Nil-Killing vector fields and Kundt structures

by

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Preface

This thesis is submitted in partial fulfilment of the requirements for the degree of Philosophiae Doctor (PhD) at the University of Stavanger, Faculty of Science and Technology, Norway. The research has been carried out at the University of Stavanger from Septermber 2015 to December 2019.

I would like to thank my wife for her love, strength, patience and many sacrifices in the process of this work and my beautiful children for making my life a meaningful, exciting, beautiful mess.

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Matthew Terje Aadne
Stavanger, December 2019

Abstract

This thesis is based on three papers, for which two have been submitted for publication and one is published. A chapter presenting relevant background material is included giving convenient access to preliminary foreknowledge for the papers. The research for which the thesis and papers are based concerns Nil-Killing vector fields, which generalize Killing vector fields in the sense that the Lie derivative of the metric is nilpotent. We study their properties and find that they form infinitesimal automorphisms of certain G-structures. Based on this we are able to express Kundt spacetimes in terms of G-structures, giving new tools for their investigation.

List of papers

Paper I

David Duncan McNutt, Matthew Terje Aadne (2019). $\mathcal{I}$-preserving diffeomorphisms of Lorentzian manifolds. *J. Math. Phys. 60, 032501* arXiv:1901.04728v2 [gr-qc]

Paper II

Matthew Terje Aadne (2019). Nil-Killing vector fields and type III deformations. *Submitted for publication in Journal of mathematical physics.* arXiv:1912.02809 [math.DG]

Paper III

Matthew Terje Aadne (2019). Kundt structures. *Submitted for publication in Differential Geometry and its Applications.* arXiv:1912.02570 [math.DG]

Table of Contents

Appendix

1 Introduction

Given a pseudo-Riemannian manifold (M, g) we define the scalar polynomial curvature invariants (spi's) to be the collection of smooth functions which can be expressed as full contractions of the Riemann tensor and its covariant derivatives to all order, i.e., functions of the form

$$\{R_{abcd}R^{abcd}, \nabla_e R_{abcd}\nabla^e R^{abcd}, \dots\}. \tag{1.1}$$

In the Riemannian case the spi's completely characterize the orbit of each metric, meaning that each isometry class of metrics can be seperated by them. In addition any Riemannian manifold for which the spi's are constant across the manifold are locally homogeneous [10], and any Riemannian manifold for which the spi's all vanish must be flat.

In Lorentzian signature the situation is not so simple. We can find examples of metrics for which there exists deformations exiting the orbit of the metric whilst leaving the spi's fixed, Lorentzian metrics with all spi's constant having no local Killing vector fields, and non-flat spacetimes for which the spi's vanish.

At present time all examples exhibiting the above behaviour have been found to belong to the rich class of Kundt spacetimes [1, 11, 9] which are also presented in chapter 2 section 2.2.

In this thesis we present three papers which all study various aspects of Kundt spacetimes, namely their deformations and natural Lie algebras of vector fields. We start our presentation in chapter 2 by giving some helpful foreknowledge. Here we present the basics of boost-weight decompositon, Kundt spacetimes and principal G-bundles.

In paper I, "$\mathcal{I}$-preserving diffeomorphisms of Lorentzian manifolds", we investigate the properties of nil-Killing vector fields [4]. They generalize Killing vector fields in the sense that the Lie derivative of the metric is nilpotent when regarded as an operator. They can also be regarded as generalizations of Kundt vector fields since all Kundt vector fields in turn are nil-Killing. Our findings show that any nil-Killing vector field is of type III with respect to some null-distribution. In addition we found that they constitute a Lie algebra when grouped together as such. This Lie algebra is shown to have an ideal given by the vector fields in this collection belonging to the orthogonal compliment of the null-distribution.

Paper II, "Nil-Killing vector fields and type III deformations", continues the investigation of these vector fields. Here we find that the flows of nil-Killing vector fields can be characterized by deforming the metric along the direction of a type III tensors. We extend our focus by obtaining results concerning deformations of tensors of type II in the direction of tensors of type III. This gives us criteria for when the scalar polynomial invariants of such tensors are constant under deformations, and with respect to the integral curves of nil-Killing vector fields. We proceed by generalizing the Lie algebra construction from [7], showing that we can form Lie algebras by fixing the datum of a boost-order s and a collection of tensors, by requiring that the Lie derivative of the tensors be of boost order $\leq s$. Next we find results concerning Kundt spacetimes. We give a characterization for when the transverse metric of Kundt spacetime is locally homogeneous. This happens exactly when there is a transverse collection of nil-Killing vector fields. We go on to give a theorem giving a classification of the tensors that are algebraically stable under covariant derivatives to any order. This allows us to characterize the deformations of degenerate Kundt spacetimes considered in [2, 3] which leave the spi's fixed, as those whose deformation tensor and its covariant derivatives are of type III to all orders. Lastly, for degenerate Kundt spacetimes we obtain a class of nil-Killing vector fields which preserve the spi's given by the metric.

Paper III is concerned with the "Kundt structures". Here we use the flow properties of nil-Killing vector fields in order to define a class of G-structure. We find that they have a number of amenable properties: They give rise to an algebraic classification of tensors in such a way that we can perform full contractions of even ranked type II tensors. In addition they have a natural class of Lorentzian metrics associated to them which form an affine space with respect to symmetric rank two tensors of type III. The nil-Killing vector fields of each metric belonging to the G-structure are shown to coincide with the Lie algebra of infinitesimal automorphisms of the G-structure. Since any Kundt vector field is nil-Killing this shows that by an integrability assumption and by demanding the existence of a certain class of infinitesimal autormorphisms, any metric belonging to such a G-structure is automatically Kundt. Using this we define Kundt structures, and show that a GN-structure is Kundt if and only if it admits local torsion-free connections. Motivated by the idea that Kundt metrics with constant scalar curvature invariants can be classified through the existence of nil-Killing vector fields satisfying additional properties, we characterize all left-invariant Kundt structures on homogeneous spaces and present them in terms of the underlying Lie algebras for the homogeneous space. Lastly we classify left-invariant Kundt structures on three dimensional Lie groups, and find necessary conditions that a nilpotent Lie group must satisfy in order to support such a structure.

2 Preliminaries

2.1 Boost-weight classification

In this section we present the notions of boost-weight classification [8] which are used extensively in papers $I - III$.

If (V, g) is an n-dimensional real vector space with a Lorentzian inner product g, a null-basis $\{k, l, m_i\}_{i=1,\dots n-2}$ is a collection of vectors such that

$$\begin{gathered} g(k,k) = g(l,l) = g(k,m_i) = g(l,m_i) = 0, \\ g(k,l) = 1, \quad g(m_i,m_j) = \delta_{ij}, \end{gathered} \tag{2.1}$$

for all $i, j = 1, \dots n-2$.

Such a null-basis induces a group action $\mathbb{R}^* \times V \to V$, which in terms of the null-basis is given by

$$(\lambda, k) \mapsto \lambda^{-1}k, \quad (\lambda, l) \mapsto \lambda l, \quad (\lambda, m_i) \mapsto m_i, \tag{2.2}$$

for all $c \in \mathbb{R}$ and $i = 1, \dots n-2$.

We can extend this to give an action $\mathbb{R}^* \times \mathcal{T}(V) \to \mathcal{T}(V)$, where $\mathcal{T}(V)$ is the tensor algebra.

Letting s be some integer, a tensor $T \in \mathcal{T}(V)$ is said to be of boost-weight s if

$$(\lambda, T) = \lambda^s T, \tag{2.3}$$

for all $\lambda \in \mathbb{R}^*$.

Now considering the space $\mathcal{T}^r(V)$ of tensors of rank r, we can define the subspace $T^r(V)_s$ of tensors of boost-weight s.

This gives us a decomposition

$$\mathcal{T}^r(V) = \bigoplus_{-r \le s \le r} \mathcal{T}^r(V)_s. \tag{2.4}$$

Given a tensor $T \in \mathcal{T}^r(V)$, we therefore have a decomposition

$$T = \sum_{-r \leq s \leq r} T_s, \tag{2.5}$$

of T into tensors of boost-weight ranging from $-r$ to r.

With respect to the given null-basis, each component of T must necessarily be a tensor with a well-defined boost-weight. Hence this allows us to refer to the components of T belonging to $\mathcal{T}^r(V)_s$ as the boost-weight s components of T.

Lastly, given an integer s we say that a tensor T is of boost-order $\leq s$ if the boost-weight $s+t$ components vanish, for all $t \geq 1$. The classification of tensors into boost-orders only depends on the direction of the null-vector k, i.e., it remains unchanged with respect to any null-basis $\{k', l', m'_i\}$ for which k and k' are linearly dependent.

A tensor T of boost-order ≤ 0 is said to of type II. Likewise we say that T is of type III if it has boost-order ≤ -1.

2.2 Kundt spacetimes

Here we briefly present the main features of Kundt spacetimes found in [1, 9, 11] in addition to providing some helpful characterisations.

If (M, g) is a Lorentzian manifold and Z is some vector field on M, then its shear and divergence are given by

$$\nabla_{(a}Z_{b)}\nabla^a Z^b \quad \text{and} \quad \nabla_a Z^a, \tag{2.6}$$

respectively.

A triple (M, g, λ) consisting of a Lorentzian manifold (M, g) with a rank 1 null-distribution λ is said to be a Kundt spacetime if the orthogonal compliment $\lambda^\perp$ is an integrable distribution and about any point $p \in M$, there exists a neighborhood U with a vector field X belonging to λ such that X is affinely geodesic, shear-free and divergence-free. We shall refer to such a vector field as being Kundt.

This can be seen to be equivalent to requiring integrability of $\lambda^{\perp}$ and that any vector field Z belonging to λ satisfies

$$\mathcal{L}_Z g(W, W') = 0, \tag{2.7}$$

for all $W, W' \in \lambda^{\perp}$.

Now we shall portray some properties of null-frames $\{k, l, m_i\}$ such that k belongs to λ and their connection coeffisients.

Recall that if e_a is s frame such that $g_{ab} = g(e_a, e_b)$ are constant, for all $a, b = 1 \ldots n$, then setting

$$\Gamma_{abc} = g(e_a, \nabla_{e_c} e_b), \tag{2.8}$$

and

$$C_{abc} = g_{ae} C^{e}{}_{bc} = g([e_b, e_c], e_a), \tag{2.9}$$

we have the identities

$$\Gamma_{(ab)c} = 0, \tag{2.10}$$

and

$$\Gamma_{abc} = \frac{1}{2}(C_{acb} - C_{cba} - C_{bca}). \tag{2.11}$$

Characterization 1. *Let (M, g, λ) be a Lorentzian manifold with a null-distribution λ. (M, g, λ) is Kundt iff. the connection coefficients of each null-frame $\{k, l, m_i\}$ with k belonging to λ satisfy $\Gamma_{0ij} = 0$ and $\Gamma_{i00} = 0$, for all ij.*

Proof. By the above considerations we see that (M, g, λ) is Kundt iff. for each null-frame $\{k, l, m_i\}$ such that k belongs to λ, the following are satisfied:

$$g([m_i, m_j], k) = 0, \quad g([k, m_i], k) = 0 \tag{2.12}$$

and

$$\mathcal{L}_k g(m_i, m_j) = -g([k, m_i], m_j) - g(m_i, [k, m_j]) = 0, \tag{2.13}$$

for all $i, j = 1 \dots n-2$.

Letting $\{e_0, \dots, e_{n-1}\} = \{k, l, m_i\}$ we see from (2.11) that

$$\begin{aligned} \Gamma_{0ij} &= \frac{1}{2}(-g(k, [m_i, m_j]) + g(m_j, [k, m_i]) + g(m_i, [k, m_j])), \\ \Gamma_{i00} &= g(k, [m_i, k]). \end{aligned} \tag{2.14}$$

From this we see that

$$\Gamma_{0ij} - \Gamma_{0ji} = -g(k, [m_i, m_j]) \tag{2.15}$$

and

$$\Gamma_{0ij} + \Gamma_{0ji} = g([k, m_i], m_j) + g(m_i, [k, m_j]), \tag{2.16}$$

and therefore equations (2.12) and (2.13) hold if and only if

$$\Gamma_{0ij} = 0 \quad \text{and} \quad \Gamma_{i00} = 0, \tag{2.17}$$

for all i, j.

□

About each point in a Kundt spacetime (M, g, λ) we can find a neighborhood with coordinates (u, v, x^k) such that $\frac{\partial}{\partial v}$ belongs to λ and is Kundt, du is the metric dual of $\frac{\partial}{\partial v}$ and the metric can be expressed by

$$g = 2du(dv + Hdu + W_i dx^i) + \tilde{g}_{ij}(u, x^k)dx^i dx^j, \tag{2.18}$$

for some smooth functions H, W_i, for $i = 1, \dots n-2$.

A Kundt spacetime (M, g, λ) is said to be degenerate if the Riemannian curvature and all its covariant derivatives $\{\nabla^m R\}_{m\geq 0}$ are of type II. This was shown in [1] to be equivalent to the functions H, W_i in the local expression (2.18) taking the form

$$H(u,v,x^k) = v^2 H^{(2)}(u,x^k) + vH^{(1)}(u,x^k) + H^{(0)}(u,x^k) \qquad (2.19)$$

and

$$W_i(u,v,x^k) = vW_i^{(1)}(u,x^k) + W_i^{(0)}(u,x^k), \qquad (2.20)$$

respectively.

Given a Kundt space-time (M, g, λ) we can use the coordinate expression in (2.18) to construct a null-coframe $\{\mathbf{k}, \mathbf{l}, \mathbf{m}^i\}$, referred to as the canonical Kundt frame, by

$$\mathbf{k} = du, \quad \mathbf{l} = dv + Hdu + A_i\mathbf{m}^i, \quad \delta_{ij}\mathbf{m}^i\mathbf{m}^j = \tilde{g}_{ij}dx^i dx^j. \qquad (2.21)$$

Letting $\{k, l, m_i\}$ be the corresponding null frame, then it satisfies

i) $[k, l] \in \{k\}^\perp$,

ii) $[k, m_i] \propto k$,

iii) $[m_i, m_j] \in \{k\}^\perp$.

If in addition (M, g, λ) is degenerate Kundt, then the canonical Kundt frame furthermore satisfies

a) $[k, [k, l]] \propto k$,

b) $[k, [k, [k, l]]] = 0$,

c) $[k, [k, m_i]] = 0$.

In general we have the following useful characterization of frames satisfying properties $i) - iii)$ and $a) - c)$ in terms of connection coefficients:

Characterization 2. *Suppose that (M, g, λ) is a Lorentzian manifold with a null-distribution λ. A frame $\{k, l, m_i\}$ belonging to λ satisfies $i) - iii)$ and $a) - c)$ iff. $\Gamma_{\alpha\beta\gamma}$=0, for all strictly positive* *boost-weight indices* $\alpha\beta\gamma$ *and for each integer* $s \geq 0$, $k^{(s+1)}\Gamma_{\alpha\beta\gamma} = 0$, *whenever* $\alpha\beta\gamma$ *is an index of boost-weight* $-s$.

Such frames will be useful to us in ensuing chapters.

2.3 Principal G-bundles

In this section we shall give a short introduction to the fundamentals of principal G-bundles. Our goal is to give basic foreknowledge in order to understand the content of paper III. An extensive treatment of this topic can be found in [5, 6].

Fibre bundles

We shall start by recalling the definition of a fibre bundle. Suppose that M and F are smooth manifolds, then a fibrebundle over M with typical fibre F is a smooth manifold E along with a map

$$\begin{array}{c} E \\ \big\downarrow{\scriptstyle \pi} \\ M \end{array} \tag{2.22}$$

such that at each point $p \in M$ there exists a neighborhood U and a map $\phi : \pi^{-1}(U) \to U \times F$ making the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle pr} & \\ & U & \end{array} \tag{2.23}$$

commute, such that for each $q \in U$, the restriction

$$\pi^{-1}(q) \xrightarrow{\phi_q} \{q\} \times F \tag{2.24}$$

is a diffeomorphism. The maps ϕ are referred to as local trivializations and given a point $p \in M$ we say that $\pi^{-1}(p)$ is the fibre above p and denote it by E_p.

Let $\mathbb{F}$ be the field of either the real or the complex numbers. If V is a finite dimensional vector space over $\mathbb{F}$, then we define a vector bundle over M with typical fibre V to be a fibre bundle $E \xrightarrow{\pi} M$ such that for each $p \in M$, the fibre E_p has the structure of a vector space over $\mathbb{F}$ and about each point there exists a local trivialization such that the induced map between fibres is an isomorphism.

As an example of a vector bundle consider the tangent bundle $TM \xrightarrow{\pi} M$ of an n-dimensional manifold M. By taking any local frame $\{e_1, \ldots, e_n\}$ over an open set $U \subset M$ we get an induced map

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n \tag{2.25}$$

by associating to each vector $x \in T_pM$ the pair (p, v) consisting of the basepoint p and the coordinates $v \in \mathbb{R}^n$ of x with respect to the frame. The map ϕ gives an isomorphism on each fibre, and therefore TM is a vector bundle with typical fibre given by $\mathbb{R}^n$.

Principal bundles

A Lie group G is a differentiable manifold which is also a group, such that the map $G \times G \to G$ given by

$$(g, h) \mapsto gh^{-1}, \tag{2.26}$$

for $g, h \in G$, is smooth.

To each element $a \in G$ we have corresponding maps $L_a, R_a : G \to G$ given by left and right multiplication by a, respectively. A vector field X on G is said to be left invariant if $(L_a)^*X = X$.

The Lie algebra of G is defined as the collection of left-invariant vector fields, denoted by $\mathfrak{g}$, and it is a finite-dimensional vector space with a product given by the Lie bracket between vector fields.

A principal G-bundle is a fibre bundle $P \xrightarrow{\pi} M$ with typical fibre G together with a smooth group action on the right $P \times G \to P$ such that the following are satisfied:

i) For each $x \in M$, the group action restricts to a transitive free action $P_x \times G \to P_x$ on the fibre above x.

ii) Given a point $x \in M$, there exists a neighborhood $U \subset M$ and a local trivialization $\phi : \pi^{-1}(U) \to U \times G$ such that

$$\phi(ag) = \phi(a)g, \tag{2.27}$$

for all $g \in G$ and $a \in P$, where the action of G on $U \times G$ is the natural one given by right multiplication on the second coordinate.

A local trivialization ϕ satisfying $ii)$ is said to be equivariant and for each element $g \in G$ the induced right multiplication map is denoted by $R_g : P \to P$.

If $E \to M$ is a vector bundle with typical fibre $\mathbb{R}^n$, then we can construct a fibre bundle $P \xrightarrow{\pi} M$ as follows: If $x \in M$, let the fibre $P_x = \pi^{-1}(x)$ consist of the collection of invertible linear maps

$$GL(\mathbb{R}^n, E_x). \tag{2.28}$$

Then letting

$$P = \cup_{x \in M} P_x \tag{2.29}$$

we have a natural map $\pi : P \to M$ given by mapping elements of P_x to x, for all $x \in M$.

Now suppose that $a \in P_x = GL(\mathbb{R}^n, E_x)$, for some $x \in M$ and $g \in GL(V)$. Then we can take the composition $ag : \mathbb{R}^n \to E_x$ giving a new element in P_x. This gives a group action on the right $P \times GL(\mathbb{R}^n) \to P$, which is clearly transitive and free on each fibre.

If $x \in M$, we can find a neighborhood $U \subset M$ with a local frame $\{e_1, \dots e_n\}$. For each $y \in U$ we can define the map $h_x : E_x \to \mathbb{R}^n$, by sending each vector in E_y to its corresponding tuple of coordinates with respect to the basis $\{e_1, \dots e_n\}$ at y. Now we can define a map $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, by

$$\phi(a) = (\pi(a), h_{\pi_a} \circ a), \tag{2.30}$$

for all $a \in \pi^{-1}(U)$. Clearly then ϕ gives an equivariant local trivialization. If we endow P with the smooth structure defined by the class of such trivializations, then P is a principal $GL(\mathbb{R}^n)$-bundle. Since the elements of a fibre P_x can be identified with bases of E_x, we call $P \to M$ the principal frame bundle of E.

Connection on principal bundles

Given a Lie group G with Lie algebra $\mathfrak{g}$ and a principal G-bundle $P \to M$, then for each point $a \in P$ we have a map

$$\alpha_a : \mathfrak{g} \to T_a P, \tag{2.31}$$

which is given as the derivative of the map

$$G \ni g \mapsto ag. \tag{2.32}$$

For each $a \in P$, the image $\alpha_a(\mathfrak{g}) \subset T_a P$ gives a subspace which we refer to as the vertical vectors at a and denote by V_a.

A connection on P is a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying the following conditions:

i) $\omega \circ \alpha_a$ is the identity on $\mathfrak{g}$, for all $a \in P$.

ii) $(R_g)^*\omega = Ad(g^{-1})\omega$, for all $g \in G$.

Given a connection ω on P, we obtain a distribution

$$H_a = ker(\omega_x), \tag{2.33}$$

for all $a \in P$. The subspaces defined by H are said to be horizontal. One can use the conditions placed upon the connection ω to show that H is right invariant in the sence that

$$(R_g)_* H_a = H_{ag}, \tag{2.34}$$

for all $a \in P$ and $g \in G$.

H gives a notion of parallelism on P. If $c : I \to M$ is any curve in M and a is any point in $\pi^{-1}(c(0))$, then there exists a unique curve $a : I \to P$ such that $a(0) = a$, $\pi(a(t)) = c(t)$ and $a'(t) \in H_{a(t)}$, for all t.

we define the curvature of ω as a $\mathfrak{g}$-valued two form Ω by the relation

$$d\omega(X,Y) = -\frac{1}{2}[\omega(X),\omega(Y)] + \Omega(X,Y), \tag{2.35}$$

for all $X, Y \in TP$, where $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$.

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Paper I

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$\mathcal{I}$-Preserving Diffeomorphisms of Lorentzian Manifolds

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Abstract

We examine the existence of one parameter groups of diffeomorphisms whose infinitesimal generators annihilate all scalar polynomial curvature invariants through the application of the Lie derivative, known as $\mathcal{I}$-preserving diffeomorphisms. Such mappings are a generalization of isometries and appear to be related to nil-Killing vector fields, for which the associated Lie derivative of the metric yields a nilpotent rank two tensor. We show that the set of nil-Killing vector fields contains Lie algebras, although the Lie algebras may be infinite and can contain elements which are not $\mathcal{I}$-preserving diffeomorphisms. We then study the curvature structure of a general Lorenztian manifold, or spacetime to show that $\mathcal{I}$-preserving diffeomorphism will only exists for $\mathcal{I}$-degenerate spacetimes and to determine when the $\mathcal{I}$-preserving diffeomorphisms are generated by nil-Killing vector fields. We identify necessary and sufficient conditions for the degenerate Kundt spacetimes to admit an additional $\mathcal{I}$-preserving diffeomorphism and conclude with an application to the class of Kundt spacetimes with constant scalar polynomial curvature invariants to show that a finite transitive Lie algebra of nil-Killing vector fields always exists for these spacetimes.

1 Introduction

Unlike the Riemannian spaces where the set, $\mathcal{I}$, of all scalar polynomial curvature invariants ($SPIs$):

$$\mathcal{I} = \{R, R_{abcd}R^{abcd}, \ldots, R_{abcd;e}R^{abcd;e}, \ldots\},$$

locally characterize the manifold completely, for the pseudo-Riemannian spaces there exists classes of manifolds which cannot be uniquely characterized locally by their $SPIs$. That is, for any such metric, $\mathbf{g}$, there exists a smooth (one parameter) deformation of the metric, $\tilde{\mathbf{g}}_\tau$, with $\tilde{\mathbf{g}}_0 = \mathbf{g}$ and $\tilde{\mathbf{g}}_\tau$, $\tau > 0$ not diffeomorphic to $\mathbf{g}$ yielding the same set $\mathcal{I}$, such a space is called *$\mathcal{I}$-degenerate* [1, 2].

In the case of a spacetime, i.e., a Lorentzian manifold, $(M, \mathbf{g})$, a more practical definition of $\mathcal{I}$-degeneracy can be stated in terms of the structure of the curvature tensor and its covariant derivatives. To discuss this, we must examine the effect of a boost on the null coframe $\{\mathbf{n}, \boldsymbol{\ell}, \mathbf{m}^i\}$, $\boldsymbol{\ell}' = \lambda\boldsymbol{\ell}$, $\mathbf{n}' = \lambda^{-1}\mathbf{n}$, for which the components of an arbitrary tensor, $\mathbf{T}$, of rank n transform as

$$T'_{a_1a_2\ldots a_n} = \lambda^{b_{a_1a_2\ldots a_n}} T_{a_1a_2\ldots a_n}, \quad b_{a_1a_2\ldots a_n} = \sum_{i=1}^{n}(\delta_{a_i0} - \delta_{a_i1}), \tag{1}$$

where δ_{ab} denotes the Kronecker delta symbol. The quantity, $b_{a_1a_2\ldots a_n}$, is called the *boost weight* (b.w) of the frame component $T_{a_1a_2\ldots a_p}$. Any tensor can be decomposed in terms of the b.w. of its components and this b.w. decomposition gives rise to the *alignment classification*, by identifying null directions relative to which the components of a given tensor have a particular b.w. configuration. This classification reproduces the Petrov and Segre classifications in 4D, and also leads to a coarse classification in higher dimensions [3, 4, 5, 6].

We will define the maximum b.w. of a tensor, $\mathbf{T}$, for a null direction $\boldsymbol{\ell}$ as the boost order, and denote it as $\mathcal{B}_{\mathbf{T}}(\boldsymbol{\ell})$. The Weyl tensor and any rank two tensor, $\mathbf{T}$, can be broadly classified into five *alignment types*: **G**,**I**, **II**, **III**, and **N** if there exists an $\boldsymbol{\ell}$ such that $\mathcal{B}_{\mathbf{T}}(\boldsymbol{\ell}) = 2, 1, 0, -1, -2$ and we will say $\boldsymbol{\ell}$ is $\mathbf{T}$-aligned, while if $\mathbf{T}$ vanishes, then it belongs to alignment type **O**. For higher rank tensors, like the covariant derivatives of the curvature tensor, the alignment types are still applicable despite the possibility that $|\mathcal{B}_{\mathbf{T}}(\boldsymbol{\ell})|$ may be greater than two. Any $\mathcal{I}$-degenerate spacetime admits a null frame such that all of the positive b.w. terms of the curvature tensor and its covariant derivatives are zero in this common frame, that is they are all of alignment type **II**.

A significant subset of the $\mathcal{I}$-degenerate spacetimes are contained in a subclass of the Kundt spacetimes, for which the curvature tensors and its covariant derivatives are of alignment type **II**, known as the *degenerate Kundt spacetimes*. In the three-dimensional (3D) and four-dimensional (4D) cases, all such spacetimes are contained in the degenerate Kundt spacetimes [7]. It is conjectured that any D-dimensional $\mathcal{I}$-degenerate spacetime is a degenerate Kundt spacetime [8].

Of particular interest are those spacetimes where all elements of $\mathcal{I}$ vanish or are constant, such spacetimes are known as *vanishing scalar invariant* (VSI) or *constant scalar invariant* (CSI) spacetimes respectively [9]. The class of CSI spacetimes are applicable to many theories of gravity, as they contain a subset of spacetimes that are universal, and hence solve the vacuum equations of all gravitational theories with a Lagrangian constructed from SPIs [10, 11].

In 3D and 4D, it has been shown that all CSI spacetimes are either locally homogeneous or they belong to the degenerate Kundt class [12, 13], while in higher dimensions it is conjectured that a CSI spacetime will either be locally homogeneous or belong to the degenerate Kundt class [9]. It has been shown that the VSI spacetimes belong to the degenerate Kundt class in all dimensions [14]. The subset of CSI spacetimes belonging to the Kundt class are called *Kundt-CSI*. For Kundt-CSI metrics, the transverse space is a locally homogeneous Riemannian manifold and the metric functions must satisfy particular differential equations [12, 13, 15, 16].

In the Riemannian case, a space is CSI if and only if the space is locally homogeneous. In the Lorentzian case there are Kundt-CSI spacetimes that do not have enough Killing vector fields to determine the CSI property. However, any Kundt-CSI spacetime can be mapped to a related locally homogeneous Kundt-CSI spacetime with the same set $\mathcal{I}$ which provides an explanation for the CSI property [2, 16]. Such a metric is known as a Kundt$^{\infty}$ triple and will be defined in section 6.

The pseudo-Riemannian case admits CSI metrics, with two known classes of metrics containing CSI solutions, namely the Kundt and Walker pseudo-Riemannian metrics [2]. Unlike the Lorentzian case, there exists CSI pseudo-Riemannian spaces which are mapped to simpler CSI spaces lacking a sufficient number of Killing vector fields required to prove the metrics are CSI. In such cases, all possible $SPIs$ up to an appropriate order must be checked explicitly to prove the CSI property. As an example, consider the following neutral signature metric in $4D$:

$$ds^2 = 2du(Vdu + dv) + dU(av^4dU + dV), \tag{2}$$

where a is a constant. Any SPI constructed from the curvature tensor and its covariant derivative up to order 3 all vanish, while all $SPIs$ constructed from the covariant derivatives of the curvature tensor of order $p \geq 3$ are constant.

While this spacetime does not admit a sufficient number of Killing vector fields, it does admit a transitive set of vector fields,

$$\left\{ \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial U}, \frac{\partial}{\partial V} \right\}.$$

For each of these vector fields, the Lie derivative of the metric in the direction of the vector field produces a nilpotent rank 2 tensor, that is, they are *nil-Killing* vector fields [17]. A subset of the nil-Killing vector fields known as Kerr-Schild vector fields have been studied as generators for Kerr-Schild transformations of spacetimes [18]. The Kerr-Schild vector fields have also been used to establish the existence of trapping horizons in 4D spacetimes [19]. Generally the Kerr-schild vector fields are finite dimensional. However, in some cases the Kerr-Schild vector fields can form an infinite dimensional Lie algebra.

In comparison, the four nil-Killing vector fields of the line-element (2) form a finite abelian Lie algebra and the flows of each of the vector fields leave the elements of $\mathcal{I}$ invariant. Such a vector field generalizes the concept of an isometry by preserving SPIs without necessarily being an isometry of the metric, and so the corresponding flow of such a vector field is called an *$\mathcal{I}$-preserving diffeomorphism* (IPD). The associated vector fields of the $IPDs$ can help determine if a spacetime is CSI without explicitly checking all SPIs [20]. Motivated by this example, it is of interest to determine a simple criteria to identify nil-Killing vector fields which generate diffeomorphisms that preserve the set $\mathcal{I}$ for a given metric.

The paper is organized as follows. In section 2, we determine the general form of a nilpotent self-adjoint operator and relate the choice of frame basis to a preferred null direction, to give a more precise definition for the nil-Killing vector fields. We also show that the nil-Killing vector fields that generalize the Kerr-Schild vector fields form a Lie algebra, and that other Lie algebras can potentially exist depending on the choice of additional conditions for the nil-Killing vector fields. In section 3, we examine the structure of the curvature invariants for a generic spacetime to determine the existence of $IPDs$ and show they can only exist in $\mathcal{I}$-degenerate spacetimes [7]. In section 4, we employ a frame based approach to determine when a nil-Killing vector field gives rise to an IPD and whether $IPDs$ exist whose infinitesimal generators are not nil-Killing vector fields. In section 5, we consider a general degenerate Kundt spacetime and establish conditions that must be satisfied in order to admit an additional IPD. In section 6, we apply the results of section 5 to the Kundt-CSI spacetimes and prove a finite transitive Lie algebra of nil-Killing vector fields which generate $IPDs$ always exists. We summarize our results in section 7 and discuss the existence of $IPDs$ for $\mathcal{I}$-degenerate pseudo-Riemannian manifolds of different signatures.

2 Nilpotent Operators and Nil-Killing Vectors

In this section we will introduce some general results about nilpotent operators and relate these results to the alignment classification [3, 4, 5], in order to give a more precise definition of a nil-Killing vector field.

Proposition 2.1. *For a spacetime, $(M, \mathbf{g})$, given $\mathbf{T} : T_pM \to T_pM$, a self-adjoint endomorphism at an arbitrary point $p \in M$, then*

1. $\mathbf{T}^2 = 0$ *if and only if there exists a null vector, $\boldsymbol{\ell}$, such that* $\mathbf{T}(\{\boldsymbol{\ell}\}^\perp) = 0$ *where $\{\boldsymbol{\ell}\}^\perp$ denotes the orthogonal vector space to $\boldsymbol{\ell}$.*
2. $\mathbf{T}^3 = 0$ *if and only if there exists a null vector $\boldsymbol{\ell}$ such that* $\mathbf{T}(T_pM) \subset \{\boldsymbol{\ell}\}^\perp$ *and* $\mathbf{T}(\{\boldsymbol{\ell}\}^\perp) \subset \mathbb{R}\boldsymbol{\ell}$.
3. $\mathbf{T}$ *is nilpotent if and only if* $\mathbf{T}^3 = 0$.

Proof. 1. Supposing that $\mathbf{X} \in T_pM$, then $g(\mathbf{TX}, \mathbf{TX}) = g(\mathbf{T}^2\mathbf{X}, \mathbf{X}) = 0$, and so $\mathbf{TX}$ is a null vector with $\mathbf{TX} \propto \boldsymbol{\ell}$ for some null vector. If $\mathbf{W} \in \{\boldsymbol{\ell}\}^\perp$ and $\mathbf{Z} \in T_pM$, we can write $\mathbf{TZ} = c\boldsymbol{\ell}$ for some constant $c \in \mathbb{R}$, then

$$\mathbf{g}(\mathbf{T}\boldsymbol{W}, \mathbf{Z}) = \mathbf{g}(\boldsymbol{W}, \mathbf{TZ}) = \mathbf{g}(\mathbf{W}, c\boldsymbol{\ell}) = 0,$$

therefore $\mathbf{TW} = 0$ and $\mathbf{T}(\{\boldsymbol{\ell}\}^\perp) = 0$. To show the other direction, suppose that $\mathbf{Z} \in T_pM$ and $\mathbf{W} \in \{\boldsymbol{\ell}\}^\perp$ then $\mathbf{g}(\mathbf{Z}, \mathbf{TW}) = 0$, and so $\mathbf{T}(T_pM) \subset \mathbb{R}\boldsymbol{\ell}$ which implies $\mathbf{T}^2 = 0$.

2. We will assume $\mathbf{T}^2 \neq 0$ and $\mathbf{T}^3 = 0$ to avoid the first part of the proof. Using the fact that $(\mathbf{T}^2)^2 = 0$, there must be some null vector $\boldsymbol{\ell} \in T_pM$ such that $\mathbf{T}^2(\{\boldsymbol{\ell}\}^\perp) = 0$. Given $\mathbf{Z} \in T_pM$,

$$\mathbf{g}(\mathbf{T}^2\mathbf{Z}, \mathbf{W}) = \mathbf{g}(\mathbf{Z}, \mathbf{T}^2\mathbf{W}) = 0, \ \forall \ \mathbf{W} \in \{\boldsymbol{\ell}\}^\perp.$$

It follows that $\mathbf{T}^2(T_pM) = \mathbb{R}\boldsymbol{\ell}$, and since $\mathbf{T}^3 = 0$, it is necessary that $\mathbf{T}\boldsymbol{\ell} = 0$ and $\mathbf{T}(T_pM) \subset \{\boldsymbol{\ell}\}^\perp$ since

$$\mathbf{g}(\mathbf{TZ}, \boldsymbol{\ell}) = \mathbf{g}(\mathbf{Z}, \mathbf{T}\boldsymbol{\ell}) = 0, \forall \ \mathbf{Z} \in T_pM.$$

If $\mathbf{W} \in \{\boldsymbol{\ell}\}^\perp$ then, $g(\mathbf{T}^2\mathbf{W}, \mathbf{W}) = \mathbf{g}(\mathbf{TW}, \mathbf{TW}) = 0$, so that $\mathbf{T} : \{\boldsymbol{\ell}\}^\perp \to \{\boldsymbol{\ell}\}^\perp$ and $\mathbf{T}(\{\boldsymbol{\ell}\}^\perp) = \mathbb{R}\boldsymbol{\ell}$. To prove the other direction, we note that $\mathbf{T}(T_pM) \subset \{\boldsymbol{\ell}\}^\perp$ and so $\mathbf{g}(\mathbf{TZ}, \boldsymbol{\ell}) = \mathbf{g}(\mathbf{Z}, \mathbf{T}\boldsymbol{\ell}) = 0, \forall \ \mathbf{Z} \in T_pM$ implying $\mathbf{T}\boldsymbol{\ell} = 0$, and hence $\mathbf{T}^3(T_pM) \subset \mathbf{T}^2(\{\boldsymbol{\ell}\}^\perp) \subset \mathbb{R}\mathbf{T}\boldsymbol{\ell} = 0$.

3. Supposing that $\mathbf{T}^n = 0$ with $\mathbf{T}^{n-1} \neq 0$ for some $n \geq 3$, then $(\mathbf{T}^{(n-1)})^2 = 0$ and from (1) there is a null-vector $\boldsymbol{\ell}$ such that $\mathbf{T}^{n-1}(\{\boldsymbol{\ell}\}^\perp) = 0$. If $\mathbf{T}(T_pM) \not\subset \{\boldsymbol{\ell}\}^\perp$, then there is some non-zero $\mathbf{Z} \in T_pM$ for which $\mathbf{TZ} \notin \{\boldsymbol{\ell}\}^\perp$ giving the identity,

$$\mathbf{T}^{(n-1)}(\mathbf{TZ}) = \mathbf{T}^n\mathbf{Z} = 0.$$

This implies $\mathbf{T}^{n-1} = 0$ which is a contradiction and so $\mathbf{T}(T_pM) \subset \{\boldsymbol{\ell}\}^\perp$. Using this fact and $\mathbf{g}(\mathbf{T}\boldsymbol{\ell}, \mathbf{Z}) = \mathbf{g}(\boldsymbol{\ell}, \mathbf{TZ}) = 0$ implies that $\mathbf{T}\boldsymbol{\ell} = 0$.

With $\boldsymbol{\ell}$ we can construct a null coframe, $\{\mathbf{n}, \mathbf{m}^i, \boldsymbol{\ell}\}$ so that the metric is of the form

$$\mathbf{g} = 2\boldsymbol{\ell}\mathbf{n} + \delta_{ij}\mathbf{m}^i\mathbf{m}^j$$

The self-adjoint operator $\mathbf{T}$ with $\mathbf{T}\boldsymbol{\ell} = 0$ will have the matrix representation

$$\mathbf{T} = \begin{bmatrix} 0 & \mathbf{v}^T & c \\ \vdots & \mathbf{S} & \mathbf{v} \\ 0 & \dots & 0 \end{bmatrix}$$

where $\mathbf{v}$ is a $(n-2)$-dimensional vector, $\mathbf{S}$ is a symmetric $(n-2) \times (n-2)$ matrix and c is real-valued. For any power k of $\mathbf{T}$, there is some vector $\mathbf{v}_k$ and real number c_k such that

$$\mathbf{T}^k = \begin{bmatrix} 0 & \mathbf{v}_k^T & c_k \\ \vdots & \mathbf{S}^k & \mathbf{v}_k \\ 0 & \dots & 0 \end{bmatrix}$$

If $\mathbf{T}^n = 0$ then $\mathbf{S}^n = 0$, since $\mathbf{S}$ is symmetric and is spanned by tensor products of spatial vectors, this implies $\mathbf{S} = 0$ and so for any element of $\{\boldsymbol{\ell}\}^\perp$, $\mathbf{T}\{\boldsymbol{\ell}\}^\perp \subset \mathbb{R}\boldsymbol{\ell}$, giving $\mathbf{T}^3 = 0$. We note that if there are two linearly independent null vectors with property (1) or (2) then $\mathbf{T} = 0$.

□

From Proposition 2.1, we can give the following definition that motivates the use of the alignment classification.

Definition 2.2. A self-adjoint endomorphism $\mathbf{T}$ of the tangent space T_pM of a spacetime is nilpotent with respect to a null vector $\boldsymbol{\ell}$ if $\mathbf{T}(T_pM) \subset \{\boldsymbol{\ell}\}^\perp$ and $\mathbf{T}\boldsymbol{\ell} = 0$. For a particular null vector $\boldsymbol{\ell}$, the collection of self-adjoint nilpotent with respect to $\boldsymbol{\ell}$ will be denoted as $S_{\boldsymbol{\ell}}(T_pM, \mathbf{g})$

This idea can be extended to a symmetric rank two tensor field on a spacetime through the identity

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) = \mathbf{g}(\hat{\mathbf{T}}\mathbf{X}, \mathbf{Y}), \quad \forall\ \mathbf{X},\ \mathbf{Y} \in \mathfrak{X}(M),$$

such that $\forall\ p \in M$ the endomorphism of T_pM, $\hat{\mathbf{T}}$, is self-adjoint with respect to the metric. If $\mathbf{T}$ is nilpotent at a point $p \in M$, then there is a corresponding null vector field $\boldsymbol{\ell} \in T_pM$ for which $\mathbf{T}|_p$ can be decomposed in terms of elements of

$\{\boldsymbol{\ell}\}^{\perp}$. That is, $\mathbf{T}$ is nilpotent with respect to $\boldsymbol{\ell}$ if $\hat{\mathbf{T}}$ is nilpotent with respect to $\boldsymbol{\ell}$ which is equivalent to $\mathbf{T}(\boldsymbol{\ell}, \mathbf{Z}) = 0, \ \forall\, \mathbf{Z} \in T_pM$ and $\mathbf{T}(\mathbf{W}, \tilde{\mathbf{W}}) = 0, \ \forall\, \mathbf{W}, \tilde{\mathbf{W}} \in \{\boldsymbol{\ell}\}^{\perp}$.

Due to the smoothness of the manifold, this can be extended in a neighbourhood U of $p \in M$, and so we say a symmetric rank two tensor-field is nilpotent with respect to a null vector field $\boldsymbol{\ell}$, if $\mathbf{T}|_p$ is nilpotent with respect to $\boldsymbol{\ell}|_p$ for all $p \in U$. Completing the null coframe with $\boldsymbol{\ell}$ as a basis element, $\{\mathbf{n}, \boldsymbol{\ell}, \mathbf{m}^i\}$, any nilpotent rank 2 tensor with respect to $\boldsymbol{\ell}$ can be written as

$$\mathbf{T} = T_{11}\boldsymbol{\ell}\boldsymbol{\ell} + 2T_{1i}\boldsymbol{\ell}\mathbf{m}^i. \tag{3}$$

With this definition, we can define a more precise definition of a nil-Killing vector field [17]:

Definition 2.3. For a spacetime $(M, \mathbf{g})$, a vector field $\mathbf{X} \in \mathfrak{X}(M)$ is nil-Killing with respect to $\boldsymbol{\ell}$ if $\mathcal{L}_{\mathbf{X}}\mathbf{g} \in S_{\boldsymbol{\ell}}(T_pM, \mathbf{g})$.

Note that in the Riemannian case, this can only occur if $\mathbf{X}$ is Killing. In the literature, a specialization of the nil-Killing vector fields known as the Kerr-Schild vector fields, has been discussed [18, 19] these are defined as nil-Killing vector fields with respect to $\boldsymbol{\ell}$ for which $\mathcal{L}_{\mathbf{X}}\mathbf{g} = \mathbf{T}$ is nilpotent of order two, $\mathbf{T}^2 = 0$ with the additional condition:

$$[\mathbf{X}, \boldsymbol{\ell}] \propto \boldsymbol{\ell}. \tag{4}$$

This additional condition allows nil-Killing vector fields to act as automorphisms on $S_{\boldsymbol{\ell}}(T_pM, \mathbf{g})$.

Lemma 2.4. *Given a non-vanishing null vector field $\boldsymbol{\ell}$ in a spacetime $(M, \mathbf{g})$ and a nil-Killing vector field, $\mathbf{X}$, with respect to $\boldsymbol{\ell}$ satisfying*

$$[\mathbf{X}, \boldsymbol{\ell}] = f\boldsymbol{\ell}, f \in C^{\infty}(M).$$

If $\mathbf{T} \in S_{\boldsymbol{\ell}}(T_pM, \mathbf{g})$ then $\mathcal{L}_{\mathbf{X}}\mathbf{T} \in S_{\boldsymbol{\ell}}(T_pM, \mathbf{g})$.

Proof. Suppose that $\mathbf{W} \in \{\boldsymbol{\ell}\}^{\perp}$, then the conditions that $\mathbf{X}$ is nil-Killing in Proposition 2.1 implies that $\mathcal{L}_{\mathbf{X}}\mathbf{g}(\boldsymbol{\ell}, \{\boldsymbol{\ell}\}^{\perp}) = 0$ and $\mathcal{L}_{\mathbf{X}}\mathbf{g}(\{\boldsymbol{\ell}\}^{\perp}, \{\boldsymbol{\ell}\}^{\perp}) = 0$ (we have made a minor abuse of notation to treat $\mathcal{L}_{\mathbf{X}}\mathbf{g}$ as the corresponding nilpotent operator), the condition (4) implies that $[\mathbf{X}, \mathbf{W}] \in \{\boldsymbol{\ell}\}^{\perp}$ since

$$0 = \mathcal{L}_{\mathbf{X}}\mathbf{g}(\boldsymbol{\ell}, \mathbf{W}) = -\mathbf{g}([\mathbf{X}, \boldsymbol{\ell}], \mathbf{W}) - \mathbf{g}(\boldsymbol{\ell}, [\mathbf{X}, \mathbf{W}]) = -\mathbf{g}(\boldsymbol{\ell}, [\mathbf{X}, \mathbf{W}]).$$

For any $\mathbf{Z} \in \mathfrak{X}(M)$ and $\mathbf{W}, \tilde{\mathbf{W}} \in \{\boldsymbol{\ell}\}^{\perp}$ this implies

$$\mathcal{L}_{\mathbf{X}}\mathbf{T}(\boldsymbol{\ell}, \mathbf{Z}) = \mathbf{X}(\mathbf{T}(\boldsymbol{\ell}, \mathbf{Z})) - \mathbf{T}([\mathbf{X}, \boldsymbol{\ell}], \mathbf{Z}) - \mathbf{T}(\boldsymbol{\ell}, [\mathbf{X}, \mathbf{Z}]) = 0, \tag{5}$$

and

$$\mathcal{L}_{\mathbf{X}}\mathbf{T}(\mathbf{W}, \tilde{\mathbf{W}}) = \mathbf{X}(\mathbf{T}(\mathbf{W}, \tilde{\mathbf{W}})) - \mathbf{T}([\mathbf{X}, \mathbf{W}], \tilde{\mathbf{W}}) - \mathbf{T}(\mathbf{W}, [\mathbf{X}, \tilde{\mathbf{W}}]) = 0. \tag{6}$$

Therefore $\mathcal{L}_{\mathbf{X}}\mathbf{T}$ is also nilpotent with respect to $\boldsymbol{\ell}$. □

We can now show that the set of nil-Killing vector fields satisfying (4) form a Lie algebra and not just the Kerr-Schild vector fields.

Proposition 2.5. *For any spacetime* $(M, \mathbf{g})$ *and* $\boldsymbol{\ell}$ *a null vector field then*

$$\mathfrak{g}_{\boldsymbol{\ell}} := \{\mathbf{X} \in \mathfrak{X}(M) | [\mathbf{X}, \boldsymbol{\ell}] \propto \boldsymbol{\ell},\ \mathbf{X} \text{ is nil-Killing with respect to } \boldsymbol{\ell}\}$$

is a Lie algebra and

$$\mathfrak{h}_{\boldsymbol{\ell}} := \{\mathbf{X} \in \{\boldsymbol{\ell}\}^{\perp} | [\mathbf{X}, \boldsymbol{\ell}] \propto \boldsymbol{\ell},\ \mathbf{X} \text{ is nil-Killing with respect to } \boldsymbol{\ell}\}$$

is an ideal.

Proof. Suppose that $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}_{\boldsymbol{\ell}}$, then by assumption $\mathcal{L}_{\mathbf{Y}}\mathbf{g}$ is nilpotent with respect to $\boldsymbol{\ell}$, and so by Proposition 2.1 $\mathcal{L}_{\mathbf{X}}(\mathcal{L}_{\mathbf{Y}}\mathbf{g})$ is nilpotent since $[\mathbf{X}, \boldsymbol{\ell}] \propto \boldsymbol{\ell}$. Repeating this argument with $\mathbf{X}$ and $\mathbf{Y}$ switched gives another nilpotent operator, and the difference

$$\mathcal{L}_{[\mathbf{X},\mathbf{Y}]}\mathbf{g} = \mathcal{L}_{\mathbf{X}}(\mathcal{L}_{\mathbf{Y}}\mathbf{g}) - \mathcal{L}_{\mathbf{Y}}(\mathcal{L}_{\mathbf{X}}\mathbf{g})$$

must be nilpotent with respect to $\boldsymbol{\ell}$ as well. While $[\mathbf{X}, \mathbf{Y}]$ is nil-Killing with respect to $\boldsymbol{\ell}$ the condition (4) must be preserved also. Supposing that $f_1, f_2 \in C^{\infty}(M)$ such that $[\mathbf{X}, \boldsymbol{\ell}] = f_1\boldsymbol{\ell}$ and $[\mathbf{Y}, \boldsymbol{\ell}] = f_2\boldsymbol{\ell}$ the Jacobi identity gives

$$\begin{aligned} [[\mathbf{X}, \mathbf{Y}], \boldsymbol{\ell}] &= -[[\mathbf{Y}, \boldsymbol{\ell}], \mathbf{X}] - [[\boldsymbol{\ell}, \mathbf{X}], \mathbf{Y}] = [\mathbf{X}, f_2\boldsymbol{\ell}] - [\mathbf{Y}, f_1\boldsymbol{\ell}] \\ &= (\mathbf{X}(f_2) - \mathbf{Y}(f_1))\boldsymbol{\ell}, \end{aligned}$$

therefore $[\mathbf{X}, \mathbf{Y}] \in \mathfrak{g}_{\boldsymbol{\ell}}$ and $\mathfrak{g}_{\boldsymbol{\ell}}$ is a Lie algebra

Suppose now that $\mathbf{X} \in \mathfrak{g}_{\boldsymbol{\ell}}$ and $\mathbf{Y} \in \mathfrak{h}_{\boldsymbol{\ell}}$, then $[\mathbf{X}, \mathbf{Y}] \in \{\boldsymbol{\ell}\}^{\perp}$ and so $[\mathbf{X}, \mathbf{Y}] \in \mathfrak{h}_{\boldsymbol{\ell}}$ implying that $\mathfrak{h}_{\boldsymbol{\ell}}$ is an ideal. □

If $\{\boldsymbol{\ell}\}^{\perp}$ is integrable, the condition in equation (4) can be relaxed for $\mathfrak{h}_{\boldsymbol{\ell}}$. This will be particularly important for the degenerate Kundt spacetimes which admit an integrable $\{\boldsymbol{\ell}\}^{\perp}$ and cannot be uniquely characterized locally by their SPIs. For such spacetimes, we expect that a subset of the nil-Killing vector fields to give rise to $IPDs$ and that they should form a Lie algebra.

Corollary 2.6. *For any spacetime* $(M, \mathbf{g})$ *and* $\boldsymbol{\ell}$ *a null vector field, if* $\{\boldsymbol{\ell}\}^{\perp}$ *is integrable then*

$$\mathfrak{h}_{\boldsymbol{\ell}} := \{\mathbf{X} \in \{\boldsymbol{\ell}\}^{\perp} | \mathbf{X} \text{ is nil-Killing with respect to } \boldsymbol{\ell}\} \tag{7}$$

is a Lie algebra.

Proof. If $\{\boldsymbol{\ell}\}^{\perp}$ is integrable, and

$$\mathbf{Z} \in \{\mathbf{X} \in \{\boldsymbol{\ell}\}^{\perp} | \mathbf{X} \text{ is nil-Killing with respect to } \boldsymbol{\ell}\},$$

then $[\mathbf{Z}, \mathbf{W}] \in \{\boldsymbol{\ell}\}^{\perp},\ \forall\ \mathbf{W} \in \{\boldsymbol{\ell}\}^{\perp}$. Since $\mathbf{Z}$ is nil-Killing with respect to $\boldsymbol{\ell}$ it follows that

$$0 = \mathcal{L}_{\mathbf{Z}}\mathbf{g}(\boldsymbol{\ell}, \mathbf{W}) = -\mathbf{g}([\mathbf{Z}, \boldsymbol{\ell}], \mathbf{W}) - \mathbf{g}(\boldsymbol{\ell}, [\mathbf{X}, \mathbf{W}]) = -\mathbf{g}([\mathbf{Z}, \boldsymbol{\ell}], \mathbf{W}),$$

and so $[\mathbf{Z}, \boldsymbol{\ell}] \propto \boldsymbol{\ell}$ and $\mathbf{Z} \in \mathfrak{h}_{\boldsymbol{\ell}}$. The converse inclusion is trivial, and hence $\mathfrak{h}_{\boldsymbol{\ell}}$ is a Lie algebra. □

Alternatively, we can relax the condition in equation (4) and instead consider any nil-Killing vector field, $\mathbf{X}$, for which $\mathbf{T} = \mathcal{L}_{\mathbf{X}}\mathbf{g}$ is nilpotent with respect to $\boldsymbol{\ell}$ and $\mathbf{T}^2 = 0$. We will say $\mathbf{X}$ *is a nil-Killing vector field with respect to* $\boldsymbol{\ell}$ *of order two.*

Proposition 2.7. *Given a spacetime* $(M, \mathbf{g})$ *and a non-vanishing null vector field* $\boldsymbol{\ell}$. *If* $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$ *are nil-Killing with respect to* $\boldsymbol{\ell}$ *of order two, then* $[\mathbf{X}, \mathbf{Y}]$ *is nil-Killing with respect to* $\boldsymbol{\ell}$.

Proof. From Proposition 2.1, a vector field $\mathbf{Z}$ is nil-Killing with respect to $\boldsymbol{\ell}$ of order two if

$$\mathcal{L}_{\mathbf{Z}}\mathbf{g}(\mathbf{W}, \mathbf{P}), \quad \forall\ \mathbf{W} \in \{\boldsymbol{\ell}\}^{\perp},\ \mathbf{P} \in \mathfrak{X}(M).$$

In addition for any nil-Killing vector field, $\mathbf{Z}$, with respect to $\boldsymbol{\ell}$, $[\mathbf{Z}, \boldsymbol{\ell}] \in \{\boldsymbol{\ell}\}^{\perp}$ since

$$\mathcal{L}_{\mathbf{Z}}\mathbf{g}(\boldsymbol{\ell}, \boldsymbol{\ell}) = -2\mathbf{g}([\mathbf{Z}, \boldsymbol{\ell}], \boldsymbol{\ell}) = 0.$$

Using these facts it follows that

$$\mathcal{L}_{[\mathbf{X},\mathbf{Y}]}\mathbf{g}(\boldsymbol{\ell}, \mathbf{P}) = 0, \quad \forall\ \mathbf{P} \in \mathfrak{X}(M),$$
$$\mathcal{L}_{[\mathbf{X},\mathbf{Y}]}\mathbf{g}(\mathbf{W}, \tilde{\mathbf{W}}) = 0, \quad \forall\ \mathbf{W},\ \tilde{\mathbf{W}} \in \{\boldsymbol{\ell}\}^{\perp}$$

From Proposition 2.1, this implies that $[\mathbf{X}, \mathbf{Y}]$ is nil-Killing with respect to $\boldsymbol{\ell}$. □

While $[\mathbf{X}, \mathbf{Y}]$ is nil-Killing with respect to $\boldsymbol{\ell}$ it may no longer be a nil-Killing vector field of order two, and so nil-Killing vector fields of this type do not form a Lie algebra without imposing additional conditions on the metric or the set of nil-Killing vector fields $\mathbf{X}$ and $\mathbf{Y}$. For example, in the case of Kerr-Schild vector fields requiring $[\mathbf{X}, \boldsymbol{\ell}] \propto \boldsymbol{\ell}$ and $[\mathbf{Y}, \boldsymbol{\ell}] \propto \boldsymbol{\ell}$ forces $[\mathbf{X}, \mathbf{Y}]$ to be a nil-Killing vector field of order two and hence forms a Lie algebra.

This suggest that there are other Lie algebras within the set of nil-Killing vector fields. It is of interest to determine if a condition can be imposed to produce a finite Lie algebra for the nil-Killing vector fields. We are primarily interested in determining a finite Lie algebra of nil-Killing vector fields that generate a transitive set of $IPDs$, as such we will employ our characterization of nil-Killing vector fields to determine when a nil-Killing vector field preserves the set $\mathcal{I}$.

3 Existences of $\mathcal{I}$-Preserving Diffeomorphisms

Due to our interest in $IPDs$, we would like to find all vector fields $\mathbf{X}$ such that

$$\mathcal{L}_{\mathbf{X}} I = 0,\ \forall\ I \in \mathcal{I}$$

and which are **not** Killing vector fields, we will call $\mathbf{X}$ an *IPD infinitesimal generator*, or an *IPD vector field*. In order to do so, we will employ an alternative set of invariants that locally characterize a spacetime uniquely: the *Cartan invariants*, $\mathcal{R}^q$, which are the components of the curvature tensor and its covariant derivatives relative to a particular frame determined by the Cartan-Karlhede algorithm. A review of the Cartan-Karlhede algorithm is outside the scope of the current paper, we will refer to Chapter 9 of [21] for the 4D implementation of the algorithm and [22, 23] for a discussion of the algorithm in five and higher dimensions. An IPD vector field exists when the set of Cartan invariants $\mathcal{R}^q$ has a larger rank (i.e., the number of functionally independent components) than $\mathcal{I}$,

$$rank(\mathcal{R}^q) > rank(\mathcal{I}).$$

This condition implies that the spacetime is not locally characterized uniquely by its $SPIs$. In 3D and 4D, such metrics must belong to the degenerate Kundt class and the curvature tensor and its covariant derivatives must be of type **II** to all orders [7]. In higher dimensions it is conjectured that this will be the case as well. Denoting $[\mathcal{R}]_{b.w.0}$ as the set of components of the curvature tensor and its covariant derivatives of b.w. zero, we can introduce an alternative criteria for the existence of IPD vector fields for all $\mathcal{I}$-degenerate spacetimes using the alignment classification without generating the entire set $\mathcal{I}$.

Theorem 3.1. *Relative to the basis determined by the Cartan-Karlhede algorithm, a spacetime admits a non-trivial IPD vector field,* $\mathbf{X}$*, such that*

$$\mathcal{L}_{\mathbf{X}}\mathcal{I} = 0, \tag{8}$$

if and only if the spacetime is of alignment type **II** *to all orders and*

$$0 \leq rank([\mathcal{R}^q]_{b.w.0}) < rank(\mathcal{R}^q).$$

That is, the spacetime is $\mathcal{I}$-degenerate.

Proof. If a non-trivial IPD, $\mathbf{X}$, exists then we may choose local coordinates where $\mathbf{X} = \frac{\partial}{\partial x}$ implying that the $SPIs$ are independent of x. We note that since we have assumed that $\mathbf{X}$ is a non-trivial IPD vector field, there must exist some Cartan invariant which is dependent on the x coordinate [21]. As we cannot express the x coordinate in term of $SPIs$, the $SPIs$ are unable to distinguish orbits of $\mathbf{X}$, and so they do not uniquely characterize the spacetime. Thus, the spacetime is necessarily $\mathcal{I}$-degenerate, and Corollary 3.4 in [24] implies that the curvature tensor and its covariant derivatives cannot be of alignment type **I** or **G** at any order. That is, the spacetime is at least of alignment type **II** to all orders. For any such spacetime, the components of b.w. zero of the curvature tensor are determined by the $SPIs$ (Corollary II.11 in [8]) and hence

$$0 \leq rank([\mathcal{R}^q]_{b.w.0}) < rank(\mathcal{R}^q).$$

The opposite direction follows from the fact that when constructing a complete contraction of any tensor of type **II**, only the b.w. 0 components contribute to the resulting SPI.

□

The dimension of the Lie group of isometries, G, can be computed from the Cartan-Karlhede algorithm using the formula:

$$dim(G) = D - I_q + dim(H_q),$$

where D is the dimension of the manifold, q is the final iteration of the algorithm, I_q is the number of functionally independent Cartan invariants and H_q is the linear isotropy group. Motivated by this result we can determine the dimension, m, of the subset of the tangent space spanned by all non-trivial IPD vector fields by taking the difference:

$$m = I_q - rank([\mathcal{R}^q]_{b.w.0}).$$

For example, for a generic degenerate Kundt spacetime admitting no additional isometries, $m = 1$, whereas for a CSI spacetime that is not locally homogeneous $m = D$.

4 The Nil-Killing Condition and IPD Vector Fields

Assuming the spacetime is $\mathcal{I}$-degenerate, let us consider the condition introduced in [17] to study the set of IPD vector fields,

$$\mathcal{L}_{\mathbf{X}}\mathbf{g} = \mathbf{N},$$

where $\mathbf{N}$ is a nilpotent rank two tensor. For any nilpotent operator, there is a related null direction, $\boldsymbol{\ell}$, as illustrated in equation (3). Let us choose $\boldsymbol{\ell}$ as a coframe basis element, and complete the coframe basis $\{\boldsymbol{\theta}^a\} = \{\mathbf{n}, \boldsymbol{\ell}, \mathbf{m}^i\}$ then we can consider the effect of a Lie derivative in the direction of $\mathbf{X}$ on the coframe basis:

$$\begin{aligned} \mathcal{L}_{\mathbf{X}}\boldsymbol{\ell} &= A\boldsymbol{\ell} + \tilde{A}\mathbf{n} + B_i\mathbf{m}^i, \\ \mathcal{L}_{\mathbf{X}}\mathbf{n} &= C\boldsymbol{\ell} + \tilde{C}\mathbf{n} + D_i\mathbf{m}^i, \\ \mathcal{L}_{\mathbf{X}}\mathbf{m}^i &= E^i\boldsymbol{\ell} + \tilde{E}^i\mathbf{n} + F^i{}_j\mathbf{m}^j, \end{aligned} \tag{9}$$

where the coefficients are functions of the coordinates. Imposing the condition that $\mathcal{L}_{\mathbf{X}}\mathbf{g}$ is nilpotent implies that this symmetric tensor must only have non-zero components with negative b.w. which puts conditions on the coefficients

$$\tilde{A} = 0,\ \tilde{C} = -A,\ \tilde{E}^i = -B^i \text{ and } F_{ij} = -F_{ji}, \tag{10}$$

and so

$$\begin{aligned} \mathcal{L}_{\mathbf{X}}\boldsymbol{\ell} &= A\boldsymbol{\ell} + B_i\mathbf{m}^i, \\ \mathcal{L}_{\mathbf{X}}\mathbf{n} &= C\boldsymbol{\ell} - A\mathbf{n} + D_i\mathbf{m}^i, \\ \mathcal{L}_{\mathbf{X}}\mathbf{m}^i &= E^i\boldsymbol{\ell} - B^i\mathbf{n} + F^i{}_j\mathbf{m}^j, \quad F_{(ij)} = 0. \end{aligned} \tag{11}$$

We will now focus our attention on nil-Killing vector fields such that $\mathcal{L}_{\mathbf{X}}\mathbf{g}$ is nilpotent with respect to the vector field $\boldsymbol{\ell}$ for which the Riemann tensor and its covariant derivatives are of type **II** or higher [7]. Using abstract index notation briefly, we will consider the subset of these nil-Killing vector fields which also annihilate $SPIs$ constructed from an arbitrary rank two symmetric curvature tensor R_{ab} (such as the Ricci tensor or $C_{abcd;e}C^{abcd;f}$ as two examples) with the simplest SPI, the contraction

$$I = R^a{}_a.$$

In general, for a degenerate Kundt spacetime, the Riemann tensor and its covariant derivatives are of alignment type **II**, and so the Ricci and Weyl tensor are at least of alignment type **II**. We will assume it is possible to construct at least one rank two tensor of alignment type **II**. We note that this analysis will be restricted to degenerate Kundt spacetimes of alignment type **II**, **III** and **N** to all orders, and that the subclass of metrics which have alignment type **D** to all orders, known as type $\mathbf{D}^k$ will be excluded. Such metrics are $\mathcal{I}$-degenerate but are characterized by their $SPIs$, although not uniquely, since $rank([\mathcal{R}]_{b.w.0}) = rank(\mathcal{R})$.

If $\mathcal{L}_{\mathbf{X}} I = 0$, then the trace of $\mathcal{L}_{\mathbf{X}} R_{ab}$ is zero since the Lie derivative commutes with contraction. In order to avoid the possibility that $\mathcal{L}_{\mathbf{X}} R_{ab}$ could be trace-free for some choices of R_{ab}, we assume the following property holds:

Definition 4.1. A spacetime is *generic of type* **II**, **D**, **III** or **N** if the set of rank two curvature tensors spans the vector space of rank two tensors of alignment type **II**, **D**, **III** or **N** respectively.

For any $\mathcal{I}$-degenerate spacetime which is generic of type **II**, corollary II.11 in [8] gives a necessary condition for the vanishing of the trace of $\mathcal{L}_{\mathbf{X}} R_{ab}$:

$$[\mathcal{L}_{\mathbf{X}} R_{ab}]_{b.w.0} = 0,$$

or in standard notation,

$$[\mathcal{L}_{\mathbf{X}}(R_{ab}\boldsymbol{\theta}^a\boldsymbol{\theta}^b)]_{b.w.0} = 0.$$

That is, $\mathcal{L}_{\mathbf{X}} R_{ab}$ is of type **III** as all b.w. zero components must vanish.

Imposing the condition that

$$\mathcal{L}_{\mathbf{X}}(R_{ab}\boldsymbol{\theta}^a\boldsymbol{\theta}^b) = \mathbf{X}(R_{ab})\boldsymbol{\theta}^a\boldsymbol{\theta}^b + 2R_{ab}\boldsymbol{\theta}^a\mathcal{L}_{\mathbf{X}}\boldsymbol{\theta}^b$$

is of type **III**, we note that $\mathbf{X}(R_{ab})\boldsymbol{\theta}^a\boldsymbol{\theta}^b$ will only contribute negative b.w. terms to this tensor sum. Therefore, we have additional conditions on the Lie derivative of the basis (11) by $\mathbf{X}$:

$$\begin{aligned}
&\mathcal{L}_{\mathbf{X}}\boldsymbol{\ell} = A\boldsymbol{\ell}, \\
&\mathcal{L}_{\mathbf{X}}\mathbf{n} = C\boldsymbol{\ell} - A\mathbf{n} + D_i\mathbf{m}^i, \\
&\mathcal{L}_{\mathbf{X}}\mathbf{m}^i = E^i\boldsymbol{\ell} + F^i_{\ j}\mathbf{m}^j, \quad F_{(ij)} = 0
\end{aligned} \tag{12}$$

where F_{ij} satisfies the supplemental condition that $R_{(i|j}F^j_{\ k)} = 0$. This will hold for all symmetric rank two tensors that can be constructed from the curvature tensor and its covariant derivatives. Furthermore, since $\mathcal{L}_{\mathbf{X}} R_{ab}$ is of type **III** and R_{ab} is at least of type **II**, any SPI constructed from contractions of copies of R_{ab} will vanish under $\mathcal{L}_{\mathbf{X}}$ due to the Liebnitz property and the fact that the Lie derivative of the tensor product, $\mathcal{L}_{\mathbf{X}}(R_{ab})$, must be of type **III**.

Repeating this analysis to tensors constructed from the curvature tensor and its covariant derivatives of higher rank yield no additional constraints. However, applying the analysis for those $\mathcal{I}$-degenerate spacetimes whose Riemann tensors and their covariant derivatives are of alignment type **III** and **N** gives the following result.

Proposition 4.2. *For any $\mathcal{I}$-degenerate spacetime which is at least generic of type* **II**, *suppose that* $\mathbf{X}$ *is a nil-Killing vector field with respect to the Riemann-aligned null vector field,* $\boldsymbol{\ell}$. *If under exponentiation of* $\mathbf{X}$ *all $SPIs$ constructed from the curvature tensor and its covariant derivatives are preserved, then* $\mathcal{L}_{\mathbf{X}}$ *produces the following transformation on the coframe basis:*

- *Alignment type* **II** *and* **III***:*

$$\begin{aligned} &\mathcal{L}_{\mathbf{X}}\boldsymbol{\ell} = A\boldsymbol{\ell}, \\ &\mathcal{L}_{\mathbf{X}}\mathbf{n} = C\boldsymbol{\ell} - A\mathbf{n} + D_i\mathbf{m}^i, \\ &\mathcal{L}_{\mathbf{X}}\mathbf{m}^i = E^i\boldsymbol{\ell} + F^i{}_j\mathbf{m}^j, \quad F_{(ij)} = 0. \end{aligned} \tag{13}$$

- *Alignment type* **N***:*

$$\begin{aligned} &\mathcal{L}_{\mathbf{X}}\boldsymbol{\ell} = A\boldsymbol{\ell} + B_i\mathbf{m}^i, \\ &\mathcal{L}_{\mathbf{X}}\mathbf{n} = C\boldsymbol{\ell} - A\mathbf{n} + D_i\mathbf{m}^i, \\ &\mathcal{L}_{\mathbf{X}}\mathbf{m}^i = E^i\boldsymbol{\ell} - B^i\mathbf{n} + F^i{}_j\mathbf{m}^j, \quad F_{(ij)} = 0. \end{aligned} \tag{14}$$

In fact, using the action (9) of the Lie derivative of the coframe in the direction of a vector field **X** we may prove the following result:

Proposition 4.3. *For any $\mathcal{I}$-degenerate spacetime which is generic of type* **II***, an IPD vector field is necessarily a nil-Killing vector field with respect to $\boldsymbol{\ell}$ of the form* (12)*.*

If the $\mathcal{I}$-degenerate spacetime admits curvature tensors of type **III** or **N**, the set of IPD vector fields may not necessarily be contained within the set of nil-Killing vector fields since the action of the Lie derivative in the direction of a vector field **X** on the coframe basis will not give enough b.w. zero components to restrict the form of (9).

5 $\mathcal{I}$-Preserving Diffeomorphisms in the Kundt Spacetimes

As the Kundt spacetimes contain a subclass that are $\mathcal{I}$-degenerate, we will study the curvature structure of this subclass to determine conditions on the metric functions in order to admit an additional IPD vector field, **X**. The class of Kundt spacetimes are given by the line element:

$$ds^2 = 2du\left(dv + H(v,u,x^\delta)du + W_\alpha(v,u,x^\delta)dx^\alpha\right) + \tilde{g}_{\alpha\beta}(u,x^\delta)dx^\alpha dx^\beta. \tag{15}$$

Choosing the initial null coframe,

$$\boldsymbol{\ell} = du, \; \mathbf{n} = dv + Hdu + W_i m^i{}_\alpha dx^\alpha, \quad \mathbf{m}^i = m^i{}_\alpha dx^\alpha, \tag{16}$$

such that the metric tensor in the line element, $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, takes the form

$$g_{\alpha\beta} = 2\ell_{(\alpha}n_{\beta)} + \delta_{ij}m^i_\alpha m^i_\beta.$$

We apply a Lorentz transformation to work with the coframe arising from the Cartan-Karlhede algorithm. While there may be some isotropy remaining from this choice, this will have no effect on the resulting analysis of the b.w. zero components as $\boldsymbol{\ell}$ remains fixed. This will allow us to consider the b.w. zero components of the initial frame with $\mathbf{m}^i$ adapted to the geometry of the transverse metric.

We note that $\boldsymbol{\ell} = \frac{\partial}{\partial v}$ is a nil-Killing vector field,

$$\mathcal{L}_{\boldsymbol{\ell}}\mathbf{g} = H_{,v}\boldsymbol{\ell}\boldsymbol{\ell} + 2W_{i,v}\mathbf{m}^i\boldsymbol{\ell},$$

but it is not necessarily an IPD vector field as the linearly independent non-zero components of the Riemann tensor with b.w. 1 and 0 are:

$$\begin{aligned}
R_{121i} &= -\frac{1}{2}W_{i,vv} \\
R_{1212} &= -H_{,vv} + \frac{1}{4}\left(W_{i,v}\right)\left(W^{i,v}\right), \\
R_{12ij} &= W_{[i}W_{j],vv} + W_{[i;j],v}, \\
R_{1i2j} &= \frac{1}{2}\left[-W_jW_{i,vv} + W_{i;j,v} - \frac{1}{2}\left(W_{i,v}\right)\left(W_{j,v}\right)\right], \\
R_{ijkl} &= \tilde{R}_{ijkl}.
\end{aligned} \tag{17}$$

where $\tilde{R}_{ijkl}$ denotes the curvature tensor of the transverse space.

In order for the metric to be degenerate Kundt, it must be of type **II** to all orders. From [7, 8] this occurs if and only if both of the following quantities vanish:

$$I_0 = R^{abcd}R_{a\ c}^{\ e\ f}\mathcal{L}_{\boldsymbol{\ell}}\mathcal{L}_{\boldsymbol{\ell}}g_{bd}\mathcal{L}_{\boldsymbol{\ell}}\mathcal{L}_{\boldsymbol{\ell}}g_{ef}, \quad K_{ab} = \mathcal{L}_{\boldsymbol{\ell}}\mathcal{L}_{\boldsymbol{\ell}}\mathcal{L}_{\boldsymbol{\ell}}g_{ab}, \tag{18}$$

which gives the following conditions on the metric functions H and W_i in (16):

$$\begin{aligned}
H &= H^{(2)}(u,x^\delta)\frac{v^2}{2} + H^{(1)}(u,x^\delta)v + H^{(0)}(u,x^\delta), \\
W_i &= W_i^{(1)}(u,x^\delta)v + W_i^{(0)}(u,x^\delta).
\end{aligned} \tag{19}$$

Looking at the b.w. zero components of the Riemann tensor (17), it is clear that all of the components are now independent of v. However, since

$$\mathcal{L}_{\boldsymbol{\ell}}\mathbf{g} \neq 0$$

this implies that there are components of the Riemann tensor that are dependent on the v coordinate, namely the negative b.w. terms.

In section 4 we have shown that the condition that $\mathbf{X}$ is a nil-Killing vector field is not sufficient to prove it is an IPD vector field. If a degenerate Kundt spacetime admits an IPD vector field, we can deduce its properties from its action on the b.w. zero components treated as Cartan invariants. Supposing there is an additional IPD vector field $\mathbf{X}$, we can determine conditions from the equations (17) that the functions $H^{(2)}$ and $W_i^{(1)}$ must satisfy at zeroth order:

$$\begin{aligned}
&\sigma(u,x^\delta) = H_{,vv} - \frac{1}{4}W_{i,v}W^i_{\ ,v}, \\
&a_{ij}(u,x^\delta) = W_{[i;j],v}, \\
&s_{ij}(u,x^\delta) = W_{(i;j),v} - \frac{1}{2}W_{i,v}W_{j,v}, \\
&\mathcal{L}_{\mathbf{X}}R_{ijkl} = \mathcal{L}_{\mathbf{X}}\sigma = \mathcal{L}_{\mathbf{X}}a_{ij} = \mathcal{L}_{\mathbf{X}}s_{ij} = 0.
\end{aligned} \tag{20}$$

To determine additional conditions we compute the first covariant derivative of the curvature tensor. We note that the connection coefficients cannot contribute positive b.w terms, and the connection coefficients with zero b.w. are independent of v, this implies that the b.w. zero components of the covariant derivatives of the Riemann tensor will also be independent of v [7]. Thus, $\boldsymbol{\ell} = \frac{\partial}{\partial v}$ will be an IPD.

Due to the form of R_{ijkl} and the connection coefficients for the degenerate Kundt metrics, we can identify a simple condition any IPD vector field must satisfy in terms of the covariant derivatives of the transverse curvature tensor:

$$\mathcal{L}_{\mathbf{X}} R_{ijkl} = \mathcal{L}_{\mathbf{X}} R_{ijkl;i_1} = \mathcal{L}_{\mathbf{X}} R_{ijkl;i_1\cdots i_p} = 0.$$

This implies that $\mathbf{X}$ must be a Killing vector field for the transverse space and so $\mathcal{L}_{\mathbf{X}}\Gamma^i_{jk} = 0$ as well. The remaining first order IPD equations are then

$$\begin{aligned}
&\mathcal{L}_{\mathbf{X}}\mathcal{L}_{\mathbf{m}_i}\sigma = \mathcal{L}_{\mathbf{X}}\mathcal{L}_{\mathbf{m}_k} a_{ij} = \mathcal{L}_{\mathbf{X}}\mathcal{L}_{\mathbf{m}_k} s_{ij} = 0,\\
&\alpha_i = R_{121i;2} = \sigma W_{i,v} - \frac{1}{2}(s_{ij} + a_{ij})W^{j,v},\\
&\beta_{ijk} = R_{1ijk;2} = W^{l,v}\tilde{R}_{lijk} - W_{i,v}a_{jk} + (s_{i[j} + a_{i[j})W_{k],v},\\
&\mathcal{L}_{\mathbf{X}}\alpha_i = \mathcal{L}_{\mathbf{X}}\beta_{ijk} = 0.
\end{aligned} \tag{21}$$

Higher order covariant derivatives provide additional conditions on the vielbein of the transverse space and the b.w. 0 components of the curvature tensor and its covariant derivatives. If $\mathbf{X}$ is an IPD vector field and $H \in [\mathcal{R}^q]_{b.w.0}$, then

$$\mathcal{L}_{[\mathbf{X},\mathbf{m}_i]} H = 0.$$

This provides a consistency condition for the zeroth and first order equations. For any degenerate Kundt spacetime, the action of the Lie derivative of $\mathbf{X}$ acting on the vielbein takes the form in equation (13) or (14) and so the consistency condition implies that the anti-symmetric matrix F_{ik} accounts for elements of the isotropy group of the transverse metric $\tilde{g}_{ij}$.

In summary, we have the following theorem:

Theorem 5.1. *For a degenerate Kundt spacetime, if* $\mathbf{X}$ *is an* IPD *vector field, then the metric functions* $H^{(2)}$ *and* $W_i^{(1)}$ *must satisfy the first order and second order equations* (20) *and* (21) *while the transverse space* $\tilde{\mathbf{g}}$ *admits* $\mathbf{X}$ *as a Killing vector field.*

This result is in agreement with theorem II.7 in [8] which states that all b.w. zero components of the curvature tensor and its covariant derivatives depend on $H^{(2)}$ and $W_i^{(1)}$ alone. That is, we can ignore the lower order v coefficients $H^{(0)}$ and $W_i^{(0)}$ in the metric. Theorem 5.1 leads to the following corollary:

Corollary 5.2. *If the degenerate Kundt metric* $\mathbf{g}'$ *with* $H^{(1)} = H^{(0)} = W_i^{(0)} = 0$ *admits an* IPD *vector field,* $\mathbf{X}$*, then any related metric* $\mathbf{g}$ *with non-zero* $H^{(1)}, H^{(0)}$ *or* $W_i^{(0)}$ *will also admit* $\mathbf{X}$ *as an* IPD *vector field.*

For a degenerate Kundt spacetime, if a particular vector field $\mathbf{X}$ is chosen as an IPD vector field, for example $\tilde{\mathbf{X}} = \frac{\partial}{\partial u}$, all solutions to the equations (20) and (21) for $W_i^{(1)}$ may be difficult to determine. However, a simple solution can always be produced by requiring that

$$\tilde{g}_{\alpha\beta}(x^\delta) \text{ and } \mathcal{L}_{\tilde{\mathbf{X}}} W_i^{(1)} = 0.$$

In this case, $\tilde{\mathbf{X}}$ will be a Killing vector field for the degenerate Kundt metric, $\mathbf{g}'$. From this observation we have another corollary:

Corollary 5.3. *If the degenerate Kundt metric* $\mathbf{g}'$ *with* $H^{(1)} = H^{(0)} = W_i^{(0)} = 0$ *admits a Killing vector field* $\mathbf{X}$*, i.e.,*

$$\mathcal{L}_{\mathbf{X}} \mathbf{g}' = 0, \tag{22}$$

then any related metric $\mathbf{g}$ *with non-zero* $H^{(1)}, H^{(0)}$ *or* $W_i^{(0)}$ *will admit a nil-Killing* IPD *vector field.*

We note that for some choices of $H^{(1)}, H^{(0)}$ and $W_i^{(0)}$ the vector field $\mathbf{X}$ will still be a Killing vector field for $\mathbf{g}$.

6 Kundt-CSI Spacetimes

A Kundt-CSI spacetime is a degenerate Kundt spacetime where the transverse space $\tilde{\mathbf{g}}$ is a locally homogeneous Riemannian manifold and the metric functions $H^{(2)}$ and $W^{(1)}$ satisfy the equations (20) and (21) with $\sigma, a_{ij}, s_{ij}, \alpha_i$ and β_{ijk} constant. In 3D and 4D any CSI spacetime is either locally homogeneous or belongs to the Kundt-CSI class [12, 13] while in higher dimensions the non-flat VSI spacetimes are a subset of the Kundt-CSI spacetimes [14] and it is conjectured that all higher dimensional CSI spacetimes are either locally homogeneous or belong to the Kundt-CSI class as well.

Using Proposition 4.3 and Corollary 5.3 we are able to confirm the conjecture for Kundt-CSI spacetimes in [17]:

Conjecture 6.1. *Assume that a* D*-dimensional spacetime has all constant curvature invariants (*CSI*). Then there exists a set* N *of nil-Killing vector fields which is transitive; i.e.,* $dim(N|p) = D$ *for all* $p \in M$*.*

Given a Kundt-CSI spacetime, we may use a diffeomorphism ϕ_t with respect to a point p generated by the boost defined in [2] and take the limit $\lim_{t\to\infty} \phi_t^* \mathbf{g} = \mathbf{g}'$ to produce a locally homogeneous Kundt-CSI spacetime known as a *Kundt*$^\infty$ *triple* with non-zero metric functions (19) of the form:

$$(H, \mathbf{W}, \tilde{\mathbf{g}}) = (H^{(2)}(u_0, x^\delta), W_i^{(1)}(u_0, x^\delta), \tilde{g}_{\gamma\epsilon}(u_0, x^\delta) dx^\gamma dx^\epsilon). \tag{23}$$

Thus, a corresponding locally homogeneous Kundt$^\infty$ triple which is generic of type $\mathbf{D}$ can always be generated with an identical set of constant $SPIs$ as the original Kundt-CSI spacetime.

In fact, the locally homogeneous Kundt$^\infty$ triples are of alignment type $\mathbf{D}^k$, i.e., the curvature tensor and its covariant derivatives are of type $\mathbf{D}$ to all orders

[16]. As the $SPIs$ fully determine the Cartan invariants of a type $\mathbf{D}^k$ spacetime [7, 16, 8], the Cartan invariants must be constant, ensuring the existence of a fully transitive set of Killing vector fields. Thus, for any Kundt-CSI spacetime, corollary 5.3 and proposition 4.3 give the following proposition:

Proposition 6.2. *For any Kundt-CSI spacetime, the Killing vector fields of the corresponding Kundt$^\infty$ triple act as a finite transitive set of nil-Killing IPD vector fields for the original spacetime.*

7 Discussion and Future Work

In this paper we have examined the general form of the nilpotent operators and introduced a new definition for the nil-Killing vector fields. Using this definition we have shown that the nil-Killing vector fields, which generalize the Kerr-Schild vector fields, form a Lie algebra. We have also argued that other Lie algebras can be formed by imposing additional conditions on the nil-Killing vector fields. Since the existence of a nil-Killing vector field does not ensure that it will be an IPD vector field, we then studied the existence of IPD vector fields using a frame based approach. By considering the form of the curvature tensor and its covariant derivatives arising from the Cartan-Karlhede algorithm we have determined the dimension of the subset of the tangent space spanned by the IPD vector fields.

Employing the stronger definition of a nil-Killing vector field and the action of the Lie derivative of the nil-Killing vector fields on the coframe, we have shown that the set of nil-Killing vector fields contain IPD vector fields. Furthermore we proved that for a spacetime which is generic of type **II** to all orders, the IPD vector fields are strictly contained in the set of nil-Killing vector fields. In the case of spacetimes which are generic of type **III** or **N** we are unable to show that an IPD vector field is necessarily a nil-Killing vector field, and so it is possible that such algebraically special $\mathcal{I}$-degenerate spacetimes can admit IPD vector fields which are *not* nil-Killing.

To determine the existence of an IPD vector field in a general degenerate Kundt spacetime we have proposed a constructive approach by assuming that an IPD vector field is given and determining the form of the metric functions. The existence of an IPD vector field influences the form of the transverse space, $\tilde{\mathbf{g}}$, and the metric functions $H^{(2)}$ and $W_i^{(1)}$. Any metric sharing these functions with differing $H^{(1)}$, $H^{(0)}$ and $W_i^{(0)}$ will admit the same IPD vector field.

From this result, we have demonstrated that a transitive set of nil-Killing vector fields exist for any Kundt-CSI spacetime. This was achieved using a mapping from an arbitrary Kundt-CSI spacetime to a unique Kundt-CSI spacetime of alignment type $\mathbf{D}^k$ with identical $SPIs$, $\mathcal{I}$, as the original spacetime but whose Cartan invariants are characterized by the set $\mathcal{I}$. Such spacetimes admit a transitive set of Killing vector fields, i.e., they are locally homogeneous, and Theorem 5.1 implies that these vector fields are nil-Killing IPD vector fields for the original Kundt-CSI spacetime.

Admittedly, this mapping will not work for other CSI pseudo-Riemannian spaces of indefinite signature, as it may yield spaces which are not locally homogeneous but are CSI. We hope the frame approach introduced here will be helpful in identifying IPD-vector fields for pseudo-Riemannian spaces. However, this task is complicated by the higher dimensional b.w. structure of the pseudo-Riemannian spaces. To illustrate the issue, we will consider an $(2k+m)$-dimensional manifold of signature $(k, k+m)$, a null coframe can be chosen such that

$$ds^2 = 2(\boldsymbol{\ell}^1\mathbf{n}^1 + \boldsymbol{\ell}^2\mathbf{n}^2 + \ldots + \boldsymbol{\ell}^k\mathbf{n}^k) + \delta_{ab}\mathbf{m}^a\mathbf{m}^b, \quad a, b \in [1, m]. \tag{24}$$

Relative to this coframe, the Abelian subgroup of the group $SO(k, k+m)$ are boosts in each of the k null planes:

$$(\boldsymbol{\ell}^i, \mathbf{n}^i) \to (e^{\lambda_i}\boldsymbol{\ell}_i, e^{-\lambda_i}\mathbf{n}_i) \tag{25}$$

for $i \in [1, k]$ where λ_i are real-valued. In analogy with the Lorentzian case, we have the concept of *boost weights* $\mathbf{b} \in \mathbb{Z}^k$ such that for an arbitrary component of a rank n tensor $\mathbf{T}$ with respect to the coframe (24), a boost in each of the k null planes gives the transformation

$$T_{\mu_1\ldots\mu_n} \to e^{(b_1\lambda_1 + b_2\lambda_2 + \ldots + b_k\lambda_k)}T_{\mu_1\ldots\mu_n},$$

where $b_1, \ldots, b_k$ are integers and $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ is the boost weight vector of the component $T_{\mu_1\ldots\mu_n}$. We can decompose the tensor $\mathbf{T}$ into the following decomposition

$$\mathbf{T} = \sum_{\mathbf{b}\in\mathbb{Z}^k} (T)_{\mathbf{b}}. \tag{26}$$

Here, $(T)_{\mathbf{b}}$ denotes the projection onto the subspace of components of boost weight $\mathbf{b}$.

With the boost weight decomposition, we can introduce properties to classify tensors in a similar manner to the alignment classification:

Definition 7.1. Consider the conditions

$$\begin{aligned} &B1)\ (T)_{\mathbf{b}} = 0,\ \text{for all } \mathbf{b} = (b_1, b_2, \ldots, b_k), b_1 > 0,\\ &B2)\ (T)_{\mathbf{b}} = 0,\ \text{for all } \mathbf{b} = (0, b_2, \ldots, b_k), b_2 > 0,\\ &\quad\vdots\\ &Bk)\ (T)_{\mathbf{b}} = 0,\ \text{for all } \mathbf{b} = (0, 0, \ldots, 0, b_k), b_k > 0, \end{aligned} \tag{27}$$

A tensor $\mathbf{T}$ possesses the $\mathbf{S}_i$ property, $i \in [1, k]$, if there exists a null coframe such that the conditions $B1) - Bi)$ holds.

Definition 7.2. A tensor $\mathbf{T}$ posses the $\mathbf{N}$ property if a null coframe exists such that $B1) - Bk)$ are satisfied and

$$(T)_{\mathbf{b}} = 0,\ \text{for all } \mathbf{b} = (0, 0, \ldots 0, 0).$$

For indefinite signatures other than Lorentzian signature there is another set of properties that must be considered. A tensor which does not have the S_i property can still have a degenerate structure, since the boost weights are a lattice $\mathbf{b} \in \mathbb{Z}^k \subset \mathbb{R}^k$, we can use a linear transformation $\mathbf{G} \in GL(k)$ to map the boost weight onto a lattice Γ in $\mathbb{R}^k$. If such a $\mathbf{G} \in GL(k)$ exists such that the image of the boost weights $\mathbf{Gb}$ of the tensor $\mathbf{T}$ now satisfies some of the properties above, we say the tensor $\mathbf{T}$ possesses the $\mathbf{S}_i^{\mathbf{G}}$ or $\mathbf{N}^{\mathbf{G}}$ property.

Any tensor satisfying *at least* the $S_1^{\mathbf{G}}$ property will not be characterized by its invariants [2]. Thus, a given pseudo-Riemannian space is $\mathcal{I}$-degenerate if the curvature tensor and its covariant derivatives satisfy the $S_1^{\mathbf{G}}$ property relative to a common null coframe. As in the Lorentzian case [16], the proof of this result relies on the limit of a diffeomorphism associated with an appropriately chosen boost in order to generate a non-diffeomorphic space with the same set $\mathcal{I}$. Motivated by this result, and theorem 3.1 we can state a simple existence theorem for IPD vector fields in pseudo-Riemannian spaces:

Theorem 7.3. *Consider a pseudo-Riemannian space, for which the curvature tensor and its covariant derivatives satisfies the $S_1^{\mathbf{G}}$ property with $\mathbf{G} \in GL(k)$ relative to a fixed coframe basis. Denoting $\mathcal{R}_{\mathbf{G}}$ as the $\mathbf{G}$-transformed components of the curvature tensor and its covariant derivatives, if*

$$0 \leq rank([\mathcal{R}^q_{\mathbf{G}}]_{b_i=0}) < rank(\mathcal{R}^q_{\mathbf{G}}),$$

then the manifold admits a non-trivial IPD vector field, $\mathbf{X}$, such that

$$\mathcal{L}_{\mathbf{X}}\mathcal{I} = 0.$$

That is, the pseudo-Riemannian space is $\mathcal{I}$-degenerate.

In principle the set of IPD vector fields can be determined using this approach. However, in practice this is too difficult to compute for a generic pseudo-Riemannian manifold due to the $S_i^{\mathbf{G}}$ property. As an alternative, theorem 7.3 can be restated in terms of differential invariants [25] by comparing the rank of $\mathcal{I}$ to the rank of the set of differential invariants. In future work, we will explore alternative approaches to finding a transitive set of nil-Killing IPD vector fields for spacetimes using the geometric evolution equations [20] with the goal of extending the approach to pseudo-Riemannian spaces of other signatures.

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N_ncp GG
K_rrfcu Rcphc ?_blc &0./7', Lgj+Igjjgle tcarmp dgcjbq _lb rwnc GGG bcdmpk_rgmlq, Qs`kgrrcb dmp ns`jga_rgml gl *Hmspl_j md k_rfck_rga_j nfwqgaq*, _pVgt8/7/0,.06.7 Yk_rf,BE[

N_ncp GGG
K_rrfcu Rcphc ?_blc &0./7', Islbr qrpsarspcq, Qs`kgrrcb dmp ns`jga_rgml gl *Bgddcpclrg_j Ecmkcrpw _lb grq ?nnjga_rgmlq*, _pVgt8/7/0,.035. Yk_rf,BE[