or no NORSOK experience. In Stavanger Aftenblad, 8th of May 2018, we can read about the decisions made in the 80's and 90's to place platform contracts to yards in Asia. These Asian yards had limited NORSOK experience and it was probably not known at the time how big impact this lack of NORSOK experience actually had on the project delivery. In retrospect we can see that there are large cultural differences between how work is managed and executed in a Norwegian and an Asian Yard.

1) *The acronym NORSOK stands for "the Norwegian shelf's competitive position". The NORSOK standards serve as references to the authorities regulations and have been in use since 1994 to ensure adequate safety, value adding and cost effectiveness within development and operation of petroleum assets and activities.*

The cultural differences will also influence how the functional requirements given in Norwegian rules and regulations are interpreted and implemented. Did the oil company consider that the prices and schedules from an Asian yard had the same predictability as for a Norwegian Yard? The oil companies were in these decision problems most likely faced with elements of un-measureable uncertainties in relation to the consequence of poor knowledge and understanding of Norwegian rules and regulations as defined by the NORSOK standards. These un-measureable uncertainties were probability not properly captured by the decision analysis available at the time. The consequences of some of these decisions were large cost overruns, significant delays and poor quality. A significant amount of rework that caused further delays also had to be done by Norwegian yards to ensure that the new platforms satisfied the mandatory functional requirements. There are also recent examples of platform projects where the implementation of mandatory functional requirements are suspended to the offshore location. These decisions to move large portions of the mechanical completion to the offshore location opens up for new both measurable and un-measureable uncertainties with an increased potential for cost overruns and delays.

One of the reasons that Norwegian Yards gets a higher share of platform contracts in the recent years could be that the un-measureable uncertainties with respect to the use of Asian yard are shifted to measureable uncertainties in recent decision analysis. Recent decision analysis can then describe random variations for this phenomena of measureable uncertainty based on the past experience of de facto negative outcomes.

With respect to choice of facility concepts we can also see that oil companies may have different preferences. A recent article in Dagens Næringsliv, 7th of May 2018, refers to a current discussion between two oil companies on the concept selection for a new field development. One of the license partners wants to have a single integrated process platform while the other license partner wants to have several smaller wellhead platforms. These different views could be caused by different economic drivers. The different views could also be caused by subjective preferences and experiences or biased views to the un-measureable uncertainties related to the execution and production of the proposed facility concepts. The decision analysis results will then largely depend on the respective oil company's assessments of the level of un-measureable uncertainties and how these are balanced with their subjective preferences and experiences.

The un-measureable uncertainties described in the above examples are, in a decision analysis context, commonly termed as ambiguity. Ambiguity can both refer to non-uniqueness, inability or lack of information to describe the uncertainty of a decision problem. The measureable uncertainties are, on the other hand, commonly termed as risk in a decision analysis context.

The objective with this thesis is to assess if quantitative models can be introduced to give preferences and margins when subject to both measureable uncertainties (risk) and un-measureable uncertainties (ambiguity). This assessment of quantitative models will be based on recent advances in decision theories and can potentially provide useful additional decision support information beyond current industry practice.

1.1 Background

A project manager may have to handle different types of decision problems and uncertainties in a project's life cycle. Common capital value processes are however implemented by the industry to capture uncertainty and ensure a staged maturing and definition of a project development. The common capital value processes include the decision gates DG1, DG2 and DG3. Decision gate DG1 is based on the uncertainties and the technical and cost accuracy described by the feasibility study for a project development. Decision gate DG2 is based on the uncertainties and the technical and cost accuracy described by the concept study for a project development. Decision gate DG3 is based on the uncertainties and the technical and cost accuracy described by the FEED study for a project development.

The types of decision problems that need to be assessed at the decision gates may involve economic risks and accidental risks (Aven, 2012). The decision process that the project manager is expected to follow is a value process for decision making under uncertainty (Aven, 2015). This decision process starts with a definition of the decision problem and a description of the decision alternatives. The next step in the decision process is to perform analysis and evaluations which may include risk analyses and decision analyses.

In decision analysis, the element of risk and uncertainty are assessed by the use of probability distributions and Monte Carlo simulations. The analysis results are normally represented by expected values and confidence intervals that describe an uncertainty range of the observable quantities. Normally the decision analysis stops with the description of the expected value and the uncertainty range and a qualitative judgement is performed to define a margin to address the uncertainty in the results. The risk and decision analyses and evaluations are subject to value based assessments in the form of a management review and judgement process and a stakeholders review process. The value based assessments are also evaluating and expected to take due account of uncertainties that are not covered by the risk and decision analyses. These uncertainties can lead to the implementation of additional precautionary or cautionary measures or other actions that will be part of the final risk picture which will form the basis for the decision making.

In this thesis, the objective is to go one step further in the quantitative decision analysis and include analytical models that define margins or premiums for risk and ambiguity. These margins or premiums can be used to define a single equivalent sure amount as additional information to the expected value and uncertainty range. This information about an equivalent sure amount can potentially be used as a guide for rational decision making when qualitative judgements are performed.

1.2 Scope and limitations

This thesis includes a detailed review of relevant decision theory, qualitative research in the form of peer group interviews and quantitative assessments of selected decision problems. The assessments performed are referring to the analytical and bureaucratic decision setting (Aven, 2012) where decisions are made based on strategic decision analysis which may include detailed processes of identifying alternatives and based on analysis of probabilities and consequences.

The concepts and theories relating to decision analysis with risk and ambiguity have been reviewed and included in section 2. References have been made to relevant sources of information consisting of articles and books found through the UiS library databases. The detailed review is performed with an emphasis on identifying quantiative models and methods that can be used by a project manager to perform a rational subjective assessment along with an estimate of the uncertainty for his decision alternatives. The relevant decision theories are selected based on the ability to identify preferences in outcome sets where the occurrence of only some of the outcomes have a defined probability. The other outcomes are then subject to ambiguity in the form of undefined probability. The descriptive theories of decision making is only briefly described as these need to be supported by extensive specific qualitative research that is beyond the scope of this thesis.

The research question that has been driving the qualitative research is basically to assess if decision theory under ambiguity is known and used by the industry. The quality research performed has been in the form of in-depth interviews of a selected peer group. The results of the qualitative research described in section 3 are based on a subjective assessment of the responses from the members of the peer group.

Quantiative assessments of selected decision problems in relation to economic risk and accident risk have been performed. In section 4, a business risk decision problem related to running or withdrawing from a car race has been analysed. In section 5, analysis is performed for a typical cost risk decision problem of defining the project contingency for a project cost estimate. In section 6, a production risk decision problem is analysed regarding the ranking of project development scenarios. In section 7, a decision problem associated with accident risk is included that analyse whether or not a subsea safety isolation valve (SSIV) need to be installed.

Analysis and discussions to the above assessments of decision theory and their application are included in section 8. A conclusion of the assessment of ambiguity in decision analysis is given in section 9.

2. Theoretical models for decision analysis with ambiguity

A literature review is performed to identify potential decision theories that address uncertainties in a decision problem. A detailed review and screening of the identified potential decision theories is done to see whether these can define preferences in a three-color decision problem (Ellsberg, 1961) with elements of both risk and ambiguity.

2.1 Literature review

The literature review identify the relevant theoretical basis for decision analysis under uncertainty and how these theories correspond and interface with general theories of risk and uncertainty. The literature review also identify the relevant basis and standards for decision problems in relation to business risk, production risk, cost risk and accidental risk.

2.1.1 Decision theories

The expected value method is commonly used in decision analysis and originates from portfolio theory (Markowitz, 1952), (Abrahamsen, et al., 2004), (Aven, 2012) . The expected value is represented by the average value of the portfolio. The variance of the portfolio is represented by the sum of the average variance of the non-systematic risk and the covariance of the systematic risk. The expected value method is recommended used in decision problems associated with low uncertainty (Soerskaar & Abrahamsen, 2017). The expected value method has remained as a pillar in value propositions and net-present value analysis (Bratvold & Begg, 2009), (Willigers, et al., 2017).

The expected utility theory (von Neumann & Morgenstern, 1947) and the subjective expected utility theory (Savage, 1954) also referred to as Bayesian decision theory (Lindley, 1985) have for a long period been the governing framework of decision theory when associated with rational decision making under uncertainty. The subjective expected utility was defined as a function of the subjective probabilities (Ramsey, 1931) and the utility functions (von Neumann & Morgenstern, 1947) for an outcome set. A reduced state space for the subjective expected utility was introduced by a two-staged lottery and horse-race definition of the subjective probabilities (Anscombe & Aumann, 1963).

The Wald maximin criterion (Wald, 1949) define the preferences for an extreme risk and ambiguity averse decision maker. The maximin criterion has been further generalised in maxmin expected utility which define a function of multiple priors and outcomes (Gilboa & Schmeidler, 1989).

The axioms and postulates that represent the basis for the subjective expected theory of rational decision making have been challenged over the years. Case studies were presented in 1953 by Allais and in 1961 by Ellsberg (1961) that showed that a decision maker may struggle to select his preferences when faced with un-measureable uncertainty in his decision problems.

Descriptive decision theories were developed to capture the cognitive thinking behind human action based on decision weighting and non-additive probabilities (Quiggin, 1982), (Schmeidler, 1989), (Kahneman & Tversky, 1992), (Tuthill & Frechette, 2002), (Hampel, 2009), (Aerts & Sozzo, 2015), (dos Santos, et al., 2018). It was found that people in both experimental and real life situations do not conform to the axioms and postulates that the expected utility and the subjective expected utility are based on (Quiggin, 1982). Anticipated utility theory (Quiggin, 1982) and the non-expected utility theories (Tuthill & Frechette, 2002); weighted expected utility, rank dependent utility, also called Choquet expected utility, and cumulative prospect theory was therefore introduced. These theories are associated with a weaker set of axioms than the expected utility axioms. The weaker axioms permits the use of decision weights of non-additive probabilities. Tests of the Choquet expected utility (Mangelsdorff & Weber, 1994) concluded that the Choquet expected utility was not superior to the expected utility theory when defining preferences in the Ellsberg three-color problem (Ellsberg, 1961).

The normative and descriptive decision theories seem to have very different perspectives. The normative decision theories are based on a set of axioms and postulates that define rational decision making (Howard, 1988). The descriptive decision theories are based on cognitive testing on how people actual make their decisions (Howard, 1988). The descriptive decision theories are therefore based on tests and case studies on how decision makers *do* behave as opposed to the normative decision theory that prescribe how a decision maker *should* behave. The independence axiom and the sure-thing principle defined in the normative decision theories are replaced by weaker axioms in the descriptive decision theories to align with tested human behaviour (Howard, 1988). As noted in Howard (1988), it is a descriptive fact that most of us can make mistakes in arithmetic calculations, but it is the normative rules of arithmetic that allows us to recognize a mistake. A similar relationship exists between normative and descriptive theories in decision analysis (Howard, 1988). The acceptance of the normative decision theory thus allows us to recognize our decision mistakes (Howard, 1988).

In recent decision theory literature (Klibanoff, et al., 2005), (Eichberger & Kelsey, 2007), (Gilboa & Marinacci, 2011), (Etner, et al., 2012), (Klibanoff, et al., 2012), (Cerreia-Vioglio, et al., 2013a), (Maccheroni, et al., 2013), (Borgonovo & Marinacci, 2015), (Hansen & Marinacci, 2016) uncertainties are divided into two categories. Uncertainties that can be defined by a probability distribution is termed “risk” while uncertainties where the decision maker is not able to specify a unique probability distribution is termed “ambiguity”.

The Smooth Ambiguity Functional (Klibanoff, et al., 2005) and the Extended Arrow-Pratt quadratic estimation method (Maccheroni, et al., 2013) are advances to the subjective expected utility. These are developed to incorporate the elements of both “risk” and “ambiguity” and the decision maker’s risk and ambiguity aversion in the analysis of a decision problem. The Smooth Ambiguity Functional (Klibanoff, et al., 2005) introduces second order probabilities of the predicted probability for an uncertain outcome. The second order probabilities are a probability distribution that introduces a variation in the definition of the predicted probability. The Smooth Ambiguity Functional calculates a risk and ambiguity premium for a decision problem (Klibanoff, et al., 2005). The Extended Arrow-Pratt Quadratic estimation (Maccheroni, et al., 2013) can be used to provide an estimate of the split between a premium of risk and a premium of ambiguity of the risk and ambiguity premium found by the Smooth Ambiguity Functional.

Quantitative methods are introduced (Borgonovo & Marinacci, 2015) by the use of the Smooth Ambiguity Functional (Klibanoff, et al., 2005) and the Extended Arrow-Pratt Quadratic estimation

(Maccheroni, et al., 2013) to resolve the decision maker's preferences in the Ellsberg three-color-problem (Ellsberg, 1961) and the Carter racing decision problem (Brittain & Sitkin, 1990).

2.1.2 The bridge between risk and decision theory

In risk analysis literature (Ramsey, 1931), (Kaplan & Garrick, 1981), (Aven, 2012), (Aven, et al., 2014), (Aven, 2014), (Aven, 2015), (Soerskaar & Abrahamsen, 2017) the term "risk" is recommended to have a broader definition. Risk is here defined by the two main dimensions consequences, (C), and uncertainties, (U) (Aven, 2015). The risk description is defined by specified consequences and a descriptive measure of the uncertainty (Q). Probability distributions are normally used as the descriptive measure of the uncertainty (Q), where subjective probabilities are assigned to both of the categories of uncertainty described in the decision theory literature.

A clear distinction is however made between "aleatory" and "epistemic" uncertainty in risk analysis (Aven, 2012), (Aven, et al., 2014). An aleatory uncertainty can be described by both subjective and frequentist probabilities in order to describe a natural variation of a phenomena. This type of phenomena can however, not necessarily be reduced as more information becomes available. An epistemic uncertainty can only be described by subjective probabilities as this uncertainty is a result of lack of information. The epistemic uncertainty can therefore be reduced if more information becomes available.

The aleatory and epistemic uncertainties defined in risk analysis do represent an important bridge between the risk analysis and the decision analysis that may result in improved decision support (Pate-Cornell & Dillon, 2006), (Pate-Cornell, 2007), (Borgonovo, et al., 2015), (Borgonovo, et al., 2016), (Borgonovo, et al., 2018).

2.1.3 Risk and ambiguity aversion

In the past, risk attitudes or risk tolerance have been subject to empirical studies and analysis within the petroleum industry (Spetzler, 1968), (Walls, et al., 1995), (Walls & Dyer, 1996), within large corporations (Howard, 1988), (Pate-Cornell & Fischbeck, 1992), (Smith, 2004) and within the health sector (Treich, 2010). A decision-theoretic status on risk attitudes is presented by Baccelli (2017).

Ambiguity and a decision maker's attitude towards ambiguity were found by the experiments introduced by Allais and Ellsberg (1961). Ambiguity refers to a decision situation under uncertainty when there is incomplete information about the likelihood of events (Eichberger & Kelsey, 2007). In the Ellsberg experiments it was found that the decision maker would have a preference for betting on events with defined probability distributions (Ellsberg, 1961), (Eichberger & Kelsey, 2007). The decision maker can be either ambiguity neutral or ambiguity averse (Etner, et al., 2012). The dominating behavior for a decision maker in the gain domain (upsides) is ambiguity aversion while the dominating behavior for a decision maker in the loss domain (downside) is ambiguity neutrality (Etner, et al., 2012). Negative attitude or aversion towards ambiguity do however not seem to hold in situations where the decision maker feels comfortable with the situation despite the presence of unknown probabilities (Eichberger & Kelsey, 2007).

Risk and ambiguity aversion are included in the recent advances in decision theory (Klibanoff, et al., 2005), (Maccheroni, et al., 2013), (Borgonovo & Marinacci, 2015).

2.1.4 Business risk decisions

The challenger launch decision (Brittain & Sitkin, 1990), (Liedtka, 1990) has been used by many organisations as a case study for training in decision making exposed to risk and ambiguity. The objective with this training in decision making is to address how actions derived from quantitative analysis are implemented using organisational mechanisms and behavioural interventions (Brittain & Sitkin, 1990).

Quantitative analysis performed for the Carter racing case (Borgonovo & Marinacci, 2015) is a stylised example of a business risk decision that refers to the Challenger Launch Decision.

2.1.5 Production risk decisions

Production risk refers to uncertainties associated with a certain performance measure (Aven, 2012) and can refer to potentially reduced production efficiency as a result of downtime caused by equipment failures and maintenance or as a result of subsurface production issues. The production system for a field development consists of complex subsystems for reservoir, wells and facilities. These subsystems are typically treated independently in both design and operations (Chow & Arnondin, 2000) and the relevant components of the system have historically been optimised on the basis of the local subsystem instead of the overall global production system (Chow & Arnondin, 2000). Risk based integrated production models are introduced (Chow & Arnondin, 2000), (Chow, et al., 2000) (Fassihi, et al., 2000) to quantify and manage uncertainty associated with field-development design, implementation and operation.

2.1.6 Cost risk decisions

Cost risk refers to uncertainties associated with project cost estimates (Aven, 2012). Cost risk has been analysed in cost engineering forums with the objective to improve the predictability of project cost estimates (Dillon, et al., 2002), (Burger, 2003), (Sauser, et al., 2009), (Howell, et al., 2010), (Olumide, et al., 2010), (Idrus, et al., 2011), (van Niekerk & Bekker, 2014). Standard practices on the definitions and guidelines on the use of allowance, contingency and reserves in project cost estimates for the building industry have been developed (ASTM-E1946, 2012), (ASTM-E1369, 2015), (ASTM-E2168, 2016). Deviation in the definition of allowance is found between standard practice (ASTM-E2168, 2016) and the previous paper presented by Karlsen & Lereim (2005).

2.1.7 Accident risk decisions

An extended risk and performance perspective is recommended (Aven, 2014) to address accidental risk associated with large uncertainties and potential high consequences. An extreme safety perspective is defined for decisions associated with large uncertainties and potential high consequences (Soerskaar & Abrahamsen, 2017). Pre-cautionary and cautionary measures are the governing principle in an extreme safety perspective (Soerskaar & Abrahamsen, 2017).

2.2 A closer look at relevant theoretical models

The normative and descriptive models identified in the literature review originates from the first principles of portfolio theory and expected utility theory. Table 2.1 give an overview of the first generation models that have been developed and the recent advances.

<table>
<tr><th>First principles</th><th colspan="3">First generation models</th><th>Recent advances</th></tr>
<tr><td rowspan="2">Portfolio theory[1)] (1952)</td><td colspan="4">Expected value method[1)]</td></tr>
<tr><td colspan="3">Arrow-Pratt quadratic estimation[1)]</td><td>Extended Arrow-Pratt quadratic estimation[1)] (2013)</td></tr>
<tr><td rowspan="3">Expected utility theory[1)] (1947)</td><td colspan="3">Wald maximin functional[1)] (1949)</td><td>Maxmin expected utility[1)] (2011)</td></tr>
<tr><td colspan="3">Subjective expected utility[1)] (1954)</td><td>Smooth ambiguity functional[1)] (2005)</td></tr>
<tr><td></td><td>Choquet expected utility theory[2)](1989)</td><td>Cumulative prospect theory[2)] (1994)</td><td></td></tr>
</table>

1) *Normative*
2) *Descriptive*

Table 2.2.1 Overview of theoretical models for decision analysis with ambiguity

The above theories and models are supposed to define the preferences in a decision problem based on the decision maker's attitudes to risk and ambiguity. The decision maker's attitude to risk can be risk neutral, risk averse or risk seeking (Lindley, 1985). A decision maker has a risk neutral attitude if negative or positive outcomes are given the same weighting. A decision maker has a risk averse attitude if negative outcomes are given higher weighting than positive outcomes. A decision maker has a risk seeking attitude if positive outcomes are given higher weighting than negative outcomes. Ambiguity and ambiguity aversion are found in the famous case studies of two urns with known and unknown numbers of white and black balls (Ellsberg, 1961) and the three-color-problem (Ellsberg, 1961). The decision maker's attitude to ambiguity can be ambiguity neutral or ambiguity averse (Etner, et al., 2012). Ambiguity aversion is found to be the dominating behavior for a decision maker in the gain domain (upsides) while ambiguity neutrality is found to be the dominating behavior for a decision maker in the loss domain (downside) (Etner, et al., 2012).

The expected value method is derived from the normative portfolio theory (Markowitz, 1952) and represents a linear combination of payoffs and probabilities. The expected value method is considered to be a risk neutral approach. The expected utility theory (von Neumann & Morgenstern, 1947) is a value representation that can include aversion to risk or loss by a utility function. A rational decision maker is then supposed to maximize the expected utility by a linear combination of utilities and lotteries described by objective probabilities. The expected utility is a normative theory which means that it gives a prescription of a rational decision maker. Subjective expected utility (Savage, 1954) is a further development of expected utility theory where the lotteries are redefined as acts and the objective probabilities are redefined as subjective probabilities. A rational decision maker is, in the framework of subjective expected utility, also maximizing expected utility as a linear combination of utilities and probabilities, and can also have utility functions that includes aversion to risk. The subjective expected utility is considered to be an ambiguity neutral approach (Etner, et al., 2012). The preferences to an extreme risk and ambiguity averse or maxmin decision maker is the minimum of the worst consequences of an outcome set (Wald, 1949). The maxmin expected utility (Gilboa & Schmeidler, 1989) is a further generalization of the Wald maximin criterion. Maxmin expected utility is based on multiple priors for the outcome set where the decision maker will make his preferences by comparing the minimal expected utility of two decision alternatives (Etner, et al., 2012). The Extended Arrow-Pratt Quadratic Estimation method (Maccheroni, et al., 2013) is based on the portfolio theory of mean, variance and covariance and introduce parameters for risk aversion and ambiguity aversion. The Smooth Ambiguity Functional (Klibanoff, et al., 2005) is a further development of the subjective expected utility theory, where the ambiguity aversion is introduced as a second order function of the utility function. The Choquet Expected Utility theory (Schmeidler, 1989) (Mangelsdorff & Weber, 1994) is an extension to subjective expected utility theory that is based on non-additive probabilities. The Choquet Expected Utility theory is a descriptive theory that cover situations where the outcomes are only negative or only positive. The cumulative prospect theory (Kahneman & Tversky, 1992) is based on the descriptive Choquet Expected Utility theory. The cumulative prospect theory describes how a person makes his choice based on his individual reference point and attitude towards loss and gain.

In Soerskar & Abrahamsen (2017), two extreme perspectives are defined that reflects the high and low levels of uncertainty and consequence in a decision problem. These perspectives are shown in Figure 2.1. An extreme economic perspective reflects a decision context with low uncertainties and low consequences. An extreme safety perspective reflects a decision context with high uncertainties and high consequences. Decision problems that belongs to an extreme economic perspective can thus be analysed by use of cost-benefit analysis that are based on expected values. Cost-benefit analysis based on expected values should, however, not be used for decision problems that belongs to an extreme safety perspective. The cautionary principle should then be followed and relevant cautionary and pre-cautionary measures should be introduced with no references to cost-benefit analysis nor cost-effectiveness analysis according to Soerskar & Abrahamsen (2017).

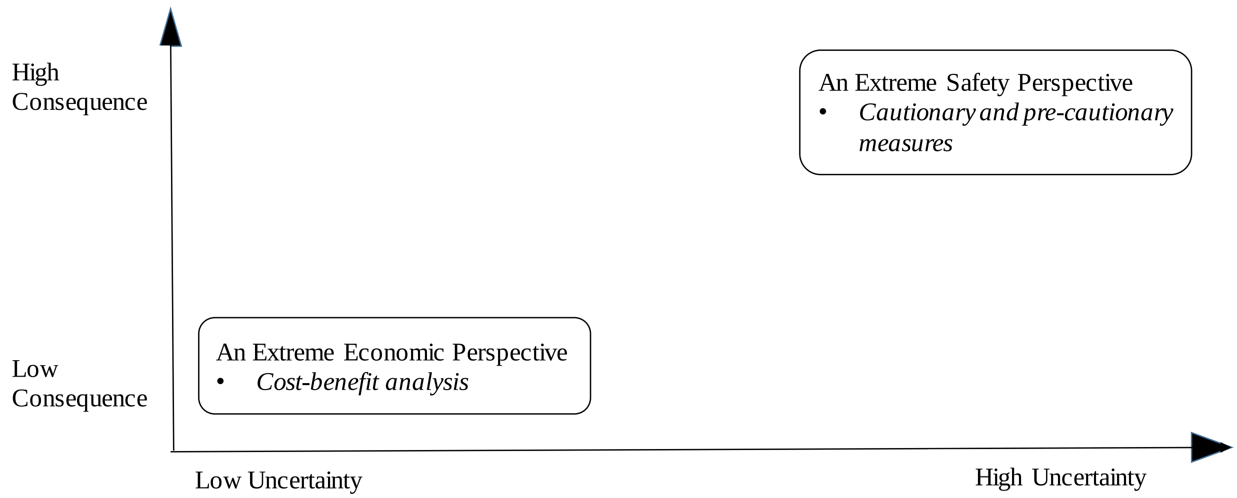


Figure 2.2.1 Extreme Perspectives on Uncertainty and Consequence

As shown in the above figure, there seem to be a gap between contexts with low uncertainty, referred to as an extreme economic perspective (Soerskaar & Abrahamsen, 2017), and contexts with high uncertainty referred to as an extreme safety perspective where no quantitative models can be used according to Soerskaar & Abrahamsen (2017).

Can quantitative models be developed based on the decision theoretical models derived from portfolio theory or expected utility theory, and then be used to analyse the uncertainties that are referred to in the extreme safety perspective? Maybe some portion of the uncertainty in an extreme safety perspective can be better understood if quantitative models were introduced?

An assessment of the theoretical models given in table 2.2.1 is therefore performed to find out if some of these models can capture uncertainty associated with risk and ambiguity, and define preferences in the decision problems described by the Ellsberg three-color problem (Ellsberg, 1961). The Ellsberg three-color problem (Ellsberg, 1961) is described in the next chapter and is a decision problem where the decision maker is faced with elements of both risk and ambiguity as a result of insufficient information.

2.2.1 The Ellsberg three-color problem

The table below show the Ellsberg three-color problem, with the acts f, g, f' and g' and the corresponding payoff if you draw a specified ball from an urn consisting of red, blue and yellow balls. We know that the number of red balls in the urn is $\frac{1}{3}$ while the number of blue and yellow balls are not known.

Act	Payoff		
	Red	Blue	Yellow
f	100	0	0
g	0	100	0
f'	100	0	100
g'	0	100	100

The outcome space S is defined as S ϵ (Red ball, Blue ball, Yellow ball) and the probability to draw a red ball are by the information given described as, P(Red) = $\frac{1}{3}$. The probability to draw a blue ball P(Blue) or the probability to draw a yellow ball P(Yellow) is however not possible to describe based on the information given. We do however know that the probability of drawing either a blue or yellow ball is described by P(Blue) + P(Yellow) = $1 - P(Red) = 1 - \frac{1}{3} = \frac{2}{3}$.

In the following, the probability of drawing a blue ball is designated, P(Blue) = p. The probability of drawing a yellow ball is then a dependency on p given as; P(Yellow) = $\frac{2}{3} - p$.

The decision maker wants to perform a rational choice. He therefore performs an assessment of available normative and descriptive theoretical models to see whether these can help him to decide his preferences. He also want to assess whether these models can be used to describe the certainty equivalent for the different alternatives. The certainty equivalent is defined as the sure amount of an uncertain monetary value (Borgonovo & Marinacci, 2015).

The decision theoretical models given in table 2.1 have been described and discussed in relation to the Ellsberg three-color problem in the subsequent chapters. The objective with this review was to see whether some of these models could be used for a decision maker that have a set of alternatives and where he needs to incorporate risk and ambiguity in his rational decision making.

2.2.2 Portfolio theory

Expected monetary value are often used to support decision making under uncertainty. This principle originates from portfolio theory (Markowitz, 1952). As described in Abrahamsen, et al. (2004), a portfolio of projects consists of N different projects where each of the projects have a weight $\frac{1}{N}$ in the portfolio. The expected value of the return r_i is $E(r_i)$ and the variance for the return r_i can is $Var(r_i)$.

The expected value and variance for the portfolio is expressed:

$$E_p = \frac{1}{N} \sum_{i=1}^{N} E_i$$

The variance for the portfolio is expressed:

$$VAR_p = \sum_{i=1}^{N} (\tfrac{1}{N})^2 VAR_i + \sum_{i=1}^{N} \sum_{j\neq i, j=1}^{N} (\tfrac{1}{N})^2 COV_{i,j} = \tfrac{1}{N}\overline{VAR} + (1 - \tfrac{1}{N})\overline{COV}$$

Where;

$COV_{i,j} = E\{(r_i - E_i) \cdot (r_j - E_j)\}$

$\overline{VAR} = \frac{1}{N} \sum_{i=1}^{N} VAR_i$.

$\overline{COV} = \frac{1}{N^2 - N} \sum_{i=1}^{N} \sum_{j\neq i, j=1}^{N} COV_{i,j}$

The unsystematic risk which refers to specific project uncertainties is in the above described by the average variance $\overline{VAR}$ and the systematic risk which refers to general market movements is in the above described by the average covariance $\overline{COV}$. In a portfolio with a large number of projects, we then see that the systematic risk represented by the average covariance will dominate since the average variance will go to zero when N is large.

The project portfolio perspective is therefore relevant when assessing the risk and return for a company with several projects. For a single project however, the unsystematic risk will dominate and the expected values can have large deviations from the true values.

A broader perspective than the expected values is therefore needed in order to take account of uncertainties. To calculate the expected monetary values is however a very common method to use in decision analysis. The problem with this method is that positive and negative outcomes are given the same weight and that the decision maker therefore is neutral to a potential negative or positive outcome. This is commonly termed as being risk neutral.

For the Ellsberg three-color problem, the expected monetary value for each of the alternatives can be calculated as the sum of the products of probability and payoff. We then see that the expected monetary value for alternative f and alternative g' can be determined while the expected monetary value for alternative g and alternative f' depends on the unknown probability *p*.

$\mathbb{E}_p(f, w) = \frac{1}{3} \cdot (100) + \left(\frac{2}{3} - p\right) \cdot (0) + p \cdot (0) = \frac{1}{3} \cdot (100) = 33.33$

$\mathbb{E}_p(g, w) = \frac{1}{3} \cdot (0) + \left(\frac{2}{3} - p\right) \cdot (100) + p \cdot (0) = \left(\frac{2}{3} - p\right) \cdot (100)$

$\mathbb{E}_p(f', w) = \frac{1}{3} \cdot (100) + \left(\frac{2}{3} - p\right) \cdot (0) + p \cdot (100) = \left(\frac{1}{3} + p\right) \cdot (100)$

$\mathbb{E}_p(g', w) = \frac{1}{3} \cdot (0) + \left(\frac{2}{3} - p\right) \cdot (100) + p \cdot (100) = \left(\frac{2}{3}\right) \cdot (100) = 66.66$

The expected monetary value for option g and f' will vary depending on the probability value of p which could range between 0 and $\frac{2}{3}$. If p is close to $\frac{2}{3}$, then option f' will have the maximum expected value. If p is close to 0 , then option g and option g' will have the maximum expected value. The preferred option in the Ellsberg three-color problem is therefore not possible to decide for the decision maker based on the principle of expected monetary value since the probability *p* is non-uniquely defined.

2.2.3 Expected utility theory

The expected utility theory assumes that a decision maker's choice and behavior is based on rational decision making. A set of axioms were defined by von Neumann and Morgenstern (1947) that defined the rational behavior and the decision maker's preference over lotteries (Abrahamsen & Aven, 2008). A lottery can be described as a set of outcomes where the probability of occurrence for each of the outcomes can be described by objective probabilities. Lotteries are defined in a state space S = $\{X, Y, Z\}$ where the decision maker may have a preference over lotteries X, Y and Z. The "Weak order" axiom defines the decision maker's preference over lotteries based on the properties of completeness, transitive and reflexive. Completeness means that the decision maker can prefer X over Y, or Y over X or can be indifferent between X and Y. Transitive means that the decision maker prefer X over Y and Y over Z, it is also then given that the decision maker prefers X over Z. Reflexive means that the decision maker is indifferent between two identical lotteries X and X. The "Continuity" axiom defines that there exists only one value of p between 0 and 1 which makes the decision maker indifferent between lottery Y and a compound lottery of X and Z, this implies that Y~ pX + (1-p)Z. The "Preference increasing with probability" axiom means that a decision maker preference of two lotteries X and Y with the same outcomes would be the lottery with the highest probability. The "Compound probabilities" axiom defines that any lottery which has further lotteries as outcomes can be reduced to a one stage lottery. The "independence" axiom states that if a decision maker has a preference to lottery X over lottery Y this preference should not change if a common outcome in both lotteries are changed.

The important feature of utility theory is that the utility value function can have a shape that represents the decision maker's attitude or weighting to gain and loss. The utility can be defined as a linear function or in a concave or convex shape. If the utility function is linear the decision maker is neutral, which means that a negative or positive outcome are given the same weighting. If the utility function has a concave shape this means that negative outcomes are given higher weighting than the positive outcomes. If the utility function has a convex shape, this means that the positive outcomes are given higher weighting than the negative outcomes.

If we have a defined state space S = $\{X, Y, Z\}$ and have defined the utilities and the probabilities of occurrence for the outcome space, then the expected utility is found as a linear combination of the utilities and the corresponding probabilities. The rational decision maker will then have a preference for the lottery that maximizes his expected utility.

The state space for the Ellsberg three-color problem is however not lotteries where the success of all outcomes are described by objective probabilities. An objective probability for red balls exists but not objective probabilities for the blue balls and respectively for yellow balls, only for their union. The basis for the use of the expected utility is therefore not satisfied for the Ellsberg three-color problem and the decision maker is not able to perform a rational choice of his preferences by the use of expected utility.

2.2.4 Subjective expected utility theory

Savage (1954) described the subjective expected utility (SEU) as a function of the subjective probabilities defined by Ramsey (1931) and de Finetti and the expected utility as defined by von Neumann and Morgenstern (1947). Several postulates were defined by Savage (1954) that represents a further refinement of the axioms defined for expected utility by von Neumann and Morgenstern. The most important postulate is the sure-thing principle, which refer back to the "independence" axiom. Another important postulate by Savage (1954) relate to the preference of an outcome space, which means that if the outcome space {0, 100} is changed to {0, 1000} this should not result in change in preference.

Savage also changed the state space from a set of lotteries to a set of acts which could include lotteries but also other actions where the subjective belief of the decision maker is described by subjective probabilities. Assume that preferences satisfy Savage postulates and that there exist a probability measure μ on S and a utility function u for act f and g. The value representation for the preference of act f over act g is then expressed as follows:

$$\mathrm{f} \gtrsim \mathrm{g} \leftrightarrow \int_S u\big(f(s)\big)d\mu(s) \gtrsim \int_S u\big(g(s)\big)d\mu(s)$$

μ is a subjective probability distribution over the state space and u is a utility function over the outcome set. The Savage construction of subjective expected utility includes the subjective beliefs of the decision maker which he then uses in a linear manner to find his preferences by maximizing expected utility. Savage representation means that the behaviour of a decision maker is to maximise the expected utility, with the restriction being that his beliefs must be described by a probability distribution. His beliefs may be strange or unreasonable, but the basis for subjective expected utility theory is still satisfied as long as his beliefs are described by a probability distribution. (Etner, et al., 2012).

Anscombe and Aumann (1963) defined a reduced state space for subjective expected utility consisting of vectors of two-stage acts or compound lotteries that were a combination of lotteries with objective probabilities also called roulette-wheel lotteries, and acts with subjective probabilities also called horse-race lotteries. The two-stage acts are based on the assumption of monotonicity in the prizes and reversal of order. Monotonicity in the prizes means that if the prize for one outcome differs between two otherwise identical horse lotteries then your preference associated with the price of that outcome will also govern your preference between the two horse races. Reversal of order means that your preference between a set of two-stage acts consisting of a compound horse race and roulette-wheel lottery is not affected by the order of these lotteries, i.e. whether the horse race starts before or after the spinning of the roulette-wheel.

The subjective expected theory is a normative theory where the preferred act is to be selected based on maximizing the subjective expected utility (Savage, 1954). The subjective expected utility allow the decision maker to include his aversion to risk in a utility function. The utility function for risk aversion is normally described as a concave exponential function where the negative payoffs are given a much higher weight than a positive payoff and where a high payoff is reduced in relation to a lower payoff.

The subjective expected utilities for the acts in the Ellsberg three-color problem are expressed below by the use of the utility function u(a,w)=$-\ e^{-a \cdot w}$ (Borgonovo & Marinacci, 2015) with a risk aversion constant, a = $\frac{1}{100}$ and with a wealth w ranging from 0 to 100.

$$U(f,p) = \frac{1}{3}u(100) + \left(\frac{2}{3} - p\right)u(0) + p \cdot u(0) = \frac{1}{3} \cdot u(100) + \frac{2}{3} \cdot u(0) = -0.79$$

$$U(g,p) = \frac{1}{3}u(0) + \left(\frac{2}{3} - p\right)u(100) + p \cdot u(0) = \left(\frac{1}{3} + p\right) \cdot u(0) + \left(\frac{2}{3} - p\right) \cdot u(100) =$$

$$U(g,p) = \left(\frac{1}{3} + p\right) \cdot (-1) + \left(\frac{2}{3} - p\right) \cdot (-0.37)$$

$$U(f',p) = \frac{1}{3}u(100) + \left(\frac{2}{3} - p\right)u(0) + p \cdot u(100) = \left(\frac{1}{3} + p\right) \cdot u(100) + \left(\frac{2}{3} - p\right) \cdot u(0) =$$

$$U(f',p) = \left(\frac{1}{3} + p\right) \cdot (-0.37) + \left(\frac{2}{3} - p\right) \cdot (-1)$$

$$U(g',p) = \frac{1}{3}u(0) + \left(\frac{2}{3} - p\right)u(100) + p \cdot u(100) = \left(\frac{1}{3}\right) \cdot u(0) + \left(\frac{2}{3}\right) \cdot u(100) = -0.58$$

As seen in the above, the maximum expected utility is not uniquely defined since the expected utility for act g and act f' will depend on the probability value p. The rational decision maker following the SEU set-up would, in such a situation, assign his subjective probabilities based on his judgement and subjective belief of the number of blue and yellow balls in the urn. A reasonable assumption could be to go for a symmetry argument with the same number of blue and yellow balls. The subjective probability assigned by the decision maker then becomes p = $\frac{1}{3}$. With this value of p, the decision maker would be indifferent in preference between act f' and act g'. This conclusion does however seem to ignore or miss out some important information. These two acts are associated with uncertainties that cannot be sufficiently described by subjective probabilities, he can only give a guess. The rational decision maker should thus, in contrast to the SEU behavior, accept that he is faced with ambiguity in the results for act g and act f', and that he would need supplementary models in order to decide the preferred act.

The Ellsberg three-color problem (Ellsberg, 1961) has been used as an example of a violation of the sure-thing principle which is one of the key axioms in subjective expected utility. A reframing of the Ellsberg three-color problem does however show that a preference for act f over act g and a preference for act g' over act f' is consistent with the sure-thing principle (Bradley, 2015). The Ellsberg three-color problem does however also show that the subjective expected utility have its limitations in decision settings where the uncertainties cannot be sufficiently described by assigning subjective probabilities. Several extended models have therefore been proposed as a further development of the subjective expected utility, in order to cater for such types of uncertainty or ambiguity that cannot be described by a probability distribution.

2.2.5 Wald maximin functional

The Wald maximin criterion (Wald, 1949), (Gilboa & Marinacci, 2011), (Etner, et al., 2012) is a very conservative model where preference is based exclusively by the worst possible consequences. If the outcome set is $\{x, y, z\}$ where x represent the worst consequence, then the decision maker would make his preference based on u(x) from this outcome set. (Etner, et al., 2012).

For the Ellsberg three-color problem, the use of the Wald maximin functional would give the following results:

$$U_{min}(f) = U(f, p) = \frac{1}{3} \cdot (-1.00) + \frac{2}{3} \cdot (-0.37) = -0.79$$

$$U_{min}\left(g, p = \frac{2}{3}\right) = \left(\frac{1}{3} + p\right) \cdot (-1) + \left(\frac{2}{3} - p\right) \cdot (-0.37) = -1.00$$

$$U_{min}(f', p = 0) = \left(\frac{1}{3} + p\right) \cdot (-0.37) + \left(\frac{2}{3} - p\right) \cdot (-1) = -0.79$$

$$U_{min}(g') = U(g', p) = \left(\frac{1}{3}\right) \cdot (-1.00) + \left(\frac{2}{3}\right) \cdot (-0.37) = -0.58$$

The decision maker preferences can thus be defined based on this method, but it is considered to represent the extreme ambiguity aversion situation.

2.2.6 Maxmin expected utility theory

In maxmin expected utility theory (Gilboa & Schmeidler, 1989), an act f is preferred to an act g if the minimum expected utility for a set of multiple priors for act f is larger than the minimum expected utility for a similar set of multiple priors for act g. The multiple set of priors is then supposed to be used in the absence of precise subjective beliefs. (Etner, et al., 2012). The multiple priors can then be a set of probability distributions that are used to calculate the maxmin expected utility for the state space S = {f, g, f’, g’} in the Ellsberg three-color problem. The decision maker then prefers act g’ over act f’ if the minimum expected utility for act g’ is higher than the minimum expected utility for act f’ for each of the probability distributions. This is however not the case in the Ellsberg three-color problem, since the preference between act f’ and act g’ varies depending on the probability distribution between p_2 and p_3 as shown in figure 2.6. Variations to the preferences between act f’ and act g’ therefore result in incomplete preferences for the Ellsberg three-color problem.

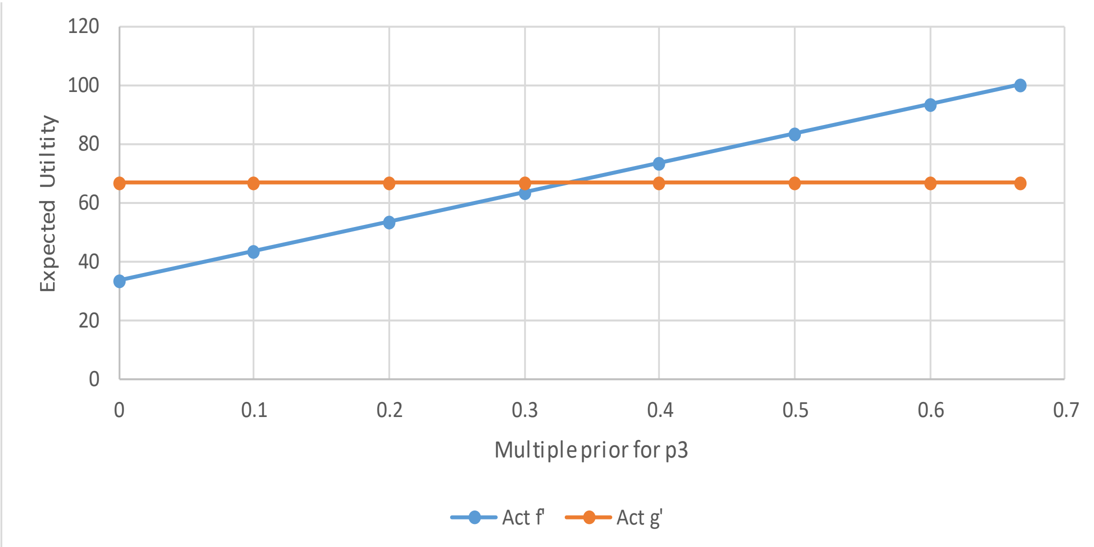


Fig. 2.6 Minimum Expected Utility vs Multiple priors for probability of blue balls (p_2) and probability of yellow balls (p_3)

The α-Maxmin (Etner, et al., 2012) is a combination of the worst and best consequences. If the outcome set is $\{x, y, z\}$ where x represent the worst consequence and z the best consequence, then the value for a decision with this outcome set would be, $\alpha \cdot u(x) + (1\text{-}\alpha) \cdot u(z)$. If $\alpha = 1$, then we are back to the maxmin expected utility. If $\alpha = 0$, then we have the maxmax expected utility. The idea of having this factor α as a value between 0 and 1 is to have a measure of the decision maker's pessimism. According to Gilboa & Marinacci (2011) however, the theoretical basis for $\alpha \in (0,1)$ remains unresolved.

2.2.7 Cumulative prospect theory

The cumulative prospect theory was developed by Kahneman and Tversky (1992). The cumulative prospect theory describes the decision makers's choice as a function of five defined behavioral phenomena; framing, nonlinear preferences, source dependence, risk seeking and loss aversion. The framing of the options and their outcome will have an impact on the choice as the decision maker has different attitudes to gains and losses. This is also in line with the expected utility theory where the expected utility is a weighted product of gain and loss with a higher weight on loss than gain. Nonlinear preferences states that difference in probabilities between 0.99 and 1.00 have more impact than differences in probabilities between say 0.10 and 0.11. This is in contrast to the expectation principle in expected utility theory which states that the expected utility of a risky prospect have a linear relation to the outcome probabilities.

Source dependence describe that a decision maker is more willing to choose an uncertain event or prospect in his area of competence over a matched risky event or prospect that lies outside his are of comtpetence. Risk seeking is the dominating behavior when decision maker's need to choose between a sure loss and a substantial probability of a larger loss. Loss aversion describes that the decision maker is found to be more sensitive to losses than to gains. The loss aversion, risk seeking and nonlinear preferences are in cumulative prospect theory represented by a value and weighting function. The value of each outcome are separated for gains and losses and these are then multiplied by non-additive decision weights, which is different from gains and losses.

The cumulative prospect theory (Kahneman & Tversky, 1992) is a further development of rank dependent utility also called Choquet expected utility (Etner, et al., 2012). Choquet expected utility (Schmeidler, 1989) is then again a further development of the subjective expected theory, but where the beliefs of the decision maker are not described by non-additive probabilities, also described as non-additive capacities. An act f is then preferred to an act g if there exists a utility function u and a capacity v that satisfies the following utility value representations:

$$\int_{Ch} u(f)dv \geq \int_{Ch} u(g)dv$$

With the outcome space S=$\{s_1, s_2, \ldots., s_n\}$, the Choquet integral is further defined by:

$$\int_{Ch} u(f)dv \geq u(x_1) + \big(u(x_2) - u(x_1)\big) \cdot v(\{s_2, s_3, \ldots., s_n\}) + \cdots + \big(u(x_{i+1}) - u(x_i)\big) \cdot v(\{s_{i+1}, \ldots., s_n\}) + \cdots + \big(u(x_n) - u(x_{n-1})\big) \cdot v(\{s_n\})$$

The Choquet integral consider first the lowest outcome and then add the positive increments that are weighted with decision maker's belief represented by the capacities v(s) over the outcome space S.

The cumulative prospect theory (Kahneman & Tversky, 1992) also uses capacities to describe the decision maker's beliefs. The difference between Choquet expected utility and cumulative prospect theory is however that the latter has one set of capacities for gains, and another set of capacities for losses. The utility value representation in cumulative prospect theory for an act f is then described by a utility function u and a capacity for gain v^+ and for loss v^- as follows:

$$V(f) = u(x_1) + \sum_{i=2}^{k} v^-(U_{j=1}^{k}\{s_j\})(u(x_i) - u(x_{i-1})) + \sum_{i=k+1}^{k} v^+(U_{j=1}^{k}\{s_j\})(u(x_i) - u(x_{i-1}))$$

An issue with the cumulative prospect theory is that the model supposes that there is a reference point where the decision maker treats outcomes above as gains and outcomes below as losses.

In the Ellsberg three-color problem, the reference point for the decision maker is 0 since there are only outcomes with a gain of 0 or 100. The negative portion of the value representation for the cumulative prospect theory is therefore not relevant and the cumulative prospect theory value representation is reduced to a Choquet value representation. In Mangelsdorff & Weber (1994), the Ellsberg three-color problem has been assessed by the use of Choquet expected utility. The Choquet expected utility is here used to formulate a value representation of the prospects based on the decision maker's initial preferences, or by empirical surveys.

2.2.8 Extended Arrow-Pratt Quadratic Estimation

The certainty equivalent for an expected utility maximizer with utility u, wealth w and investment h can be expressed by the Arrow-Pratt approximation (Maccheroni, et al., 2013) given by:

$$c(w+h,P) \approx w + E_P(h) - \frac{1}{2}\lambda_u(w)\sigma_p^2(h)$$

$E_P(h)$ is the expected wealth of the investment h based on the probabilistic model P. $\sigma_p^2(h)$ is the statistical variation of the investment h with respect to the probabilistic model P. The coefficient $\lambda_u(w) = -\frac{u''(w)}{u'(w)}$ which is the ratio between the double derivative and the derivative of the u function, describe the agent's or decision maker's aversion to risk.

By setting f = w + h and $\lambda = \lambda_u(w)$, the value representation or certainty equivalent for a prospect f becomes:

$$C(f) = E_P(f) - \frac{\lambda}{2}\sigma_p^2(f)$$

The risk premium for prospect f is therefore given by:

$$\Pi_\lambda = \frac{\lambda}{2}\sigma_p^2(f)$$

A premium for ambiguity for prospect f can further be found by quadratic approximation of an extension to the Arrow-Pratt analysis to account for model uncertainty (Maccheroni, et al., 2013). Model uncertainty then refers to situations where the decision maker is uncertain about the probabilistic model P.

The quadratic approximation takes an exact form as given below if the investment h has a normal cumulative distribution $\varphi(\cdot; m, \sigma)$ with unknown mean m and known variance σ^2 (Maccheroni, et al., 2013). It is also then supposed that the prior on the unknown means m is given by a normal cumulative distribution $\varphi(\cdot; \overline{\mu}, \sigma)$. Due to the normal distribution of the prior, this implies that $\overline{\mu}$ is the mean of the unknown means m and σ_μ^2 is the variance of the unknown means.

$$\begin{aligned} \mathrm{C(w+h)} &= v^{-1}\Big(\int v\,\Big(u^{-1}\Big(\int u(w+x)d\varphi(m;\overline{\mu},\sigma_\mu)\Big)\Big) \\ &= v^{-1}\Big(\int v\,\Big(u^{-1}\Big(w+m-\frac{1}{2}\lambda_u(w)\cdot\sigma^2\Big)d\varphi(m;\overline{\mu},\sigma_\mu)\Big)\Big) \\ &= w+\overline{\mu}-\frac{1}{2}\lambda_u(w)\sigma^2-\frac{1}{2}\lambda_v(w)\sigma_\mu^2 \end{aligned}$$

The standard mean-variance quadratic approximation or certainty equivalent to account for model uncertainty or ambiguity is given below and are determined by the parameters, λ, ɵ and μ at a constant w such that $\lambda = \lambda_u(w)$ and $\theta = \lambda_v(w) - \lambda_u(w)$. The parameters λ and ɵ represents the decision maker's negative attitude towards risk (λ) and ambiguity (ɵ). $\sigma_\mu^2(E(f))$ is the variance of the averages of prospect f or the variance of the expected value of prospect f.

$$C(f) = E_P(f) - \frac{\lambda}{2}\sigma_P^2(f) - \frac{\theta}{2}\sigma_\mu^2(E(f))$$

The ambiguity premium for the prospect f is therefore given by:

$\Pi_\theta = \frac{\theta}{2}\sigma_\mu^2(E(f))$

A summary of the analysis results for the exact solution of the Ellsberg three-color problem is given below and refers to the detailed analysis in appendix A1. $\sigma_\mu^2(-)$ is the statistical variation of the investment (-) with respect to the predicted probabilistic model μ and is equivalent to $\sigma_p^2(-)$ in the above description. $\sigma_{AP}^2(-)$ is the variance of the averages of prospect (-) or the variance of the expected value of prospect (-) and is equivalent to $\sigma_\mu^2(E(-))$ in the above description.

Act	$\sigma_\mu^2(-)$	$\sigma_{AP}^2(-)$	Risk premium	Ambiguity premium	Certainty equivalent
f	0.00	2222.22	11.11	0.00	22.22
g	370.37	2222.22	11.11	1.85	20.37
f’	370.37	2222.22	11.11	1.85	53.70
g’	0.00	2222.22	11.11	0.00	55.56

The Ellsberg three-color problem can also be solved by the numerical quadratic approximation method as described in appendix A2. The analysis results from this numerical estimation is given below and show very close correlation to the exact solution given above.

Act	$\sigma_\mu^2(-)$	$\sigma_{AP}^2(-)$	Risk premium	Ambiguity premium	Certainty equivalent
f	0.00	2222.22	11.11	0.00	22.22
g	363.96	2213.95	11.07	1.82	20.20
f’	358.28	2237.69	11.19	1.79	53.22
g’	0.00	2222.22	11.11	0.00	55.56

2.2.9 Smooth ambiguity functional

A smooth ambiguity functional (Klibanoff, et al., 2005) is defined by two utility functions. The utility function u is the same as used in subjective expected utility and is a transformation of the outcomes with a weighting on negative and positive outcomes that reflects the decision maker’s risk aversion. The other utility function ϕ does a further transformation of the utility function u and adds another layer of weighting that reflects the decision maker’s ambiguity aversion. This smooth ambiguity model is depending on the selected shape of the utility functions which can be very different for different decision makers.

The value representation of smooth ambiguity model (Klibanoff, et al., 2005) is described as an increasing concave function of the subjective utility function.

$$V(f) = \int_{\Delta(S)} v\big(c(f,p)\big)d\mu(p) = \int_{\Delta(S)} v\big(u^{-1}(u(f,p))\big)d\mu(p)$$

$$V(g) = \int_{\Delta(S)} v\big(c(g,p)\big)d\mu(p) = \int_{\Delta(S)} v\big(u^{-1}(u(g,p))\big)d\mu(p)$$

$$V(f') = \int_{\Delta(S)} v\big(c(f',p)\big)d\mu(p) = \int_{\Delta(S)} v\big(u^{-1}(u(f',p))\big)d\mu(p)$$

$$V(g') = \int_{\Delta(S)} v\big(c(g',p)\big)d\mu(p) = \int_{\Delta(S)} v\big(u^{-1}(u(g',p))\big)d\mu(p)$$

The full analysis for the Ellsberg three-color problem are included in appendix A.3 and the results are summarized in the table below. The decision maker can select act g' as the preferred act since this act have the highest KMM utility and therefore also the highest certainty equivalent.

Act	KMM Utility, V(-)	Certainty Equivalent	Risk and ambiguity premium
f	-0.62	23.66	9.67
g	-0.64	22.15	11.18
f'	-0.35	52.11	14.55
g'	-0.33	54.72	11.94

The risk premium for act f and act g' are now defined since we know that there are no ambiguity and therefore no ambiguity premium for act f and act g'. The portion of the risk premium and the ambiguity premium for act g and f' do however need to be further analysed by the use of the Extended Arrow-Pratt Quadratic Estimation (Maccheroni, et al., 2013). This method is presented in appendix A2 and the results from the numerical estimation is summarized in section 2.8.

2.2.10 Analysis and discussion of theoretical models

The expected value is a linear combination of payoffs and probabilities that do not include any effects due to risk aversion or ambiguity aversion. A decision maker who make his preferences based on expected values is therefore considered to be neutral to both risk and ambiguity. Expected utility can include aversion to risk or loss and a rational decision maker is then supposed to maximize the expected utility by a linear combination of objective probabilities and utilities that include risk aversion. The use of multiple priors as an extension of expected utility theory may represent a challenge to the decision maker as some priors may show that his preference changes. The subjective expected utility is a linear combination of subjective probabilities and utilities that include risk aversion. The Choquet expected utility and the cumulative prospect theory seem to be tailored to how a person makes his choice based on his individual reference point and attitude towards loss and gain. The Choquet expected utility and prospect theory seem to be less relevant for a decision maker that has to find a preference between project alternatives.

The Wald Maximin Functional is an extreme version of the subjective expected utility. The preferences to an extreme risk and ambiguity averse decision maker can be found by the Wald Maximin Functional as done for the Ellsberg three-color problem.

The Extended Arrow-Pratt Quadratic Estimation method defines a risk aversion and an ambiguity aversion parameter. The decision maker is then able to define his preferences in the Ellsberg three-color problem if he is able to define these two parameters. For the Ellsberg three-color problem it was possible to find both an exact and approximate solution that both relies on the decision maker's risk and ambiguity aversion.

The Smooth Ambiguity Functional is a further development of the subjective expected utility theory, where the ambiguity aversion is introduced as a second order function. This method also defines the parameters that defines the decision maker's aversion to risk and ambiguity. The decision maker is then able to define his preferences in the Ellsberg three-color problem if he is able to define these two parameters.

The review shows that the Wald Maximin Functional, the Extended Arrow-Pratt Quadratic Estimation and the Smooth Ambiguity Functional are models that can be used to define decision maker's preferences when faced with both risk and ambiguity. The basis for the quadratic estimation and smooth ambiguity analysis is however the parameters that describe the decision maker's aversion to risk and ambiguity.

The parameter that describes the decision maker's risk aversion in a Smooth Ambiguity Model needs to be a function of the value of the payoff. If the payoff is {-100, 0, 100} and the risk aversion parameter a = $\frac{1}{100}$ and the utility function have a negative exponential representation then the utilities becomes:

$u(-100) = -e^{-a\cdot(-100)} = -e^{1}$ = - 2.72

$u(0) = -e^{-a\cdot 0} = -e^{-0}$ = - 1.00

$u(100) = -e^{-a\cdot 100} = -e^{-1}$ = - 0.37

If the payoffs were {-1000, 0, 1000} and the corresponding risk aversion parameter were a = $\frac{1}{1000}$, then a similar relation can be found between the utilities of the payoff when the negative exponential representation of the utility function is used. Similarly if the payoffs were for example {-500, 0, 500}, then the risk aversion parameter of a = $\frac{1}{500}$ would give the same results.

The ambiguity aversion parameter b is the other value that needs to be included in the quadratic estimation and smooth ambiguity analysis. The ambiguity aversion parameter has to be defined in an iterative process where the certainty equivalent found by the Smooth Ambiguity Functional (Klibanoff, et al., 2005) is correlated with the certainty equivalent found by the Extended Arrow-Pratt Quadratic Estimation method (Maccheroni, et al., 2013).

3 Qualitative research of decision making practice

Decisions that are associated with uncertainties need to be based on our subjective judgement and our attitude towards risk. Some of us may have a high aversion to risk or losses while others may be risk seekers. The uncertainty in our decision making may include elements of both risk and ambiguity. Risk is defined as uncertainty that we are confident to describe by assigning a subjective probability distribution (Borgonovo & Marinacci, 2015). Ambiguity is defined as uncertainty that we are not confident to describe by assigning subjective probabilities (Borgonovo & Marinacci, 2015). The uncertainty associated with a decision therefore seems to be very dependent on the decision maker's subjective judgement and tolerance to risk and ambiguity.

The research question that has been driving the qualitative research of decision making practice is to assess if decision theory under ambiguity is known and used by the industry. How decisions are analysed and assessed and how these align with the theoretical models of decision analysis have therefore formed part of the qualitative research. The qualitative research of decision making practice has also assessed how decision maker's subjective judgement and risk and ambiguity tolerance are incorporated in a decision process. Another aspect of the qualitative research has been to see whether decision analysis models available from recent advances in decision theory can be calibrated and translated into practical methods that can be used to assist in rational decision making.

3.1 Methodology

In-depth interviews of a peer group of decision makers have been selected as the methodology for the qualitative research. The experts in this peer group represent a diverse base of relevant knowledge and practice in the oil and gas industry. Areas of their primary expertise include attributes of project management, project assurance, subsurface, facilities engineering, decommissioning, field development, modifications and portfolio management. The respondents have experience ranging from 28 to 35 years. In total 15 professionals were interviewed, where 13 respondents have key roles in oil and gas companies and 2 respondents have key roles in project services companies. The members of the peer group were interviewed on a one-to-one basis and were not introduced to the questions before the individual interviews. Neither did the selected members know the composition of the peer group. The average duration of the one-to-one interviews were typically one hour. Notes from the interviews were sent to the respondents after each of the interviews. The respondents subsequently replied with minor corrections and/or acceptance of the notes made from the interviews.

3.1.1 Interview questions

In the in-depth interviews, the same set of interview questions were posed to explore and identify current practice relating to the application of uncertainties in decision making and decision analysis. The interview question protocol is included in Appendix C.

The interview questions were centered on the use of probabilistic analysis and the processes and methods used to translate these into reliable budgets for a project, a portfolio of projects or for a value proposition for a field development.

The interview questions posed in the in-depth interviews also included scenarios where the respondents were requested to choose their preferences among two sets of stylized projects. These projects are described in chapter 3.1.2 below and were selected in order to illustrate some of the dilemmas and conflicts a decision maker may be faced with when uncertainty is present in analysis and assessments. The objective with these scenarios was to assess if the respondents makes decisions that are consistent with normative decision theories.

3.1.2 Framing of scenarios

The in-depth interviews were performed with reference to two project cases with positive outcomes that are referred to as project A and B. The two projects have three possible outcomes that may be associated with either risk or a combination of risk and ambiguity. The probability of outcome x_1 is given as $p_1=1/3$. The probability is not described for the specific outcome of x_2 or x_3. The probability of either outcome x_2 or outcome x_3 is however $p_2 + p_3 = 2/3$.

	Outcome sets		
Projects	x_1	x_2	x_3
A	100	0	100
B	0	100	100

The expected values for the projects are the sum of products of probabilities and outcomes and is expressed by $\mathbb{E} = p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3$. The expected value for project A is however not uniquely defined since $p_2 \in (0, 2/3)$ due to the uncertain probability distribution between p_2 and p_3. The expected value for project A may therefore range between 33 and 100. The expected value for project B is uniquely defined despite $p_2 \in (0, 2/3)$ due to the defined probability distribution between p_1 and the sum of p_2 and p_3. The expected value for project B is 67.

The in-depth interviews were also performed with reference to two project cases that are referred to as project C and D. The description of project C and D for the two first interviews included a combination of positive and negative outcomes. The description used for project C and D for the subsequent interviews only referred to negative outcomes as shown in the table below. The projects C and D then have three possible outcomes that may be associated with either risk or a combination of risk and ambiguity. The probability of outcome x_1 is given as $p_1=1/3$. The probability is not described for the specific outcome of x_2 or x_3. The probability of either outcome x_2 or outcome x_3 is however $p_2 + p_3 = 2/3$.

	Outcome sets		
Projects	x_1	x_2	x_3
C	-100	0	-100
D	0	-100	-100

The expected values for the projects are the sum of products of probabilities and outcomes and is expressed by $\mathbb{E} = p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3$. The expected value for project C is however not uniquely defined since $p_2 \in (0, 2/3)$ due to the uncertain probability distribution between p_2 and p_3. The expected value for project C may therefore range between -33 and -100. The expected value for project D is uniquely defined despite $p_2 \epsilon (0, 2/3)$ due to the defined probability distribution between p_1 and the sum of p_2 and p_3. The expected value for project D is -67.

3.2 Results

The results indicate that the current practice and methods for the assessment of uncertainties for a project are very similar across the oil and gas companies. There are probabilistic analysis performed which describe an uncertainty range or confidence interval in the form of the P10 and P90 confidence levels. The probabilistic analysis also define a P50 confidence level which for most of the respondents is considered to be the expected estimate or the base estimate. Qualitative assessments are performed in order to establish a project contingency. A specific method for the definition of contingency has not been identified. The process to define contingency generally include peer reviews and management reviews to go through the base estimate and the identified risk elements and then by qualitative judgements establishing a level of contingency for the project.

Detailed review processes are introduced to ensure conservative estimates and prognosis before a project is sanctioned. These detailed processes are introducing technical allowances in the scope elements to address the variation of known unknowns in the estimates. The technical allowances are then to be added to the base estimate if a deterministic approach is used and alternatively modelled as variations to the estimate if a probabilistic approach is used. A definition of allowance is given in ASTM E2168 (2016) by the following primary classification:

> *"5.2.1 Allowance – a sum of money that is intended to be spent on the planned scope of work. Used in the absence of precise knowledge, and estimated to the best of one's abilities, to ensure a full and complete estimate. Allowances cover events and activities that are normally internal and so are directly controllable within the project plan."*

There seem to be some cultural differences between the oil and gas companies when it comes to the weighting of positive and negative outcomes. The larger operator would tend to have a more linear focus on the upsides and the downsides, while the smaller operator would tend to have a relative higher focus or weighting of the downsides. This has some correlation with the analysis in Howard (1988) that indicated a higher risk tolerance in companies with a higher financial turnover and profit base. The probabilistic analysis done in the oil and gas companies appears to be reasonably advanced. However, no analytical methods appear to be used to translate the probabilistic analysis into weighting functions as basis for contingency for cost and income.

The results from the scenarios are listed in table 3.2.1. These results indicate that there do not seem to be a consistent approach among the peers of experts in selecting positive and negative outcomes. Four of the respondents selected a reliable outcome space for both the positive and negative outcomes that is supported by the normative decision theory. Some respondents chose to chase the outcomes of project A and are willing to accept that there are some risk and ambiguity to the positive outcomes. Others will choose project C and are willing to accept some risk and ambiguity on the negative outcomes. None of the respondents selected both project A and project C. This is somewhat reassuring since it can be seen that the respondents are not taking a risk seeking approach. The approach taken is rather an active strategic position towards either the downside or the upside by implementing necessary actions in order to either maximize the results of project A or minimize the results in project C.

Response	Positive outcomes				Negative outcomes			
	Prefer A	Prefer B	Indifferent A=B	E[A]	Prefer C	Prefer D	Indifferent C=D	E[C]
1			v	67				
2			v	75				
3		v		50	v			-50
4			v	67			v	-67
5		v		67	v			-50
6	v			67		v		-67
7		v		67		v		-67
8		v		67		v		-67
9		v		50	v			-50
10		v		45		v		-50
11	v			-		v		-
12		v		67	v			-80
13		v		33	v			0
14		v		50	v			-80
15		v		67		v		-67

Table 3.2.1. Responses on the Case Studies

To select projects based on only the expected values does not seem to give enough information in cases such as project A and C where the expected values is not uniquely defined due to the undefined probabilities for p_2 and p_3. Analysis results using the Arrow-Pratt quadratic estimation is shown in table 3.2.2 and use the risk premiums and the ambiguity premiums for the various projects as a guide for preference between the projects. These analysis then show that project B and project D would be the projects with highest certainty equivalent and therefore these should be the preference for a rational decision maker according to the normative decision theory. An assessment of parameters for risk aversion and ambiguity aversion were planned to be part of the case studies. A method or test case for this type of assessment does however need to be more comprehensive than the stylized projects that were used.

Project	Predicted Value	Risk premium	Ambiguity premium	Certainty equivalent (CE)
A	66.67	11.11	16.67	38.89
B	66.67	11.11	0.00	55.56
C	-66.67	11.11	16.67	-94.44
D	-66.67	11.11	0.00	-77.78

Table 3.2.2. Certainty Equivalent for the Case Studies

The project contingency can have elements of both aleatory and epistemic uncertainty. It is normal to have a high contingency at the early phase of a project, which then is reduced as the project definition is more matured and the epistemic uncertainty then reduced. This staged development of contingency is defined by the project stages DG1, DG2 and DG3 which is commonly used in the industry. There may however be large uncertainties also at project sanction (after DG3). This may be aleatory uncertainties related to weather sensitive activities such as offshore hook-up work, marine operations or epistemic uncertainties related to project performance of new contractors or the use of novel technology and solutions. A definition of contingency is given in ASTM E2168 (2016) by the following primary classification:

> *"5.2.2 Contingency – A sum of money that is provided to cover the occurrence of unintended departures from the planned scope of work. Used in the absence of precise knowledge, and estimated to the best of one's knowledge to ensure that a financial buffer is available within a budget. Contingencies assist in mitigating the effects of unplanned events and other risks that are external to, and are not directly controllable within, a project plan."*

The contingency of the cost estimate seem to be based on deterministic based guidelines that may range from 40% for DG1 to 15% for DG3. The research results indicate that there is a mixture of probabilistic and deterministic approaches used. The P10, P50 and P90 confidence levels are probabilistic based while the contingency is deterministic based. The research results also indicate some inconsistency in how contingency is introduced. The contingency is either added to the P50 confidence level or included in the values of P10, P50 and P90 confidence levels.

The results confirm that there are large uncertainties in the prediction of the oil and gas production of a field development. These uncertainties could for example be related to subsurface issues in the form of optimistic reservoir characteristics or facility issues in the form of insufficient processing capacity or low production efficiency. Risk and ambiguity aversion and corresponding premiums do however not seem to be used in the value proposition for the overall net present value for a life of field estimate. The large uncertainties and the large consequences for a company in the event of large negative outcome would however warrant the use of risk and ambiguity premiums or contingencies. It should therefore be further assessed if risk and ambiguity premiums should be introduced to account for uncertainties related to the oil and gas production. Uncertainties in the income of the oil and gas production will also be affected by the external factors in the form of a volatile oil price that would warrant the use of ambiguity premiums or management reserves. A definition of reserve is given in ASTM E2168 (2016) by the following primary classification:

> *"5.2.3 Reserve – A sum, usually held by management (client) to be disbursed only when project requirements are changed. Used to provide insurance against a project or program failing to complete on budget of for the revision of a budget in the case of changed management or program direction and requirement."*

The results confirm that there are situations where potential large negative outcomes can result in a decision to stop an activity or to turn down or delay a project development. These types of decision problems with high consequence are also normally associated with ambiguity that need to be mitigated by pre-cautionary or cautionary measures before the continuation of an activity or a project.

3.2.1 The use of expected values and confidence intervals

The expected value or mean value represents the weighted average or the center of gravity of a probability distribution of the outcomes. The mean value is also referred to as the P50 confidence level although these may differ if the probability distribution is skewed. The P50 confidence level gives equal probability of underrun and overrun and the area under the probability curve is thus the same on both sides of the P50 value.

The questions included in the in-depth interviews were as follows:

- Are you familiar with the concepts of expected value or P50 confidence level?
- Do you know where the concepts of expected value or P50 confidence level are used in the decision process?

All responses from the oil and gas companies confirmed familiarization with the use of a P50 confidence level and also confirmed that the term expected value is less frequently used. The respondents further described that the P50 confidence level is normally used to determine a base estimate within cost and plan for a project. One of the respondents that is managing partner-operated assets used the P70 confidence level as the base estimate which then was considered to be close to the P50 confidence level added with the contingency reported by the asset operator. One respondent however referred to contingency as the difference between the base estimate and the P50 estimate. This shows that there are some differences in the interpretation and the build- up of the project cost estimates between the oil companies.

A respondent that has a role as portfolio manager for capital projects described that the term expected values is often misused. There can be occasions where managers refer to expected values with no underlying probabilistic basis or where the expected value is a mixture of probabilistic and deterministic approaches.

A respondent that is managing field developments described the assessments he performs before a project sanction. He then performs a detailed process where all potential downsides and upsides are challenged and conservative estimates and prognosis are included in order to take ownership and maximise the probability of success in the execution of the project.

Several respondents described that there can be large uncertainties and corresponding widespread confidence intervals in the subsurface results and advanced probabilistic analysis of uncertainties are therefore performed within this area.

The responses describe that the uncertainty represented by the P10 and P90 confidence levels is subject to detailed assessments of the elements that give the highest contribution to the confidence interval variation. Sensitivity analysis with more refined or skewed probability distributions may then be included in the detailed assessments in order to reduce the variation and give a better description of the relevant phenomena.

3.2.2 Are there any weighting of consequences?

A decision maker is neutral to loss or gain if a negative or positive outcome is given the same weighting. This also means that the decision maker has a linear relationship between potential negative and potential positive outcomes.

The questions included in the in-depth interviews were as follows:

- In a decision process, do you differentiate between a potential positive and a potential negative outcome?
- Do you know how this differentiation of outcomes can be done?
- Are you familiar with the term of being risk neutral?
- Do you agree that decisions based on expected values are a risk neutral approach?

The respondents that are managing projects describe that the weighting of potential upsides and downsides is based on a balanced judgement of the inputs from the members of the project team. A well performing project team will have members with different attitudes and judgements, some may have an optimistic attitude that pushes potential upsides and some may have a more skeptical attitude which can be used to address and find potential downsides. A mixture of these types of personalities are required in a project team in order to ensure a balanced view and steering of the outcomes from a project.

All responses confirmed that there is a high focus on how to reduce or mitigate the downsides or negative outcomes. One of the respondents from a larger operator considered however that there is a linear weighting relationship and high focus on both the negative and positive outcomes. Only one of the respondents was aware and familiar with analytical methods for a non-linear weighting of negative and positive outcomes. This respondent has previous experience as decision analyst.

Most of the other respondents described the introduction of skewed probability distributions, technical reviews, management reviews and cautionary measures to reduce or mitigate the potential negative outcomes.

3.2.3 Familiarity with risk and ambiguity premium and certainty equivalent

Tolerance of risk or aversion to risk is a subjective judgement and expresses the decision maker's aversion to losses or a potential negative outcome. The potential outcomes are then based on uncertainty assessments of the known unknowns of estimated quantities, rates and norms of the technical solution or scope. Ambiguity is a term used to describe uncertainty when you are not confident in assigning a probability distribution. This could refer to uncertainty of unspecified or unknown elements of a technical solution or scope. But it could also include externalities that lie outside of the project's control. When faced with risk and ambiguity the decision maker may therefore be willing to pay a premium in order to mitigate or neutralize his uncertainty exposure. The certainty equivalent for a project subject to risk and ambiguity is then the difference between a predicted expected value and the risk and ambiguity premiums.

The questions included in the in-depth interviews were as follows:

- Do you know how a decision maker's risk aversion or contingency is implemented in the decision process?
- Do you know how a decision maker's ambiguity aversion or budget reserve is implemented in the decision process?
- Are you familiar with the term certainty equivalent value and the use of contingency and budget reserves in a decision process?

Only one respondent confirmed that he was familiar with the terminology and use of risk premium, ambiguity premium and certainty equivalent. This respondent was the one with experience as decision analyst. All responses confirmed that contingency and reserves are commonly used in project cost definition. The contingency would then include elements of both risk and ambiguity for the defined technical solution and scope. The respondents describe that the project cost definition are normally well defined by detailed base cost estimates that is matured during the project stages leading up to the decision gate for project sanction and these are also updated at different stages during the project execution. The respondents describe the process of defining the contingency and reserve as a qualitative process where the 80% confidence interval and the corresponding elements with uncertainty are subject to detailed reviews by technical peers group and finally decided in management reviews. Contingency ranging from 5% to 20% of the base estimate is used at project sanction. Examples referred to by one of the respondents were related to the contingency cost of jacket fabrication and jacket installation. Jacket fabrication performed in a new fabrication yards are given a higher contingency contribution than the familiar fabrication yards with a strong track record of previous jacket fabrication deliveries to company. For the jacket installation, the company has experience with launched jackets and no or limited experience with lifted jackets. According to the respondent, the lifted jacket concept is therefore given a contingency contribution of 20% and the launched jacket is given a contingency contribution of 12%.

Reserve is also by some respondents referred to as the not-to-exceed target or management reserves. This reserve is normally not within the control of a project manager and is intended to account for external factors that lie outside of the project. This could be elements such as fluctuating market situations that will impact the cost and availability of contractor services or assets or major changes to the framing of the project. The respondents describe that the reserve is defined by a qualitative judgement of the members of the management review.

The respondents describe that the net present value of a field development normally includes cost based on the P50 confidence level. The respondents emphasise the importance of the management and coordination of the interface between the subsurface and facility groups in order to develop an optimum field development solution. One respondent noted that the project execution and project control may have been delivered within cost and plan, but the throughput in the form of oil and gas production were significant lower than the prognosis that formed the basis for the value proposition for the field development. The uncertainties in the oil and gas production for a field development are many and can be related to uptime, processing capacity or the presence of process bottlenecks or it can be related to subsurface issues relating to for example reservoir volumes, pressure or permeability.

3.2.4 Scenarios with positive outcomes

Project A and B were introduced to the respondents as two projects that only have three possible positive outcomes that may be associated with either risk or a combination of risk and ambiguity. Project A and B can for example be related to the potential upsides related to the improvement in reliability of a production plant.

It was explained to the respondents that the probability of outcome x_1 is defined and given a probability of p_1=1/3. It was also explained to the respondents that the probability cannot be uniquely described for the specific outcome of x_2 or x_3. The respondent was however reminded that the probability of either outcome x_2 or outcome x_3 is $p_2 + p_3 = 2/3$ due to the law of probability for a defined outcome set.

	Outcome sets		
Projects	x_1	x_2	x_3
A	100	0	100
B	0	100	100

The respondents were informed that the expected values for the projects are the sum of products of probabilities and outcomes expressed by $\mathbb{E} = p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3$. The respondent were also informed that the expected value for project A cannot be mathematically defined by a single value since $p_2 \in (0, 2/3)$ due to the uncertain probability distribution between p_2 and p_3. It was therefore explained to the respondents that the expected value for project A may range between 33 and 100. The respondents were then informed that the expected value for project B is uniquely defined despite $p_2 \in (0, 2/3)$ due to the defined probability distribution between p_1 and the sum of p_2 and p_3. The respondents were informed that the calculated expected value for project B is 67.

The questions included in the in-depth interviews were as follows:

- Please have a look at project A and project B, would you prefer one out of these two projects or do you consider them to be indifferent?
- Can you explain how you arrived at your choice between project A and project B?
- What would you consider as an appropriate expected value for project A?
- If you had more information about P10 and P90 for the projects would you then assign a reserve or buffer to the expected values to account for the statistical variation in project B?
- Would you assign a reserve or buffer to the expected values to account for the unknown unknowns in project A?
- Can you recall a similar decision setting as for project A where you are not able to describe the probabilities for a specific outcome?

Eight of the responses were a preference for project B which would indicate a preference for a reliable or predictable opportunity space. Three of the responses were indifferent between the projects A and B. The reason for this indifference was explained by a prediction of equal probabilities for all outcomes. Two of the responses were a preference for project A as the opportunity for maximizing the positive results was considered highest for project A. The responses on the expected value prediction for project A resulted in a mean value of 62.6 and a standard deviation of 9.6.

All responses confirmed that a margin for risk and ambiguity would be introduced. More background information on the decision setting and basis for the numbers presented for the projects is however required in order to assess the appropriate level of the required risk and ambiguity margin. Several of the respondents confirmed that projects with elements of variable or undefined probabilities are common. There can be many subprojects that have probabilities of variable degrees of precision that the project manager has to understand and have a feel or intuition on what to challenge and monitor.

3.2.5 Scenarios with negative outcomes

Project C and D were introduced to the respondents as two projects that only have three possible negative outcomes that may be associated with either risk or a combination of risk and ambiguity. Project C and D were explained to the respondents as an example related to potential cost increases for a contracted delivery. The respondents were informed that the probability of outcome x_1 is defined and given a probability of $p_1=1/3$. The respondents were informed that the probability is not defined for the specific outcome of x_2 or x_3. It was explained to the respondents that the probability of either outcome x_2 or outcome x_3 is $p_2 + p_3 = 2/3$ due to the law of probability for a defined outcome set.

Projects	Outcome sets		
	x_1	x_2	x_3
C	-100	0	-100
D	0	-100	-100

The respondents were given the same information as for the scenarios with the positive outcomes that the expected values for the projects are the sum of products of probabilities and outcomes and is expressed by $\mathbb{E} = p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3$. The respondents were reminded that the expected value for project C is not uniquely defined since $p_2 \in (0, 2/3)$ due to the uncertain probability distribution between p_2 and p_3. It was explained to the respondents that the expected value for project C may range between -33 and -100. The expected value for project D was informed to the respondents as being uniquely defined despite $p_2 \in (0, 2/3)$ due to the defined probability distribution between p_1 and the sum of p_2 and p_3. The respondents were informed that the calculated expected value for project D is -67.

The questions included in the in-depth interviews were as follows:

- Please have a look at project C and project D, would you prefer one out of these two projects or do you consider them to be indifferent?
- Can you explain how you arrived at your choice between project C and project D?
- What would you consider as an appropriate expected value for project C?
- If you had more information about P10 and P90 for the projects would you then assign a reserve or buffer to the expected values to account for the statistical variation in project B and D?
- Would you assign a reserve or buffer to the expected values to account for the unknown unknowns in project C?
- Can you recall a similar decision setting as for project C where you are not able to describe the probabilities for a specific outcome?
- Can you recall a past decision setting where you gave potential negative outcomes more weighting or in an extreme case only considered a negative outcome?

Five of the responses were a preference for project C. The reason for the preference for project C was explained as a higher opportunity for improving the result than for project D. One of the respondents were indifferent between the projects C and D. The reason for this indifference was explained by a prediction of equal probabilities for all outcomes. Five of the responses were a preference for project D. The reason for the preference for project D was higher reliability. The responses on the expected value prediction for project C is shown resulted in a mean value of -60.9 and a standard deviation of 11.1.

All responses confirmed that a margin for risk and ambiguity would be introduced. More background information on the decision setting and basis for the numbers presented for the projects are however required in order to assess the appropriate level of the required risk and ambiguity margin. The responses confirmed that a potential large negative outcome may be a cause to stop a project or a concept development. This could be presented in the form of a likelihood of exceeding a certain amount. As an example, barriers for the proximity of LQ to the process facilities resulted in a low expected safety factor which then turned this concept down. This focus on negative outcomes is typical when addressing safety, environment and quality issues. Within subsurface, one of the respondents noted that the decision to proceed with a field development project largely depends on the results from the appraisal wells. Poor results from appraisal wells can increase the uncertainty of a reservoir that can lead to a stop or delay of a field development project.

4 A business risk decision problem- Race or withdraw

The Challenger Launch decision (Brittain & Sitkin, 1990), (Liedtka, 1990) has by many organisations been used as a business risk case study for decision making with elements of risk and ambiguity. The quantitative analysis performed for the Carter racing case (Borgonovo & Marinacci, 2015) is a stylised example of a business risk decision that refers to the Challenger Launch Decision.

This business risk decision example could also be seen as a stylised example of the important business decisions an oil company often have to make. These decisions can for example be related to the selection of a Norwegian or an Asian yard for a platform contract or the trade-off between execution of the mechanical completion of a platform on an onshore fabrication site or alternatively at the offshore location.

4.1 Introduction

Carter is a professional racing driver that has to decide whether to run or withdraw from the next race. This decision problem is associated with business risk and been subject to decision analysis under uncertainty. The Carter Racing model (Borgonovo & Marinacci, 2015) address the probability p_2 of engine failure and the conditional probability p_1 of coming top five in the race given no engine failure. The outcome m_1 is the reward of top five, outcome m_2 is the result of ending the race below top five and outcome m_3 is the downside in the event of engine failure.

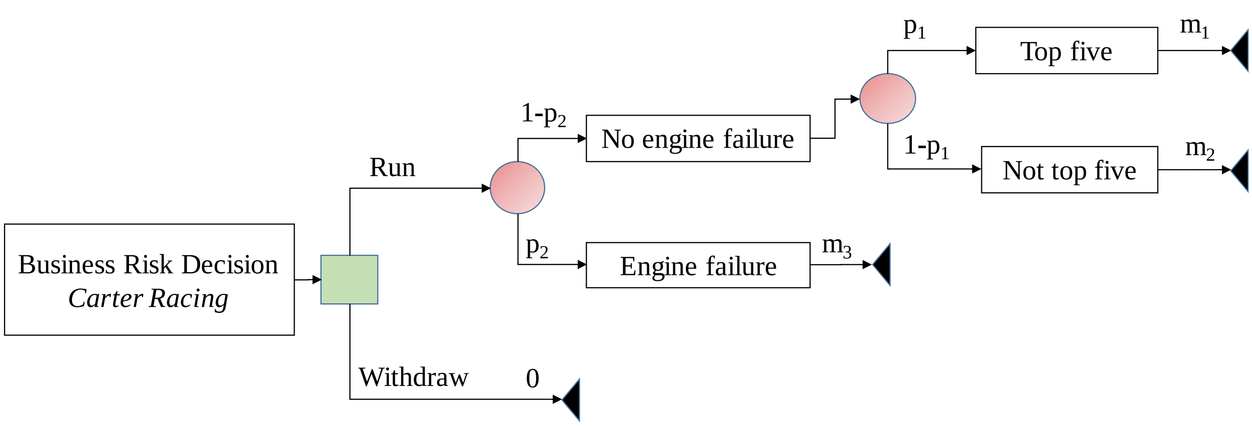


Figure 4.1.1 Carter racing model

4.2 Expected Value

In Borgonovo & Marinacci (2015), the outcome set is $\{m_1, m_2, m_3\} = \{1000, 0, -500\}$, the predicted probability of engine failure given as $p_2 = 0.30$ and the conditional probability of coming in top five given as $p_1 = 5/7$. The expected value of the decision to run or not to run can then be calculated as follows:

$H_{Run}=(1-p_2) \cdot [p_1 \cdot m_1+(1-p_1) \cdot m_2]+p_2 \cdot m_3=(1-0.3) \cdot [5/7 \cdot (1000)+(1-5/7) \cdot (0)]+0.3 \cdot (-500)= 350$

$H_{Withdraw}=0$

4.3 Smooth ambiguity aversion

In Borgonovo & Marinacci (2015), the smooth ambiguity aversion model is included. This is also referred to as the KMM functional (Klibanoff, et al., 2005). This model can be used to estimate risk and ambiguity premiums and the corresponding certainty equivalent. A second order probability distribution for the predicted probability is then first included in the Carter Racing model by a combination of two sets of beta distributions where the expected probability value is similar to predicted probability value. The expected value for the beta distribution with a=2 and b=6 is ¼ and the expected value for the beta distribution with a=6 and b=2 is ¾. The expected probability value for the second order probability distribution is adjusted to be similar to the predicted probability value by a factor m as given by the formula; $p_2^{Pred} = (1-m) \cdot beta(2,6) + m \cdot beta(6,2) = (1-m)\cdot 1/4 + m\cdot 3/4$. The factor m can therefore be calculated as; $m = (p_2^{Pred} - ¼) \cdot 2 = 0.10$.

The calculation of the risk and ambiguity premiums is an iterative process where the risk tolerance factor and the ambiguity aversion parameter have to be balanced by the use of the smooth ambiguity model (KMM functional) and the Arrow-Pratt (AP) quadratic estimation method. These methods are described in Appendix B.

The risk and ambiguity premiums are calculated by use of both the smooth ambiguity model (KMM functional) and the extended form of the AP quadratic estimation. The extended AP quadratic estimation also estimate the portion of risk premium and ambiguity premium. The certainty equivalent represents the difference between the predicted mean value and the risk and ambiguity premiums. The risk and ambiguity premium and corresponding certainty equivalent is shown in the table below and is balanced with a risk aversion parameter of 1/1000 and an ambiguity aversion parameter of 1/100.

	KMM functional		AP Quadratic Estimation	
Certainty equivalent, C(Run)=	-143.3[1]	-147.5[2]	-148.77[1]	-162.9[2]
Risk premium, Π_{AM}(Run)=	495.46[1]	497.56[2]	226.66[1]	226.25[2]
Ambiguity premium, Π_{AM}(Run)=			274.42[1]	286.65[2]

1) *Analysis results derived from model developed in Excel*
2) *Analysis results given in Borgonovo & Marinacci* (2015)

4.4 Risk aversion

The decision three for the Carter Racing decision problem has been modelled by use of the Precision Tree Software. A concave exponential function with a risk tolerance factor of 1000 have been used. The certainty equivalent that include risk aversion have then been calculated to C(Run)= 129.5. This is close to the certainty equivalent with risk aversion of C(Run)=123.75 as given in Borgonovo & Marinacci (2015).

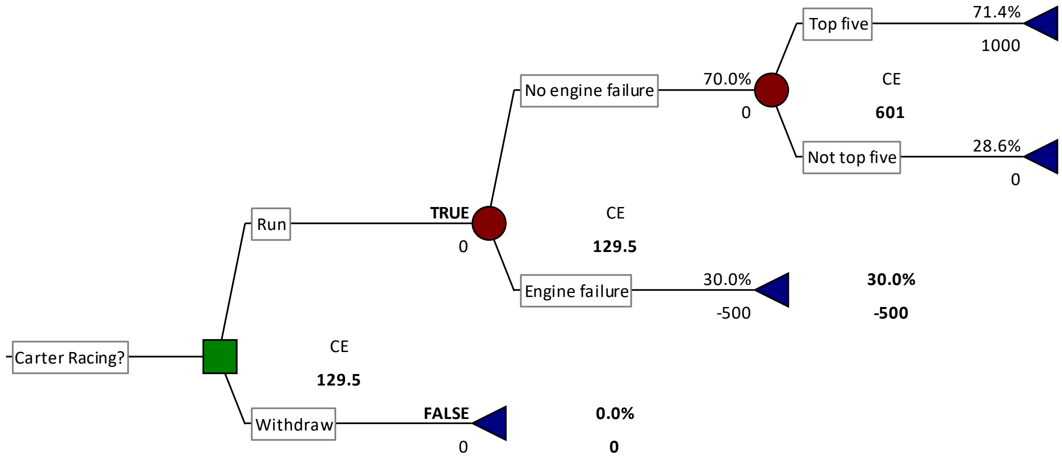


Figure 4.4.1 CE for Carter Racing with risk aversion

4.5 Extreme Risk and Ambiguity aversion

The certainty equivalent for the Wald Maximin Functional is the worst possible outcome for the decision alternatives. The certainty equivalent values for Wald Maximin Functional is therefore C(Run) = -500 and C(Withdraw) = 0.

4.6 Decision maker's preferences for the Carter Racing

The objective with the analysis was to assess the whether to run or withdraw from the race. Analysis have been performed to assess the expected values and the certainty equivalent values for the decision alternatives. Several methods for analysis the certainty equivalent values have been applied that addresses the preferences for a decision maker with risk aversion, risk and ambiguity aversion and extreme risk and ambiguity aversion. The results from these analysis methods can be summarised as follows:

- The expected value is maximised for the decision to run
- The risk and ambiguity averse certainty equivalent is maximised for the decision to withdraw from the race
- The risk averse certainty equivalent is maximised for the decision to run
- The extreme risk and ambiguity averse certainty equivalent is maximised for the decision to withdraw from the race

A risk neutral decision maker will have a preference to run since expected value is highest for this alternative. A risk and ambiguity averse decision maker will have a preference to withdraw since the certainty equivalent for the KMM functional and Arrow-Pratt Approximation is highest for this alternative. A risk averse decision maker will have a preference to run since the certainty equivalent calculated by use of the Precision Tree software is highest for this alternative. An extreme risk and ambiguity decision maker will have a preference to withdraw since the maximum worst consequence is minimised for this alternative.

5 A Cost Risk Decision Problem - Project contingency

Uncertainties associated with project cost estimates is defined as cost risk (Aven, 2012). Project contingency is normally introduced as a buffer or margin to address the cost risk in a project cost estimate. The project contingency for a project cost estimate has been analysed by the use of theoretical models that capture various levels of risk and ambiguity aversion.

5.1 Introduction

There may be uncertainties in the execution of a project that would warrant the use of contingency and reserve. This could relate to uncertainties in the design of solutions, uncertainties in the complexity and performance of fabrication or uncertainties in the marine operations, subsea construction or hook-up and completion activities. A subsea support structure has been developed through the feasibility, concept and FEED stages and a DG3 cost estimate need to be defined that address the uncertainties in the subsequent project execution. The objective with the decision analysis in this example is therefore to define the risk and ambiguity premium for the project execution phase. The risk and ambiguity premium would in this case be the same as the project contingency. The assessment will be based on methods for maximising expected utility. The expected utility will then further translate into certainty equivalents where risk and ambiguity premiums are introduced to address the uncertainties related to the distribution of the input variables for the cost estimate.

5.2 Probability distributions of input variables

The technical solution for the subsea support structure has been matured during the concept and FEED stages but there are still some fatigue sensitive details that has to be detailed in the detail engineering stage. The estimated weight of the subsea support structure is based on detailed drawings developed from the structural global and local analysis. The main members is therefore not likely to change during detail engineering but it is expected that more local stiffeners will be added both on the clamp and the leg sleeves. The installation method for the subsea support structure was defined during the FEED phase and was subject to HAZID review with participation from offshore operations supervisors from one of the potential EPCI contractors. The installation methods for the handling and design of required installation aids have therefore been defined but not developed in sufficient detail. The subsea installation is planned to be installed in the summer season and the critical lifting operations has to be performed in a weather restricted window with low sea states. The cost elements are listed in the table below and their minimum, most likely (ml) and maximum values have been included with triangular probability distributions.

Input parameters				Distributions		
Category	Average	Downside	Upside	Min	Ml	Max
Company Project Management	7.6	10%	-10%	6.84	7.6	8.36
Contractors Project Management	7.9	10%	-10%	7.11	7.9	8.69
Installation Engineering[1]	16.1	10%	-10%	14.49	16.1	17,71
Design Engineering and Verification[1]	11.5	10%	-10%	10.35	11.5	12.65
Production of Clamps and Bracings[1]	40	30%	-10%	36	40	52
FAT	2.8	10%	-10%	2.52	2.8	3.08
Production of temporary Clamps	1.2	50%	-10%	1.08	1.2	1.8
Vessel Mobilisation	4.9	10%	-10%	4.41	4.9	5.39
Subsea Installation of Clamps and Bracings[1]	43.5	140%	-10%	39.15	43.5	104.4
Waiting on Weather (WOW)	6.1	140%	-60%	2.44	6.1	14.64
Base Estimate	141.6					

1) *The subsea construction cost and the cost elements related to installation engineering, design engineering and verification and production of clamps and bracings have been included in the analysis with a positive correlation.*
2) *Units in MNOK*

Table 5.2 Input variables to the cost estimate

5.3 Expected value and confidence interval

Stochastic simulations are performed by the use of @Risk in order to define the mean value and the 100% confidence interval represented by the minimum and maximum levels of the cost estimate.

	Min	Max	Mean
Cost estimate (MNOK)	130.1	218.9	164.7

5.4 Smooth ambiguity aversion model

The Carter Racing model (Borgonovo & Marinacci, 2015) is a numerical model of the smooth ambiguity aversion model also referred to as the KMM functional (Klibanoff, et al., 2005). This model can be used to estimate risk and ambiguity premiums and the corresponding certainty equivalent.

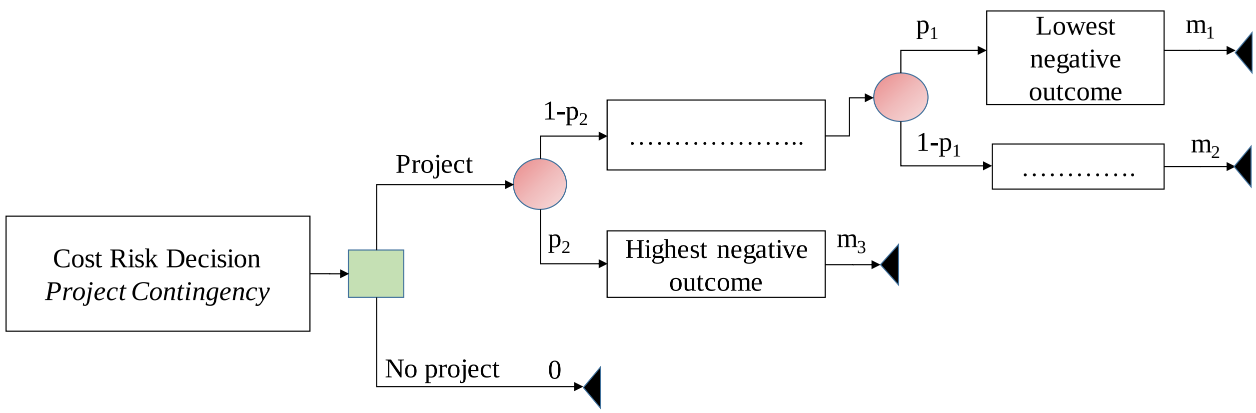


Figure 5.3.1 Analysis Model for Project Contingency

The outcome set is $\{m_1, m_2, m_3\} = \{-130.1, 0, -218.9\}$, where m_1 is the minimum negative value and m_3 is the maximum negative value of the cost estimate. The probability p_1=1 and the predicted probability p_2 is calculated based on the relationship between the minimum, mean and maximum values by the formulae: $p_2^{Pred} = \frac{(Mean-m1)}{(m3-m1)} = 0.389$. The secondary probability distribution for the predicted probability is included in the Carter Racing model by a combination of two sets of beta distributions where the expected probability value is similar to predicted probability value. The expected value for the beta distribution with a=2 and b=6 is ¼ and the expected value for the beta distribution with a=6 and b=2 is ¾. The expected probability value for the secondary probability distribution is adjusted to be similar to the predicted probability value by a factor m as given by the formula; $p_2^{Pred} = (1-m) \cdot beta(2,6) + m \cdot beta(6,2) = (1-m) \cdot 1/4 + m \cdot 3/4$. The factor m can therefore be calculated as; $m = (p_2^{Pred} - ¼) \cdot 2 = 0.279$.

The calculation of the risk and ambiguity premiums is an iterative process where the risk tolerance factor and the ambiguity aversion parameter have to be balanced by the use of the smooth ambiguity model (KMM functional) and the Arrow-Pratt (AP) quadratic estimation method. The risk and ambiguity premiums are calculated by use of both the smooth ambiguity model (KMM functional) and the extended form of the Arrow-Pratt quadratic estimation. The extended Arrow-Pratt quadratic estimation also estimate the portion of risk premium and ambiguity premium. The certainty equivalent represents the difference between the predicted mean value and the risk and ambiguity premiums. The risk and ambiguity premium and corresponding certainty equivalent is shown in the table below and is balanced with a risk aversion parameter of 1/220 and an ambiguity aversion parameter of 5/220.

	KMM functional	AP Quadratic Estimation
Certainty equivalent	-174.5	-174.1
Risk premium	9.8	4.3
Ambiguity premium		5.4

5.5 Risk aversion model

The predicted probabilities and the minimum and maximum cost estimate values have been used to construct a decision three by the use of the Precision Three Software. Risk aversion is included in the Precision Three Software by the use of a risk tolerance factor of 220 and an exponential concave utility function. The certainty equivalent derived from the precision three software analysis is -169.1 as shown in the decision tree below.

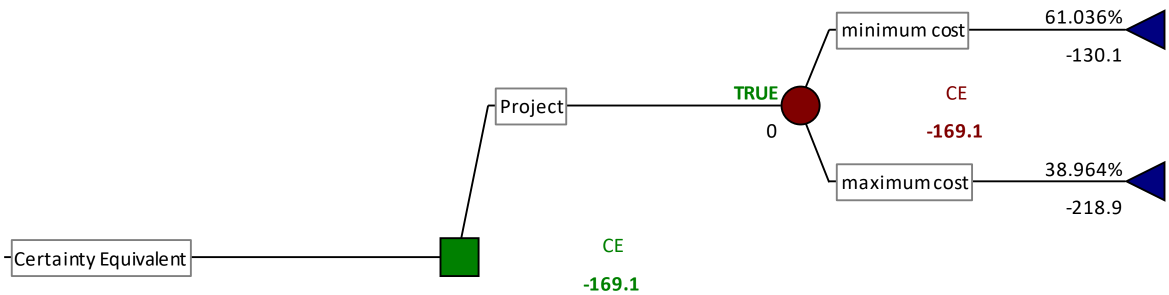


Figure 5.4.1 Decision Tree for CE with Risk Aversion

5.6 Resulting confidence levels for the project cost estimate

A risk and ambiguity premium of 9.8 MNOK and the corresponding Certainty Equivalent of 174.5 MNOK is found by the smooth ambiguity aversion model which is similar to the P70 confidence level for the cost estimate as shown in figure 5.5.1. The Certainty Equivalent of 169.1 MNOK found by the use of the risk aversion model in the Precision Tree Software is similar to the P62 confidence level of the cost estimate as shown in figure 5.5.2. In Rothwell (2004), the project cost contingency of a cost estimate has been evaluated and found to be approximated by the standard deviation of the cost estimate. The mean value plus the standard deviation then becomes 184.3 MNOK which is similar to the P82 confidence level as shown in figure 5.5.3.

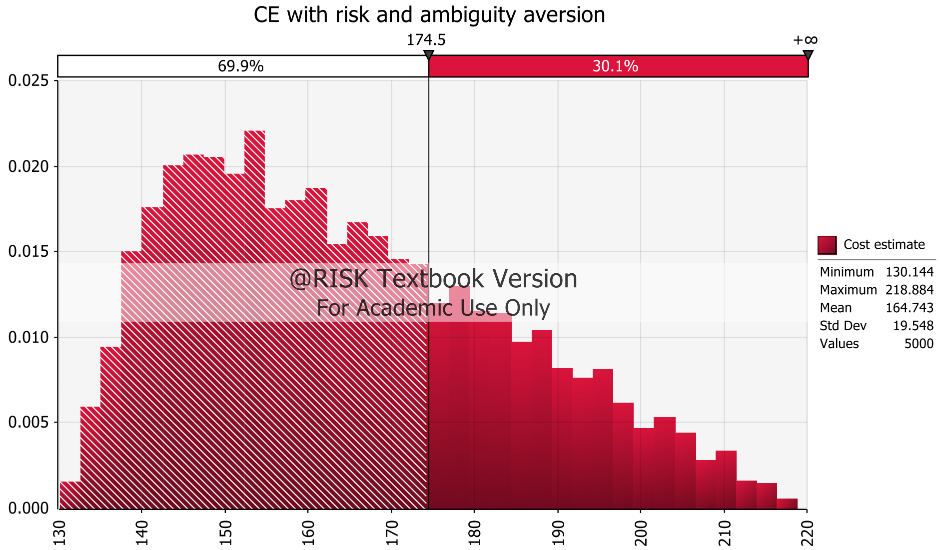


Figure 5.5.1 Confidence Level and Certainty Equivalent with Risk and Ambiguity Aversion

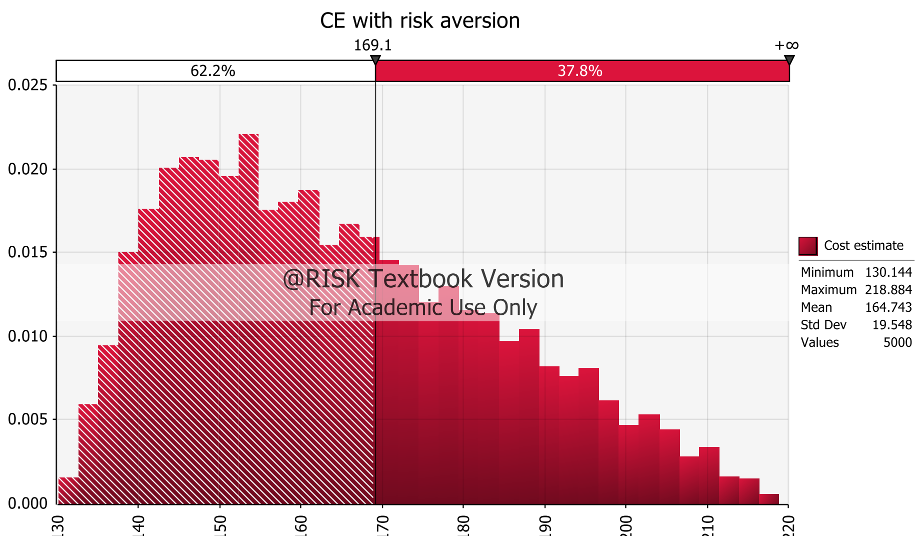


Figure 5.5.2 Confidence Level and Certainty Equivalent with Risk Aversion

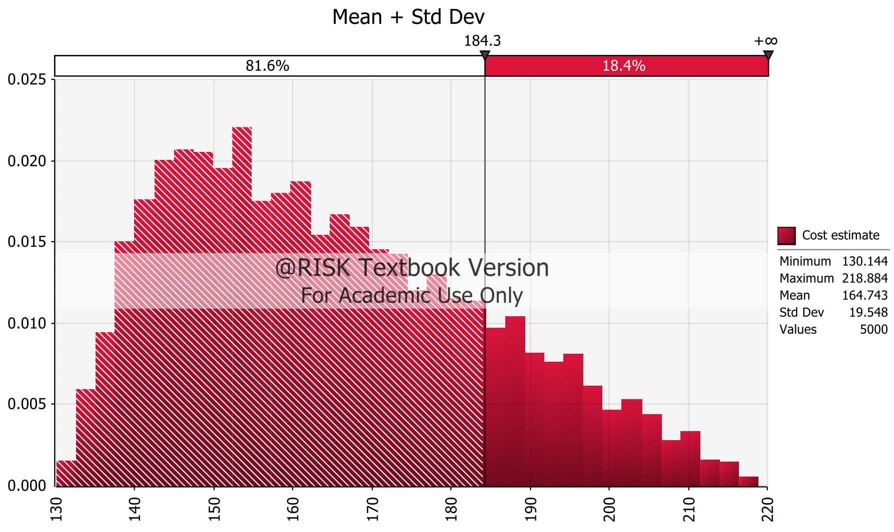


Figure 5.5.3 Confidence Level and Standard Deviation

6 A Production Risk Decision Problem - Ranking of scenarios

Production risk may refer to uncertainties associated with a certain performance measure (Aven, 2012) and can refer to potentially reduced production efficiency as a result of downtime caused by equipment failures and maintenance.

In this example, a ranking of project development scenarios have been performed that address Capex, Opex and production efficiency and income. The scenario with the highest certainty equivalent net present value has been given the highest ranking.

6.1 Introduction

There may be uncertainties in the oil and gas production that would warrant the use of contingency and reserve to account for uncertainties and risk and ambiguity aversion. This could relate to uncertainties in the subsurface characteristics, uncertainties in the facility performance or uncertainties in the oil price. How to introduce the method of risk and ambiguity premiums in the net present value of the production for a marginal field is described.

A marginal field in the North Sea is planned to be developed where the production from the field will be transferred to an existing process platform through a subsea pipeline. The production from the marginal field can be performed by a subsea system or by two alternative wellhead platform solutions.

The following project development scenarios are defined:

Scenario 1: Production from a subsea system with tie-back to the host processing platform (Subsea)

Scenario 2: Production from a wellhead platform with sea access and tie-back to the host processing platform (SAW)

Scenario 3: Production from a wellhead platform with helicopter access and production tie-back to the host processing platform (HAW)

The objective with the decision analysis in this example is to perform a ranking of the above scenarios. The ranking of the scenarios will be based on methods for maximising expected value and maximising expected utility of the net present values. The expected utility will then further translate into certainty equivalents where risk and ambiguity premiums are introduced to address the uncertainties related to distribution of the net present value.

6.2 Probability distributions of input variables

The future oil price used in the analysis is assumed to be approximately represented by the statistical variation of the oil price between 1990 and 2018. In this period the minimum oil price was 12.3 USD and the maximum oil price was 109.5 USD. The mean oil price in the period was 46.5 USD with a standard deviation of 32.2. The above parameters has been included in the analysis for all three scenarios by the use of the beta distribution. This distribution is shown in figure E.1.

The predicted average and the predicted potential downside and upside of the cost elements for Capex and Opex have been provided as input to the analysis. These cost elements have been included with the use of triangular probability distributions. The predicted minimum and maximum uptime for each of the scenarios have been provided as input to the analysis. The probability distribution between the minimum and maximum uptime is not defined and uniform probability distributions are therefore included in the analysis for range in uptime. The input variables are summarised in table E.1.

6.3 Production profiles

Production profiles based on the predicted uptime for the each of the scenarios have been provided as input to the decision analysis and shown in figure E.2. The production profiles have been multiplied with the uptime probability distribution and divided by the predicted uptime value in order to include the stochastic variation of uptime.

6.4 Expected value and confidence interval for the net present values

Stochastic simulations are performed by the use of @Risk in order to define the mean value and the P10 and P90 confidence levels of the net present value for the scenarios. Tornado graphs are presented in figure E.3. These show the sensitivity of the NPV mean values as a function of the input distribution. The tornado graphs show the dominating effect of the oil price. The analysis results for the 80% confidence interval for the net present value are presented in the table below.

Scenario	P10	P90	Mean
1	-1441	10122	3985
2	-1383	10394	4165
3	-1967	9795	3625

6.5 Smooth ambiguity aversion model

The Carter Racing model (Borgonovo & Marinacci, 2015) is a numerical model of the smooth ambiguity aversion model also referred to as the KMM functional (Klibanoff, et al., 2005). This model can be used to estimate risk and ambiguity premiums and the corresponding certainty equivalents.

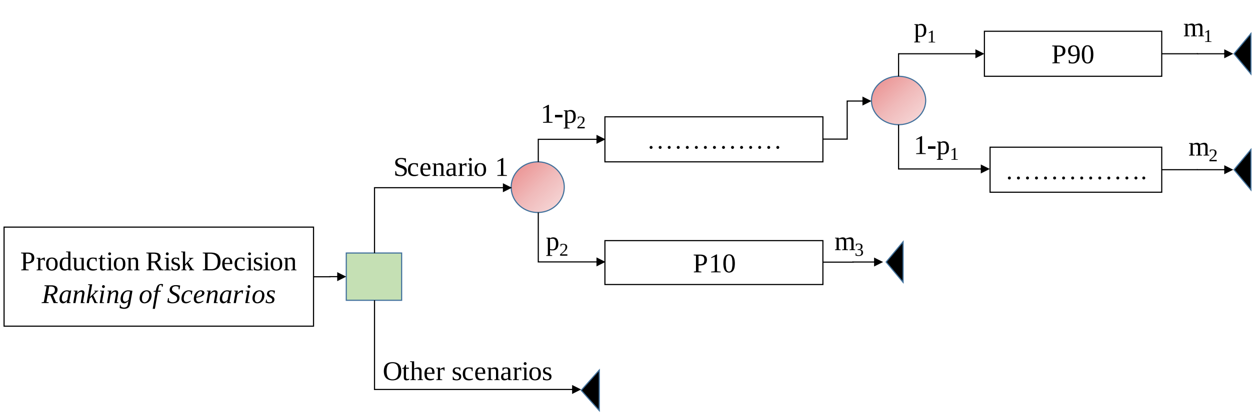


Figure 5.4.1 Analysis Model for Scenario 1

The outcome sets X_1, X_2 and X_3 in the model represents the calculated net present values for the P10 and P90 confidence levels.

$X_1 = \{m_1, m_2, m_3\} = \{10122, 0, -1441\}$

$X_2 = \{m_1, m_2, m_3\} = \{10394, 0, -1383\}$

$X_3 = \{m_1, m_2, m_3\} = \{9795, 0, -1967\}$

The probability p_1=1 and the predicted probability for p_2 is calculated based on the relationship between the values for P10, Mean and P90 by the formulae: $p_2^{Pred} = \frac{(Mean - P90)}{(P10 - P90)}$. The calculated predicted probabilities are then found to be 0.531, 0.529 and 0.525 for scenario 1, 2 and 3.

The secondary probability distribution for the predicted probability is included in the Carter Racing model by a combination of two sets of beta distributions where the expected probability value is similar to predicted probability value. The expected value for the beta distribution with a=2 and b=6 is ¼ and the expected value for the beta distribution with a=6 and b=2 is ¾. The expected probability value for the secondary probability distribution is adjusted to be similar to the predicted probability value by a factor m as given by the following formula:

$p_2^{Pred} = (1-m) \cdot beta(2,6) + m \cdot beta(6,2) = (1-m) \cdot 1/4 + m \cdot 3/4$

The factor m can therefore be expressed as; $m = (p_2^{Pred} - ¼) \cdot 2$ and the m factors are then found to be 0.561, 0.557 and 0.549 for scenario 1, 2 and 3. The calculation of the risk and ambiguity premiums is an iterative process where the risk tolerance factor and the ambiguity aversion parameter have to be balanced by the use of the smooth ambiguity model (KMM functional) and the Extended Arrow-Pratt quadratic estimation method. The risk and ambiguity premiums are calculated by use of both the smooth ambiguity model (KMM functional) and the extended form of the Arrow-Pratt quadratic estimation. The extended Arrow-Pratt quadratic estimation also estimate the portion of risk premium and ambiguity premium. The certainty equivalent for each of the scenarios represents the difference between the predicted net present value and the risk and ambiguity premiums. The results using these two methods are shown in the table below.

Method/Scenario	1	2	3
KMM functional			
- Certainty equivalent	2393[1]	2458[1]	1864[2]
- Risk and ambiguity premium	1592[1]		
Ext. AP quadratic estimation			
- Certainty equivalent	2328[1]	2456[1]	1852[2]
- Risk premium	1631[1]	1691[1]	1689[2]
- Ambiguity premium	27[1]	28[1]	112[2]

1) *Results are based on a risk aversion parameter of a= 1/10200 and an ambiguity aversion parameter of b=1.05/10200*
2) *Results are based on a risk aversion parameter of a= 1/10200 and an ambiguity aversion parameter of b=1.2/10200*

6.6 Risk aversion model

The predicted probabilities and the P10 and P90 confidence values have been used to construct a decision three by the use of the Precision Three Software. Risk aversion is included in the Precision Three Software by the use of a risk tolerance factor and an exponential concave utility function. The risk tolerance factor used is consistent with the Carter Racing model. The decision three is shown in figure E.5. The certainty equivalents derived from the precision three analysis are summarised below.

Risk aversion model /Scenario	1	2	3
Certainty equivalent	2439	2562	2020

6.7 Extreme risk and ambiguity aversion model

The preferences based on the Wald Maximin Functional are based on a comparison of the worst possible consequences. The certainty equivalents for the Wald Maximin Functional can therefore be represented by the minimum net present values as shown in the table below.

Maxmin model /Scenario	1	2	3
Certainty equivalent	-4729	-4968	-5076

6.8 Decision maker's preferences for the production scenarios

Analysis have been performed addressing both a beta distribution and a uniform distribution of the oil price. The minimum and maximum values for these distributions are the same and are based on the past statistical variation of the oil price. The selected beta distribution has a higher weighting on the low values and do therefore give more conservative results. The results of the analysis have been performed by the use of stochastic simulations of the net present value of the production. The objective with the analysis was to define a ranking of the three alternative scenarios. Analysis have therefore been performed by to assess the expected values (mean values) and the certainty equivalent values. Several methods for analysis the certainty equivalent values have been applied that addresses the preferences for a decision maker with risk aversion, risk and ambiguity aversion and extreme risk and ambiguity aversion.

The results from these analysis methods can be summarised as follows:

- The expected value is maximised for scenario 2
- The risk and ambiguity averse certainty equivalent is maximised for scenario 2
- The risk averse certainty equivalent is maximised for scenario 2
- The maximum worst consequence is minimised for scenario 1

A risk neutral decision maker will have a preference for scenario 2 since expected value is highest for this alternative.

A risk and ambiguity averse decision maker will also have a preference for scenario 2 since the certainty equivalent for the KMM functional is highest for this alternative.

A risk averse decision maker will have a preference for scenario 2 since the certainty equivalent for the risk aversion model is highest for scenario 2.

An extreme risk and ambiguity decision maker will struggle to take a decision since the extreme negative outcomes reflects negative net present values which would indicate that none of the alternatives should be further pursued. If he however has to give his preference he would prefer scenario 1.

Input parameters					Distributions		
Category	Average	Upside	Downside	Unit	Min	Ml	Max
Subsea Capex facility	3000	-10%	20%		2700	3000	3600
SAW Capex facility	2800	-10%	20%	MMNOK	2520	2800	3360
HAW Capex facility	3000	-10%	10%		2700	3000	3300
Subsea Capex wells	3200	-15%	20%		2720	3200	3840
SAW Capex wells	3200	-15%	20%	MMNOK	2720	3200	3360
HAW Capex wells	3200	-10%	15%		2880	3200	3680
Subsea Opex facility	40	-15%	10%		34	40	44
SAW Opex facility	40	-10%	30%	MMNOK per Year	36	40	52
HAW Opex facility	60	-10%	20%		54	60	72
Subsea Opex wells	40	-20%	30%		32	40	52
SAW Opex wells	30	-10%	20%	MMNOK per Year	27	30	36
HAW Opex wells	20	-10%	10%		18	20	22
Subsea maintenance	500	-10%	10%		450	500	550
SAW maintenance	3000	-10%	10%	Mhr per Year	2700	3000	3300
HAW maintenance	6000	-10%	10%		5400	6000	6600
Subsea Uptime	0.95	2%	-3%		92%	95%	97%
SAW Uptime	0.96	2%	-2%		94%	96%	98%
HAW Uptime	0.98	1%	-1%		97%	98%	99%
Oil price				USD/BOE	12.3	46.5	109.8
Conversion	7.94	15%	-15%	NOK/USD	6.75	7.94	9.13
Discount rate	0.1	-2%	2%		8%	10%	12%
Manhour rate	1500	-10%	10%	NOK/hour	1350	1500	1650

Table 6.1 Input parameter to the stochastic analysis

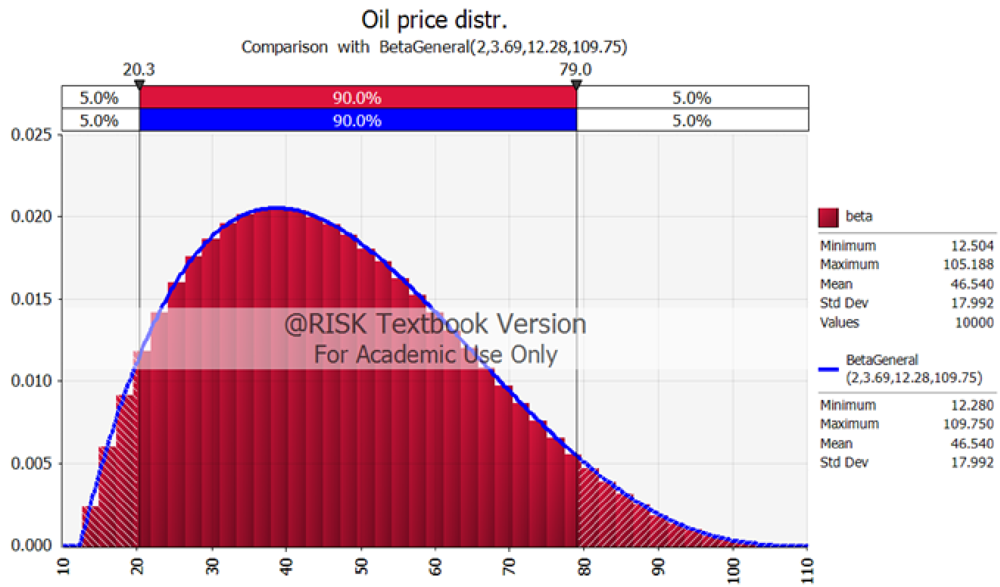
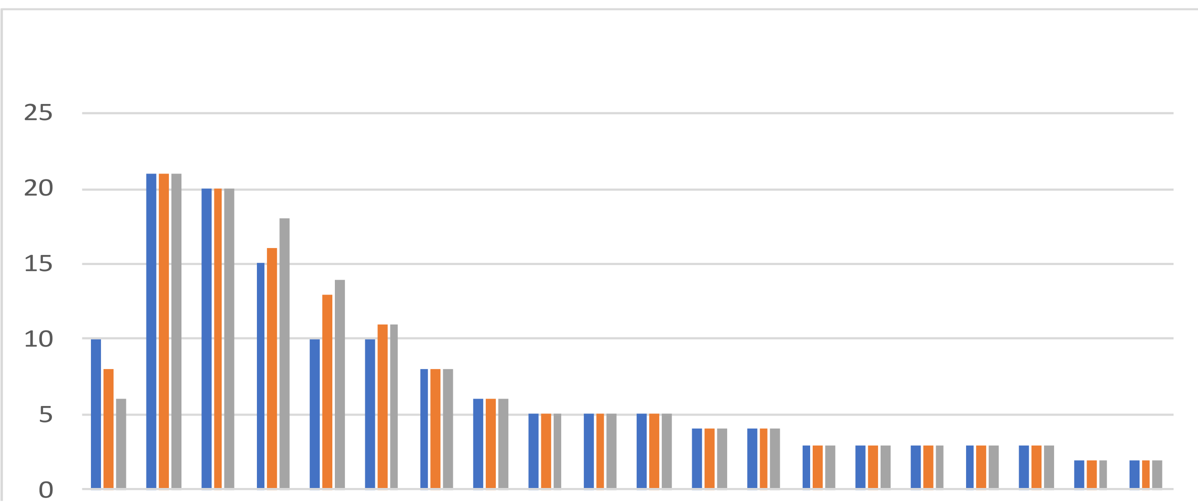


Figure 6.1 Beta distribution of oil price



	Yr1	Yr2	Yr3	Yr4	Yr5	Yr6	Yr7	Yr8	Yr9	Yr10	Yr11	Yr12	Yr13	Yr14	Yr15	Yr16	Yr17	Yr18	Yr19	Yr20	
Subsea	10	21	20	15	10	10	8	6	5	5	5	4	4	3	3	3	3	3	2	2	142
SAW	8	21	20	16	13	11	8	6	5	5	5	4	4	3	3	3	3	3	2	2	145
HAW	6	21	20	18	14	11	8	6	5	5	5	4	4	3	3	3	3	3	2	2	146

Figure 6.2 Production profiles with predicted uptime

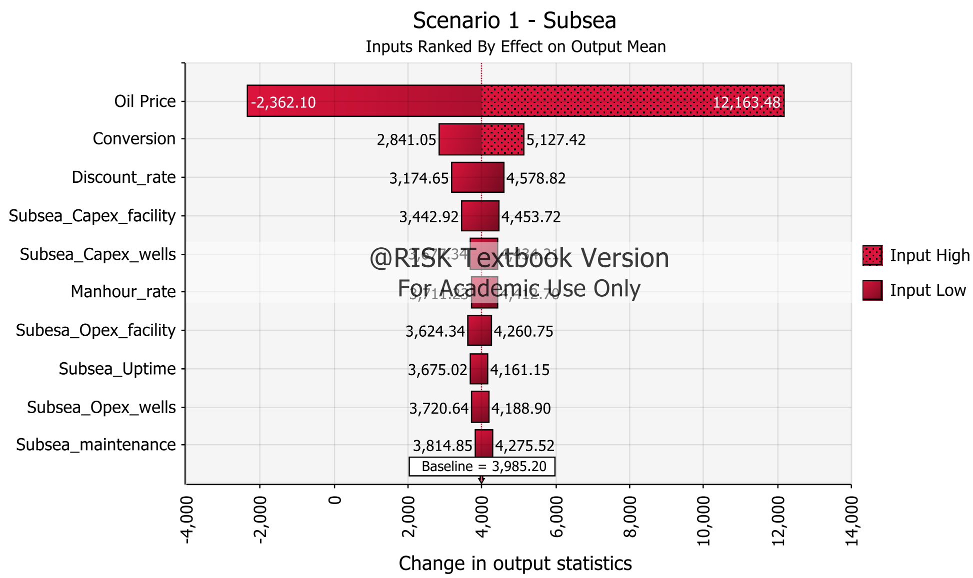


Figure 6.3 Tornado diagram for scenario 1

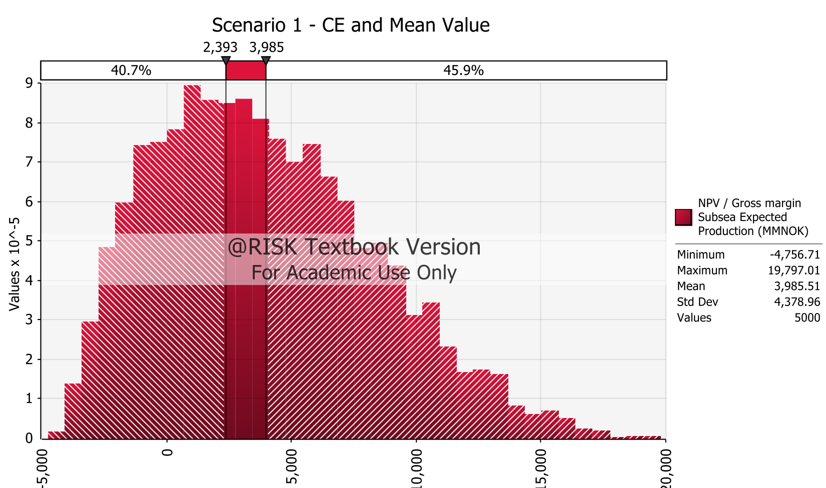


Figure 6.4 Scenario 1 – Certainty equivalent with risk and ambiguity aversion

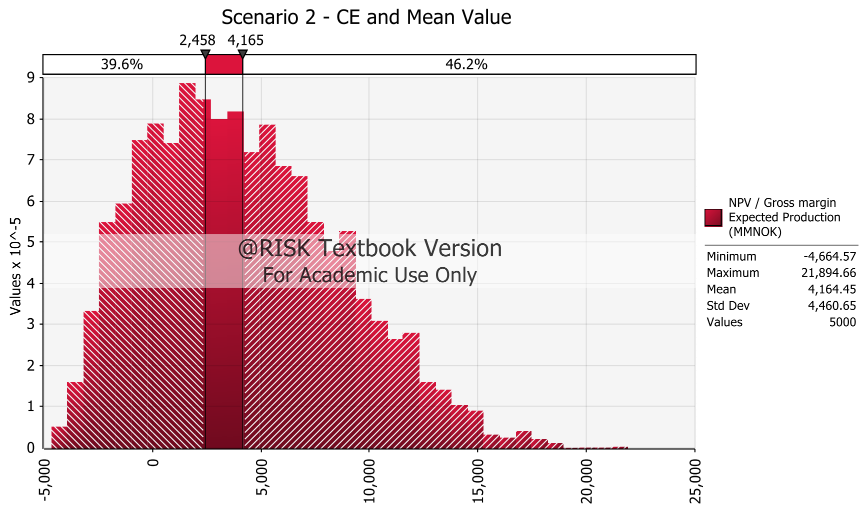


Figure 6.5 Scenario 2 – Certainty equivalent with risk and ambiguity aversion

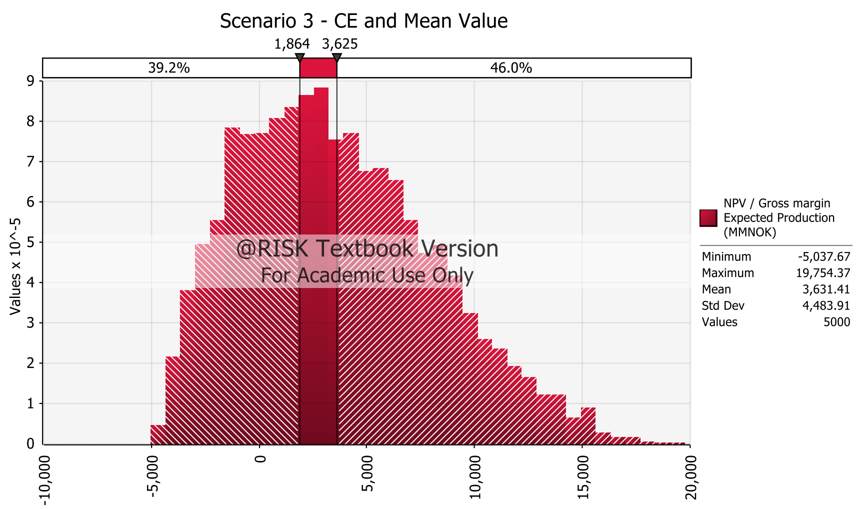


Figure 6.6 Scenario 3 - Certainty equivalent with risk and ambiguity aversion

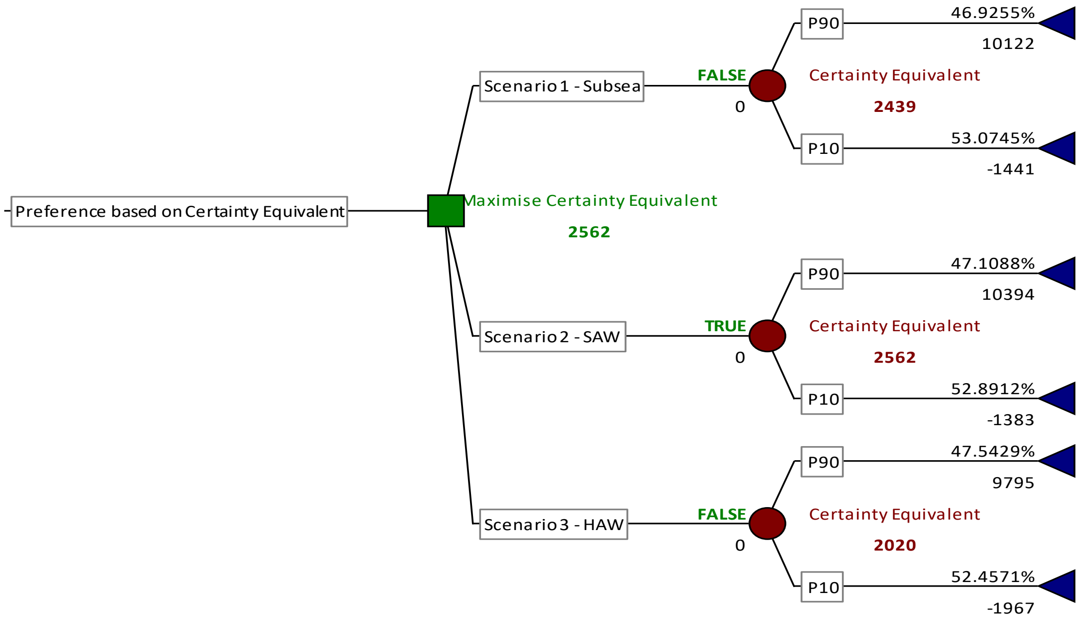


Figure 6.7 Decision tree for the Certainty equivalent with Risk Aversion

7 An Accident Risk Decision Problem – The need for safety valve

The assessment of the need to have a safety valve on the pipeline is a decision problem that is associated with an accident risk with large potential consequences.

This accident risk decision problem can be categorized to fall within an extreme safety perspective (Soerskaar & Abrahamsen, 2017). The certainty equivalent for various levels of risk and ambiguity aversion with and without safety valve has been addressed.

7.1 Introduction

There may be undesired events associated with an oil and gas production that have a potential for large negative outcomes. These types of events have a low probability of occurrence and could for example relate to the integrity of the oil & gas facilities. A marginal field in the North Sea is planned to be developed where the production from the field will be transferred to an existing process platform through a subsea pipeline. The production from the marginal field is planned performed based on scenario 2 which includes production from a wellhead platform with sea access (SAW) and tie-back to the host processing platform. The objective with the decision analysis in this example is to assess whether or not to install a subsea isolation valve (SSIV) on the export pipeline between the SAW and the Host Platform. The assessment will be based on methods for maximising expected value and maximising expected utility of the net present values. The expected utility will then further translate into certainty equivalents where risk and ambiguity premiums are introduced to address the uncertainties related to distribution of the net present value.

7.2 Probability distributions of input variables

The future oil price used in the analysis is assumed to be approximately represented by the statistical variation of the oil price between 1990 and 2018. In this period the minimum oil price was 12.3 USD and the maximum oil price was 109.5 USD. The mean oil price in the period was 46.5 USD with a standard deviation of 32.2. The above parameters has been included in the analysis by the use of the beta distribution. (Figure 6.1)

The predicted average and the predicted potential downside and upside of the cost elements for Capex and Opex and the predicted minimum and maximum uptime for each of the scenarios have been provided as input to the analysis. The Capex and Opex cost elements have been included with the use of triangular probability distributions. The probability distribution between the minimum and maximum uptime is not defined and uniform probability distributions are therefore included in the analysis for the range in uptime. The input parameter of the net present value for scenario 2 without SSIV is the same as presented in the previous chapter that describe the ranking of the production scenarios. The additional input parameters for the net present value for scenario 2 with SSIV installed include an expected investment cost for the SSIV (Abrahamsen, et al., 2004) and an annual expected cost for inspection and maintenance of the SSIV (Abrahamsen, et al., 2004). The input variables are summarised in table 7.2 below.

Input parameters				Distributions		
Category	Average	Upside	Downside	Min	Ml	Max
SAW Capex facility[1]	2800	-10%	20%	2520	2800	3360
SAW Capex wells[1]	3200	-15%	20%	2720	3200	3360
SAW Opex facility[2]	40	-10%	30%	36	40	52
SAW Opex wells[2]	30	-10%	20%	27	30	36
SAW maintenance[3]	3000	-10%	10%	2700	3000	3300
SAW Uptime	0.96	2%	-2%	94%	96%	98%
Oil price[4]				12.3	46.5	109.8
Conversion[5]	7.94	15%	-15%	6.75	7.94	9.13
Discount rate	0.1	-2%	2%	8%	10%	12%
Manhour rate[6]	1500	-10%	10%	1350	1500	1650
SSIV Capex[1]	75					
SSIV Opex[2]	2					

1) *Units in MNOK*
2) *Units in MNOK/Year*
3) *Units in Mhrs/Year*
4) *Units in USD/BOE*
5) *Units in NOK/USD*
6) *Units in NOK/Mhr*

Table 7.2 Input Variables

7.3 Production profiles

Production profiles based on the predicted uptime for the each of the scenarios have been provided as input to the decision analysis and shown in figure 6.2. The production profiles have been multiplied with the uptime probability distribution and divided by the predicted uptime value in order to include the stochastic variation of uptime.

7.4 Expected value and confidence interval given no riser failure

Stochastic simulations are performed by the use of @Risk in order to define the mean value and the P10 and P90 confidence levels of the net present value for scenario 2 with and without SSIV installed. The stochastic simulation results of the net present value for scenario 2 without SSIV is the same as presented in the previous chapter that describe the ranking of the production scenarios.

Scenario	Net present value without SSIV			Net present value with SSIV		
	P10	P90	Mean	P10	P90	Mean
2	-1383	10394	4165	-1509	10295	4075

7.5 Consequence of riser failure

A failure of a pipeline between the producing unit and the host platform could have large consequences (Abrahamsen, et al., 2004). If there is no SSIV installed, the continued production from the SAW after a pipeline failure could result in many fatalities, severe damage to the host platform and years of lost production. If a SSIV is installed the consequence of a pipeline failure is reduced. An estimate of the overall cost impact in the event of a pipeline failure is based on the expected costs described in Abrahamsen, et al. (2004) and shown in the table below with and without SSIV. The monetary values are given in MNOK.

	Consequence without SSIV	Consequence with SSIV
Expected number of fatalities	5	0.5
Expected damage cost	800	200
Expected loss of income	40000	8000
Value of a Statistical Life (VSL)	30	30
Total	40950	8215

7.6 Certainty Equivalent using the Expected Monetary Value

The expected monetary value (EMV) is the certainty equivalent when the expected value method is used. The expected values is shown in the table below and also shown in the decision tree shown overleaf.

	Without SSIV	With SSIV
CE – Expected value	4160	4074

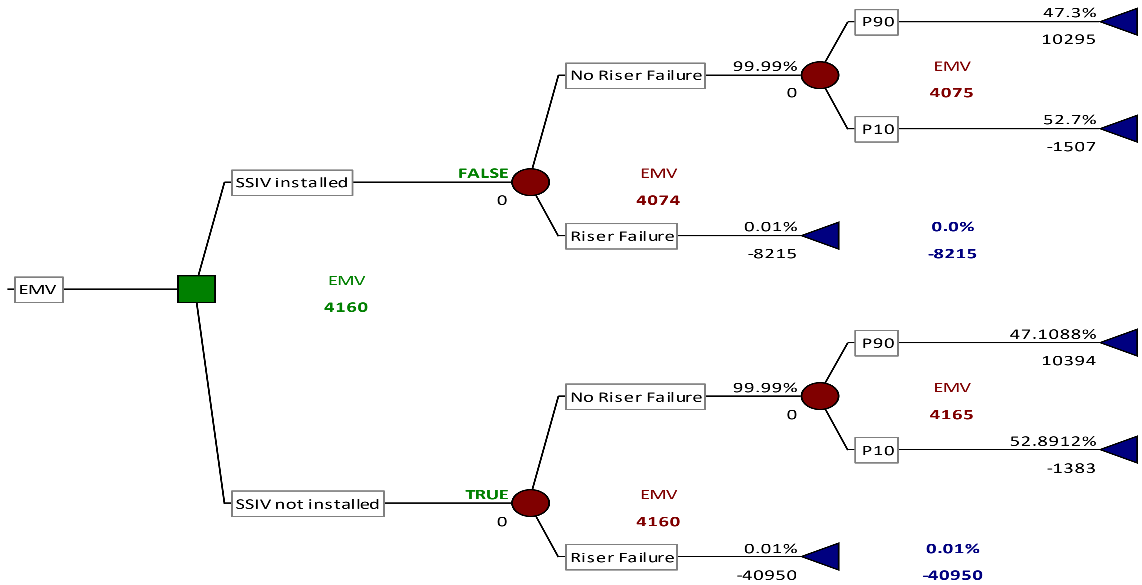


Figure 7.6.1 Decision tree - Expected Monetary value

7.7 Certainty Equivalent using the Smooth Ambiguity Aversion Model

The Carter Racing model (Borgonovo & Marinacci, 2015) is a numerical model of the smooth ambiguity aversion model also referred to as the KMM functional (Klibanoff, et al., 2005). This model is used to estimate risk and ambiguity premiums and the corresponding certainty equivalents.

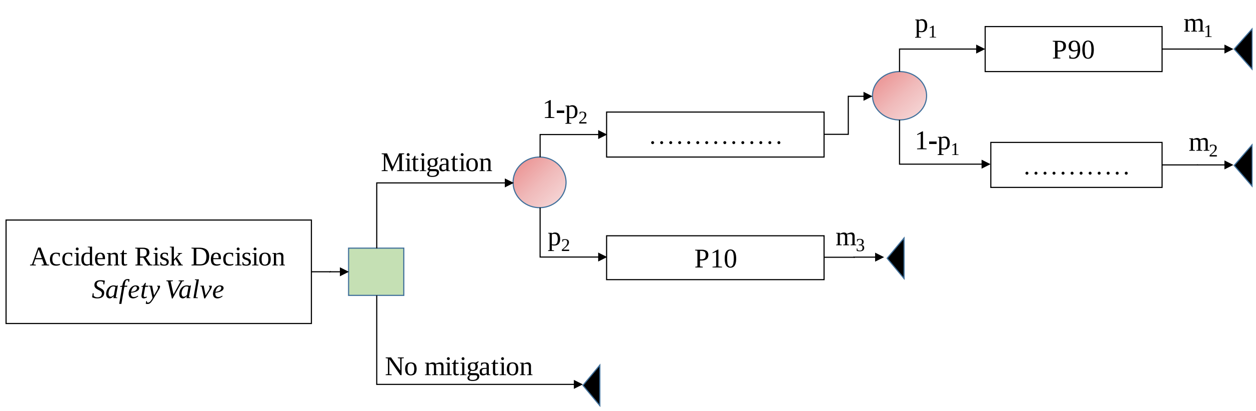


Figure 7.7.1 Analysis Model for Safety Valve

7.7.1 Certainty equivalent conditional of no riser failure

The outcome set given no riser failure is the calculated net present values for the P10 and P90 confidence levels. The outcome set without SSIV is therefore $\{m_1, m_2, m_3\} = \{10394, 0, -1383\}$ and outcome set without SSIV is $\{m_1, m_2, m_3\} = \{10295, 0, -1509\}$. The probability of p_1=1 and the predicted probability for the outcome set is calculated based on the relationship between the values for P10, Mean and P90 by the formulae; $P10_{Predicted} = ((Mean - P90))/((P10 - P90))$. The predicted probability for p_2 is thus 0.529 without SSIV and 0.527 with SSIV. The second order probability distribution for the predicted probability is included in the Carter Racing model by a combination of two sets of beta distributions where the expected probability value is similar to predicted probability value. The expected value for the beta distribution with a=2 and b=6 is ¼ and the expected value for the beta distribution with a=6 and b=2 is ¾. The expected probability value for the secondary probability distribution is adjusted to be similar to the predicted probability value by a factor m as given by the following formula $P10_{Predicted} = (1\text{-}m) \cdot \text{beta}(2,6) + m \cdot \text{beta}(6,2) = (1\text{-}m) \cdot 1/4 + m \cdot 3/4$. The factor m is therefore found by the following formula: $m = (P10_{Predicted} - ¼) \cdot 2$ and is 0.558 without SSIV and 0.554 with SSIV.

The calculation of the risk and ambiguity premiums is an iterative process where the risk tolerance factor and the ambiguity aversion parameter have to be balanced by the use of the smooth ambiguity model (KMM functional) and the Arrow-Pratt (AP) quadratic estimation method. The risk and ambiguity premiums are calculated by use of both the smooth ambiguity model (KMM functional) and the extended form of the AP quadratic estimation. The extended AP quadratic estimation also estimate the portion of risk premium and ambiguity premium. The certainty equivalent for each of the scenarios represents the difference between the predicted net present value and the risk and ambiguity premiums. The results using these two methods are shown in the table below.

	Without SSIV given no riser failure	With SSIV given no riser failure
CE - KMM functional	2458[1]	2346[1]
CE - AP quadratic estimation	2456[1]	2350[1]

3) *The above results are based on a risk aversion parameter of a= 1/10200 and an ambiguity aversion parameter of b=1.05/10200*

7.7.2 Certainty equivalent unconditional of riser failure

The outcome set unconditional of riser failure without and with SSIV is the calculated certainty equivalent given no riser and the worst possible negative outcome given a riser failure. The outcome set without SSIV is therefore $\{m_1, m_2, m_3\} = \{2419, 0, -40950\}$ and outcome set without SSIV is $\{m_1, m_2, m_3\} = \{2346, 0, -8215\}$. The probability for $p_1=1$ and predicted probability for p_2 is given as 1e-4. The second order probability distribution for the predicted probability included in the Carter Racing model is modified to a single beta distribution where the expected probability value is similar to predicted probability value. The reason for this adjustment is due to the low predicted probability of 1E-4. The expected value for the beta distribution with a=2 and b=6 is ¼. The expected probability value for the secondary probability distribution is adjusted to be similar to the predicted probability value by a factor m as given by the following formula; $P(\text{predicted}) = (1\text{-}m) \cdot \text{beta}(2,6) = (1\text{-}m) \cdot 1/4$. The factor m is therefore found to be; $m = 1 - 4 \cdot p_2(\text{predicted}) = 0.9996$.

The calculation of the risk and ambiguity premiums is an iterative process where the risk tolerance factor and the ambiguity aversion parameter have to be balanced by the use of the smooth ambiguity model (KMM functional) and the Arrow-Pratt (AP) quadratic estimation method. The risk and ambiguity premiums are calculated by use of both the smooth ambiguity model (KMM functional) and the extended form of the AP quadratic estimation. The extended AP quadratic estimation also estimate the portion of risk premium and ambiguity premium. The certainty equivalent for each of the scenarios represents the difference between the predicted net present value and the risk and ambiguity premiums. The results using these two methods are shown in the table below.

	Without SSIV unconditional on riser failure	With SSIV unconditional on riser failure
CE - KMM functional	2450[1]	2344[2]
CE - AP quadratic estimation	2450[1]	2344[2]

1) *The above results without SSIV are based on a risk aversion parameter of a= 1/40950 and an ambiguity aversion parameter of b=12/40950*
2) *The above results with SSIV are based on a risk aversion parameter of a= 1/8215 and an ambiguity aversion parameter of b=18/8215*

7.8 Certainty Equivalent using the Risk Aversion model

The predicted probabilities and the P10 and P90 confidence values have been used to construct a decision three by the use of the Precision Three Software. Risk aversion is included in the Precision Tree Software with a risk tolerance factor and an exponential concave utility function. The certainty equivalent given no riser failure is 2491 MNOK when SSIV is installed and 2590 MNOK when SSIV is not installed based on a risk tolerance of R=10200. The certainty equivalent without SSIV and unconditional of riser failure is 2582 MNOK based on a risk tolerance of R=40950. The certainty equivalent with SSIV and unconditional of riser failure is 2489 MNOK based on a risk tolerance of R=8215.

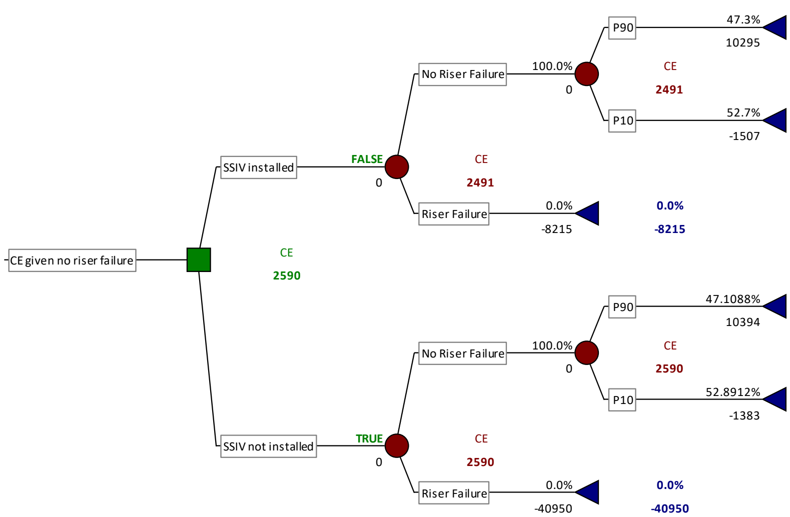


Figure 7.8.1 CE conditional of no riser failure

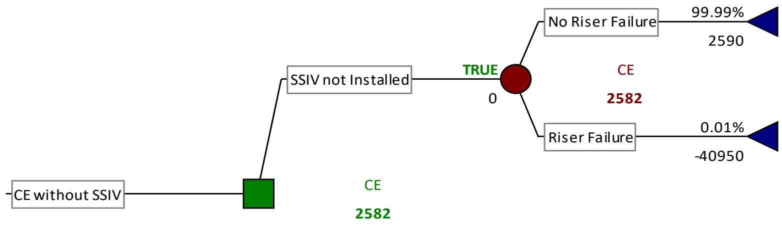


Figure 7.8.2 CE without SSIV and unconditional of riser failure

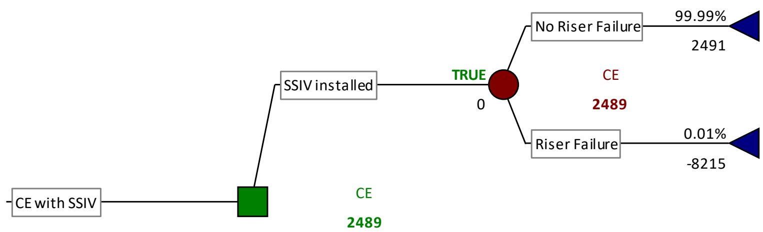


Figure 7.8.3 CE without SSIV and unconditional of riser failure

7.9 Certainty Equivalent using the Extreme Risk and Ambiguity Aversion model

The preferences based on the Wald Maximin Functional are based on a comparison of the worst possible consequences. The certainty equivalents for the Wald Maximin Functional can therefore be represented by the possible severe consequences of a riser failure as shown in the table below.

	Without SSIV	With SSIV
Certainty equivalent (MNOK)	-40950	-8215

7.10 Decision maker's preferences for SSIV

The results of the analysis have been performed by the use of stochastic simulations of the net present value of the production. The objective with the analysis was to assess the need for a safety valve on the pipeline between the producing unit and the host platform. Analysis have therefore been performed to assess the expected values (mean values) and the certainty equivalent values with and without a safety valve (SSIV). Several methods for analysis the certainty equivalent values have been applied that addresses the preferences for a decision maker neutral to risk and a decision maker with risk aversion, risk and ambiguity aversion and extreme risk and ambiguity aversion.

The results from these analysis methods can be summarised as follows:

- The expected value is maximised without SSIV

- The risk and ambiguity averse certainty equivalent is maximised without SSIV
- The risk averse certainty equivalent is maximised without SSIV
- The maximum worst consequence is minimised with SSIV

A risk neutral decision maker will have a preference to not have SSIV since expected value is highest for this alternative. A risk and ambiguity averse decision maker will have a preference to not have a SSIV since the certainty equivalent for the KMM functional and Arrow-Pratt Approximation is highest for this alternative. A risk averse decision maker will there also have a preference to not have a SSIV since the certainty equivalent calculated by use of the Precision Tree software is highest without SSIV. The ALARP principle however states that a safety measure has to be implemented unless it can be demonstrated that there is a gross disproportion between the cost and the benefit. I.e. that the expected cost of implementation is much larger than the expected benefit. The

An extreme risk and ambiguity decision maker will however have a preference for the alternative with SSIV since the maximum worst consequence is minimised for this alternative.

8 Analysis and discussions

As shown by the previous examples, there are theoretical models for decision analysis with ambiguity that can be used to analyse several types of decision problems that the industry faces.

It is however a concern that the industry do not seem to be aware of these theoretical models. Theoretical models for decision analysis with ambiguity are also not included in the textbooks currently used in the lecturing of decision analysis. Models of risk aversion and calculation of corresponding certainty equivalent values are covered in the decision analysis textbook "Practical Management Science" by Winston & Albright (2018), although ambiguity and ambiguity aversion is not covered. It is also stated in Winston & Albright (2018) that the concept of risk aversion and certainty equivalent is rarely used in the industry. In the text book "Making good decision" by Bratvold & Begg (2009), the concept of risk aversion and certainty equivalent is very briefly mentioned. In the papers (Abrahamsen, et al., 2004), (Abrahamsen & Aven, 2008), (Soerskaar & Abrahamsen, 2017) used in the course RIS620 "Economic analysis in Risk Management" there is also no description given of theoretical models associated with ambiguity. The theoretical models for decision analysis with ambiguity do therefore neither seem to be known nor aligned among peers of experts within the field of decision theory. Further research on the practical applications of the theoretical models represented by the Smooth Ambiguity Functional (Klibanoff, et al., 2005) and the Extended Arrow-Pratt Quadratic Estimation (Maccheroni, et al., 2013) should therefore be done at several institutions in order to have these theories accepted and included in decision analysis textbooks. This is a challenge that I believe needs to be resolved in order to introduce quantitative models for decision analysis with ambiguity to the industry.

The balance between normative decision theory and decision making practice, how the decision problems that are analysed can be framed in an uncertainty perspective and how these perspectives can assist in rational decision making are further discussed in the subsections.

8.1 Decision theory versus decision making practice

Relevant models considered to be used to define decision maker's preferences under risk and ambiguity have been assessed with reference to a three-color decision problem (Ellsberg, 1961). The decision theoretical models that the decision maker can use to find his preferences in the three-color decision problem were found to be the Smooth Ambiguity Functional (Klibanoff, et al., 2005), the Extended Arrow-Pratt Quadratic Estimation (Maccheroni, et al., 2013) and the Wald Maximin Functional (Wald, 1949).

Qualitative research of decision making practice have been performed with reference to a selected peer group that have project management functions within the oil and gas industry. The objective with the qualitative research was to assess if the decision theory under ambiguity is known and used by the industry. The responses indicate that the decision theory under ambiguity and the use of certainty equivalent, risk premiums and ambiguity premiums are generally not known by the members of the peer group. Findings from the case studies in the qualitative research do however seem to correspond with previous findings (Kahneman & Tversky, 1992) that decision makers have a tendency to think differently about positive and negative outcomes. The decision maker may have a tendency to a risk seeking and ambiguity neutral behavior towards negative outcomes and a tendency to a risk and ambiguity averse behavior for positive outcomes.

The normative decision theory under ambiguity (Klibanoff, et al., 2005), (Maccheroni, et al., 2013), (Wald, 1949) is based on risk and ambiguity aversion. The preference found by the use of the normative decision theory would therefore provide guidance to the decision maker, since risk and ambiguity aversion are introduced for both positive and negative outcomes.

8.2 Decision analysis of business risk

The analytical methods of the Smooth Ambiguity Functional (Klibanoff, et al., 2005), the Extended Arrow-Pratt Quadratic Estimation (Maccheroni, et al., 2013) and the Wald Maximin Functional (Wald, 1949) have been documented by a stylized business risk decision problem. The business risk decision problem is whether to run or withdraw from a race. The analysis shows that a risk neutral decision is to run, while a risk and ambiguity averse decision and an extreme risk and ambiguity averse decision is to withdraw.

An oil company are sometimes faced with similar business risk decision problems as illustrated by the Carter racing dilemma. These decisions could be related to the trade-off between the selection of a familiar Norwegian fabrication yard or an unknown foreign fabrication yard. The business risk decision could also be related to the trade-off between onshore and offshore execution of the mechanical completion of a platform. Following the logic from the Carter racing, a risk neutral decision would be to select the alternative with the lowest cost estimate and shortest delivery, while a risk and ambiguity averse decision would be to select the alternative with the highest predictability of cost and schedule.

8.3 Decision analysis of cost risk

The research results indicate that project contingency is used as a way to account for uncertainties in project cost estimates. Project contingency is thus a similar concept as risk and ambiguity premiums for an uncertainty associated with cost risk. The decision makers perform a cost risk assessment of the uncertainty range or confidence interval and assigns a margin on top of the base estimate based on qualitative assessments. The research results indicate that there are differences in how project contingency is introduced in a project cost estimate. The project contingency is either included as an deterministic margin in the P50 confidence level or the project contingency is introduced by a qualitative judgment of the uncertainty range. The difference between the P70 confidence level and the P50 confidence level is normally used as the project contingency in cases where an uncertainty range or confidence interval is defined. To have project contingency as the difference between the P70 confidence level and the P50 confidence level is also in alignment with the analysis performed in the project contingency example by the use of the Smooth Ambiguity Functional and the Extended Arrow-Pratt Quadratic Estimation method.

8.4 Decision analysis of production risk

The responses from the in-depth interviews indicate that the P10, P50 and P90 confidence levels are used as the decision parameters for the production risk related to the income prognosis of an oil and gas production. It is however not clear from the responses how the confidence levels are used to define a margin or buffer in the value proposition for the income prognosis.

In the production risk example, the net present value of three alternative project development scenarios are compared. The production profiles provided for these scenarios are based on expected annual production. Information of the uncertainty range for the expected annual production is not defined. The analysis for the production risk example do therefore not include the uncertainties associated with the subsurface performance.

Stochastic analysis has been performed on the variation in Capex, Opex and uptime in order to define the 80% confidence interval for the net present value for the three scenarios. The uncertainty range represented by the 80% confidence interval have been analysed in order to define the preference based on the certainty equivalent. The analysis shows that a wellhead platform with sea access (SAW) is the overall preferred alternative as this gives the highest certainty equivalent both for a risk neutral decision maker (expected value), for a decision maker with risk aversion (Precision tree) and for a decision maker with risk and ambiguity aversion (Smooth Ambiguity Functional/Extended Arrow-Pratt Quadratic Estimation Method).

This production risk decision problem of ranking of different concepts do show the benefit of having a single equivalent value (certainty equivalent) to compare. To perform ranking of the concepts based on the expected value and the uncertainty range could also be done, but would most likely be based on a qualitative judgement of the uncertainty range.

8.5 Decision analysis of accidental risk

The responses from the in-depth interviews indicate that impact assessments are used for decision problems that relates to accident risk. An example of a decision problem associated with accident risk is whether or not to install a SSIV on the pipeline between the wellhead platform and the host platform. The second order probability distribution defined by the two beta distributions for the Smooth Ambiguity Functional do not work when the probability of riser failure is as low as 1E-4. The preferences for this type of accident risk can however be defined by the use of the Wald Maximin Functional (Wald, 1949). To install a SSIV is therefore the preferred alternative as this gives the highest certainty equivalent for the extreme undesired event. There is also a relatively small difference in the expected net present value with and without SSIV. The expected cost with SSIV is therefore not in gross disproportion to the benefit gained. A SSIV should therefore be installed to satisfy the ALARP principle that states that a safety measure is to be installed unless it can be demonstrated that the expected cost is grossly disproportionate to the benefit gained (Soerskaar & Abrahamsen, 2017).

8.6 Use of risk aversion models

The decision problems were also analysed by the risk aversion model included in the Precision Tree software provided by Palisade Ltd. This risk aversion model refers to the subjective expected utility and do not include ambiguity aversion. In the decision problems for business risk, the risk aversion model gave the same preference as the expected value method. In the cost risk decision problem, the risk averse model would include a project contingency equivalent to the difference between the P62 confidence level and the P50 confidence level. This project contingency is lower than recommended practice. The risk aversion model seem to capture only part of uncertainty in these decision problems. These cost risk and business risk decision problems are therefore not recommended to be analysed by the use of the risk aversion model included in the Precision Tree software.

8.7 Risk and ambiguity aversion parameters

The analysis performed by the use of the Smooth Ambiguity Functional and the Extended Arrow-Pratt Quadratic Estimation require a definition of a risk aversion parameter and an ambiguity aversion parameter. The Smooth Ambiguity Functional analysis is found to give reasonable results when the risk aversion parameter is set to the inverse of the largest outcome for each of the decision problems. The same risk and ambiguity aversion parameters are used for the both the Smooth Ambiguity Functional and the Extended Arrow-Pratt Quadratic Estimation. The ambiguity aversion parameter is defined in an iterative process where the calculated risk and ambiguity premium found by the Smooth Ambiguity Functional is balanced with the corresponding risk premium and ambiguity premium found by the Extended Arrow-Pratte Quadratic Estimation.

How to select the risk aversion parameter and how to find the ambiguity aversion parameter is not covered in detail in the literature describing the Smooth Ambiguity Functional and the Extended Arrow-Pratt Quadratic Estimation. It is recommended that the selection of risk aversion parameters and the relationship between the risk aversion parameter and the ambiguity aversion parameter are subject to further research.

8.8 Beyond the economic perspective

The quantitative methods recommended for the decision analysis for the production risk, cost risk, business risk and accident risk decision problems are shown in figure 8.8.1 in relation to high and low uncertainty and consequence. Figure 8.8.1 indicates that there are quantitative models that can be used to assess the uncertainties beyond the extreme economic perspective. The Smooth Ambiguity aversion and Extended Arrow-Pratt estimation can be used for ranking of scenarios, project contingency and business risk decisions. The Wald maximin criterion would be the method to use when assessing accident risks.

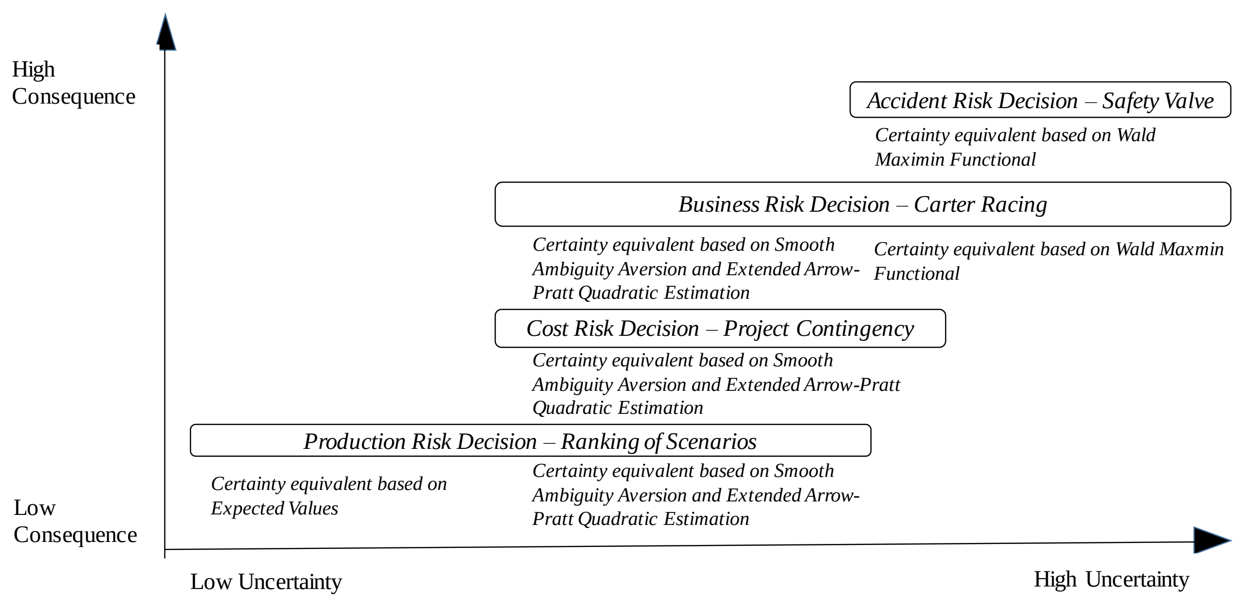


Figure 8.8.1 Decision problems in relation to analysis methods, uncertainty and consequence

8.9 A project manager's recommendation

The quantitative methods given in figure 8.8.1 can assist the project manager in his decision making as a supplement and input to his recommendations and decision support. A project manager's recommendation is therefore proposed included in the decision value process (Aven, 2015) shown in figure 8.9.1.

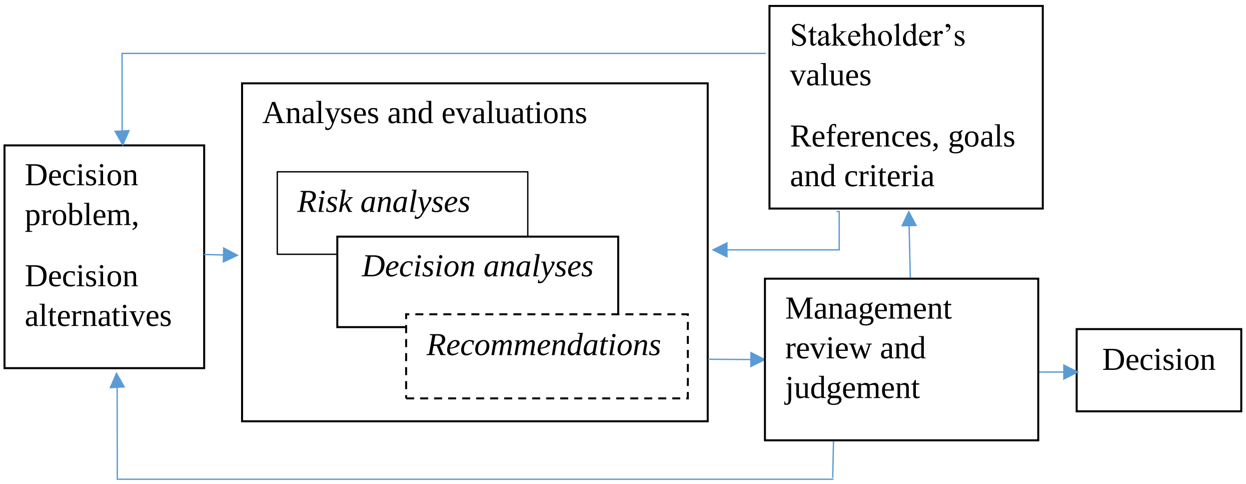


Fig.8.9.1 Decisions as a value process (Aven, 2015)

There is a relatively high expectation that a lot of issues are resolved and thought through in the above value based assessment involving the management review and judgement process and the stakeholders review process. For an oil company the management review and judgement process would typically be held at an asset management level and the stakeholder review process would typically be facilitated by technical and management partner committees. In such a setting, the decision support that is subject to a value based assessment needs to be established by the project manager based on performed analysis and evaluations. This means that the project manager needs to make up his own mind before he presents key messages and recommendations to stakeholders and management, as his personal view will always be part of the discussion in the management and stakeholder reviews.

The responses from the qualitative research do however show that project managers may have different preferences when faced with ambiguity in specific decision problems. Some project managers prefer a predictable outcome set while others prefer a combination of predictable positive outcomes and less predictable negative outcomes. The introduction of quantitative methods for analysising the decision problems is therefore recommended. This could give guidance to the decision maker on a consistent risk and ambiguity averse approach for both postive and negative outcomes.

9 Conclusion

The objective with this thesis was to assess if quantitative models can be introduced to give preferences and margins when subject to both measureable uncertainties (risk) and un-measureable uncertainties (ambiguity). The assessment shows that quantitative models can be developed based on the theoretical models described by the recent advances in decision theory as an extension to current decision analysis practice. Large uncertainties referred to in the extreme safety perspective is however recommended to be based on the precautionary and cautionary principle.

A balanced economic and safety perspective is recommended introduced that lies between the extreme economic perspective and the extreme safety perspective (Soerskaar & Abrahamsen, 2017). The uncertainty perspectives can then be described by an extreme economic perspective, a balanced economic and safety perspective and an extreme safety perspective. Analytical methods relevant for these uncertainty perspectives are shown in figure 9.1 and these are recommended used as a guide for a project manager in his analysis and evaluation of specific decision problems.

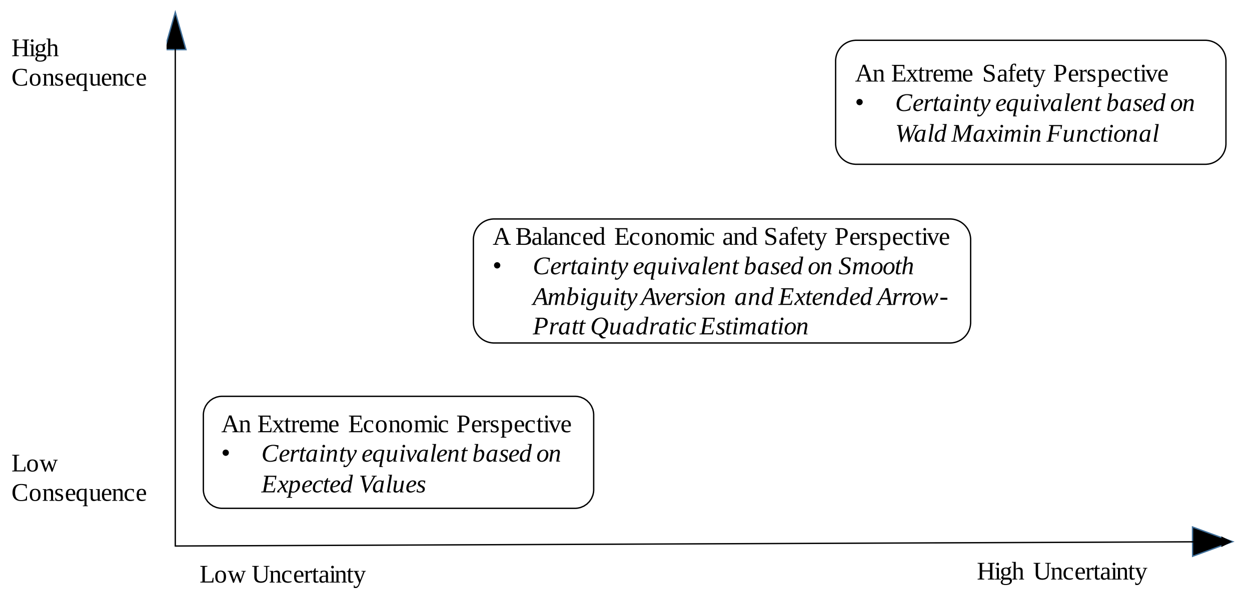


Figure 9.1 Perspectives and recommended analysis methods for uncertainty and consequence

The quantitative decision analysis performed for the balanced economic and safety perspective can define margins or premiums for risk and ambiguity. These margins or premiums can then be used to define a single equivalent sure amount or certainty equivalent as additional information to the expected value and the uncertainty range. This information about an equivalent sure amount or certainty equivalent is recommended used as a guide for rational decision making when qualitative judgements are performed.

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Appendix A – Detailed analysis of the Ellsberg three-color problem

A detailed stepwise description of the analytical methods given in Borgonovo & Marinacci (2015) for the Ellsberg three-color problem is included.

A.1 Exact quadratic solution of the variance of the expected payoff

The Ellsberg three color problem can be expressed by the following expected utility formulations:

$$U(f,p) = \frac{1}{3}u(100) + \frac{1}{3}u(0) + \frac{1}{3}u(0) = \frac{1}{3}u(100) + \frac{2}{3}u(0)$$

$$U(g,p) = \frac{1}{3}u(0) + \left(\frac{2}{3} - p\right)u(0) + p \cdot u(0) = \left(\frac{1}{3} + p\right)u(0) + \left(\frac{2}{3} - p\right)u(100)$$

$$U(f',p) = \frac{1}{3}u(100) + \left(\frac{2}{3} - p\right)u(0) + p \cdot u(100) = \left(\frac{1}{3} + p\right)u(100) + \left(\frac{2}{3} - p\right)u(0)$$

$$U(g',p) = \frac{1}{3}u(0) + \left(\frac{2}{3} - p\right)u(100) + p \cdot u(100) = \left(\frac{1}{3}\right)u(0) + \left(\frac{2}{3}\right)u(100)$$

A predictive probability of p can then be assigned. For the Ellsberg paradox it seem natural to assign a predictive probability of $p_\mu = \frac{1}{3}$. The probabilities then becomes $p_R = \frac{1}{3}$, $p_B = \frac{1}{3}$ and $p_Y = \frac{1}{3}$ and these will then result in the following expected utilities which is the certainty equivalent for a decision maker neutral to both ambiguity and risk (Borgonovo & Marinacci, 2015):

$$H(p_\mu, f) = \sum_{q=1}^{Q} p_\mu^f\,(q) \cdot w_q$$

$$H(f) = U(f, p_\mu) = \frac{1}{3}u(100) + \frac{1}{3}u(0) + \frac{1}{3}u(0) = \frac{1}{3} \cdot (100) + \frac{2}{3} \cdot (0) = 33.33$$

$$H(g) = U(g, p_\mu) = \frac{1}{3}u(0) + \frac{1}{3}u(100) + \frac{1}{3}u(0) = \frac{2}{3} \cdot (0) + \frac{1}{3} \cdot (100) = 33.33$$

$$H(f') = U(f', p_\mu) = \frac{1}{3}u(100) + \frac{1}{3}u(0) + \frac{1}{3}u(100) = \frac{1}{3} \cdot (0) + \frac{2}{3} \cdot (100) = 66.66$$

$$H(g') = U(g', p_\mu) = \frac{1}{3}u(0) + \frac{1}{3}u(100) + \frac{1}{3}u(100) = \frac{1}{3} \cdot (0) + \frac{2}{3} \cdot (100) = 66.66$$

As shown the preference is then: $f \simeq g \precsim f' \simeq g'$

The risk and ambiguity premium of alternative f is in Borgonovo & Marinacci (2015) defined as $\Pi(f) = H(p_{\mu}, f) - C(f)$

The certainty equivalent for act f (Borgonovo & Marinacci, 2015) is defined as:

$$C(f) \simeq C_{AP}(f) + \Pi_{AM}(\mathrm{f})$$

The classic Arrow-Pratt approximation (Borgonovo & Marinacci, 2015) is defined as:

$$C_{AP}(f) = w + \mathbb{E}_{p_{\mu}}(w_f) - \frac{1}{2} \cdot \lambda_y(w) \cdot \sigma^2_{p_u}(w_f)$$

$C_{AP}(f)$ is based on the predicted probability distribution p_{μ}. $\mathbb{E}_{p_{\mu}}(w_f)$ is the expected payoff of investment f under the predicted probability distribution p_{μ}. $\sigma^2_{p_u}(w_f)$ is the corresponding variance of the investment f under the predicted probability distribution p_{μ}.

The quadratic ambiguity premium for act f (Borgonovo & Marinacci, 2015) is then defined as:

$$\Pi_{AM}(f) = -\frac{1}{2} \cdot (\lambda_v(w) - \lambda_u(w)) \cdot \sigma^2_{\mu}(\mathbb{E}_{p_{\mu}}(w_f))$$

$\sigma^2_{\mu}(\mathbb{E}_{p_{\mu}}(w_f))$ is the scope of epistemic uncertainty and $\lambda_v(w) - \lambda_u(w)$ captures the decision makers different attitudes toward aleatory and epistemic uncertainty. Theorem 2 in Borgonovo & Marinacci (2015) summarize these formulas as follows:

$$C_{AP}(f) = w + \sum_{q=1}^{Q} p^f_{\mu}(q) \cdot w_q - \frac{1}{2} \cdot \lambda_u \cdot \sigma^2_{p_{\mu}}(f)$$

$$\sigma^2_{p_{\mu}}(f) = \sum_{q=1}^{Q} p^f_{\mu}(q) \cdot w_q^2 - (\sum_{q=1}^{Q} p^f_{\mu}(q) \cdot w_q)^2$$

$$\Pi_{AM}(f) = -\frac{1}{2} \cdot (\lambda_v - \lambda_u) \cdot (\sum_{q,r=1}^{Q} Cov_{\mu}\, [p^f(q), p^f(r)] \cdot w_q \cdot w_r)$$

$$Cov[p^f(q), p^f(r)] = \mathbb{E}_{\mu}[\left(p^f(q) - p^f_{\mu}(q)\right) \cdot \left(p^f(r) - p^f_{\mu}(r)\right)]$$

$$\sigma^2_{\mu}\left(\mathbb{E}_{p_{\mu}}(w_{f'})\right) = \sum_{q,r=1}^{Q} Cov_{\mu}\, [p^{f'}(q), p^{f'}(r)] w_q w_r$$

$$Cov\left[p^{f'}(q), p^{f'}(r)\right] = \mathbb{E}_{\mu}\left[\left(p^{f'}(q) - p^{f'}_{\mu}(q)\right)\left(p^{f'}(r) - p^{f'}_{\mu}(r)\right)\right] =$$

$$Cov\left[p^{f'}(q), p^{f'}(r)\right] = \mathbb{E}_{\mu}[\left(p^{f'}(q)p^{f'}(r) - p^{f'}_{\mu}(q)p^{f'}(r) - p^{f'}(q)p^{f'}_{\mu}(r) + p^{f'}_{\mu}(q)p^{f'}_{\mu}(q)\ \right) =$$

$$Cov\left[p^{f'}(q), p^{f'}(r)\right] = \mathbb{E}_{\mu}[\left(p^{f'}(q)p^{f'}(r) - p_{\mu}^{f'}(q)p^{f'}(r) - p^{f'}(q)p_{\mu}^{f'}(r) + p_{\mu}^{f'}(q)p_{\mu}^{f'}(q)\right)$$

$$Cov\left[p^{f'}(1), p^{f'}(1)\right]=\mathbb{E}_{\mu}[\left(p^{f'}(1)p^{f'}(1) - p_{\mu}^{f'}(1)p^{f'}(1) - p^{f'}(1)p_{\mu}^{f'}(1) + p_{\mu}^{f'}(1)p_{\mu}^{f'}(1)\right)=$$

$$Cov\left[p^{f'}(1), p^{f'}(1)\right] = \mathbb{E}_{\mu}[\left(p^{f'}(1)p^{f'}(1) - 2p_{\mu}^{f'}(1)p^{f'}(1) + p_{\mu}^{f'}(1)p_{\mu}^{f'}(1)\right)=$$

$$Cov\left[p^{f'}(1), p^{f'}(1)\right] = \mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)^2\right] - 2\mathbb{E}_{\mu}\left[\left(p_{\mu}^{f'}(1)p^{f'}(1)\right)\right] + \mathbb{E}_{\mu}\left[\left(p_{\mu}^{f'}(1)\right)^2\right] =$$

The difference $(\lambda_v(w) - \lambda_u(w))$ = (b – a), captures the decision maker attitudes toward aleatory and epistemic uncertainty. (Borgonovo & Marinacci, 2015). In the following calculation we use $\lambda_v(w) = b = \frac{2}{100}$ and $\lambda_u(w) = a = \frac{1}{100}$.

w = initial wealth

w_f = wealth added by investment f

Letting w=0, we then have;

$$C_{AP}(f) = \sum_{q=1}^{Q} p_{\mu}^{f}(q)w_q - \frac{1}{2}\lambda_u \sigma_{p_{\mu}}^2(f)$$

$$\sum_{q=1}^{Q} p_{\mu}^{f}(q) \cdot w_q = \frac{1}{3} \cdot 100 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 = \frac{100}{3}$$

$$\sigma_{p_{\mu}}^2(f) = \sum_{q=1}^{Q} p_{\mu}^{f}(q)w_q^2 - \left(\sum_{q=1}^{Q} p_{\mu}^{f}(q)w_q\right)^2$$

$$\sigma_{p_{\mu}}^2(f) == \left[\frac{1}{3} \cdot 100^2 + \frac{1}{3} \cdot 0^2 + \frac{1}{3} \cdot 0^2\right] - \left[\frac{1}{3} \cdot 100 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0\right]^2 = \frac{10000}{3} - \frac{10000}{9} = 2222.22$$

$$C_{AP}(f) = \frac{100}{3} - \frac{1}{2} \cdot \lambda_u \cdot \left(\frac{10000}{3} - \frac{10000}{9}\right) = \frac{100}{3} - \frac{1}{2} \cdot \frac{1}{100} \cdot \left(\frac{10000}{3} - \frac{10000}{9}\right) = 22.22$$

$$C_{AP}(g) = \sum_{q=1}^{Q} p_{\mu}^{g}(q)w_q - \frac{1}{2} \cdot \lambda_u \sigma_{p_{\mu}}^2(g)$$

$$\sum_{q=1}^{Q} p_{\mu}^{g}(q)w_q = \frac{1}{3} \cdot 100 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 = \frac{100}{3}$$

$$\sigma^2_{p_\mu}(g) = \sum_{q=1}^{Q} p_\mu^{g}(q) w_q^2 - \left(\sum_{q=1}^{Q} p_\mu^{g}(q) w_q\right)^2$$

$$\sigma^2_{p_\mu}(g) = \left[\frac{1}{3}\cdot 0^2 + \frac{1}{3}\cdot 100^2 + \frac{1}{3}\cdot 0^2\right] - \left[\frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 100 + \frac{1}{3}\cdot 0\right]^2 = \frac{10000}{3} - \frac{10000}{9} = 2222.22$$

$$C_{AP}(g) = \frac{100}{3} - \frac{1}{2}\lambda_u\left(\frac{10000}{3} - \frac{10000}{9}\right) = \frac{100}{3} - \frac{1}{2}\cdot\frac{1}{100}\left(\frac{10000}{3} - \frac{10000}{9}\right) = 22.22$$

$$C_{AP}(f') = \sum_{q=1}^{Q} p_\mu^{f'}(q) w_q - \frac{1}{2}\lambda_u \sigma^2_{p_\mu}(f')$$

$$\sum_{q=1}^{Q} p_\mu^{f'}(q) w_q = \frac{1}{3}\cdot 100 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 100 = \frac{200}{3}$$

$$\sigma^2_{p_\mu}(f') = \sum_{q=1}^{Q} p_\mu^{f'}(q) w_q^2 - \left(\sum_{q=1}^{Q} p_\mu^{f'}(q) w_q\right)^2$$

$$\sigma^2_{p_\mu}(f') = \left[\frac{1}{3}\cdot 100^2 + \frac{1}{3}\cdot 0^2 + \frac{1}{3}\cdot 100^2\right] - \left[\frac{1}{3}\cdot 100 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 100\right]^2 = \frac{20000}{3} - \frac{40000}{9} =$$

$$\sigma^2_{p_\mu}(f') = 2222.22$$

$$C_{AP}(f') = \frac{200}{3} - \frac{1}{2}\lambda_u\left(\frac{20000}{3} - \frac{40000}{9}\right) = \frac{200}{3} - \frac{1}{2}\cdot\frac{1}{100}\left(\frac{20000}{3} - \frac{40000}{9}\right) = 55.55$$

$$C_{AP}(g') = \sum_{q=1}^{Q} p_\mu^{g'}(q) w_q - \frac{1}{2}\lambda_u \sigma^2_{p_\mu}(g')$$

$$\sum_{q=1}^{Q} p_\mu^{g'}(q) w_q = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 100 + \frac{1}{3}\cdot 100 = \frac{200}{3}$$

$$\sigma^2_{p_\mu}(g') = \sum_{q=1}^{Q} p_\mu^{g'}(q) w_q^2 - \left(\sum_{q=1}^{Q} p_\mu^{g'}(q) w_q\right)^2$$

$$\sigma^2_{p_\mu}(g') = \left[\frac{1}{3}\cdot 0^2 + \frac{1}{3}\cdot 100^2 + \frac{1}{3}\cdot 100^2\right] - \left[\frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 100 + \frac{1}{3}\cdot 100\right]^2 = \frac{20000}{3} - \frac{40000}{9} =$$

$$\sigma^2_{p_\mu}(g') = 2222.22$$

$C_{AP}(g') = \frac{200}{3} - \frac{1}{2}\lambda_u\left(\frac{20000}{3} - \frac{40000}{9}\right) = \frac{200}{3} - \frac{1}{2}\cdot\frac{1}{100}\cdot\left(\frac{20000}{3} - \frac{40000}{9}\right) = 55.55$

$$\Pi_{AM}(f) = -\frac{1}{2}(\lambda_v - \lambda_u)\sigma_\mu^2(f) = -\frac{1}{2}(\lambda_v - \lambda_u)\left(\sum_{q,r=1}^{Q} Cov_\mu\,[p^f(q), p^f(r)]w_q w_r\right)$$

$$Cov[p^f(q), p^f(r)] = \mathbb{E}_\mu\left[\left(p^f(q) - p_\mu^f(q)\right)\left(p^f(r) - p_\mu^f(r)\right)\right]$$

r,q=1,2,……….,Q

$$\sigma_\mu^2(f) = \sum_{q,r=1}^{Q} Cov_\mu\,[p^f(q), p^f(r)]w_q w_r$$

For the acts in Ellsberg there are three possible states: $p_1=\frac{1}{3}$, $p_2=\frac{1}{3}$, $p_3=\frac{1}{3}$

$$\sigma_\mu^2(f) = Cov\left(\frac{1}{3},\frac{1}{3}\right)w_1^2(f) + Cov(p,p)w_2^2(f) + Cov\left(\frac{2}{3}-p,\frac{2}{3}-p\right)w_3^2(f) +$$
$$Cov\left(\frac{1}{3},p\right)w_1(f)w_2(f) + Cov\left(p,\frac{1}{3}\right)w_2(f)w_1(f) + Cov\left(\frac{1}{3},\frac{2}{3}-p\right)w_1(f)w_3(f) +$$
$$Cov\left(\frac{2}{3}-p,\frac{1}{3}\right)w_3(f)w_1(f) + Cov\left(p,\frac{2}{3}-p\right)w_2(f)w_3(f) + Cov\left(\frac{2}{3}-p,p\right)w_3(f)w_2(f)$$

Since $Cov\left(\frac{1}{3},p\right) = Cov\left(p,\frac{1}{3}\right)$ and $Cov\left(\frac{1}{3},\frac{2}{3}-p\right) = Cov\left(\frac{2}{3}-p,\frac{1}{3}\right)$ and $Cov\left(p,\frac{2}{3}-p\right) = Cov\left(\frac{2}{3}-p,p\right)$ we then have:

$$\sigma_\mu^2(f) = Cov\left(\frac{1}{3},\frac{1}{3}\right)w_1^2(f) + Cov(p,p)w_2^2(f) + Cov\left(\frac{2}{3}-p,\frac{2}{3}-p\right)w_3^2(f) +$$
$$2Cov\left(\frac{1}{3},p\right)w_1(f)w_2(f) + 2Cov\left(\frac{1}{3},\frac{2}{3}-p\right)w_1(f)w_3(f) + 2Cov\left(p,\frac{2}{3}-p\right)w_2(f)w_3(f)$$

$$Cov\left(\frac{1}{3},\frac{1}{3}\right) = \mathbb{E}_\mu\left[\left(\frac{1}{3}-\frac{1}{3}\right)\left(\frac{1}{3}-\frac{1}{3}\right)\right] = 0$$

$$Cov\left(\frac{1}{3},p\right) = \mathbb{E}_\mu\left[\left(\frac{1}{3}-\frac{1}{3}\right)\left(\frac{1}{3}-p\right)\right] = 0$$

$$Cov\left(\frac{1}{3},\frac{2}{3}-p\right) = \mathbb{E}_\mu\left[\left(\frac{1}{3}-\frac{1}{3}\right)\left(\frac{2}{3}-p-\frac{1}{3}\right)\right] = 0$$

$$\sigma_\mu^2(f) = Cov(p,p)w_2^2(f) + Cov\left(\frac{2}{3}-p,\frac{2}{3}-p\right)w_3^2(f) + 2Cov\left(p,\frac{2}{3}-p\right)w_2(f)w_3(f)$$

$w_2(f) = 0$

$w_3(f) = 0$

$\sigma_\mu^2(f) = 0$

$w_2(g) = 100$

$w_3(g) = 0$

$\sigma_{\mu}^2(g) = Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) w_2^2(g)$

$w_2(f') = 0$

$w_3(f') = 100$

$\sigma_{\mu}^2(f') = Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) w_3^2(f')=$

$w_2(g') = 100$

$w_3(g') = 100$

$\sigma_{\mu}^2(g') = Cov(p,p)w_2^2(g') + Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) w_3^2(g') + \ 2 \cdot Cov\left(p, \frac{2}{3} - p\right) w_2(g')w_3(g')$

$Cov(p,p) = \mathbb{E}_{\mu}\left[\left(p - \frac{1}{3}\right) \cdot \left(p - \frac{1}{3}\right)\right] = \mathbb{E}_{\mu}[p^2 - \frac{1}{3} \cdot p - \frac{1}{3} \cdot p + \frac{1}{9}] = \mathbb{E}_{\mu}[p^2 - \frac{2}{3} \cdot p + \frac{1}{9}] =$

$Cov(p,p) = \mathbb{E}_{\mu}[p^2] - \frac{2}{3} \cdot \mathbb{E}_{\mu}[p] + \frac{1}{9}$

$\mathbb{E}_{\mu}[p] = \int_0^{2/3} p \cdot \frac{3}{2} \cdot dp = \left[\frac{1}{2} \cdot p^2 \cdot \frac{3}{2}\right]_0^{\frac{2}{3}} = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 \cdot \frac{3}{2} = \frac{1}{3}$

$\mathbb{E}_{\mu}[p^2] = \int_0^{2/3} p^2 \cdot \frac{3}{2} \cdot dp = \left[\frac{1}{3} \cdot p^3 \cdot \frac{3}{2}\right]_0^{\frac{2}{3}} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^3 \cdot \frac{3}{2} = \frac{4}{27}$

$Cov(p,p) = \frac{4}{27} - \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{9} = \frac{4}{27} - \frac{6}{27} + \frac{3}{27} = \frac{1}{27}$

$Cov\left(p, \frac{2}{3} - p\right) = \mathbb{E}_{\mu}\left[\left(p - \frac{1}{3}\right)\left(\frac{2}{3} - p - \frac{1}{3}\right)\right] = \mathbb{E}_{\mu}\left[\left(p - \frac{1}{3}\right)\left(\frac{1}{3} - p\right)\right] = \mathbb{E}_{\mu}[\frac{1}{3} \cdot p - p^2 - \frac{1}{9} + \frac{1}{3} \cdot p] =$

$Cov\left(p, \frac{2}{3} - p\right) = \mathbb{E}_{\mu}[\frac{2}{3} \cdot p - p^2 - \frac{1}{9}] = \frac{2}{3}\mathbb{E}_{\mu}[p] - \ \mathbb{E}_{\mu}[p^2] - \frac{1}{9} = \frac{2}{3} \cdot \frac{1}{3} - \frac{4}{27} - \frac{1}{9} = \frac{6}{27} - \frac{4}{27} - \frac{3}{27} =$

$Cov\left(p, \frac{2}{3} - p\right) = -\frac{1}{27}$

$Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) = \mathbb{E}_{\mu}\left[\left(\frac{2}{3} - p - \frac{1}{3}\right) \cdot \left(\frac{2}{3} - p - \frac{1}{3}\right)\right] = \mathbb{E}_{\mu}\left[\left(\frac{1}{3} - p\right) \cdot \left(\frac{1}{3} - p\right)\right] =$

$Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) = \mathbb{E}_{\mu}[\frac{1}{9} - \frac{1}{3} \cdot p - \frac{1}{3} \cdot p + p^2] = \mathbb{E}_{\mu}[\frac{1}{9} - \frac{2}{3} \cdot p + p^2] = \frac{1}{9} - \frac{2}{3} \cdot \mathbb{E}_{\mu}[p] + \mathbb{E}_{\mu}[p^2] =$

$Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) = \frac{1}{9} - \frac{2}{3} \cdot \frac{1}{3} + \frac{4}{27} = \frac{3}{27} - \frac{6}{27} + \frac{4}{27} = \frac{1}{27}$

$\sigma_{\mu}^2(g) = Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) w_2^2(g) = (\frac{1}{27})100^2$

$\sigma_{\mu}^2(f') = Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) w_3^2(f') = (\frac{1}{27})100^2$

$\sigma_{\mu}^2(g') = \left(\frac{1}{27}\right) w_2^2(g') + \left(\frac{1}{27}\right) w_3^2(g') + \ 2 \cdot \left(-\frac{1}{27}\right) w_2(g')w_3(g') = 0$

$\Pi_{AM}(f) = -\frac{1}{2}(\lambda_v - \lambda_u)\sigma_{\mu}^2(f) = -\frac{1}{2}(\lambda_v - \lambda_u) \cdot (0) = 0$

$\Pi_{AM}(g) = -\frac{1}{2}(\lambda_v - \lambda_u)\sigma_{\mu}^2(g) == -\frac{1}{2}(\lambda_v - \lambda_u)\left(Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) w_2^2(g)\right) =$

$\Pi_{AM}(g) = -\frac{1}{2}(\lambda_v - \lambda_u)\left(\left(\frac{1}{27}\right) \cdot 100^2\right) = -\frac{1}{2}(b - a)\left(\left(\frac{1}{27}\right) \cdot 100^2\right) = -\frac{1}{2}(0{,}01)\left(\left(\frac{1}{27}\right) \cdot 100^2\right) =$

$\Pi_{AM}(g) = -1.85$

$\Pi_{AM}(f') = -\frac{1}{2}(\lambda_v - \lambda_u)\sigma_{\mu}^2(f') = -\frac{1}{2}(\lambda_v - \lambda_u)(Cov\left(\frac{2}{3} - p, \frac{2}{3} - p\right) w_3^2(f')) =$

$\Pi_{AM}(f') = -\frac{1}{2}(\lambda_v - \lambda_u)\left(\left(\frac{1}{27}\right) \cdot 100^2\right) = -\frac{1}{2}(b - a)\left(\left(\frac{1}{27}\right) \cdot 100^2\right) =$

$\Pi_{AM}(f') = -\frac{1}{2}(0.01)\left(\left(\frac{1}{27}\right) \cdot 100^2\right) = -1.85$

$\Pi_{AM}(g') = -\frac{1}{2}(\lambda_v - \lambda_u)\sigma_{\mu}^2(g') = -\frac{1}{2}(\lambda_v - \lambda_u) \cdot (0) = 0$

A.2 Approximate quadratic solution of the variance of the expected payoff

The approximate quadratic solution for the variance of the expected payoff are shown in the following for act f'.

$$\sigma_{\mu}^{2}\left(\mathbb{E}_{p_{\mu}}(w_{f'})\right) = sig11 \cdot w_1 \cdot w_1 + sig21 \cdot w_2 \cdot w_1 + sig31 \cdot w_3 \cdot w_1 + sig12 \cdot w_1 \cdot w_2 + sig22 \cdot w_2 \cdot w_2 + sig32 \cdot w_3 \cdot w_2 + sig13 \cdot w_1 \cdot w_3 + sig23 \cdot w_2 \cdot w_3 + sig33 \cdot w_3 \cdot w_3$$

$$p^{f'}(1) = \frac{1}{3}$$

$$\mathbb{E}_{\mu}[p^{f'}(1)] = \int_0^{2/3} p^{f'}(1)\mu(p)dp = \left(\frac{1}{N}\sum_{N=1}^{N}\frac{1}{3}\cdot\frac{3}{2}\right)\cdot\frac{2}{3} = 0.33$$

$$p^{f'}(2) = p = rand()\cdot\frac{2}{3}$$

$$\mathbb{E}_{\mu}[p^{f'}(2)] = \int_0^{2/3} p^{f'}(2)\mu(p)\,dp = \left(\frac{1}{N}\sum_{N=1}^{N} rand()\cdot\frac{2}{3}\cdot\frac{3}{2}\right)\cdot\frac{2}{3} = 0.33$$

$$p^{f'}(3) = \frac{2}{3} - p = \frac{2}{3} - rand()\cdot\frac{2}{3}$$

$$\mathbb{E}_{\mu}[p^{f'}(3)] = \int_0^{2/3} p^{f'}(3)\,\mu(p)dp = \left(\frac{1}{N}\sum_{N=1}^{N}(\frac{2}{3} - rand()\cdot\frac{2}{3})\cdot\frac{3}{2}\right)\cdot\frac{2}{3} = 0.33$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)^2\right] = \int_0^{2/3}\left(p^{f'}(1)\right)^2\cdot\mu(p)dp = \left(\frac{1}{N}\sum_{N=1}^{N}(\frac{1}{3}\cdot\frac{1}{3})\cdot\frac{3}{2}\right)\cdot\frac{2}{3}$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(2)\right)^2\right] = \int_0^{2/3}\left(p^{f'}(2)\right)^2\cdot\mu(p)dp = \left(\frac{1}{N}\sum_{N=1}^{N}(rand()\cdot\frac{2}{3})(rand()\cdot\frac{2}{3})\frac{3}{2}\right)\frac{2}{3}$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(3)\right)^2\right] = \int_0^{2/3}\left(p^{f'}(3)\right)^2\cdot\mu(p)dp =$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(3)\right)^2\right] = \left(\frac{1}{N}\sum_{N=1}^{N}(\frac{2}{3} - rand()\cdot\frac{2}{3})\cdot(\frac{2}{3} - rand()\cdot\frac{2}{3})\frac{3}{2}\right)\frac{2}{3}$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)\left(p^{f'}(2)\right)\right] = \int_0^{\frac{2}{3}}\left(p^{f'}(1)\right)\left(p^{f'}(2)\right)\cdot\mu(p)\cdot dp =$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)\left(p^{f'}(2)\right)\right] = \left(\frac{1}{N}\sum_{N=1}^{N}(\frac{1}{3})\cdot(rand()\cdot\frac{2}{3})\cdot\frac{3}{2}\right)\cdot\frac{2}{3}$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)\left(p^{f'}(3)\right)\right] = \int_0^{\frac{2}{3}}\left(p^{f'}(1)\right)\left(p^{f'}(3)\right)\cdot\mu(p)\cdot dp =$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)\left(p^{f'}(3)\right)\right] = \left(\frac{1}{N}\sum_{N=1}^{N}(\frac{1}{3})\cdot(\frac{2}{3}-\ rand()\cdot\ \frac{2}{3})\cdot\ \frac{3}{2}\right)\cdot\frac{2}{3}$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(2)\right)\left(p^{f'}(3)\right)\right] = \int_{0}^{2/3}\left(p^{f'}(2)\right)\left(p^{f'}(3)\right)\cdot \mu(p)\cdot dp=$$

$$\mathbb{E}_{\mu}\left[\left(p^{f'}(2)\right)\left(p^{f'}(3)\right)\right] = \left(\frac{1}{N}\sum_{N=1}^{N}(rand()\cdot\ \frac{2}{3})\cdot(\frac{2}{3}-\ rand()\cdot\ \frac{2}{3})\cdot\ \frac{3}{2}\right)\cdot\frac{2}{3}$$

sig11=$\mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)^{2}\right]-\ [\mathbb{E}_{\mu}[p^{f'}(1)]^{2}$= *0*

sig22=$\mathbb{E}_{\mu}\left[\left(p^{f'}(2)\right)^{2}\right]-\ [\mathbb{E}_{\mu}[p^{f'}(2)]^{2}$= *0.03667 versus 0.03704 (1% deviation)*

sig33= $\mathbb{E}_{\mu}\left[\left(p^{f'}(3)\right)^{2}\right]-\ [\mathbb{E}_{\mu}[p^{f'}(3)]^{2}$=*0.03667 versus 0.03704 (1% deviation)*

sig12 = sig21= $\mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)\left(p^{f'}(2)\right)\right]-\ \mathbb{E}_{\mu}[p^{f'}(1)]\cdot\mathbb{E}_{\mu}[p^{f'}(2)]$ = *0*

sig13=sig31= $\mathbb{E}_{\mu}\left[\left(p^{f'}(1)\right)\left(p^{f'}(3)\right)\right]-\ \mathbb{E}_{\mu}[p^{f'}(1)]\cdot\mathbb{E}_{\mu}[p^{f'}(3)]$ = *0*

sig23=sig32= $\mathbb{E}_{\mu}\left[\left(p^{f'}(2)\right)\left(p^{f'}(3)\right)\right]-\ \mathbb{E}_{\mu}[p^{f'}(2)]\cdot\mathbb{E}_{\mu}[p^{f'}(3)]$ =*-0.03667 versus 0.03704=1/27 (i.e. 1% deviation)*

A.3 Solving the smooth ambiguity functional

The smooth ambiguity aversion (Klibanoff, et al., 2005) is described as an increasing concave function of the subjective utility function.

$$V(f) = \int_{\Delta(S)} v\big(c(f,p)\big) \cdot d\mu(p) = \int_{\Delta(S)} v\big(u^{-1}(u(f,p))\big) \cdot d\mu(p)$$

$$V(g) = \int_{\Delta(S)} v\big(c(g,p)\big) \cdot d\mu(p) = \int_{\Delta(S)} v\big(u^{-1}(u(g,p))\big) \cdot d\mu(p)$$

$$V(f') = \int_{\Delta(S)} v\big(c(f',p)\big) \cdot d\mu(p) = \int_{\Delta(S)} v\big(u^{-1}(u(f',p))\big) \cdot d\mu(p)$$

$$V(g') = \int_{\Delta(S)} v\big(c(g',p)\big) \cdot d\mu(p) = \int_{\Delta(S)} v\big(u^{-1}(u(g',p))\big) \cdot d\mu(p)$$

First we find the utility function for the acts that also includes risk aversion by the exponential function and the risk aversion constant a:

$$u(\mathrm{f,p}) = \tfrac{1}{3}u(100) + \tfrac{2}{3}u(0) = \tfrac{1}{3}\big(-\mathrm{e}^{-\mathrm{a}\cdot(100)}\big) + \tfrac{2}{3}\big(-\mathrm{e}^{-\mathrm{a}\cdot(0)}\big) = \tfrac{1}{3}\big(-\mathrm{e}^{-\mathrm{a}\cdot(100)}\big) + \tfrac{2}{3}(\text{-}1) =$$

$$u(\mathrm{f,p}) = -\tfrac{1}{3}[\mathrm{e}^{-\mathrm{a}\cdot(100)} + 2]$$

$$u(\mathrm{g,p}) = (\frac{1}{3} + p)u(0) + (\frac{2}{3} - p)u(100) = (\frac{1}{3} + p)\big(-\mathrm{e}^{-\mathrm{a}\cdot(0)}\big) + (\frac{2}{3} - p)\big(-\mathrm{e}^{-\mathrm{a}\cdot(100)}\big) =$$

$$u(\mathrm{g,p}) = -[\tfrac{1}{3} + p + (\tfrac{2}{3} - p)\mathrm{e}^{-\mathrm{a}\cdot(100)}]$$

$$u(\mathrm{f',p}) = (1-p)u(100) + p\cdot u(0) = (1-p)\big(-\mathrm{e}^{-\mathrm{a}\cdot(100)}\big) + p\big(-\mathrm{e}^{-\mathrm{a}\cdot(0)}\big) =$$

$$u(\mathrm{f',p}) = -[(1-p)\mathrm{e}^{-\mathrm{a}\cdot(100)} + p]$$

$$u(\mathrm{g',p}) = \tfrac{1}{3}u(0) + \tfrac{2}{3}u(100) = \tfrac{1}{3}\big(-\mathrm{e}^{-\mathrm{a}\cdot(0)}\big) + \tfrac{2}{3}\big(-\mathrm{e}^{-\mathrm{a}\cdot(100)}\big) = -[\tfrac{1}{3} + \tfrac{2}{3}\mathrm{e}^{-\mathrm{a}\cdot(100)}]$$

The certainty equivalent function of the subjective expected utility function is then found as the inverse of the utility function:

$$c(f,p) = u^{-1}(u(f,p)) = \text{-}\tfrac{1}{a}\ln(-u(f,p)) = \text{-}\tfrac{1}{a}\ln[\tfrac{1}{3}\big(\mathrm{e}^{-\mathrm{a}\cdot(100)} + 2\big)]$$

$$c(g,p) = u^{-1}(u(g,p)) = \text{-}\tfrac{1}{a}\ln(-u(g,p)) = \text{-}\tfrac{1}{a}\ln[\tfrac{1}{3} + p + (\tfrac{2}{3} - p)\mathrm{e}^{-\mathrm{a}\cdot(100)}]$$

$$c(f',p) = u^{-1}(u(f',p)) = \text{-}\tfrac{1}{a}\ln(-u(f',p)) = \text{-}\tfrac{1}{a}\ln[(1-p)\mathrm{e}^{-\mathrm{a}\cdot(100)} + p]$$

$$c(g',p) = u^{-1}(u(g',p)) = \text{-}\tfrac{1}{a}\ln(-u(fg',p)) = \text{-}\tfrac{1}{a}\ln[\tfrac{1}{3} + \tfrac{2}{3}\mathrm{e}^{-\mathrm{a}\cdot(100)}]$$

The ambiguity aversion function can now be introduced as an exponential function of the subjective expected utility function.

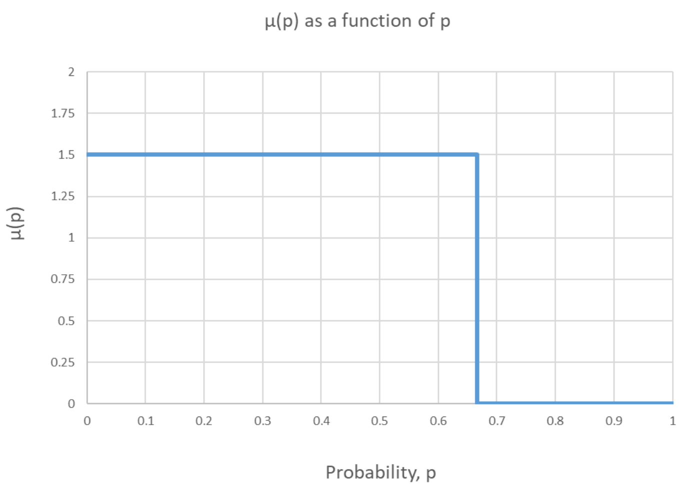
$$v(c(f,p)) = v(u^{-1}u(f,p)) = -\, e^{-b \cdot u^{-1}(u(f,p))} \cdot \mu(p) = -\, e^{-b[-\frac{1}{a}\ln(\frac{1}{3}(e^{-a\cdot(100)}+2))]}\mu(p)$$

$$v(c(g,p)) = v(u^{-1}u(g,p)) = -\, e^{-b \cdot u^{-1}(u(g,p))} \cdot \mu(p) = -\, e^{-b[-\frac{1}{a}\cdot\ln(\frac{1}{3}+p+\left(\frac{2}{3}-p\right)\cdot e^{-a\cdot(100)})]}\mu(p)$$

$$v(c(f',p)) = v(u^{-1}u(f',p)) = -\, e^{-b \cdot u^{-1}(u(f',p))} \cdot \mu(p) = -\, e^{-b[-\frac{1}{a}\ln((1-p)e^{-a\cdot(100)}+p)]}\mu(p)$$

$$v(c(g',p)) = v(u^{-1}u(g',p)) = -\, e^{-b \cdot u^{-1}(u(g',p))} \cdot \mu(p) = -\, e^{-b[-\frac{1}{a}\ln\left(\frac{1}{3}+\frac{2}{3}e^{-a\cdot(100)}\right)]}\mu(p)$$

$\mu(p)$ is a second order probability density for p. For the Ellsberg case described in Borgonovo & Marinacci (2015), the second order probability density is defined as uniform distributed with $\mu(p)= \frac{3}{2}$ for p between 0 and $\frac{2}{3}$ and $\mu(p)= 0$ for $p > \frac{2}{3}$. This distribution is shown in the figure below.



The predictive utility function is defined by:

$$G(f)=\int_{\Delta(S)}\left(\int_S u(f(s))dp(i)\right)d\mu(p)$$

$u: X \rightarrow \mathbb{R}$ is a utility function that captures risk attitude (i.e. uncertainties toward aleatory uncertainty). *X* is the space of consequences, where the consequences are real numbers (i.e. $X \subseteq \mathbb{R}$). Real numbers are denoted $\mathbb{R}$.

$\mu: \sigma(\Delta(s)) \longrightarrow [0,1]$ is a subjective prior distribution that quantifies the epistemic uncertainty about models.

$$p_\mu(f) = p_\mu(g) = p_\mu(f) = p_\mu(g') = \int_{\Delta(S)} p(f)d\mu(p) = \int_0^{\frac{2}{3}} p\frac{3}{2}dp = \left[\frac{1}{2}\cdot\frac{3}{2}p^2\right]_0^{\frac{2}{3}} =$$

$$\mathrm{p}_\mu(\mathrm{f}) = \mathrm{p}_\mu(\mathrm{g}) = \mathrm{p}_\mu(\mathrm{f}) = \mathrm{p}_\mu(\mathrm{g}') = \left[\frac{3}{4}\cdot\frac{4}{9} - 0\right] = \frac{1}{3}$$

p_μ is the predictive probability $p_\mu \in \Delta(S)$. The probabilities then becomes P(Red) = $\frac{1}{3}$, P(Blue) = $\frac{1}{3}$ and P(Yellow) = $\frac{1}{3}$ and these will then result in the following predicted subjective expected utilities:

$$G(f) = U(f, p_\mu) = \frac{1}{3}(100) + \frac{1}{3}(0) + \frac{1}{3}(0) = \frac{1}{3}(100) + \frac{2}{3}(0) = 33.33$$

$$G(g) = U(g, p_\mu) = \frac{1}{3}(0) + \frac{1}{3}(100) + \frac{1}{3}(0) = \frac{2}{3}(0) + \frac{1}{3}(100) = 33.33$$

$$G(f') = U(f', p_\mu) = \frac{1}{3}(100) + \frac{1}{3}(0) + \frac{1}{3}(100) = \frac{1}{3}(0) + \frac{2}{3}(100) = 66.66$$

$$G(g') = U(g', p_\mu) = \frac{1}{3}(0) + \frac{1}{3}(100) + \frac{1}{3}(100) = \frac{1}{3}(0) + \frac{2}{3}(100) = 66.66$$

The total risk and ambiguity aversion is then found by integration for p ϵ (0, $\frac{2}{3}$):

$$\mathrm{V(f)}=\int_0^{2/3} \mathrm{v(u(w))d\mu p} = \int_0^{2/3} -\ \mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(\frac{1}{3}(\mathrm{e}^{-\mathrm{a}\cdot(100)}+2))]}\cdot \frac{3}{2}\mathrm{dp}$$

$$\mathrm{V(g)}=\int_0^{2/3} \mathrm{v(u(w))d\mu p} = \int_0^{2/3} -\ \mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(\frac{1}{3}+p+\left(\frac{2}{3}-p\right)\mathrm{e}^{-\mathrm{a}\cdot(100)})]}\cdot \frac{3}{2}\mathrm{dp}$$

$$\mathrm{V(f')}=\int_0^{2/3} \mathrm{v(u(w))d\mu p} = \int_0^{2/3} -\ \mathrm{e}^{-\mathrm{b}\cdot[-\frac{1}{a}\cdot\ln((1-p)\cdot\mathrm{e}^{-\mathrm{a}\cdot(100)}+p)]}\cdot \frac{3}{2}\mathrm{dp}$$

$$\mathrm{V(g')}=\int_0^{2/3} \mathrm{v(u(w))d\mu p} = \int_0^{2/3} -\ \mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(\frac{1}{3}+\frac{2}{3}\mathrm{e}^{-\mathrm{a}\cdot(100)})]}\cdot \frac{3}{2}\mathrm{dp}$$

The above integrals are numerically estimated. The v(u(w)) function has been calculated N=5000 times by the use of the constants a = $\frac{1}{100}$ and b=$\frac{2}{100}$ and the probability p simulated by the Excel function, p = rand() $\cdot \frac{2}{3}$.

$$\mathrm{V(f)}=(\frac{1}{N}\sum_{N=1}^{N} -\ \mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(\frac{1}{3}(\mathrm{e}^{-\mathrm{a}\cdot(100)}+2))]}\cdot \frac{3}{2})\,\frac{2}{3} = -0.62$$

$$\mathrm{V(g)}=(\frac{1}{N}\sum_{N=1}^{N} -\ \mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(\frac{1}{3}+rand()\cdot\frac{2}{3}+\left(\frac{2}{3}-p\right)\mathrm{e}^{-\mathrm{a}\cdot(100)})]}\cdot \frac{3}{2})\,\frac{2}{3} = -0.64$$

$$\mathrm{V(f')}=(\frac{1}{N}\sum_{0}^{2/3} -\ \mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(-\frac{1}{a}\ln((1-rand()\cdot\frac{2}{3})\mathrm{e}^{-\mathrm{a}\cdot(100)}+rand()\cdot\frac{2}{3}))]}\cdot \frac{3}{2})\frac{2}{3} = -0.35$$

$$\mathrm{V(g')}=(\frac{1}{N}\sum_{0}^{2/3} -\ \mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(-\frac{1}{a}\ln(\frac{1}{3}+\frac{2}{3}\mathrm{e}^{-\mathrm{a}\cdot(100)}))]}\cdot \frac{3}{2})\,\frac{2}{3} = -0.33$$

The certainty equivalent using the smooth ambiguity aversion have the following general expression:

$$C(f) = v^{-1}\big(V(f)\big) = v^{-1}\left[\int_{\Delta(S)} v\big(c(f,p)\big)d\mu(p)\right] = v^{-1}\left[\int_{\Delta(S)} v\big(u^{-1}(u(f,p))\big)d\mu(p)\right]$$

$$C(f) = v^{-1}\big(V(f)\big) = -\frac{1}{b}\ln[-V(f)] = -\frac{1}{b}\ln[-\int_0^{2/3} -\mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(\frac{1}{3}[\mathrm{e}^{-\mathrm{a}\cdot(100)}+2])]}\cdot\frac{3}{2}\mathrm{dp}]$$

$$C(g) = v^{-1}\big(V(f)\big) = v^{-1}\left[\int_{\Delta(S)} v\big(c(f,p)\big)d\mu(p)\right] = v^{-1}\left[\int_{\Delta(S)} v\big(u^{-1}(u(f,p))\big)d\mu(p)\right]$$

$$C(g) = v^{-1}\big(V(f)\big) = -\frac{1}{b}\ln[-V(f)] =$$

$$C(g) = -\frac{1}{b}\ln[-\int_0^{2/3} -\mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(\frac{1}{3}+rand()\cdot\frac{2}{3}+\left(\frac{2}{3}-p\right)\mathrm{e}^{-\mathrm{a}\cdot(100)})]}\cdot\frac{3}{2}\mathrm{dp}]$$

$$C(f') = v^{-1}\big(V(f)\big) = v^{-1}\left[\int_{\Delta(S)} v\big(c(f,p)\big)d\mu(p)\right] = v^{-1}\left[\int_{\Delta(S)} v\big(u^{-1}(u(f,p))\big)d\mu(p)\right]$$

$$C(f') = v^{-1}\big(V(f)\big) = -\frac{1}{b}\ln[-V(f)] =$$

$$C(f') = -\frac{1}{b}\ln[-\int_0^{2/3} -\mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(-\frac{1}{a}\ln((1-rand()\cdot\frac{2}{3})\mathrm{e}^{-\mathrm{a}\cdot(100)}+rand()\cdot\frac{2}{3}))]}\cdot\frac{3}{2}\mathrm{dp}]$$

$$C(g') = v^{-1}\big(V(f)\big) = v^{-1}\left[\int_{\Delta(S)} v\big(c(f,p)\big)d\mu(p)\right] = v^{-1}\left[\int_{\Delta(S)} v\big(u^{-1}(u(f,p))\big)d\mu(p)\right]$$

$$C(g') = v^{-1}\big(V(f)\big) = -\frac{1}{b}\ln[-V(f)] =$$

$$C(g') = -\frac{1}{b}\ln[-\int_0^{2/3} -\mathrm{e}^{-\mathrm{b}[-\frac{1}{a}\ln(-\frac{1}{a}\ln(\frac{1}{3}+\frac{2}{3}\mathrm{e}^{-\mathrm{a}\cdot(100)})]}\cdot\frac{3}{2}\mathrm{dp}]$$

By the use of the results of the numerical estimation of the KMM utility V(-) and b = $\frac{2}{100}$, we then get the following results:

Act	Predicted value	KMM Utility, V(-)	Certainty Equivalent	Risk and ambiguity premium
f	33.33	-0.62	23.66	9.67
g	33.33	-0.64	22.27	11.06
f'	66.67	-0.35	52.02	14.65
g'	66.67	-0.33	54.72	11.95

Based on these results, the decision maker can select act g' as the preferred act since this act have the highest KMM utility and therefore also the highest certainty equivalent. The risk premium for act f and act g' are now defined since we know that there are no ambiguity and therefore no ambiguity premium for act f and act g'.

The portion of the risk premium and the ambiguity premium for act g and f' is further analysed by the use of the Arrow-Pratt Quadratic estimation (Maccheroni, et al., 2013) as shown below.

Arrow-Pratt Quadratic Estimation					
			Certainty Equivalent (AP)		
Act			$C_v(-)$	Risk premium	Ambiguity premium
f	$\sigma^2_\mu(f)$	0.00	22.22	11.11	0.00
g	$\sigma^2_\mu(g)$	371.76	20.93	11.22	1.86
f'	$\sigma^2_\mu(f')$	366.33	53.65	11.12	1.83
g'	$\sigma^2_\mu(g')$	0.00	55.56	11.11	0.00
	$\Pi_{AM}(f)$	0.00	Preferred act (Arrow-Pratt)		
	$\Pi_{AM}(g)$	-1.86	g'		
	$\Pi_{AM}(f')$	-1.86			
	$\Pi_{AM}(g')$	0.00			

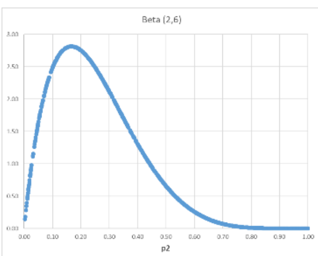
Appendix B – Detailed analysis of the Carter Racing Model

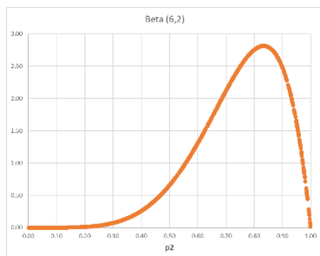
A detailed stepwise description of the analytical methods referenced in Borgonovo & Marinacci (2015) for the Carter Racing model is described.

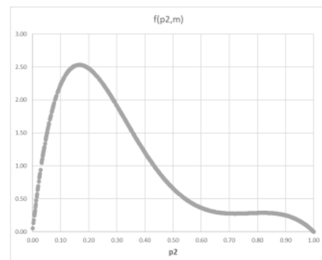
B.1 Second order probability distribution

The probability of engine failure is denoted p_2 and has a predicted probability of 0.3. The probability p_2 has been subject to uncertainties and a second order probability distribution has therefore been assigned which is a combination of the two beta distributions. These are the distributions beta (2,6) and beta (6,2) and their combined distribution f(p2,m) that are shown in the figures below.

$f(p2,m) = (1\text{-}m) \cdot beta(2,6) + m \cdot beta(6,2)$







The expected value of p_2 for the beta(2,6) distribution is $\frac{1}{4}$ and the expected value of p2 is $\frac{3}{4}$ for the beta(6,2) distribution. The factor m is then chosen as $\frac{1}{10}$ as this give the predicted value of $\tilde{p}$ = 0.3. The predicted value of p2 = $\tilde{p}$ is also calculated by the following approximation in the simulations:

$$\text{ptilde} = \int_0^1 \text{p2} \cdot \text{f(p2, m)} \cdot \text{dp2} = \frac{1}{N}\sum_{N=1}^{N} rand() \cdot [\,(1-\text{m}) \cdot \text{beta}(2,6) + \text{m} \cdot \text{beta}(6,2)] = 0.3$$

The conditional probability of being the first 5 given no engine failure has been assigned a value of $p_1=\frac{5}{7}$. The value of being top 5 is estimated to m_1=1000. The value of being below top 5 is estimated to m_2 = 0. The value of engine failure is estimated to m_3 = - 500.

B.2 Expected value

The expected value H of the decision problem can be calculated as follows for a risk neutral decision maker;

$y0(p2)= (1\text{-}p2) \cdot [p1 \cdot m1+(1\text{-}p1)\cdot m2]+p2 \cdot m3$

$$\text{H}=\int_0^1 \text{y0(p2)} \cdot \text{f(p2, m)} \cdot \text{dp2} = \frac{1}{N}\sum_{N=1}^{N}[1-rand()] \cdot [\text{p1} \cdot \text{m1} + (1-\text{p1}) \cdot \text{m2}] + \text{p2} \cdot \text{m3}] =$$

$$\text{H} = \frac{1}{N}\sum_{N=1}^{N}[1-rand()] \cdot [\,[\,\tfrac{5}{7} \cdot (1000) + (1-\tfrac{5}{7}) \cdot (0)] + \text{rand}() \cdot (-500)] = 352.3$$

B.3 Risk aversion

A risk averse decision maker will give a lower weighting to positive outcomes and a higher weighting to negative outcomes. This can be introduced by the use of a negative exponential function and a risk aversion parameter a. A risk aversion parameter of a = $\frac{1}{1000}$ have been used in the analysis. The utility values then becomes:

$u(a,m1) = -e^{-a \cdot m1} = -e^{-\frac{1}{1000} \cdot 1000} = -e^{-1} = -0.37$

$u(a,m2) = -e^{-a \cdot m2} = -e^{-\frac{1}{1000} \cdot (0)} = -e^{0} = -1.00$

$u(a,m3) = -e^{-a \cdot m3} = -e^{-\frac{1}{1000} \cdot (-500)} = -e^{0.5} = -1.65$

The expected utility of the decision problem can then be calculated:

$u(p2,a) = ya(p2,a) = (1-p2) \cdot [p1 \cdot u(a,m1) + (1-p1) \cdot u(a,m2)] + p2 \cdot u(a,m3)$

$G(a) = \int_0^1 ya(p2,a) \cdot f(p2,m) \cdot dp2 =$

$G(a) = \frac{1}{N} \sum_{N=1}^{N} [(1 - rand()) \cdot \left(\frac{5}{7} \cdot (-0.37) + \left(1 - \frac{5}{7}\right) \cdot (-1.00) \right) + \text{rand}(\quad) \cdot (-1.65)][(1 - \text{m}) \cdot \text{beta}(2,6) + \text{m} \cdot \text{beta}(6,2)] = -0.88$

B.4 Wald Maximin Functional

The Wald Maximin Functional is representing an extreme ambiguity averse decision maker. The Wald Maximin Functional is in this decision problem the same as the utility for the negative outcome m_3.

$W = u(a,m3) = -e^{-a \cdot m3} = -e^{-\frac{1}{1000} \cdot (-500)} = -e^{0.5} = -1.65$

The corresponding certainty equivalent for the maxmin decision maker is therefore:

$c(p2,a) = uinv(u(a,m3)) = \frac{-\ln(-u(a,m3))}{a} = -\frac{1}{1000} \ln(-1.65) = -500$

B.5 Risk and ambiguity aversion expressed by the KMM utility

Risk and ambiguity aversion are introduced by the risk aversion parameter a = $\frac{1}{1000}$ for the utility function u(p2,a) = ya(p2,a) and the ambiguity aversion parameter b = $\frac{1}{100}$ for the second order utility function v(b,uinv(a,u(p2,a))).

$c(p2,a) = uinv(a, ya(p2,a)) = \frac{-\ln(-ya(p2,a))}{a} = \frac{-\ln(-[(1-p2)\cdot[p1\cdot u(a,m1)+(1-p1)\cdot u(a,m2)]+p2\cdot u(a,m3)])}{a} =$

$c(p2,a) = \frac{-\ln\left(-[(1-rand())\cdot[\frac{5}{7}(-0.37)+(1-\frac{5}{7})\cdot(-1.00)]+rand()\cdot(-1.65)]\right)}{a}$

$c2 = \int_0^1 c(p2,a)\, f(p2,m)dp2 = \frac{1}{N}\sum_{N=1}^{N}\frac{-\ln(-ya(p2,a))}{a}\cdot[(1-\mathrm{m})\cdot\mathrm{beta}(2,6)+\mathrm{m}\cdot\mathrm{beta}(6,2)] =$

$c2 = \frac{1}{N}\sum_{N=1}^{N}\frac{-\ln(-[(1-p2)\cdot[p1\cdot u(a,m1)+(1-p1)\cdot u(a,m2)]+p2\cdot u(a,m3)])}{a}\cdot[(1-\mathrm{m})\cdot\mathrm{beta}(2,6)+\mathrm{m}\cdot\mathrm{beta}(6,2)] =$

$c2 = \frac{1}{N}\sum_{N=1}^{N}\frac{-\ln\left(-[(1-rand())\cdot[\frac{5}{7}(-0.37)+(1-\frac{5}{7})\cdot(-1.00)]+rand()\cdot(-1.65)]\right)}{a}\cdot[(1-\mathrm{m})\cdot\mathrm{beta}(2,6)+\mathrm{m}\cdot\mathrm{beta}(6,2)] =$

$c2 = 159.68$

$y2(p2,a,b)=v(b, c(p2,a)) = -e^{(-b\cdot c(p2,a))} = -e^{\left(-b\cdot\frac{-\ln\left(-[(1-rand())\cdot[\frac{5}{7}(-0.37)+(1-\frac{5}{7})\cdot(-1.00)]+rand()\cdot(-1.65)]\right)}{a}\right)}$

$V(a,b)=\int_0^1 y2(p2,a,b)\cdot f(p2,m)\cdot dp2=$

$V(a,b)=\int_0^1 -e^{\left(-b\cdot\frac{-\ln\left(-[(1-rand())\cdot[\frac{5}{7}(-0.37)+(1-\frac{5}{7})\cdot(-1.00)]+rand()\cdot(-1.65)]\right)}{a}\right)}\cdot[(1-\mathrm{m})\mathrm{beta}(2,6)+\mathrm{m}\cdot\mathrm{beta}(6,2)]dp2=$

$V(a,b)=\frac{1}{N}\sum_{N=1}^{N} -e^{\left(-b\cdot\frac{-\ln\left(-\left[(1-rand(\ \ ))\cdot\left[\frac{5}{7}(-0.37)+\left(1-\frac{5}{7}\right)\cdot(-1.00)\right]+rand(\ \ )\cdot(-1.65)\right]\right)}{a}\right)}\left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2,6)+\left(\frac{1}{10}\right)\mathrm{beta}(6,2)\right]\mathrm{dp2} = -4.19$

The above method of calculating the KMM utility is similar to the KMM functional for the Ellsberg three-color problem. Another method is shown below which gives the same result.

$\varphi(p2,b,a)= -(-ya(p2,a))^{\frac{b}{a}}$

$V2(a,b)=\int_0^1 \varphi(p2,a,b)f(p2,m)dp2 = \int_0^1 -(-ya(p2,a))^{\frac{b}{a}}\cdot[(1-\mathrm{m})\mathrm{beta}(2,6)+\mathrm{m}\cdot\mathrm{beta}(6,2)]dp2=$

$V2(a,b)= \int_0^1 -((1-p2)\cdot [p1 \cdot u(a,m1)+(1-p1) \cdot u(a,m2)] + p2 \cdot u(a,m3))^{\frac{b}{a}} \cdot [(1-\mathrm{m})\mathrm{beta}(2,6) + \mathrm{m} \cdot \mathrm{beta}(6,2)]dp2=$

$V2(a,b)=\frac{1}{N}\sum_{N=1}^{N} -((1-p2)\cdot [p1 \cdot u(a,m1)+(1-p1) \cdot u(a,m2)] + p2 \cdot u(a,m3))^{\frac{b}{a}} \cdot [(1-\mathrm{m})\mathrm{beta}(2,6) + \mathrm{m} \cdot \mathrm{beta}(6,2)]dp2=$

$V2(a,b)=\frac{1}{N}\sum_{N=1}^{N} -((1-rand())\cdot [\frac{5}{7} (-0.37)+(1-\frac{5}{7}) \cdot (-1.00)] +rand() \cdot (-1.65))^{\frac{1/100}{1/1000}} \cdot [(1-\frac{1}{10})\mathrm{beta}(2,6) + \frac{1}{10} \cdot \mathrm{beta}(6,2)]dp2 = -4.19$

B.6 Certainty equivalent as the inverse of the KMM utility

The certainty equivalent for the risk and ambiguity averse decision maker are then calculated as the inverse of the KMM utility.

$cAmAv = vinv(b, V2(a,b)) = \frac{-\ln(-V2(a,b))}{b} = -\frac{1}{b}\ln\big(-V2(a,b)\big) = -\frac{1}{\frac{1}{100}}\ln\big(-(-4.19)\big) = -143.2$

The risk and ambiguity premium Π is the difference between the risk and ambiguity neutral expected value H and the certainty equivalent cAmAv.

Π=H-cAmAv = 352.3 – (-143.2) = 495.5

B.7 Balancing the share of risk premium and ambiguity premium

The portion of the risk premium and the ambiguity premium can be found by the use of the quadratic approximation method. The risk premium is calculated by first finding the variance sig2pmu by the formula:

$\mathrm{sig2pmu} = \sum_{s=0}^{2} (\mathrm{w}_{0,s})^2 \cdot \mathrm{pmu}_{0,s} - \left[\sum_{s=0}^{2} \mathrm{w}_{0,s} \cdot \mathrm{pmu}_{0,s}\right]^2$

$w_{0,0} = 1000$

$w_{0,1} = 0$

$w_{0,2} = -500$

$pmu(0,0)=(1-\tilde{p})\cdot p1$

$pmu(0,1)=(1-\tilde{p}) \cdot (1-p1)$

$pmu(0,2)= \tilde{p}$

$\sum_{s=0}^{2}(w_{0,s})^2 \cdot pmu_{0,s} = (1000)^2(1-\tilde{p}) \cdot p1 + (0)^2(1-\tilde{p}) \cdot (1-p1) + (-500)^2 \cdot \tilde{p} =$

$\sum_{s=0}^{2}(w_{0,s})^2 \cdot pmu_{0,s} = (1000)^2 \cdot (1-\tilde{p}) \cdot \frac{5}{7} + (0)^2 \cdot (1-\tilde{p}) \cdot \left(1-\frac{5}{7}\right) + (-500)^2 \cdot \tilde{p} =$ 573924.14

$$\sum_{s=0}^{2} w_{0,s} \cdot pmu_{0,s} = [(1000)(1-\tilde{p}) \cdot p1 + (0)(1-\tilde{p}) \cdot (1-p1) + (-500) \cdot \tilde{p}] =$$

$\sum_{s=0}^{2} w_{0,s} \cdot pmu_{0,s} = [(1000)(1-\tilde{p}) \cdot \frac{5}{7} + (0)(1-\tilde{p}) \cdot \left(1-\frac{5}{7}\right) + (-500) \cdot \tilde{p}] = 347.41$

sig2pmu = 574011.06 - $[347.41]^2$= 453314.89

The certainty equivalent CAP is the difference between the expected value H and the risk premium:

CAP = H - $\frac{1}{2} \cdot a \cdot sig2pmu = 352.3 - \frac{1}{2} \cdot \left(\frac{1}{1000}\right) \cdot (453314.89) = 125.65$

The ambiguity premium is found by the extended quadratic approximation:

CAM = $\frac{1}{2} \cdot (b-a) \cdot sig2mu$

$sig2mu = \sum_{s=0}^{2} \sum_{j=0}^{2} (SIG_{s,j} \cdot w_{0,s} \cdot w_{0,j})$

Q1(p2)=(1-p2)p1

$mQ1 = \int_0^1 Q1(p2) f(p2,m) dp2 = \int_0^1 (1-p2)p1\,[(1-\mathrm{m})\mathrm{beta}(2,6) + \mathrm{m} \cdot \mathrm{beta}(6,2)]dp2 =$

$mQ1 = \frac{1}{N}\sum_{N=1}^{N}(1-p2)p1[(1-\mathrm{m})\mathrm{beta}(2,6) + \mathrm{m} \cdot \mathrm{beta}(6,2)] =$

$mQ1 = \frac{1}{N}\sum_{N=1}^{N}(1-rand()) \cdot \frac{5}{7} \cdot \left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2,6) + \frac{1}{10} \cdot \mathrm{beta}(6,2)\right] = 0.50$

Q2(p2)=(1-p2)(1-p1)

$mQ2 = \int_0^1 Q2(p2) f(p2,m) dp2 = \int_0^1 (1-p2)(1-p1)\,[(1-\mathrm{m})\mathrm{beta}(2,6) + \mathrm{m} \cdot \mathrm{beta}(6,2)]dp2 =$

$mQ2 = \frac{1}{N}\sum_{N=1}^{N}(1-rand()) \cdot (1-\frac{5}{7}) \cdot \left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2,6) + \frac{1}{10} \cdot \mathrm{beta}(6,2)\right] = 0.20$

Q3(p2) = p2 = rand()

$mQ3 = \int_0^1 Q3(p2) f(p2,m) dp2 = \int_0^1 p2\,[(1-\mathrm{m})\mathrm{beta}(2,6) + \mathrm{m} \cdot \mathrm{beta}(6,2)]dp2 =$

$mQ3 = \frac{1}{N}\sum_{N=1}^{N}(rand()) \cdot \left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2,6) + \frac{1}{10} \cdot \mathrm{beta}(6,2)\right] = 0.30$

sig11=$\int_0^1 Q1(p2)^2$*f(p2,m)dp2* - *mQ1*^{2}=$\int_0^1((1-p2)p1)^2\,[(1-\mathrm{m})\mathrm{beta}(2{,}6)+\mathrm{m}\cdot\mathrm{beta}(6{,}2)]$*dp2* - *mQ1*^{2}=

sig11=$\frac{1}{N}\sum_{N=1}^{N}((1-rand())\cdot\,\frac{5}{7})^2\cdot\left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2{,}6)+\frac{1}{10}\cdot\mathrm{beta}(6{,}2)\right]-\;0.50^2\;=0.0197$

sig22=$\int_0^1 \mathrm{Q2(p2)}^2$f(p2,m)dp2 - mQ2^2= $\int_0^1((1-p2)(1-p1))^2\,[(1-\mathrm{m})\mathrm{beta}(2{,}6)+\mathrm{m}\cdot\mathrm{beta}(6{,}2)]$dp2 - mQ2^2=

sig22=$\frac{1}{N}\sum_{N=1}^{N}((1-rand())\cdot\left(1-\,\frac{5}{7}\right))^2\cdot\left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2{,}6)+\frac{1}{10}\cdot\mathrm{beta}(6{,}2)\right]-$ $0.20^2\;=0.0031$

sig33= $\int_0^1 Q3(p2)^2$*f(p2,m)dp2* - *mQ3*^{2}= $\int_0^1(\mathrm{p2})^2\,[(1-\mathrm{m})\mathrm{beta}(2{,}6)+\mathrm{m}\cdot\mathrm{beta}(6{,}2)]$dp2 - mQ3^2=

sig33=$\frac{1}{N}\sum_{N=1}^{N}(rand())^2\cdot\left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2{,}6)+\frac{1}{10}\cdot\mathrm{beta}(6{,}2)\right]-\;0.30^2\;=0.0413$

sig12 = sig21=$\int_0^1$ *Q1(p2) Q2(p2)f(p2,m)dp2 - mQ1mQ2=*

$sig12\;=\;sig21\;=\int_0^1(1-p2)p1$*(1-p2)(1-p1)*$[(1-\mathrm{m})\mathrm{beta}(2{,}6)+\mathrm{m}\cdot\mathrm{beta}(6{,}2)]$*dp2 - mQ1mQ2=*

$sig12\;=\;\frac{1}{N}\sum_{N=1}^{N}\left(1-rand(\quad)\right)\cdot\frac{5}{7}\cdot(1-rand())\cdot(1-\frac{5}{7})\cdot\left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2{,}6)+\frac{1}{10}\mathrm{beta}(6{,}2)\right]-\;0.50\cdot0.20=$ 7.9E-03

sig13=sig31= $\int_0^1$ *Q1(p2)Q3(p2)f(p2,m)dp2 - mQ1mQ3=*

$sig13\;=\;sig31\;=\int_0^1(1-p2)p1$*(p2)*$[(1-\mathrm{m})\mathrm{beta}(2{,}6)+\mathrm{m}\cdot\mathrm{beta}(6{,}2)]$*dp2 - mQ1mQ3=*

$sig13\;=\;\frac{1}{N}\sum_{N=1}^{N}(1-rand())\cdot\frac{5}{7}\cdot(rand())\cdot\left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2{,}6)+\frac{1}{10}\mathrm{beta}(6{,}2)\right]-\;0.50\cdot$ 0.30 = -0.0310

sig23=sig32= $\int_0^1$ *Q2(p2)Q3(p2)f(p2,m)dp2 - mQ2mQ3=*

$sig23 = sig32 = \int_0^1 (1-p2)(1-p1)(p2)[(1-\mathrm{m})\mathrm{beta}(2{,}6) + \mathrm{m}\cdot\mathrm{beta}(6{,}2)]dp2 - mQ2mQ3 =$

$sig23 = \frac{1}{N}\sum_{N=1}^{N}\left(1 - rand(\quad)\right)\cdot\left(1-\frac{5}{7}\right)\cdot\left(rand(\quad)\right)\cdot\left[\left(1-\frac{1}{10}\right)\mathrm{beta}(2{,}6) + \frac{1}{10}\mathrm{beta}(6{,}2)\right] - 0.21\cdot 0.30 =$

$sig23 =$ -0.0124

$sig2mu = \sum_{s=0}^{2}\sum_{j=0}^{2}(SIG_{s,j}\cdot w_{0,s}\cdot w_{0,j})$

$sig2mu = \sum_{s=0}^{2}\sum_{j=0}^{2}(SIG_{s,j}\cdot w_{0,s}\cdot w_{0,j}) = SIG_{0,0}\cdot w_{0,0}\cdot w_{0,0} + SIG_{0,1}\cdot w_{0,0}\cdot w_{0,1} + SIG_{0,2}\cdot w_{0,0}\cdot w_{0,2} + SIG_{1,0}\cdot w_{0,1}\cdot w_{0,0} + SIG_{1,1}\cdot w_{0,1}\cdot w_{0,1} + SIG_{1,2}\cdot w_{0,1}\cdot w_{0,2} + SIG_{2,0}\cdot w_{0,2}\cdot w_{0,0} + SIG_{2,1}\cdot w_{0,2}\cdot w_{0,1} + SIG_{2,2}\cdot w_{0,2}\cdot w_{0,2} =$

$sig2mu = sig11\cdot w(0{,}0)\cdot w(0{,}0) + sig12\cdot w(0{,}0)\cdot w(0{,}1) + sig13\cdot w(0{,}0)\cdot w(0{,}2) + sig21\cdot w(0{,}1)\cdot w(0{,}0) + sig22\cdot w(0{,}1)\cdot w(0{,}1) + sig23\cdot w(0{,}1)\cdot w(0{,}2) + sig31\cdot w(0{,}2)\cdot w(0{,}0) + sig32\cdot w(0{,}2)\cdot w(0{,}1) + sig33\cdot w(0{,}2)\cdot w(0{,}2) =$

$sig2pmu = (0.0197)\cdot(1000)\cdot(1000) + (7.9\mathrm{E}-03)\cdot(1000)\cdot(-500) + (-0.0310)\cdot(1000)\cdot(-500) + (7.9\mathrm{E}-03)\cdot(0)\cdot(1000) + (0.0031)\cdot(0)\cdot(0) + (-0.0124)\cdot(0)\cdot(-500) + (-0.0310)\cdot(-500)\cdot(1000) + (-0.0124)\cdot(-500)\cdot(0) + (0.0413)\cdot(-500)\cdot(-500) =$ 60982.21

The ambiguity premium CAM can now be calculated:

$CAM = \frac{1}{2}\cdot(b - a)\cdot sig2mu = \frac{1}{2}\cdot\left(\frac{1}{100} - \frac{1}{1000}\right)\cdot 60987.21 = 274.42$

B.8 Realisations and integration in Excel

The formulas described in chapter B.1 to B.7 for the Smooth Ambiguity Functional and the Extended Arrow-Pratt Approximation have been analysed by the use of the standard functions in Excel. The annotations used correspond to the formulas given in chapter B.1 to B.7. The number of realisations, N, used in this analysis is N=1996. This number could be higher or rounded off to 2000, but this have minimal impact on the results. The formulas for the Smooth Ambiguity Functional and the Extended Arrow-Pratt Approximation have been integrated over N realisations by the use of the AVERAGE function in Excel.

B.8.1 Realisations for the Smooth Ambiguity Functional

Table B8.1 show the set-up and annotations for the realisations for the Smooth Ambiguity Functional. The commands for row 5 and column G to V are shown below. These commands have been copied to row 6 down to 2001 in order to obtain N=1996 realisations of the formulas.

p2:	G5=RAND()
beta(2,6):	H5=BETA.DIST(G5,2,6,FALSE)
beta(6,2):	I5=BETA.DIST(G5,6,2,FALSE)
f(p2,m):	J5=(1-m)*H5+m*I5
*p2*f(p2,m):*	K5=G5*J5
y0(p2):	L5=(1-G5)*(p1_*m1_+(1-p1_)*m2_)+G5*m3_
*y0(p2)*f(p2,m):*	M5=L5*J5
y3(p2,b):	N5=-EXP(-b*L5)
*y3(p2,b)*f(p2,m):*	O5=N5*J5
ya(p2,m):	P5=(1-G5)*(p1_*(-EXP(-a*m1_))+(1-p1_)*(-EXP(-a*m2_))) +G5*(-EXP(-a*m3_))
*ya(p2,a)*f(p2,m):*	Q5=P5*J5
c(p2,a):	R5=-LN(-P5)/a
*c(p2,a)*f(p2,m):*	S5=R5*J5
y2(p2,a,b):	T5=-EXP(-b*R5)
*y2(p2,a,b)*f(p2,m):*	U5=T5*J5
ϕ(p2,b,a):	V5=-((-p5)^(b/a))
*ϕ(p2,b,a)*f(p2,m):*	W5=V5*J5

	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
4	p2	beta(2,6)	beta(6,2)	f(p2,m)	p2*f(p2,m)	y0(p2)	y0(p2)*f(p2,m)	y3(p2,b)	y3(p2,b)*f(p2,m)	ya(p2,a)	ya(p2,a)*f(p2,m)	c(p2,a)	c(p2,a)*f(p2,m)	y2(p2,a,b)	y2(p2,a,b)*f(p2,m)	φ(p2,b,a)	φ(p2,b,a)*f(p2,m)
5	0.33	1.89	0.11	1.71	0.56	316.84	543.34	-0.042	-0.072	-0.909	-1.56	95.85	164.37	-0.38	-0.66	-0.38	-0.66
6	0.30	2.15	0.07	1.94	0.57	354.17	686.22	-0.029	-0.056	-0.875	-1.69	133.79	259.22	-0.26	-0.51	-0.26	-0.51
7	0.79	0.02	2.69	0.28	0.22	-239.23	-67.67	-10.939	-3.094	-1.412	-0.40	-345.32	-97.67	-31.60	-8.94	-31.60	-8.94
8	0.17	2.81	0.00	2.53	0.43	509.06	1288.96	-0.006	-0.016	-0.734	-1.86	308.66	781.53	-0.05	-0.12	-0.05	-0.12
9	0.68	0.09	1.97	0.28	0.19	-113.49	-31.89	-3.111	-0.874	-1.299	-0.36	-261.22	-73.40	-13.63	-3.83	-13.63	-3.83
10	0.03	1.09	0.00	0.98	0.03	677.61	663.63	-0.001	-0.001	-0.582	-0.57	541.77	530.60	0.00	0.00	0.00	0.00
11	0.87	0.00	2.72	0.27	0.24	-343.57	-93.68	-31.053	-8.467	-1.507	-0.41	-410.11	-111.83	-60.41	-16.47	-60.41	-16.47
12	0.36	1.62	0.16	1.48	0.53	277.23	409.71	-0.063	-0.092	-0.944	-1.40	57.10	84.39	-0.56	-0.83	-0.56	-0.83
13	0.78	0.02	2.65	0.28	0.22	-228.43	-64.27	-9.819	-2.762	-1.403	-0.39	-338.37	-95.20	-29.48	-8.29	-29.48	-8.29
14	0.41	1.25	0.28	1.16	0.47	220.25	254.80	-0.111	-0.128	-0.996	-1.15	3.89	4.50	-0.96	-1.11	-0.96	-1.11
15	0.69	0.08	2.06	0.28	0.19	-126.88	-35.22	-3.556	-0.987	-1.311	-0.36	-270.52	-75.09	-14.96	-4.15	-14.96	-4.15
16	0.50	0.63	0.68	0.64	0.32	101.68	64.83	-0.362	-0.231	-1.104	-0.70	-98.54	-62.83	-2.68	-1.71	-2.68	-1.71
17	0.30	2.14	0.07	1.93	0.57	352.63	680.06	-0.029	-0.057	-0.876	-1.69	132.19	254.94	-0.27	-0.51	-0.27	-0.51
18	0.68	0.09	1.97	0.28	0.19	-113.64	-31.93	-3.116	-0.875	-1.299	-0.36	-261.33	-73.42	-13.64	-3.83	-13.64	-3.83
19	0.77	0.02	2.59	0.28	0.21	-216.50	-60.53	-8.715	-2.437	-1.392	-0.39	-330.63	-92.44	-27.28	-7.63	-27.28	-7.63
20	0.68	0.09	1.98	0.28	0.19	-115.34	-32.35	-3.169	-0.889	-1.300	-0.36	-262.51	-73.62	-13.81	-3.87	-13.81	-3.87
21	0.52	0.57	0.75	0.59	0.30	85.54	50.10	-0.425	-0.249	-1.118	-0.65	-111.70	-65.42	-3.06	-1.79	-3.06	-1.79
22	0.95	0.00	1.53	0.15	0.15	-443.94	-67.97	-84.720	-12.971	-1.598	-0.24	-468.70	-71.76	-108.53	-16.62	-108.53	-16.62
23	0.40	1.33	0.25	1.22	0.49	232.63	284.66	-0.098	-0.119	-0.985	-1.21	15.22	18.62	-0.86	-1.05	-0.86	-1.05
24	0.58	0.33	1.14	0.41	0.24	13.26	5.41	-0.876	-0.358	-1.184	-0.48	-168.62	-68.84	-5.40	-2.20	-5.40	-2.20
25	0.28	2.25	0.05	2.03	0.58	370.85	754.22	-0.025	-0.050	-0.860	-1.75	151.22	307.53	-0.22	-0.45	-0.22	-0.45
26	0.66	0.12	1.81	0.29	0.19	-90.96	-26.39	-2.483	-0.721	-1.278	-0.37	-245.37	-71.20	-11.63	-3.38	-11.63	-3.38
27	0.32	1.91	0.10	1.73	0.56	319.68	553.68	-0.041	-0.071	-0.906	-1.57	98.69	170.92	-0.37	-0.65	-0.37	-0.65
28	0.33	1.89	0.11	1.71	0.56	315.87	539.84	-0.042	-0.073	-0.909	-1.55	94.88	162.16	-0.39	-0.66	-0.39	-0.66
29	0.42	1.13	0.33	1.05	0.45	200.37	211.21	-0.135	-0.142	-1.014	-1.07	-14.03	-14.79	-1.15	-1.21	-1.15	-1.21
30	0.80	0.01	2.74	0.28	0.23	-251.48	-71.51	-12.364	-3.516	-1.424	-0.40	-353.15	-100.42	-34.18	-9.72	-34.18	-9.72
1980	0.29	2.23	0.06	2.01	0.58	367.26	739.41	-0.025	-0.051	-0.863	-1.74	147.44	296.84	-0.23	-0.46	-0.23	-0.46
1981	0.92	0.00	2.14	0.21	0.20	-408.38	-87.46	-59.373	-12.715	-1.566	-0.34	-448.34	-96.01	-88.53	-18.96	-88.53	-18.96
1982	0.98	0.00	0.83	0.08	0.08	-473.21	-39.22	-113.537	-9.409	-1.624	-0.13	-485.17	-40.21	-127.96	-10.60	-127.96	-10.60
1983	0.23	2.59	0.02	2.33	0.55	429.84	1003.07	-0.014	-0.032	-0.806	-1.88	215.40	502.66	-0.12	-0.27	-0.12	-0.27
1984	0.27	2.38	0.04	2.14	0.57	390.80	837.77	-0.020	-0.043	-0.842	-1.80	172.47	369.72	-0.18	-0.38	-0.18	-0.38
1985	0.88	0.00	2.67	0.27	0.24	-352.70	-94.44	-34.022	-9.109	-1.515	-0.41	-415.58	-111.27	-63.81	-17.08	-63.81	-17.08
1986	0.83	0.00	2.81	0.29	0.24	-293.42	-83.83	-18.806	-5.373	-1.462	-0.42	-379.49	-108.43	-44.47	-12.71	-44.47	-12.71
1987	0.61	0.22	1.41	0.34	0.21	-29.96	-10.25	-1.349	-0.461	-1.223	-0.42	-201.17	-68.79	-7.48	-2.56	-7.48	-2.56
1988	0.55	0.42	0.96	0.48	0.26	45.02	21.40	-0.638	-0.303	-1.155	-0.55	-144.01	-68.46	-4.22	-2.01	-4.22	-2.01
1989	0.53	0.53	0.81	0.55	0.29	74.80	41.42	-0.473	-0.262	-1.128	-0.62	-120.36	-66.64	-3.33	-1.84	-3.33	-1.84
1990	0.57	0.37	1.05	0.44	0.25	27.69	12.09	-0.758	-0.331	-1.171	-0.51	-157.51	-68.79	-4.83	-2.11	-4.83	-2.11
1991	0.98	0.00	0.69	0.07	0.07	-478.31	-32.79	-119.472	-8.191	-1.629	-0.11	-488.01	-33.46	-131.64	-9.03	-131.64	-9.03
1992	0.97	0.00	1.16	0.12	0.11	-460.35	-53.47	-99.837	-11.597	-1.613	-0.19	-477.97	-55.52	-119.07	-13.83	-119.07	-13.83
1993	0.10	2.50	0.00	2.25	0.23	591.16	1327.62	-0.003	-0.006	-0.660	-1.48	415.44	932.99	-0.02	-0.04	-0.02	-0.04
1994	0.64	0.17	1.60	0.31	0.20	-59.90	-18.63	-1.820	-0.566	-1.250	-0.39	-223.11	-69.40	-9.31	-2.90	-9.31	-2.90
1995	0.23	2.60	0.02	2.35	0.54	432.64	1014.78	-0.013	-0.031	-0.804	-1.89	218.56	512.64	-0.11	-0.26	-0.11	-0.26
1996	0.07	1.99	0.00	1.79	0.12	633.04	1132.49	-0.002	-0.003	-0.622	-1.11	474.65	849.14	-0.01	-0.02	-0.01	-0.02
1997	0.08	2.18	0.00	1.96	0.15	619.82	1215.76	-0.002	-0.004	-0.634	-1.24	455.58	893.61	-0.01	-0.02	-0.01	-0.02
1998	0.97	0.00	1.22	0.12	0.12	-457.72	-56.07	-97.239	-11.912	-1.610	-0.20	-476.49	-58.37	-117.32	-14.37	-117.32	-14.37
1999	0.10	2.53	0.00	2.28	0.24	586.90	1337.23	-0.003	-0.006	-0.664	-1.51	409.62	933.30	-0.02	-0.04	-0.02	-0.04
2000	0.26	2.43	0.04	2.19	0.57	400.53	879.03	-0.018	-0.040	-0.833	-1.83	183.00	401.62	-0.16	-0.35	-0.16	-0.35
2001	0.24	2.57	0.02	2.31	0.55	425.10	983.18	-0.014	-0.033	-0.811	-1.87	210.09	485.90	-0.12	-0.28	-0.12	-0.28

Table B8.1 Realizations for the Smooth Ambiguity Functional

B.8.2 Realisations for the Extended Arrow-Pratt Approximation

The formulas for the Extended Arrow-Pratt Approximation have been analysed by the use of the standard functions in Excel. Table B8.2 show the set-up and the annotations used where the commands for row 5 and column X to AI are shown below. These commands have been copied to row 6 down to row 2001 in order to obtain 1996 realisations of the formulas.

Q1(P2):	X5=(1-G5)*p1_
*Q1(p2)*f(p2,m):*	Y5=X5*J5
Q2(p2):	Z5=(1-G5)*(1-p1_)
*Q2(p2)*f(p2,m):*	AA5=Z5*J5
Q3(p2):	AB5=G5
*Q3(p2)*f(p2,m):*	AC5=AB5*J5
*Q1(p2)*Q1(p2)*f(p2,m):*	AD5=X5*X5*J5
*Q2(p2)*Q2(p2)*f(p2,m):*	AE5=Z5*Z5*J5
*Q3(p2)*Q3(p2)*f(p2,m):*	AF5=AB5*AB5*J5
*Q1(p2)*Q2(p2)*f(p2,m):*	AG5=X5*Z5*J5
*Q1(p2)*Q3(p2)*f(p2,.m):*	AH5=X5*AB5*J5
*Q2(p2)*Q3(p2)*f(p2,m):*	AI5=Z5*AB5*J5

	G	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI
4	p2	Q1(p2)	Q1(p2)*f(p2,m)	Q2(p2)	Q2(p2)*f(p2,m)	Q3(p2)	Q3(p2)*f(p2,m)	Q1(p2)*Q1(p2)*f(p2,m)	Q2(p2)*Q2(p2)*f(p2,m)	Q3(p2)*Q3(p2)*f(p2,m)	Q1(p2)*Q2(p2)*f(p2,m)	Q1(p2)*Q3(p2)*f(p2,m)	Q2(p2)*Q3(p2)*f(p2,m)
5	0.33	0.48	0.82	0.19	0.33	0.33	0.56	0.40	0.063	0.184	1.6E-01	0.270	0.11
6	0.30	0.50	0.97	0.20	0.39	0.30	0.57	0.49	0.078	0.170	2.0E-01	0.289	0.12
7	0.79	0.15	0.04	0.06	0.02	0.79	0.22	0.01	0.001	0.174	2.7E-03	0.034	0.01
8	0.17	0.59	1.50	0.24	0.60	0.17	0.43	0.89	0.143	0.072	3.6E-01	0.254	0.10
9	0.68	0.23	0.06	0.09	0.03	0.68	0.19	0.01	0.002	0.131	5.8E-03	0.044	0.02
10	0.03	0.69	0.68	0.28	0.27	0.03	0.03	0.47	0.075	0.001	1.9E-01	0.020	0.01
11	0.87	0.09	0.03	0.04	0.01	0.87	0.24	0.00	0.000	0.207	9.2E-04	0.022	0.01
12	0.36	0.46	0.68	0.18	0.27	0.36	0.53	0.31	0.049	0.191	1.2E-01	0.243	0.10
13	0.78	0.16	0.04	0.06	0.02	0.78	0.22	0.01	0.001	0.170	2.9E-03	0.035	0.01
14	0.41	0.42	0.49	0.17	0.20	0.41	0.47	0.21	0.033	0.191	8.3E-02	0.199	0.08
15	0.69	0.22	0.06	0.09	0.02	0.69	0.19	0.01	0.002	0.133	5.3E-03	0.042	0.02
16	0.50	0.35	0.23	0.14	0.09	0.50	0.32	0.08	0.013	0.162	3.2E-02	0.114	0.05
17	0.30	0.50	0.97	0.20	0.39	0.30	0.57	0.49	0.078	0.171	1.9E-01	0.288	0.12
18	0.68	0.23	0.06	0.09	0.03	0.68	0.19	0.01	0.002	0.131	5.8E-03	0.044	0.02
19	0.77	0.17	0.05	0.07	0.02	0.77	0.21	0.01	0.001	0.164	3.1E-03	0.036	0.01
20	0.68	0.23	0.06	0.09	0.03	0.68	0.19	0.01	0.002	0.131	5.7E-03	0.043	0.02
21	0.52	0.34	0.20	0.14	0.08	0.52	0.30	0.07	0.011	0.157	2.8E-02	0.104	0.04
22	0.95	0.03	0.01	0.01	0.00	0.95	0.15	0.00	0.000	0.139	6.7E-05	0.005	0.00
23	0.40	0.43	0.53	0.17	0.21	0.40	0.49	0.23	0.036	0.193	9.1E-02	0.209	0.08
24	0.58	0.30	0.12	0.12	0.05	0.58	0.24	0.04	0.006	0.136	1.5E-02	0.071	0.03
25	0.28	0.51	1.04	0.20	0.42	0.28	0.58	0.53	0.085	0.163	2.1E-01	0.295	0.12
26	0.66	0.24	0.07	0.10	0.03	0.66	0.19	0.02	0.003	0.128	6.7E-03	0.046	0.02
27	0.32	0.48	0.84	0.19	0.33	0.32	0.56	0.40	0.064	0.183	1.6E-01	0.271	0.11
28	0.33	0.48	0.82	0.19	0.33	0.33	0.56	0.39	0.063	0.184	1.6E-01	0.269	0.11
29	0.42	0.41	0.43	0.16	0.17	0.42	0.45	0.18	0.029	0.189	7.2E-02	0.184	0.07
30	0.80	0.15	0.04	0.06	0.02	0.80	0.23	0.01	0.001	0.180	2.4E-03	0.033	0.01
1980	0.29	0.51	1.03	0.20	0.41	0.29	0.58	0.52	0.084	0.164	2.1E-01	0.294	0.12
1981	0.92	0.05	0.01	0.02	0.00	0.92	0.20	0.00	0.000	0.183	2.5E-04	0.011	0.00
1982	0.98	0.02	0.00	0.01	0.00	0.98	0.08	0.00	0.000	0.079	8.2E-06	0.001	0.00
1983	0.23	0.55	1.28	0.22	0.51	0.23	0.55	0.70	0.112	0.128	2.8E-01	0.299	0.12
1984	0.27	0.52	1.12	0.21	0.45	0.27	0.57	0.59	0.094	0.152	2.4E-01	0.299	0.12
1985	0.88	0.09	0.02	0.03	0.01	0.88	0.24	0.00	0.000	0.207	8.0E-04	0.020	0.01
1986	0.83	0.12	0.03	0.05	0.01	0.83	0.24	0.00	0.001	0.197	1.7E-03	0.029	0.01
1987	0.61	0.28	0.09	0.11	0.04	0.61	0.21	0.03	0.004	0.128	1.0E-02	0.058	0.02
1988	0.55	0.32	0.15	0.13	0.06	0.55	0.26	0.05	0.008	0.144	2.0E-02	0.084	0.03
1989	0.53	0.34	0.19	0.14	0.07	0.53	0.29	0.06	0.010	0.154	2.5E-02	0.099	0.04
1990	0.57	0.31	0.14	0.12	0.05	0.57	0.25	0.04	0.007	0.140	1.7E-02	0.077	0.03
1991	0.98	0.01	0.00	0.01	0.00	0.98	0.07	0.00	0.000	0.066	4.5E-06	0.001	0.00
1992	0.97	0.02	0.00	0.01	0.00	0.97	0.11	0.00	0.000	0.109	2.5E-05	0.003	0.00
1993	0.10	0.64	1.44	0.26	0.58	0.10	0.23	0.93	0.148	0.023	3.7E-01	0.146	0.06
1994	0.64	0.26	0.08	0.10	0.03	0.64	0.20	0.02	0.003	0.126	8.3E-03	0.051	0.02
1995	0.23	0.55	1.29	0.22	0.51	0.23	0.54	0.71	0.113	0.126	2.8E-01	0.298	0.12
1996	0.07	0.67	1.19	0.27	0.48	0.07	0.12	0.79	0.127	0.008	3.2E-01	0.080	0.03
1997	0.08	0.66	1.29	0.26	0.52	0.08	0.15	0.85	0.136	0.012	3.4E-01	0.101	0.04
1998	0.97	0.02	0.00	0.01	0.00	0.97	0.12	0.00	0.000	0.114	3.0E-05	0.003	0.00
1999	0.10	0.64	1.46	0.26	0.58	0.10	0.24	0.93	0.149	0.025	3.7E-01	0.153	0.06
2000	0.26	0.53	1.16	0.21	0.47	0.26	0.57	0.62	0.099	0.147	2.5E-01	0.300	0.12
2001	0.24	0.54	1.26	0.22	0.50	0.24	0.55	0.68	0.110	0.131	2.7E-01	0.300	0.12

Table B8.2 Realizations for the Extended Arrow-Pratt Approximation

B.8.3 Analysis results

Table B8.3 show the analysis results of the integration over the N realisations and post-processing of the formulas given in chapter B.1 to B.7. The Excel commands used to arrive at the analysis results given in column E of table B8.3 are shown below.

ptilde:	E5=SUM(K5:K2001)/COUNT(K5:K2001)
H:	E6=AVERAGE(M5:M2001)
G(a):	E12=AVERAGE(Q5:Q2001)
W:	E13=-EXP(-a*m3_)
c2:	E14=AVERAGE(S5:S2001)
V(a,b):	E15=AVERAGE(U5:U2001)
V2(a,b):	E16=AVERAGE(W5:W2001)
cAmAv:	E17=-LN(-E16)/b
Π:	E18=E6-E17
mQ1:	E19=AVERAGE(Y5:Y2001)
mQ2:	E20=AVERAGE(AA5:AA2001)
mQ3:	E21=AVERAGE(AC5:AC2001)
sig11:	E22=AVERAGE(AD5:AD2001)-E19*E19
sig22:	E23=AVERAGE(AE5:AE2001)-E20*E20
sig33:	E24=AVERAGE(AF5:AF2001)-E21*E21
sig12:	E25=AVERAGE(AG5:AG2001)-E19*E20
sig13:	E26=AVERAGE(AH5:AH2001)-E19*E21
sig23:	E27=AVERAGE(AI5:AI2001)-E20*E21
pmu(0,0):	E31=(1-E5)*p1_
pmu(0,1):	E32=(1-E5)*(1-p1_)
pmu(0,2):	E33=E5
∑*(w(0,s)*pmu(0,s):*	E34=E28*E31+E29*E32+E30*E33
∑*(w(0,s)^2*pmu(0,s):*	E35=E28^2*E31+E29^2*E32+E30^2*E33

sig2pmu: E36=E35-E34^2

CAP: E37==E6-(1/2)*a*E36

Sig2mu: E38=E22*E28*E28+E25*E28*E29+E26*E28*E30+
E23*E29*E29+E25*E28*E29+E27*E29*E30+E26*E28*E30+E27*E29*
E30+E24*E30*E30

CAM: E39=(1/2)*(b-a)*E38

Πquad= CAP-CAM: E40=E37-E39

Certainty Equivalent KMM functional (cAmAv): E42=E17

Certainty Equivalent Arrow-Pratt Approximation (Πqua): E43=E40

Risk premium (H-CAP): E44=E6-E37

Ambiguity premium (CAM): E45=E39

Certainty Equivalent Wald Maximin utility: E46=-(1/a)*LN(-E13)

Certainty Equivalent KMM functional (V(a,b)): E47=-(1/b)*LN(-E15)

	A	B	C	D	E
4	**Constants**			**Simulation results**	
5	m	0.1		ptilde	0.3021
6	p1	0.71		H	352.31
7	m1	1000		Hmax	713.74
8	m2	0		Hmin	-498.61
9	m3	-500		ptilde-max	0.58
10	a	0.001		ptilde-min	0.00
11	b	0.01		V0(b)	-2.01
12				G(a)	-0.88
13	**Range names used**			W	-1.65
14	a	=Sheet1!$B$9		c2	159.68
15	b	=Sheet1!$B$10		V(a,b)	-4.19
16	m	=Sheet1!$B$4		V2(a,b)	-4.19
17	m1_	=Sheet1!$B$6		cAmAv	-143.2
18	m2_	=Sheet1!$B$7		Π	495.46
19	m3_	=Sheet1!$B$8		mQ1	0.50
20	p1_	=Sheet1!$B$5		mQ2	0.20
21				mQ3	0.30
22				sig11	0.0197
23				sig22	0.0031
24				sig33	0.0413
25				sig12	7.9E-03
26				sig13	-0.0310
27				sig23	-0.0124
28				w(0,0)=	1000
29				w(0,1)=	0
30				w(0,2)=	-500
31				pmu(0,0)	0.498
32				pmu(0,1)	0.199
33				pmu(0,2)	0.30
34				∑(w(0,s)*pmu(0,s)=	347.41
35				∑(w(0,s)^2*pmu(0,s)=	574011.06
36				sig2pmu=	453314.89
37				CAP	125.65
38				sig2mu	60982.21
39				CAM	274.42
40				Πquad= CAP-CAM=	-148.77
41					
42				Certainity Equivalent KMM functional (cAmAv)	-143.2
43				Certainity Equivalent Arrow-Pratt Approximation (Πqua	-148.77
44				Risk premium (H-CAP)	226.66
45				Ambiguity premium (CAM)	274.42
46				Certainty Equivalent Maxmin expected utility	-500
47				Certainty Equivalent KMM functional (V(a,b))	-143.2

Table B8.3 Excel analysis results for the Smooth Ambiguity Functional and the Extended Arrow-Pratt Approximation

Appendix C – Interview Question Protocol

The questions posed in the in-depth interviews of the selected peer group.

General questions:	
1	Expected value/mean represents the weighted average. It reflects the "centre of gravity" in a probability distribution. The P50 value gives equal probability of under run/overrun and the area under the probability curve is then the same on both sides. Are you familiar with these concepts of expected value and P50 value?
2	Do you know where the concepts of expected values/P50 values are used in a decision process?
3	In a decision process do you differentiate between a potential positive and a potential negative outcome?
4	Do you know how this differentiation or weighting of outcomes can be done?
5	A decision maker is neutral to loss or gain if a negative or positive outcome are given the same weighting. Are you familiar with this term of being risk neutral?
6	Do you agree that decisions based on expected values is a risk neutral approach?
7	Tolerance of risk or aversion to risk is a subjective judgment and expresses the decision maker's aversion to losses or potential negative outcomes. The potential outcomes are then based on uncertainty assessments of quantities, rates and norms that are related to the known elements of the described technical solution. Do you know how a decision maker's risk aversion or contingency is implemented in the decision process?
8	Ambiguity is a term used to describe uncertainty when you are not confident in assigning a probability distribution. This could refer to uncertainty of unspecified or unknown elements of a technical solution. Do you know how a decision maker's ambiguity aversion or budget reserve is implemented in the decision process?
9	The certainty equivalent value is the value you consider as being risk free or the same as having money in your bank. For a project subject to risk and ambiguity, the difference between an expected value and a risk premium and an ambiguity premium represents the certainty equivalent value. The risk and ambiguity premiums are based on a subjective judgement by the decision maker. Risk premium have some similarities to the project contingency concept and ambiguity premium have some similarities to the use of budget reserves. Are you familiar with the term certainty equivalent value and the use of contingency and budget reserves in a decision process?
Project specific questions:	
1	Please have a look at project A and project B. Would you prefer one out of these two projects or do you consider them to be indifferent?
2	Can you explain how you arrived at your choice between project A and project B?
3	What would you consider as an appropriate expected value for project A?
4	Please have a look at project C and project D. Would you prefer one out of these two projects or do you consider them to be indifferent?
5	Can you explain how you arrived at your choice between project C and project D?
6	What would you consider as an appropriate expected value for project C?

7	Can you recall a similar decision setting as for project A and project C where you are not able to describe the probabilities for a specific outcome?
8	If you had more information about P10 and P90 for the projects would you then assign a reserve or buffer to the expected values to account for the statistical variation in project B and D?
9	Would you assign a reserve or buffer to the expected values to account for the unknown unknowns in project A and C?
10	Can you recall a past decision setting where you gave potential negative outcomes more weighting or in an extreme case only considered a negative outcome?