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INVOLUTIONS OF QUADRICS

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Contents

Chapter 1

Introduction

In the introductory chapter, we will explain briefly what all this work is about. First is worthy to mention that the most included area is algebraic geometry, which of course is a combination of algebra and linear algebra. But at the same time all the work required some knowledge of projective geometry as well.

In the first chapter, the main point is defining a map between projective spaces $\mathbb{P}^1$ and $\mathbb{P}^n$, and afterwards showing that the defined map is actually an isomorphism between $\mathbb{P}^1$ and the image of it.

The second chapter will have on focus involutions from $\mathbb{P}^1$ into $\mathbb{P}^1$. In this part comes in use the map we defined in the first chapter. It will be shown that the image of the map is a conic C in $\mathbb{P}^2$, and since it is isomorphism, it means $\mathbb{P}^1$ and C are isomorphic. Using this fact, we will define a map from C to C in such way that for a chosen point $p \in \mathbb{P}^2$ outside of C, and for any point $q \in C$, we take the line through these points and find the second intersection point q' with the conic C. So the map from C to C, will reflect any point q, through p, into q'. Using it we will build a map from $\mathbb{P}^1$ into $\mathbb{P}^1$, and afterwards prove that it is an involution.

In the next chapter, we will move in projective space $\mathbb{P}^3$. we start by building a map from $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$, which is an isomorphism into its image. The image is a quadric Q in $\mathbb{P}^3$, so $\mathbb{P}^1 \times \mathbb{P}^1$ and Q are isomorphic. Using this, we will be able to build a map from $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^1 \times \mathbb{P}^1$, for which it will be shown is an involution. Similar as in the previous chapter, the same logic will be used here too, to build the map. A point $p \in \mathbb{P}^3$ but not in Q will be chosen. For any point $q \in Q$, a line through it and p will be taken, and after, the second intersection point with the quadric Q is going to be found. Using it, the wanted map will be defined, and proved it is an involution.

Last chapter will be focused in $\mathbb{P}^n$, more exactly in quadric hypersurfaces (quadrics). It will start by taking a normalized quadric Q and build a map from Q into Q. One more time, the same idea will be used. We choose a point $p \in \mathbb{P}^n$ but outside of Q. And after, for any point $q \in Q$, we build lines that contain those points. The second intersection point q', between the line and the quadric Q, will be found. So the map from Q into Q, will be built in that way that it reflects the point q, through the point p, into the point q'. At the end it will be proved this map is an involution.

The last part of the chapter shows that any quadric Q' can be normalized, and using this fact, we will be able to build maps from any Q' into Q', which again is an involution.

It is worthy to mention that mathematical language, which will be used, is the same as in [3]. And the main information needed from algebraic geometry will be taken from [3] and [4].

Chapter 2

Maps between projective spaces

As said in the introduction, the goal of this study is to give a better view of involutions between quadrics in projective space $\mathbb{P}^n$. To keep up with this, first thing is going to happen is to show that a quadric in a projective space $\mathbb{P}^n$ is actually isomorphic to projective space $\mathbb{P}^1$.

For reaching that point, we start by defining a map between projective spaces $\mathbb{P}^1$ and $\mathbb{P}^n$.

Definition 2.0.1. For projective spaces $\mathbb{P}^1$ and $\mathbb{P}^n$, let $\Theta : \mathbb{P}^1 \to \mathbb{P}^n$ be a map between them such that:

$$(x : y) \mapsto (x^n : x^{n-1}y : ... : y^n) \tag{2.1}$$

for $\forall (x : y) \in \mathbb{P}^1$.

What we really are interested to prove about this map is that actually it is an isomorphism between $\mathbb{P}^1$ and its image $\Theta(\mathbb{P}^1)$. From now one we will write its image with C. To prove that, it must be proved Θ is a bijective map, it is a morphism and its inverse is a morphism too.

This is going to be shown in two simple lemmas. Let us start by showing that our map is injective.

Lemma 2.0.1. *The map $\Theta : \mathbb{P}^1 \to \mathbb{P}^n$ defined above is an injective map.*

Proof. To show the injectivity, we will have a look on our map, locally. First we can write our projective space as union of two subsets $\mathbb{P}^1 = \{(u : v)\} = U \cup V$, where $U, V \in \mathbb{P}^1$ such that:

$$U = \{(u : v)|v \neq 0\} = \{(u : 1)\} \cong A^2$$

and

$$V = \{(u : v)|u \neq 0\} = \{(1 : v)\} \cong A^2$$

Let us choose two random points $p, q \in P^1$ with the condition that $p \neq q$. If $p, q \in U$ then we have $p = (x : 1)$ and $q = (y : 1)$ obviously $x \neq y$. So acting with our map in those points we will get:

$$\Theta(p) = \Theta(x : 1) = (x^n : x^{n-1} : ... : 1)$$

$$\Theta(q) = \Theta(y : 1) = (y^n : y^{n-1} : ... : 1)$$

which from the way map is defined means $\Theta(p) \neq \Theta(q)$.

If $p \in U$ and $q \notin U$ means that $p = (x : 1)$ and $q = (1 : 0)$ where second coordinate in the point q is zero otherwise we are in the same situation as above. Again putting those points inside Θ will gives us:

$$\Theta(p) = \Theta(x : 1) = (x^n : x^{n-1} : ... : 1)$$

$$\Theta(q) = \Theta(1 : 0) = (1 : 0 : ... : 0)$$

which again shows that $\Theta(p) \neq \Theta(q)$. Since from this is noticeable that for any two points $p, q \in \mathbb{P}^1$ which are not equal then $\Theta(p) \neq \Theta(q)$, it means that our map is injective. $\square$

From the lemma above, restricting the map $\Theta : \mathbb{P}^1 \to C$, then Θ is bijective.

Next step is trying to show that both Θ and its inverse are morphisms. To prove that, it is suficient to show that our map can be written as a regular map [Proposition 4.7, page 34 3, Lemma 7.5, page 58] locally.

Lemma 2.0.2. *The map Θ and its inverse Θ^{-1} can be represented by regular functions locally.*

Proof. We are going to analyse this, not only in $\mathbb{P}^n$ but first in $\mathbb{P}^2$ and $\mathbb{P}^3$ finally moving into general one.

For $n = 2$, the map $\Theta : \imath^1 \to C \subset \mathbb{P}^2$ will be as below:

$$(x : y) \mapsto (x^2 : xy : y^2) \tag{2.2}$$

Since our map will be checked locally, make sense writing our projective space $\mathbb{P}^2$ as union of open subsets $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$ such that U_i is defined as below:

$$U_i = \{(u_0 : u_1 : u_2) | u_i \neq 0\}$$

for $i = 0, 1, 2$. Without losing anything from generality, let us start from $i = 0$, finding the intersection between C and U_0.

$$C \cap U_0 = \{(x^2 : xy : y^2) | x \neq 0\} = \{(1 : \frac{y}{x} : \frac{y^2}{x^2})\}$$

Now using the fact that, between open subsets of $\mathbb{P}^2$ and $\mathbb{A}^2$ exists a bijective map [3, Remark 6.3], allows us looking at it as an affine situation. This would be a map $C \cap U_0 \subset U_0 \to \mathbb{A}^2$ which looks:

$$(1 : \frac{y}{x} : \frac{y^2}{x^2}) \mapsto (\frac{y}{x}, \frac{y^2}{x^2})$$

Restricting the inverse of Θ into $C \cap U_0$, it will be $\Theta^-1|_{C\cap U_0} : C \cap U_0 \to \mathbb{P}^1$, more precisely in affine space, such that:

$$(\frac{y}{x}, \frac{y^2}{x^2}) \mapsto (\frac{y}{x})$$

which obviously is a regular map. In general we get:

$$A^2 \to A^1; (s,t) \mapsto (s)$$

$$P^2 \to P^1; (1 : s : t) \mapsto (1 : s)$$

In the similar way is shown that it is given by regular maps when is analysed for $i = 1$ and $i = 2$. This proves that our map $\Theta : \mathbb{P}^1 \to \mathbb{P}^2$ and its inverse is given by regular maps.

Moving into projective space $\mathbb{P}^3$, similar with the proof above, the map $\Theta : \mathbb{P}^1 \to \mathbb{P}^3$ is:

$$(x : y) \mapsto (x^3 : x^2y : xy^2 : y^3) \tag{2.3}$$

We start by writing the projective space $\mathbb{P}^3$ as union of open subsets $\mathbb{P}^3 = U_0 \cup U_1 \cup U_2 \cup U_3$ such that U_i is a set of the form:

$$U_i = \{(u_0 : u_1 : u_2 : u_3) | u_i \neq 0\}$$

for $i = 0, 1, 2, 3$. Again we start from $i = 0$, by first finding the intersection between C and U_0:

$$C \cap U_0 = \{(x^3 : x^2y : xy^2 : y^3) | x \neq 0\} = \{(1 : \frac{y}{x} : \frac{y^2}{x^2} : \frac{y^3}{x^3})\}$$

Using the same fact that between U_i and $\mathbb{A}^3$ can be build a bijective map, is allowed to move the problem in affine space, and the map $C \cap U_0 \subset U_0 \to \mathbb{A}^3$ has the form:

$$(1 : \frac{y}{x} : \frac{y^2}{x^2} : \frac{y^3}{x^3}) \mapsto (\frac{y}{x}, \frac{y^2}{x^2}, \frac{y^3}{x^3})$$

Restricting the inverse map $\Theta^{-1}|_{C\cap U_0} : C \cap U_0 \to \mathbb{P}^1$, or in affine charts in this situation we get:

$$(\frac{y}{x}, \frac{y^2}{x^2}, \frac{y^3}{x^3}) \mapsto \frac{y}{x}$$

which is clear is a regular map. So this gives the general situation:

$$A^3 \to A^1; (s,t,w) \mapsto s$$

$$P^3 \to P^1; (1 : s : t : w) \mapsto (1 : s)$$

Showing this in other open subspaces U_1, U_2 and U_3, is similar with what we already did above. This let quite clear that Θ and its inverse are given by regular maps, which is what we were trying to proof in $\mathbb{P}^3$.

After doing the proof for $\mathbb{P}^2$ and $\mathbb{P}^3$, is time to move on and finish the proof in projective space $\mathbb{P}^n$. We already have defined our map in Definition 1.1. Let us start by writing the projective space $\mathbb{P}^n$ as union of open subsets $P^n = U_0 \cup U_1 \cup U_2 \cup ... \cup U_n$, where U_i is defined as below:

$$U_i = \{(u_0 : u_1 : ... : u_n) | u_i \neq 0\}$$

for $i = 0, 1, 2, ..., n$.Let us start by finding the intersection $C \cap U_i$:

$$C \cap U_i = \{(x^n : ... : x^{n-i}y^i : ... : y^n) | x^{n-i}y^i \neq 0\}$$

$$= \{(\frac{x^i}{y^i} : ... : \frac{x}{y} : 1 : \frac{y}{x} : ... : \frac{y^{n-i}}{x^{n-i}})\}$$

Similar as in the previous case, we use the fact that between open subsets U_i and $\mathbb{A}^n$ exist a bijective map, which allows to look at our problem in affine space. This map $U_i \to \mathbb{A}^n$ looks like:

$$(\frac{x^i}{y^i} : ... : \frac{x}{y} : 1 : \frac{y}{x} : ... : \frac{y^{n-i}}{x^{n-i}}) \mapsto (\frac{x^i}{y^i}, ..., \frac{x}{y}, \frac{y}{x}, ..., \frac{y^{n-i}}{x^{n-i}})$$

The inverse map of Θ restricted on $C \cap U_i$, such $\Theta^{-1}|_{C \cap U_i} : C \cap U_i \to \mathbb{P}^1$, more precisely in affine chart will look as below:

$$(\frac{x^i}{y^i}, ..., \frac{x}{y}, \frac{y}{x}, ..., \frac{y^{n-i}}{x^{n-i}}) \mapsto \frac{x}{y}$$

or

$$(\frac{x^i}{y^i}, ..., \frac{x}{y}, \frac{y}{x}, ..., \frac{y^{n-i}}{x^{n-i}}) \mapsto \frac{y}{x}$$

depends on the situation, but which is a regular map for both as we were trying to prove. The general view would be:

$$A^n \to A^1; (u_0, ..., u_{i-1}, u_{i+1}, ..., u_n) \mapsto u_{i-1}(u_{i+1})$$

$$P^n \to P^1; (u_0 : ... : u_{i-1} : 1 : u_{i+1} : ... : u_n) \mapsto (u_{i-1} : 1)((1 : u_{i+1}))$$

or again depending on the situation:

$$A^n \to A^1; (u_0, ..., u_{i-1}, u_{i+1}, ..., u_n) \mapsto u_{i+1}$$

$$P^n \to P^1; (u_0 : ... : u_{i-1} : 1 : u_{i+1} : ... : u_n) \mapsto (1 : u_{i+1})$$

$\square$

From Lemma above, where is proved that the map Θ and its inverse Θ^{-1} are represented by regular maps locally, and from the fact that if a map is regular map then it is a morphism, we came to the conclusion that Θ is a morphism. Combining this fact and the Lemma 1.1 is clear that the theorem bellow is true:

Theorem 2.0.3. *The map $\Theta : \mathbb{P}^1 \to C \subset \mathbb{P}^n$ is an isomorphism. So $\mathbb{P}^1$ and C are isomorphic.*

Chapter 3

Conics

3.1 Isomorphy between projective space $\mathbb{P}^1$ and conics

In the first chapter, we defined a function $\Theta : \mathbb{P}^1 \to \mathbb{P}^n$ and managed to prove that Θ is an isomorphism between $\mathbb{P}^1$ and its image $\Theta(\mathbb{P}^1) = C$ making them isomorphic.

This chapter will be mostly focused in projective space $\mathbb{P}^2$, starting by showing that C is actually a conic. After that we will analyse a function that takes one point of this conic and reflects it into another point in that conic too.

Before proving that our C is a conic, let us give the general equation of a conic in an affine space $\mathbb{A}^2$ [7, Section: Definition and basic properties]. A conic is a curve obtained as an intersection of the surface of a cone with a plane the general equation of it will be:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{3.1}$$

The projective closure of it in $\mathbb{P}^2$ will be:

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0 \tag{3.2}$$

Now recalling the map $\Theta : \mathbb{P}^1 \to \mathbb{P}^2$, we will prove the lemma below:

Lemma 3.1.1. *The image $C = \Theta(\mathbb{P}^1) \subset \mathbb{P}^2$ is a conic and its equation is:*

$$u_1^2 - u_0u_2 = 0 \tag{3.3}$$

Proof. This lemma is going to be proved by checking the projective space locally, so let us write it as union of open subsets $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$, where

$U_i = \{(u_0 : u_1 : u_2)|u_i \neq 0\}$ for $i = 0, 1, 2$ and as seen before $U_i \cong \mathbb{A}^2$. Now let $(x : y)$ be any point from $\mathbb{P}^1$. Acting on this point with our map Θ we will get $(x^2 : xy : y^2) \in \mathbb{P}^2$

Without losing anything from generality let assume that the point is inside U_0, which means that $x \neq 0$. This gives the opportunity to rewrite it:

$$(x^2 : xy : y^2) = (1 : \frac{y}{x} : \frac{y^2}{x^2}) = (1 : z : z^2)$$

where $z = \frac{y}{x}$. Since $U_0 \cong \mathbb{A}^2$, exists the possibility to try proving this problem in affine space and that is what we are going to do:

$$(1 : z : z^2) \to (z, z^2)$$

That we are trying to do here is to build a function $q(u_1, u_2)$ such that this function will be zero for every point of the form $(z, z^2) \in \mathbb{A}^2$. So:

$$q(u_1, u_2) = 0 \leftrightarrow (u_1, u_2) = (z, z^2)$$

Referring the general formula of a conic, it is easy to see that a function which will fit perfectly to our problem is:

$$q(u_1, u_2) = u_1^2 - u_2$$

In affine chart let

$$C' = \{(z, z^2)|z \in \mathbb{A}^1\}$$

this means that C' is the affine variety of $q(u_1, u_2) = u_1^2 - u_2$, so:

$$C' = V_a(u_1^2 - u_2)$$

Finding the projective closure of the affine variety we get

$$C' = V_a(u_1^2 - u_2) \to C = V_p(u_1^2 - u_0u_2)$$

where u_0 is the new coordinate that we introduce to be able to homogenise the equation, and C is a projective variety. The function which fits perfectly for our problem is:

$$Q(u_0, u_1, u_2) = u_1^2 - u_0u_2$$

In the case when the point belongs to U_2 is proven in similar way as above, So let us assume that $(x^2 : xy : y^2) \in U_2$, which means that $y \neq 0$ and the point can be rewritten as bellow:

$$(x^2 : xy : y^2) = (\frac{x^2}{y^2} : \frac{x}{y} : 1) = (t^2 : t : 1)$$

where $t = \frac{x}{y}$. Again we can move the problem into affain space and treat it there.

$$(t^2 : t : 1) \to (t^2, t)$$

Again we want to find a function $q(u_0, u_1)$ which will be zero for every point $(t^2, t) \in \mathbb{A}^2$, so:

$$q(u_0, u_1) = 0 \leftrightarrow (u_0, u_1) = (t^2, t)$$

Based on the general formula of conic in $\mathbb{A}^2$, a function that fits to our problem is:

$$q(u_0, u_1) = u_1^2 - u_0$$

In affine chart let

$$C' = \{(t^2, t) | t \in \mathbb{A}^1\}$$

which means C' is an affine variate of $q(u_0, u_1) = u_1^2 - u_0$, giving:

$$C' = V_a(u_1^2 - u_0)$$

Finding the projective closure of this affine variety, we reach the same projective variety as before

$$C' = V_a(u_1^2 - u_0) \to C = V_p(u_1^2 - u_0 u_2)$$

where u_2 is the new coordinate that we introduce to be able to homogenise the equation. And the function of C is:

$$Q(u_0, u_1, u_2) = u_1^2 - u_0 u_2$$

If our point is $(x^2 : xy : y^2) \in U_1$, meaning that $xy \neq 0$ this directly will show that $x \neq 0$ and $y \neq 0$, and we can discuss this case in totally the same way as with one of the cases before.

So the function that represents in perfect way our projective variety $C = V_p(u_1^2 - u_0 u_2)$:

$$Q(u_0, u_1, u_2) = u_1^2 - u_0 u_2 \tag{3.4}$$

and for any point $(u_0 : u_1 : u_2) = (x^2 : xy : y^2)$, we will have:

$$u_1^2 - u_0 u_2 = 0 \tag{3.5}$$

The equation above, without any doubt, is a conic, which was the purpose of this lemma. $\square$

3.2 Involutions of projetive space $\mathbb{P}^1$

Since we already found the equation of the conic $C \subset \mathbb{P}^2$, it can be really interesting and at the same time helpful to analyse it more. Since the conic is inside $\mathbb{P}^2$, we will try to build a map that takes a point from it and reflects it into another point which belongs to the conic too. The right way is to start this is explaining the geometric idea of the function.

Let us select a point $p = (u_0, u_1, u_2) \in \mathbb{P}^2$ which does not belong to the conic. For any point $q \in C$ is possible to build a line which passes through the selected point p, and through the point q, intersecting with the conic in another point too. The second intersection point will represent the reflection point of the map we are trying to build.

Now let $\varphi : C \to C$ be the map we are trying to define. We will use this map to build a map from $\mathbb{P}^1$ and itself, using the fact that $C \cong \mathbb{P}^1$. To do all of this, let us prove the theorem below:

Theorem 3.2.1. *Between $\mathbb{P}^1$ and itself exists a map $f : \mathbb{P}^1 \to \mathbb{P}^1$, which is given by a matrix A, such that $\forall (x : y) \in \mathbb{P}^1$:*

$$f((x : y)) = A(x : y) \tag{3.6}$$

where

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_2 & -u_1 \end{pmatrix} \tag{3.7}$$

and where u_i, $i = 0, 1, 2$ are the coordinates of the selected point $p \in \mathbb{P}^2$ through which, the reflection is done.

Proof. First of all, let $(x : y)$ be any point that belongs to $\mathbb{P}^1$. It is obvious that we are trying to build a two by two matrix which multiplies the chosen point giving as a result another point that belongs to $\mathbb{P}^1$ too. So we will try to find a matrix of the form as below:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

And the way our map $f : \mathbb{P}^1 \to \mathbb{P}^1$ will work is:

$$f((x : y)) = A(x : y)$$

such that:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

so the result we get would be:

$$A(x:y) = (ax+by:cx+dy)$$

Now using the map Θ, we reflect the point $(x:y)$ to $q=(x^2:xy:y^2)$ in $C \subset \mathbb{P}^2$. Let us choose a point $p=(u_0:u_1:u_2) \in \mathbb{P}^2$ which does not belong to the conic C. The theorem will be proven, checking the projective space $\mathbb{P}^2$ locally, so let us write it as union of open subsets $P^2 = U_0 \cup U_1 \cup U_2$ with $U_i = \{(u_0:u_1:u_2) | u_i \neq 0\}$ for $i=0,1,2$.

First let us assume that $p, q \in U_0$, which means that the first coordinate is different from zero, so $u_0 \neq 0$ and $x \neq 0$. This gives the opportunity to change the form of them as below:

$$p = (u_0:u_1:u_2) = (1:\frac{u_1}{u_0}:\frac{u_2}{u_0}) = (1:v_1:v_2)$$

$$q = (x^2:xy:y^2) = (1:\frac{y}{x}:\frac{y^2}{x^2}) = (1:z:z^2)$$

We are gonna move this problem from projective space into affine space and try to prove it there. The equation of our conic C, where q and $\phi_p(q)$ belong, is:

$$C : Q(w_0, w_1, w_2) = w_1^2 - w_0 w_2$$

and moving it to affine space we get:

$$C' : q(w_1, w_2) = w_1^2 - w_2$$

The points p and q in affine space will be:

$$p = (1:v_1:v_2) \rightarrow (v_1, v_2) \in \mathbb{A}^2$$

$$q = (1:z:z^2) \rightarrow (z, z^2) \in \mathbb{A}^2$$

The equation of the line in A^2 through p and q is:

$$\frac{w_2 - v_2}{w_2 - z^2} = \frac{w_1 - v_1}{w_1 - z}$$

Solving the system of equations that contains the equation of conic $w_1^2 - w_2 = 0$ and the line above will give the point that we are looking for.
From the equation of line we get:

$$w_2 = -\frac{z^2 - v_2}{v_1 - z} w_1 + \frac{z^2 v_1 - z v_2}{v_1 - z}$$

Replacing w_2 with the expression above in the equation of conic will give us:

$$w_1^2 + \frac{z^2 - v_2}{v_1 - z} w_1 + \frac{z^2 v_1 - z v_2}{v_1 - z} = 0$$

One solution of this equation is already known, and it is $w_1 = z$. Dividing the expression above by $(w_1 - z)$, gives us the other solution. So:

$$(w_1^2 + \frac{z^2 - v_2}{v_1 - z} w_1 + \frac{z^2 v_1 - z v_2}{v_1 - z}) : (w_1 - z) = w_1 + \frac{v_1 z - v_2}{v_1 - z}$$

$$(w_1 - z)(w_1 + \frac{v_1 z - v_2}{v_1 - z}) = 0$$

According to this, the second solution is:

$$w_1 = -\frac{v_1 z - v_2}{v_1 - z}$$

Moving back to projective space, by replacing z with $\frac{y}{x}$ an v_i with $\frac{u_i}{u_0}$, z' is going to be:

$$w_1 = \frac{u_2 x - u_1 y}{u_1 x - u_0 y}$$

In the other side the other coordinate w_2 will be:

$$w_2 = \frac{(u_2 x - u_1 y)^2}{(u_1 x - u_0 y)^2}$$

So the second intersection point $\varphi(q) \in C$ will be:

$$\varphi(q) = ((u_1 x - u_0 y)^2 : (u_1 x - u_0 y)(u_2 x - u_1 y) : (u_2 x - u_1 y)^2)$$

Using the inverse of the map Θ, we move this point back to $\mathbb{P}^1$, getting:

$$((u_1 x - u_0 y)^2 : (u_1 x - u_0 y)(u_2 x - u_1 y) : (u_2 x - u_1 y)^2) \mapsto (u_1 x - u_0 y : u_2 x - u_1 y)$$

Now comparing it with the general form we get the coefficients of the matrix: $a = u_1$, $b = -u_0$, $c = u_2$ and $d = -u_1$. so the matrix we are looking for is:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_2 & -u_1 \end{pmatrix}$$

Let us check the other possibility when $p, q \in U_2$, which means that the third coordinate is different from zero, so $u_2 \neq 0$ and $y \neq 0$ and we have:

$$p = (u_0 : u_1 : u_2) = (\frac{u_0}{u_2} : \frac{u_1}{u_2} : 1) = (v_0 : v_1 : 1)$$

$$q = (x^2 : xy : y^2) = (\frac{x^2}{y^2} : \frac{x}{y} : 1) = (z^2 : z : 1)$$

We are gonna move this problem from projective space into affine space and solve it there. The equation of our conic C, where q and $\phi_p(q)$ belong, is:

$$C : Q(w_0, w_1, w_2) = w_1^2 - w_0 w_2$$

and moving it to affine space we get:

$$C' : q(w_0, w_1) = w_1^2 - w_0$$

The points p and q in affine space will be:

$$p = (v_0 : v_1 : 1) \to (v_0, v_1) \in \mathbb{A}^2$$

$$q = (z^2 : z : 1) \to (z^2, z) \in \mathbb{A}^2$$

The equation of the line in $\mathbb{A}^2$ through p and q is:

$$\frac{w_1 - v_1}{w_1 - z} = \frac{w_0 - v_0}{w_0 - z^2}$$

Solving the system of equations that contains the equation of conic $w_1^2 - w_0 = 0$ and the line above will give the point that we are looking for.
From the equation of line we get:

$$w_0 = -\frac{v_0 - z^2}{z - v_1} w_1 + \frac{v_0 z - v_1 z^2}{z - v_1}$$

Replace w_0 with the expression above in the equation of conic and we get

$$w_1^2 + \frac{v_0 - z^2}{z - v_1} w_1 - \frac{v_0 z - v_1 z^2}{z - v_1} = 0$$

One solution of this equation is already known, and it is $w_1 = z$. Dividing the expression above by $(w_1 - z)$, gives us the other solution. So:

$$(w_1^2 + \frac{v_0 - z^2}{z - v_1} w_1 - \frac{v_0 z - v_1 z^2}{z - v_1}) : (w_1 - z) = w_1 + \frac{v_0 - v_1 z}{z - v_1}$$

$$(w_1 - z)(w_1 + \frac{v_0 - v_1 z}{z - v_1}) = 0$$

According to this the solution is:

$$w_1 = -\frac{v_0 - v_1 z}{z - v_1}$$

Moving back to projective space, by replacing z with $\frac{x}{y}$ an v_i with $\frac{u_i}{u_2}$, z' is going to be:

$$w_1 = \frac{u_1x - u_0y}{u_2x - u_1y}$$

and the other coordinate w_0 will be:

$$w_0 = \frac{(u_1x - u_0y)^2}{(u_2x - u_1y)^2}$$

The second intersection point $\varphi(q)$ will be:

$$\varphi(q) = ((u_1x - u_0y)^2 : (u_1x - u_0y)(u_2x - u_1y) : (u_2x - u_1y)^2)$$

Again using the inverse map of Θ, we get:

$$((u_1x-u_0y)^2 : (u_1x-u_0y)(u_2x-u_1y) : (u_2x-u_1y)^2) \mapsto (u_1x-u_0y : u_2x-u_1y)$$

so the coefficients of matrix are: $a = u_1$; $b = -u_0$; $c = u_2$ and $d = -u_1$. The matrix itself is

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_2 & -u_1 \end{pmatrix}$$

In the similar way as above, we show when the points $p, q \in U_1$, and again we get as a result the same matrix:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_2 & -u_1 \end{pmatrix}$$

But this is not the whole proof of the theorem, because we have checked only the cases when p and q belong to the same subset U_i. What happens if the points do not belong to the same subset. In this case we have to analyse $(1:0:0) \in U_0$, $(0:1:0) \in U_1$ and $(0:0:1) \in U_2$.

Let us assume that $q = (1:0:0)$ and $p = (0:u_1:u_2)$. Since they do not belong to the same subset, is not possible to try and prove it in affine space. The parametric equation of a line in $\mathbb{P}^2$ will be:

$$(w_0 : w_1 : w_2) = (x^2 : xy : y^2) + t(u_0 - x^2 : u_1 - xy : u_2 - y^2)$$

where t is the parameter. Replacing our points in the equation, gives us the results below:

$$\varphi(q) = (w_0 : w_1 : w_2) = (1:0:0) + t(-1 : u_1 : u_2) = (1 - t : u_1t : u_2t)$$

Replacing the expressions of w_i inside the equation of conic $w_1^2 - w_0w_2 = 0$, gives us the expression of parameter t:

$$t = \frac{u_2}{u_1^2 + u_2}$$

from which we can find the second intersection point $\varphi(q)$:

$$\varphi(q) = (u_1^2 : u_1u_2 : u_2^2)$$

Moving this point to $\mathbb{P}^1$ using the inverse map Θ, gives as result the point $(u_1 : u_2)$. What would have happen if the matrix above is used? Well that is something that we are about to find out. Since the first coordinate of p is zero our matrix is like below:

$$A = \begin{pmatrix} u_1 & 0 \\ u_2 & -u_1 \end{pmatrix}$$

and multiplying it with $(1 : 0)$ gives:

$$\begin{pmatrix} u_1 & 0 \\ u_2 & -u_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

In the similar way as above it can be shown that the same idea works for $q = (0 : 0 : 1)$ and $p = (u_0 : u_1 : 0)$, from which we get the matrix:

$$A = \begin{pmatrix} u_1 & -u_0 \\ 0 & -u_1 \end{pmatrix}$$

and the result:

$$\begin{pmatrix} u_1 & u_0 \\ 0 & -u_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -u_0 \\ -u_1 \end{pmatrix}$$

and as we already know $(-u_0 : -u_1) = (u_0 : u_1)$. At the end the point $(0 : 1 : 0)$ does not belong to our conic so is not taken into consideration.

Finally all this proves that our function $f : \mathbb{P}^1 \to \mathbb{P}^1$ is just a multiplication of each point in $\mathbb{P}^1$ with the matrix A:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_2 & -u_1 \end{pmatrix} \tag{3.8}$$

□

The theorem above is all based in analysing the way how a line and the conic C interact with each other. Having this picture in mind, and finding the intersection points between them we have three possible scenarios. If the intersection between the line and the conic contains two points, then the line is called secant line. If the intersection between them is only one point, then the line is called tangent line. And the last possibility is that the line and the conic does not have intersection points and the line is called exterior line.

While formulating and proving the above theorem, always was mentioned that the chosen point p belongs to $\mathbb{P}^2$, but not to the conic C. A question that could be asked here: is it allowed that the point p to be in the conic C, and why or why not? Let us start assuming that the point $p \in C$. In this case $p = (u_0 : u_1 : u_2) = (w_1^2 : w_1w_2 : w_2^2)$. The matrix A, that represents the map f, will be:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_2 & -u_1 \end{pmatrix} = \begin{pmatrix} w_1w_2 & -w_1^2 \\ w_2^2 & -w_1w_2 \end{pmatrix}$$

The best way to show if this actually is allowed is by finding the determinant of our matrix:

$$detA = det \begin{pmatrix} w_1w_2 & -w_1^2 \\ w_2^2 & -w_1w_2 \end{pmatrix} = -w_1^2w_2^2 + w_1^2w_2^2 = 0$$

The fact that $detA = 0$ means that the matrix is not non-singular, which immediately implicates that the inverse map f^{-1}, does not exist. The answer of our question is that p is not allowed to be inside our conic C.

The map f from the theorem above, is not just a random map from $\mathbb{P}^2$ into itself. It can be shown that this map is an involution [5], which means that for every point $x \in \mathbb{P}^2$, $f(f(x)) = AAx = x$. Let us show this. Let $X = (x : y)$ be any point in $\mathbb{P}^2$. Acting on it with the map f twice we will get:

$$f(X) = AX = \begin{pmatrix} u_1 & -u_0 \\ u_2 & -u_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_1x - u_0y \\ u_2x - u_1y \end{pmatrix}$$

$$f(f(X)) = AAX = \begin{pmatrix} u_1 & -u_0 \\ u_2 & -u_1 \end{pmatrix} \begin{pmatrix} u_1x - u_0y \\ u_2x - u_1y \end{pmatrix} = \begin{pmatrix} (u_1^2 - u_0u_2)x \\ (u_1^2 - u_0u_2)y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix A belong to $PGL(2)$.

3.3 Involutions of normalised conics

Any non-singular conic C in the projective space $\mathbb{P}^2$, according to Sylvester's law of inertia [8], can be brought into a normal form:

$$Q(x) = \pm x_0^2 \pm x_1^2 \pm x_2^2$$

.

For a normalised conic $Q = \sum_{i=0}^{2} x_i^2$ we will define involutions using the secants. So we take a point p outside the conic and through it we take lines which will have two intersection points with the conic. For the two intersection points, a map will be build in that way that takes the first one and maps it into the second one. This process will be shown in the theorem below.

Theorem 3.3.1. *Let* $Q = V_p(\sum_{i=0}^{2} x_i^2)$ *be the normalized conic. A map* $g : Q \to Q_n$ *can be found such that for every* $q \in Q$*:*

$$g(q) = A_p q \tag{3.9}$$

where A_p *is an* $3\times$ *matrix, which is:*

$$\begin{pmatrix} -u_0^2 + u_1^2 + u_2^2 & -2u_0u_1 & -2u_0u_2 \\ -2u_0u_1 & u_0^2 - u_1^2 + u_2^2 & -2u_1u_2 \\ -2u_0u_2 & -2u_1u_2 & u_0^2 + u_1^2 - u_2^2 \end{pmatrix}$$

The coefficients of matrix A_p*, are coefficients of the reflection point* $p \in \mathbb{P}^2$*, but not in* Q*.*

Proof. The process of proving this theorem will have the same idea as in the theorem above. A point $p \in \mathbb{P}^2$ is chosen under the condition that it does not belong to the normalised conic. For any point $q \in Q$, a line through p and q will be build, and the intention is to find the other intersection point between the line and the quadric Q, since the first intersection point is q. The second intersection point will be written by q'.

Using this process and the connection between q and q', the map $g : Q \to Q$ will be build by defining the matrix A_p that represents it. The proof here will be done in a slightly different way from what we have seen in the previous theorem. There, while we proved similar theorem, the problem was analysed locally moving into affine spaces. Here we will not do the same, so the whole problem will be analysed and proved in projective space.

So, as can be noticed above for $q \in Q$, $q' = g(q) = A_p q$ or shown as in terms of matrices:

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{00}x_0 + a_{01}x_1 + a_{02}x_2 \\ a_{01}x_0 + a_{11}x_1 + a_{12}x_2 \\ a_{02}x_0 + a_{12}x_1 + a_{22}x_2 \end{pmatrix}$$

Let the point $p = (u_0 : u_1 : u_2)$ be the chosen point, and the point $q = (x_0 : x_1 : u_2)$ any point in Q. This means the coordinates of q satisfy the equation of the conic Q.

$$\sum_{i=0}^{2} x_i^2 = x_0^2 + x_1^2 + x_2^2 = 0$$

Through those two points, a line is built and the parametric equation of this line will be:

$$(x'_0 : x'_1 : x'_2) = (x_0 : x_1 : x_2) + t(u_0 - x_0 : u_1 - x_1 : u_2 - x_2)$$

Since the point we are trying to find belongs to the line and the conic, the below system must be solved in terms of finding it.

$$\begin{aligned} x'^2_0 + x'^2_1 + x'^2_2 &= 0 \\ x'_i = x_i + t(u_i - x_i), i &= 0, 1, 2 \\ t &\neq 0 \end{aligned}$$

By replacing the values of x'_i in the equation of the quadric we get:

$$(x_0 + t(u_0 - x_0))^2 + (x_1 + t(u_1 - x_1))^2 + (x_2 + t(u_2 - x^2))^2 = 0$$

and from here, is possible to be found the expression for the parameter t, which will be:

$$t = \frac{-2(u_0x_0 + u_1x_1 + u_2x_2)}{u_0^2 + u_1^2 + u_2^2 - 2(u_0x_0 + u_1x_1 + u_2x_2)}$$

Replacing it, on the equation $x'_i = x_i + t(u_i - x_i)$, the coordinates of the intersection point q' will be:

$$x'_0 = \frac{(-u_0^2 + u_1^2 + u_2^2)x_0 - 2u_0u_1x_1 - 2u_0u_2x_2}{u_0^2 + u_1^2 + u_2^2 - 2(u_0x_0 + u_1x_1 + u_2x_2)}$$

$$x'_1 = \frac{-2u_0u_1x_0 + (u_0^2 - u_1^2 + u_2^2)x_1 - 2u_1u_2x_2}{u_0^2 + u_1^2 + u_2^2 - 2(u_0x_0 + u_1x_1 + u_2x_2)}$$

$$x'_2 = \frac{-2u_0u_2x_0 - 2u_1u_2x_1 + (u_0^2 + u_1^2 - u_2^2)x_2}{u_0^2 + u_1^2 + u_2^2 - 2(u_0x_0 + u_1x_1 + u_2x_2)}$$

Since q' belongs to a projective space, it means that $q' = aq'$ for any constant a. At the same time from the equations of the coordinates, it can be seen that the denominator is the same for all of them, so nothing will be changed if the point is multiplied by it. As a result we will have:

$$x'_0 = (-u_0^2 + u_1^2 + u_2^2)x_0 - 2u_0u_1x_1 - 2u_0u_2x_2$$

$$x'_1 = -2u_0u_1x_0 + (u_0^2 - u_1^2 + u_2^2)x_1 - 2u_1u_2x_2$$

$$x'_2 = -2u_0u_2x_0 - 2u_1u_2x_1 + (u_0^2 + u_1^2 - u_2^2)x_2$$

By comparing the general form of the point q' with the results above, is simple to define the coefficients of the matrix A_p.

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} (-u_0^2+u_1^2+u_2^2)x_0 - 2u_0u_1x_1 - 2u_0u_2x_2 \\ -2u_0u_1x_0 + (u_0^2-u_1^2+u_2^2)x_1 - 2u_1u_2x_2 \\ -2u_0u_2x_0 - 2u_1u_2x_1 + (u_0^2+u_1^2-u_2^2)x_2 \end{pmatrix}$$

and the matrix A_p will be:

$$\begin{pmatrix} -u_0^2+u_1^2+u_2^2 & -2u_0u_1 & -2u_0u_2 \\ -2u_0u_1 & u_0^2-u_1^2+u_2^2 & -2u_1u_2 \\ -2u_0u_2 & -2u_1u_2 & u_0^2+u_1^2-u_2^2 \end{pmatrix}$$

□

The map g, is not just a simple map, it is an involution, which means that for any point $x \in Q$, $g(g(x)) = A_pA_px = x$. This will be shown below. Let $X = (x_0 : x_1 : x_2) \in Q$, then:

$$A_pX = \begin{pmatrix} (-u_0^2+u_1^2+u_2^2)x_0 - 2u_0u_1x_1 - 2u_0u_2x_2 \\ -2u_0u_1x_0 + (u_0^2-u_1^2+u_2^2)x_1 - 2u_1u_2x_2 \\ -2u_0u_2x_0 - 2u_1u_2x_1 + (u_0^2+u_1^2-u_2^2)x_2 \end{pmatrix}$$

$$A_p(A_pX) = \begin{pmatrix} (u_0^2+u_1^2+u_2^2)^2x_0 \\ (u_0^2+u_1^2+u_2^2)^2x_1 \\ (u_0^2+u_1^2+u_2^2)^2x_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = X$$

Furthermore, the matrix not only belongs to $PGL(3)$, but it belongs to $PO(3)$ (the set of all non-singular matrices which are orthogonal at the same time).

3.4 Automorphisms of conics

In the last part of this chapter, we will focus on something a bit more different. According to a theorem in linear algebra [2, Theorem 4.5.1, page 116], every quadratic form can be converted into a canonic form. We are going to use this in our conic C, and try to bring it into a canonic form. Not just that, we will find a matrix that actually makes all this transformation possible.

Theorem 3.4.1. *The conic C with the quadratic form $Q(u_0, u_1, u_2) = u_1^2 - u_0u_2$ can be transformed into a canonic form, and the result is:*

$$Q(u_0, u_1, u_2) = -\frac{1}{2}u_0^2 + \frac{1}{2}u_1^2 + u_3^2$$

Proof. We will start proving this theorem, by first finding the symmetric matrix B which represents our conic C. This means that we will find a matrix that will allow to write the function of conic as a product between matrices (vectors), so:

$$Q(u_0, u_1, u_2) = u_1^2 - u_0 u_2 = \begin{pmatrix} u_0 & u_1 & u_2 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}$$

The matrix B in our case will be:

$$B = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

Since we got the matrix B, in the next part we will use the spectral theorem [6] and [2, section 2.8, pages 63, 64] from linear algebra, so these steps will be followed :

1. Find eigenvalues of the matrix B,
2. Find eigen vectors for each of the eigenvalues above,
3. Find the matrix P, which transforms matrix B into the diagonal matrix D, with eigenvalues as diagonal coefficients. The transformation form will be $D = P^{-1}BP$.

First of all, for the matrix B we build the matrix $B - \lambda I$:

$$B - \lambda I = \begin{pmatrix} -\lambda & 0 & -\frac{1}{2} \\ 0 & 1-\lambda & 0 \\ -\frac{1}{2} & 0 & -\lambda \end{pmatrix}$$

and to find the eigenvalues, the characteristic equation $det(B - \lambda I) = 0$ should be solved.

$$det(B - \lambda I) = (1-\lambda)(\lambda^2 - \frac{1}{4}) = 0$$

The solutions of the equation above, which are the eigenvalues of B, are: $\lambda_0 = -\frac{1}{2}$ and $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$.

Next step is finding the eigen vectors for each eigenvalue by solving the system of equations $Bx = \lambda x$. For $\lambda_0 = -\frac{1}{2}$, the system to be solved is $Bv_0 = -\frac{1}{2}{}_0$ so:

$$Bv_0 + \frac{1}{2}v_0 = \begin{pmatrix} -\frac{1}{2}x_2 + \frac{1}{2}x_0 \\ x_1 + \frac{1}{2}x_1 \\ -\frac{1}{2}x_0 + \frac{1}{2}x_2 \end{pmatrix} = 0$$

And as a result we get the vector $v_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

For $\lambda_1 = \frac{1}{2}$, solving the system $Bv_1 = \frac{1}{2}v_1$, so:

$$Bv_1 - \frac{1}{2}v_1 = \begin{pmatrix} -\frac{1}{2}x_2 - \frac{1}{2}x_0 \\ x_1 - \frac{1}{2}x_1 \\ -\frac{1}{2}x_0 - \frac{1}{2}x_2 \end{pmatrix} = 0$$

giving as a result, the vector $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

For $\lambda_2 = 1$, we should solve the system $Bv_2 = v_2$, so:

$$Bv_2 - v_2 = \begin{pmatrix} -\frac{1}{2}x_2 - x_0 \\ x_1 - x_1 \\ -\frac{1}{2}x_0 - x_2 \end{pmatrix} = 0$$

which gives as a result the vector $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Moving on in the third and the last step of the theorem, we will build the matrix P, mentioned in the third step of the process, using the eigen vectors. The characteristic of our matrix P is that its columns actually are made from eigen vectors, so the result will be:

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

which obviously is a non-singular matrix with the inverse matirx, which is just the transposed matrix:

$$P^{-1} = P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

From $D = P^T BP$, the diagonal matrix D that we have been looking for will be:

$$D = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, the canonic form of our conic will be:

$$Q(u_0, u_1, u_2) = -\frac{1}{2}u_0^2 + \frac{1}{2}u_1^2 + u_3^2 \tag{3.10}$$

which is what we have been trying to show with this theorem. □

In the last part of this section, the intention is to build a map between the normalized conic $Q = -x_0^2 + x_1^2 + x_2^2$ and the conic C. After that to build an automorphism on C. To do this, we start by trying to build a map between the normalized conic Q and the conic defined in the theorem above, which we will write with Q''.

Comparing the polynomials of those two conics:

$$f(x) = -x_0^2 + x_1^2 + x_2^2$$

$$f(x'') = -\frac{1}{2}x_0''^2 + \frac{1}{2}x_1''^2 + x_2''^2$$

with $x \in Q$ and $x'' \in Q''$, for a point of Q to be reflected into a point of Q'', it has to be multiplied by the matrix:

$$G = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 3.4.1. There exists a map $\alpha : Q \to Q''$ such that for any point $x \in Q$, $\alpha(x) = Gx \in Q''$ for G defined above.

We continue the process by building a map between Q'' and C. To do that, the theorem 1.2 will be used. Since $D = P^TBP$ then $B = PDP^T$. So if we want to map any point of the conic Q'' into the conic C, is is enough to multiply it by P^T. Let $x \in Q''$, then:

$$f(P^Tx) = (P^Tx)^TD(P^Tx) = x^TPDP^Tx = x^TBx$$

Definition 3.4.2. There exists a map $\gamma : Q'' \to C$, such that for any point $x \in Q''$, $\gamma(x) = P^Tx \in C$, where the matrix P^T is:

$$P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

From two definitions above, it is possible to build a map that connects the normalized conic Q and our conic C. Let us analyse the situation:

$$Q \underset{\alpha}{\rightarrow} Q'' \underset{\gamma}{\rightarrow} C$$

such that, for any $x \in Q$, $\alpha(x) = Gx \in Q''$ and $\gamma(Gx) = P^T Gx \in C$.

Definition 3.4.3. There exists a map $\beta : Q \rightarrow C$, such that for every $x \in Q$, $\beta(x) = Mx$, where M is the matrix which we get as a product between matrices P^T and G. So:

$$M = P^T G = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

The inverse matrix M^{-1} of the matrix M is:

$$M^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \frac{2}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Finally, for our normalized conic $Q = -x_0^2 + x_1^2 + x_2^2$ it is possible to construct a map in the same way as the map $g : Q \rightarrow Q$, which we will write with g_1, and which is involution too. The matrix A_p that represents this map is:

$$A_p = \begin{pmatrix} u_0^2 + u_1^2 + u_2^2 & -2u_0u_1 & -2u_0u_2 \\ 2u_0u_1 & -u_0^2 - u_1^2 + u_2^2 & -2u_1u_2 \\ 2u_0u_2 & -2u_1u_2 & -u_0^2 + u_1^2 - u_2^2 \end{pmatrix}$$

Using it and the map $\beta : Q \rightarrow C$, is possible to define an automorphism [1] in C. Let p' be a chosen point in $\mathbb{P}^2$ and q' any point in C. The reflection of the point q' through the point p' will be happening as below:

$$C \xrightarrow[\beta^{-1}]{} Q \underset{g_1}{\rightarrow} Q \underset{\beta}{\rightarrow} C$$

Definition 3.4.4. There exists an automorphism ψ in the conic C, such that for any point $q' \in C$, it will be reflected through the chosen point p' into another point $\psi(q')$ in C, where:

$$\psi(q') = MA_pM^{-1} = MA_{Mp'}M^{-1} \tag{3.11}$$

where the matrix $MA_{Mp'}M^{-1} \in PGL(3)$.

Let us now show that ψ is an involution. Let $q' \in Q$ be any point.

$$\psi(q') = MA_{Mp'}M^{-1}q'$$

Acting again with the map ψ, gives:

$$\psi(\psi(q')) = MA_{Mp'}M^{-1}MA_{Mp'}M^{-1}q' = MA_{Mp'}A_{Mp'}M^{-1}q' = MM^{-1}q' = q'$$

In the same way automorphisms can be build between any conic Q of the projective space $\mathbb{P}^2$.

Chapter 4

Quadric surfaces

4.1 Isomorphy between quadric surfaces and $\mathbb{P}^1 \times \mathbb{P}^1$

This chapter will have in focus the quadrics in projective space $\mathbb{P}^3$ [7] and [4, Chapter 1], before everything moves into the general case $\mathbb{P}^n$. At first, we will be analysing a specifics quadric in $\mathbb{P}^3$, and how the points inside it will interact if a line is built between them (secants). After that, the focus will be in how two quadrics interact with each other, more specifically any quadric with a quadric written in canonic form.

Similarly with the process explained in two first chapters, here too everything will start by defining a map between the product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$ and the projective space $\mathbb{P}^3$.

Definition 4.1.1. For the product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$ and the projective space $\mathbb{P}^3$, there exists a map $h : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ such that acting with it on $((x_1 : y_1), (x_2 : y_2)) \in \mathbb{P}^1 \times \mathbb{P}^1$ gives as a result:

$$((x_1 : y_1), (x_2 : y_2)) \mapsto (x_1x_2 : x_1y_2 : x_2y_1 : y_1y_2) \tag{4.1}$$

for every $(x_1 : y_1), (x_2 : y_2) \in \mathbb{P}^1$.

Let $Q = h(\mathbb{P}^1 \times \mathbb{P}^1)$ be the image of the map h. Before moving to the main part of the chapter, it will be really important showing the map is an isomorphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to Q by proving the bijectivity.

Lemma 4.1.1. *[3, Proposition 7.11, page 60] The map $h : \mathbb{P}^1 \times \mathbb{P}^1 \to Q$, defined above, is an isomorphism.*

Proof. Let $z = (z_{00} : z_{01} : z_{10} : z_{11}) \in Q$, where at least one of the coordinates is different from zero. Without loosing anything from generality, we assume

that $z_{00} \neq 0$. If $h(x, y) = z$ for $x = (x_0 : x_1), y = (y_0 : y_1) \in \mathbb{P}^1 \times \mathbb{P}^1$, then $x_0 \neq 0$ and $y_0 \neq 0$. So we can move to affine coordinate locally, or have $x_1 = y_1 = 1$, which means $\exists((1 : x_1), (1 : y_1)) \in P^1 \times P^1$ such that $x_1 = z_{10}$ and $y_1 = z_{01}$. This means that h is injective so it is bijective.
From all the information above, is not only proven the bijectivity, but the fact that h and h^{-1} can be given by affine coordinates locally too, so h is an isomorphism.

This isomorphism is known as Segre embedding [3, Construction 7.10, page 60] and $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to Q, and is going to play an important role on the rest of the chapter.

The image of the map, written with Q, even that is mentioned, is not yet defined what actually it represents. That is the intention in the part we are going to discuss. We will prove this represents a quadric hypersurface.

First of all a quadric hypersurface or a quadric [7] is just a generalisation of conics discussed in second chapter. The general equation of a quadric in affine space $\mathbb{A}^3$, is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0 \qquad (4.2)$$

Since our work takes place in projective space, the equation of the quadric in projective space ı^3 will be:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gxt + Hyt + Izt + Jt^2 = 0 \qquad (4.3)$$

$\square$

Lemma 4.1.2. *[3, Example 7.12, page 61] The image of the map $h(\mathbb{P}^1 \times \mathbb{P}^1) = Q$ is a quadric with the equation:*

$$u_1u_2 - u_0u_3 = 0 \qquad (4.4)$$

Proof. To find the equation of Q, we will check at it locally. To do that the projective space $\mathbb{P}^3$ will be written as union of open subsets $U_i \subset \mathbb{P}^3$, $\mathbb{P}^3 = U_0 \cup U_1 \cup U_2 \cup U_3$, where $U_i = \{(u_0 : u_1 : u_2 : u_3) | u_i \neq 0\}$, for $i = 0, 1, 2, 3$.

It must be checked that Q is quadric hypersurface, and find the equation that fits to our case. Let us focus on $Q \cap U_0$ first and see what happens there first. after that, is possible to check the rest.

Let $(x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) \in Q \cap U_0 \subset P^3$, which means $x_1y_1 \neq 0$ so $x_1 \neq 0$ and $y_1 \neq 0$. Using this fact, is possible to rewrite points:

$$(x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) = (1 : \frac{y_2}{y_1} : \frac{x_2}{x_1} : \frac{x_2y_2}{x_1y_1}) = (1 : z_1 : z_2 : z_1z_2)$$

where $z_1 = \frac{y_2}{y_1}$ and $z_2 = \frac{x_2}{x_1}$. Moving from projective space to affine space, gives:

$$(1 : z_1 : z_2 : z_1 z_2) \to (z_1, z_2, z_1 z_2) \in \mathbb{A}^3$$

The intention is to find a function which will represents our Q' in $\mathbb{A}^3$, for which the below relation is true:

$$q(u_1, u_2, u_3) = 0 \leftrightarrow (u_0, u_1, u_2) = (z_1, z_2, z_1 z_2)$$

for some z_1, z_2.

A function that fits to our case is:

$$q(u_1, u_2, u_3) = u_1 u_2 - u_3$$

Moving back to projective space, we get:

$$Q(u_0, u_1, u_2, u_3) = u_1 u_2 - u_0 u_3$$

which is obviously a quadric in $\mathbb{P}^3$

In completely similar way, is possible to show it for other subsets. Let $(x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) \in Q \cap U_1 \subset P^3$, which means $x_1y_2 \neq 0$ so $x_1 \neq 0$ and $y_2 \neq 0$. We get:

$$(x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) = (\frac{y_1}{y_2} : 1 : \frac{x_2y_1}{x_1y_2} : \frac{x_2}{x_1}) = (z_1 : 1 : z_1z_2 : z_2)$$

where $z_1 = \frac{y_1}{y_2}$ and $z_2 = \frac{x_2}{x_1}$. We move from projective space to affine space, getting:

$$(z_1 : 1 : z_1z_2 : z_2) \to (z_1, z_1z_2, z_2) \in A^3$$

A function that fits to our case is:

$$q(u_0, u_2, u_3) = u_2 - u_0 u_3$$

Moving this back to projective space we get:

$$Q(u_0, u_1, u_2, u_3) = u_1 u_2 - u_0 u_3$$

Let $(x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) \in Q \cap U_2 \subset P^3$, which means $x_2y_1 \neq 0$ so $x_2 \neq 0$ and $y_1 \neq 0$. We get:

$$(x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) = (\frac{x_1}{x_2} : 1 : \frac{x_1y_2}{x_2y_1} : \frac{y_2}{y_1}) = (z_1 : 1 : z_1z_2 : z_2)$$

where $z_1 = \frac{x_1}{x_2}$ and $z_2 = \frac{y_2}{y_1}$. We move from projective space to affine space, getting:

$$(z_1 : z_1z_2 : 1 : z_2) \to (z_1, z_1z_2, z_2) \in A^3$$

A function that fits to this case is:

$$q(u_0, u_1, u_3) = u_1 - u_0u_3$$

Moving this back to projective space we get:

$$Q(u_0, u_1, u_2, u_3) = u_1u_2 - u_0u_3$$

For $(x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) \in Q \cap U_3 \subset P^3$, which means $x_1y_2 \neq 0$ so $x_1 \neq 0$ and $y_2 \neq 0$. We get:

$$(x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) = (\frac{x_1y_1}{x_2y_2} : \frac{x_1}{x_2} : \frac{y_1}{y_2} : 1) = (z_1z_2 : z_1 : z_2 : 1)$$

where $z_1 = \frac{x_1}{x_2}$ and $z_2 = \frac{y_1}{y_2}$. We move from projective space to affine space, getting:

$$(z_1z_2 : z_1 : z_2 : 1) \to (z_1z_2, z_1, z_2) \in A^3$$

A function that fits here is:

$$q(u_0, u_1, u_2) = u_1u_2 - u_0$$

Moving this back to projective space we get:

$$Q(u_0, u_1, u_2, u_3) = u_1u_2 - u_0u_3$$

As can be seen from above, after checking our Q locally, we came in conclusion that the image of the map h, is a projective variety V_p:

$$Q = V_p(u_1u_2 - u_0u_3) \tag{4.5}$$

or as a function:

$$Q(u_0, u_1, u_2, u_3) = u_1u_2 - u_0u_3 \tag{4.6}$$

which obviously is a quadric in projective space $\mathbb{P}^3$. □

4.2 Involutions of the product $\mathbb{P}^1 \times \mathbb{P}^1$

As said in the beginning of the chapter, now, the focus will be in our quadric Q, and the way how the points in it interact between each other. Since the map h maps lines from $\mathbb{P}^1 \times \mathbb{P}^1$ into lines that belong to Q, we will try to analyze how two points in Q, are connected with a line that contains both of them, and use that to build a map between $\mathbb{P}^1 \times \mathbb{P}^1$ and itself.

What we are going to do is to use the fact that $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q$ and find two matrices $A, B \in PGL(2)$, which will define the map $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$.

This whole situation will be discussed in details in the next theorem.

Theorem 4.2.1. *For a map $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, there exist two matrices $A, B \in PGL(2)$, where:*

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_3 & -u_2 \end{pmatrix} \tag{4.7}$$

and

$$B = \begin{pmatrix} u_2 & -u_0 \\ u_3 & -u_1 \end{pmatrix} \tag{4.8}$$

such that:

$$f(X, Y) = (AY, BX)$$

with $X, Y \in \mathbb{P}^1$, and u_i coefficients of the reflection point $p \in \mathbb{P}^3$,$i = 0, 1, 2, 3$, where p is not in the quadric Q.

Proof. Let us start proving the theorem by showing what we are trying to find here. We are trying to build a function f between $P^1 \times P^1$ and itself such that for every $(X, Y) \in P^1 \times P^1$ where $X = (x_1 : y_1)$ and $Y = (x_2 : y_2)$, we will have:

$$f(X, Y) = (AY, BX)$$

where $A, B \in PGL(2)$. At first let's take two general two by two matrices and see how they act in any point in $P^1 \times P^1$.

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}; B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \tag{4.9}$$

Now

$$Ay = (a_1x_2 + b_1y_2 : c_1x_2 + d_1y_2)$$
$$Bx = (a_2x_1 + b_2y_1 : c_2x_1 + d_2y_1)$$

This point (Ay, Bx) can be reflected to a point in Q using the map h, giving us:

$$h(Ay, Bx) = h((a_1x_2 + b_1y_2 : c_1x_2 + d_1y_2), (a_2x_1 + b_2y_1 : c_2x_1 + d_2y_1)) =$$
$$((a_1x_2 + b_1y_2)(a_2x_1 + b_2y_1) : (a_1x_2 + b_1y_2)(c_2x_1 + d_2y_1) :$$
$$(c_1x_2 + d_1y_2)(a_2x_1 + b_2y_1) : (c_1x_2 + d_1y_2)(c_2x_1 + d_2y_1))$$

Lets find the coefficients of matrices.

Using the map h we have:

$$q = h((x_1 : y_1), (x_2 : y_2)) = (x_1x_2 : x_1y_2 : y_1x_2 : y_1y_2) = q \in Q \subset \mathbb{P}^3$$

At the same time let $p = (u_0 : u_1 : u_3 : u_4) \in \mathbb{P}^3$, be any point. A line which contains points p and q will be build, and afterwards we will try to find a second point q' which is going to be an intersection between that line and the quadric Q. As is noticed q' will be the second intersection point between them, since q is already the first one.

To analyze the whole situation, the projective space $\mathbb{P}^3$ will be written as union of open subsets: $P^3 = U_0 \cup U_1 \cup U_2 \cup U_3$, where $U_i = \{(u_0 : u_1 : u_2 : u_3)|u_i \neq 0\}$, $i = 0, 1, 2, 3$. So the whole process will be checked locally in $U_i \cap Q$.

Let us assume that $q \in Q \cap U_0$, which means that $x_1 x_2 \neq 0$ giving us:

$$x_1 \neq 0 \Rightarrow X = (x_1 : y_1) = (1 : \frac{y_1}{x_1}) = (1 : z_1)$$

$$x_2 \neq 0 \Rightarrow Y = (x_2 : y_2) = (1 : \frac{y_2}{x_2}) = (1 : z_2)$$

Using this inside our function h gives us:

$$q = h((1 : z_1), (1 : z_2)) = (1 : z_2 : z_1 : z_1 z_2) \in Q \subset P^3$$

Let $p = (u_0 : u_1 : u_2 : u_3) \in \mathbb{P}^3$ such that $p \in U_0$, but not in Q, which leads to:

$$p = (1 : \frac{u_1}{u_0} : \frac{u_2}{u_0} : \frac{u_3}{u_0}) = (1 : v_1 : v_2 : v_3)$$

moving those to points from the projective into affine space we get:

$$q = (1 : z_2 : z_1 : z_1 z_2) \to (z_2, z_1, z_1 z_2) \in A^3$$

$$p = (1 : v_1 : v_2 : v_3) \to (v_1, v_2, v_3) \in A^3$$

What we are going to do here is to build the line through those two points p and q, and afterwards try to find the intersection point between this line and the quadric Q. First step lets write the parametric equation of line.

$$\gamma(t) = (z_2, z_1, z_1 z_2) + t(v_1 - z_2, v_2 - z_1, v_3 - z_1 z_2) = (w_1, w_2, w_3)$$

where (w_1, w_2, w_3) is the intersection point that we are trying to find.

The equation of the quadric in this case is:

$$Q(w_1, w_2, w_3) = w_1 w_2 - w_3$$

and to find the point that we need, the system of equations below should be solved:

$$\begin{aligned} w_1 w_2 &= w_3 \\ t &\neq 0 \end{aligned}$$

From the equation of line we get:

$$(w_1, w_2, w_3) = (z_2 + t(v_1 - z_2), z_1 + t(v_2 - z_1), z_1z_2 + t(v_3 - z_1z_2))$$

Replacing the expressions for w_1, w_2 and w_3, in the equation of quadric gives as a result:

$$(z_2 + t(v_1 - z_2))(z_1 + t(v_2 - z_1)) = z_1z_2 + t(v_3 - z_1z_2)$$

After doing the necessary calculations we get the expression for t:

$$t = \frac{v_3 - v_1z_1 - v_2z_2 + z_1z_2}{(v_1 - z_2)(v_2 - z_1)}$$

The expression for t we replace at the coordinates w_i and we will find the point we have been looking for. So:

$$w_1 = z_2 + t(v_1 - z_2) = z_2 + \frac{v_3 - v_1z_1 - v_2z_2 + z_1z_2}{(v_1 - z_2)(v_2 - z_1)}(v_1 - z_2) = \frac{v_3 - v_1z_1}{v_2 - z_1}$$

$$w_2 = z_1 + t(v_2 - z_1) = z_1 + \frac{v_3 - v_1z_1 - v_2z_2 + z_1z_2}{(v_1 - z_2)(v_2 - z_1)}(v_2 - z_1) = \frac{v_3 - v_2z_2}{v_1 - z_2}$$

$$\begin{aligned} w_3 = z_1z_2 + t(v_3 - z_1z_2) &= z_1z_2 + \frac{v_3 - v_1z_1 - v_2z_2 + z_1z_2}{(v_1 - z_2)(v_2 - z_1)}(v_3 - z_1z_2) \\ &= \frac{(v_3 - v_21z_1)(v_3 - v_2z_2)}{(v_1 - z_2)(v_2 - z_1)} \end{aligned}$$

So we get our point $(w_1, w_2, w_3) \in A^3$:

$$(w_1, w_2, w_3) = (\frac{v_3 - v_1z_1}{v_2 - z_1}, \frac{v_3 - v_2z_2}{v_1 - z_2}, \frac{(v_3 - v_21z_1)(v_3 - v_2z_2)}{(v_1 - z_2)(v_2 - z_1)})$$

Using the expressions for z_i and v_i we have:

$$w_1 = \frac{v_3 - v_1z_1}{v_2 - z_1} = \frac{u_3x_1 - u_1y_1}{u_2x_1 - u_0y_1}$$

$$w_2 = \frac{v_3 - v_2z_2}{v_1 - z_2} = \frac{u_3x_2 - u_2y_2}{u_1x_2 - u_0y_2}$$

$$w_3 = \frac{(v_3 - v_21z_1)(v_3 - v_2z_2)}{(v_1 - z_2)(v_2 - z_1)} = \frac{(u_3x_1 - u_1y_1)(u_3x_2 - u_2y_2)}{(u_2x_1 - u_0y_1)(u_1x_2 - u_0y_2)}$$

So the point is:

$$(w'_0, w'_1, w'_2, w'_3) = ((u_2x_1 - u_0y_1)(u_1x_2 - u_0y_2) : (u_2x_1 - u_0y_1)(u_3x_1 - u_1y_1)$$

$$: (u_1x_2 - u_0y_2)(u_3x_2 - u_2y_2) : (u_3x_1 - u_1y_1)(u_3x_2 - u_2y_2)) \tag{4.10}$$

Comparing the general equation of the point and the equation above, we can find the values of coefficients in matrices A and B, so this gives:

$$a_1 = u_1, b_1 = -u_0, c_1 = u_3, d_1 = -u_2$$

$$a_2 = u_2, b_2 = -u_0, c_2 = u_3, d_2 = -u_1$$

and as a final result we are able so have the real matrices:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_3 & -u_2 \end{pmatrix}; B = \begin{pmatrix} u_2 & -u_0 \\ u_3 & -u_1 \end{pmatrix}$$

The whole process is pretty similar in cases when the points p and q belong to U_1, U_2 and U_3, so everything will be shown with less details.

Let's assume that $q \in Q \cap U_1$, which means that $x_1y_2 \neq 0$, and $p \in U_1$ but not in Q so $u_1 \neq 0$. So:

$$q = h((1 : z_1), (\frac{1}{z_2} : 1)) = (\frac{1}{z_2} : 1 : \frac{z_1}{z_2} : z_1) \in Q \subset \mathbb{P}^3$$

$$p = (\frac{u_0}{u_1} : 1 : \frac{u_2}{u_1} : \frac{u_3}{u_1}) = (v_0 : 1 : v_2 : v_3)$$

Moving those to points from the projective into affine space we get:

$$q = (\frac{1}{z_2} : 1 : \frac{z_1}{z_2} : z_1) \to (\frac{1}{z_2}, \frac{z_1}{z_2}, z_1) \in \mathbb{A}^3$$

$$p = (v_0 : 1 : v_2 : v_3) \to (v_0, v_2, v_3) \in \mathbb{A}^3$$

The parametric equation of line.

$$\gamma(t) = (\frac{1}{z_2}, \frac{z_1}{z_2}, z_1) + t(v_0 - \frac{1}{z_2}, v_2 - \frac{z_1}{z_2}, v_3 - z_1) = (w_0, w_2, w_3)$$

where (w_0, w_2, w_3) is the intersection point that we are trying to find.

The system of equations below should be solved:

$$w_0w_3 = w_2$$
$$t \neq 0$$

After doing the necessary calculations, the expression for t is:

$$t = \frac{v_2 - v_3\frac{1}{z_2} - v_0z_1 + \frac{z_1}{z_2}}{(v_0 - \frac{1}{z_2})(v_3 - z_1)}$$

At the end, the coordinates of the point we have been looking for are:

$$w_0 = \frac{v_2 - v_0 z_1}{v_3 - z_1} = \frac{u_2 x_1 - u_0 y_1}{u_3 x_1 - u_1 y_1}$$

$$w_2 = \frac{(v_0 z_1 - v_2)(v_3 \frac{1}{z_2} - v_2)}{(v_0 - \frac{1}{z_2})(v_3 - z_1)} = \frac{(u_2 x_1 - u_0 y_1)(u_3 x_2 - u_2 y_2)}{(u_1 x_2 - u_0 y_2)(u_3 x_1 - u_1 y_1)}$$

$$w_3 = \frac{(v_2 - v_3 \frac{1}{z_2})}{(v_0 - \frac{1}{z_2})} = \frac{u_3 x_2 - u_2 y_2}{u_1 x_2 - u_0 y_2}$$

So the point is:

$$\begin{aligned}(w_0', w_1', w_2', w_3') = ((u_1 x_2 - u_0 y_2)(u_2 x_1 - u_0 y_1) : (u_1 x_2 - u_0 y_2)(u_3 x_1 - u_1 y_1) \\ : (u_2 x_1 - u_0 y_0)(u_3 x_2 - u_2 y_2) : (u_3 x_1 - u_1 y_1)(u_3 x_2 - u_2 y_2)) \qquad (4.11)\end{aligned}$$

Comparing the equations together, the values of coefficients in matrices A and B are:

$$a_1 = u_1, b_1 = -u_0, c_1 = u_3, d_1 = -u_2$$

$$a_2 = u_2, b_2 = -u_0, c_2 = u_3, d_2 = -u_1$$

and as a final result we are able so have the real matrices:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_3 & -u_2 \end{pmatrix}; B = \begin{pmatrix} u_2 & -u_0 \\ u_3 & -u_1 \end{pmatrix}$$

If $q \in Q \cap U_2$, which means that $x_2 y_1 \neq 0$, and $p \in U_2$ not in q, so $u_2 \neq 0$, gives us:

$$y_1 \neq 0 \Rightarrow x = (x_1 : y_1) = (\frac{x_1}{y_1} : 1) = (\frac{1}{z_1} : 1)$$

$$x_2 \neq 0 \Rightarrow x = (x_2 : y_2) = (1 : \frac{y_2}{x_2}) = (1 : z_2)$$

using this inside our function h we get:

$$q = h((\frac{1}{z_1} : 1), (1 : z_2)) = (\frac{1}{z_1} : \frac{z_2}{z_1} : 1 : z_2) \in Q \subset \mathbb{P}^3$$

$$p = (\frac{u_0}{u_2} : \frac{u_1}{u_2} : 1 : \frac{u_3}{u_2}) = (v_0 : v_1 : 1 : v_3)$$

Moving those to points from the projective into affine space we get:

$$q = (\frac{1}{z_1} : \frac{z_2}{z_1} : 1 : z_2) \to (\frac{1}{z_1}, \frac{z_2}{z_1}, z_2) \in \mathbb{A}^3$$

$$p = (v_0 : v_1 : 1 : v_3) \to (v_0, v_1, v_3) \in \mathbb{A}^3$$

The parametric equation of line is

$$\gamma(t) = (\frac{1}{z_1}, \frac{z_2}{z_1}, z_2) + t(v_1 - \frac{1}{z_1}, v_2 - \frac{z_2}{z_1}, v_3 - z_2) = (w_0, w_1, w_3)$$

where (w_0, w_1, w_3) is the intersection point that we are trying to find. The system of equations below should be solved:

$$\begin{aligned} w_0 w_3 &= w_1 \\ t &\neq 0 \end{aligned}$$

After doing the necessary calculations we get the expression for t:

$$t = \frac{v_1 - v_0 z_2 - v_3 \frac{1}{z_1} + \frac{z_2}{z_1}}{(v_0 - \frac{1}{z_1})(v_3 - z_1)}$$

The expression for t is replaced at the coordinates w_i and we will find the point we have been looking for:

$$w_0 = \frac{v_1 - v_0 z_2}{v_3 - z_2} = \frac{u_1 x_2 - u_0 y_2}{u_3 x_2 - u_2 y_2}$$

$$w_1 = \frac{(v_1 - v_3 \frac{1}{z_1})(v_1 - v_0 z_2)}{(v_0 - \frac{1}{z_1})(v_3 - z_2)} = \frac{(u_3 x_1 - u_1 y_1)(u_1 x_2 - u_0 y_2)}{(u_2 x_1 - u_0 y_1)(u_3 x_2 - u_2 y_2)}$$

$$w_3 = \frac{v_1 - v_3 \frac{1}{z_1}}{v_0 - \frac{1}{z_1}} = \frac{u_3 x_1 - u_1 y_1}{u_2 x_1 - u_0 y_1}$$

So the point is:

$$\begin{aligned} (w_0', w_1', w_2', w_3') = ((u_1 x_2 - u_0 y_2)(u_2 x_1 - u_0 y_1) : (u_3 x_1 - u_1 y_1)(u_1 x_2 - u_0 y_2) \\ : (u_2 x_1 - u_0 y_1)(u_3 x_2 - u_2 y_2) : (u_3 x_1 - u_1 y_1)(u_3 x_2 - u_2 y_2)) \end{aligned} \tag{4.12}$$

Comparing equations, we can find the values of coefficients in matrices A and B, which are:

$$a_1 = u_1, b_1 = -u_0, c_1 = u_3, d_1 = -u_2$$

$$a_2 = u_2, b_2 = -u_0, c_2 = u_3, d_2 = -u_1$$

and as a final result we are able so have the real matrices:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_3 & -u_2 \end{pmatrix}; B = \begin{pmatrix} u_2 & -u_0 \\ u_3 & -u_1 \end{pmatrix}$$

For $q \in Q \cap U_3$, which means that $y_1 y_2 \neq 0$, and $p \in U_3$ but not in Q, which means $u_3 \neq 0$, we will get:

$$q = h((\frac{1}{z_1} : 1), (\frac{1}{z_2} : 1)) = (\frac{1}{z_1 z_2} : \frac{1}{z_1} : \frac{1}{z_2} : 1) \in Q \subset \mathbb{P}^3$$

$$p = (\frac{u_0}{u_3} : \frac{u_1}{u_3} : \frac{u_2}{u_3} : 1) = (v_0 : v_1 : v_2 : 1)$$

Moving those to points from the projective into affine space we get:

$$q = (\frac{1}{z_1 z_2} : \frac{1}{z_1} : \frac{1}{z_2} : 1) \to (\frac{1}{z_1 z_2}, \frac{1}{z_1}, \frac{1}{z_2}) \in \mathbb{A}^3$$

$$p = (v_0 : v_1 : v_2 : 1) \to (v_0, v_1, v_2) \in \mathbb{A}^3$$

The parametric equation of line.

$$\gamma(t) = (\frac{1}{z_1 z_2}, \frac{1}{z_1}, \frac{1}{z_2}) + t(v_0 - \frac{1}{z_1 z_2}, v_1 - \frac{1}{z_1}, v_2 - \frac{1}{z_2}) = (w_0, w_1, w_2)$$

where (w_0, w_1, w_2) is the intersection point that we are trying to find. The system of equations below should be solved:

$$w_1 w_2 = w_0$$
$$t \neq 0$$

After doing the necessary calculations we get the expression for t:

$$t = \frac{v_0 - v_2 \frac{1}{z_1} - v_1 \frac{1}{z_2} + \frac{1}{z_1 z_2}}{(v_1 - \frac{1}{z_1})(v_2 - \frac{1}{z_2})}$$

The expression for t we replace at the coordinates w_i and we will find the point we have been looking for:

$$w_0 = \frac{(v_0 - v_1 \frac{1}{z_2})(v_0 - v_2 \frac{1}{z_1})}{(v_1 - \frac{1}{z_1})(v_2 - \frac{1}{z_2})} = \frac{(u_0 y_2 - u_1 x_2)(u_0 y_1 - u_2 x_1)}{(u_1 y_1 - u_3 x_1)(u_2 y_2 - u_3 x_2)}$$

$$w_1 = \frac{v_0 - v_1 \frac{1}{z_2}}{v_2 - \frac{1}{z_2}} = \frac{u_0 y_2 - u_1 x_2}{u_2 y_2 - u_0 x_2}$$

$$w_2 = \frac{v_0 - v_2 \frac{1}{z_1}}{v_1 - \frac{1}{z_1}} = \frac{u_0 y_1 - u_2 x_1}{u_1 y_1 - u_3 x_1}$$

So the point is:

$$(w_0', w_1', w_2', w_3') = ((u_1 x_2 - u_0 y_2)(u_2 x_1 - u_0 y_1) : (u_1 x_2 - u_0 y_2)(u_3 x_1 - u_1 y_1)$$

$$: (u_2x_1 - u_0y_1)(u_3x_2 - u_2y_2) : (u_3x_1 - u_1y_1)(u_3x_2 - u_2y_2)) \tag{4.13}$$

Comparing equations, we can find the values of coefficients in matrices A and B:

$$a_1 = u_1, b_1 = -u_0, c_1 = u_3, d_1 = -u_2$$

$$a_2 = u_2, b_2 = -u_0, c_2 = u_3, d_2 = -u_1$$

and as a final result we are able so have the real matrices:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_3 & -u_2 \end{pmatrix}; B = \begin{pmatrix} u_2 & -u_0 \\ u_3 & -u_1 \end{pmatrix}$$

The proof which is done till now, is not yet complete. As it can be noticed, there have been considered only the cases where p and q belong to the same subset U_i. The question is, if the theorem is true even when p and q do not belong to the same subset.

For each of U_i-s, it exists only one point that does not belong to other subsets. Those points are: $(1 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$

Let us start with $q = (1 : 0 : 0 : 0)$ and $p = (0 : u_1 : u_2 : u_3)$. The parametric equation of the line through p and q will be:

$$(w_0 : w_1 : w_2 : w_3) = (1 : 0 : 0 : 0) + t(-1 : u_1 : u_2 : u_3)$$

which after the right calculations will be:

$$(w_0 : w_1 : w_2 : w_3) = (1 - t : u_1t : u_2t : u_3t)$$

The system of equations below should be solved:

$$\begin{aligned} w_1w_2 &= w_0w_3 \\ t &\neq 0 \end{aligned}$$

Solving this system, gives us the parameter t:

$$t = \frac{u_3}{u_3 + u_1u_2}$$

Replacing the parameter t inside the equation of line gives us the second intersection point between the line and the quadric Q, which is:

$$\begin{aligned} (w_0 : w_1 : w_2 : w_3) &= (\frac{u_1u_2}{u_3 + u_1u_2} : \frac{u_1u_3}{u_3 + u_1u_2} : \frac{u_2u_3}{u_3 + u_1u_2} : \frac{u_3^2}{u_3 + u_1u_2}) \\ &= (u_1u_2 : u_1u_3 : u_2u_3 : u_3^2) \end{aligned}$$

Now let us show that the same point will be given if the matrices A and B will be used. Moving the point q back to $\mathbb{P}^1 \times \mathbb{P}^1$, we will get as a result the points $X = (1 : 0), Y = (1 : 0) \in \mathbb{P}^1$. Using the coordinates of the point p, our matrices will be:

$$A = \begin{pmatrix} u_1 & 0 \\ u_3 & -u_2 \end{pmatrix}, B = \begin{pmatrix} u_2 & 0 \\ u_3 & -u_1 \end{pmatrix}$$

Reacting in our points X and Y, with the matrices B and A respectively, gives the results:

$$AY = \begin{pmatrix} u_1 & 0 \\ u_3 & -u_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$$
$$BX = \begin{pmatrix} u_2 & 0 \\ u_3 & -u_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

Using our map h on the points AY and BX gives us the result:

$$h(AY, BX) = h((u_1 : u_3), (u_2 : u_3)) = (u_1u_2 : u_1u_3 : u_2u_3 : u_3^2)$$

which is the same as the intersection point between the line and the quadric Q, we found above. This proves that the map f can be expressed by matrices A and B.

Similar with what is done above, the theorem can be proven for the other points, and because of that the detalis will be avoided.

Let $q = (0 : 1 : 0 : 0)$ and $p = (u_0 : 0 : u_2 : u_3)$. The equation of the line will be:

$$(w_0 : w_1 : w_2 : w_3) = (u_0t : 1 - t : u_2t : u_3t)$$

Solving the system of equations:

$$\begin{aligned} w_1w_2 &= w_0w_3 \\ t &\neq 0 \end{aligned}$$

we are able to find the parameter t:

$$t = \frac{u_2}{u_2 + u_0u_3}$$

So the second intersection point between the line and the quadric will be:

$$(w_0 : w_1 : w_2 : w_3) = (u_0u_2 : u_0u_3 : u_2^2 : u_2u_3)$$

For our point q, we get $X = (1 : 0)$ and $Y = (0 : 1)$. And for the point p, matrices A and B will be:

$$A = \begin{pmatrix} 0 & -u_0 \\ u_3 & -u_2 \end{pmatrix}, B = \begin{pmatrix} u_2 & -u_0 \\ u_3 & 0 \end{pmatrix}$$

So the point AY and BX are:

$$AY = \begin{pmatrix} 0 & -u_0 \\ u_3 & -u_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -u_0 \\ -u_2 \end{pmatrix}$$

$$BX = \begin{pmatrix} u_2 & -u_0 \\ u_3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

The final result will be:

$$h(AY, BX) = h((u_0 : u_2), (u_2 : u_3)) = (u_0u_2 : u_0u_3 : u_2^2 : u_2u_3)$$

which is the same point we found above.

For $q = (0 : 0 : 1 : 0)$ and $p = (u_0 : u_1 : 0 : u_3)$, the equation of the line will be:

$$(w_0 : w_1 : w_2 : w_3) = (u_0t : u_1t : 1 - t : u_3t)$$

Solving the system of equations

$$\begin{aligned} w_1w_2 &= w_0w_3 \\ t &\neq 0 \end{aligned}$$

the parameter t is found:

$$t = \frac{u_1}{u_1 + u_0u_3}$$

giving the second intersection point between the line and the quadric:

$$(w_0 : w_1 : w_2 : w_3) = (u_0u_1 : u_2^2 : u_0u_3 : u_1u_3)$$

For our point q, we get $X = (0 : 1)$ and $Y = (1 : 0)$. For the point p, the matrices A and B will be:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_3 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -u_0 \\ u_3 & -u_1 \end{pmatrix}$$

So the point AY and BX are:

$$AY = \begin{pmatrix} u_1 & -u_0 \\ u_3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$$

$$BX = \begin{pmatrix} 0 & -u_0 \\ u_3 & -u_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -u_0 \\ -u_1 \end{pmatrix}$$

The final result will be:

$$h(AY, BX) = h((u_1 : u_3), (u_0 : u_1)) = (u_0u_1 : u_1^2 : u_0u_3 : u_1u_3)$$

which is the same point we found above.

Completely similar, for $q = (0 : 0 : 0 : 1)$ and $p = (u_0 : u_1 : u_2 : 0)$, the equation of line will be:

$$(w_0 : w_1 : w_2 : w_3) = (u_0 t : u_1 t : u_2 t : 1 - t)$$

By solving the system of equations

$$\begin{aligned} w_1 w_2 &= w_0 \\ t &\neq 0 \end{aligned}$$

the parameter t will be:

$$t = \frac{u_0}{u_0 + u_1 u_2}$$

which gives the intersection point:

$$(w_0 : w_1 : w_2 : w_3) = (u_0^2 : u_0 u_1 : u_0 u_2 : u_1 u_2)$$

For our point q, we get $X = (0 : 1)$ and $Y = (0 : 1)$. For the point p, the matrices A and B will be:

$$A = \begin{pmatrix} u_1 & -u_0 \\ 0 & -u_2 \end{pmatrix}, B = \begin{pmatrix} u_2 & -u_0 \\ 0 & -u_1 \end{pmatrix}$$

So the point AY and BX are:

$$AY = \begin{pmatrix} u_1 & -u_0 \\ 0 & -u_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -u_0 \\ -u_2 \end{pmatrix}$$

$$BX = \begin{pmatrix} u_2 & -u_0 \\ 0 & -u_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -u_0 \\ -u_1 \end{pmatrix}$$

The final result will be:

$$h(AY, BX) = h((u_0 : u_2), (u_0 : u_1)) = (u_0^2 : u_0 u_1 : u_0 u_2 : u_1 u_2)$$

which is the same point we found above.

Finally the proof is complete, and the map $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is:

$$f(X, Y) = (AY, BX)$$

where:

$$A = \begin{pmatrix} u_1 & -u_0 \\ u_3 & -u_2 \end{pmatrix}, B = \begin{pmatrix} u_2 & -u_0 \\ u_3 & -u_1 \end{pmatrix}$$

$\square$

After the theorem is proved, is worthy to mention that the lines, built between points of the quadric Q, are called secants, if the intersection points are different from each other, and tangent line, if the intersection points are equal.

There is something else that should be discussed about the theorem. The whole time was mentioned that the point p belongs to the projective space $\mathbb{P}^3$, but it is not contained in the quadric Q. The question is: Is it allowed to be chosen a point p which belongs to Q? Well check it out. Let us assume that $p \in Q$. This means that the coordinates of $p = (u_0 : u_1 : u_2 : u_3)$ satisfy the equation of our quadric:

$$u_1u_2 - u_0u_3 = 0$$

So how does this effects our matrices A and B? This can be seen by calculating the determinant of them.

$$detA = det(\begin{pmatrix} u_1 & -u_0 \\ u_3 & -u_2 \end{pmatrix}) = -u_1u_2 + u_0u_3 = 0$$

$$detB = det(\begin{pmatrix} u_2 & -u_0 \\ u_3 & -u_1 \end{pmatrix}) = -u_1u_2 + u_0u_3 = 0$$

From this results is clear that matrices would be singular, which means the inverse matrices do not exist. In that case A and B would not belong to $PGL(2)$. So it is not allowed to choose the point p that belongs to Q.

The beautiful part of this map is that it can be proved it is an involution. Let $X = (x_0 : x_1)$ and $Y = (y_0 : y_1)$ be two points in $\mathbb{P}^1$, so $(X, Y) \in \mathbb{P}^1 \times \mathbb{P}^1$. From $f(f((X, Y))) = (ABX, BAY)$ we get:

$$ABX = \begin{pmatrix} u_1 & -u_0 \\ u_3 & -u_2 \end{pmatrix} \begin{pmatrix} u_2x_0 - u_0x_1 \\ u_3x_0 - u_1x_1 \end{pmatrix} = \begin{pmatrix} (u_1u_2 - u_0u_3)x_0 \\ (u_1u_2 - u_1u_3)x_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = X$$

$$BAY = \begin{pmatrix} u_2 & -u_0 \\ u_3 & -u_1 \end{pmatrix} \begin{pmatrix} u_1y_0 - u_0y_1 \\ u_3y_0 - u_2y_1 \end{pmatrix} = \begin{pmatrix} (u_1u_2 - u_0u_3)y_0 \\ (u_1u_2 - u_1u_3)y_1 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = Y$$

which means that $f(f((X, Y))) = (ABX, BAY) = (X, Y)$.

4.3 Involutions of normalized quadric surfaces

According to Sylvester's law of inertia [8], any non-singular quadric Q can be brought into a normal form:

$$Q_n(x) = \pm x_0^2 \pm x_1^2 \pm x_2^2 \pm x_3^2$$

.

For a normalised quadric $Q_n(x) = \sum_{i=0}^{3} x_i^2$ we will define involutions using the secants. So we take a point p outside the quadic, and through it take lines which will have two intersection points with the quadric. For the two intersection points, a map will be build in that way that takes the first one and maps it into the second one. This process will be shown in the theorem below.

Theorem 4.3.1. *Let $Q = V_p(\sum_{i=0}^{3} x_i^2)$ be the normalized quadric. A map $g : Q_n \to Q_n$ can be found such that for every $q \in Q_n$:*

$$g(q) = A_p q \tag{4.14}$$

where A_p is an 4×4 matrix which is:

$$\begin{pmatrix} -u_0^2 + \sum_{i=1}^{3} u_i^2 & -2u_0u_1 & -2u_0u_2 & -2u_0u_3 \\ -2u_0u_1 & u_0^2 - u_1^2 + u_2^2 + u_3^2 & -2u_1u_2 & -2u_1u_3 \\ -2u_0u_2 & -2u_1u_2 & u_0^2 + u_1^2 + u_2^2 + u_3^2 & -2u_2u_3 \\ -2u_0u_3 & -2u_1u_3 & -2u_2u_3 & \sum_{i=0}^{2} u_i^2 - u_3^2 \end{pmatrix}$$

The coefficients of matrix A_p, are coefficients of a chosen point $p \in \mathbb{P}^3$, but not in Q_n.

Proof. The process of proving this theorem will have the same idea as in the previous chapters. A point $p \in \mathbb{P}^3$ is chosen under the condition that it does not belong to our quadric. For any point $q \in Q_n$, a line through p and q will be built, and the intention is to find the other intersection point between the line and the quadric Q_n, since the first intersection point is q. The second intersection point will be written by q'.

Using this process and the connection between q and q', the map $g : Q_n \to Q_n$ will be build by defining the matrix A_p that represents it. The proof here will be done in a slightly different way from what we have seen in the previous theorem. There, while we proved similar theorem, the problem was analysed locally moving into affine spaces. Here we will not do the same, so the whole problems will be analysed and proved in projective space.

So, as can be noticed above for $q \in Q_n$, $q' = g(q) = A_p q$ or shown as in terms of matrices:

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + a_{03}x_3 \\ a_{01}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{02}x_0 + a_{12}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{03}x_0 + a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \end{pmatrix}$$

Let the point $p = (u_0 : u_1 : u_2 : u_3)$ be the chosen point, and the point $q = (x_0 : x_1 : u_2 : x_3)$ any point in Q_n. This means the coordinates of q satisfy the equation of the quadric Q_n.

$$\sum_{i=0}^{3} x_i^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$$

Through those two points, a line is built and the parametric equation of this line will be:

$$(x_0' : x_1' : x_2' : x_3') = (x_0 : x_1 : x_2 : x_3) + t(u_0 - x_0 : u_1 - x_1 : u_2 - x_2 : u_3 - x_3)$$

Since the point we are trying to find belongs to the line and the quadric, the below system must be solved in terms of finding it.

$$\begin{aligned} x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2 = 0 \\ x_i' = x_i + t(u_i - x_i), i = 0, 1, 2, 3 \\ t \neq 0 \end{aligned}$$

By replacing the values of x_i' in the equation of the quadric we get:

$$(x_0+t(u_0-x_0))^2+(x_1+t(u_1-x_1))^2+(x_2+t(u_2-x^2))^2+(x_n+t(u_n-x_n))^2 = 0$$

and from here, is possible to be found the expression for the parameter t, which will be:

$$t = \frac{-2(u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3)}{u_0^2 + u_1^2 + u_2^2 + u_3^2 - 2(u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3)}$$

Replacing it, on the equation $x_i' = x_i + t(u_i - x_i)$, the coordinates of the intersection point q' will be:

$$x_0' = \frac{(-u_0^2 + u_1^2 + u_2^2 + u_3^2)x_0 - 2u_0u_1x_1 - 2u_0u_2x_2 - 2u_0u_3x_3}{u_0^2 + u_1^2 + u_2^2 + u_3^2 - 2(u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3)}$$

$$x_1' = \frac{-2u_0u_1x_0 + (u_0^2 - u_1^2 + u_2^2 + u_3^2)x_1 - 2u_1u_2x_2 - 2u_1u_3x_3}{u_0^2 + u_1^2 + u_2^2 + u_3^2 - 2(u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3)}$$

$$x_2' = \frac{-2u_0u_2x_0 - 2u_1u_2x_1 + (u_0^2 + u_1^2 - u_2^2 + u_3^2)x_2 - 2u_2u_3x_3}{u_0^2 + u_1^2 + u_2^2 + u_3^2 - 2(u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3)}$$

$$x_3' = \frac{-2u_0u_3x_0 - 2u_1u_3x_1 - 2u_2u_3x_2 + (u_0^2 + u_1^2 + u_2^2 - u_3^2)x_3}{u_0^2 + u_1^2 + u_2^2 + u_3^2 - 2(u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3)}$$

Since q' belongs to a projective space, it means that $q' = aq'$ for any constant a. At the same time from the equations of the coordinates, it can be seen that the denominator is the same for all of them, so nothing will be changed if the point is multiplied by it. As a result we will have:

$$\begin{aligned}
x'_0 &= (-u_0^2 + u_1^2 + u_2^2 + u_3^2)x_0 - 2u_0u_1x_1 - 2u_0u_2x_2 - 2u_0u_3x_3 \\
x'_1 &= -2u_0u_1x_0 + (u_0^2 - u_1^2 + u_2^2 + u_3^2)x_1 - 2u_1u_2x_2 - 2u_1u_3x_3 \\
x'_2 &= -2u_0u_2x_0 - 2u_1u_2x_1 + (u_0^2 + u_1^2 - u_2^2 + u_3^2)x_2 - 2u_2u_3x_3 \\
x'_3 &= -2u_0u_3x_0 - 2u_1u_3x_1 - 2u_2u_3x_2 + (u_0^2 + u_1^2 + u_2^2 - u_3^2)x_3
\end{aligned}$$

By comparing the general form of the point q' with the results above, is simple to define the coefficients of the matrix A_p.

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} (-u_0^2 + u_1^2 + u_2^2 + u_3^2)x_0 - 2u_0u_1x_1 - 2u_0u_2x_2 - 2u_0u_3x_3 \\ -2u_0u_1x_0 + (u_0^2 - u_1^2 + u_2^2 + u_3^2)x_1 - 2u_1u_2x_2 - 2u_1u_3x_3 \\ -2u_0u_2x_0 - 2u_1u_2x_1 + (u_0^2 + u_1^2 - u_2^2 + u_3^2)x_2 - 2u_2u_3x_3 \\ -2u_0u_3x_0 - 2u_1u_3x_1 - 2u_2u_3x_2 + (u_0^2 + u_1^2 + u_2^2 - u_3^2)x_3 \end{pmatrix}$$

and the matrix A_p will be:

$$\begin{pmatrix} -u_0^2 + \sum_{i=1}^3 u_i^2 & -2u_0u_1 & -2u_0u_2 & -2u_0u_3 \\ -2u_0u_1 & u_0^2 - u_1^2 + u_2^2 + u_3^2 & -2u_1u_2 & -2u_1u_3 \\ -2u_0u_2 & -2u_1u_2 & u_0^2 + u_1^2 + u_2^2 + u_3^2 & -2u_2u_3 \\ -2u_0u_3 & -2u_1u_3 & -2u_2u_3 & \sum_{i=0} 2u_i^2 - u_3^2 \end{pmatrix}$$

□

Now it will be shown that this map is an involution, so for any point $X = (x_0 : x_1 : x_2 : x_3) \in Q_n$, $g(g(X)) = A_pA_pX = X$.

$$A_pX = \begin{pmatrix} (-u_0^2 + u_1^2 + u_2^2 + u_3^2)x_0 - 2u_0u_1x_1 - 2u_0u_2x_2 - 2u_0u_3x_3 \\ -2u_0u_1x_0 + (u_0^2 - u_1^2 + u_2^2 + u_3^2)x_1 - 2u_1u_2x_2 - 2u_1u_3x_3 \\ -2u_0u_2x_0 - 2u_1u_2x_1 + (u_0^2 + u_1^2 - u_2^2 + u_3^2)x_2 - 2u_2u_3x_3 \\ -2u_0u_3x_0 - 2u_1u_3x_1 - 2u_2u_3x_2 + (u_0^2 + u_1^2 + u_2^2 - u_3^2)x_3 \end{pmatrix}$$

$$A_p(A_pX) = \begin{pmatrix} (u_0^2 + u_1^2 + u_2^2 + u_3^2)^2x_0 \\ (u_0^2 + u_1^2 + u_2^2 + u_3^2)^2x_1 \\ (u_0^2 + u_1^2 + u_2^2 + u_3^2)^2x_2 \\ (u_0^2 + u_1^2 + u_2^2 + u_3^2)^2x_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = X$$

Furthermore, the matrix not only belongs to $PGL(4)$, but it belongs to $PO(4)$

4.4 Automorphisms of quadric surfaces

The last section is about automorphisms of quadrics, more exactly the quadric $Q = V_p(u_1u_2 - u_0u_3)$ will be transformed into canonic form [2, theorem 4.5.1, page 116], and afterwards it will be normalized. Using this whole process it is possible to build to build maps between quadric and the normalized quadric.

We will start this process by transforming the quadric Q into canonic form in the theorem below.

Theorem 4.4.1. *The quadric Q with the quadric form $Q(u_0, u_1, u_2, u_3) = u_1u_2 - u_0u_3$ can be transformed into a canonic form, and the result will be:*

$$Q = -\frac{1}{2}u_0 - \frac{1}{2}u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_3$$

Proof. First we will find the symmetric matrix B, which represents our quadric Q. Writing the quadric form as product of matrices, gives:

$$Q(u_0, u_1, u_2, u_3) = u_1u_2 - u_0u_3 = \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

The matrix B will be:

$$B = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

Using the spectral theorem, the matrix B will be converted into a diagonal matrix, that represents the base of the canonic form [2, Section 2.8, pages 63, 64] and [6]. So similar with the process we have used in the chapter one, again three main steps are going to be followed :

1. Find eigenvalues of the matrix B,
2. For each eigenvalue, eigen vector will be found,
3. Build the invertible matrix P, which will transform the matrix B into a diagonal matrix D, with the eigenvalues in the diagonal. The transformation will be of the form $D = P^{-1}BP$.

Let us find the eigen values, first by building a matrix $B - \lambda I$:

$$B - \lambda I = \begin{pmatrix} -\lambda & 0 & 0 & -\frac{1}{2} \\ 0 & -\lambda & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\lambda & 0 \\ -\frac{1}{2} & 0 & 0 & -\lambda \end{pmatrix}$$

and after that, by solving the characteristic equation, the eigenvalues will be found. The characteristic equation will be:

$$det(B-\lambda I) = det(\begin{pmatrix} -\lambda & 0 & 0 & -\frac{1}{2} \\ 0 & -\lambda & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\lambda & 0 \\ -\frac{1}{2} & 0 & 0 & -\lambda \end{pmatrix}) = \lambda^4 - \frac{1}{16} = 0$$

The solutions of the equation, which at the same time are the eigenvalues, are $\lambda_0 = -\frac{1}{2}$ and $\lambda_1 = \frac{1}{2}$.

Next step is finding the eigen vectors for both eigenvalues, by solving the system of equations $Bx = \lambda x$.

For the eigenvalue $\lambda_0 = -\frac{1}{2}$, the system of equations will be:

$$Bx + \lambda_0 x = \begin{pmatrix} -\frac{1}{2}x_3 + \frac{1}{2}x_0 \\ \frac{1}{2}x_2 + \frac{1}{2}x_1 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ -\frac{1}{2}x_0 + \frac{1}{2}x_3 \end{pmatrix} = 0$$

From the system above, the eigen vectors are $v_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $v_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

For the eigenvalue $\lambda_1 = \frac{1}{2}$, the system of the equations will be:

$$Bx + \lambda_1 x = \begin{pmatrix} -\frac{1}{2}x_3 + \frac{1}{2}x_0 \\ \frac{1}{2}x_2 + \frac{1}{2}x_1 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ -\frac{1}{2}x_0 + \frac{1}{2}x_3 \end{pmatrix} = 0$$

Solving the system, gives as a result the eigen vectors that correspond to the eigenvalue, and they are $v_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

The eigen vectors that we found above, will be used to finish the last part of the proof, by building the matriw P, which help the diagonalisation process. The columns of this matrix will contain those vectors, so the matrix is:

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Obviously our matrix P is invertible, and the inverse matrix is:

$$P^{-1} = P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

From the rule $D = P^{-1}BP = P^TBP$, is possible for us to find the diagonal matrix D, which is:

$$D = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Finally, it is possible to write the canonic form of the quadric, and the final result is:

$$Q(u_0, u_1, u_2, u_3) = -\frac{1}{2}u_0^2 - \frac{1}{2}u_1^2 + \frac{1}{2}u_2^2 + \frac{1}{2}u_3^2$$

what was asked to be proved. □

In the final part of the chapter, our intention is to build a map between our quadric Q and the normalized quadric Q_n. The process will begin by defining a map between the normalized quadric Q_n and the canonic form of our quadric that we will write with the symbol Q''. If we compare their polynomials:

$$Q_n(x) = -x_0^2 - x_1^2 + x_2^2 + x_3^2$$

$$Q''(x'') = -\frac{1}{2}x_0''^2 - \frac{1}{2}x_1''^2 + \frac{1}{2}x_2''^2 + \frac{1}{2}x_3''^2$$

for any point $x \in Q_n$, if we multiply it with a diagonal matrix with the coefficients to be μ_i, such that $\mu_i = \lambda_i$, $i = 0, 1, 2, 3$, the point we get will be inside Q''. Let this matrix be the matrix G of the form:

$$G = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Definition 4.4.1. There exists a map $\alpha : Q_n \to Q''$, such that:

$$\alpha(x) = Gx$$

for any $x \in Q_n$, and the matrix G the one defined above.

On the other side, using the theorem above, it is possible to build a map between the quadics Q and Q''. Since $D = P^T BP$, then $B = PDP^T$. Based on it, for every $x'' \in Q''$, multiplying it by the matrix P^T, the point will be reflected into Q:

$$f(P^T x'') = (P^T x'')^T D(P^T x'') = x''^T PDP^T x'' = x''^T Bx''$$

Definition 4.4.2. There exists a map $\gamma : Q'' \to Q$, such that for any $x'' \in Q''$

$$\gamma(x'') = P^T x''$$

with the matrix P^T defined in the theorem 1.2

Now since the map between Q_n and Q'' and the map between Q'' and Q have been defined, it can be built the map between Q_n and Q.

$$Q_n \underset{\alpha}{\to} Q'' \underset{\gamma}{\to} Q$$

Definition 4.4.3. There exists a map $\beta : Q_n \to Q$, which is a composition between maps α and γ, such that for every $x \in Q_n$:

$$\beta(x) = Mx$$

where M is a 4×4 matrix, defined as a product between matrices G and P^T. So:

$$M = GP^T = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

For our normalized quadric $Q_n(u_0, u_1, u_2, u_3) = -u_0^2 - u_1^2 + u_2^2 + u_3^2$, same as in the previous section, there can be build a map $g_1 : Q_n \to Q_n$, which is involution. The matrix A_p that represents this map will be:

$$\begin{pmatrix} u_0^2 - u_1^2 + u_2^2 + u_3^2 & +2u_0u_1 & -2u_0u_2 & 2u_0u_3 \\ 2u_0u_1 & -u_0^2 + \sum_{i=1}^{3} u_i^2 & -2u_1u_2 & -2u_1u_3 \\ 2u_0u_2 & 2u_1u_2 & -\sum_{i=0}^{2} u_i^2 + u_3^2 & -2u_2u_3 \\ 2u_0u_3 & 2u_1u_3 & -2u_2u_3 & -u_0^2 - u_1^2 + u_2^2 + u_3^2 \end{pmatrix}$$

Using this map and the map β, a map $\psi : Q \to Q$ will be defined, which will reflect any point $q' \in Q$ through a chosen point $p' \in \mathbb{P}^2$, but not in Q, into another point again in Q. Let us analyse this situation:

$$Q \xrightarrow[\beta^{-1}]{} Q_n \underset{g_1}{\to} Q_n \underset{\beta}{\to} Q$$

Definition 4.4.4. There Exists an automorphism $\psi : Q \to Q$, such that for any point $q' \in Q$, it is reflected into another point $\psi(q')$ through a chosen point p', where:

$$\psi(q') = MA_pM^{-1}q' = MA_{Mp'}M^{-1}q' \tag{4.15}$$

where the matrix $MA_{Mp'}M^{-1} \in PGL(4)$

Let us now show that ψ is an involution. Let $q' \in Q$ be any point.

$$\psi(q') = MA_{Mp'}M^{-1}q'$$

Acting again with the map ψ, gives:

$$\psi(\psi(q')) = MA_{Mp'}M^{-1}MA_{Mp'}M^{-1}q' = MA_{Mp'}A_{Mp'}M^{-1}q' = MM^{-1}q' = q'$$

Is worthy to mention that for any quadric inside $\mathbb{P}^2$, can be build automorphisms [1] in the same way as for quadric Q.

Chapter 5

Quadric hypersurfaces

5.1 Involutions of normalized quadric hypersurfaces

The previous chapters were focused on conics in projective space $\mathbb{P}^2$ and quadrics in projective space $\mathbb{P}^3$, showing the maps from them to themself, and after that even between different conics respectivly between quadrics. In this chapter, we will move in more general problems. First of all everything that will be shown here is going to take place in the projective space $\mathbb{P}^n$. A quadric in this projective space is just a generalisation of a conic (ellipse, parabola and hyperbola), and sometimes is called quadric hypersurface too [7].

The first part will be oriented on the normalized quadrics. According to the Sylvester's law of inertia [8], any non-singular quadric form in a real projective space can be put into a normal form:

$$Q(x_0, x_1, ..., x_n) = \pm x_0^2 \pm x_1^2 \pm ... \pm x_n^2$$

We will build a map, between the quadric and itself, by taking lines through a chosen point p that belongs to the projective space $\mathbb{P}^2$, and finding the the intersections between the lines and the quadric. And after that the purpose will be to find the matrix which will represent the map.

After that, by choosing any quadric in projective space $\mathbb{P}^n$, the process of transforming it into the normalized form will be shown. This way we will be able to build maps between quadrics, and not only that, at the same time see if it is possible to generalise them into maps that takes points from $\mathbb{P}^n$ into itself.

Let us start the whole process by proving the theorem above.

Theorem 5.1.1. *Let $Q = V_p(\sum_{i=0}^n x_i^2)$ be the normalized quadric. A map $g : Q \to Q$ can be found such that for every $q \in Q$:*

$$g(q) = A_p q \tag{5.1}$$

where A_p is an $(n+1) \times (n+1)$ matrix which is:

$$\begin{pmatrix} -u_0^2 + u_1^2 + ... + u_n^2 & -2u_0u_1 & ... & -2u_0u_n \\ -2u_0u_1 & u_0^2 - u_1^2 + ... + u_n^2 & ... & -2u_1u_2 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ -2u_0u_n & -2u_1u_n & ... & u_0^2 + u_1^2 + ... - u_n^2 \end{pmatrix} \tag{5.2}$$

The coefficients of matrix A_p, are coefficients of a chosen point $p \in \mathbb{P}^n$, but not in Q.

Proof. The process of proving this theorem will have the same idea as in the previous chapters. A point $p \in \mathbb{P}^2$ is chosen under the condition that it does not belong to our quadric. For any point $q \in Q$, a line through p and q will be built, and the intention is to find the other intersection point between the line and the quadric Q, since the first intersection point is q. The second intersection point will be written by q'.

Using this process and the connection between q and q', the map $g : Q \to Q$ will be build by defining the matrix A_p that represents it. The proof here will be done in a bit different way from what we have seen in the previous chapters. There, while we proved similar theorems, the problem was analysed locally moving into affine spaces. Here we will not do the same, so the whole problems will be analysed and proved in projective space.

So, as can be noticed above for $q \in Q$, $q' = g(q) = A_p q$ or shown as in terms of matrices:

$$\begin{pmatrix} x'_0 \\ x'_1 \\ . \\ . \\ . \\ x'_n \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & ... & a_{0n} \\ a_{01} & a_{11} & ... & a_{1n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{0n} & a_{1n} & ... & a_{nn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} a_{00}x_0 + a_{01}x_1 + ... + a_{0n}x_n \\ a_{01}x_0 + a_{11}x_1 + ... + a_{1n}x_n \\ . \\ . \\ . \\ a_{0n}x_0 + a_{1n}x_1 + ... + a_{nn}x_n \end{pmatrix}$$

Let the point $p = (u_0 : u_1 : ... : u_n)$ be the chosen point, and the point $q = (x_0 : x_1 : ... : x_n)$ any point in Q. This means the coordinates of q satisfy the equation of the quadric Q.

$$\sum_{i=0}^n x_i^2 = x_0^2 + x_1^2 + ... + x_n^2 = 0$$

Through those two points, a line is built and the parametric equation of this line will be:

$$(x'_0 : x'_1 : ... : x'_n) = (x_0 : x_1 : ... : x_n) + t(u_0 - x_0 : u_1 - x_1 : ... : u_n - x_n)$$

Since the point we are trying to find belongs to the line and the quadric, the below system must be solved in terms of finding it.

$$\begin{aligned} x_0'^2 + x_1'^2 + ... + x_n'^2 = 0 \\ x'_i = x_i + t(u_i - x_i), i = 0, 1, ..., n \\ t \neq 0 \end{aligned}$$

By replacing the values of x'_i in the equation of the quadric we get:

$$(x_0 + t(u_0 - x_0))^2 + (x_1 + t(u_1 - x_1))^2 + ... + (x_n + t(u_n - x_n))^2 = 0$$

and from here, is possible to be found the expression for the parameter t, which will be:

$$t = \frac{-2(u_0x_0 + u_1x_1 + ... + u_nx_n)}{u_0^2 + u_1^2 + ... + u_n^2 - 2(u_0x_0 + u_1x_1 + ... + u_nx_n)}$$

Replacing it, on the equation $x'_i = x_i + t(u_i - x_i)$, the coordinates of the intersection point q' will be:

$$x'_0 = \frac{(-u_0^2 + u_1^2 + ... + u_n^2)x_0 - 2u_0u_1x_1 - ... - 2u_0u_nx_n}{u_0^2 + u_1^2 + ... + u_n^2 - 2(u_0x_0 + u_1x_1 + ... + u_nx_n)}$$

$$x'_1 = \frac{-2u_0u_1x_0 + (u_0^2 - u_1^2 + ... + u_n^2)x_1 - ... - 2u_1u_nx_n}{u_0^2 + u_1^2 + ... + u_n^2 - 2(u_0x_0 + u_1x_1 + ... + u_nx_n)}$$

$$\vdots$$

$$x'_n = \frac{-2u_0u_nx_0 - 2u_1u_nx_1 - ... + (u_0^2 + u_1^2 + ... - u_n^2)x_0}{u_0^2 + u_1^2 + ... + u_n^2 - 2(u_0x_0 + u_1x_1 + ... + u_nx_n)}$$

Since q' belongs to a projective space, it means that $q' = aq'$ for any constant a. At the same time from the equations of the coordinates, it can be seen that the denominator is the same for all of them, so nothing will be changed if the point is multiplied by it. As a result we will have:

$$x'_0 = (-u_0^2 + u_1^2 + ... + u_n^2)x_0 - 2u_0u_1x_1 - ... - 2u_0u_nx_n$$

$$x'_1 = -2u_0u_1x_0 + (u_0^2 - u_1^2 + ... + u_n^2)x_1 - ... - 2u_1u_nx_n$$

$$.$$

$$.$$

$$.$$

$$x'_n = -2u_0u_nx_0 - 2u_1u_nx_1 - ... + (u_0^2 + u_1^2 + ... - u_n^2)x_0$$

By comparing the general form of the point q' with the results above, is simple to define the coefficients of the matrix A_p.

$$\begin{pmatrix} x'_0 \\ x'_1 \\ . \\ . \\ . \\ x'_n \end{pmatrix} = \begin{pmatrix} (-u_0^2 + u_1^2 + ... + u_n^2)x_0 - 2u_0u_1x_1 - ... - 2u_0u_nx_n \\ -2u_0u_1x_0 + (u_0^2 - u_1^2 + ... + u_n^2)x_1 - ... - 2u_1u_nx_n \\ . \\ . \\ . \\ -2u_0u_nx_0 - 2u_1u_nx_1 - ... + (u_0^2 + u_1^2 + ... - u_n^2)x_0 \end{pmatrix}$$

and the matrix will be:

$$A_p = \begin{pmatrix} -u_0^2 + u_1^2 + ... + u_n^2 & -2u_0u_1 & ... & -2u_0u_n \\ -2u_0u_1 & u_0^2 - u_1^2 + ... + u_n^2 & ... & -2u_1u_2 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ -2u_0u_n & -2u_1u_n & ... & u_0^2 + u_1^2 + ... - u_n^2 \end{pmatrix}$$

□

This map g that we just defined, is not just a simple map, it can be proven that it is an involution [5]. This means that for any $x \in Q$, $g(g(x)) = A_p(A_px) = x$, and it will be proved in the next lemma.

Lemma 5.1.2. *The map $g : Q \to Q$ is an involution.*

Proof. Let $x = (x_0 : x_1 : ... : x_n)$ be any point in Q. Now $g(x) = A_px$ will be:

$$A_px = \begin{pmatrix} (-u_0^2 + u_1^2 + ... + u_n^2)x_0 - 2u_0u_1x_1 - ... - 2u_0u_nx_n \\ -2u_0u_1x_0 + (u_0^2 - u_1^2 + ... + u_n^2)x_1 - ... - 2u_1u_nx_n \\ . \\ . \\ . \\ -2u_0u_nx_0 - 2u_1u_nx_1 - ... + (u_0^2 + u_1^2 + ... - u_n^2)x_0 \end{pmatrix}$$

Acting with the map g one more time in that point the result will be:

$$A_p(A_px) = \begin{pmatrix} (u_0^2 + u_1^2 + ... + u_n^2)^2 x_0 \\ (u_0^2 + u_1^2 + ... + u_n^2)^2 x_1 \\ . \\ . \\ . \\ (u_0^2 + u_1^2 + ... + u_n^2)^2 x_n \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} = x$$

which proves that $g(g(x)) = x$. So the map g is an involution. □

Furthermore, the matrix not only belongs to $PGL(n+1)$, but it belongs to $PO(n+1)$.

5.2 Automorphisms of quadric hypersurfaces

Now let us move into something bigger than just the quadric above. Let $Q' \in \mathbb{P}^n$ be any quadric. The general form of this quadric will be $Q = V_p(f)$, where f is:

$$f(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} x_i x_j$$

The polynomial f can be written as product between vectors and matrices, by taking the coefficients a_{ii} be coefficients in the diagonal b_{ii} and the rest will be given by $b_{ij} = b_{ji} = \frac{1}{2} a_{ij}$. So in this case f will be: $f(x) = x^T B x$ and as it can be seen the matrix B is symmetric.

What we are going to do, is transform the matrix B into a diagonal matrix D, and at the same time find the matrix P which makes this transformation possible. So what we are supposed to do is use the spectral theorem [6] and be able to figure out $D = P^T B P$.

Before we move into the process, two lemmas will be prove first.

Lemma 5.2.1. *[6, Theorem 17.2] Symmetric matrices have only real eigenvalues.*

Proof. The proof will be done not only for real, but complex matrices too. First let us define the dot product between two vectors x and y as:

$$(x, y) = \bar{x} y = \sum_{i=0}^{n} \bar{x}_i y_i$$

On the other side when a matrix A is symmetric, it is called self-adjoint too, and $A^* = \bar{A}^T = A$. Now the properties of the dot product defined above are: $(A^*x, y) = (x, Ay)$, $(\lambda x, y) = \bar{\lambda}(x, y)$ and $(x, \lambda y) = \lambda(x, y)$, where λ is an eigenvalue. So:

$$\bar{\lambda}(x, x) = (\lambda x, x) = (Ax, x) = (A^*x, x) = (x, Ax) = (x, \lambda x) = \lambda(x, x)$$

From above it is obvious that $\bar{\lambda} = \lambda$, which is possible only if λ is real number. □

Lemma 5.2.2. *[6, Theorem 17.3] If the matrix is symmetric, then different eigenvectors of different eigenvalues are perpendicular.*

Proof. Assume that $Ax = \lambda x$ and $Ay = \mu y$, where $\lambda \neq \mu$. So:

$$\lambda(x, y) = (\lambda x, y) = (Ax, y) = (x, Ay) = (x, \mu y) = \mu(x, y)$$

From $\lambda(x, y) = \mu(x, y)$ and since $\lambda \neq \mu$ there is only one possibility, that $(x, y) = 0$. Which means the eigenvectors are perpendicular. □

Back to the matrix B, it is obvious that is symmetric. Both of the lemmas above apply of it. Now we follow the steps of the spectral theorem [2, Section 2.8, pages 63, 64] to find the diagonal matrix.

1. First we build the matrix $B - \lambda I$. To find the eigenvalues of the matrix, the characteristic equation should be written and afterwards it must be solved. The characteristic equation is $det(B - \lambda I) = 0$. The solutions of the equation which will be written by λ_0, λ_1,..., λ_n, are the eigenvalues and according to the lemma above they are all real numbers.

2. For the found eigenvalues of the matrix B, the eigenvectors should be found. Te eigenvectors are found by solving the equations $(B - \lambda_i I)v_i = 0$, for $i = 0, 1, ..., n$. According to the lemma 1.2, the eigenvectors v_0, v_1,..., v_n, found by solving the equations, are perpendicular: $v_i^T v_j = 0$. Not only that, we have the possibility to choose the vectors in that way such the dot product $v_i^T v_i = 1$.

3. According to the spectral theorem, using the eigenvectors v_i, $i = 1, 1, ..., n$, a matrix P can be build by having the vectors as columns of the matrix:

$$P = \begin{pmatrix} v_0 & v_1 & ... & v_n \end{pmatrix}$$

Characteristic of this matrix is that it has an inverse P^{-1}, which actually is the same as the transposed matrix P^T. So $PP^T = P^T P = I$.

After the matrix P has been defined, then the final step is to find the diagonal matrix D, which will be $D = P^T BP$, and not only that, but the coefficients in the diagonal are the eigenvalues λ_i. So the diagonal matrix is:

$$D = \begin{pmatrix} \lambda_0 & 0 & ... & 0 \\ 0 & \lambda_1 & ... & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$$

The matrix P, will take any point in Q' and will map it to another point into a new quadric Q'' which will have a canonic form. Let $x \in Q'$ be any point and for it is true:

$$f(x) = x^T Bx$$

Now, multiplying it with the matrix P, the result will be:

$$f(Px) = (Px)^T B(Px) = x^T P^T BPx = x^T Dx$$

Writing it as an equation we get:

$$f(Px) = \lambda_0 x_0^2 + \lambda_1 x_1^2 + ... + \lambda_n x_n^2$$

Finally, $f(Px) = 0$ if and only is $\sum_{i=0}^n \lambda_i x_i^2 = 0$, which means that the new quadric $Q'' = V_p(\sum_{i=0}^n \lambda_i x_i^2 = 0)$.

Next part is really interesting, and important at the same time. Our intention is to try and build a connection between the quadrics Q and Q'', and afterwards to use the connection we have between quadrics Q'' and Q', so the quadrics Q and Q' can be connected. By connection is meant a map that will take points from Q and map them into points in Q'.

Let us choose any point $x = (x_0 : x_1 : ... : x_n) \in Q$, it means that it satisfies the equation $\sum_{i=0}^n x_i^2 = 0$. A matrix G that maps this point into a point $x'' \in Q''$, should be built. If we have a look in the matrical form of our quadrics:

$$f(x) = x^T Ix$$
$$f(x'') = x''^T Dx''$$

is noticable that what should be done is to transform the matrix I into the matrix D. this is something easy to do since $D = GIG$ and the matrix G will be a diagonal matrix of the form:

$$G = \begin{pmatrix} \mu_0 & 0 & ... & 0 \\ 0 & \mu_1 & ... & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & ... & \mu_n \end{pmatrix}$$

where the coefficients in the diagonal are defined in that way such $\mu_i^2 = \lambda_i$. So finally a map between Q and Q'' can be defined.

Definition 5.2.1. Between the quadrics Q and Q'' exists a map $\alpha : Q \to Q''$ such that for every $x \in Q$, $\alpha(x) \in Q''$ and:

$$\alpha(x) = Gx$$

G the diagonal matrix defined above.

From the part where the way how Q' and Q'' are connected is explained, is said that $D = P^T BP$ and by multiplying any point x' by the matrix P, a new point $x'' = Px'$ is built, which belong to Q''. Well every process has the inverse process so $B = PDP^T$ and if the point $x'' \in Q''$ then $x' = P^T x''$ is in Q'. So a map between Q'' and Q' can be defined.

Definition 5.2.2. Between the quadrics Q'' and Q' exists a map $\gamma : Q'' \to Q'$ such that for every $x'' \in Q''$, $\gamma(x'') \in Q'$ and:

$$\gamma(x'') = P^T x''$$

From both maps defined above, it is possible to define a third map $\beta : Q \to Q'$ which will be a composition of two other maps.

Definition 5.2.3. Between the quadrics Q and Q' exists the map $\beta : Q \to Q'$ which is a composition of the α and γ, such that for every $x \in Q$, $x' = \beta(x) \in Q'$ and:

$$\beta(x) = Mx$$

where M is a matrix which we get by multiplying G and P^T.

A question that can be asked here is: Can this map be generalised into a map between projective space $\mathbb{P}^n$ and itself? What should be done to give an answer to this question is to try and prove is lines are mapped into lines.

Lemma 5.2.3. *The matrix M which defines the map β, maps lines into lines.*

Proof. Let L be any line in projective space $\mathbb{P}^2$ with the general form:

$$L = \{(x_0 : x_1 : ... : x_n) | a_0x_0 + a_1x_1 + ... + a_nx_n = 0\}$$

where a_i are the coefficients, $i = 0, 1, ..., n$. Writing it in a matrical form, the result will be:

$$L = \{ \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} \mid \begin{pmatrix} a_0 & a_1 & ... & a_n \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} = 0\}$$

Acting with the matrix M on the line, gives:

$$ML = \{M \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} \mid \begin{pmatrix} a_0 & a_1 & ... & a_n \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} = 0\}$$

which again will be a line:

$$L = \{ \begin{pmatrix} x'_0 \\ x'_1 \\ . \\ . \\ . \\ x'_n \end{pmatrix} \mid \begin{pmatrix} a_0 & a_1 & ... & a_n \end{pmatrix} M^{-1} \begin{pmatrix} x'_0 \\ x'_1 \\ . \\ . \\ . \\ x'_n \end{pmatrix} = 0\}$$

$$= \{(x'_0 : x'_1 : ... : x'_n) | a'_0 x'_0 + a'_1 x'_1 + ... + a'_n x'_n = 0\}$$

proving the lemma. □

Now this gives the possibility to generalize the map $\beta : \mathbb{P}^n \to \mathbb{P}^n$.

After the whole process and all the maps that have been defined above, in the last part of the chapter, we will construct the automorphism [1] for any Q'. The automorphism will be a matrix that belongs to $PGL(n+1)$. First of all let $p' \in \mathbb{P}^n$ be a chosen point that does not belong to Q'. By the definition of the map β, knowing that it has an inverse map β^{-1}, acting with it on the point p' gives as result the point $p = \beta^{-1}p' = M^{-1}p'$, which again belong to $\mathbb{P}^n$.

At the same time for any point $q' \in Q'$, acting on it with the inverse map β^{-1}, will give as a result the point $q = \beta^{-1}q' = M^{-1}q'$, which belongs to the quadric Q.

In the first theorem, the map $g : Q \to Q$ was defined with the matrix A_p. Using this map, for any point $q \in Q$, we get a new point $g(q) = A_n q$ which again belongs to the quadric Q.

And finally, again using the map β, the points from Q will be mapped again in the Q'. so the whole process will be as below:

$$Q' \xrightarrow[\beta^{-1}]{} Q \xrightarrow[g_p]{} Q \xrightarrow[\beta]{} Q'$$

and for a point $q' \in Q'$:

$$q = \beta^{-1}(q') = M^{-1}q' \in Q$$

$$g_p(q) = A_p q = A_p M^{-1} q' \in Q$$
$$\beta(g_p(q)) = M A_p M^{-1} q' \in Q'$$

So it is possible do define the automorphism for the quadric Q'

Definition 5.2.4. The reflection $\psi : Q' \to Q'$, of any quadric, through a chosen point $p' \in \mathbb{P}^n$, which does not belong to Q', is given by a matrix as below:

$$\psi(q') = M A_p M^{-1} q' = M A_{M^{-1}p'} M^{-1} q'$$

for any point q', and the matrix $M A_{M^{-1}p'} M^{-1} \in PGL(n+1)$.

The question here is: Is the map ψ an involution? The answer of this question is yes and it will be proved now. Let $q' \in Q'$ be any point. Acting on it with the map ψ gives as result:

$$\psi(q') = M A_{Mp'} M^{-1} q'$$

Now acting one more time with the map on the new point, we get:

$$\psi(\psi(q')) = M A_{Mp'} M^{-1} (M A_{Mp'} M^{-1} q') = M A_{Mp'} A_{Mp'} M^{-1} q' = M M^{-1} q' = q'$$

What answers to our question.

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