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Poncelet's Theorem

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Introduction

This thesis will be concerned with different questions related to the theorem of Poncelet.

Poncelet's Theorem. Let C_1 and C_2 be two real conics, with C_1 inside C_2. Suppose there is an n-sided polygon inscribed in C_2 and circumscribed about C_1. This means that we have a closed polygon where the n sides $L_0, L_1, ..., L_{n-1}$ are tangents to the inner circle and the vertices $P_0 = L_{n-1} \bigcap L_0, P_1 = L_0 \bigcap L_1, ..., P_{n-1} = L_{n-2} \bigcap L_{n-1}$ are points on the outer circle. Then for any other point of C_2, there exists an n-sided polygon, inscribed in C_2 and circumscribed about C_1, which has this point for one of its vertices [8, p.1].

We will use most of our time studying this theorem for smooth conics C_1 and C_2 in general position in complex projective plane.

This theorem has kept a lot of mathematicians busy in different periods of human history. New ideas in geometry and other fields of mathematics have during the years been used to find new proofs. It also has a long prehistory that we will study in chapter 1. Here we will also look into how Poncelet contributed to the development of projective geometry.

To understand how it is possible to prove Poncelet's theorem, we first need to study conics in projective plane. Our treatment will mainly be of complex projective conics, but we will also take a look at the real ones. We will see what the projective plane is, and how this environment is a great advantage when working with conics. Thereafter we discuss whether a proof in complex projective plane will imply that the theorem holds in the real projective case.

Understanding a modern proof will also require a lot more than the theory of conics. We will take a journey through interesting ideas and theorems, starting with the concept of divisors. This is needed to state the theorem of Riemann-Roch which we will need several times on our way. Hurwitz formula will also play an important role in the proof that we will use most of our time learning about. This formula will make it possible to show that the curve that Poncelet described is a curve of genus 1, an elliptic curve. The group structure of an elliptic curve will be important for the modern proof that we concentrate on. This latter proof was done by Griffiths and Harris in 1977. After all the tools we need are presented, we are ready to explain the proof of this reformulation of the theorem:

If η^n has a fixed point for some positive integer n, then η^n is the identity map on E.
Here E is the Poncelet curve and η a Poncelet map which is composition of two involutions of E.

After this, we will present a proof in real projective plane that was done recently by Halbeisen and Hungerbuhler. Since it uses Pascal's theorem, we will take some time to look at that first.

At the end we remind ourselves that this theorem is not only belonging to a great history book. Mathematicians of today work on generalisations and related problems.

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Contents

1 Historical background **6**
- 1.1 Early history of the theorem 6
  - 1.1.1 Triangles 6
  - 1.1.2 "Chapple-Euler"-formula 7
- 1.2 The development of Projective geometry 8
  - 1.2.1 Perspective and central projection 9
  - 1.2.2 Poncelet's principal of central projection. 11
  - 1.2.3 What is an ideal chord? 14
  - 1.2.4 What kind of principle is the principle of continuity? . 15

2 Conics in the projective plane. **18**
- 2.1 Projective n-space 18
- 2.2 Conics and their structure 20
- 2.3 Conics in $\mathbb{P}^2_{\mathbb{C}}$ 22
- 2.4 Conics in $\mathbb{P}^2_{\mathbb{R}}$ 24
- 2.5 Formulation of Poncelet's Theorem 26

3 Algebrogeometric reformulation of Poncelet's Theorem **29**

4 Divisors on a curve **31**
- 4.1 Divisor classes 31
- 4.2 Riemann-Roch Theorem 34
- 4.3 Hurwitz formula 36

5 Elliptic curves **37**
- 5.1 E is isomorphic to a curve on Weierstrass form. 37
- 5.2 Geometric construction of group law on an elliptic curve . . . 38
- 5.3 $E \simeq Pic^0(E)$ 39
- 5.4 A synthetic construction of the group law on E 40

6 Proof of Poncelet's Theorem **44**

7 A proof by Pascal's Theorem **46**
- 7.1 Cayley-Bacharach Theorem 46
- 7.2 Triangles 51
- 7.3 The case of a n-sided polygon interscribed between two conics 53

8 New generations will be inspired by Poncelet's Theorem. **57**

1 Historical background

In this chapter we will look at the prehistory of Poncelet's theorem and how Poncelet contributed to the developement of projective geometry. Most of the material is taken from [2], [3] and [16].

1.1 Early history of the theorem

1.1.1 Triangles

One can debate what is the "prehistory" of this great theorem and what lead Poncelet to think about it. Even in prison he could not get his mind of this problem. Learning about mathematicians interested in geometry before him, it is natural to look at works on triangles and circles. Given two circles, one inside the other, when is it possible to draw a triangle inscribed in the outer circle and circumscribed about the inner.

Theorem 1.1. Given a circle C_1 with radius r inside a circle C_2 with radius R. Denote by d the distance between their centers. Then there exists a triangle inscribed in C_2 and circumscribed about C_1 if and only if

$$d^2 = R^2 - 2rR$$

Before a presentation of different men working on this, we take time for a proof.

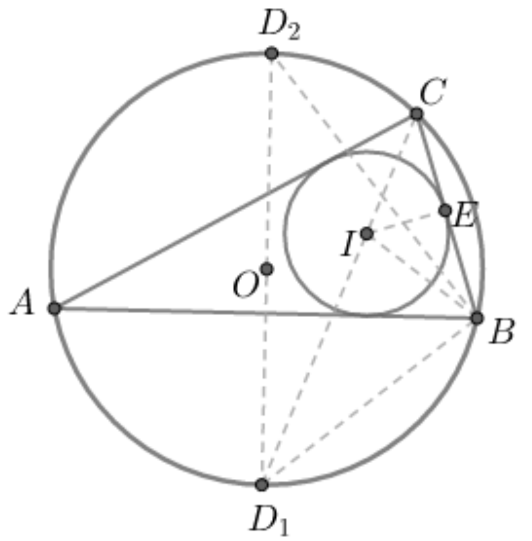


figure 1.1. [3, p. 65].

Proof. Given a triangle ABC, a circumscribed circle with center O and an inscribed circle with center I. Like in the picture above, the midpoints of the arc AB, D_1 and D_2, are marked out. E is the projection of I onto BC. We have that: $\angle D_1BI = \angle D_1BA + \frac{1}{2}B$. $\angle D_1BA = \angle D_1CA$, which means that $\angle D_1BI = \angle BID_1$ and so $|D_1I| =|D_1B|$. We also see $D_1D_2B \sim ICE$, which means $\frac{|D_1D_2|}{|D_1B|} = \frac{|IC|}{|IE|}$ so that $2Rr =|IC|\cdot|D_1B| =|IC|\cdot|ID_1|$. This is minus the power of I with respect to the circumscribed circle and the same as $R^2 - d^2$ if $|OI| = d$.

For the other direction, given $d^2 = R^2 - 2rR$. Let C_1 be a circle inside the circle C_2 and M_1 and M_2 their centers. Let C be one of the intersection points of the line M_1M_2 and the circle C_2. Make a tangent to the inner circle C_1 that is at the same time perpendicular to M_1M_2. Let A and B be the names of the intersections of C_2 and this tangent line. And let D be the name of $M_1M_2 \cap AB$. We then have the following relation between the length of line segments:

$$|CD| = R + r + d$$

$$|AD|^2 = R^2 - (r + d)^2$$

$$|AC|^2 = 2R(R + r + d).$$

Let x be the distance from M_1 to the line AC. The triangles ADC and M_1EC are similar, so we get $2Rx^2 = (R + d)^2(R - r - d)$. Using $d^2 = R^2 - 2rR$ we end up with $x^2 = r^2$. Then we know that AC and BC are tangent to C_1, and we have found the triangle that we wanted [3, p. 65]. □

Like we will soon see, quite a few mathematicians within a period of about hundred years starting from early eighteen century, studied this problem. Some of them also considered the relation between the sides x, y, z of such a triangle. This suggests that if one such triangle exists, it will not be the only one. It is possible to look at R and r as functions of x, y, z : $R = R(x, y, z)$ and $r = r(x, y, z)$. The distance $d = (R^2 - 2rR)^{\frac{1}{2}}$ will be fixed if R and r is fixed. Since we have two conditions and three variables, the functions for R and r leave one degree of freedom for the values of x, y and z. Each of the possible triangles is congruent to an interscribed one between our two circles.

1.1.2 "Chapple-Euler"-formula

The formula $d^2 = R^2 - 2rR$ was given by William Chapple in an article in 1746, thereof the name "Chapple's formula". Apparently, not a lot of people read his article though. As we will soon see, this formula was also discovered by others. There is one thing that distinguishes Chapple's work from most of the others working on the same problem. It seems that he assumes that if this relation is valid for two circles, the triangle will not depend on any special starting point. He did not prove this.

The formula is also known as "Euler-Chapple"-formula. Euler wrote a paper in 1765, among others treating special points of a triangle. Two such points are the center of the inscribed and the center of the circumscribed circles. In his paper he included the formula for the distance between these:

$$d^2 = \frac{(xyz)^2}{16A^2} - \frac{xyz}{x+y+z}$$

x, y, z are the sides of the triangle. Since Euler did not express the distance d by the relation between R and r, some have the opinion that his name should not be glued to the formula.

Nicolas Fuss also proved the formula in 1797, and his article was read by far more people than in Chapple's case. So for a while, he was known as the person to first discover this relation. He also studied inscribed and circumscribed 5-,6-,7- and 8-gons, but did not in these cases find a general relation between d, r, R.

Mathematicians who still had not read about this formula, continued to find $d^2 = R^2 - 2rR$. One of them was Lhuillier in 1810. He was one of the first to explicitly point out the existence of closure in these cases. [2, p. 296].

Poncelet proved the closure theorem and it was published in 1822. Talking about the formula $d^2 = R^2 - 2rR$, Poncelet did not write anything about the relation of his closure theorem with this. Jacobi was the first to comment on that.

Steiner (1827) gave formulas on relation between d, r, R for n-gons in the case n=4,5,6 and 8. He did not give proofs or discussed closure property [2, p. 297].

1.2 The development of Projective geometry

We will now look at the development of projective geometry and how Poncelet tackled intersection points that are imaginary. In the Euclidean geometry we have 5 postulates [5, p. 4]:

1. A straight line may be drawn from any point to any other point.
2. A finite straight line may be extended continuously in a straight line.
3. A circle may be described with any center and any radius.
4. All right angles are equal to one another.
5. If a straight line meets two other lines so as to make the two interior angles on one side of it together less than two right angles, the other straight lines, if extended indefinitely, will meet on that side on which the angles are less than two right angles.

In projective geometry, also parallel lines meet.

1.2.1 Perspective and central projection

The end of the 1700-century was dominated by analytical methods in mathematics and science. Not too many had an interest in descriptive geometry. Some exceptions were the french mathematician Monge and his students Poncelet, Brianchon and Chasles who started to develop what we today know as projective geometry.

In the world of art, creating a realistic perspective in paintings, occupied quite some people from the renaissance and onward. Albrecht Durer (1471-1528) was a German artist who was seriously devoted to recreate the three dimensional world into two dimensions helped by a central projection. The idea is a canvas between the artist and the object of interest. The illusion of that object still being there when removed is made by the rays of light from points on the object, intersecting the canvas, to the eye of the artist (the projection center).

We will take time for an analytic formulation of this [16, p. 322]. Start by a point $\tilde{x}$ for the artists eye and a vector a which represents the shortest distance from $\tilde{x}$ to the canvas. We want to project the real object (point) x onto the plane (the picture), so we want to find the coordinates (u_1, u_2) there. Expressing the plane by the orthogonal vectors h and g, which are also orthogonal to a, we write for the horizontal $h = a \times (0, 0, 1)^\top$ and $g = h \times a$. h and g are normalised.

$$h = \frac{1}{\sqrt{{a_1}^2 + {a_2}^2}} \begin{bmatrix} a_2 \\ -a_1 \\ 0 \end{bmatrix}$$

.

$$g = \frac{1}{\sqrt{({a_1}^2 + {a_2}^2)({a_1}^2 + {a_2}^2 + {a_3}^2)}} \begin{bmatrix} -a_1 a_3 \\ -a_2 a_3 \\ a_1^2 + a_2^2 \end{bmatrix}$$

The vector from $\tilde{x}$ to (u_1, u_2) we call w, $w = \lambda(x - \tilde{x})$. Since $w - a$ is orthogonal to a, that gives us λ.

$$\langle w - a, a \rangle = 0 \Rightarrow \lambda = \frac{\langle a, a \rangle}{\langle x - \tilde{x}, a \rangle}$$

Then we can compute $u_1 = w \cdot h$ and $u_2 = w \cdot g$.

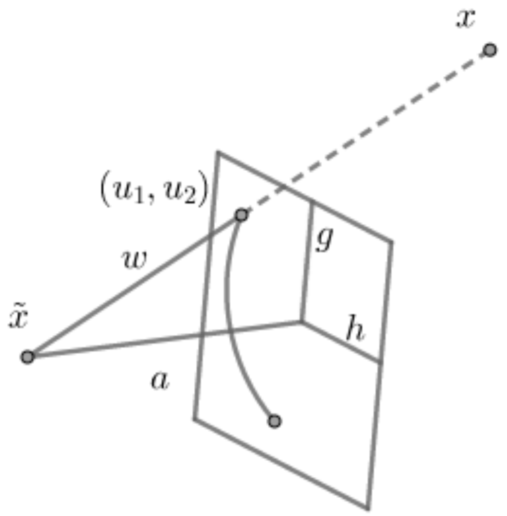


figure 1.2. [16, p. 322]

Drawing an object on a canvas by a central projection will result in the following. Given two points A and B on the three dimensional object. Imagine moving A on the line AB until infinity. Then the line from the projection centre to "the A moving to infinity" will be closer and closer to a parallel line of the AB we started with. The point of intersection of this parallel line with the canvas is called the vanishing point of this direction.

All lines that are parallel to AB on the object will meet in the same vanishing point on the canvas. It contributes to the illusion that the painting has three dimensions since it mimics the phenomenon in real world that parallel lines appear to meet in a point very far from where you stand.

1.2.2 Poncelet's principal of central projection.

History tells that Poncelet particularly enjoyed the studies under Monge. After his studies he joined the military forces in Napoleon's invasion of Russia. He got a lot of time to contemplate the lectures of Monge, staying two years in prison during the war. There by the river Volga, important thinking on projective geometry were unfolding. Transforming figures into simpler form by central projection, he managed to prove old and new theorems. In the following we will take a look at some of his results. We look for a (central) projection that maps a perspective image in a Cartesian plane to the given image. Two of his main concepts were "principle of continuity" and "ideal chords". The analytic approach he used in the early days of his career was no longer apparent. From now on he approached geometrical questions in a purely synthetic manner.

Lemma 1.2. *Principal lemma on perspective.* Let OPQ be an arbitrary triangle and U be an arbitrary "unit point" inside OPQ. Then there exists a central projection which maps the line PQ to infinity and for which P and Q are the vanishing points of a pair of orthogonal axes centered at O', the image of O. The image U' of U is a unit point i.e. $O'U'$ is the diagonal of a square with sides on the axes $O'P'$ and $O'Q'$ [16, p. 324].

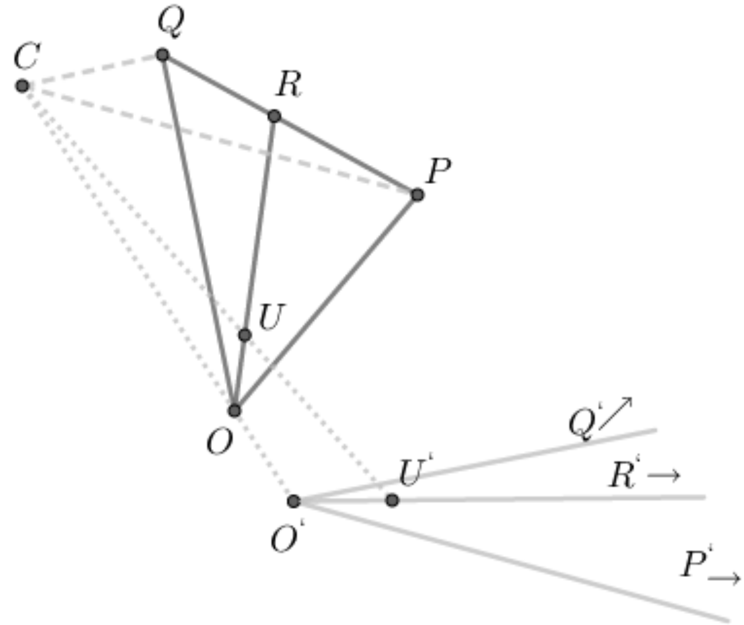


figure 1.3. [16, p. 324]

Proof. To find the center of projection, C, let us first set that we want it to be in the same horizontal plane as the line PQ, the triangle PQO is placed vertical. We choose a point U in our triangle that in the projection will be the diagonal of a unit square. Draw OU and call its intersection with PQ R. Now our projection of $OPRQ$ shall lie in a plane parallel to CPQ. P', Q' and R' will be points at infinity. C must be constructed so that PCQ is a right angle (see picture below), and $P'O'Q'$ is thereby a right angle. C must lie on a circle that has QP as diagonal to make this happen. Now we need only one more restriction to choose C. PCR must be constructed to be 45°, if the point U' shall have the same distance from the axis and thereby be a unit point in a Cartesian grid. Now our center of projection is determined. □

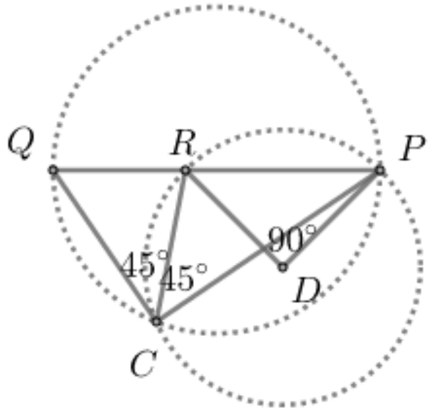


figure 1.4. [16, p. 324]

We will in the following study how Poncelet proved different theorems. Let us first state one of them to discuss some obstacles that he met.

Theorem 1.3. *Projection theorem.* Any pair D_1, D_2 of conics in a (real) plane V is the projective image of a pair of circles [2, p. 298].

Let us first make some comments on how Poncelet approached this. Starting with two ellipses D_1 and D_2 in a plane V in the real space, he tried to find a point C (center of projection) and plane W in that space, so that $V \to W$ is a perspective projection through C that maps both the ellipses to circles. Poncelet showed how this can be done when D_1 and D_2 have at most two points of intersection. He constructed a line d in V by endpoints R_1 and R_2, found the midpoint Q on it, and drew the line CQ . CQ was drawn in the plane perpendicular to d. The length of CQ is then $r = \frac{1}{2}|R_1R_2|$. K is the circle with center Q and radius r. Still we have not explained where the points R_1 and R_2 comes from, but we soon will. They are points with complex values that are translated and given meaning in the real plane. Poncelet proved that any point C on the circle K that is not in the original plane V, and any plane W (parallel to the plane determined by C and d), can be the center (C) and plane (W) of the desired projection.

If D_1, D_2 intersect in more than two points, the value of r (radius of K) is imaginary and we do not have a real center C for the projection. Imaginary centers of projection were not in his mind, but he moved forward in his work focusing on the two points R_1 and R_2. When the number of intersection points exceeds two, R_1 and R_2 and the other intersection points are the same. After realizing that, he looked at R_1 and R_2 as somehow also representing the two other intersection points when they were missing. He called R_1R_2 the "common chord" of the two ellipses. Finally, out of prison, he found out others had been debating reliability of among others infinitely distant points, negative quantities and calculations with $\sqrt{-1}$ while he was away. Equipped with the constructions "ideal chords" and "principle of continuity" his commitment to a synthetic approach turned into a theory where geometry in real space should take into account results using imaginary quantities but spared for analytic method.

1.2.3 What is an ideal chord?

Instead of adding complex points to the plane, Poncelet invented "ideal chords". These chords were used when a conic and a line do not intersect in the real plane.

If the line L and the conic C do intersect in 2 points in the real plane, then we have a chord between these two points. Call these points P_1 and P_2. Let M be the conjugate diameter of C with respect to L and call its intersection points with the conic O_1 and O_2. Q is the intersection between M and L. The two diameters L and M are conjugate if the midpoints of chords parallel to L lie on M. The tangent of O_1 and O_2 are parallel to L and vice versa. From this we have the relation $|QP_1|^2 = \gamma|O_1Q||O_2Q|$ for some constant γ.

If the line L and the conic C do not intersect in the real plane, the two endpoints of the ideal chord, R_1 and R_2, are found in the following way. Let M be the conjugate diameter of C with respect to L and name $L \cap M = Q_2$. Make parallel lines to L that intersect C in 2 points. Choose one of these parallels, and call its intersection points with C : P_1 and P_2. Now we have the information we need to find γ in the relation $|QP_1|^2 = \gamma|O_1Q||O_2Q|$. We use it to find the line segment Q_2R_1. $|Q_2R_1|^2 = \gamma|O_1Q||O_2Q|$.

This can be translated to two complex points $T_i = S + / - iD$. Since T_1 and T_2 are conjugate, their midpoint S is real and the distance D is real. We have $R_1 = S + D, R_2 = S - D$. Poncelet was able to show that for any pair of conics (real), there are two "common chords" I_1I_2 and I_3I_4 which will be real or ideal chords to both conics. I_1, I_2, I_3, I_4 correspond to the four real or complex intersection points of the conics. Poncelet also introduced the term "circular points" to cover that two imaginary points lie at the line at infinity. All circles pass through this line. He knew that the chords and their endpoints could lie here [2, p. 301].

1.2.4 What kind of principle is the principle of continuity?

The main point of this principle is that when a figure in the plane is continuously deformed such that some data and theorems from the figures remain, the properties of the figures remain as well, even when sign changes or certain magnitudes vanish. These changes will also be predictable (for example when two conics have two points of intersection, but have four after transformation). How he used this principle will be demonstrated by the proofs of the next theorems. We can see how it is used in the end of the next proof to conclude that it holds in general.

Theorem 1.4. Let D be a conic and d a straight line. Then D and d are the projective images of a circle and the line at infinity [2, p. 302].

Proof. The conic D lies in the plane V and is embedded in the real space E. Let us first look at the case were d does not intersect the conic. By theory of conic sections we can find a point C that is the center of the projection and a plane W such that from the point C one can project the conic D onto a circle in W, and also the plane through C and d is parallel to W. The line d will thereby be projected to the line at infinity. A point C and a plan W cannot be found in E if d intersects D. But Poncelet argued that generality follows from the principle of continuity. □

Here we have our first example of how Poncelet used the principle of continuity to generalise his proofs. It is maybe the best way to try to understand what is meant by this principle. We will also comment this proof with a picture and a remark that links it directly to the construction in the principal lemma on perspective, that we proved above.

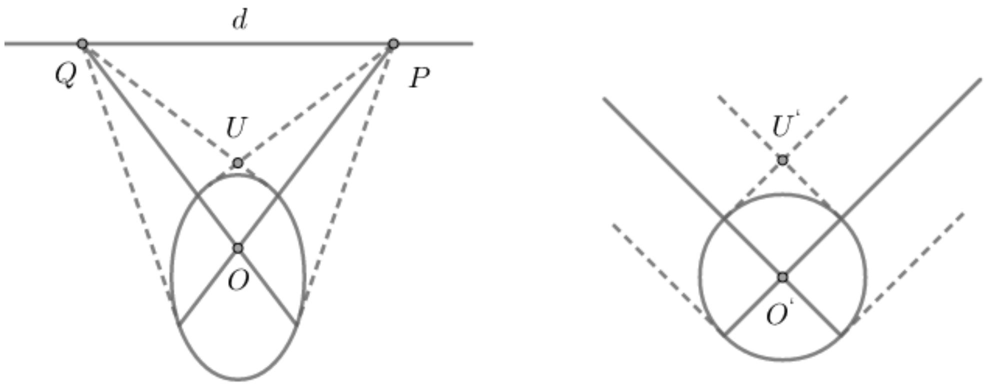


figure 1.5. [16, p. 325]

The polar line to the arbitrary point P at d meet d in Q. We have the point O which is the intersection of the polars of P and Q, and the point U which is the intersection point of their tangents. From here we can apply lemma 1.2, principal lemma on perspective, to obtain tangents that restrict this conic to the form of a circle.

Poncelet used his insights to find new proofs of geometrical relations, like for example in this proof of Pascal's theorem.

Theorem 1.5. Let $P_1P_2P_3P_4P_5P_6$ be a hexagon inscribed in a conic. Then the intersection points $K = P_1P_2 \cap P_4P_5, L = P_2P_3 \cap P_5P_6, M = P_3P_4 \cap P_6P_1$ of opposite sides are collinear [16, p. 327].

Proof. Map the line through K, L to infinity by (1.2) above. This projection turns the ellipse into a circle and makes the two pairs of opposite sides parallel so that the arcs $P_2P_3P_4$ and $P_5P_6P_1$ are of same length. Then the angle at P_3 equals that of P_6, and the third pair of opposite sides will be parallel as well. M is also mapped to a point at the line at infinity and we are done. □

We see again that the principle of continuity is needed to make the proof work in the case when the line containing K, L, M passes through the conic in Pascal's theorem.

In a similar manner Poncelet proved Brianchon's theorem, the projective dual of Pascal's theorem.

Theorem 1.6. Let $Q_1Q_2Q_3Q_4Q_5Q_6$ be a hexagon circumscribing a conic. Then the three diagonals joining pairs of opposite vertices are concurrent [16, p. 327].

Proof. Label the points where the edges of the hexagon are tangent to the conic $P_1, P_2, P_3, P_4, P_5, P_6$. Now we have the same situation as in Pascal's theorem and we make the same projection. Now the triangles $P_iQ_iP_{i+1}$ are isosceles. Two opposite triangles have parallel basis. Their altitudes are concurrent. They all meet in the circle center. □

Now we will see how Poncelet proved that two conics are the projective image of a pair of circles. The proof is made for two circles that intersects in at most two real points. If the conics have more than two real intersection points, the center of projection would be imaginary. Poncelet said that one can consider two conics to be the projective image of two circles also in this case [2, p. 299].

Theorem 1.7. Let C_1 and C_2 be two conics. Then the pair C_1, C_2 is the projective image of a pair of circles [2, p. 303].

Proof. If the number of real intersection points of C_1 and C_2 are at most two, they also have an "ideal common chord" along a line d. The endpoints of the chord, R_1 and R_2, represents imaginary intersection points. We then use (1.3) to map the conic C_1 to a circle C_{1*} and d to d_0 at infinity. The points where d_0 intersects C_{1*} will also lie on C_{2*} (the image of C_2). Those points are the so called circular ones, and this shows C_{2*} must also be a circle. If C_1 and C_2 have four real intersection points the principle of continuity ensures that this theorem will still be valid. □

Poncelet's first proof of his closure theorem was analytic. Later he made a new proof using his synthetic geometry constructions. His results are true even though parts of his arguments relied on informal ideas such as the principle of continuity.

2 Conics in the projective plane.

In this chapter we will first look at the projective n-space and at some algebraic curves in the projective plane. In particular we study conics. After we have looked at conics in general, we look at intersections between a line and a conic and intersection of two conics in complex projective plane and then in real projective plane. Most of the results are from [8].

2.1 Projective n-space

We will soon see that working in projective rather that affine plane has a lot of advantages for our purpose. Parabolas, hyperbolas and ellipses are not distinct conics in the projective plane. Also, if K is an algebraically closed field, two curves in $\mathbb{P}^n_K$ of degree m and n intersects in mn points, counted with multiplicities.

Definition 2.1. The projective n-space over K is the set of equivalence classes of points $\mathbb{P}^n_K = (K^{n+1} \backslash 0)/\sim$, where $(x_0, ..., x_n) \sim (y_0, ..., y_n)$ if $(x_0, ..., x_n) = \lambda(y_0, ..., y_n)$, for some $\lambda \in K \setminus \{0\}$.

In other words, $\mathbb{P}^n_K$ is the collection of the 1-dimensional linear subspaces of the vector space K^{n+1}. For a point $(x_0, ..., x_n)$ in affine space $\mathbb{A}^{n+1}_K$, its equivalence class is denoted $(x_0 : ... : x_n)$. The coordinates $x_0, .., x_n$ are called the homogeneous coordinates of that point.

$\mathbb{A}^n_K$ is embedded in K^{n+1} by the injective map $f : \mathbb{A}^n_K \to \mathbb{P}^n, \quad (x_1, ..., x_n) \mapsto (1 : x_1 : ... : x_n)$. $U_0 = \{(x_0 : \cdots : x_n) | x_0 \neq 0\}$. The inverse is then $f^{-1} : U_0 \to \mathbb{A}^n, \quad (x_0 : ... : x_n) \mapsto (\frac{x_1}{x_0}, ..., \frac{x_n}{x_0})$. The points at infinity are those which are not in U_0. They are of the form $(0 : x_1 : ... : x_n)$. After this extension we have a meeting point also for parallel lines, which lies at infinity.

In this text we will consider lines, conics and cubics in $\mathbb{P}^2$. They are algebraic curves, which are the set of points $x = (x_0 : x_1 : x_2)$ satisfying $P(x) = 0$ for a homogeneous polynomial, of degree one, two and three respectively.

Definition 2.2. A polynomial is homogeneous if every monomial term has the same total degree, that is, if the sum of the exponents in every monomial is the same. The degree of the homogeneous polynomial is the total degree of any of its monomials. An equation is homogeneous if every non-zero monomial has the same total degree [9, p. 27].

We will show by an example why it is necessary to work with homogeneous polynomials. Given the polynomial $F(x_0, x_1) = x_1 - x_0^2$. $f(1,1) = 0$ but $f(2,2) \neq 0$ even if $(1:1) \sim (2:2)$. This problem is solved by a homogenization of the polynomial: $F(x_0, x_1, x_2) = x_1x_2 - x_0^2$. Now $f(2,4,1) = 0$ and also $f(\lambda 2, \lambda 4, \lambda) = 0$ for any λ since $f(\lambda 2, \lambda 4, \lambda)$ gives $\lambda 4 \cdot \lambda - (\lambda 2)^2 = \lambda^2(4-4) = 0$.

Definition 2.3. For a set S of homogeneous polynomials in $K[x_0, ..., x_n]$,

$$V(S) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in S\}$$

is the projective zero locus of S in $\mathbb{P}^n$. Subsets of $\mathbb{P}^n$ that are of this form are called projective varieties [10, p. 49].

In projective n-space there is a one-to-one correspondence between points and hyperplanes. We write $\mathbb{P}^{n*}$ and call this set the dual projective space. For a point $(a_0 : ... : a_n)$, the dual is the zero set of the linear equation $a_0x_0 + ... + a_nx_n = 0$, $x_i \in \mathbb{P}^n$. By

The principle of duality we have that to each theorem in projective geometry, there corresponds a dual theorem in which line and point, pass through and lie on, intersection point of two lines and line connecting two points, concurrent and collinear, polar and pole and points on conics and lines tangent to conics are interchanged [16, p. 339].

We consider maps from $\mathbb{P}^2$ to $\mathbb{P}^2$. They are called projectivities, and map lines to lines.

Definition 2.4. For a 3×3 matrix A with $\det A \neq 0$, and x a homogeneous coordinate vector in $\mathbb{P}^2$,

$$T(x) = Ax$$

is called a projectivity of $\mathbb{P}^2$, or a projective transformation of the plane [16, p. 338].

These invertible matrices form the general linear group and give the set of projective transformations its group structure.

Like we said, a projectivity is a mapping of lines to lines. Let us see what the induced projectivity on $\mathbb{P}^{2*}$ looks like.

A line in $\mathbb{P}^2_{\mathbb{C}}$ will be, $l_a = \{(x_0 : x_1 : x_2) : a_0x_0 + a_1x_1 + a_2x_2 = 0\}$, $a_i \in \mathbb{C}$, not all can be zero.

Theorem 2.5. The projectivity $y = Ax$ maps the line l_a to l_b, where $b = (A^T)^{-1}a$. Thus the projectivity $y = Ax$ on $\mathbb{P}^2$ induces the projectivity $b = (A^T)^{-1}a$ on $\mathbb{P}^{2*}$ [8, p. 20].

Proof. $a \cdot x = 0$ is the equation of the line l_a. The transformation $a \cdot A^{-1}y = 0$ is the same as $(A^T)^{-1}a \cdot y = 0$. So l_a will be mapped to l_b by $y = Ax$. □

2.2 Conics and their structure

A homogeneous polynomial of degree two in three variables has the general form $Q(x_0, x_1, x_2) = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_2^2$. The corresponding curve is called a conic, $C = \{x \in \mathbb{P}^2 \mid Q(x_0, x_1, x_2) = 0\}$.

Note that $Q(x) = \sum a_{ij}x_ix_j$ can be expressed in matrix notation: $Q = x^TAx$, where $A \neq \mathbf{0}$ is a symmetric 3×3 matrix.

A projective transformation $x = Tx'$ transforms $Q = x^TAx$ into

$$(x')^TA'x' = 0,$$

where $A' = T^TAT$. Ellipses, parabolas and hyperbolas are no longer preserved.

A real symmetric matrix A has a basis of orthogonal eigenvectors and real eigenvalues, so there exist a non-singular matrix T such that $T^TAT =$ diag $(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i \in \{0, \pm 1\}$. We have the following classification of conics in the real projective plane.

$(\lambda_1, \lambda_2, \lambda_3)$	Equation	conic
$(0, 0, 0)$	$0 = 0$	projective plane
$(1, 0, 0)$	$x_1^2 = 0$	(double) line
(1,1,0)	$x_1^2 + x_2^2 = 0$	point
$(1, -1, 0)$	$x_1^2 - x_2^2 = 0$	two crossing lines
$(1, 1, 1)$	$x_1^2 + x_2^2 + x_3^2 = 0$	empty set
$(1, 1, -1)$	$x_1^2 + x_2^2 - x_3^2 = 0$	circle

Table 2.1 [16, p. 341]

In $\mathbb{P}^2_{\mathbb{C}}$, rank is the only invariant modulo a projective transformation. $x_1^2 + x_2^2 + x_3^2 = 0$ is the proper conic, $x_1^2 + x_2^2 = 0$ two lines crossing and $x_1^2 = 0$ one double line.

Definition 2.6. A conic with equation $x^T Ax = 0$ is called non-degenerate if A is non-singular, and it is called degenerate if A is singular [8, p. 31].

Up to a linear change of coordinates there exist in $\mathbb{R}^2$ exactly 3 non-degenerate conics; parabolas, ellipses and hyperbolas. In $\mathbb{C}^2$ we have only 2. Here ellipses and hyperbolas are not distinct. When in $\mathbb{P}^2$ all non-degenerate conics are projectively equivalent.

Non-degenerate conics are also called smooth. For every point there is a tangent line. So we can determine whether a conic is smooth or not in a point by looking at the partial derivatives of the curve at this point.

Definition 2.7. Let $a \in \gamma$, where γ is the algebraic curve in $\mathbb{P}^2$ with equation $P(x) = 0$. The curve γ is non-singular at a if the partial derivatives $P_i(a), 1 \leqslant i \leqslant 3$, are not all 0. In this case the tangent line to γ at a is defined to be the line with equation $\sum_{i=1}^{3} P_i(a)x_i = 0$ [8, p. 26].

If an algebraic curve has no singular points, it is called smooth. Being smooth is an important property of a conic. In the following we sometimes want a projectively transformed version of some conic. Therefore we would like to know if a non-singular point on a curve is mapped to a non-singular point.

Theorem 2.8. Let the projectivity $y = Ax$ map the algebraic curve γ in $\mathbb{P}^2$ to $\tilde{\gamma}$. Then $\tilde{\gamma}$ is an algebraic curve in $\mathbb{P}^2$. The projectivity maps non-singular points of γ to non-singular points of $\tilde{\gamma}$ and tangent lines of γ to tangent lines of $\tilde{\gamma}$ [8, p. 28].

Proof. $\tilde{\gamma}$ can be expressed as $= Q(y) := P(By) = 0$, $B = A^{-1}$ when γ has equation $P(x)$. P is a homogeneous polynomial in $x = (x_0 : x_1 : x_2)$ and so is Q. Also deg Q = deg P. So $\tilde{\gamma}$ is an algebraic curve in $\mathbb{P}^2$. $\nabla P(a) := (P_1(a), P_2(a), P_3(a)) \neq \mathbf{0}$ for a simple point $a \in \gamma$. For $b = Aa \in \tilde{\gamma}$ we have by the chain rule $\nabla Q(b) = B'(\nabla P(a)) = (A^T)^{-1}(\nabla P(a))$. $\nabla Q(b) \neq \mathbf{0}$ since $\nabla P(a) \neq \mathbf{0}$ and $(A^T)^{-1}$ is non-singular. b is a simple point of $\tilde{\gamma}$.
Let γ have the tangent l at a, and $\tilde{\gamma}$ have the tangent l' at b. Then l can be expressed by $\nabla P(a) \cdot x = 0$ and l' by $\nabla Q(b) \cdot y = (A^T)^{-1}(\nabla P(a)) \cdot y = 0$. From this, and theorem 2.5, we can see that $y = Ax$ maps l to l'. $\square$

Theorem 2.9. A conic is non-degenerate if and only if it consists only of non-singular points [8, p. 32].

Proof. We have the partial derivatives $\frac{\delta Q}{\delta x_i} = 2\sum_{j=1}^{3} a_{ij}x_j$ for the conic C with equation $Q(x) = x'Ax$. If $\frac{\delta Q}{\delta x_i}(x) = 0$ for $1 \leq i \leq 3$ and $x \neq 0$ we have

a singular point, and $\frac{\delta Q}{\delta x_i} = 0$ is equal to $Ax = \mathbf{0}$. C contains no singular points if A is non-singular, because then $x = \mathbf{0}$ is the only solution.
In the case when A is singular, $Ax = \mathbf{0}$ has non-zero solutions which corresponds to singular points. $\square$

2.3 Conics in $\mathbb{P}^2_{\mathbb{C}}$

By Bezout's theorem, that we will soon state, a line in $\mathbb{P}^2_{\mathbb{C}}$ intersects a conic in two points counting multiplicities. Also, we have that through any point x in $\mathbb{P}^2_{\mathbb{C}}$ there are two distinct lines tangent to a smooth conic C, except when x is in C and we have one tangent.

Theorem 2.10. A line in $\mathbb{P}^2_{\mathbb{C}}$ intersects a smooth conic in two distinct points, except when the line is a tangent to the conic and intersects the conic only in the point of tangency [8, p. 34].

Proof. Let $Q(x) = ax_0^2 + bx_0x_1 + cx_1^2 + x_2L = 0$, where L is a linear form in x_0, x_1, x_2 which after a projective change of coordinates has equation $x_2 = 0$ and C the smooth conic with equation $Q(x) = 0$. Q is irreducible since it is smooth. $Q(x_0, x_1, 0) = ax_0^2 + bx_0x_1 + cx_1^2 = 0$ is satisfied by the points $(x_0 : x_1 : x_2)$ in $l \cap C$. $Q(x_0, x_1, 0)$ factors into two linear forms. We have either

$$Q(x_0, x_1, 0) = (\alpha x_0 + \beta x_1)^2, \quad \text{where } (\alpha, \beta) \neq (0, 0) \qquad (2.1)$$

or

$$Q(x_0, x_1, 0) = (\alpha x_0 + \beta x_1)(\gamma x_0 + \delta x_1), \quad \text{where } \alpha\delta - \beta\gamma \neq 0 \qquad (2.2)$$

In the case of (2.1), then $(-\beta : \alpha : 0)$ is the one point in $(l \cap C)$. We get $\frac{\partial Q}{\partial x_0} = \frac{\partial Q}{\partial x_1} = 0$ when differentiating Q at $(-\beta : \alpha : 0)$, which means that l is tangent to C at this point. In the other case, differentiating at the two points of $(l \cap C)$, $(-\beta : \alpha : 0), (-\delta : \gamma : 0)$ we will get $\frac{\partial Q}{\partial x_0} = (\alpha(\alpha\delta - \beta\gamma)$ and $\frac{\partial Q}{\partial x_1} = \beta(\alpha\delta - \beta\gamma)$. l is not tangent to C at $(-\beta : \alpha : 0)$, since the two derivatives are not both zero. The same is true for $(-\delta : \gamma : 0)$. $\square$

Since in projective geometry points and lines are interchangeable objects, we have that tangents to a conic are a dual conic. We have as the dual of (2.10) that through any point x in $\mathbb{P}^2_{\mathbb{C}}$ there are two distinct lines tangent to a smooth conic C except when x is in C, in which case there is only one such line.

We will now look at how two conics in $\mathbb{P}^2_{\mathbb{C}}$ intersect.

Bezout's Theorem Let $F = 0$, $G = 0$ be two curves in $\mathbb{P}^2_{\mathbb{K}}$, where F and G are homogeneous polynomials without a common factor and where $\mathbb{K}$ is an algebraically closed field. Then the number of intersections of the two curves equals deg $F\cdot$ deg G, provided one counts multiplicities [8, p. 43].

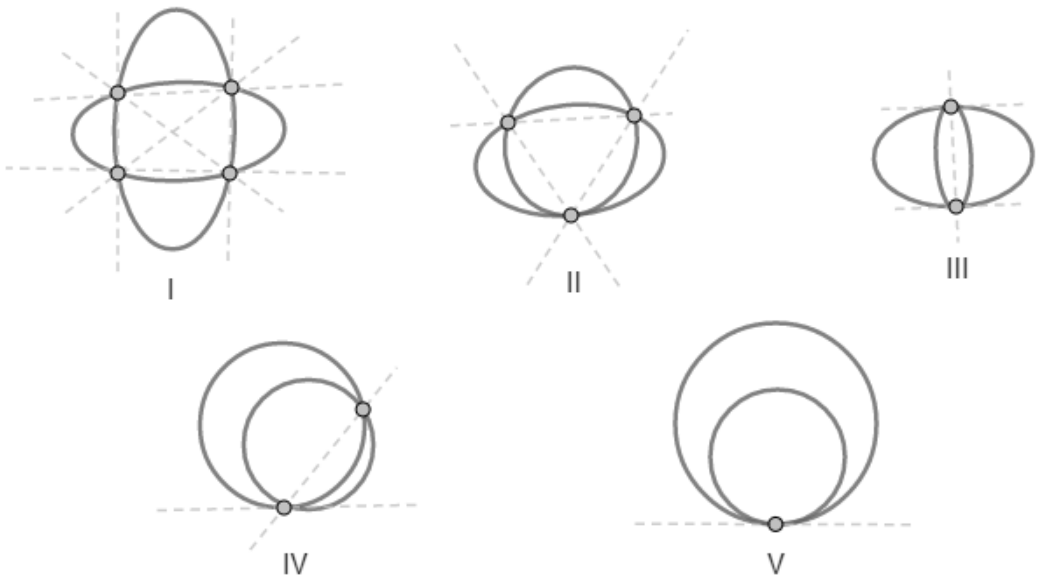


figure 2.1 [8, p. 52]

The picture above illustrates the different possibilities we have with respect to intersection of two conics in $\mathbb{P}^2_{\mathbb{C}}$. This classification can be done by looking at the degenerated conics of a pencil. The degenerate conics in $\{C, D\}$ are given in the table below (2.1). We will therefore first define a conic pencil.

Definition 2.11. Let $C(x)$ and $D(x)$ be two non-proportional quadratic forms in $x = (x_0 : x_1 : x_2)$. The set of conics $\lambda C(x) + \mu D(x) = 0$, where λ and μ are arbitrary numbers not both 0, is called the conic pencil generated by the conics $C(x) = 0$ and $D(x) = 0$. The conic pencil is denoted $\{C, D\}$

[8, p. 53].

The base points of the pencil are the points of intersection of $C_1 = 0$ and $D_1 = 0$. We have between one and four such, by Bezout's theorem. In the following we consider conic pencils in which there exists at least one non-degenerate conic.

Theorem 2.12. A non-degenerated pencil contains at least one and at most three degenerate conics [8, p. 55].

Proof. $\lambda(x'Cx) + x'Dx = x'(\lambda C + D) = 0$ is the equation of the conics of the pencil generated by C and D, C smooth so that $|C| \neq 0$. This conic is degenerate if $|\lambda C + D| = 0$ That is a cubic equation in λ with leading coefficient $|C| \neq 0$, so it has between one and three distinct roots. $\square$

In the table below $C \boxplus D = \{(p_i, m_i), i \leq j \leq k\}$, where $p_1, ..., p_k$ are the points of $C \cap D$, C, D two distinct smooth conics, and $m_1, ..., m_k$ the multiplicities of these points. l_{ab} is the line joining the a and b, t_a is the tangent at a to C and D, $l_1 \cup l_2$ the union of those two lines and $2l$ is the line l counted twice.

type	$C \boxplus D$	degenerate conics in $\{C, D\}$
I	$(a,1),(b,1),(c,1),(d,1)$	$l_{ab} \cup l_{cd}, l_{ab} \cup l_{cd}, l_{ab} \cup l_{cd}$
II	$(a,2),(b,1),(c,1)$	$l_{ab} \cup l_{ac}, t_a \cup l_{bc}$
III	$(a,2),(b,2)$	$t_a \cup t_b, 2l_{ab}$
IV	(a,3),(b,1)	$l_{ab} \cup t_a$
V	$(a,4)$	$2t_a$

Table 2.1 [8, p. 56]

In the proof of Poncelet's theorem that we will later look at, the two conics are in general position. That is, they intersect in four different points like illustrated as type number 1 above.

2.4 Conics in $\mathbb{P}^2_{\mathbb{R}}$

Turning towards conics in $\mathbb{P}^2_{\mathbb{R}}$, of main interest will be those issues that are relevant in the modern proof of Poncelet's theorem that we will soon focus on. We first look at how a line intersects a conic in $\mathbb{P}^2_{\mathbb{R}}$ and from where in $\mathbb{P}^2_{\mathbb{R}}$ it is possible to draw a tangent. Thereafter intersection points for two real projective conics are discussed.

Above (2.10) we showed that a line in $\mathbb{P}^2_{\mathbb{C}}$ intersects a smooth conic in two distinct points, except when the line is a tangent to the conic and intersects the conic only in the point of tangency. In $\mathbb{P}^2_{\mathbb{R}}$ we also have the possibility that a line may not intersect the conic at all.

We assumed above that l has the equation $x_3 = 0$ and looked at $Q(x_0, x_1, 0) = ax_0^2 + bx_0x_1 + cx_1^2 = 0$ which is satisfied by the points $(x_0 : x_1 : x_2)$ in $l \cap C$. Since we cannot have that both x_0 and x_1 are 0, the points are of the form $(x_0 : 1 : 0)$. Like we know, the equation $ax_0^2 + bx_0 + c = 0$ gives us three possibilities when solved over real numbers, depending on the discriminant $b^2 - 4ac$, when $a \neq 0$. This corresponds to the number of intersection points that the line will have with the conic. If $D < 0$ there are none, if $D = 0$ there is 1 and if $D > 0$ there are 2 intersection points.

In $\mathbb{P}^2_{\mathbb{R}}$ it is also not true that from every point that does not lie on a conic there are exactly two tangent lines to the conic from this point. If the point lies inside the conic, there is no way to construct a tangent to the conic form this point.

We will now look at how two conics in $\mathbb{P}^2_{\mathbb{R}}$ intersects. Like in $\mathbb{P}^2_{\mathbb{C}}$ we can in $\mathbb{P}^2_{\mathbb{R}}$ have the situation that two conics intersect in 1, 2, 3 or 4 points. But unlike in $\mathbb{P}^2_{\mathbb{C}}$, we might have that the conics do not intersect at all.

figure 2.2.

In the following we will study conics in general position. What are two conics in general position in real projective plane like?

For two real projective conics that intersect in four distinct points we can have the following: They might intersect in 4 real points, 2 real and 2 imaginary points or in 4 imaginary points.

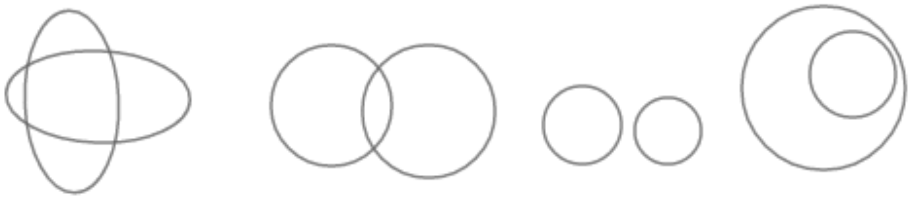


figure 2.3.

In the two first cases it will be possible to draw a tangent to C_1 from points on C_2, as long as the points of C_2 are not inside those of C_1. In the last situation we need to make sure that if the conics are nested, C_2 is not inside C_1, but the other way around. The three possibilities of type 1 intersections of two smooth conics in $\mathbb{P}^2_{\mathbb{R}}$ is represented by Levy into the categories 1, 1a and 1b [4, p. 7].

orbit	f_0	g_0	real	im
1	$x^2 - y^2$	$x^2 - z^2$	1111	
1a	$x^2 + y^2 + z^2$	xz		1111
1b	$x^2 + y^2 - z^2$	xz	11	11

Table 2.2 [4, p. 7]

2.5 Formulation of Poncelet's Theorem

We will state Poncelet's Theorem for conics in real and complex projective plane, but first we define what a polygon that is interscribed between two conics, C_1 and C_2, is.

Definition 2.13. Let $p_{2_1}, ..., p_{2_n}$ be points on C_2 such that all the lines joining the pair of points $(p_{2_1}, p_{2_2}), (p_{2_2}, p_{2_3}), ..., (p_{2_n}, p_{2_1})$ are tangent to C_1. The tangent lines $(p_{2_i}, p_{2_{i+1}})$ are the edges of a polygon that is inscribed in C_2 and circumscribed about C_1. A vertex of the polygon is the point p_{2_i} on C_2 where two edges meet.

This definition corresponds to the construction of a Poncelet polygon in the language we will use in the next chapter. Let p_1 be a point on C_1 and p_2 a point on C_2, so that the line $l_{p_1 p_2}$ is tangent to C_1. This tangent intersect

C_2 also in p_2'. Give the name ι_1 to the action of interchanging between the two pair of points (p_1, p_2) and (p_1, p_2'). From any point p_2 on C_2, there are two tangent lines to C_1, one to the point p_1 and one to the point p_1'. Give the name ι_2 to the action of interchanging between the two pair of points (p_1, p_2) and (p_1', p_2). Applying $\iota_2 \circ \iota_1$ will take an element (p_1, p_2) to (p_1', p_2'). We call this composition η. If we after applying η a finite number of times are back at our starting point, we have a closed polygon.

In the introduction we stated a version of Poncelet's theorem where C_1 is inside C_2.

Poncelet's Theorem for real conics Let C_1 and C_2 be two real conics, with C_1 inside C_2. Suppose there is an n-sided polygon inscribed in C_2 and circumscribed about C_1. Then for any other point of C_2, there exists an n-sided polygon, inscribed in C_2 and circumscribed about C_1, which has this point for one of its vertices [8, p.1].

The proof of Poncelet's theorem done by Griffiths and Harris concerns two conics in $\mathbb{P}^2_{\mathbb{C}}$ in general position.

Poncelet's Theorem for complex conics Let C_1 and C_2 be two complex conics in general position. If there is an n-sided polygon inscribed in C_2 and circumscribed about C_1. Then for any other point of C_2, there exists an n-sided polygon, inscribed in C_2 and circumscribed about C_1, which has this point for one of its vertices.

Does this implies that the theorem holds for two real conics in the projective plane?

Remark Poncelet's Theorem for complex conics implies Poncelet's Theorem for real conics.

Given two real conics, C_1 inside C_2, for which there exists an interscribed polygon. The two conics are a real part of two complex conics and the real polygon is a real part of a complex polygon. The two real conics are in general position. So we are in the same situation as that of the conditions from the proof done in complex projective plane. Will there then exist a real polygon for any real point on C_2, that have this point as a vertex?

The points on the real part of the complex conic C_2 is a subset of the points on the complex conic, and will therefore by Poncelet's theorem for complex conics also be vertices of interscribed polygons, if we can also be sure that the edges exist and are real parts of tangents to C_1.

For a tangent to C_1 from one point of C_2 to exist over real numbers, we must have that the point of C_2 lies outside of C_1. This is true for all points of C_2 in our situation.

Like we have seen, a line and a conic over the real numbers meet in one point if the algebraic expression for $C \cap l_\infty$ over the real numbers gives us $D = 0$. This corresponds to the algebraic formulation of the same line and the same conic meeting in exactly one point over the complex numbers.

If we have a closed n-sided polygon interscribed between two real conics, C_1 inside C_2, then Poncelet's theorem for complex conics gives us that we have infinitely many such n-sided polygons, with one of the points of C_2 as one of its vertices, interscribed between the two real conics.

We can argue in the same manner for the situation where the two conics have 4 imaginary intersection points but are not nested. For the cases where the two real conics have either 4 real or 2 real and 2 imaginary intersection points, we can modify to say that Poncelet's theorem for complex conics implies Poncelet's theorem for real conics for all points on C_2 that lie outside of C_1.

3 Algebrogeometric reformulation of Poncelet's Theorem

Now we will approach the proof of Poncelet's theorem by Griffiths and Harris. We have so far studied conics in complex projective plane. Two such will intersect in four points. We will now reformulate the theorem. In the following two chapters we study the theory that is needed to understand it.

We will now use a model to label the points that we need and then give a definition of our curve E.

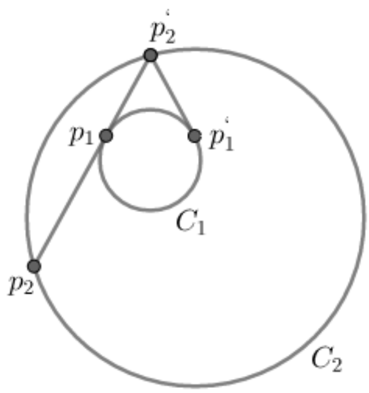


figure 3.1

Definition 3.1. $E = \{(p_1, p_2) \in C_1 \times C_2 : l_{p_1 p_2} \text{ is tangent to } C_1\}$

E is an algebraic curve with surjective maps to the two conics:

$$\varphi_1 : E \to C_1, (p_1, p_2) \mapsto p_1$$
$$\varphi_2 : E \to C_2, (p_1, p_2) \mapsto p_2$$

The preimages of these maps:

$$\varphi_1^{-1}(\{p_1\}) = \{(p_1, p_2), (p_1, p_2')\}$$
$$\varphi_2^{-1}(\{p_2\}) = \{(p_1, p_2), (p_1', p_2)\}$$

has 2 points, but with 4 exceptions. For these exceptions the preimage contains only 1 element. 4 is the number of intersection points and 4 is the number of common tangents to two circles in general position. Elsewhere φ is a $2:1$ map, E is a two-sheeted cover of $\mathbb{P}^1$.

We have the involutions ι_1 of φ_1 on E and ι_2 of φ_2 on E. They interchange the two elements of the preimages of φ.

$$\iota_1 : E \to E \quad (p_1, p_2) \mapsto (p_1, p_2')$$
$$\iota_2 : E \to E \quad (p_1, p_2) \mapsto (p_1', p_2)$$

$$\iota_1 \circ \iota_1 = id$$
$$\iota_2 \circ \iota_2 = id$$

We call the composition of these two involutions $\eta : E \to E$

In the following we will study the proof of this reformulation of Poncelet's theorem:

Theorem 3.2. If η^n has a fixed point for some positive integer n, then η^n is the identity map on E.

This is equivalent to the Poncelet's Theorem above. Applying η one time will correspond to

$$\eta(p_1, p_2) = (\iota_2 \circ \iota_1)(p_1, p_2) = \iota_2(p_1, p_2') = (p_1', p_2').$$

If we have, for some n, that $\eta^n(p_1, p_2) = (p_1, p_2)$, we are back at our starting point after n iterations of η. The aim of the proof will be to show that having $\eta^n(p_1, p_2) = (p_1, p_2)$ is not dependent on the choice of starting point.

4 Divisors on a curve

After giving a definition of the group operation and choosing a zero, the group structure on an elliptic curve can be described by adding points on the curve. The group structure on elliptic curves can also be described by divisors. To do so we need to understand what divisors on irreducible smooth projective curves are. We will use the map between addition of points on an elliptic curve and addition of divisors on an elliptic curve in the proof of Poncelet's theorem.

Even though we need Hurwitz formula to know that the Poncelet correspondence is an elliptic curve, we state it a bit later. Hurwitz formula is a corollary of a theorem that compares special divisors on curves.

Also to be able to use the Riemann-Roch theorem, we must know what a divisor is. It is a statement that involves, among other things, the degree of a divisor on a curve. In this text we use Riemann-Roch to prove a theorem about involutions on elliptic curves which will be important for the modern proof of Poncelet's theorem.

4.1 Divisor classes

Definition 4.1. A *divisor* on an irreducible smooth projective curve X, is a formal sum $D = k_1p_1 + ... + k_np_n$ where $p_1, ..., p_n$ are distinct points of X and $k_1, ..., k_n$ integer coefficients for some $n \in \mathbb{N}$.

The number $\deg D := k_1 + ... + k_n \in \mathbb{Z}$ is the *degree* of a divisor D.

A divisor $D = k_1p_1 + ... + k_np_n$ is called *effective*, written $D \geq 0$, if $k_i \geq 0$ for all $i = 1, ..., n$. [10, p. 113].

Expanding to look at rational functions, we will see the non-effective divisors.

$\mathrm{Div}X$, the divisors of X is an Abelian group. Addition of the coefficients of the points of X gives a group of maps: $\mathrm{Div}X \to \mathbb{Z}$.

Soon we will be looking at a divisor of a curve where the components are the intersection points between a curve X and the zero set of a polynomial $f \in S(X)$, were $S(X) := k(x_0, ..., x_n)/I(X)$ is the homogeneous coordinate ring of X.

First we will include the definitions of a rational function on a projective curve X and of v_p, the valuation of a function at a point.

Definition 4.2. A rational function $\varphi = \frac{g}{h}$ on a projective curve X is a function $\varphi : X \to k$ where $f, g \in S(X)$ are homogeneous polynomials of same degree.

The set of rational functions on X is denoted $k(X)$. A rational function is regular at a point p if there is a representation $\varphi = \frac{g}{h}$ with $h(p) \neq 0$.

Definition 4.3. For every function $g \in O_{X,p}$ regular at p, the valuation or multiplicity of g at p is given by

$$v_p(g) := \max\{k \mid g \in m_p^k.\}$$

where $m_p = \{g \in O_{X,p} \mid g(p) = 0\}$ is the maximal ideal of the local ring $O_{X,p}$.

A function g vanishes at p if and only if $v_p(g) \geq 1$. For every rational function $f \neq 0 \in k(X)$ the multiplicity of f at p is defined by

$$v_p(f) := v_p(g) - v_p(h),$$

where $f = g/h$ for some $g, h \in O_{X,p}$. If $v_p(f) > 0$, then one says that f has a zero of order $v_p(f)$ in P. If $v_p(f) < 0$, then one says that f has a pole of order $-v_p(f)$ in P. [14, p. 169].

We also have that

$$v_p(fg) = v_p(f) + v_p(g)$$

for all $p \in X$. In particular, we have $\mathrm{div}(fg) = \mathrm{div}f + \mathrm{div}g$ in $\mathrm{Div}X$ [10, p. 114].

What is then a divisor of a non-zero homogeneous polynomial f?

Definition 4.4. Let $X \subset \mathbb{P}^n$ be an irreducible smooth curve. For a non-zero homogeneous polynomial $f \in S(X)$ the divisor of f it is defined to be

$$\mathrm{div}f := \Sigma_{p \in V_X(f)}\ v_p(f) \cdot p \in \mathrm{Div}X.$$

$V_X(f)$ is the set of points for which f is zero on X. Then we will have that the degree of the divisor of f is equal to $\deg X \cdot \deg f$. [10, p. 114].

As an example, if we have a projective curve C of degree d in $\mathbb{P}^2_{\mathbb{C}}$ and a line L, the sum $C \cap L = \Sigma k_i p_i$ is a divisor of degree d on C. By considering all such lines and the divisors they give rise to on C, we obtain a linear system on C [12, p.129].

Above we defined what a rational function on a curve is. We also need to define what a divisor of a rational function is:

Definition 4.5. The divisor of the rational function $\varphi = \frac{g}{h}$ is defined to be

$$\mathrm{div}\varphi = \Sigma_{p \in V_X(g) \cup V_X(h)}\ v_p(\varphi) \cdot p = \mathrm{div}g - \mathrm{div}h.$$

[10, p. 114].

The following will be useful.

Lemma 4.6. For $\varphi = \frac{g}{h}$, $\mathrm{div}\varphi$ has degree 0.

Proof. $\deg \mathrm{div}\varphi = \deg(\mathrm{div}g - \mathrm{div}h) = \deg \mathrm{div}g - \deg \mathrm{div}h = \deg X \cdot \deg g - \deg X \cdot \deg h = 0$ □

We have just seen that divisors of rational functions have degree 0. Adding a divisor of degree 0 to any divisor D will not affect the degree of D.

Definition 4.7. A divisor on an irreducible smooth projective curve X is called *principal* if it is the divisor of a (non-zero) rational function. $\mathrm{Prin}X$ is the notation for the set of all principal divisors.

The quotient $\mathrm{Pic}X := \mathrm{Div}X/\mathrm{Prin}X$ is called the *Picard group* or *group of divisor classes* on X. And $\mathrm{Pic}^0X := \mathrm{Div}^0X/\mathrm{Prin}X$ [10, p. 115].

Two divisors D and D' of a curve X are linearly equivalent if $D' = D + \mathrm{div}f$, where f is a rational function on X.

Next comes a lemma on the non-triviality of the Picard group of a smooth cubic curve in $\mathbb{P}^2$. We use it later when we prove that there is a bijective map that takes a point a on a elliptic curve E to a divisor $a - a_0$ in Pic^0E.

Proposition 4.8. Let $X \subset \mathbb{P}^2$ be a smooth cubic curve. Then for all distinct $a, b \in X$ we have $a - b \neq 0$ in Pic^0X, that is there is no non-zero rational function φ on X with $\mathrm{div}\varphi = a - b$ [10, p. 118].

4.2 Riemann-Roch Theorem

For the last step of the proof of Poncelet's Theorem, we need a result (5.4) concerning an involution of a map from an elliptic curve to $\mathbb{P}^1$ which relies on the Riemann-Roch theorem. We will not prove Riemann-Roch, but we want to understand the components of the formula, so that we can use it. We will also need the concept of a base point free linear system. So we include some definitions and propositions that will reveal for us the meaning of this concept. Not all proofs are included, but can be found in [14].

Riemann-Roch theorem states the connection between the degree of a divisor of a curve, the dimension of the vector space of rational functions on the curve that are linearly equivalent to effective divisors and the genus of the curve. The genus can be described as the number of "holes" in the surface being the topological space of the curve.

Curves can be classified by their genus. The Riemann-Roch theorem is often used to compute the dimension of the vector space we mentioned above. So first we need to know what number the genus is. For a non-singular curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$ it can be computed by the following formula: $\binom{d-1}{2}$ [10, p. 111]. So a line or conic will have genus 0 and a cubic will have genus 1.

Above we looked at divisors of curves. Two divisors D and D' of a curve X are linearly equivalent if $D' = D + \text{div} f$, where f is a rational function on X. Every divisor together with the linearly equivalent ones that are effective, defines a complete linear system.

Definition 4.9. A divisor D defines a complete linear system

$$|D| := \{D' \geq 0 \mid D' \sim D\}$$

Soon we will see that this complete linear system corresponds to the points of projective space.

Let us first give a name to the collection of all rational functions on a curve C that give rise to linearly equivalent effective divisors.

Definition 4.10. Given a divisor D.

$$L(D) := \{0 \neq f \in k(C) \mid \text{div} f \geq -D\} \cup \{0\}.$$

$L(D)$ is a vector space. Its dimension is denoted $l(D)$.

Before our next result on the equivalence of $|D|$ and $\mathbb{P}(L(D))$, we need the following theorem:

Theorem 4.11. If V is an irreducible projective variety defined over an algebraically closed field k, then every regular function on V is constant, that is $O(V) \cong k$ [14, p. 78].

We can now move on to an important map.

Proposition 4.12. There is a bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(L(D))$. [14, p. 185].

Proof. Consider the map

$$\mathbb{P}(L(D)) \to |D|, \quad f \mapsto D_f,$$

where $D_f := \text{div} f + D$, $0 \neq f \in L(D)$. $\text{div} f = \text{div}(\lambda f), \lambda \in k^*$. By definition $D_f \geq 0$ and $D_f \sim D$, so the map is well defined. The map is injective because two rational functions f, g with $\text{div} f = \text{div} g$ would give rise to an everywhere regular function f/g which has to be a constant by (4.11) and $f = \lambda g$. We see that the map is surjective supposing $D' \geq 0$, $D' \sim D$ and letting f be a rational function with $\text{div} f = -D + D'$. Since $D' \geq 0$, $f \in L(D)$. $\square$

We will soon consider subspaces of complete linear systems.

Definition 4.13. A linear system on a curve C is a projective subspace of a complete linear system $|D|$ [14, p. 186].

Like mentioned above, we will also need the definition of a base point free linear system.

Definition 4.14. Given a linear system ϑ contained in $|D|$. A point $P \in C$ is called a base point in the linear system ϑ if $\vartheta = \vartheta \cap |D - P|$ [14, p. 186].

In other words ϑ is base point free if $\forall\ P \in C\ \exists\ B \in \vartheta : P \notin B$

We just showed (4.9) that there is a bijection between a complete linear system and projective space $\mathbb{P}(L(D))$. Because of this we can study maps defined by a complete linear system.

$$\varphi_D : C \to \mathbb{P}(L(D)), \quad P \mapsto H_D = \{s \in L(D) : s(x) = 0\}.$$

$\varphi(P)$ are hyperplanes in projective space. $C \cap H$ are divisors on C. If C is not contained in a hyperplane, these divisors $D = Z(s) = \{x \in X :$

$s(x) = 0\}$ are effective. For $s_1, s_2 \in L(D)$, $\frac{s_1(x)}{s_2(x)}$ is a rational function. This means that $z(s_1) \sim z(s_2)$. This collection of hyperplane sections is a base point free linear system if there is no point P that is contained in all the divisors of the linear system.

The Riemann-Roch theorem also involves the canonical divisor, denoted K below. It is a unique divisor class of every curve given by class of differential form.

Now we will state the Riemann-Roch Theorem.

Theorem 4.15. Let D be a divisor on a curve X of genus g. Then

$$l(D) - l(K - D) = \deg D + 1 - g.$$

4.3 Hurwitz formula

We are in the following restricting our attention to the cases where char(k)=0. Before we are ready to state Hurwitz formula which concerns a morphism $f : X \to Y$, we need to define the ramification index e_P for a point $P \in X$.

Definition 4.16. Let $Q = f(P)$, let $t \in O_Q$ be a local parameter at Q, consider t as an element of O_p via the natural map $f^\sharp : O_Q \to O_P$ and define $e_P = v_P(t)$, where v_P is the valuation associated to the ring O_P. If $e_P > 1$ we say that f is ramified at P, and that Q is a branch point of f. If $e_P = 1$, we say that f is unramified at P [12, p. 299].

Theorem 4.17. Let $f : X \to Y$ be a morphism of curves. Let $n = \deg f$. Then

$$2g(X) - 2 = n(2g(Y) - 2) + \Sigma_{P \in X}(e_P - 1)$$

[12, p. 301].

Corollary 4.18. Let $f : X \to Y$ be a 2:1 map with ramification being points which have only one element in their fiber. Let t be the number of ramification points. Then

$$2g(X) = 4g(Y) - 2 + t$$

We also observe that if we use Riemann-Roch for $D = 0$ we get that $l(K) = g$. Using that and letting $D = K$, we see again from Riemann-Roch that deg$K = 2g - 2$. Then we can see that Hurwitz formula is comparing canonical divisors. n is the degree of the morphism and the last part of the formula, $+\Sigma_{P \in X}(e_P - 1)$ is measuring the effect of the points of exception.

5 Elliptic curves

An elliptic curve is a curve of genus 1.

5.1 E is isomorphic to a curve on Weierstrass form.

Proposition 5.1. Let E be an elliptic curve over k, with **char** $k \neq 0$, and let $P_0 \in E$ be a given point. Then E is isomorphic to a cubic curve of the form

$$y^2 = x(x-1)(x-\lambda)$$

for some $\lambda \in k$ and the point P_0 goes to infinity $(0, 1, 0)$ on the y-axis [12, p. 319].

Proof. By the linear system $|3P_0|$, E is embedded in $\mathbb{P}^2$. Consider the vector spaces

$$k \subseteq L(2P_0) \subseteq L(3P_0) \subseteq ...$$

By Riemann-Roch we have

$$\dim\ L(nP_0) = n$$

for $n > 0$. Now choose $x \in L(2P_0)$ and $y \in L(3P_0)$ so that $1, x$ and $1, x, y$ respectively are the basis for the spaces. $1, x, y, x^2, xy, x^3, y^2$ are then in $L(6P_0)$. They are not linearly independent, since $\dim\ L(6P_0) = 6$. x^3 and y^2 have a 6-fold pole at P_0, and therefore have non-zero coefficients. For some $a_i \in k$ we have

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

On the left side we replace y by

$$y' = y + \frac{1}{2}(a_1x + a_3)$$

The new equation can be written as $y^2 = (x-a)(x-b)(x-c)$, and by a linear change of x we have $y^2 = x(x-1)(x-\lambda)$. x and y have a pole at P_0. So P_0 goes to $(0, 1, 0)$, the point at infinity.

□

5.2 Geometric construction of group law on an elliptic curve

We will look at how we by a geometrical approach can define addition of points on an elliptic curve and show that the construction is a group under this operation. Consider a cubic curve from the description above. We draw a line through two points P, Q on the curve. It will intersect the curve in one more point, or one of the points will be an intersection of multiplicity 2 of the cubic curve and the line. Let us call the third intersection point PQ. The reflection of PQ about the x-axis is defined to be $P \oplus Q$. $y^2z = x^3 + axz^2 + bz^3$ is the homogenized version of our equation. Acting in $\mathbb{P}^2$, reflecting about the x-axis can be viewed as drawing the line from one point through $O = (0 : 1 : 0)$ and end up in the third intersection point with the cubic. So we can look at it as "applying O", or joining a point by another with a line: $P \oplus Q = O(PQ)$. We will now see that the addition of points in this manner defines a commutative group.

$P \oplus Q = Q \oplus P$. There is only one line that contains both P and Q simultaneously.

Take any point R on the cubic. Draw a line through R and O and call the last intersection point RO. To find $R \oplus O$ we need to take RO and "reflect in about the x-axis", or "apply" O. $O(RO) = R \oplus O = O \oplus R = R$. So O is the identity element.

To find the inverse of a point R, draw a line through R and O to obtain RO. What would be $R \oplus RO$ in this situation? The third point of intersection between the line through R and RO with the cubic is O. Reflecting O by the x-axis, we get O again. So $R \oplus RO = RO \oplus O = O$. So R and RO, the points reflecting each other by the x-axis are inverses.

Associativity, $P \oplus (Q \oplus R) = (P \oplus Q) \oplus R$, takes a bit more to show. Here we will sketch a geometric proof from [6, p. 48] using the Cayley-Bacharach theorem: Let $X_1, X_2 \subset \mathbb{P}^2$ be cubic plane curves meeting in nine points $p_1, ..., p_9$. If $X \subset \mathbb{P}^2$ is any cubic containing $p_1, ..., p_8$, then X contains p_9 as well [7, p. 301] (proof in last chapter).

We are now ready to show that the associative property holds in this construction. Start by a degenerated conic $(P, Q) \cup (QR, Q \oplus R) \cup (R, P \oplus Q)$ and another one $(Q, R) \cup (PQ, P \oplus Q) \cup (P, Q \oplus R)$. $y^2 = 4x^3 + ax + b$ and the two curves just described, have the eight point: $P, Q, R, PQ, QR, Q \oplus P, Q \oplus R, O$ in common. They will then by Cayley-Bacharach theorem also have a ninth point in common, that is $P(Q \oplus R) = (P \oplus Q)R$.

5.3 $E \simeq Pic^0(E)$

We have already mentioned that the addition of points on an elliptic curve, with a choice of zero and addition as described above is isomorphic to the divisor class containing divisors of degree 0 together with the operation of addition. Above the zero of the geometric construction was the point $(0 : 1 : 0)$. We could have chosen any random point on the curve. In our proof below we call it a_0.

Proposition 5.2. Let $E \subset \mathbb{P}^2$ be an elliptic curve, and let $a_0 \in X$ be a point. Then the map $\Phi : E \to \mathrm{Pic}^0 E, \quad a \mapsto a - a_0$ is a bijection that maps $+_E$ to $+_{\mathrm{Pic}^0 E}$ [10, p. 120].

Proof. The map is well defined since $\deg(a - a_0) = 0$. If for $a, b \in X$, $\Phi(a) = \Phi(b)$, then $a - a_0 = b - a_0$ and so $a - b = 0 \in \mathrm{Pic}^0 E$. Since there is no non-zero rational function φ on X with $\mathrm{div}\varphi = a - b$, $a = b$. The map is injective.

$D = a_1 + ... + a_m - b_1 - ... - b_m$ is an arbitrary element of $\mathrm{Pic}^0 E$. First we look at the situation $m \geq 2$. A line through the points a_1, a_2, a_1a_2 (in accordance with our notation above a_1, a_2 are two points on E and a_1a_2 is the third intersection point of the line through a_1 and a_2 and the curve E) has the divisor: $a_1 + a_2 + a_1a_2$. And let $b_1 + b_2 + b_1b_2$ be the divisor of the homogeneous linear polynomial that intersect E in b_1, b_2, b_1b_2. $a_1 + a_2 + a_1a_2 - b_1 - b_2 - b_1b_2$ is then the divisor of the quotients of the above linear polynomials. Since this quotient is a rational function, the divisor is zero in $\mathrm{Pic}^0 E$. Substituting into our arbitrary element of $\mathrm{Pic}^0 E$, we get $D = b_1b_2 + a_3 + ... + a_m - a_1a_2 - b_3 - ... - b_m \in \mathrm{Pic}^0 E$. Continue to reduce the number of points in D, we will eventually reach $m = 1$. That is the same as to say that for some $a_1, b_1 \in X$, $D = a_1 - b_1$. In $\mathrm{Pic}^0 E$ we also have $a_0 + a_1 + a_0a_1 - b_1 - a_0a_1 - b_1(a_0a_1) = 0$. Because of that $D = a_1 - b_1 = b_1(a_0a_1) - a_0 \in \mathrm{Pic}^0 E$. Then we are ready to conclude that Φ is surjective, since $D = \Phi(b_1(a_0a_1))$.
Addition is preserved by this map. That 3 points on E are collinear means that their sum is in $|3P_0|$. $|3P_0|$ was the system giving the embedding of E to $\mathbb{P}^2$. This is the same as saying that the sum of the images of the points in $Pic^0(E)$ is zero. □

We can also prove proposition 5.2 by the Riemann-Roch theorem [12, p. 297].

We must show that for any divisor D of degree 0, there exists a unique point $a \in X$ such that $D \sim a - a_0$. By Riemann-Roch we get

$$l(D + a_0) - l(K - D - a_0) = 1 + 1 - 1.$$

Deg $K = 0$ and so $\deg(K - D - a_0) = -1$. Then $l(K - D - a_0) = 0$ which gives $l(D+a_0) = 1$. That is the same as to say that dim $|D+a_0| = 0$, which means we have a unique effective divisor linearly equivalent to $D+a_0$. That must be a point, a, since the degree of the divisor is 1. So there is a unique point $a \sim D + a_0$, that is $D \sim a - a_0$.

We have now seen that the points on E are isomorphic to elements of $\mathrm{Pic}^0(E)$. For 2 points $p, q \in E$ consider the linear system $|p + q|$ on E. It is base point free and by Riemann-Roch it has dimension 1. This linear system therefore defines a morphism $\varphi : E \to \mathbb{P}^1$ of degree 2. Every 2:1 map from E to $\mathbb{P}^1$ is given by a linear system of degree 2. The lemma below will be used in the proof of Poncelet's theorem. First we define a covering involution.

Definition 5.3. Given a 2:1 map $\varphi : X \to Y$. A covering involution of φ on X interchanges the two elements in the fibers of φ.

Lemma 5.4. If $\varphi : E \to \mathbb{P}^1$ is a 2:1 covering map, and $\iota : E \to E$ the covering involution of φ on E, then $\exists r \in E$, such that $\iota(x) = r - x$, $\forall x \in E$.

Proof. Since φ is 2:1 it is defined by a system of degree 2. A general element of this system is of the form $|p + q|$ with p, q being two points of a chosen fiber. ι interchanges the two points in the fibers of φ, so $\iota(p) = q$. Also for any $x \in E$, $x + \iota(x)$ is a fiber of φ. So $x + \iota(x) \in |p+q|$, which is the same as to say $x+\iota(x) \sim p+q$. Then we have the required form: $\iota(x) \sim p+q-x$. □

5.4 A synthetic construction of the group law on E

We will also take time for a synthetic construction of the group law on E [15, p. 60]. By using the labeling below:

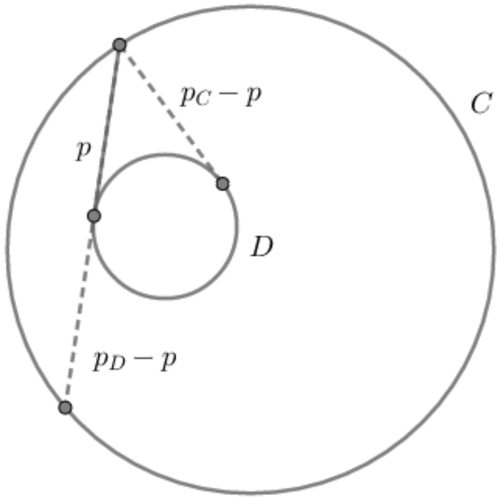


figure 5.1

The trajectory of tangents can be described as $p \to p_C - (p_D - p) = p + (p_C - p_D)$. If $n(p_C - p_D) = 0$ then we are back at our starting point after n steps.

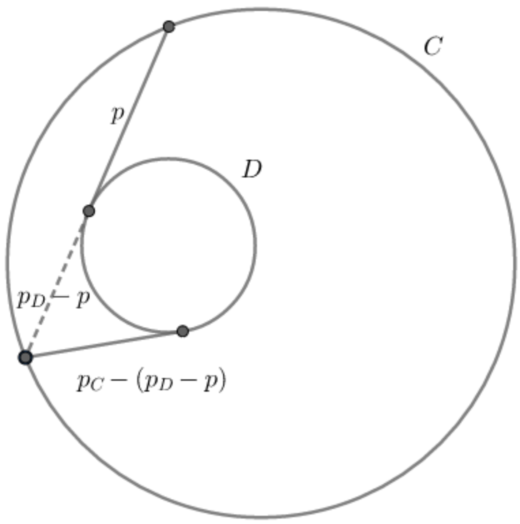


figure 5.2

Start with two circles C and D that, for simplicity, intersects in two real points, O and M (figure below). Let O be the zero of the group law on E.

We will not prove here that this construction, with this choice of zero, fulfil the group axioms of a commutative group.

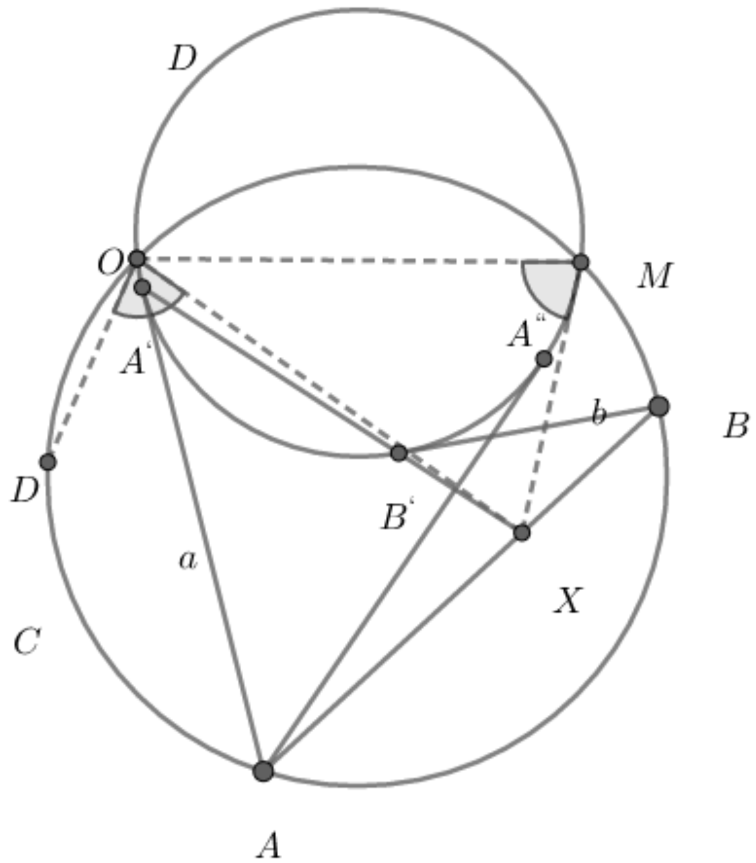


figure 5.3

Draw from the point A on C the two tangents to D, AA' and AA''. They share a point at C and will be negatives to each other since from the choice of zero, we have $p_C = 0 + 0 = 0$. Then let $a = AA'$ and similarly $b = BB'$. $a - b$ and $b - a$ will be negatives of each other and therefore share a point at C, let us call it D. The following construction will give us this D. Find the intersection point X of AB and $A'B'$. Mark out the $\angle XMO$. Copy it to $\angle XOD$. Here we have our point D, which is the point on C which corresponds to $a - b$ and $b - a$. $a - b$ and $p_D - (a - b) = p_D - a - (-b)$ have a common endpoint on D. We can find $p_D - a$ and $-b$, by using the transformation described above, we can construct their difference, their common endpoint at C. $a - b$ is the point D with the line through $(p_D - a) - (-b)$.

6 Proof of Poncelet‘s Theorem

We can now pass to the proof of Poncelet's theorem. For that we keep the notation from section 3. We have the map φ_1 which is a double cover branched in 4 points, the intersection points of the two conics. Hence by corollary 4.18 with $t = 4$ we get

$$\begin{aligned} 2g(E) &= 4g(\mathbb{P}^1) - 2 + 4 \\ g(E) &= 1. \end{aligned}$$

E is an elliptic curve.

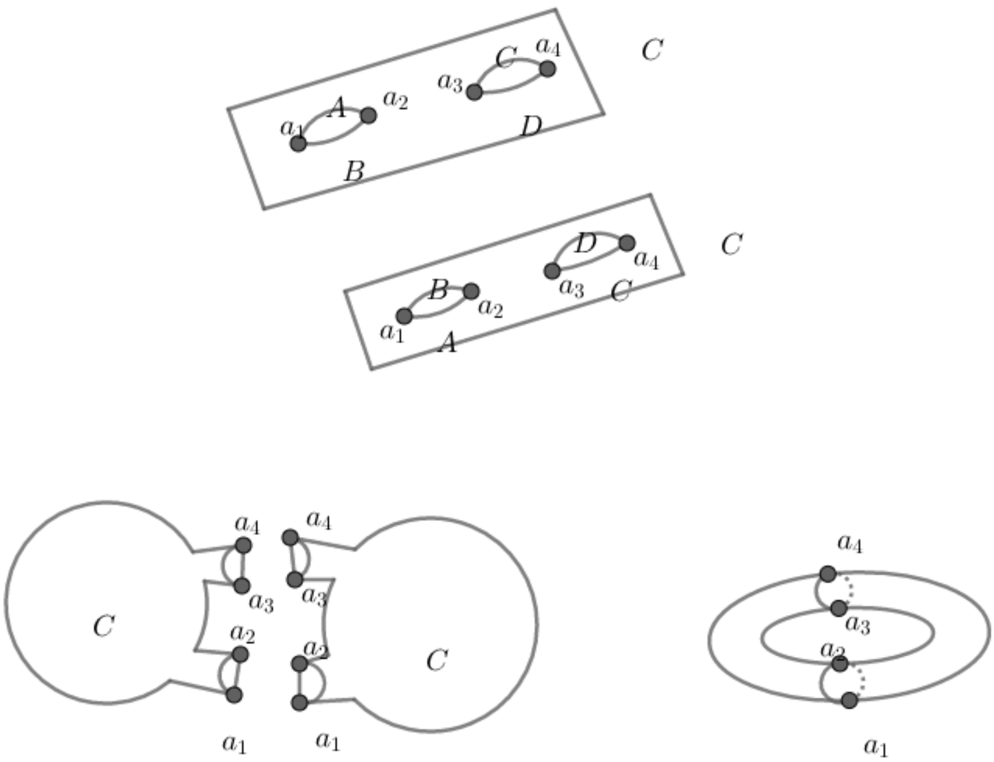


figure 6.1

To see that E is a curve without singularities, we need to pay special attention to the points of intersection of C_1 and C_2. When $(p_1, p_2) \notin C_1 \cap C_2, (p_1, p_2) \mapsto p_1$ is a 2:1 map from E to C. For the points $(p_1, p_2) \in C_1 \cap C_2$, there is a 2:1 map from (p_1, p_2) to one of the conics. So for every point

$(p_1, p_2) \in E$, $E \to \mathbb{P}^1$ is a 2:1 map and therefore smooth.

We are now ready to show the proof of the following reformulation of Poncelet‘s theorem:

If η^n has a fixed point for some positive integer n, then η^n is the identity map on E.

Proof. We have defined ι_1 and ι_2 and the maps φ_1 and φ_2. ι_1 is the covering involution of φ_1 on E and ι_2 is the covering involution of φ_2 on E. By lemma 5.4, for a covering involution of φ on E, there exists a point $r \in E$ so that we for any $x \in E$ have that $\iota(x) = r - x$.

For our ι_1, ι_2, let $\iota_1 = r_1 - x$ and $\iota_2 = r_2 - x$.

Then the composition $\iota_2 \circ \iota_1$ will result in:

$$\eta(x) = (\iota_2 \circ \iota_1)(x) = \iota_2(r_1 - x) = r_2 - (r_1 - x) = x + (r_2 - r_1)$$

and

$$\eta^n(x) = x + n(r_2 - r_1)$$

Let x be the point $(p_1, p_2) \in E$. If we have that after n iterations of η, we are back at our starting point (p_1, p_2) :

$$\eta^n(p_1, p_2) = (p_1, p_2)$$

then

$$n(r_2 - r_1) = 0.$$

Since $n(r_2 - r_1)$ being 0 does not depend on the starting point, the proof is finished. □

7 A proof by Pascal's Theorem

In his book about Poncelet's theorem, Leopold Flatto writes that among all the proofs of this theorem, none are elementary. This statement has inspired someone to try to find one. Lorenz Halbeisen and Norbert Hungerbuhler present in their paper a proof which relies on Pascal's theorem in the projective plane. The material in this section is from [7] and [11].

Before we look at their proof, we will take some time for Pascal's Theorem.

Pascal's Theorem. Let l_{ab} be the line joining the points a and b. Any six points 1,...,6 lie on a conic if and only if

$$l_{12} \cap l_{45}$$

$$l_{23} \cap l_{56}$$

$$l_{34} \cap l_{61}$$

are collinear.

7.1 Cayley-Bacharach Theorem

Pascal's theorem has several forms. In the following we will include a proof of a theorem that covers all the variations of it.

In our section on the history of projective geometry, we looked at how Poncelet used his principal of central projection to prove Pascal's theorem. We now use the Cayley-Bacharach theorem to give a formal proof. We will study the proof of Chasles's theorem together with a proposition concerning the necessary and sufficient conditions for the set of points for it to impose independent conditions on the curves in question. We first study this proposition. For a set of points $\Omega = \{p_1, ..., p_n\}$, we want to know if they impose independent conditions on a curve. For a polynomial f, substitute the coordinates of a point $p_i \in \Omega$ for the variables of f. Then f vanishing at p_i is a linear condition on the coefficients of f. Doing the same for all $p_i \in \Omega$, we want to know if we have n linearly independent conditions or not. $\Omega = \{p_1, ..., p_n\} \subset \mathbb{P}^2$ impose l conditions on curves of degree d if in $\mathbb{P}^{\binom{d+2}{2}}$ the subspace of those containing Ω has codimension l.

Proposition 7.1. Let $\Omega = \{p_1, ..., p_n\} \subset \mathbb{P}^2$ be any collection of $n \leq 2d+2$ distinct points. The points of Ω fail to impose independent conditions on curves of degree d if and only if either i) $d+2$ of the points of Ω are collinear or ii) $n = 2d+2$ and Ω is contained in a conic [7, p. 302].

Proof. Let us first show that in the situation where either $d+2$ of the points of Ω are collinear or $n = 2d+2$ and Ω is contained in a conic, then the points of Ω fail to impose independent conditions on curves of degree d.
i) If a line L contains $d+2$ of the $p_i \in \Omega$, a degree d curve containing Ω must by Bezout's theorem contain L. Calculating the codimension of the curves of degree d that contains L, we get: $\binom{d+2}{2} - \binom{(d-1)+2}{2} = d+1$. We have the other $n-(d+2)$ points, and that is the maximal number of independent conditions they can contribute with. So $n-(d+2)+d+1 = n-1$ is then the highest number of conditions Ω can impose.
ii) If a conic C contains Ω, a degree d curve containing Ω must by Bezout's theorem contain C. Calculating the codimension of the curves of degree d that contains C, we get: $\binom{d+2}{2} - \binom{(d-2)+2}{2} = 2d+1$. That is then the highest number of conditions Ω can impose in this case. Since the condition for ii) is $n = 2d+2$, Ω fails to impose independent conditions on curves of degree d.

For the other direction we use induction for the degree and the number of points. We first observe that when $d = 1$ and Ω consists of 4 or less points, the proposition holds. This set of points fail to impose independent conditions on lines if and only if i) $n = d+2 = 3$ and the points lies on the same line, or ii) $n = 4$.

Now we look at, for any d, the situation $n \leq d+1$. That a set of points fail to impose independent conditions, may be expressed equally as the situation that any plane degree d curve containing $n-1$ of the n points, contain them all. So now we show that, in the case when we have $d+1$ points or less, and therefore not $d+2$ collinear points, the $d+1$ points do not fail to impose independent conditions on the curve of degree d. We use a curve of degree d that contain every p_i but one, by taking the union of lines through all p_i, $i = 1, ..., n-1$. Then we choose a curve of degree $d-n+1$, which has the only restriction that it is not containing the last point of Ω. Then the points do not fail to impose independent conditions on curves of degree d.

When $n > d+1$ we can first think of the case when $d+1$ points of Ω lie on a line L_1. Call this set of $d+1$ points Θ. The set $\Omega \setminus \Theta \leq d+1$. Now

we can reformulate our question and claim: $\Omega \setminus \Theta$ do not impose independent conditions on $d-1-$degree curves. If they did, we could also find a $d-1$-degree curve X containing all points of $\Omega \setminus \Theta$ unless one. The union of X with the line would be a degree d curve containing all points of Ω but one. With the assumptions that we made, $\Omega \setminus \Theta$ consists of $d+1$ points on a line L_2. Then we have that i) either L_1 contains $d+2$ of the points ii) or $n = 2d+2$ and all are contained in the conic $L_1 \cap L_2$.
Consider a line, l containing 3 or more points of Ω. The $n-l$ points left, fail to impose independent conditions on $d-1$- degree curves and must consist of at least $d+1$ collinear points.

What is left to prove is the situation when Ω contains no three points that are collinear. Choose a set $\Upsilon = \{p_1, p_2, p_3\}$ of any three points from Ω. If $(\Omega \setminus \Upsilon) \cup \{p_i\}$, for any i, impose independent conditions on curves of degree $d-1$, then we can find a $d-1$-degree curve X that contain $\Omega \setminus \Upsilon$ but not p_i. Let L_{jk} be a line joining p_j and p_k. $X \cup L_{jk}$ is a degree d curve containing all but one of the points from Ω. We can continue by thinking $(\Omega \setminus \Upsilon) \cup \{p_i\}$ will not impose independent conditions on degree d-1 curves. $(\Omega \setminus \Upsilon) \cup \{p_i\}$ cannot contain $d+1$ points on the same line. By induction $n = 2d+2$. For every i, $(\Omega \setminus \Upsilon) \cup \{p_i\}$ lies on a conic X_i. We are finished if $d = 2$. If $d \geq 3$, $\Omega \setminus \Upsilon$ contains five or more points where no three of them lies on the same line. There can then be not more than one conic containing Ω, so all X_i are the same and then contain Ω. □

Theorem 7.2. (Chasles) Let $X_1, X_2 \subset \mathbb{P}^2$ be cubic plane curves meeting in nine points $p_1, ..., p_9$. If $X \subset \mathbb{P}^2$ is any cubic containing $p_1, ..., p_8$, then X contains p_9 as well [7, p. 301].

Proof. Let us consider the set of m distinct points $\Gamma = \{p_1, ..., p_m\} \subset \mathbb{P}^2$ and the vector space of polynomials of degree d on $\mathbb{P}^2$. We shall give the name l to the number of conditions that Γ imposes on polynomials of degree d. Then l is defined equivalent as above to be the codimension of the degree d polynomials that vanish on $p_1, ..., p_m$. Cubics in $\mathbb{P}^2$ live in a 10-dimensional vector space. Let F_1, F_2 be the equations of X_1, X_2. These two equations, vanishing on Γ, spans a 2-dimensional subspace of the vector space of polynomials of degree 3 in $\mathbb{P}^2$ of polynomials vanishing on Γ. The 9 points cannot impose independent conditions.

We will show that if a cubic X contains exactly 8 points from $\Gamma = \{p_1, ..., p_9\}$, then, on a cubic, this set of 8 points and Γ imposes the same number of conditions. By Bezout's theorem we have that two curves of degree 3 cannot meet in more than 9 points unless they have a component in

common. So by assumption the situation that X_1, X_2 would have multiple components is already not an option. F_1, F_2 has no repeated factors. From the proposition above we have seen the only condition for our eight points, remembering that they come from intersection of two cubics, to fail to impose independent conditions on curves of degree 3 is i) either that 5 of the points are collinear or ii) $n = 8$ and the all the points are contained in a conic.
i) In this case, the points on that line, if it existed, would have to be contained in both X_1 and X_2. We already assumed that they have no components in common, so it cannot be that 5 points lie on the same line.
ii) If all the 8 points were lying on a conic, then both X_1 and X_2 would have to contain a component of this conic. Like above, that would have contradicted our assumption. So the 8 points impose independent conditions on our curve of degree 3. $\square$

Now Pascal's theorem will follow from the above.

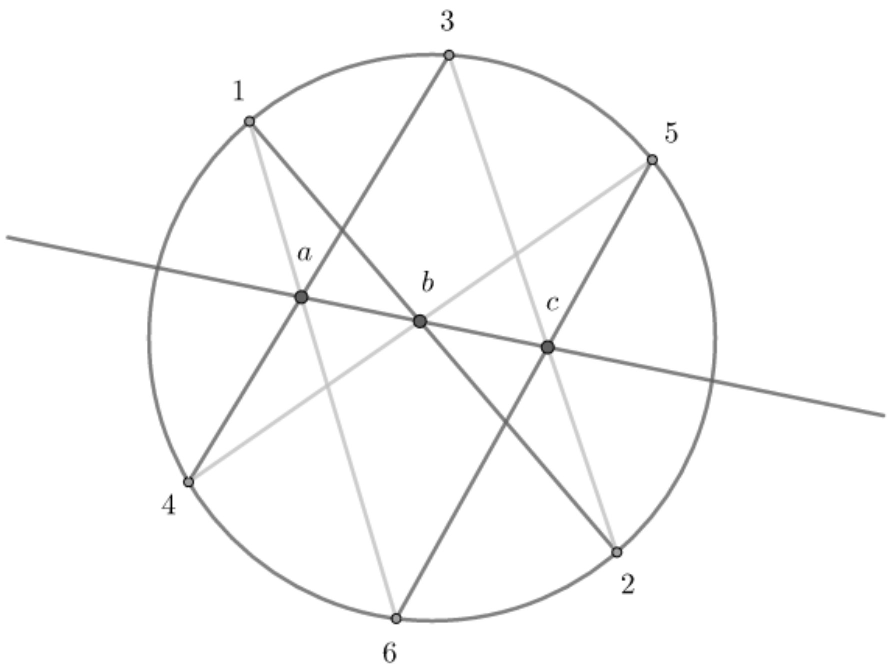


figure 7.1

Proof. This is a construction where the cubic which consists of the three lines l_{12}, l_{34} and l_{56} and the other cubic which consists of the lines l_{16}, l_{23} and l_{45} have the points lying on the circle in our picture in common. By Bezout's theorem they also have three more points in common, call them a, b, c. These three points must lie on a line, since the conic containing the six points together with a line passing through any two of the points a, b, c is a cubic that shares 8 points with our first two cubics.
For the other direction, we also use Chasles's theorem. We have that the 3 points $a = l_{34} \cap l_{61}$, $b = l_{12} \cap l_{45}$ and $c = l_{23} \cap l_{56}$ are collinear and we have a conic through $1, ..., 5$. The point 6 is not on the line, so it must be on this conic.

□

A reformulation of Pascal's theorem that we will use several times is Carnot's theorem.

Carnot's Theorem. Any six points 1,...,6 lie on a conic if and only if

$$l_{(l_{12}\cap l_{34})(l_{45}\cap l_{61})}$$

$$l_{25}$$

$$l_{36}$$

are concurrent.

We also have Brianchon's theorem, which is equivalent to Pascal's, by exchanging points by lines. Also, by exchanging points with lines in Carnot's theorem we get its dual, we can call it Carnot's* theorem. All of these variations is covered by the proof of Cayley-Bacharach.

7.2 Triangles

Halbeisen and Hungerbuhler first demonstrate their approach in the special case where the polygon is a triangle. The below result is needed.

Theorem 7.3. If two triangles are inscribed in a conic and the two triangles do not have a common vertex, then the sides of the triangles are tangent to a conic [11, p.3].

Proof. The triangles $a_1a_2a_3$ and $b_1b_2b_3$ are both inscribed in C_2. They share no vertexes. To find the three collinear intersection points by which we can apply Pascal‘s theorem, consider the following points:

$$I := l_{a_1-a_2} \cap l_{b_1-b_2}$$

$$X := l_{a_2-b_3} \cap l_{b_2-a_3}$$

$$I' := l_{a_3-a_1} \cap l_{b_3-b_1}$$

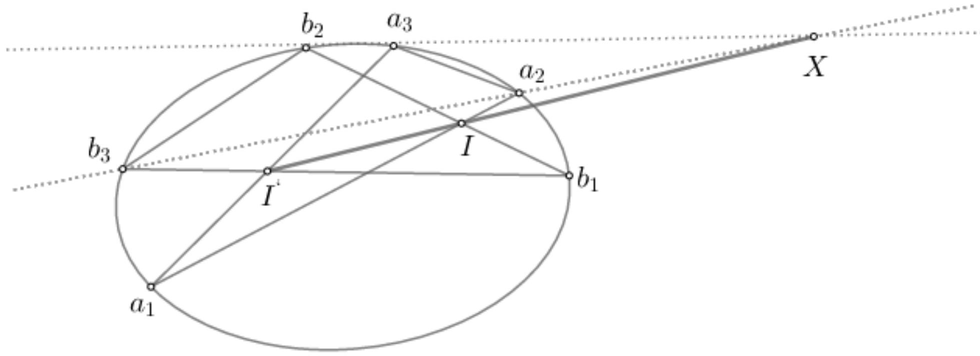


figure 7.2

Then again label:

$$a_1 := 1$$
$$a_2 := 2$$
$$a_3 := 6$$
$$b_1 := 4$$
$$b_2 := 5$$
$$b_3 := 3$$

The three intersection points:

$$l_{12} \cap l_{45}$$
$$l_{23} \cap l_{56}$$
$$l_{34} \cap l_{61}$$

are by Pascal‘s theorem collinear, so I, X and I’ are collinear.

Now the proof continues in the same fashion with labeling of edges to make use of Carnot‘s theorem.

$$l_{a_1 a_2} := l_2$$
$$l_{a_2 a_3} := l_1$$
$$l_{a_3 a_1} := l_6$$
$$l_{b_1 b_2} := l_5$$
$$l_{b_2 b_3} := l_4$$

$$l_{b_3b_1} := l_3$$

These six sides are tangent to another conic C, by Carnot's, iff

$$l_{(l_1\cap l_2)(l_3\cap l_4)} \cap l_{(l_4\cap l_5)(l_6\cap l_1)}$$

$$l_2 \cap l_5$$

$$l_3 \cap l_6$$

lie on the same line. That is equivalent to collinearity of the points X, I and I'. Like just pointed out above, this again is the same as saying that the points a_1, a_2, a_3, b_1, b_2 and b_3 are lying on a conic. □

From this we get Poncelet's theorem for the case of triangles as a corollary.

Poncelet's Theorem for triangles Let C_1 and C_2 be smooth conics which neither meet nor intersect. Suppose there is a triangle $a_1a_2a_3$ inscribed in C_2 and circumscribed about C_1. Then for any point b_1 of C_2, there exists a triangle $b_1b_2b_3$ which is also inscribed in C_2 and circumscribed about C_1 [11, p.6].

Proof. Given the above situation and that the point b_1 is not equal to any of the points a_1, a_2 or a_3. b_2 and b_3 are chosen so that $l_{b_1b_2}$ and $l_{b_1b_3}$ are tangents to C_1. Now five of the six sides of the two triangles must be tangent to C_1. From theorem (7.3) above we also have that since the two triangles do not have a common vertex, the sides of the triangles must be tangent to some conic. This must be our C_1 since five tangents given already define the conic. So we can conclude that $b_1b_2b_3$ is also circumscribed about C_1. □

7.3 The case of a n-sided polygon interscribed between two conics

We start with the situation as above, only now we have a n-gon with sides $a_1, ..., a_n$. The question is, if we make another $(n-1)-$polygonal chain starting in a point $b_1 \neq a_i$, will $l_{b_nb_1}$ be tangent to C_1?

We will now need the following result.

Lemma 7.4. For $n \geq 4$, the three intersection points

$$I := l_{a_1a_2} \cap l_{b_1b_2}$$

$$X := l_{a_2 b_{n-1}} \cap l_{b_2 a_{n-1}}$$
$$I' := l_{a_{n-1} a_n} \cap l_{b_{n-1} b_n}$$

are collinear [Hal, p. 7].

Proof. We first consider how our starting point will be like depending on whether n is odd or even. If n is even, $k = \frac{n}{2}$. We set:

$$I := l_{a_1 a_2} \cap l_{b_1 b_2} = l_{a_{k-1} a_k} \cap l_{b_{k-1} b_k} := l_2 \cap l_5$$
$$I' := l_{a_{n-1} a_n} \cap l_{b_{n-1} b_n} = l_{a_{k+1} a_{k+2}} \cap l_{b_{k+1} b_{k+2}} := l_6 \cap l_3$$
$$X := l_{a_2 b_{n-1}} \cap l_{b_2 a_{n-1}} = l_{a_k b_{k+1}} \cap l_{b_k a_{k+1}}$$
$$= l_{(l_2 \cap l_1)(l_4 \cap l_3)} \cap l_{(l_5 \cap l_4)(l_1 \cap l_6)},$$
$$\text{where } l_{a_k a_{k+1}} := l_1, l_{b_k - b_{k+1}} := l_4$$

We then have that $I - X - I'$ are collinear by Carnot's* theorem. When n is odd, $k = \frac{n+1}{2}$, we have:

$$I := l_{a_1 a_2} \cap l_{b_1 b_2} = l_{a_{k-1} a_k} \cap l_{b_{k-1} b_k} := l_{12} \cap l_{45}$$
$$X := l_{a_2 b_{n-1}} \cap l_{b_2 a_{n-1}} = l_{a_{k-1} b_{k+1}} \cap l_{b_{k-1} a_{k+1}} := l_{16} \cap l_{43}$$
$$I' := l_{a_{n-1} a_n} \cap l_{b_{n-1} b_n} = l_{a_k a_{k+1}} \cap l_{b_k b_{k+1}} := l_{23} \cap l_{56}$$

And here, the collinearity of $I - X - I'$ is given by Pascal's theorem.

There are two steps that need to be proven. Five points, $I_{p-1}, I_p, X, I_q, I_{q+1}$, are identified. It is required that $2 \leq p < q \leq a_{n-1}$. The first thing that needs to be shown is:

Lemma 7.5. If $I_p - X - I_q$ are collinear, then so will $I_{p-1} - X - I_{q+1}$. The five points of consideration are:

$$I_{p-1} := l_{a_{p-1} a_p} \cap l_{b_{p-1} b_p}$$
$$I_p := l_{a_p a_{p+1}} \cap l_{b_p b_{p+1}}$$
$$I_q := l_{a_{q-1} a_q} \cap l_{b_{q-1} b_p}$$
$$I_{q+1} := l_{a_q a_{q+1}} \cap l_{b_q b_{q+1}}$$
$$X := l_{a_p b_q} \cap l_{b_p a_q}$$

Proof. X is defined as the intersection of the lines $\beta = l_{a_p b_q}$ and $\alpha = l_{b_p a_q}$. The three points $I_p - X - I_q$ lie on one line, γ, that is given in the claim. So α, β, γ meet in X.
Label the lines that meet in I_p, l_2 and l_3 and the lines that meet in I_q, l_6 and l_5. Let l_1 be the line that intersects l_6 in a_q. And let l_4 be the line that intersects l_3 in b_p. Then let ϵ be the line $l_{(l_2 \cap l_1)(l_4 \cap l_5)}$. We will have that $\alpha, \gamma, \varepsilon$ are concurrent by Brianchon's theorem.
Now label the lines that meet in the point I_{p-1}, l_1 and l_6 and label the lines through I_{q+1}, l_3 and l_4. The line through I_{p-1} and I_{q+1} is called γ. The line $l_{(l_2 \cap l_3)(l_6 \cap l_5)}$ has the name ϵ. We can conclude, again thanks to Brianchon's theorem, that β, γ, δ meet in one point. Then we can state that $I_{p-1} - X - I_{q+1}$ are collinear, since both $\alpha, \beta, \varepsilon$ and α, β, δ meet in X in the above construction. $\square$

Now we have one more claim to prove before proceeding to the proof of Poncelet. In addition to the points I_{p-1}, X, I_{q+1} above, now also consider $X' := l_{(a_{p-1} b_{q+1}) \cap (b_{p-1} a_{q+1})}$ What we want to show is:

Lemma 7.6. If $I_{p-1} - X - I_{q+1}$ are collinear, then also $I_{p-1} - X^{‘} - I_{q+1}$ will be so.

Proof. The collinearity of $I_{p-1} - X - I_{q+1}$ is given. Let $l_{12} \cap l_{45}$ be the lines that meet in I_{p-1}. Also $a_p = 2, a_q = 6, b_p = 5, b_q = 3$. J is the point $l_{16} \cap l_{43}$ and $X = l_{23} \cap l_{56}$. This situation is exactly as described in the theorem of Pascal, and we can conclude that $I_{p-1} - X - J$ are collinear.

The next move is to again label the points. $J = l_{12} \cap l_{45}$. $a_q = 2$ and $b_q = 5$. Her we include the points X' and I_{q+1}. $I_{q+1} = l_{23} \cap l_{56}$ and $X' = l_{16} \cap l_{43}$. Again collinearity of tree points is the result by Pascal's theorem, this time $X' - J - I_{q+1}$. By this we have what is needed to prove that $I_{p-1} - X^{‘} - I_{q+1}$ are collinear. Our last step gives that $X^{‘}$ is on $J - I_{q+1}$ and our two previous steps shows that $I_{p-1} - J - I_{q+1}$ lies on the same line. $\square$

So far we have shown that if $I_p - X - I_q$ are collinear, then so will $I_{p-1} - X - I_{q+1}$ and if $I_{p-1} - X - I_{q+1}$ are collinear, then also $I_{p-1} - X^{‘} - I_{q+1}$ will be. Together this gives us the collinearity of $I - X - I^{‘}$.

$\square$

Is $l_{b_n b_1}$ also tangent to C_1? That is the big question which will prove Poncelet's theorem if the answer is yes.

Proof.

$$I := l_{a_1a_2} \cap l_{b_1b_2}$$

$$X := l_{a_2b_{n-1}} \cap l_{b_2a_{n-1}}$$

$$I' := l_{a_{n-1}a_n} \cap l_{b_{n-1}b_n}$$

$$J := l_{a_{n-1}a_1} \cap l_{b_{n-1}b_1}$$

$$X' := l_{a_nb_1} \cap l_{b_na_1}$$

The three first points are the same as earlier in the text, and we have shown in lemma (7.4) that they are collinear.

If we manage to show that $I - X^{‘} - I^{‘}$ are collinear, Carnot$*$ will give that all six lines, $1 = l_{a_na_1}, 2 = l_{a_{n-1}a_n}, 3 = l_{b_1b_2}, 4 = l_{b_nb_1}, 5 = l_{b_{n-1}b_n}, 6 = l_{a_1a_2}$, are tangent to some conic C. C and C_1 must be the same conic and we will know that $l_{b_n-b_1}$ is tangent to C_1, since five tangents determines a conic.

Since we already know by lemma (7.4) that $I-X-I'$ lies on the same line, if we were able to show the same for $I - X - J$, we would obtain collinearity for $I - X - J - I'$ and if also $I' - J - X'$ are so, the conclusion would be that $I - X' - I'$ are collinear. This will be in reach by labeling our points first in the following way: $a_1 = 1, a_2 = 2, a_{n-1} = 6, b_1 = 4, b_2 = 5, b_{n-1} = 3$ and

$$I = l_{12} \cap l_{45}$$

$$X = l_{23} \cap l_{56}$$

$$J = l_{16} \cap l_{34}$$

Then $I - X - J$ are collinear by the theorem of Pascal. In a similar manner, by the labeling below:
$a_{n-1} = 1, a_n = 2, a_1 = 6, b_{n-1} = 4, b_n = 5, b_1 = 3$ and

$$I' = l_{12} \cap l_{45}$$

$$X' = l_{23} \cap l_{56}$$

$$J = l_{16} \cap l_{34}$$

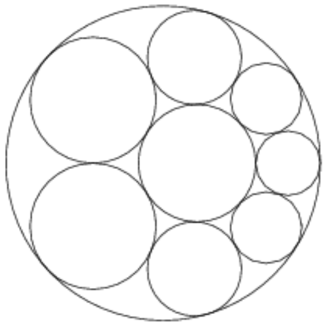
again by Pascal‘s theorem, we will get the wanted collinearity of $I' - J - X'$.

And then we can conclude that $l_{b_nb_1}$ is also a tangent to C_1, whereby it is showed that if we have one n-sided polygon inscribed in C_2 and circumscribed about C_1, we can start at any point of C_2 and obtain the same. □

8 New generations will be inspired by Poncelet's Theorem.

From the days of Poncelet and until today, his work has led people to ask new questions. We will end this text by mentioning only a few examples.

One related problem that was posed by Jakob Steiner (1796-1863) is the following: For two circles, C inside D, if it is possible to "fill" the space between C and D by circles, W_i, for which all have exactly one point in common with both C and D, and exactly one point in common with W_{i-1} and W_{i+1}, then this will happen for any choice of first circle W_i.



As a mechanical system, billiards in an ellipse is also related to Poncelet's theorem. Start with a point moving from one point of the interior of an ellipse to another point of the interior of that ellipse, and continuing doing so obeying the laws of reflection. Every line segment of this route will be tangent to another ellipse which is confocal to the one we started with. This was recently used to prove the existence of all knots in any elliptic cylinder billiard.

A follow up question to Poncelet's theorem that has been studied is: What is the condition that makes it possible to have a n-polygon interscribed between two given conics? Cayley found an analytic answer to this question in 1853. Almost 100 years later Lebesgue explained this conditions using projective geometry and algebra. A modern proof is given by Griffiths and Harris (1987) [6, p. 3].

Griffiths and Harris are two of the mathematicians that have worked on the generalizing of Poncelet's theorem and related problems to higher dimensions and other curves called Poncelet curves. This is still an area of active research.

In this text we have looked at some parts of the long history of Poncelet's theorem. It has not only inspired a lot of people to find new proofs and solve interesting related problems in the past, it still is. Today researchers in different areas of mathematics study its generalisations.

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