FACULTY OF SCIENCE AND TECHNOLOGY

MASTER'S THESIS

Universitetet i Stavanger

Study programme/specialisation: Mathematics and physics	Autumn / Spring semester, 2019/2020 Open
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Title of master's thesis: Introduction to Supersymmetry	
Credits: 60	
Keywords: Lie algebras, Clifford algebras, Superalgebras, Killing vectors, Poincaré and conformal algebras, Field theory, Wess-Zumino model	Number of pages: 73 + supplemental material/other: none Stavanger, June 15, 2020 date/year

Introduction to Supersymmetry

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June 2020

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where stated otherwise by reference or acknowledgement, the work presented is entirely my own.

Acknowledgement

Many thanks to my supervisor Dr. Paul Francis de Medeiros for valuable instruction during this thesis.

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1 Abstract

This thesis covers the mathematical foundations of supersymmetry, and looks at the simplest non-trivial example of supersymmetry in physics, the Wess-Zumino model. On the way we will also explicitly calculate the Poincaré and conformal superalgebras.

2 Introduction

The enlightenment philosophers who started the scientific revolution formulated the metaphor of the laws of nature, building on the image of a divine legislator, judge and enforcer of natural laws. This metaphor slowly gave way to the more metaphysically neutral concept of symmetry, which is the idea of quantities that are conserved under transformation. In the early 20th century, Noether showed that conservation laws of physics are always associated with symmetries of the action. [9]

By the 1960s, the physicist community had become accutely interested in the mathematics of symmetry, namely group theory, especially Lie groups and their Lie algebras. (See Chapter 3.2) The mathematical basis for this research was created and discovered by Lie in the late 19th century.

Physicists were hoping to be able to identify an overarching group structure that related spacetime and internal symmetries. Coleman and Mandula poured a bucket of ice into those ambitions by proving that any attempt at unifying the symmetries in a Lie group beyond a direct product overconstrains the structure, leading to unphysical behavior.

They showed that such additional constraints would cause the scattering angle of two interacting particles to only take discrete values. This violates the observed range of continuous scattering angles.

In practice, the commutator relation $[S_{ext}, S_{int}]$ between an external and internal symmetry S_{ext} andS_{int} must equal 0.

Unfazed by this so-called "no-go" theorem, Haag, Sohnius, and Łopuszański[11] in 1975 proved that there was a backdoor around the problem. While Lie groups and algebras are verboten, an anticommutator relation, which is the basis of Clifford algebras, can circumvent the barrier while still respecting the Coleman-Mandula Theorem.

They did this by constructing a $\mathbb{Z}_2$-graded superalgebra, which we will revisit in chapter 5, exhibiting supersymmetry, lovingly called SUSY.

The key feature of supersymmetry is that it unifies the internal and spacetime symmetries by introducing a new fermionic charge Q, whose super-commutator yields a spacetime translation $[Q, Q] \propto P_\mu$. This is where the love happens. Supersymmetry is then guaranteed to impact all symmetries because everything in physics is affected by translation.

As an extraordinary side-effect, by making the supersymmetry charge a function of spacetime coordinates $Q(x)$ one gets a gauge theory of translations, which is precisely General Relativity and gravity. Supersymmetric gravity (SUGRA) is beyond the scope of this thesis as we will only be dealing with rigid (flat) spacetime with a constant metric.

Supersymmetry is not without its problems. One of its predictions is that every particle has a super-version of the same mass. No such super particle has ever been observed. If history is a judge, this means that supersymmetry is almost certainly false. In 2006, for instance, Lisi proposed an interesting model [7] based on the exceptional Lie group E8. Initially, the model received a flurry of interest because of its mathematical elegance and simplicity, but it predicts a host of particles that have never been observed in nature and a decade later, his theory is all but forgotten. Similar fates have befallen countless other hopeful theories.

Supersymmetry still clings to the hope that it will be saved by a similar mechanism of symmetry breaking that gives mass to particles in the Standard Model at high energies, the Higgs mechanism. It has one piece of indirect imperical evidence in its favor, namely the fact that all forces of nature unite at the same energy level in supersymmetry models, whereas this does not occur in the Standard Model. Also, some have proposed that the hypothesized dark matter in the universe might in fact be stable superparticles.

However, even if it should turn out that supersymmetry is wrong, it might very well be wrong *in the right neighborhood*. That is, it may have most of the ingredients of the correct solution and its errors – if they exist – are constructive and instructive, leading to the tweaks necessary to formulate a correct theory.

Such informative flaws are not at all uncommon in physics. In fact, they are the norm. Consider the luminous ether. The popular story today is that it has been falsified, discarded and replaced by something new and much shinier. The truth is that the luminous ether is still mostly intact, but today we call it a *quantum field*, which can be thought of as nothing other than an ether that obeys special relativity.

Thus, even if supersymmetry is not without its problems, it has so many attractive features that suggest that even if it should turn out to be wrong, it can lead to the right solution.

3 Mathematical Preliminaries

3.1 Notation

I will be using standard index notation for tensors, with upper indexes X^μ representing vectors and lower indexes X_μ dual vectors. It will be useful to reserve upper case letters for vectors (e.g. X, Y, Z), and lower case letters (e.g. x, y, z) for points in a manifold, represented by calligraphic uppercase letters (e.g. $\mathcal{M}, \mathcal{N}, \mathcal{P}$).

Using the Einstein summation convention, the summation sign is dropped on repeated upper and lower indices: $\Sigma X_\mu X^\mu = X_\mu X^\mu$.

By convention, the derivative operator $\partial/\partial x_\mu$ is written ∂_μ. ∂^μ is shorthand for $g^{\mu\nu}\partial_\nu$, where g is the metric. Derivative operators are coordinate basis vectors, but will in most cases be omitted as the basis vectors can be restored from the components. For instance, $X^\mu \partial_\mu$ will typically be abbreviated to X^μ. ∂^μ is not to be confused with dx^μ, which is the 1-form basis dual vector.

In the rare case of multiple coordinate systems in the same context, it may be useful to write $\partial/\partial x_\mu$ and the different coordinate basis $\partial/\partial y_\rho$ as ∂x_μ and ∂y_ρ respectively.

A coordinate function is denoted by indexed lower case letters, mapping to points in the manifold. For instance, $X(x)$ should be read as "the vector X at point x" whereas $X(x^\mu)$ is to be read as "the vector X at coordinates x^μ." This in turn will typically be abbreviated to X^μ. Whenever there is a need to distinguish between points and coordinates, explicit evaluation is used, e.g. $X(x^\mu)|_p$ which reads "X evaluated at point p with coordinates x^μ."

In this thesis, the variable n is reserved for dimension. k is reserved for the counter or length of a series.

We may define the signed n-dimensional diagonal matrix with signature (s,t) as $\eta(t, s)$ as:

$$\eta(t, s) = \text{diag}(\underbrace{-1, \ldots, -1}_{t}, \underbrace{1, \ldots, 1}_{s}) \tag{3.1}$$

3.2 Lie Algebras

Although we will not be working with Lie groups directly, all physics, including supersymmetry is built on smooth manifolds and we will therefore briefly recap its definition before we turn to Lie algebras[10, Ch. 1].

Definition 1. A **Lie group** G is a finite, n-dimensional **smooth** (=infinitely differentiable, C^∞) **manifold**. That is, every $g \in G$ can be mapped locally onto $\mathbb{R}^n, \mathbb{C}^n$, or $\mathbb{H}^n$ and group multiplication is a smooth, invertible function.

Definition 2. An **vector space** V on a field $\mathbb{K}$ is a set equipped with a vector addition operation $+ : V \times V \to V$ and a scalar multiplication operation $\cdot : \mathbb{K} \times V \to V$. $x, y, z \in V$ called **vectors** and $a, b \in \mathbb{K}]$ called **scalars** satisfy the following properties:

$x + 0 = x$ (identity)

x + (-x) = 0 (inverse)

$x + y = y + x$ (commutativity)

$(x + y) + z = x + (y + z)$ (associativity)

$(a + b)(x + y) = ax + ay + bx + by$ (distributivity)

Definition 3. [8, Ch. 5.6.2] An **algebra** A on a field is a linear vector space equipped with a binary operation $\circ$: $A \times A \to A$.

Definition 4. A **Lie algebra** L is an algebra with the binary operation $[-,-]$ called the **Lie bracket**. It fulfills the properties that for $x, y, z \in L$ and $a, b \in \mathbb{R}$:

$[ax + by, z] = a[x, z] + b[y, z]$ (bilinearity)

$[z, ax + by] = a[z, x] + b[z, y]$

$[x, y] = -[y, x]$ (anti-commutativity)

$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity)

Definition 5. Given a vector space V over a field $\mathbb{K}$ the **dual vector space** V^* of V is the set of linear functions $f : V \to \mathbb{K}$. An element of V^* is called a **dual vector**.

Definition 6. A **homomorphism** is a map f of $x, y \in A$ into B, $f : A \to B$ that preserves the operations of the algebra, i.e. $f(x \cdot y) = f(x) \cdot f(x)$. An **endomorphism** is a map from V onto itself. $f : V \to V$. If f is invertible, the map is called an **automorphism**.

Definition 7. Let $L(V)$ be the Lie algebra consisting of all linear endomorphisms of the vector space V. A **representation** of the Lie algebra A on V is the Lie algebra homomorphism $f : A \to L$. The representation is said to be faithful if its kernel is zero. Ado's theorem ensures that every finite-dimensional Lie algebra has a faithful representation on a finite-dimensional vector space[6].

To distinguish between vectors and dual vectors, vectors are given lower indices and dual vectors upper indices. In index notation, the linear combination of basis vectors e_a is implemented with Einstein sum where the scalar componenents $c \in \mathbb{K}$ are given opposite index position to indicate contraction: $x = c^a e_a$. The same goes for dual vectors: $x = c_a e^a$. We may also combine vectors and dual vectors with the tensor product. $c^a_{\ b} e^a \otimes e_b$. For brevity, we often drop the basis vectors because they can be recovered from the components. Familiar operations and objects can now be written in index notation:

$$\begin{aligned}(A \cdot v)_a &= A^a_{\ b} v^b \\ (A \cdot v)^b &= A_a^{\ b} v_b \\ (A \cdot B)^a_{\ b} &= A^a_{\ c} B^c_{\ b} \\ \eta &= \eta_{ab} e^a \otimes e^b \\ \eta_{ab} v^b &= v_a\end{aligned} \tag{3.2}$$

A basis vector e_a of a Lie algebra A is called a **generator**, and any element $x \in A$ can be written as a linear combination of the generators. If the Lie bracket of two generators is **closed**, it produces a linear combination of other generators $[e_a, e_b] = f_{ab}^{\ \ c} e_c$. f is called a **structure constant** and with its index notation and Einstein sum, it facilitates the linear combination while conserving free indices.

The faithful representation of the Lie algebra we will be encountering in this thesis is the commutator $[X, Y] = XY - YX$.

Let ρ be a map such that $\rho(X) \cdot Y$ is a a faithful representation of $[X, Y]$. Then $\rho([X, Y]) = \rho(\rho(X) \cdot Y) - \rho(\rho(Y) \cdot X)$ This in turn equals $\rho(X)\rho(Y) - \rho(Y)\rho(X) = [\rho(X), \rho(Y)]$, i.e. the desired homomorphism.

3.2.1 Killing Vectors

In the context of spacetime symmetries, we define the basis vectors of a spacetime vector space as the partial derivative operators ∂_μ in each of the n directions, such that they span $\mathbb{R}^n$. A vector X is thereby given by $X^\mu \partial_\mu$. The commutator tof two vectors then become:

$$[X, Y] = X^\mu \partial_\mu Y^\nu \partial_\nu - Y^\nu \partial_\nu X^\mu \partial_\mu \tag{3.3}$$

The metric tensor $g_{\mu\nu}$ is an object that encodes the distance measurement of a space. As such, the metric defines its geometrical shape. An isometry is a transformation of the metric that keeps it unchanged along some direction. Isometries therefore provide information about spatial symmetries. In general, the metric tensor transforms as $g_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu} \frac{\partial y^\sigma}{\partial x^\nu} g_{\rho\sigma}$. Concretely, for an infinitesimal change ϵ along some vector Y an isometry takes the following form:

$$\partial_\mu (x^\alpha + \epsilon Y^\alpha) \partial_\nu (x^\beta + \epsilon Y^\beta) g_{\alpha\beta}(x^\rho + \epsilon Y^\rho) = g_{\mu\nu}(x^\rho)$$

To solve for isometries, first expand each derivative term and approximate the metric with a first order Taylor expansion, evaluated at coordinate x^ρ:

$$(\delta^\alpha_\mu + \epsilon \partial_\mu Y^\alpha)(\delta^\beta_\nu + \epsilon \partial_\nu Y^\beta)(g_{\alpha\beta} + \epsilon Y^\rho \partial_\rho g_{\alpha\beta})$$

Expand terms up to first order of ϵ:

$$\delta^\alpha_\mu \delta^\beta_\nu (g_{\alpha\beta} + \epsilon Y^\rho \partial_\rho g_{\alpha\beta}) + \epsilon(\partial_\mu Y^\alpha \delta^\beta_\nu + \partial_\nu Y^\alpha \delta^\beta_\mu) g_{\alpha\beta}$$

Contract and simplify:

$$g_{\mu\nu} + \epsilon(\underbrace{Y^\rho \partial_\rho g_{\mu\nu} + \partial_\mu Y^\alpha g_{\alpha\nu} + \partial_\nu Y^\beta g_{\mu\beta}}_{(\mathcal{L}_Y g)_{\mu\nu}})$$

The term in the bracket is the Lie derivative[8, Ch. 5.4.3] of the metric $\mathcal{L}_Y g$. The expression will be equal to $g_{\mu\nu}$ iff the Lie derivative vanishes. The vector fields Y that satisfy this requirement are called **Killing vector fields**. Since the derivatives of the metric in linear Minkowski space are zero, the Lie derivative of the Killing vectors reduce to the Killing equation:

$$\partial_\mu Y^\alpha g_{\alpha\nu} + \partial_\nu Y^\beta g_{\mu\beta} = \partial_\mu Y_\nu + \partial_\nu Y_\mu = 0$$

The only way for the equation to be satisfied is for the solution to be anti-symmetric in μ and ν.

In general, the Lie derivative of a smooth function f along a vector X is defined as:

$$\begin{aligned} \mathcal{L}_X f &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}(f(x^\mu + \epsilon X^\mu) - f(x^\mu)) \\ &= X^\mu \partial_\mu f = X[f] \end{aligned} \tag{3.4}$$

Thus, the commutator $[X, Y]f$ is in fact $\mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f$. First, let us see if this bracket is closed:

$$\begin{aligned} [X, Y]f &= X^\mu \partial_\mu (Y^\nu \partial_\nu f) - Y^\nu \partial_\nu (X^\mu \partial_\mu f) \\ &= X^\mu (\partial_\mu Y^\nu \partial_\nu f + Y^\nu \partial_\mu \partial_\nu f) \\ &- Y^\nu (\partial_\nu X^\mu \partial_\mu f + X^\mu \partial_\nu \partial_\mu f) \\ &= (^\mu \partial_\mu Y^\nu \partial_\nu - Y^\nu \partial_\nu X^\mu \partial_\mu) f \end{aligned} \tag{3.5}$$

The result is a linear combination of first order partial derivatives, because those nasty higher order terms cancels due to the commutativity of partial derivatives. So $[X, Y]$ is closed and does indeed form the product in a Lie algebra. A useful identity is:

$$\mathcal{L}_{[X,Y]} f = [X, Y]f = \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f \tag{3.6}$$

If X and Y are two killing vectors, it follows that their Lie bracket must also be a killing vector.

3.2.2 The Poincaré Algebra

Next we will identify the Killing vectors[8, Ch. 7.7] for $\mathcal{M} = \mathbb{R}^{s,t}, g = \eta(t, s))$.

We start by differentiating the Killing equation by ∂_ρ.

$$\partial_\rho \partial_\mu Y_\nu + \partial_\rho \partial_\nu Y_\mu = 0$$

Then cyclically permute the indexes to produce three equations[8, Ch. 5.4.3].

$$\partial_\rho \partial_\mu Y_\nu + \boxed{\partial_\rho \partial_\nu Y_\mu} = 0$$

$$\boxed{\partial_\mu \partial_\nu K_\rho} + \partial_\mu \partial_\rho Y_\nu = 0$$

$$\boxed{\partial_\nu \partial_\rho Y_\mu} + \boxed{\partial_\nu \partial_\mu Y_\rho} = 0$$

Add the two first and subtract the third equation and utilize that partial derivatives commute to get:

$$\partial_\rho \partial_\mu Y_\nu = 0$$

Integrating over ρ yields $\partial_\mu Y_\nu = b_{\mu\nu}$, an anti-symmetric constant. Integrating once more over μ gives the Killing dual vector solutions in their most general form:

$$Y_\nu = a_\nu + b_{\mu\nu} x^\mu$$

Setting b to zero, we find n Killing vectors a_ν corresponding to translations along the coordinate basis vectors ∂_ν. This implies that there are a total of n symmetric and and $n(n-1)/2$ anti-symmetric solutions, making a total of $n(n+1)/2$ linearly independent Killing vectors.

n-dimensional manifolds that have this number of Killing vectors are said to be **maximally symmetric**. All flat spaces have this property, in addition to spaces with constant positive curvature ($\mathbf{S}^n$) and constant negative curvature ($\mathbf{H}^n$). These are the only possible maximally symmetric spaces.

Setting a to 0, we find the Killing vectors corresponding to rotations and boosts: $Y_\mu = b_{\mu\nu} x^\nu$. This yields the Lorentz Lie algebra generator $M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$.

Similarly, the translation killing vector $b^\mu \partial_\mu$ gives the translation generator $P_\mu = \partial_\mu$.

Together P_μ and $M_{\mu\nu}$ form what is called the Poincaré Lie algebra.

When investigating the commutator relations, we can exploit the fact that higher order partial derivative terms cancel, as shown earlier.We can safely ignore them in the calculations, which will greatly simplify the expressions. A trivial example of this is $[P_\mu, P_\nu]$ which only contain these commutative higher order terms and therefore equals zero.

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, M_{\rho\sigma}] = \partial_\mu (x_\rho \partial_\sigma - x_\sigma \partial_\rho) = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = (x_\mu \partial_\nu - x_\nu \partial_\mu)(x_\rho \partial_\sigma - x_\sigma \partial_\rho) - (x_\rho \partial_\sigma - x_\sigma \partial_\rho)(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

$$= x_\mu (g_{\nu\rho} \partial_\sigma - g_{\nu\sigma} \partial_\rho) - x_\nu (g_{\mu\rho} \partial_\sigma - g_{\mu\sigma} \partial_\rho) - x_\rho (g_{\sigma\mu} \partial_\nu - g_{\sigma\nu} \partial_\mu) + x_\sigma (g_{\rho\mu} \partial_\nu - g_{\rho\nu} \partial_\mu)$$

$$= \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho}$$

To verify that they really form a Lie algebra we must check the Jacobi identity.

$$
\begin{aligned}
&[P_\mu,[P_\nu,P_\rho]]+[P_\nu,[P_\rho,P_\mu]]+[P_\rho,[P_\mu,P_\nu]] \\
&\qquad =[P_\mu,0]+[P_\nu,0]+[P_\rho,0] \\
&\qquad =0
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&[P_\mu,[P_\nu,M_{\rho\sigma}]]+[P_\nu,[M_{\rho\sigma},P_\mu]]+[M_{\rho\sigma},[P_\mu,P_\nu]] \\
&\qquad =[P_\mu,\propto P]+[P_\nu,\propto P]+[M_{\rho\sigma},0]
\end{aligned} = 0
\tag{3.8}
$$

$$
\begin{aligned}
&[P_\kappa,[M_{\mu\nu},M_{\rho\sigma}]]+[M_{\mu\nu},[M_{\rho\sigma},P_\kappa]]+[M_{\rho\sigma},[P_\kappa,M_{\mu\nu}]] \\
&=[P_\kappa,\eta_{\nu\rho}M_{\mu\sigma}-\eta_{\nu\sigma}M_{\mu\rho}-\eta_{\mu\rho}M_{\nu\sigma}+\eta_{\mu\sigma}M_{\nu\rho}] \\
&+[M_{\mu\nu},-\eta_{\kappa\rho}P_\sigma+\eta_{\kappa\sigma}P_\rho]+[M_{\rho\sigma},\eta_{\kappa\mu}P_\nu-\eta_{\kappa\nu}P_\mu] \\
&=\eta_{\nu\rho}(\eta_{\kappa\mu}P_\sigma-\eta_{\kappa\sigma}P_\mu)-\eta_{\nu\sigma}(\eta_{\kappa\mu}P_\rho-\eta_{\kappa\rho}P_\mu) \\
&-\eta_{\mu\rho}(\eta_{\kappa\nu}P_\sigma-\eta_{\kappa\sigma}P_\nu)+\eta_{\mu\sigma}(\eta_{\kappa\nu}P_\rho-\eta_{\kappa\rho}P_\nu) \\
&+\eta_{\kappa\rho}(\eta_{\sigma\mu}P_\nu-\eta_{\sigma\nu}P_\mu)-\eta_{\kappa\sigma}(\eta_{\rho\mu}P_\nu-\eta_{\rho\nu}P_\mu) \\
&-\eta_{\kappa\mu}(\eta_{\nu\rho}P_\sigma-\eta_{\nu\sigma}P_\rho)+\eta_{\kappa\nu}(\eta_{\mu\rho}P_\sigma-\eta_{\mu\sigma}P_\rho) \\
&=0
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&[M_{\kappa\lambda},[M_{\mu\nu},M_{\rho\sigma}]]+[M_{\mu\nu},[M_{\rho\sigma},M_{\kappa\lambda}]]+[M_{\rho\sigma},[M_{\kappa\lambda},M_{\mu\nu}]] \\
&=[M_{\kappa\lambda},\eta_{\nu\rho}M_{\mu\sigma}-\eta_{\nu\sigma}M_{\mu\rho}-\eta_{\mu\rho}M_{\nu\sigma}+\eta_{\mu\sigma}M_{\nu\rho}] \\
&+[M_{\mu\nu},\eta_{\sigma\kappa}M_{\rho\lambda}-\eta_{\sigma\lambda}M_{\rho\kappa}-\eta_{\rho\kappa}M_{\sigma\lambda}+\eta_{\rho\lambda}M_{\sigma\kappa}] \\
&+[M_{\rho\sigma},\eta_{\lambda\mu}M_{\kappa\nu}-\eta_{\lambda\nu}M_{\kappa\mu}-\eta_{\kappa\mu}M_{\lambda\nu}+\eta_{\kappa\nu}M_{\lambda\mu}] \\
&=-\eta_{\nu\rho}(\eta_{\sigma\kappa}M_{\mu\lambda}-\eta_{\sigma\lambda}M_{\mu\kappa}-\eta_{\mu\kappa}M_{\sigma\lambda}+\eta_{\mu\lambda}M_{\sigma\rho}) \\
&+\eta_{\nu\sigma}(\eta_{\rho\kappa}M_{\mu\lambda}-\eta_{\rho\lambda}M_{\mu\kappa}-\eta_{\mu\kappa}M_{\rho\lambda}+\eta_{\mu\lambda}M_{\rho\lambda}) \\
&+\eta_{\mu\rho}(\eta_{\sigma\kappa}M_{\nu\lambda}-\eta_{\sigma\lambda}M_{\nu\kappa}-\eta_{\nu\kappa}M_{\sigma\lambda}+\eta_{\nu\lambda}M_{\sigma\kappa}) \\
&-\eta_{\mu\sigma}(\eta_{\rho\kappa}M_{\nu\lambda}-\eta_{\rho\lambda}M_{\nu\kappa}-\eta_{\nu\kappa}M_{\rho\lambda}+\eta_{\nu\lambda}M_{\rho\kappa}) \\
&+\eta_{\sigma\kappa}(\eta_{\nu\rho}M_{\mu\lambda}-\eta_{\nu\lambda}M_{\mu\rho}-\eta_{\mu\rho}M_{\nu\lambda}+\eta_{\mu\lambda}M_{\nu\rho}) \\
&-\eta_{\sigma\lambda}(\eta_{\nu\rho}M_{\mu\kappa}-\eta_{\nu\kappa}M_{\mu\rho}-\eta_{\mu\rho}M_{\nu\kappa}+\eta_{\mu\kappa}M_{\nu\rho}) \\
&-\eta_{\rho\kappa}(\eta_{\nu\sigma}M_{\mu\lambda}-\eta_{\nu\lambda}M_{\mu\sigma}-\eta_{\mu\sigma}M_{\nu\lambda}+\eta_{\mu\lambda}M_{\nu\sigma}) \\
&+\eta_{\rho\lambda}(\eta_{\nu\sigma}M_{\mu\kappa}-\eta_{\nu\kappa}M_{\mu\sigma}-\eta_{\mu\sigma}M_{\nu\kappa}+\eta_{\mu\kappa}M_{\nu\sigma}) \\
&-\eta_{\lambda\mu}(\eta_{\nu\rho}M_{\kappa\sigma}-\eta_{\nu\sigma}M_{\kappa\rho}-\eta_{\kappa\rho}M_{\nu\sigma}+\eta_{\kappa\sigma}M_{\nu\rho}) \\
&+\eta_{\lambda\nu}(\eta_{\mu\rho}M_{\kappa\sigma}-\eta_{\mu\sigma}M_{\kappa\rho}-\eta_{\kappa\rho}M_{\mu\sigma}+\eta_{\kappa\sigma}M_{\mu\rho}) \\
&+\eta_{\kappa\mu}(\eta_{\nu\rho}M_{\lambda\sigma}-\eta_{\nu\sigma}M_{\lambda\rho}-\eta_{\lambda\rho}M_{\nu\sigma}+\eta_{\lambda\sigma}M_{\nu\rho}) \\
&-\eta_{\kappa\nu}(\eta_{\mu\rho}M_{\lambda\sigma}-\eta_{\mu\sigma}M_{\lambda\rho}-\eta_{\lambda\rho}M_{\mu\sigma}+\eta_{\lambda\sigma}M_{\mu\rho}) \\
&=0
\end{aligned}
\tag{3.10}
$$

It does indeed satisfy the Jacobi identity and we can therefore with certainty say that P_μ and $M_{\mu\nu}$ are the generators of a Lie algebra.

To summarize,

$$\begin{aligned}[P_\mu, P_\nu] &= 0 \\ [P_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho \\ [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho}\end{aligned} \tag{3.11}$$

3.2.3 Conformal Killing Vectors (CKVs)

It is sometimes useful to be able to formulate physical theories based on transformations that preserve the metric up to some scale factor, typically encoded as $e^{2\sigma}$. The solution to such scale-relaxed isometries are called conformal Killing vectors[8, Ch. 7.7.2] (CKVs), and they are solutions solutions of the form:

$$\partial_\mu (x^\alpha + \epsilon Y^\alpha) \partial_\nu (x^\beta + \epsilon Y^\beta) \eta_{\alpha\beta}(x^\rho + \epsilon Y^\rho) = e^{2\sigma} \eta_{\mu\nu}(x^\rho)$$

Noting that ϵ and σ are proportional, it proves useful to set $\sigma = \epsilon\psi/2$, where ψ is a scalar. We then repeat the calculation of expanding the equation and Taylor expand $e^{\epsilon\psi}$ up to first order in ϵ to find:

$$\eta_{\mu\nu} + \epsilon(\underbrace{Y^\rho \partial_\rho \eta_{\mu\nu} + \partial_\mu Y_\nu + \partial_\nu Y_\mu}_{\mathcal{L}_Y \eta}) = (1 + \epsilon\psi)\eta_{\mu\nu}$$

Thus,

$$\mathcal{L}_Y \eta = Y^\rho \partial_\rho \eta_{\mu\nu} + \partial_\mu Y_\nu + \partial_\nu Y_\mu = \psi \eta_{\mu\nu} \tag{3.12}$$

We solve by multiplying both sides by $g^{\mu\nu}$ and note that $g_{\mu\nu} g^{\mu\nu} = \delta^\mu_\mu = Dim(\mathcal{M}) = n$. We obtain:

$$\psi = \frac{g^{\mu\nu} Y^\rho \partial_\rho g_{\mu\nu} + \partial_\mu Y^\mu + \partial_\nu Y^\nu}{n} = \frac{g^{\mu\nu} Y^\rho \partial_\rho g_{\mu\nu} + 2\partial_\mu Y^\mu}{n}$$

For a metric where all components are constant, all partial derivatives are zero. For the n-dimensional Minkowski metric the expression for ψ then reduces to:

$$\psi = \frac{2\partial_\rho Y^\rho}{n} \tag{3.13}$$

Putting this back into (3.12) gives:

$$\partial_\mu X_\nu + \partial_\nu X_\mu - \eta_{\mu\nu} \frac{2}{n} \partial_\rho X^\rho = 0 \tag{3.14}$$

The strategy for solving the equation is to repeat the steps from the Poincaré solution, namely a specific linear combination of cyclic permutations $\gamma \to \mu \to \nu \to \gamma$ of the derivative ∂_γ of (3.14).

$$\begin{aligned}\partial_\gamma \partial_\mu X_\nu + \partial_\gamma \partial_\nu X_\mu &= \eta_{\mu\nu} \frac{2}{m} \partial_\gamma \partial_\rho X^\rho \\ \partial_\mu \partial_\nu X_\gamma + \partial_\mu \partial_\gamma X_\nu &= \eta_{\nu\gamma} \frac{2}{m} \partial_\mu \partial_\rho X^\rho \\ -\partial_\nu \partial_\gamma X_\mu - \partial_\nu \partial_\mu X_\gamma &= -\eta_{\gamma\mu} \frac{2}{m} \partial_\nu \partial_\rho X^\rho\end{aligned} \tag{3.15}$$

Summing them yields:

$$\partial_\gamma \partial_\mu X_\nu = \frac{1}{n}(\eta_{\mu\nu}\partial_\gamma + \eta_{\nu\gamma}\partial_\mu - \eta_{\gamma\mu}\partial_\nu)\partial_\rho X^\rho \tag{3.16}$$

We can now constrain the solution by testing the order of derivatives. Act on (3.14) with ∂^μ and obtain:

$$\Box X_\nu + \partial_\nu \partial^\mu X_\mu - \frac{2}{n}\partial_\nu \partial_\rho X^\rho = 0 \tag{3.17}$$

Act again with ∂^ν:

$$\Box \partial^\nu X_\nu + \Box \partial^\mu X_\mu - \frac{2}{n}\Box \partial^\rho X_\rho = 0$$
$$(1-n)\Box \partial^\rho X_\rho = 0$$

Thus, if $n \neq 1$:

$$\Box \partial^\rho X_\rho = 0 \tag{3.18}$$

Now, rearrange (3.17) to find for $n \neq 2$:

$$\partial_\nu \partial^\rho X_\rho = \frac{n}{2-n}\Box X_\nu \tag{3.19}$$

Acting on it with ∂_μ gives an equation that is symmetric in μ and ν because partial derivatives commute.

$$\partial_\mu \partial_\nu \partial^\rho X_\rho = \frac{n}{2-n}\Box \partial_\mu X_\nu \tag{3.20}$$

We can exploit this by acting on (3.14) with $\Box$ to obtain another equation which therefore must also be symmetric in μ and ν:

$$\Box \partial_\mu X_\nu + \Box \partial_\nu X_\mu = g_{\mu\nu}\frac{2}{n}\Box \partial^\rho X_\rho$$
$$\Box \partial_\mu X_\nu = \eta_{\mu\nu}\frac{1}{n}\Box \partial^\rho X_\rho \tag{3.21}$$

Inserting (3.21) into (3.20) gives together with (3.18):

$$\partial_\mu \partial_\nu \partial^\rho X_\rho = \frac{1}{2-n}\eta_{\mu\nu}\Box \partial^\rho X_\rho = 0 \tag{3.22}$$

Relabeling μ and ν to κ and λ in (3.14) and acting with $\partial_\mu \partial_\nu$yields:

$$\partial_\mu \partial_\nu \partial_\kappa X_\lambda = -\partial_\mu \partial_\nu \partial_\lambda X_\kappa + \eta_{\kappa\lambda}\frac{2}{n}\partial_\mu \partial_\nu \partial^\rho X_\rho \tag{3.23}$$

From (3.22) the last term is zero:

$$\partial_\mu \partial_\nu \partial_\kappa X_\lambda = -\partial_\mu \partial_\nu \partial_\lambda X_\kappa \tag{3.24}$$

With the third term gone, we now see that the index of the partial derivative and X anticommute. Therefore,

$$\partial_\mu\partial_\nu\partial_\kappa X_\lambda = -\partial_\mu\partial_\lambda\partial_\nu X_\kappa$$
$$= \partial_\mu\partial_\lambda\partial_\kappa X_\nu$$
$$= -\partial_\mu\partial_\nu\partial_\kappa X_\lambda = 0$$

So the third derivative of X is zero, and hence X is at most quadratic in x:

$$X_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{3.25}$$

3.2.4 The Conformal Algebra

Now it is time to see if the conformal Killing vectors form a Lie algebra. The constants a and the antisymmetric part of b yield the same solutions as the Poincaré algebra, P_μ and $M_{\mu\nu}$ respectively. Therefore the Poincaré algebra must be a sub-algebra of the conformal algebra.

Let us find the solution for the symmetric part of b by inserting $b_{\nu\alpha}x^\alpha$ into (3.14).

$$\partial_\mu b_{\nu\alpha}x^\alpha + \partial_\nu b_{\mu\alpha}x^\alpha = \eta_{\mu\nu}\frac{2}{n}\partial_\rho b_{\sigma\alpha}x^\alpha\eta^{\rho\sigma}$$
$$b_{\nu\mu} + b_{\mu\nu} = \eta_{\mu\nu}\frac{2}{n}b_{\sigma\rho}\eta^{\rho\sigma}$$
$$b_{\mu\nu} = \eta_{\mu\nu}\frac{1}{n}b^\sigma_\sigma$$

Thus, b is proportional to the metric. The corresponding vector fields $X^\mu\partial_\mu$is therefore:

$$X^\mu\partial_\mu = \eta^{\mu\nu}X_\nu\partial_\mu = \eta^{\mu\nu}\eta_{\nu\alpha}x^\alpha\partial_\mu =$$
$$x^\mu\partial_\mu = D \tag{3.26}$$

D is called a *dilatation* generator. Next, we find c by inserting $c_{\nu\alpha\beta}x^\alpha x^\beta$ into (3.16).

$$\partial_\gamma\partial_\mu c_{\nu\alpha\beta}x^\alpha x^\beta = \frac{1}{n}(\eta_{\mu\nu}\partial_\gamma + \eta_{\nu\gamma}\partial_\mu - \eta_{\gamma\mu}\partial_\nu)\partial_\rho c_{\sigma\alpha\beta}x^\alpha x^\beta\eta^{\sigma\rho}$$
$$\partial_\gamma(c_{\nu\alpha\mu}x^\alpha + c_{\nu\mu\beta}x^\beta) = \frac{1}{n}(\eta_{\mu\nu}\partial_\gamma + \eta_{\nu\gamma}\partial_\mu - \eta_{\gamma\mu}\partial_\nu)(c^\rho_{\alpha\rho}x^\alpha + c^\rho_{\rho\beta}x^\beta)$$
$$c_{\nu\gamma\mu} + c_{\nu\mu\gamma} = \frac{1}{n}(\eta_{\mu\nu}(c^\rho_{\gamma\rho} + c^\rho_{\rho\gamma}) + \eta_{\nu\gamma}(c^\rho_{\mu\rho} + c^\rho_{\rho\mu}) - \eta_{\gamma\mu}(c^\rho_{\nu\rho} + c^\rho_{\rho\nu}))$$

We see that if c is antisymmetric in its two last indices, all terms cancel. The surviving symmetric components yield:

$$c_{\nu\gamma\mu} = \frac{1}{n}(\eta_{\mu\nu}c^\rho_{\rho\gamma} + \eta_{\nu\gamma}c^\rho_{\rho\mu} - \eta_{\gamma\mu}c^\rho_{\rho\nu})$$

We can drop the constant $1/n$ and incorporate it into the c. It follows that the vector fields are:

$$X^\kappa \partial_\kappa = \eta^{\kappa\nu} X_\nu \partial_\kappa = \eta^{\kappa\nu}(\eta_{\mu\nu} c^\rho_{\rho\gamma} + \eta_{\nu\gamma} c^\rho_{\rho\mu} - \eta_{\gamma\mu} c^\rho_{\rho\nu}) x^\gamma x^\mu \partial_\kappa$$
$$= \eta^{\kappa\nu}(\eta_{\mu\nu} c^\rho_{\rho\gamma} + \eta_{\nu\gamma} c^\rho_{\rho\mu} - \eta_{\gamma\mu} c^\rho_{\rho\nu}) x^\gamma x^\mu \partial_\kappa$$
$$= (\delta^\kappa_\mu c^\rho_{\rho\gamma} + \delta^\kappa_\gamma c^\rho_{\rho\mu} - \eta_{\gamma\mu} c^{\rho\kappa}_\rho) x^\gamma x^\mu \partial_\kappa$$
$$= c^\rho_{\rho\gamma} x^\gamma x^\kappa \partial_\kappa + c^\rho_{\rho\mu} x^\kappa x^\mu \partial_\kappa - c^{\rho\kappa}_\rho x_\mu x^\mu \partial_\kappa$$

Now, in the first term, relabel κ to μ and γ to κ. In the second term relabel κ to μ and vice versa. We can safely do this because they are summed over.

$$= c^{\rho\kappa}_\rho x_\kappa x^\mu \partial_\mu + c^{\rho\kappa}_\rho x^\mu x_\kappa \partial_\mu - c^{\rho\kappa}_\rho x_\mu x^\mu \partial_\kappa$$
$$= c^{\rho\kappa}_\rho (2x_\kappa x^\mu \partial_\mu - x_\mu x^\mu \partial_\kappa)$$

And therefore, the basis vector $K_\kappa = 2x_\kappa x^\mu \partial_\mu - x^2 \partial_\kappa$. K_κ is called a *special conformal transformation*.

To summarize, the basis vectors of the conformal algebra are:

$$\begin{aligned} P_\mu &= \partial_\mu \\ M_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu \\ D &= x^\mu \partial_\mu \\ K_\mu &= 2x_\mu x^\kappa \partial_\kappa - x^2 \partial_\mu \end{aligned} \tag{3.27}$$

The non-vanishing commutator relations:

$$[P_\mu, D] = \partial_\mu x^\nu \partial_\nu = \partial_\mu = P_\mu$$

$$[P_\mu, K_\nu] = 2(g_{\mu\nu} x^\rho + x_\nu \delta^\rho_\mu)\partial_\rho - 2x_\mu \partial_\nu = 2(g_{\mu\nu} D - M_{\mu\nu})$$

$$[D, K_\nu] = x^\mu 2(g_{\mu\nu} x^\rho + x_\nu \delta^\rho_\mu)\partial_\rho - x^\mu 2x_\mu \partial_\nu - 2x_\nu x^\rho \delta^\mu_\rho \partial_\mu + x^2 \delta^\mu_\nu \partial_\mu$$
$$= 4x_\nu x^\rho \partial_\rho - 2x^2 \partial_\nu - 2x_\nu x^\rho \partial_\rho + x^2 \partial_\nu = K_\nu$$

$$[M_{\mu\nu}, K_\kappa] = x_\mu(2(g_{\nu\kappa} x^\rho + x_\kappa \delta^\rho_\nu)\partial_\rho - 2x_\mu x_\nu \partial_\kappa - x_\nu(2(g_{\mu\kappa} x^\rho + x_\kappa \delta^\rho_\mu)\partial_\rho + 2x_\nu x_\mu \partial_\kappa$$
$$-2x_\kappa x^\rho(g_{\mu\rho}\partial_\nu - g_{\nu\rho}\partial_\mu) + x^2(g_{\mu\kappa}\partial_\nu - g_{\nu\kappa}\partial_\mu)$$
$$= g_{\nu\kappa} 2x_\mu x^\rho \partial_\rho - g_{\mu\kappa} 2x_\nu x^\rho \partial_\rho + g_{\mu\kappa} x^2 \partial_\nu - g_{\nu\kappa} x^2 \partial_\mu = g_{\nu\kappa} K_\mu - g_{\mu\kappa} K_\nu$$

The vanishing relations:

$$[M_{\mu\nu}, D] = x_\mu \delta^\rho_\nu \partial_\rho - x_\nu \delta^\rho_\mu \partial_\rho - x^\rho g_{\rho\mu} \partial_\nu + x^\rho g_{\rho\mu} \partial_\mu = x_\mu \partial_\nu - x_\nu \partial_\mu - x_\mu \partial_\nu + x_\nu \partial_\mu = 0$$

$$[D, D] = x^\mu \delta^\nu_\mu \partial_\nu - x^\nu \delta^\mu_\nu \partial_\mu = 0$$

$$[K_\mu, K_\nu] = 2x_\mu x^\rho(2(g_{\rho\nu} x^\sigma + x_\nu \delta^\sigma_\rho)\partial_\sigma - 2x_\rho \partial_\nu) - x^2(2(g_{\mu\nu} x^\sigma + x_\nu \delta^\sigma_\mu)\partial_\sigma - 2x_\mu \partial_\nu)$$

$$-2x_\nu x^\sigma(2(g_{\sigma\mu}x^\rho + x_\mu\delta^\rho_\sigma)\partial_\rho - 2x_\sigma\partial_\mu) + x^2(2(g_{\nu\mu}x^\rho + x_\mu\delta^\rho_\nu)\partial_\rho - 2x_\nu\partial_\mu)$$
$$= 8x_\mu x_\nu x^\sigma\partial_\sigma - 4x_\mu x^2\partial_\nu - 2x^2 g_{\mu\nu}x^\sigma\partial_\sigma - 2x^2 x_\nu\partial_\mu - 2x^2 x_\mu\partial_\nu)$$
$$-8x_\nu x_\mu x^\rho\partial_\rho + 4x_\nu x^2\partial_\mu + 2x^2 g_{\nu\mu}x^\rho\partial_\rho + 2x^2 x_\mu\partial_\nu + 2x^2 x_\nu\partial_\mu) = 0$$

To summarize:

$$\begin{aligned}
[P_\mu, D] &= P_\mu \\
[P_\mu, K_\nu] &= 2(\eta_{\mu\nu}D - M_{\mu\nu}) \\
[D, K_\nu] &= K_\nu \\
[M_{\mu\nu}, K_\kappa] &= \eta_{\nu\kappa}K_\mu - \eta_{\mu\kappa}K_\nu \\
[M_{\mu\nu}, D] &= [D, D] = [K_\mu, K_\nu] = 0
\end{aligned} \tag{3.28}$$

We check if the conformal generators satisfy the Jacobi identity to see if they form a Lie algebra.

$$\begin{aligned}
&[P_\mu, [P_\nu, D]] + [P_\nu, [D, P_\mu]] + [D, [P_\mu, P_\nu]] \\
&= [P_\mu, P_\nu] + [P_\nu, -P_\mu] + 0 \\
&= 0
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&[P_\mu, [D, D]] + [D, [D, P_\mu]] + [D, [P_\mu, D]] \\
&= 0 + [D, -P_\mu] + [D, P_\mu] \\
&= 0
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&[M_{\mu\nu}, [M_{\rho\sigma}, D]] + [M_{\rho\sigma}, [D, M_{\mu\nu}]] + [D, [M_{\mu\nu}, M_{\rho\sigma}]] \\
&= 0 + 0 + [D, \propto M] \\
&= 0
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
&[M_{\mu\nu}, [D, D]] + [D, [D, M_{\mu\nu}]] + [D, [M_{\mu\nu}, D]] \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
&[D, [D, D]] + [D, [D, D]] + [D, [D, D]] \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
&[P_\mu, [P_\nu, K_\rho]] + [P_\nu, [K_\rho, P_\mu]] + [K_\rho, [P_\mu, P_\nu]] \\
&= [P_\mu, 2(\eta_{\nu\rho}D - M_{\nu\rho})] + [P_\nu, -2(\eta_{\rho\mu}D - M_{\rho\mu})] + 0 \\
&= 2\eta_{\nu\rho}P_\mu - 2\eta_{\mu\nu}P_\rho + 2\eta_{\mu\rho}P_\nu \\
&\quad -2\eta_{\rho\mu}P_\nu - 2\eta_{\nu\rho}P_\mu + 2\eta_{\nu\mu}P_\rho \\
&= 0
\end{aligned} \tag{3.34}$$

$$
\begin{aligned}
&[P_\mu,[K_\nu,K_\rho]]+[K_\nu,[K_\rho,P_\mu]]+[K_\rho,[P_\mu,K_\nu]] \\
&=0+[K_\nu,-2(\eta_{\mu\rho}D-M_{\mu\rho})]+[K_\rho,2(\eta_{\mu\nu}D-M_{\mu\nu})] \\
&=2\eta_{\mu\rho}K_\nu-2\eta_{\rho\nu}K_\mu+2\eta_{\mu\nu}K_\rho \\
&-2\eta_{\mu\nu}K_\rho+2\eta_{\nu\rho}K_\mu-2\eta_{\mu\rho}K_\nu \\
&=0
\end{aligned}
\tag{3.35}
$$

$$
3[K_\mu,[K_\nu,K_\rho]]=0 \tag{3.36}
$$

$$
\begin{aligned}
&[P_\mu,[D,K_\nu]]+[D,[K_\nu,P_\mu]]+[K_\nu,[P_\mu,D]] \\
&=[P_\mu,K_\nu]+[D,-2(\eta_{\mu\nu}D-M_{\mu\nu})]+[K_\nu,P_\mu] \\
&=0
\end{aligned}
\tag{3.37}
$$

$$
\begin{aligned}
&[M_{\mu\nu},[D,K_\rho]]+[D,[K_\rho,M_{\mu\nu}]]+[K_\rho,[M_{\mu\nu},D]] \\
&=[M_{\mu\nu},K_\rho]+[D,-(\eta_{\nu\rho}K_\mu-\eta_{\mu\rho}K_\nu)]+0 \\
&=\eta_{\nu\rho}K_\mu-\eta_{\mu\rho}K_\nu-\eta_{\nu\rho}K_\mu+\eta_{\mu\rho}K_\nu \\
&=0
\end{aligned}
\tag{3.38}
$$

Thus, all generators satisfy the Jacobi identity and the conformal algebra is therefore closed.

3.3 Clifford Algebra

3.3.1 Introduction

When Hamilton discovered the third real normed division algebra, the quaternions ((H)), in 1843, efforts were made to generalize them to higher dimensions. Clifford succeeded in creating an associative generalization which he called geometric algebra, but which later came to bear his name – Clifford algebras. The quaternion famously gave rise to the notion of the dot product and the cross product. Instead of the cross product, which only works in 3 and 7 dimensions, Clifford used the exterior (wedge) product $\wedge$, which is the completely generalized anti-symmetric binary product in any dimension.

In its modern incarnation, a Clifford algebra is defined as an associative algebra over a vector space with a quadratic form q with signature (s,t):

$$
q(x)=x_1^2+x_2^2+\dots x_s^2-x_{s+1}^2-x_{s+2}^2+\dots-x_{s+t}^2 \tag{3.39}
$$

This vector space is real and of dimension $n=s+t$, $\mathbb{R}^{s,t}$. A Clifford algebra is said to be generated by n generators γ_a, called gamma matrices, and in the olden days, this algebra was simply called $Cl(n)$. Today, we conventinally label it $Cl(s,t)$ to not only reflect its dimension but also its signature. The n generators of $Cl(s,t)$ satisfy the **Clifford relation**:

Definition 8.

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}\mathbb{1} \tag{3.40}$$

Hidden in this innocent-looking equation is the fact that these gamma matrices contain a completely antisymmetric product, which conveniently cancels in the in the Clifford relation. That is, for any $a \neq b$, $\gamma_a\gamma_b = -\gamma_b\gamma_a$. This is the building block of the previously mentioned wedge product.

3.3.2 Some Instructive Examples

There are seven famous examples that satisfy the Clifford relation, namely the real composition algebras[1], which should come as no surprise given the origins of Clifford algebras. A composition algebra A over the field $\mathbb{R}$ has a non-degenerate quadratic form N that satisfies the relation:

$$N(xy) = N(x)N(y), \forall x, y \in A \tag{3.41}$$

$N(x)$ is called the **norm** and is defined as $x \cdot x^*$ where x^* is the conjugate of x.

The trivial case is $\mathbb{R}$. It is isomorphic to $Cl(0,0)$, i.e. the Clifford algebra with zero generators, only equipped with the identity element, $\mathbb{1}$. The three next ones are the complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and, if we temporarily relax the associativity requirement for the sake of completion, the octonions ($\mathbb{O}$). These four algebras are called the real normed division algebras and have a positive definite norm. They have 0,1,2 and 3 generators respectively. The three remaining algebras are the split-complex numbers ($\hat{\mathbb{C}}$), split-quaternions ($\hat{\mathbb{H}}$) and split-octonions ($\hat{\mathbb{O}}$), which have the same dimensions as the corresponding division algebra, but with a split signature. They are sometimes colloquially referred to as the split-algebras. For $n = 1, 2, 3$ the composition algebras satisfy the following special case of the Clifford relation:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\sigma\delta_{\mu\nu}\mathbb{1} \tag{3.42}$$

$\sigma = -1$ yields to the three normed division algebras above the reals, whereas $\sigma = 1$ gives us the split-algebras. The case of $n = 3$ is included for completeness, but the octonions and split-octonions do not correspond to $Cl(0,3)$ and $Cl(3,0)$ due to their non-associativity, even if they satisfy the Clifford relation.

The most famous and well-known of these algebras apart from $\mathbb{R}$ is the complex numbers, which corresponds to $Cl(0,1)$. It has only one generator, γ_0 corresponding to the complex root $\boldsymbol{i}$, whose signature is negative, such that $\gamma_0^2 = -1$.

Although most students of physics have not heard of split-complex numbers, which are isomorphic to $Cl(1,0)$, they should be deeply familiar with them. throught their use of hyperbolic numbers with *cosh* and *sinh*. While most textbooks gloss over the unnecessary complexity of defining a hyperbolic imaginary $\boldsymbol{j} = \gamma_0$ such that $\boldsymbol{j}^2 = \gamma_0^2 = 1$, it is sometimes explicitly used in Lorentz transformations in undergraduate textbooks.

The quaternions are less known but are of great importance in physics. Notably, Maxwell used them to formulate his famous equations of electromagnetism. After having been ousted during the vector wars in the late 19th century, quaternions stubbornly reappeared in quantum mechanics in the form of Pauli matrices to describe spin. Today they live happily on in physics and mathematics under the guise of $Cl(0,2)$, and in group theory as the Symplectic group.

Its unruly sibling, the split-quaternion, isomorphic to $Cl(2,0)$, is not used for much, although the fact that they can perform both Lorentz transformations and ordinary rotations, makes them of interest to some physicists. However, they provide an instructive illustration of the machinations of Clifford algebras.

The two generators of $Cl(2,0)$ are γ_0 and γ_1 and square to 1. They correspond to the split-quaternion basis vector $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$. The quaternion basis vector $\boldsymbol{i}$ is a composite $\hat{\boldsymbol{j}}\hat{\boldsymbol{k}}$, corresponding to $\gamma_0\gamma_1$. We can now show that $\boldsymbol{i}$ squares to -1:

$$\begin{aligned} \boldsymbol{i}^2 &= (\gamma_0\gamma_1)(\gamma_0\gamma_1) = -\gamma_1\gamma_0\gamma_0\gamma_1 \\ &= -\gamma_1\gamma_0^1\gamma_1 = -\gamma_1^1 = -1 \end{aligned} \tag{3.43}$$

If we let $x = x_1 + x_2\boldsymbol{i} + x_3\hat{\boldsymbol{j}} + x_4\hat{\boldsymbol{k}}$, the norm is:

$$N(x) = xx^* = x_1^2 + x_2^2 - x_3^2 - x_4^2 \tag{3.44}$$

Thus, the split-quaternion has signature $(2,2)$ and is isomorphic to $Cl(2,0)$. Notice that the split-algebra has 2^2 elements while $Cl(2,0)$ has only 2. This is no coincidence. Together with the identity and the n basis vectors of $Cl(n)$, the exterior product generates n^2 independent elements that form the basis of the exterior algebra. Thus, $\hat{\mathbb{H}}$ is the exterior algebra of $Cl(2,0)$.

Amazingly, $Cl(1,1)$ also gives us the split-quaternion, except that here γ_0 corresponds to $\boldsymbol{i}$ and γ_1 to $\hat{\boldsymbol{j}}$. Therefore, all the six first Clifford algebras for $n = 0,1,2$ correspond exactly to the five associate real composition algebras.

That's convenient, because with the following proposition, we can use them to build Clifford algebras of any size.

$$\begin{aligned} Cl(n,0) \otimes Cl(0,2) &\cong Cl(0,n+2) \\ Cl(0,n) \otimes Cl(2,0) &\cong Cl(n+2,s) \\ Cl(s,t) \otimes Cl(1,1) &\cong Cl(s+1,t+1) \end{aligned} \tag{3.45}$$

The proof is provided in the Appendix (8.1).

As a final note on the division algebras, Bott's periodicity theorem[2] demonstrates a deep relationship between them and the Clifford algebras. They exhibit a periodicity of 8 ([4, Ch 3.]).

In this thesis we will only be working with $n = 4$ and so will not encounter this periodicity.

$s, t \bmod 8$	$Cl(s,t)$	N
$0, 6$	$\mathrm{Mat}_N(\mathbb{R})$	$2^{n/2}$
$2, 4$	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(n-2)/2}$
$1, 5$	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(n-1)/2}$
3	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(n-3)/2}$
7	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_R(\mathbb{H})$	$2^{(n-1)/2}$

3.3.3 The k-Form

In the split-quaternion, $\boldsymbol{i}$ is a composite of two orthogonal basis vectors. Such a composite is classically referred to as a bi-vector and is in Clifford algebra interpreted as an oriented surface element. Similarly a composite of three orthogonal basis vectors is called a tri-vector and so forth. In modern language, the completely anti-symmetrized binary combination of k gamma matrices is called a k-form $\gamma_{\mu_1\mu_2\ldots\mu_k}$ and is defined as [3, A.4]:

$$\gamma_{\mu_1\mu_2\ldots\mu_k} := \frac{1}{k!}\sum \mathrm{sign}(\sigma)\gamma_{\sigma_{(1)}}\gamma_{\sigma_{(2)}}\ldots\gamma_{\sigma_{(k)}} =: \gamma_{[\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_k]} \tag{3.46}$$

The sum is over all permutations of $\{1, 2, \ldots k\}$. If we had infinitely many generators to choose from, we could create a k-form of any size. However, with only n generators to play with, an immediate consequence is that an n-dimensional Clifford algebra has no k-forms greater than $k = n$. This follows from the fact that a higher k-form would require repeated indices, which vanish in the anti-symmetrization process. The n-form is called the *volume form*, and also a *pseudoscalar*. In our context, it is also called the *chirality matrix* of the n-dimensional Clifford algebra, conventionally named γ_{n+1} and defined as:

$$\gamma_{n+1} := \gamma_0\gamma_1\ldots\gamma_{n-1} = \gamma_{01\ldots(n-1)} \tag{3.47}$$

For 4-dimensionial spacetime, which is most relevant to this thesis, it becomes:

$$\gamma_5 := \gamma_0\gamma_1\gamma_2\gamma_3$$

A useful identity for the k-form is:

$$\gamma_{\mu_1\mu_2\ldots\mu_k} = (-1)^{k(k-1)/2}\gamma_{\mu_k\ldots\mu_2\mu_1} \tag{3.48}$$

This follows from the fact that reversing the order of $\gamma_{\mu_1\mu_2\ldots\mu_k}$ requires $(k-1) + \ldots + (k-n) = k/(k-1)/2$ anticommutating permutations.

Since the chirality matrix contains every gamma matrix of a Clifford algebra, it has special properties. First, let is inquire γ_{n+1}^2.

$$\gamma_{n+1}^2 = \gamma_0\gamma_1\ldots\gamma_{n-1}\gamma_0\gamma_1\ldots\gamma_{n-1} = (-1)^{n(n-1)/2}\gamma_0\gamma_1\ldots\gamma_{n-1}\gamma_{n-1}\ldots\gamma_0\gamma_1$$

First note that every gamma matrix γ_μ appears exactly twice in the expression. By reversing the order of the second γ_{n+1} using (3.48), we can rearrange the matrices so that note that the two γ_{n-1} appear adjacent to each other by a number of antisymmetric permutations. We evaluate this to $\eta_{(n-1)(n-1)}$ and iterate this for every remaining gamma matrix. This yields for $\eta = \eta(t, s)$:

$$\gamma_{n+1}^2 = (-1)^{n(n-1)/2} \eta_{00}\eta_{11}...\eta_{(n-1)(n-1)}\mathbb{1}$$

Thus,

$$\gamma_{n+1}^2 = (-1)^{n(n-1)/2+t}\mathbb{1} \tag{3.49}$$

For 4-dimensional Minkowski spacetime this means:

$$\gamma_5^2 = (-1)^{4(4-1)/2+1}\mathbb{1} = -\mathbb{1} \tag{3.50}$$

Let us consider $\{\gamma_\mu, \gamma_{n+1}\}$. We note that γ_μ will match and contract with exactly one of the γ-elements of γ_{n+1}. Suppose that it matches the rightmost element,i.e. $\mu = n - 1$. Then, if we multiply γ_{n+1} with γ_μ from the left, we need $n-1 = \mu$ antisymmetric permutations to make the two elements adjacent. If we instead multiply γ_{n+1} with γ_μ from the right, it is already adjacent so we need $0 = n - 1 - \mu$ permutations to make them adjacent. This now also holds for all other values of μ. The difference in the number of permutations to reach the same position is: $(n - 1 - \mu) - \mu = -2\mu + n - 1$. Note that 2μ is an even number and it will therefore not contribute to a sign change. The number of permutations separating $\gamma_\mu\gamma_{n+1}$ from $\gamma_{n+1}\gamma_\mu$ is therefore $n-1$ anticommutative permutations. Thus,

$$\gamma_\mu\gamma_{n+1} = (-1)^{n-1}\gamma_{n+1}\gamma_\mu \tag{3.51}$$

For n=4 (and any other even dimension), which we will be working with in this thesis:

$$\{\gamma_\mu, \gamma_5\} = 0 \tag{3.52}$$

A useful consequence of this in 4 dimensions is:

$$\gamma_{\mu_1}\gamma_{\mu_2}...\gamma_{\mu_k}\gamma_5 = (-1)^k\gamma_5\gamma_{\mu_1}\gamma_{\mu_2}...\gamma_{\mu_k}, k = 1, 2, 3, 4 \tag{3.53}$$

A corollary is that it is also true for the antisymmetrized version:

$$\gamma_{\mu_1\mu_2...\mu_k}\gamma_5 = (-1)^k\gamma_5\gamma_{\mu_1\mu_2...\mu_k}, k = 1, 2, 3, 4 \tag{3.54}$$

This follows from the definition of the antisymmetric matrix as the sum where each of the $k!$ terms in the sum is the product of k gamma matrices. Moving the γ_5 to the other side therefore leads to a term-wise common factor of $(-1)^k$ due to k permutations in each term.

A related useful identity is[3, A.4]:

$$\begin{aligned}\gamma_{\mu_1\mu_2...\mu_k}\gamma_\nu = \gamma_{\mu_1\mu_2...\mu_k\nu} + \eta_{\nu\mu_k}\gamma_{\mu_1\mu_2...\mu_{k-1}} - \eta_{\nu\mu_{k-1}}\gamma_{\mu_1\mu_2...\widehat{\mu_{k-1}}\mu_k} \\ + ...(-1)^{k-1}\eta_{\nu\mu_1}\gamma_{\widehat{\mu_1}\mu_2...\mu_{k-1}\mu_k}\end{aligned} \tag{3.55}$$

The wide hat means that the index is omitted.

There exists a long-winded algebraic proof, but it can more concisely be formulated in words: All the indices in the k-form are different. γ_ν therefore shares either zero or one index with it. In case ν is different and $k < n$, the product forms a new antisymmetric matrix $\gamma_{\mu_1\mu_2\ldots\mu_k\nu}$. Otherwise it shares one index in the ith position. γ_ν then anticommutes with all the other elements and based on the same logic as in (3.54) we can permute it to become adjacent with μ_i so that they can contract. This requires $k-i$ permutations. An odd number of permutations gives a minus, and an even number gives a plus. The product is then equal to sum of the $k+1$ elements in the identity above.

3.3.4 Spinors

So why are Clifford algebras important in physics and supersymmetry? It turns out by some possibly magical coincidence that they can be used to construct half-spin representations of the Spin group[4, p.6], namely by the construction Σ whose elements are defined as:

$$\Sigma_{\mu\nu} := \frac{1}{4}[\gamma_\mu, \gamma_\nu] = \frac{1}{2}\gamma_{\mu\nu} \tag{3.56}$$

Σ satisfies the commutation relation:

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = \eta_{\nu\rho}\Sigma_{\mu\sigma} - \eta_{\nu\sigma}\Sigma_{\mu\rho} + \eta_{\mu\rho}\Sigma_{\sigma\nu} - \eta_{\mu\sigma}\Sigma_{\rho\nu} \tag{3.57}$$

This is a representation of lie algebra so(s,t) found in (X). The proof is straightforward.

$$\begin{aligned} \gamma_\mu\gamma_\rho &= 2\eta_{\mu\rho} - \gamma_\rho\gamma_\mu \\ \implies \gamma_\mu\gamma_\rho\gamma_\sigma &= 2\eta_{\mu\rho}\gamma_\sigma - \gamma_\rho\gamma_\mu\gamma_\sigma \\ &= 2\eta_{\mu\rho}\gamma_\sigma - 2\eta_{\mu\sigma}\gamma_\rho + \gamma_\rho\gamma_\sigma\gamma_\mu \\ \implies [\gamma_\mu, \gamma_\rho\gamma_\sigma] &= 2(\eta_{\mu\rho}\gamma_\sigma - \eta_{\mu\sigma}\gamma_\rho) \end{aligned} \tag{3.58}$$

Since this expression anticommutes in ρ and σ, it follows that $[\gamma_\mu, [\gamma_\rho, \gamma_\sigma]] = 2[\gamma_\mu, \gamma_\rho\gamma_\sigma]$. Furthermore, the identity $[AB, C] = A[B, C] + [A, C]B$ gives:

$$\begin{aligned} [\gamma_\mu\gamma_\nu, [\gamma_\rho, \gamma_\sigma]] &= \gamma_\mu[\gamma_\nu, [\gamma_\rho, \gamma_\sigma]] + [\gamma_\mu, [\gamma_\rho, \gamma_\sigma]]\gamma_\nu \\ &= 4(\eta_{\nu\rho}\gamma_\mu\gamma_\sigma - \eta_{\nu\sigma}\gamma_\mu\gamma_\rho + \eta_{\mu\rho}\gamma_\sigma\gamma_\nu - \eta_{\mu\sigma}\gamma_\rho\gamma_\nu) \end{aligned} \tag{3.59}$$

Swapping μ and ν gives:

$$[\gamma_\nu\gamma_\mu, [\gamma_\rho, \gamma_\sigma]] = 4(\eta_{\nu\rho}\gamma_\sigma\gamma_\mu - \eta_{\nu\sigma}\gamma_\rho\gamma_\mu + \eta_{\mu\rho}\gamma_\nu\gamma_\sigma - \eta_{\mu\sigma}\gamma_\nu\gamma_\rho) \tag{3.60}$$

Thus,

$$[[\gamma_\mu, \gamma_\nu], [\gamma_\rho, \gamma_\sigma]] = 4(\eta_{\nu\rho}[\gamma_\mu, \gamma_\sigma] - \eta_{\nu\sigma}[\gamma_\mu, \gamma_\rho] + \eta_{\mu\rho}[\gamma_\sigma, \gamma_\nu] - \eta_{\mu\sigma}[\gamma_\rho, \gamma_\nu]) \tag{3.61}$$

When divided by 16, it equals (3.57).

A Clifford algebra of dimension n is isomorphic to the algebra of $n \times n$ real matrices, meaning that it has a unique irreducible real n-dimensional representation[3, A.4] called *Majorana spinors*. That is, the spacetime Majorana spinor is a 4-tuplet with real entries and can be thought of as a column vector. The complexified Clifford algebra has a unique irreducible complex n-dimensional representation called *Dirac spinors*. The spacetime Diract spinor is a duplet with complex entries. We may recover one from the other by equating their conjugates $\bar{\psi} := \bar{\psi}_D = \psi^\dagger \boldsymbol{i}\gamma^0 = \bar{\psi}_M = \psi^t C$, where C is the *charge conjugation matrix*. Although Dirac spinors are practical in many situations, this thesis will only make use of the Majorana type.

3.3.5 The Charge Conjugation Matrix

Since the metric of a Clifford algebra is always diagonal, it follows from the Clifford relation that there exists an algebra A with elements γ_μ that is isomorphic to an algebra A' with the elements γ_μ^t. It satisfies the Clifford relation.

$$\begin{aligned}(\gamma_\mu\gamma_\nu)^t + (\gamma_\nu\gamma_\mu)^t &= (2\eta_{\mu\nu}\mathbb{1})^t \\ \implies \gamma_\nu^t\gamma_\mu^t + \gamma_\mu^t\gamma_\nu^t &= 2\eta_{\mu\nu}\mathbb{1} \\ \implies \gamma_\mu^t\gamma_\nu^t + \gamma_\nu^t\gamma_\mu^t &= 2\eta_{\mu\nu}\mathbb{1} \\ \implies A \cong A'\end{aligned} \tag{3.62}$$

Since they are isomorphic, we can use a change of basis to transform γ_μ into γ_μ^t, up to sign. We do this using a *charge conjugation matrix* C, which despite its name is not a matrix but a bilinear form.

$$\gamma_\mu^t = \pm C\gamma_\mu C^{-1} \tag{3.63}$$

We repeat the calculation using C.

$$\begin{aligned}\gamma_\mu^t\gamma_\nu^t + \gamma_\nu^t\gamma_\mu^t &= (C\gamma_\mu C^{-1})(C\gamma_\nu C^{-1}) + (C\gamma_\nu C^{-1})(C\gamma_\mu C^{-1}) \\ &= C\gamma_\mu\gamma_\nu C^{-1} + C\gamma_\nu\gamma_\mu C^{-1} \\ &= C(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)C^{-1} \\ &= C(2\eta_{\mu\nu}\mathbb{1})C^{-1} \\ &= 2\eta_{\mu\nu}\mathbb{1}\end{aligned} \tag{3.64}$$

This proves that γ_μ^t also satisfies the Clifford relation. Let us now do some further manipulation.

$$\begin{aligned}\gamma_\mu = (\gamma_\mu^t)^t = \pm(C\gamma_\mu C^{-1})^t = \pm(C^{-1})^t\gamma_\mu^t C^t = \\ \pm(C^{-1})^t\gamma_\mu^t C^t = \pm(C^{-1})^t(\pm C\gamma_\mu C^{-1})C^t = \\ (C^{-1}C^t)^{-1}\gamma_\mu(C^{-1}C^t) \\ \implies \gamma_\mu(C^{-1}C^t) = (C^{-1}C^t)\gamma_\mu\end{aligned}$$

Schur's lemma[12] then implies that $(C^{-1}C^t)$ is equal to the identity up to some scalar σ.

$$\begin{aligned} (C^{-1}C^t) &= \sigma\mathbb{1} \\ \implies C^t &= \sigma C \\ \implies C = \sigma C^t &= \sigma(\sigma C) \\ \implies \sigma &= \pm 1 \end{aligned} \tag{3.65}$$

Let, $\tau = \pm 1$. Then, (3.63) yields:

$$\begin{aligned} \gamma_\mu^t &= \tau C \gamma_\mu C^{-1} \\ \implies \tau C\gamma_\mu = \gamma_\mu^t C = \gamma_\mu^t C^t \sigma &= (C\gamma_\mu)^t \sigma \\ \implies (C\gamma_\mu)^t \sigma^2 = (C\gamma_\mu)^t &= \tau\sigma C\gamma_\mu \end{aligned} \tag{3.66}$$

This means that C is either symmetric or skew-symmetric. In general, for an antisymmetric gamma matrix with k elements, we find:

$$\begin{aligned} (\gamma_{\mu_1\mu_2...\mu_k})^t = (\gamma_{[\mu_1}\gamma_{\mu_2}...\gamma_{\mu_k]})^t &= \gamma^t_{[\mu_k}...\gamma^t_{\mu_2}\gamma^t_{\mu_1]} \\ = \tau^k (C\gamma_{[\mu_k}C^{-1})...(C\gamma_{\mu_2}C^{-1})&(C\gamma_{\mu_1]}C^{-1}) \\ = \tau^k C\gamma_{[\mu_k}...\gamma_{\mu_2}\gamma_{\mu_1]}C^{-1} = \tau^k(-1)^{k(k-1)/2}&C\gamma_{\mu_1\mu_2...\mu_k}C^{-1} \\ \implies \gamma^t_{\mu_1\mu_2...\mu_k}C = \sigma(C\gamma_{\mu_1\mu_2...\mu_k})^t = \tau^k(-1)^{k(k-1)/2}&C\gamma_{\mu_1\mu_2...\mu_k} \\ \implies (C\gamma_{\mu_1\mu_2...\mu_k})^t = \sigma\tau^k(-1)^{k(k-1)/2}&C\gamma_{\mu_1\mu_2...\mu_k} \end{aligned} \tag{3.67}$$

We have a degree of freedom in choosing the value of τ but based on foresight, set $\tau = -1$ in $3+1$-dimensional spacetime. The conjugation relation then reduces to:

$$(C\gamma_{\mu_1\mu_2...\mu_k})^t = \sigma(-1)^{k(k+1)/2}C\gamma_{\mu_1\mu_2...\mu_k} \tag{3.68}$$

Concretely, for $k = 1, 2, 3, 4$ this becomes:

$$\begin{aligned} (C\mathbb{1})^t &= \sigma C\mathbb{1} \\ (C\gamma_\mu)^t &= -\sigma C\gamma_\mu \\ (C\gamma_{\mu\nu})^t &= -\sigma C\gamma_{\mu\nu} \\ (C\gamma_{\mu\nu\rho})^t &= \sigma C\gamma_{\mu\nu\rho} \\ (C\gamma_{\mu\nu\rho\sigma})^t &= \sigma C\gamma_{\mu\nu\rho\sigma} \end{aligned} \tag{3.69}$$

In a Clifford algebra of dimension n, there are $\binom{n}{k}$ antisymmetric elements of length k. For n=4, we get the following table of elements and the sign of the conjugation.

Element	Sign	Dimension
$\mathbb{1}$	σ	1
γ_μ	$-\sigma$	4
$\gamma_{\mu\nu}$	$-\sigma$	6
$\gamma_{\mu\nu\rho}$	σ	4
$\gamma_{\mu\nu\rho\sigma}$	σ	1

In a Real 4-by-4 matrix antisymmetric matrices have 6 dimensions and symmetric dimensions have 10. To achieve this, we need to set $\sigma = -1$. Then, $\mathbb{1}, \gamma_{\mu\nu\rho}, \gamma_{\mu\nu\rho\sigma}$ are antisymmetric, while $\gamma_\mu, \gamma_{\mu\nu}$ are symmetric. Thus, we have proven that:

$$\begin{aligned} C^t &= -C \\ (C\gamma_\mu)^t &= C\gamma_\mu \\ (C\gamma_{\mu\nu})^t &= C\gamma_{\mu\nu} \\ (C\gamma_{\mu\nu\rho})^t &= -C\gamma_{\mu\nu\rho} \\ (C\gamma_{\mu\nu\rho\sigma})^t &= -C\gamma_{\mu\nu\rho\sigma} \end{aligned} \tag{3.70}$$

First note that by limiting ourselves to four dimensions in the analysis above, we are developing specialized mathematical tools that cannot be assumed to work in other-dimension Clifford algebras. That is, whenever the charge conjugation matrix is used, the result is only valid for $n = 4$.

Having made this choice, and worked out the values of τ and σ. let us now investigate some of its consequences.

$$\begin{aligned} \gamma_\mu^t &= C^t\gamma_\mu C^{-1} \\ \gamma_\mu^t C &= C^t\gamma_\mu \\ C^t &= -C \\ (C\gamma_\mu)^t &= C\gamma_\mu \end{aligned} \tag{3.71}$$

If we define C with indices C_{ab} we notice that $C^t = -C_{ba}$ and due to this antisymmetry some care is needed in how we use C to raise and lower indices. We use the *North-West* and *South-East* conventions[3, A.4] such that $\psi^b C_{ba} = (C^t\psi)_a = \psi_a$ and $C^{ab}\psi_b = ((C^t\psi)C^{-1})^a = \psi^a$. Here the built in assumption is that $\psi^t = \psi$ since from (3.71) $\psi^t = C^t\psi C^{-1}$.

Gamma matrices with indices can be defined as $\gamma_\mu = (\gamma_\mu)^a_{\ b}$ and from (3.71) it follows that:

$$\gamma_\mu^t = C^t\gamma_\mu C^{-1} = C^{ac}(\gamma_\mu)^d_{\ c}C_{bd} = (\gamma_\mu)_a^{\ b} \tag{3.72}$$

We can now summarize the interaction of the k-forms with the charge conjugate matrix using the convention that $(C\chi)_{ab}$ notationally can be written as $(\chi)_{ab}$.

$$\begin{aligned}
\mathbb{1}_{ab} &= -\mathbb{1}_{ba} \\
(\gamma_\mu)_{ab} &= (\gamma_\mu)_{ba} \\
(\gamma_{\mu\nu})_{ab} &= (\gamma_{\mu\nu})_{ba} \\
(\gamma_{\mu\nu\rho})_{ab} &= -(\gamma_{\mu\nu\rho})_{ba} \\
(\gamma_{\mu\nu\rho\sigma})_{ab} &= -(\gamma_{\mu\nu\rho\sigma})_{ba}
\end{aligned} \tag{3.73}$$

Matrices with lowered (or raised) spinor indices have useful properties. One of these is the clever use of transposition to cancel antisymmetric components. Consider the product $(\Sigma)_a^{\ c}(\Gamma)_{cb} = (\Sigma\Gamma)_{ab}$. Notice that from (refclifford-conjugate-transpose-final3) $(\Gamma)_{cb} = \kappa(\Gamma)_{bc}$ where $\kappa = \pm 1$ if Γ is symmetric/antisymmetric. Furthernore, decompose $\Sigma\Gamma$ into its symmetric and antisymmetric parts S and A respectively: $(\Sigma\Gamma)_{ab} = (S + A)_{ab}$. Then we can show the following:

$$\begin{aligned}
(S + A)_{ab} &= (\Sigma)_a^{\ c}(\Gamma)_{cb} \\
&= \kappa(\Sigma)_a^{\ c}(\Gamma)_{bc} \\
&= \kappa(S + A)_{ba} \\
&= \kappa(S - A)_{ab}
\end{aligned} \tag{3.74}$$

When $\kappa = +1$ we find that $S + A = S - A \implies A = 0$. Similarly, when $\kappa = -1$, $S = 0$. Thus, parity is preserved under contraction when the indices are lowered. This is a powerful technique to cancel terms in a complicated expression. An important example which we will utilize later is:

$$\begin{aligned}
(\gamma_{\mu\nu})_a^{\ c}(\gamma^{\rho\sigma})_{cb} &= (\gamma_{\mu\nu}^{\ \ \ \rho\sigma} + c\mathbb{1} + 2\delta^\sigma_{[\mu}\gamma_{\nu]}^{\ \ \rho} - 2\delta^\rho_{[\mu}\gamma_{\nu]}^{\ \ \sigma})_{ab} \\
&= 2(\delta^\sigma_{[\mu}\gamma_{\nu]}^{\ \ \rho} - \delta^\rho_{[\mu}\gamma_{\nu]}^{\ \ \sigma})_{ab}
\end{aligned} \tag{3.75}$$

Since both $(\gamma_{\mu\nu}^{\ \ \ \rho\sigma})_{ab}$ and $(\mathbb{1})_{ab}$ are antisymmetric (and $\gamma^{\rho\sigma}$ is symmetric), they vanish in the above equation.

3.3.6 Other Useful Identities

One common usage of gamma matrices in quantum field theory is the $\not\partial$ operator defined as:

$$\not\partial := \gamma^\mu \partial_\mu \tag{3.76}$$

One useful identity involving the $\not\partial$ is:

$$\not\partial^2 = \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu = (\gamma^{[\mu}\gamma^{\nu]} + \gamma^{(\mu}\gamma^{\nu)})\partial_\mu\partial_\nu = \eta^{\mu\nu}\mathbb{1}\partial_\mu\partial_\nu = \Box \tag{3.77}$$

It follows from the fact that an antisymmetric tensor that contracts with a symmetric one is zero. That is, $\gamma^{[\mu}\gamma^{\nu]}\partial_\mu\partial_\nu = -\gamma^{[\nu}\gamma^{\mu]}\partial_\nu\partial_\mu = 0$.

We will also encounter composite conjugates of the form:

$$\begin{aligned}
&\overline{\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_{k-1}}\gamma_{\mu_k}} = \gamma^t_{\mu_k}\gamma^t_{\mu_{k-1}}\ldots\gamma^t_{\mu_2}\gamma^t_{\mu_1}C \\
&= \gamma^t_{\mu_k}\gamma^t_{\mu_{k-1}}\ldots\gamma^t_{\mu_2}(C\gamma_{\mu_1})^t = \gamma^t_{\mu_k}\gamma^t_{\mu_{k-1}}\ldots\gamma^t_{\mu_2}C\gamma_{\mu_1} \\
&= \gamma^t_{\mu_k}\gamma^t_{\mu_{k-1}}\ldots(C\gamma_{\mu_2})^t\gamma_{\mu_1} = \gamma^t_{\mu_k}\gamma^t_{\mu_{k-1}}\ldots C\gamma_{\mu_2}\gamma_{\mu_1} \\
&\ldots \\
&= \gamma^t_{\mu_k}C\gamma_{\mu_{k-1}}\ldots\gamma_{\mu_2}\gamma_{\mu_1} = \overline{\gamma_{\mu_k}}\gamma_{\mu_{k-1}}\ldots\gamma_{\mu_2}\gamma_{\mu_1}
\end{aligned} \tag{3.78}$$

Finally, the k-forms can be expressed more compactly using the chirality matrix. First let us (re)define γ_5 as $\frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}$ and the totally antisymmetric tensor ϵ is initalized as $\epsilon^{0123} = -\epsilon_{0123} = 1$. Then:

$$\begin{aligned}
\gamma_{\mu\nu} &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\gamma_5 \\
\gamma_{\mu\nu\rho} &= \epsilon_{\mu\nu\rho\sigma}\gamma^{\sigma}\gamma_5 \\
\gamma_{\mu\nu\rho\sigma} &= -\epsilon_{\mu\nu\rho\sigma}\gamma_5
\end{aligned} \tag{3.79}$$

Again, this can be calculated by straightforward but longwinded calculations. However, the sketch of the proof is that since γ_5 contains all 4 gamma basis matrices, it will contract with a k-form in such a way that only the $4-k$ non-contained indices will remain. For the 4-form, the calculation is straightforward: $\gamma_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}\frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma} = 4!\frac{1}{4!}\gamma_{\mu\nu\rho\sigma}$.

Thus, a perfectly valid and equivalent basis for the exterior algebra is:

$$\{\mathbb{1}, \gamma_\mu, \gamma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5\} \tag{3.80}$$

4 Field Theory

All this work so far on describing and identifying the Poincaré and conformal algebras is nice and dandy, but how does it relate to physics? The answer is the principle of least action[8, Ch. 1.1.2]. This is where Noether's theorem[9] enter's the picture. She discovered that every differential symmetry of the action implies a conservation law.

The **action**, which is a functional, meaning that it takes a set of functions (corresponding to different worldlines) as input and outputs a number. It is done by integrating over the volume element in n-dimensional space, which is an n-form. However, since we are limiting our analysis to flat space with a constant metric, the volume element reduces to the familiar d^nx.

The set of functions we feed the action is the Lagrangian density $\mathscr{L}$. Much of the murky business of theoretical physicists is to construct Lagrangians that reproduce the known conservation laws of nature. A key ingredient, which is the foundation of Quantum Field Theory (QFT), is that it has to be Lorentz invariant, meaning that it must be a scalar.

$\mathscr{L}$ is a function of a field (or fields) Φ and its derivatives. Φ, in turn, is a function of the spacetime coordinates x. In its most general form it can be written as:

$$\mathscr{L} = f(\Phi(x), \partial_{\mu_1}\Phi(x), \ldots, \partial_{\mu_1 \ldots}\partial_{\mu_k}\Phi(x)) \tag{4.1}$$

For brevity, I will write $\Phi(x)$ as Φ. The action on Φ is then defined as :

$$S[\phi] = \int_{\mathcal{M}} \mathscr{L} d^n x$$

For brevity, I also drop the $\mathcal{M}$, $d^n x$ since they only add clutter. The action is extremized if an infinitesimal deviation $S[\Phi + \delta\Phi]$ from the wordline is zero. Since the derivative is linear, the following holds:

$$\Phi \to \Phi + \delta\Phi \implies \begin{cases} \partial_{\mu_1}\Phi \to \partial_{\mu_1}\Phi + \delta\partial_{\mu_1}\Phi \\ \ldots \\ \partial_{\mu_1 \ldots}\partial_{\mu_k}\Phi \to \partial_{\mu_1 \ldots}\partial_{\mu_k}\Phi + \delta\partial_{\mu_1 \ldots}\partial_{\mu_k}\Phi \end{cases}$$

We then have that

$$S[\Phi + \delta\Phi] = \int \mathscr{L}(\Phi + \delta\Phi, \partial_{\mu_1}\Phi + \delta\partial_{\mu_1}\Phi, \ldots, \partial_{\mu_1 \ldots}\partial_{\mu_k}\Phi + \delta\partial_{\mu_1 \ldots}\partial_{\mu_k}\Phi)$$

We define the variation of the action as:

$$\delta S[\Phi] := S[\Phi + \delta\Phi] - S[\Phi]$$

By first order Taylor expansion of the $k+1$ variables in $\delta\Phi$ we obtain:

$$\delta S = \int \delta\Phi \frac{\partial \mathscr{L}}{\partial \Phi} + \partial_{\mu_1}(\delta\Phi)\frac{\partial \mathscr{L}}{\partial(\partial_{\mu_1}\Phi)} + \ldots\partial_{\mu_1 \ldots}\partial_{\mu_k}(\delta\Phi)\frac{\partial \mathscr{L}}{\partial(\partial_{\mu_1 \ldots}\partial_{\mu_k}\phi)} + \mathscr{O}(\delta\Phi^2)$$

It is now time to limit Φ to scalar fields ϕ, because integration by parts does not work the same for fermionic fields. Utilizing integration by parts and the Divergence theorem, meaning that the total derivative vanishes at the boundary, we can effectively move the derivative from $\delta\phi$ to the other term by a change of sign. Doing this for each of the k derivatives and omitting higher order Taylor terms in $\delta\phi$ gives:

$$\delta S = \int \delta\phi \left\{ \frac{\partial \mathscr{L}}{\partial \phi} - \partial_{\mu_1}\frac{\partial \mathscr{L}}{\partial(\partial_{\mu_1}\phi)} + \ldots(-1)^k \partial_{\mu_1 \ldots}\partial_{\mu_k}\frac{\partial \mathscr{L}}{\partial(\partial_{\mu_1 \ldots}\partial_{\mu_k}\phi)} \right\}$$

In this thesis, we will never encounter more than first order derivatives ($k = 1$) in the Lagrangian, and the variation of S therefore reduces to the familiar:

$$\delta S = \int \delta\phi \left\{ \frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right\} \tag{4.2}$$

The fermionic Lagrangian density will be dealt with later.

4.1 Dimensional Analysis

Dimensional analysis is the study of the relationship between physical quantities as measured by base units. (meter, second, kilogram etc.) It can be a useful tool in identifying and verifying the correct form of an equation.

Notationwise, square brackets are used to gauge the dimensionality of a quantity. So if A is the quantity, $[A]$ equals the dimension. The square bracket can be thought of as a dimensional logarithm so that $[A^n] = n[A]$. A mere number c has dimension $[c] = 0$. In general,

$$[cA_1^{n_1}A_2^{n_2}...A_k^{n_k}] = n_1[A_1] + n_2[A_2] + ... + n_k[A_k] \tag{4.3}$$

In QFT the convention is to employ natural units where the action then becomes unitless, $[S] = 0$. We may choose to set $[\partial_\mu] = 1$. Then, since $[\partial_\mu x^\nu] = [\partial_\mu] + [x^\nu] = [\delta_\mu^\nu] = 0$, it follows that $[x^\nu] = -1$. Consequently, $[\int d^n x] = -n$. $[S] = [\int d^n x \mathscr{L}] = [\int d^n x] + [\mathscr{L}] = 0 \implies [\mathscr{L}] = n$.

We can now use these numbers to deduce the dimension of ϕ, ψ, and m from the kinetic and mass terms.

$$\begin{aligned}
[\mathscr{L}_{kin}] = [\partial_\mu^2 \phi^2] = 2[\partial_\mu] + 2[\phi] &\implies [\phi] = (n-2)/2 \\
[\mathscr{L}_{kin}] = [\partial_\mu \psi^2] = [\partial_\mu] + 2[\psi] &\implies [\psi] = (n-1)/2 \\
[\mathscr{L}_{mass}] = [m^2\phi^2] = 2[m] + 2[\phi] &\implies [m] = 1
\end{aligned} \tag{4.4}$$

For $n = 4$, this means that scalars and spinors have dimension 2 and 3/2, respectively.

4.2 Free Massless Lagrangian

4.2.1 Scalar Fields

The free massless Lagrangian density $\mathscr{L}$ of a spin 0 scalar field ϕ is $-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. This gives $\frac{\delta\mathscr{L}}{\delta\phi} = 0 - \partial^\mu(-\frac{1}{2}(2\partial^\mu\phi)) = \Box\phi$. Thus,

$$\delta S = \int \delta\phi\Box\phi$$

We now first investigate the variation of Poincare group symmetries:

$$\delta_a\phi = a^\mu P_\mu\phi = a^\mu\partial_\mu\phi$$

$$\delta_b\phi = b^{\mu\nu}M_{\mu\nu}\phi = b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi = b^{\mu\nu}2x_{[\mu}\partial_{\nu]}\phi$$

Abbreviations in the following calculations: IBP = integration by parts. $\Box = \partial_\rho\partial^\rho, n = \delta_\mu^\mu = Dim(\mathcal{M})$.

$$
\begin{aligned}
\delta_a S &= \int \delta_a \phi \Box \phi = a^\mu \int \partial_\mu \phi \Box \phi \\
&\underbrace{=}_{IBP\times 2} a^\mu \int (\Box \partial_\mu \phi)\phi \\
&\underbrace{=}_{IBP} -a^\mu \int (\Box \phi)\partial_\mu \phi \\
&= -\int \delta_a \phi \Box \phi = -\delta_a S = 0
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
\delta_b S &= \int \delta_b \phi \Box \phi = b^{\mu\nu} \int (x_{[\mu}\partial_{\nu]}\phi)\Box\phi \\
&= -b^{\mu\nu} \int \partial_\rho (2x_{[\mu}\partial_{\nu]}\phi)\partial^\rho \phi \\
&= b^{\mu\nu} \int \partial^\rho (2(\eta_{\rho[\mu}\partial_{\nu]} + x_{[\mu}\partial_{\nu]}\partial_\rho)\phi)\phi \\
&= b^{\mu\nu} \int (2(\underbrace{\partial_{[\mu}\partial_{\nu]}}_{=0} + \delta^\rho_{[\mu}\partial_{\nu]}\partial_\rho + \underbrace{x_{[\mu}\partial_{\nu]}}_{IBP}\Box)\phi)\phi \\
&= b^{\mu\nu} \int (2(\underbrace{\partial_{[\nu}\partial_{\mu]}}_{=0} - \underbrace{\eta_{[\nu\mu]}}_{=0}\Box - x_{[\mu}\partial_{\nu]}\Box)\phi)\phi \\
&= -\int \delta_b \phi \Box \phi = -\delta_b S = 0
\end{aligned}
\tag{4.6}
$$

We have thus found that the Poincaré algebra extremizes the action, which is great news because otherwise the universe would have been in dire straits. Now, let us turn our attention to the conformal algebra. As we have already established, the Poincaré algebra is a subalgebra of the conformal algebra so this we have already checked. What is needed is to investigate D and K_μ.

$$
\begin{aligned}
\delta_c \phi &= c x^\mu \partial_\mu \\
\delta_d \phi &= d^\mu K_\mu = 2x_\mu x^\kappa \partial_\kappa - x^2 \partial_\mu
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\delta_c S &= \int \delta_c \phi \Box \phi = c \int (x^\mu \partial_\mu + \Delta)\phi \Box \phi \\
&= -c \int \partial_\rho ((x^\mu \partial_\mu + \Delta)\phi) \partial^\rho \phi \\
&= c \int \partial^\rho ((\delta^\mu_\rho \partial_\mu + x^\mu \partial_\mu \partial_\rho + \Delta \partial_\rho)\phi)\phi \\
&= c \int (\Box + g^{\rho\mu}\partial_\mu \partial_\rho + \underbrace{x^\mu \partial_\mu}_{IBP} \Box + \Delta \Box)\phi)\phi \\
&= c \int (2 - \delta^\mu_\mu - x^\mu \partial_\mu + \Delta)\Box \phi)\phi \\
&= -c \int (x^\mu \partial_\mu + (\underbrace{n - 2 - \Delta}_{=\Delta}))\phi)\Box \phi \\
&= - \int \delta_c \phi \Box \phi = -\delta_c S = 0
\end{aligned} \tag{4.8}
$$

$\Delta = n - 2 - \Delta$ implies that $\Delta = (n-2)/2$.

$$
\begin{aligned}
\delta_d S &= \int \delta_d \phi \Box \phi = d^\mu \int (2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu + E x_\mu)\phi \Box \phi \\
&= -d^\mu \int \partial_\rho ((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu + E x_\mu)\phi) \partial^\rho \phi \\
&= d^\mu \int \partial^\rho ((2g_{\rho\mu} x^\nu \partial_\nu + 2x_\mu \delta^\nu_\rho \partial_\nu + 2x_\mu x^\nu \partial_\nu \partial_\rho - g_{\rho\nu} x^\nu \partial_\mu \\
&- x_\nu \delta^\nu_\rho \partial_\mu - x_\nu x^\nu \partial_\mu \partial_\rho + E g_{\rho\mu} + E x_\mu \partial_\rho)\phi)\phi \\
&= d^\mu \int ((2(\delta^\nu_\mu \partial_\nu + x^\nu \partial_\nu \partial_\mu + \delta^\nu_\mu \partial_\nu + x_\mu \Box + x^\nu \partial_\nu \partial_\mu + x_\mu \Box + x_\mu x^\nu \partial_\nu \Box \\
&- \delta^\nu_\nu \partial_\mu - x^\nu \partial_\nu \partial_\mu - x^\nu \partial_\nu \partial_\mu) - x_\nu x^\nu \partial_\mu \Box + 2E\partial_\mu + E x_\mu \Box \phi)\phi \\
&= d^\mu \int ((4 - 2m + 2E)\partial_\mu + (4 + E)x_\mu \Box + \underbrace{2x_\mu x^\nu \partial_\nu \Box - x_\nu x^\nu \partial_\mu \Box}_{IBP})\phi)\phi \\
&= d^\mu \int ((4 - 2m + 2E)\partial_\mu + (4 + E)x_\mu \Box - 2g_{\nu\mu} x^\nu \Box - 2x_\mu \delta^\nu_\nu \Box - 2x_\mu x^\nu \partial_\nu \Box \\
&+ g_{\mu\nu} x^\nu \Box + x_\nu \delta^\nu_\mu \Box + x_\nu x^\nu \partial_\mu \Box)\phi)\phi \\
&= d^\mu \int ((\underbrace{4 - 2m + 2E}_{=0})\partial_\mu + (4 - 2m + E)x_\mu \Box - 2x_\mu x^\nu \partial_\nu \Box + x_\nu x^\nu \partial_\mu \Box)\phi)\phi \\
&= -d^\mu \int (2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu + (\underbrace{-4 + 2m - E}_{=E})x_\mu)\phi)\Box \phi \\
&= - \int \delta_d \phi \Box \phi = -\delta_d S = 0
\end{aligned} \tag{4.9}
$$

$4 - 2n + 2E = 0$ is consistent with $-4 + 2n - E = E$ and implies that $E = n - 2 = 2\Delta$.

In summary, for $n = 4$

$$\begin{aligned}\delta_a\phi &= a^\mu\partial_\mu\phi \\ \delta_b\phi &= b^{\mu\nu}2x_{[\mu}\partial_{\nu]}\phi \\ \delta_c\phi &= c(x^\mu\partial_\mu + 1)\phi \\ \delta_d\phi &= d^\mu(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2)\phi\end{aligned} \tag{4.10}$$

So far, so good.

4.2.2 Fermionic Fields

Before venturing into spinor fields, some identities might be useful. Let χ be a rank 2 dual tensor, where $\chi^t = (-1)^v\chi$, and $v = 0, 1$ if it symmetric or antisymmetric under transposition respectively. ∂^r is shorthand for $\partial_{\mu_1}...\partial_{\mu_r}$. We assume that ∂^r, ∂^s, and χ together contract all indices so that the expression is Lorentz invariant. Then:

$$\begin{aligned}&\int \partial^r\psi^a\chi_{ab}\partial^s\psi^b \\ &\underbrace{=}_{IBP\times(r+s)} (-1)^{r+s}\int \partial^s\psi^a\chi_{ab}\partial^r\psi^b \\ &\underbrace{=}_{permute} (-1)^{r+s+1}\int \partial^r\psi^b\chi_{ab}\partial^s\psi^a \\ &\underbrace{=}_{\text{relabel } a\Leftrightarrow b} (-1)^{r+s+1}\int \partial^r\psi^a\chi_{ba}\partial^s\psi^b \\ &\underbrace{=}_{\text{transpose } a\Leftrightarrow b} (-1)^{r+s+v+1}\int \partial^r\psi^a\chi_{ba}\partial^s\psi^b\end{aligned} \tag{4.11}$$

"Permute" is shorthand for permuting the order of the fields, leading to a change of sign. "Relabel" is shorthand for relabeling contracted indices such that they harmonize with the labels of other similar terms. "Transpose" is shorthand for χ^t, which potentially leads to a sign change. In case of an even number of sign changes, the integral does not vanish. If it is odd, it is identically zero.

$$r + s + v + 1 = even \implies \int \partial^r\psi^a\chi_{ab}\partial^s\psi^b = \int \partial^r\psi^a\chi_{ab}\partial^s\psi^b \neq 0 \tag{4.12}$$

$$r + s + v + 1 = odd \implies \int \partial^r\psi^a\chi_{ab}\partial^s\psi^b = -\int \partial^r\psi^a\chi_{ab}\partial^s\psi^b = 0 \tag{4.13}$$

Let us just first give a brief dummy example for illustration. Consider $\int \psi^a(\gamma^\mu)_{ab}\partial_\mu\psi^b$. Here $r=0, s=1, v=0$.Thus, integration by parts yields $(-1)^{(0+1+0+1)}\int \psi^a(\gamma^\mu)_{ab}\partial_\mu\psi^b$. Since it is even, the integral does not vanish.

We are now ready to consider the Poincaré algebras. First we note from (3.70) that $\not\partial = (C\gamma^\mu)_{ab}\partial_\mu = (C\gamma^\mu)_{ba}\partial_\mu = (\not\partial)^t$. We calculate δS to the first order in $\delta\psi$.

$$\begin{aligned}
\delta S &= S[\psi+\delta\psi] - S[\psi] \\
&= \frac{1}{2}\int (\overline{(\psi+\delta\psi)}\not\partial(\psi+\delta\psi) - \bar{\psi}\not\partial\psi) \\
&= \frac{1}{2}\int (\delta\bar{\psi}\not\partial\psi + \underbrace{\bar{\psi}\not\partial\delta\psi)}_{(4.11)} \\
&= \frac{1}{2}\int (\delta\bar{\psi}\not\partial\psi + (-1)^{0+1+0+1}\bar{\psi}\not\partial\delta\psi) \\
&= \int \delta\bar{\psi}\not\partial\psi
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
\delta_a\psi &= a^\mu P_\mu\psi = a^\mu\partial_\mu\psi \\
\delta_a\bar{\psi} &= a^\mu\partial_\mu\bar{\psi} \\
\delta_b\psi &= b^{\mu\nu}M_{\mu\nu}\psi = 2b^{\mu\nu}x_{[\mu}\partial_{\nu]}\psi \\
\delta_b\bar{\psi} &= 2b^{\mu\nu}x_{[\mu}\partial_{\nu]}\bar{\psi}
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\delta_a S = \int \delta_a\bar{\psi}\not\partial\psi = a^\mu\int\partial_\mu\bar{\psi}\not\partial\psi \\
= a^\mu\int\partial_\mu\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b \underbrace{=}_{(4.13)} -a^\mu\int\partial_\mu\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b \\
= -\int\delta_a\bar{\psi}\not\partial\psi = 0
\end{aligned} \tag{4.16}$$

Thus, $\delta_a\psi$ is a symmetry of the Lagrangian. In premonition of the calculation of $\delta_b S$, it is useful to note the following relation:

$$\begin{aligned}
\int \psi^a(C\Sigma_{\mu\nu}\gamma^\rho)_{ab}\partial_\rho\psi^b \\
= \frac{1}{2}\int\psi^a(\underbrace{C(\gamma_{\mu\nu}{}^\rho}_{(4.13)\implies=0} + \delta^\rho_\nu\gamma_\mu - \delta^\rho_\mu\gamma_\nu))_{ab}\partial_\rho\psi^b \\
= \int\psi^a(C\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b
\end{aligned} \tag{4.17}$$

Then:

$$
\begin{aligned}
\delta_b S = \int \delta_b \bar{\psi} \partial\!\!\!/ \psi = 2b^{\mu\nu} \int x_{[\mu}\partial_{\nu]}\bar{\psi}\partial\!\!\!/\psi = 2b^{\mu\nu} \int x_{[\mu}\partial_{\nu]}\psi^a (C\gamma^\rho)_{ab}\partial_\rho \psi^b \\
\underbrace{=}_{IBP} -2b^{\mu\nu} \int (\underbrace{\eta_{[\mu\nu]}}_{=0} \psi^a (C\gamma^\rho)_{ab}\psi^b + \psi^a (C\gamma^\rho)_{ab} x_{[\mu}\partial_{\nu]})\partial_\rho \psi^b \\
\underbrace{=}_{IBP} 2b^{\mu\nu} \int \partial_\rho \psi^a (C\gamma^\rho)_{ab} x_{[\mu}\partial_{\nu]}\psi^b + \psi^a (C\gamma^\rho)_{ab}\eta_{\rho[\mu}\partial_{\nu]}\psi^b \\
= -2b^{\mu\nu} \int x_{[\mu}\partial_{\nu]}\psi^b (C\gamma^\rho)_{ab}\partial_\rho \psi^a - \psi^a (C\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b \\
= -2b^{\mu\nu} \int x_{[\mu}\partial_{\nu]}\psi^a (C\gamma^\rho)_{ab}\partial_\rho \psi^b - \psi^a (C\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b \\
= -\delta_b S + 2b^{\mu\nu} \int \psi^a (C\Sigma_{\mu\nu}\gamma^\rho)_{ab}\partial_\rho \psi^b \neq 0
\end{aligned}
\tag{4.18}
$$

We try adding a term $\delta_\Sigma \psi := b^{\mu\nu}\Sigma_{\mu\nu}\psi$ to $\delta_b \psi$ to see if we can cancel the extra term.

$$
\begin{aligned}
\delta_b \psi = b^{\mu\nu}(2x_{[\mu}\partial_{\nu]} + \Sigma_{\mu\nu})\psi \\
\delta_b \bar{\psi} = b^{\mu\nu}(2x_{[\mu}\partial_{\nu]}\bar{\psi} + \overline{\Sigma_{\mu\nu}\psi})
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
\delta_\Sigma \bar{\psi} := \overline{\Sigma_{\mu\nu}\psi} = (\Sigma_{\mu\nu}\psi)^t C = \psi^t \Sigma^t_{\mu\nu} C \\
= -\psi^t \Sigma^t_{\mu\nu} C^t = -\psi^t (\underbrace{C\Sigma_{\mu\nu}}_{(3.70)})^t \\
= -\psi^t C \Sigma_{\mu\nu} = -\bar{\psi}\Sigma_{\mu\nu}
\end{aligned}
\tag{4.20}
$$

Thus,

$$
\delta_b \bar{\psi} = b^{\mu\nu}(2x_{[\mu}\partial_{\nu]}\bar{\psi} - \bar{\psi}\Sigma_{\mu\nu})
\tag{4.21}
$$

We now only calculate the new part. We already found that without $\delta_\Sigma \bar{\psi}$, $\delta_{b_{\text{old}}} S = -2\delta_\Sigma S$. For it to vanish we need to find that $\delta_\Sigma S = \delta_\Sigma S$.

$$
\begin{aligned}
\delta_\Sigma S = \int \delta_\Sigma \bar{\psi} \partial\!\!\!/ \psi = -b^{\mu\nu} \int \psi^a (C\Sigma_{\mu\nu}\gamma^\rho)_{ab}\partial_\rho \psi^b \\
\underbrace{=}_{(4.17)} -b^{\mu\nu} \int \psi^a (C\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b \\
\underbrace{=}_{(4.12)} -b^{\mu\nu} \int \psi^a (C\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b \\
= -b^{\mu\nu} \int \psi^a (C\Sigma_{\mu\nu}\gamma^\rho)_{ab}\partial_\rho \psi^b \\
= \int \delta_\Sigma \bar{\psi} \partial\!\!\!/ \psi
\end{aligned}
\tag{4.22}
$$

Thus, with this extra term $\delta_b S = \delta_{b_{\text{old}}} S + \delta_\Sigma S = -\delta_b S = 0$.

$$\delta_c\psi = cx^\mu\partial_\mu\psi$$

$$\begin{aligned}
\delta_c S = \int \delta_c\bar{\psi}\partial\!\!\!/\psi = c\int x^\mu\partial_\mu\bar{\psi}\partial\!\!\!/\psi = c\int x^\mu\partial_\mu\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b \\
\underbrace{=}_{IBP} -c\int \partial_\rho\psi^a(C\gamma^\rho)_{ab}\psi^b + x^\mu\partial_\mu\partial_\rho\psi^a(C\gamma^\rho)_{ab}\psi^b \\
\underbrace{=}_{IBP} c\int \psi^b(C\gamma^\rho)_{ab}\partial_\rho\psi^a + x^\mu_\mu\partial_\rho\psi^a(C\gamma^\rho)_{ab}\psi^b + \partial_\rho\psi^a(C\gamma^\rho)_{ab}x^\mu\partial_\mu\psi^b \\
= c\int \psi^b(C\gamma^\rho)_{ab}\partial_\rho\psi^a - n\psi^b(C\gamma^\rho)_{ab}\partial_\rho\psi^a - x^\mu\partial_\mu(C\gamma^\rho)_{ab}\partial_\rho\psi^a\psi^b \\
\underbrace{=}_{relabel+transpose} -c\int (n-1)\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b + x^\mu\partial_\mu\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b \\
= -\delta_c S - c\int (n-1)\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b \neq 0
\end{aligned} \tag{4.23}$$

Like before, we can anticipate what factor must be added to $\delta_c S$ for the variation of the action to vanish, namely some constant Δ_ψ multiplying the variation: $\delta_\Delta\psi = c\Delta_\psi\psi$.

$$\begin{aligned}
\delta_\Delta S = \int \delta_\Delta\bar{\psi}\partial\!\!\!/\psi = c\int \Delta_\psi\bar{\psi}\partial\!\!\!/\psi = c\int \Delta_\psi\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b \\
\underbrace{=}_{(4.12)} c\int \Delta_\psi\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b = \delta_\Delta S
\end{aligned} \tag{4.24}$$

Therefore:

$\delta_c S + \delta_\Delta S = -\delta_c S - c\int(\underbrace{n-1-\Delta_\psi}_{=\Delta_\psi})\psi^a(C\gamma^\rho)_{ab}\partial_\rho\psi^b = 0$ The action vanishes if $\Delta_\psi = (n-1)/2$. Since we have used relations from () which assumes $n=4$, $\Delta_\psi = 3/2$

$$\delta_d\psi = d_\mu(2x^\mu x^\nu\partial_\nu - x^2\partial_\mu)\psi$$

$$
\begin{aligned}
\delta_d S = \int \delta_d \bar{\psi} \partial\!\!\!/ \psi = d^\mu \int (2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \bar{\psi} \partial\!\!\!/ \psi = \\
d^\mu \int (2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \psi^a (C\gamma^\rho)_{ab} \partial_\rho \psi^b \\
\underbrace{=}_{IBP(\rho)} -d^\mu \int (2\eta_{\rho\mu} x^\nu \partial_\nu + 2x_\mu \partial_\rho - 2\eta_{\rho\nu} x^\nu \partial_\mu) \psi^a (C\gamma^\rho)_{ab} \psi^b \\
+ \underbrace{(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \partial_\rho \psi^a (C\gamma^\rho)_{ab} \psi^b}_{IBP(\mu,\nu)} \\
= -d^\mu \int (4x^\nu \eta_{\rho[\mu} \partial_{\nu]} + 2x_\mu \partial_\rho) \psi^a (C\gamma^\rho)_{ab} \psi^b - (2x_\mu + 2x_\mu \delta^\nu_\nu - 2x_\mu) \partial_\rho \psi^a (C\gamma^\rho)_{ab} \psi^b \\
-\partial_\rho \psi^a (C\gamma^\rho)_{ab} (2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \psi^b \\
\underbrace{=}_{permute} -d^\mu \int -4x^\nu \psi^b (C\gamma^\rho)_{ab} \eta_{\rho[\mu} \partial_{\nu]} \psi^a + 2x_\mu (n-1) \psi^b (C\gamma^\rho)_{ab} \partial_\rho \psi^a \\
+(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \psi^b (C\gamma^\rho)_{ab} \partial_\rho \psi^a \\
\underbrace{=}_{relabel+transpose} -d^\mu \int -4x^\nu \psi^a (C\gamma_{[\mu})_{ab} \partial_{\nu]} \psi^b + 2x_\mu (n-1) \psi^a (C\gamma^\rho)_{ab} \partial_\rho \psi^b \\
+(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \psi^a (C\gamma^\rho)_{ab} \partial_\rho \psi^b \\
= -\delta_d S - d^\mu \int -4x^\nu \psi^a (C\Sigma_{\mu\nu} \gamma^\rho)_{ab} \partial_\rho \psi^b + 2x_\mu (n-1) \psi^a (C\gamma^\rho)_{ab} \partial_\rho \psi^b \neq 0
\end{aligned}
\tag{4.25}
$$

We note that one of the non-vanishing terms is up to a constant the same as (4.20) multiplied with x^v and the other is also a familiar expression multiplied with x_μ. We therefore continue the strategy of adding terms.

$\delta_{x\Delta} \psi = d^\mu E_{x\Delta} x_\mu \psi$

$\delta_{x\Sigma} \psi = d^\mu E_{x\Sigma} x^\nu \Sigma_{\mu\nu} \psi$ $\delta_{x\Sigma} \bar{\psi} = d^\mu E_{x\Sigma} x^\nu \overline{\Sigma\psi} = -d^\mu E_{x\Sigma} x^\nu \bar{\psi} \Sigma_{\mu\nu}$

$$
\begin{aligned}
\delta_{x\Delta} S = \int \delta_{x\Delta} \bar{\psi} \partial\!\!\!/ \psi = d^\mu E_{x\Delta} \int x_\mu \psi^a (C\gamma^\rho)_{ab} \partial_\rho \psi^b \\
\underbrace{=}_{IBP} -d^\mu E_{x\Delta} \int \eta_{\rho\mu} \psi^a (C\gamma^\rho)_{ab} \psi^b + x_\mu \partial_\rho \psi^a (C\gamma^\rho)_{ab} \psi^b \\
\underbrace{=}_{comm.+relabel+transp.} \delta_{x\Delta} S - d^\mu E_{x\Delta} \int \psi^a (C\gamma_\mu)_{ab} \psi^b
\end{aligned}
\tag{4.26}
$$

$\delta_{x\Delta} S$ on both sides of the equation implies that the extra integration term is zero.

$$
\begin{aligned}
\delta_{x\Sigma}S = \int \delta_{x\Delta}\bar{\psi}\partial\!\!\!/\psi = -d^{\mu}E_{x\Sigma}\int x^{\nu}\psi^{a}(C\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^{b} \\
\underbrace{=}_{IBP} d^{\mu}E_{x\Sigma}\int \frac{1}{2}(\delta^{\nu}_{\nu}\psi^{a}(C\gamma_{\mu})_{ab}\psi^{b} - \delta^{\nu}_{\mu}\psi^{a}(C\gamma_{\nu})_{ab}\psi^{b}) + x^{\nu}\partial_{[\nu}\psi^{a}(C\gamma_{\mu]})_{ab}\psi^{b} \\
\underbrace{=}_{comm.+relabel+transp.} \delta_{x\Sigma}S - d^{\mu}E_{x\Sigma}\int -\frac{(n-1)}{2}\psi^{a}(C\gamma_{\mu})_{ab}\psi^{b}
\end{aligned}
\tag{4.27}
$$

$\delta_{x\Sigma}S$ on both sides of the equation implies that the extra integration term is zero, but we will keep both these extra terms for consistency check.

$\delta_d S = \delta_{d_{old}}S + \delta_{x\Delta}S + \delta_{x\Sigma}S = -\delta_d S - d^{\mu}\int(\underbrace{-4+E_{x\Sigma}}_{=-E_{x\Sigma}})x^{\nu}\psi^{a}(C\Sigma_{\mu\nu}\gamma^{\rho})_{ab}\partial_{\rho}\psi^{b} + x_{\mu}(\underbrace{2(n-1)-E_{x\Delta}}_{=E_{x\Delta}})\psi^{a}(C\gamma^{\rho})_{ab}\partial_{\rho}\psi^{b}$

The action vanishes if $E_{x\Sigma} = 2$ and $E_{x\Delta} = n-1 = 2\Delta_{\psi}$. For consistency we also check $E_{x\Delta} - E_{x\Sigma}(n-1)/2 = (n-1) - 2(n-1)/2 = 0$, which agrees with the finding from the integration.

Setting $n = 4$, for which this calculation is valid, we find, in summary, that the action is invariant for:

$$
\begin{aligned}
\delta_a\psi &= a^{\mu}\partial_{\mu}\psi \\
\delta_b\psi &= b^{\mu\nu}(2x_{[\mu}\partial_{\nu]} + \Sigma_{\mu\nu})\psi \\
\delta_c\psi &= c(x^{\mu}\partial_{\mu} + 3/2)\psi \\
\delta_d\psi &= d^{\mu}(2x^{\mu}x^{\nu}\partial_{\nu} - x^{2}\partial_{\mu} + 3x_{\mu} + 2x^{\nu}\Sigma_{\mu\nu})\psi
\end{aligned}
\tag{4.28}
$$

We therefore conclude that the Poincaré and the conformal algebras are symmetries of the massless free Lagrangian for for both bosonic and fermionic fields.

4.3 Massive Lagrangian

The massive Lagrangian densities $\mathscr{L}$ of a spin 0 scalar field ϕ and a spin 1/2 fermionic field ψ are:

$$\mathscr{L}_{mass}(\phi) = -\frac{1}{2}m^{2}\phi^{2} \tag{4.29}$$

$$\mathscr{L}_{mass}(\psi) = -\frac{1}{2}\bar{\psi}m\psi \tag{4.30}$$

This gives:

$\frac{\delta\mathscr{L}}{\delta\phi} = -\partial^{\mu}(-\frac{1}{2}(2\partial^{\mu}\phi)) = -m^{2}\phi$. Similar for ψ. Since the action is linear and we have already investigated the kinetic terms, we can focus on only the massive term.

Before we do so it is worth noting that integration by parts give the same result for the massive scalar fields as for the fermionic fields. First we note that by doing integration by parts $r+s$ times for the massive scalar term we obtain:

$$\int \partial^r \phi m^2 \partial^s \phi = (-1)^{r+s} \partial^r \phi m^2 \partial^s \phi \tag{4.31}$$

Doing the same calculation using (4.11) and noting that $\bar{\psi} = \psi^a C_{ab}$ and $C_{ab} = -C_{ba}$, i.e. $v = 1$, we obtain:

$$\int \partial^r \bar{\psi} m \partial^s \psi = (-1)^{r+s+1+1} \partial^r \bar{\psi} m \partial^s \psi = (-1)^{r+s} \partial^r \bar{\psi} m \partial^s \psi \tag{4.32}$$

Therefore, the scalar and fermionic mass terms will yield the same results except for in the terms $\delta_\Sigma \psi$, $\delta_\Delta \psi$, $\delta_{x\Delta} \psi$, and $\delta_{x\Sigma}$.

4.3.1 Scalar Fields

$$\delta S = -\int \delta \phi m^2 \phi$$

We start by investigating the Poincaré algebra.

$$\begin{aligned}
\delta_a S &= -\int \delta_a \phi m^2 \phi = -a^\mu \int \partial_\mu \phi m^2 \phi \\
&= a^\mu \int \phi m^2 \partial_\mu \phi = \int \delta_a \phi m^2 \phi \\
&= -\delta_a S = 0
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\delta_b S &= -\int \delta_b \phi m^2 \phi = -b^{\mu\nu} \int x_{[\mu} \partial_{\nu]} \phi m^2 \phi \\
&= b^{\mu\nu} \int (\underbrace{\eta_{[\nu} x_{\mu]}}_{=0} + x_{[\mu} \partial_{\nu]}) \phi m^2 \phi \\
&= \int \delta_b \phi m^2 \phi = -\delta_b S = 0
\end{aligned} \tag{4.34}$$

So the Poincaré algebra is invariant under the massive scalar action.

Let us now consider the conformal transformations.

First, note that $\partial_\mu (x^\mu \phi m^2 \phi) = (\partial_\mu x^\mu) \phi m^2 \phi + x^\mu (\partial_\mu \phi) m^2 \phi) + x^\mu \phi m^2 \partial_\mu \phi$. Thus, $(x^\mu \partial_\mu \phi) m^2 \phi = (-m - x^\mu \partial_\mu) \phi m^2 \phi + \partial_\mu (x^\mu \phi m^2 \phi)$. The total derivative vanishes in the integral when evaluated at the boundary. Now,

$$\begin{aligned}
\delta_c S &= -\int \delta_c \phi m^2 \phi = -c \int (x^\mu \partial_\mu + \Delta) \phi m^2 \phi \\
&= c \int (\delta^\mu_\mu + x^\mu \partial_\mu + \Delta) \phi m^2 \phi \\
&= c \int (x^\mu \partial_\mu + (\underbrace{n - \Delta}_{=\Delta})) \phi m^2 \phi \\
&= \int \delta_c \phi m^2 \phi = -\delta_c S = 0
\end{aligned} \tag{4.35}$$

$\delta_c S$ only vanishes when $n - \Delta = \Delta \implies \Delta = n/2$, which does not agree with the value for the kinetic term $\Delta = (n-2)/2$. Thus, the action is only invariant under dilatation in the mass term when there is no kinetic component. Since it is not invariant under $\delta_c S$, there is no need to investigate $\delta_d S$. The conformal algebra is not invariant under the massive scalar action.

4.3.2 Fermions

For the fermionic case, δ_a is the same for ψ as for ϕ. Therefore, from (4.31) and (4.32) it follows that $\delta_a S = 0$. $\delta_b \psi$ has the extra term $\delta_\Sigma \psi$, which needs to be checked.

$$\delta_\Sigma \psi = b^{\mu\nu} \Sigma_{\mu\nu} \psi$$
$$\delta_\Sigma \bar{\psi} = b^{\mu\nu} \overline{\Sigma_{\mu\nu}\psi} \underbrace{=}_{(4.20)} -\psi^a (C\Sigma_{\mu\nu})_{ab} \tag{4.36}$$

$$\begin{aligned}
\delta_\Sigma S = \int \delta_\Sigma \bar{\psi} m \psi &= -b^{\mu\nu} \int \psi^a (C\Sigma_{\mu\nu})_{ab} m \psi^b \\
&\underbrace{=}_{(4.11)} - \int (-1)^{0+0+0+1} \psi^a (C\Sigma_{\mu\nu})_{ab} m \psi^b \\
&= -\delta_\Sigma S = 0
\end{aligned} \tag{4.37}$$

Thus, $\delta_b S$ vanishes.

The only difference between $\delta_c \phi$ and $\delta_c \psi$ is the value of the constant Δ versus Δ_ψ. As with the scalar case, $\delta_c S$ will only vanish if $\Delta_\psi = n/2$, which does not agree with the value $(n-1)/2$ found in (4.28). Thus, the massive fermionic Lagrangian is not conformally invariant.

4.4 Interaction terms

What about higher order interaction terms? We try:

$$\mathscr{L}_{int}(\phi) = \lambda \phi^p \tag{4.38}$$

$$\delta S_{int} = \int \delta\phi \lambda \phi^{p-1}$$

We have already performed this calculation for $p = 2$ and now repeat the steps for p¿2:

$$\partial_\mu (x^\mu \phi \lambda \phi^{p-1}) = (\partial_\mu x^\mu + x^\mu \partial_\mu + (p-1) x^\mu \partial_\mu) \phi \lambda \phi^{p-1}$$
$$\implies x^\mu \partial_\mu \phi \lambda \phi^{p-1} = \partial_\mu (x^\mu \phi \lambda \phi^{p-1}) - (n + (p-1) x^\mu \partial_\mu) \phi \lambda \phi^{p-1}$$

$$\delta_c S = \int \delta_c \phi \lambda \phi^{p-1} = c \int (x^\mu \partial_\mu + \Delta) \phi \lambda \phi^{p-1} = -c \int ((n-\Delta) + (p-1) x^\mu \partial_\mu) \phi \lambda \phi^{p-1}$$

$$= -c(p-1)\int(x^\mu\partial_\mu + (\underbrace{\frac{n-\Delta}{p-1}}_{=\Delta}))\phi\lambda\phi^{p-1} = -(p-1)\int\delta_c\phi\lambda\phi^{p-1}$$

$$p\delta_c S = 0$$

Thus, $\delta_c S = 0$ if $\Delta = n/p$. This agrees with the kinetic term if $p = 2n/(n-2)$. As $n \to \infty, p \to 2$ from above, but never reaches it, so for $n > 2, p > 2$. Similarly, $n > 6 \implies p < 3$. Thus, for $n > 6, 2 < p < 3$. Thus, p can only be a whole number for $n = 3, 4, 6$ yielding $p = 6, 4, 3$ respectively.

Let us now verify if we obtain a similar result for the special conformal transformation. Again, we investigate the total derivatives:

$$\partial_\nu(2x_\mu x^\nu\phi\lambda\phi^{p-1}) = ((2+2n)x_\mu + 2x_\mu x^\nu\partial_\nu + (p-1)2x_\mu x^\nu\partial_\nu)\phi\lambda\phi^{p-1}$$

$$\implies 2x_\mu x^\nu\partial_\nu\phi\lambda\phi^{p-1} = \partial_\nu(2x_\mu x^\nu\phi\lambda\phi^{p-1}) - ((2+2n)x_\mu + (p-1)2x_\mu x^\nu\partial_\nu)\phi\lambda\phi^{p-1}$$

$$\partial_\mu(-x_\nu x^\nu\phi\lambda\phi^{p-1}) = (-2x_\mu - x_\nu x^\nu\partial_\mu - (p-1)x_\nu x^\nu\partial_\mu)\phi\lambda\phi^{p-1}$$

$$\implies -x_\nu x^\nu\partial_\mu\phi\lambda\phi^{p-1} = \partial_\mu(-x_\nu x^\nu\phi\lambda\phi^{p-1}) - (-2x_\mu - (p-1)x_\nu x^\nu\partial_\mu)\phi\lambda\phi^{p-1}$$

Therefore,

$$\delta_d S = \int\delta_d\phi\lambda\phi^{p-1} = d^\mu\int(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + Ex_\mu)\phi\lambda\phi^{p-1}$$

$$= -(p-1)d^\mu\int(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + (\underbrace{\frac{2+2n-2-E}{p-1}}_{=E})x_\mu)\phi\lambda\phi^{p-1}$$

$$= p\delta_d S$$

Thus, $\delta_d S = 0$ if $E = 2n/p$. This agrees with the kinetic term if $p = 2n/(n-2)$, the same solution as for $\delta_c S$. p is a whole number for $n = 3, 4, 6$ yielding $p = 6, 4, 3$ respectively.

4.4.1 Fermions

For fermions, $\mathscr{L}_{int} = \lambda(\bar{\psi}\psi)^p$. Since spinors anticommute there can for $n = 4$ be at most four different spinors with different indices in a product. Therefore for all values of $p > 2$, anticommutativity ensures that the term will vanish. For $p = 2$ there is just enough room to squeeze in four different spinors. $C_{12}C_{34}\psi^1\psi^2\psi^3\psi^4$. We know that this contraction is non-zero because the Pfaffian[5] squared of a skewsymmetric matrix equals the determinant. Since C is invertible, its determinant is non-zero and hence also the Pfaffian is non-zero and the expression does not vanish.

$\mathscr{L}_{int} = \lambda(\bar{\psi}\psi)^2 = \lambda\psi^a C_{ab}\psi^b\psi^c C_{cd}\psi^d$

First we recall that $C = -C^t$.We calcultate δS_{int} to the first order in $\delta\psi$ using (4.11).

$$\begin{aligned}
\delta S_{int} &= S_{int}[\psi + \delta\psi] - S_{int}[\psi] \\
&= \lambda(\overline{(\psi + \delta\psi)}(\psi + \delta\psi)\overline{(\psi + \delta\psi)}(\psi + \delta\psi) - \bar{\psi}\psi\bar{\psi}\psi) \\
&= \lambda(\delta\psi^a C_{ab}\psi^b\bar{\psi}\psi + \psi^a C_{ab}\delta\psi^b\bar{\psi}\psi + \bar{\psi}\psi\delta\psi^c C_{cd}\psi^d + \bar{\psi}\psi\psi^c C_{cd}\delta\psi^d) \\
&= \lambda(1 + (-1)^{0+0+1+1} + (-1)^{0+0+0+4} + (-1)^{0+0+1+5})\delta\bar{\psi}\psi\bar{\psi}\psi \\
&= 4\lambda\delta\bar{\psi}\psi\bar{\psi}\psi
\end{aligned} \tag{4.39}$$

Now we are ready to check the variation.

$$\begin{aligned}
\delta_a S_{int} &= \int 4\lambda\delta_a\bar{\psi}\psi\bar{\psi}\psi = 4\lambda a^\mu \int \partial_\mu\bar{\psi}\psi\bar{\psi}\psi \\
&= -4\lambda a^\mu \int \psi^a C_{ab}\partial_\mu\psi^b\bar{\psi}\psi + \bar{\psi}\psi\partial_\mu\psi^c C_{cd}\psi^d + \bar{\psi}\psi\psi^c C_{cd}\partial_\mu\psi^d \\
&= -4\lambda a^\mu \int (-1)^{1+1}\partial_\mu\psi^b C_{ba}\psi^a\bar{\psi}\psi + (-1)^4\partial_\mu\psi^c C_{cd}\psi^d\bar{\psi}\psi \\
&\qquad + (-1)^{5+1}\partial_\mu\psi^d C_{dc}\psi^c\bar{\psi}\psi \\
&\underbrace{=}_{relabel} -4\lambda a^\mu \int 3\partial_\mu\bar{\psi}\psi\bar{\psi}\psi \\
&= -3\delta_a S_{int} = 0
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
\delta_b S_{int} &= \int 4\lambda\delta_b\bar{\psi}\psi\bar{\psi}\psi = 4\lambda b^{\mu\nu} \int (2x_{[\mu}\partial_{\nu]}\bar{\psi} - \bar{\psi}\Sigma_{\mu\nu})\psi\bar{\psi}\psi \\
&= -4\lambda b^{\mu\nu} \int 2(\underbrace{\eta_{[\nu\mu]}}_{=0}\bar{\psi}\psi\bar{\psi}\psi + \psi^a C_{ab}x_{[\mu}\partial_{\nu]}\psi^b\bar{\psi}\psi \\
&\qquad + \bar{\psi}\psi x_{[\mu}\partial_{\nu]}\psi^c C_{cd}\psi^d + \bar{\psi}\psi\psi^c C_{cd}x_{[\mu}\partial_{\nu]}\psi^d) \\
&\qquad + \psi^a(C\Sigma_{\mu\nu})_{ab}\psi^b\bar{\psi}\psi \\
&= -4\lambda b^{\mu\nu} \int 2((-1)^{1+1}x_{[\mu}\partial_{\nu]}\psi^b C_{ba}\psi^a\bar{\psi}\psi \\
&\qquad + (-1)^4 x_{[\mu}\partial_{\nu]}\psi^c C_{cd}\psi^d\bar{\psi}\psi + (-1)^{5+1}x_{[\mu}\partial_{\nu]}\psi^d C_{dc}\psi^c\bar{\psi}\psi) \\
&\qquad + (-1)^{1+0}\psi^b(C\Sigma_{\mu\nu})_{ba}\psi^a\bar{\psi}\psi \\
&\underbrace{=}_{relabel} -4\lambda b^{\mu\nu} \int (6x_{[\mu}\partial_{\nu]}\bar{\psi} - \bar{\psi}\Sigma_{\mu\nu})\psi\bar{\psi}\psi = 0
\end{aligned} \tag{4.41}$$

$$
\begin{aligned}
\delta_c S_{int} &= \int 4\lambda\delta_c\bar{\psi}\psi\bar{\psi}\psi = 4\lambda c\int (x^\mu\partial_\mu + \frac{3}{2})\bar{\psi}\psi\bar{\psi}\psi \\
&= -4\lambda c\int \delta^\mu_\mu\bar{\psi}\psi\bar{\psi}\psi + \psi^a C_{ab}x^\mu\partial_\mu\psi^b\bar{\psi}\psi \\
&\quad + \bar{\psi}\psi x^\mu\partial_\mu\psi^c C_{cd}\psi^d + \bar{\psi}\psi\psi^c C_{cd}x^\mu\partial_\mu\psi^d \\
&\quad - \frac{3}{2}\bar{\psi}\psi\bar{\psi}\psi \\
&= -4\lambda c\int 2((-1)^{1+1}x^\mu\partial_\mu\psi^b C_{ba}\psi^a\bar{\psi}\psi \\
&\quad + (-1)^4 x^\mu\partial_\mu\psi^c C_{cd}\psi^d\bar{\psi}\psi + (-1)^{5+1}x^\mu\partial_\mu\psi^d C_{dc}\psi^c\bar{\psi}\psi) \\
&\quad + (4 - \frac{3}{2})\bar{\psi}\psi\bar{\psi}\psi \\
\underbrace{=}_{relabel} & -4\lambda c\int (3x^\mu\partial_\mu + \underbrace{\frac{5}{2}}_{\neq\frac{3}{2}})\bar{\psi}\psi\bar{\psi}\psi \neq 0
\end{aligned}
\tag{4.42}
$$

$\delta_c S_{int}$ does not vanish. Hence, S_{int} is not invariant under conformal transformations.

5 Lie Superalgebras

A Lie superalgebra is a natural extension of the concept of Lie algebras consisting of a $\mathbb{Z}_2$ graded real vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is said to be even, and $\mathfrak{g}_1$ is odd. The vector space is equipped with a bilinear operation called the Lie superbracket:

$$[-,-] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j} \tag{5.1}$$

where $i, j, i+j \in \mathbb{Z}_2$. For the purposes of this thesis, we can limit the analysis to elements that are homogenous. That is, elements only in either $\mathfrak{g}_0$ or in $\mathfrak{g}_1$. Bars are used to measure the grade of a homogeneous element.

$$|X| = i \Leftrightarrow X \in \mathfrak{g}_i \tag{5.2}$$

The superbracket then satisfies the following relation:

$$[X, Y] = -(-1)^{|X||Y|}[Y, X] \tag{5.3}$$

In addition it also satisfied the superized Jacobi identity:

$$(-)^{|X||Z|}[X, [Y, Z]] + (-)^{|Z||Y|}[Z, [X, Y]] + (-)^{|Y||X|}[Y, [Z, X]] = 0 \tag{5.4}$$

For convenient calculations, we can rearrange this using (5.3) to get:

$$
\begin{aligned}
&0 = \\
&(-)^{|X||Z|}[X,[Y,Z]] - (-)^{|Z|(|X|+|Y|)+|Z||Y|}[[X,Y],Z] - (-)^{|Z||X|+|Y||X|}[Y,[X,Z]] \\
&= (-)^{|X||Z|}\left([X,[Y,Z]] - (-)^{2|Z||Y|}[[X,Y],Z] - (-)^{|Y||X|}[Y,[X,Z]]\right) \\
&= [X,[Y,Z]] - [[X,Y],Z] - (-)^{|Y||X|}[Y,[X,Z]] = 0
\end{aligned} \tag{5.5}
$$

This gives use the form of the Jabobian used in ([3, App. A]).

$$
[X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z] \tag{5.6}
$$

5.1 Poincaré Superalgebra

Let us now briefly consider the free massless Wess-Zumino model. The supercharge transformations are given by [3]. Here we introduce a new field π which is a pseudoscalar.

$$
\begin{aligned}
Q_a \cdot \phi &= \psi_a \\
Q_a \cdot \pi &= (\gamma_5)^b{}_a \psi_b = \gamma_5 \psi_a \\
Q_a \cdot \psi_b &= -(\gamma^\rho)_{ab}\partial_\rho \phi + (\gamma^\rho \gamma_5)_{ab}\partial_\rho \pi
\end{aligned} \tag{5.7}
$$

$$
\mathscr{L}_{kin} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\pi)^2 - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi \tag{5.8}
$$

$$
\delta_\epsilon S_{kin} = \int \delta_\epsilon \phi \Box \phi + \delta_\epsilon \pi \Box \pi - \delta_\epsilon \bar{\psi}\partial\!\!\!/\psi \tag{5.9}
$$

$$
\begin{aligned}
\delta_\epsilon \phi &= \bar{\epsilon}\psi \\
\delta_\epsilon \pi &= \bar{\epsilon}\gamma_5 \psi \\
\delta_\epsilon \psi &= \partial\!\!\!/(\phi + \pi\gamma_5)\epsilon \\
\delta_\epsilon \bar{\psi} &= \overline{\partial\!\!\!/(\phi + \pi\gamma_5)\epsilon} \\
&= -\bar{\epsilon}\partial\!\!\!/\phi - \bar{\epsilon}\gamma_5\partial\!\!\!/\pi \\
&= -\bar{\epsilon}\partial\!\!\!/(\phi - \gamma_5\pi)
\end{aligned} \tag{5.10}
$$

$$
\begin{aligned}
\delta_\epsilon S_{kin} &= \int \bar{\epsilon}(\psi\Box\phi + \gamma_5\psi\Box\pi + \partial\!\!\!/(\phi - \gamma_5\pi)\partial\!\!\!/\psi) \\
&\underbrace{=}_{IBP} \int \bar{\epsilon}((\Box\psi)\phi + \gamma_5(\Box\psi)\pi - \gamma^\rho(\phi - \gamma_5\pi)\partial_\rho\partial\!\!\!/\psi) \\
&= \int \bar{\epsilon}(\phi\Box\psi + \gamma_5\pi\Box\psi - (\phi + \gamma_5\pi)\partial\!\!\!/^2\psi) \\
&= \int \bar{\epsilon}(\phi\Box\psi + \gamma_5\pi\Box\psi - (\phi + \gamma_5\pi)\Box\psi) \\
&= 0
\end{aligned}
\tag{5.11}
$$

Indeed, the variation of the action vanishes. Let us now investigate the Poincaré superalgebra. Since P_μ and $M_{\mu\nu}$ are even, while Q_a is odd, we expect the supercommutator between them to produce an odd generator. Since there is only one such generator, Q, the commutator $[A, Q_a]$ should equal $c_a{}^b Q_b$, where A is even.

$$
\begin{aligned}
[P_\mu, Q_a] \cdot \phi &= \partial_\mu\psi_a - Q_a\partial_\mu\phi = 0 \\
[P_\mu, Q_a] \cdot \pi &= \gamma_5\partial_\mu\psi_a - Q_a\partial_\mu\phi = 0 \\
[P_\mu, Q_a] \cdot \psi_b &= (\gamma^\rho)_{ab}\partial_\rho\partial_\mu(-\phi + \gamma_5\pi) - Q_a\partial_\mu\psi_b \\
&= (\gamma^\rho)_{ab}\partial_\rho\partial_\mu(-\phi + \gamma_5\pi) - \partial_\mu(\gamma^\rho)_{ab}\partial_\rho(-\phi + \gamma_5\pi) = 0 \\
\implies [P_\mu, Q_a] \cdot \Phi &= 0
\end{aligned}
\tag{5.12}
$$

For P_μ, we find that the bracket is closed and $c = 0$.

$$
\begin{aligned}
[M_{\mu\nu}, Q_a] \cdot \phi &= M_{\mu\nu}\psi_a - Q_a M_{\mu\nu}\phi \\
&= (2x_{[\mu}\partial_{\nu]} + \Sigma_{\mu\nu})\psi_a - Q_a 2x_{[\mu}\partial_{\nu]}\phi \\
&= \Sigma_{\mu\nu}\psi_a = \Sigma_{\mu\nu}Q_a \cdot \phi \\
[M_{\mu\nu}, Q_a] \cdot \pi &= M_{\mu\nu}\gamma_5\psi_a - Q_a M_{\mu\nu}\pi \\
&= (2x_{[\mu}\partial_{\nu]} + \Sigma_{\mu\nu})\gamma_5\psi_a - Q_a 2x_{[\mu}\partial_{\nu]}\pi \\
&= \Sigma_{\mu\nu}\gamma_5\psi_a = \Sigma_{\mu\nu}Q_a \cdot \pi \\
[M_{\mu\nu}, Q_a] \cdot \psi_b &= (\gamma^\rho)_{ab}\partial_\rho M_{\mu\nu}(-\phi + \gamma_5\pi) - (2x_{[\mu}\partial_{\nu]} + \Sigma_{\mu\nu})Q_a\psi_b \\
&= (\gamma^\rho)_{ab}2\eta_{\rho[\mu}\partial_{\nu]}(-\phi + \gamma_5\pi) \\
&\quad - (\Sigma_{\mu\nu})_b^{\ c}(-(\gamma^\rho)_{ac}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ac}\partial_\rho\pi) \\
&= 2(\gamma_{[\mu})_{ab}\partial_{\nu]}(-\phi + \gamma_5\pi) \\
&\quad + (\Sigma_{\mu\nu})_b^{\ c}((\gamma^\rho)_{ca}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ca}\partial_\rho\pi) \\
&= 2(\gamma_{[\mu})_{ab}\partial_{\nu]}(-\phi + \gamma_5\pi) \\
&\quad + \frac{1}{2}(\gamma_{\mu\nu}^{\ \ \rho} + \gamma_\mu\delta_\nu^\rho - \gamma_\nu\delta_\mu^\rho)_{ba}\partial_\rho\phi \\
&\quad + \frac{1}{2}(\gamma_{\mu\nu}^{\ \ \rho}\gamma_5 + \gamma_\mu\gamma_5\delta_\nu^\rho - \gamma_\nu\gamma_5\delta_\mu^\rho)_{ba}\partial_\rho\pi \\
&= 2(\gamma_{[\mu})_{ab}\partial_{\nu]}(-\phi + \gamma_5\pi) \\
&\quad - \frac{1}{2}(\gamma_{\mu\nu}^{\ \ \rho} - \gamma_\mu\delta_\nu^\rho + \gamma_\nu\delta_\mu^\rho)_{ab}\partial_\rho\phi \\
&\quad + \frac{1}{2}(\gamma_{\mu\nu}^{\ \ \rho}\gamma_5 - \gamma_\mu\gamma_5\delta_\nu^\rho + \gamma_\nu\gamma_5\delta_\mu^\rho)_{ab}\partial_\rho\pi \\
&= 2(\gamma_{[\mu})_{ab}\partial_{\nu]}(-\phi + \gamma_5\pi) \\
&\quad - \frac{1}{2}((\gamma_{\mu\nu}\gamma^\rho - \gamma_\mu\delta_\nu^\rho + \gamma_\nu\delta_\mu^\rho) - \gamma_\mu\delta_\nu^\rho + \gamma_\nu\delta_\mu^\rho)_{ab}\partial_\rho\phi \\
&\quad + \frac{1}{2}((\gamma_{\mu\nu}\gamma^\rho\gamma_5 - \gamma_\mu\gamma_5\delta_\nu^\rho + \gamma_\nu\gamma_5\delta_\mu^\rho) - \gamma_\mu\gamma_5\delta_\nu^\rho + \gamma_\nu\gamma_5\delta_\mu^\rho)_{ab}\partial_\rho\pi \\
&= 2(\gamma_{[\mu})_{ab}\partial_{\nu]}(-\phi + \gamma_5\pi) \\
&\quad - (\Sigma_{\mu\nu}\gamma^\rho)_{ab}\partial_\rho\phi + (\gamma_{[\mu})_{ab}\partial_{\nu]}\phi \\
&\quad + (\Sigma_{\mu\nu}\gamma^\rho\gamma_5)_{ab}\partial_\rho\pi + (\gamma_{[\mu})_{ab}\partial_{\nu]}\gamma_5\pi \\
&= (\Sigma_{\mu\nu})_b^{\ c}(-(\gamma^\rho)_{ac}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ac}\partial_\rho\pi) \\
&= \Sigma_{\mu\nu}Q_a \cdot \psi_b \\
\implies [M_{\mu\nu}, Q_a] \cdot \Phi &= \Sigma_{\mu\nu}Q_a \cdot \Phi
\end{aligned}
\tag{5.13}
$$

For $M_{\mu\nu}$, we find that the bracket is closed and $c = \Sigma_{\mu\nu}$. When both generators are odd, we expect the bracket to produce an even generator. That is, $[Q_a, Q_b]$ should equal some linear combination $c(\gamma^\mu)_{ab}P_\mu + d(\Sigma^{\rho\sigma})_{ab}M_{\rho\sigma}$. Since this is an anticommutator, we expect the result to be symmetrical in all its indices [13, Ch. 2], indicating that $d = 0$. Using the super-Jacobian on Q_a, Q_b, P_μ, we find that:

$$
\begin{aligned}
[P_\mu, [Q_a, Q_b]] &= [[P_\mu, Q_a], Q_b] - [Q_a, [P_\mu, Q_b]] \\
\implies [P_\mu, c(\gamma^\mu)_{ab} P_\mu &+ d(\Sigma^{\rho\sigma})_{ab} M_{\rho\sigma}] = 0 \\
\implies [P_\mu, d(\Sigma^{\rho\sigma})_{ab} M_{\rho\sigma}] &= d(\Sigma^{\rho\sigma})_{ab}(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho) = 0 \\
&\implies d = 0
\end{aligned}
\tag{5.14}
$$

Thus, $[Q_a, Q_b] = c(\gamma^\mu)_{ab} P_\mu$. To find the value of c, we can investigate $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \cdot \Phi$. [3, p. 11]

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \cdot \Phi &= (\delta_{\epsilon_1} \cdot \epsilon_2^b Q_b - \delta_{\epsilon_2} \cdot \epsilon_1^a Q_a) \cdot \Phi \\
&= (\epsilon_2^b Q_b \epsilon_1^a Q_a - \epsilon_1^a Q_a \epsilon_2^b Q_b) \cdot \Phi \\
&= (\epsilon_1^a \epsilon_2^b Q_b Q_a + \epsilon_1^a \epsilon_2^b Q_a Q_b) \cdot \Phi \\
&= \epsilon_1^a \epsilon_2^b [Q_a, Q_b] \cdot \Phi \\
\implies [\delta_{\epsilon_1}, \delta_{\epsilon_2}] &= \epsilon_1^a \epsilon_2^b [Q_a, Q_b]
\end{aligned}
\tag{5.15}
$$

To find $[Q_a, Q_b]$ we can calculate $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ using the Fierz identity.[3, A.4]

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \cdot \phi &= \delta_{\epsilon_1} \cdot \epsilon_2^b \psi_b - \delta_{\epsilon_2} \cdot \epsilon_1^a \psi_a \\
&= \epsilon_2^b \not{\partial}(\phi + \pi\gamma_5)\epsilon_1^a \\
&\quad - \epsilon_1^a \not{\partial}(\phi + \pi\gamma_5)\epsilon_2^b \\
&= \epsilon_2^b((\gamma^\rho)_{ab} \partial_\rho \phi) + (\gamma^\rho \gamma_5)_{ab} \partial_\rho \pi)\epsilon_1^a \\
&\quad - \epsilon_1^a((\gamma^\rho)_{ba} \partial_\rho \phi + (\gamma^\rho \gamma_5)_{ba} \partial_\rho \pi)\epsilon_2^b \\
&= -\epsilon_1^a((\gamma^\rho)_{ab} \partial_\rho \phi) + (\gamma^\rho \gamma_5)_{ab} \partial_\rho \pi)\epsilon_2^b \\
&\quad - \epsilon_1^a((\gamma^\rho)_{ab} \partial_\rho \phi - (\gamma^\rho \gamma_5)_{ab} \partial_\rho \pi)\epsilon_2^b \\
&= -2\epsilon_1^b (\gamma^\rho)_{ab} \partial_\rho \phi \epsilon_2^b \\
&= \epsilon_1^b \epsilon_2^b (-2(\gamma^\rho)_{ab} \partial_\rho \phi) \\
\implies [Q_a, Q_b] \cdot \phi &= -2(\gamma^\rho)_{ab} \partial_\rho \phi
\end{aligned}
\tag{5.16}
$$

Note that $\gamma_5 \gamma_\mu \gamma_5 = -\gamma_\mu \gamma_5 \gamma_5 = \gamma_\mu$.

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \cdot \pi &= \delta_{\epsilon_1} \cdot \epsilon_2^b \gamma_5 \psi_b - \delta_{\epsilon_2} \cdot \epsilon_1^a \gamma_5 \psi_a \\
&= \epsilon_2^b \gamma_5 \not{\partial}(\phi + \pi\gamma_5)\epsilon_1^a \\
&\quad - \epsilon_1^a \gamma_5 \not{\partial}(\phi + \pi\gamma_5)\epsilon_2^b \\
&= \epsilon_2^b((\gamma_5\gamma^\rho)_{ab}\partial_\rho\phi) + (\gamma^\rho)_{ab}\partial_\rho\pi)\epsilon_1^a \\
&\quad - \epsilon_1^a((\gamma_5\gamma^\rho)_{ba}\partial_\rho\phi + (\gamma^\rho)_{ba}\partial_\rho\pi)\epsilon_2^b \\
&= -\epsilon_1^a((\gamma_5\gamma^\rho)_{ab}\partial_\rho\phi) + (\gamma^\rho)_{ab}\partial_\rho\pi)\epsilon_2^b \\
&\quad - \epsilon_1^a(-(\gamma_5\gamma^\rho)_{ab}\partial_\rho\phi + (\gamma^\rho)_{ab}\partial_\rho\pi)\epsilon_2^b \\
&= -2\epsilon_1^b(\gamma^\rho)_{ab}\partial_\rho\pi\epsilon_2^b \\
&= \epsilon_1^b\epsilon_2^b(-2(\gamma^\rho)_{ab}\partial_\rho\pi) \\
\implies [Q_a, Q_b] \cdot \pi &= -2(\gamma^\rho)_{ab}\partial_\rho\pi
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \cdot \psi &= \delta_{\epsilon_1} \cdot \not{\partial}(\phi + \pi\gamma_5)\epsilon_2 \\
&\quad - \delta_{\epsilon_2} \cdot \not{\partial}(\phi + \pi\gamma_5)\epsilon_1 \\
&= \gamma^\rho \underbrace{\bar{\epsilon_1}\partial_\rho(\psi + \gamma_5\psi}_{\text{scalar}}\gamma_5)\epsilon_2 \\
&\quad - \gamma^\rho \underbrace{\bar{\epsilon_2}\partial_\rho(\psi + \gamma_5\psi}_{\text{scalar}}\gamma_5)\epsilon_1 \\
&= \gamma^\rho\partial_\rho((\epsilon_2\bar{\epsilon_1} - \epsilon_1\bar{\epsilon_2}) + \gamma_5(\epsilon_2\bar{\epsilon_1} - \epsilon_1\bar{\epsilon_2})\gamma_5)\psi \\
&= \gamma^\rho\partial_\rho((\frac{1}{2}\bar{\epsilon_2}\gamma^\mu\epsilon_1\gamma_\mu - \frac{1}{4}\bar{\epsilon_2}\gamma^{\mu\nu}\epsilon_1\gamma_{\mu\nu}) \\
&\quad + \gamma_5(\frac{1}{2}\bar{\epsilon_2}\gamma^\mu\epsilon_1\gamma_\mu - \frac{1}{4}\bar{\epsilon_2}\gamma^{\mu\nu}\epsilon_1\gamma_{\mu\nu})\gamma_5)\psi \\
&= \gamma^\rho\partial_\rho((\frac{1}{2}\bar{\epsilon_2}\gamma^\mu\epsilon_1\gamma_\mu - \frac{1}{4}\bar{\epsilon_2}\gamma^{\mu\nu}\epsilon_1\gamma_{\mu\nu}) \\
&\quad + (\frac{1}{2}\bar{\epsilon_2}\gamma^\mu\epsilon_1\gamma_\mu + \frac{1}{4}\bar{\epsilon_2}\gamma^{\mu\nu}\epsilon_1\gamma_{\mu\nu}))\psi \\
&= \gamma^\rho\partial_\rho \underbrace{\bar{\epsilon_2}\gamma^\mu\epsilon_1}_{\text{scalar}}\gamma_\mu\psi \\
&= \bar{\epsilon_2}\gamma^\mu\epsilon_1\gamma^\rho\gamma_\mu\partial_\rho\psi \\
&= \bar{\epsilon_2}\gamma^\mu\epsilon_1(2\delta_\mu^\rho - \gamma_\mu \quad \underbrace{\gamma^\rho)\partial_\rho\psi}_{\text{on-shell } \not{\partial}\psi=0} \\
&= 2\bar{\epsilon_2}\gamma^\rho\epsilon_1\psi \\
&= \epsilon_1^b\epsilon_2^b(-2(\gamma^\rho)_{ab}\partial_\rho\psi \\
\implies [Q_a, Q_b] \cdot \psi &= -2(\gamma^\rho)_{ab}\partial_\rho\psi \\
\implies [Q_a, Q_b] = -2(\gamma^\rho)_{ab}P_\rho &
\end{aligned}
\tag{5.18}
$$

Since we had to appeal to the equations of motions, the bracket is only valid on-shell.

To verify that the Poincaré superalgebra is closed, we need to check the super-Jacobi identity.

$$\begin{aligned} [P_\mu, [P_\nu, Q_a]] - [[P_\mu, P_\nu], Q_a] - [P_\nu, [P_\mu, Q_a]] \\ = 0 - 0 - 0 \\ = 0 \end{aligned} \tag{5.19}$$

$$\begin{aligned} [P_\mu, [Q_a, Q_b]] - [[P_\mu, Q_a], Q_b] - [Q_a, [P_\mu, Q_b]] \\ = [P_\mu, \propto P]] - 0 - 0 \\ = 0 \end{aligned} \tag{5.20}$$

$$\begin{aligned} [Q_a, [Q_b, Q_c]] - [[Q_a, Q_b], Q_c] + [Q_b, [Q_a, Q_c]] \\ = [Q_a, \propto P]] - [\propto P, Q_c] + [Q_b, \propto P]] \\ = 0 \end{aligned} \tag{5.21}$$

$$\begin{aligned} [P_\rho, [M_{\mu\nu}, Q_a]] - [[P_\rho, M_{\mu\nu}], Q_a] - [M_{\mu\nu}, [P_\rho, Q_a]] \\ = [P_\rho, \propto Q_a] - [\propto P, Q_a] - 0 \\ = 0 \end{aligned} \tag{5.22}$$

$$\begin{aligned} & [M_{\mu\nu}, [M_{\rho\sigma}, Q_a]] - [[M_{\mu\nu}, M_{\rho\sigma}], Q_a] - [M_{\rho\sigma}, [M_{\mu\nu}, Q_a]] \\ & = [M_{\mu\nu}, \Sigma_{\rho\sigma} Q_a] - [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] Q_a - [M_{\rho\sigma}, \Sigma_{\mu\nu} Q_a] \\ & = (\Sigma_{\mu\nu}\Sigma_{\rho\sigma} - [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] - \Sigma_{\rho\sigma}\Sigma_{\mu\nu}) Q_a \\ & = [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] - [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] \\ & = 0 \end{aligned} \tag{5.23}$$

$$\begin{aligned} & [M_{\mu\nu}, [Q_a, Q_b]] - [[M_{\mu\nu}, Q_a], Q_b] - [Q_a, [M_{\mu\nu}, Q_b]] \\ & = [M_{\mu\nu}, -2(\gamma^\rho)_{ab} P_\rho] - [\Sigma_{\mu\nu} Q_a, Q_b] - [Q_a, \Sigma_{\mu\nu} Q_b] \\ & = -2(\gamma^\rho)_{ab}(-\eta_{\rho\mu} P_\nu + \eta_{\rho\nu} P_\mu) + (\gamma_{\mu\nu}) 2(\gamma^\rho)_{ab} P_\rho \\ & = -2(\gamma_\mu P_\nu - \gamma_\nu P_\mu) + 2(\gamma_\mu \delta^\rho_\nu - \gamma_\nu \delta^\rho_\mu) P_\rho \\ & = 0 \end{aligned} \tag{5.24}$$

All the super-Jacobians are zero. Thus, the Poincaré superalgebra is closed.

In the Appendix, the conformal superalgebra has also been calculated. All the brackets are calculated, except $[S, S]$ and $[Q, S]$ for which time became an issue. All super-Jacobians are checked except the ones involves the two mentioned brackets.

6 Avenues of Future Research

In the introduction, we discussed the disturbing fact that no super particle has ever been observed, although dark matter has been speculated to be a potential candidate. One distinct possibility is that supersymmetry is simply wrong, and there is no guarantee that the supersymmetry breaking option that is pursued by many researchers today will be fruitful.

There is, however, a third avenue out of the quagmire, namely some hiterto undiscovered reality constraint which excludes the possibility of observing super particles in ordinary spacetime.

If such a reality constraint exists, it would be nice if it had some familiar overarching algebraic structure in which supersymmetry would coherently fit. Let us briefly consider some candidates.

First, it is worth noting that all the composition algebras have a $\mathbb{Z}_2$ graded structure.

Let $\mathbb{F} = \mathbb{R}^+, \mathbb{R}, \mathbb{C}$, and $\mathbb{Q}$ and $\hat{e} = -1, i, j$, and E. i, j, E are (split-)complex, quaternionic and octonionic imaginaries respectively, where the (split)-imaginaries squared equal (-)+1. Then the composition algebras can be written as $\mathbb{F} \oplus \hat{e}\mathbb{F}$. Then, if $a, b \in \mathbb{Z}_2$, $\mathbb{F} \in \mathfrak{g}_0$ and $\mathbb{F}\hat{i} \in \mathfrak{g}_1$ they form a $\mathbb{Z}_2$ graded structure where $\mathfrak{g}_a \times \mathfrak{g}_b = \mathfrak{g}_{a+b}$.

Notice that the $\mathbb{Z}_2$ structure is nested hierarchically with four levels at the level of octonions. The composition algebras are therefore an obvious candidate for an overarching algebraic structure of supersymmetry.

It is worth noting that Clifford algebras by the demand for associativity somewhat unnaturally leave out the octonions for consideration.

As we already saw in the section on Clifford Algebras, the octonions fulfill the Clifford relation. Due to the amazing spherical symmetry of S^7, octonionic bivectors and trivectors also function as vectors. The octonionic anti-associativity can be described as a form of anti-commutativity between vectors and bivectors. That is, $\gamma_{12}\gamma_3 = -\gamma_3\gamma_{12}$ in the octonions, whereas in the associative Clifford algebras, $\gamma_{12}\gamma_3 = \gamma_3\gamma_{12}$. Thus, the octonions are able to algebraically distinguish between left-handed and right handed volume elements whereas they are smeared together as one in the Clifford algebras. If algebraic chirality is an important reality constraint, the octonions are potentially able to weed out half of the solutions that are found using ordinary Clifford algebras.

Another possible avenue using octonions is to extend the commutator relations to also include associators and anti-associators. The octonions satisfy the following anti-associator relation:

$$\gamma_\mu(\gamma_\nu\gamma_\rho) + (\gamma_\mu\gamma_\nu)\gamma_\rho = \sigma(\delta_{\mu\nu}\gamma_\rho - \delta_{\mu\rho}\gamma_\nu + \delta_{\nu\rho}\gamma_\mu) \tag{6.1}$$

For the split-octionons $\sigma = 1$ and -1 for the ordinary octionons. Granted, octonions are far more difficult to work with than the other composition algebras, but there exists only two real algebras that satisfy a completely antisymmetric triple product (anti-associativity), namely the split-octonions and the octonions. This promises that if there exists an octonionic solution, it will be unique and have a narrow solve path. That appears to be a worthy avenue of future research.

7 Summary

In this thesis, we've covered the mathematical preliminaries of supersymmetries and calculated the Poincaré and conformal superalgebras, and checked the simplest non-trivial supersymmetry model, the Wess-Zumino model.

8 Appendix

8.1 Proof of Clifford Period

The following is a general proof. To find the proof of (), set $s = n, t = 0$ for case 1, and $s = 0, t = n$ for case 2.

$$\begin{aligned} Cl(s,t) \otimes Cl(0,2) &\cong Cl(t, s+2) \\ Cl(s,t) \otimes Cl(2,0) &\cong Cl(t+2, s) \\ Cl(s,t) \otimes Cl(1,1) &\cong Cl(s+1, t+1) \end{aligned} \tag{8.1}$$

We start by defining a set of matrices Γ from two Clifford algebras, $\gamma = Cl(s,t), s+t = n$ with a metric $\eta = \text{Diag}(t,s)$ and σ=Cl(a,b),a+b=2 with metric $g = \text{Diag}(a,b)$, and $\sigma_3 = \sigma_0\sigma_1$, the chirality matrix of $Cl(a,b)$. The elements of Γ are defined as follows:

$$\Gamma_i := \begin{cases} \gamma_i \otimes \sigma_3 & 0 \leq i \leq n-1 \\ \mathbb{1}_n \otimes \sigma_{i-n} & n \leq i \leq n+1 \end{cases}$$

The strategy is to prove that $\Gamma_i\Gamma_j + \Gamma_j\Gamma_i$ satisfies the Clifford relation and investigate its metric. The first step is to calculate $\Gamma_i\Gamma_j$ which is shown in the table below. First note that $\sigma_3^2 = (-1)^{2(2-1)/2}(-1)^b = (-1)^{b+1}$. Second, note from (3.51) that $\gamma_\mu\gamma_3 = -\gamma_3\gamma_\mu$.

$\Gamma_i\Gamma_j$	$0 \leq i \leq n-1$	$n \leq i \leq n+1$
$0 \leq j \leq n-1$	$\gamma_i\gamma_j \otimes (-1)^{b+1}\mathbb{1}_2 =: AA_{ij}$	$\gamma_j \otimes \sigma_{i-n}\sigma_3 =: AB_{ij}$
$n \leq j \leq n+1$	$\gamma_i \otimes \sigma_3\sigma_{j-n} =: BA_{ij}$	$\mathbb{1}_n \otimes \sigma_{i-n}\sigma_{j-n} =: BB_{ij}$

Let us now investigate the cross elements:

$$AB_{ij} + BA_{ji} = \gamma_j \otimes \sigma_{i-n}\sigma_3 + \gamma_j \otimes \sigma_3\sigma_{i-n} = 2\gamma_j \otimes \{\sigma_{i-n}, \sigma_3\}$$

For $Cl(a,b)$,$\{\sigma_{i-n}, \sigma_3\} = 0$. Thus,

$$AB_{ij} + BA_{ji} = 0$$

By index permutation, we therefore also have:

$$AB_{ji} + BA_{ij} = 0$$

$AA_{ij} + AA_{ji}$	$2\eta_{ij}(-1)^{b+1}\mathbb{1}_{n+2}$	$0 \leq i,j \leq n-1$
$AB_{ij} + BA_{ji} + BA_{ij} + AB_{ji}$	0	mixed
$BB_{ij} + BB_{ji}$	$2g_{(i-n)(j-n)}\mathbb{1}_{n+2}$	$n \leq i,j \leq n+1$
$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i$	$2h_{ij}\mathbb{1}_{n+2}$	$0 \leq i,j \leq n+1$

Notice that for $b = 0,2 \implies (-1)^{b+1} = -1$. The effect is that $Cl(0,2)$ and $Cl(2,0)$ change the sign of the metric in AA, thereby mapping $Cl(s,t)$ into $Cl(t,s)$ in this subsection of Γ. Consequently, since $(-1)^{b+1} = 1$ for $b = 1$, $Cl(s,t)$ remains unchanged for $Cl(1,1)$. Similarly, BB shows that the σ metric is the same in all cases. Therefore, for $b = 0,2$ it follows that $h = (-\eta) \otimes g \cong \text{Diag}(s+b, t+a)$, while for $a = 1, h = \eta \otimes g \cong \text{Diag}(t+b, s+a)$. Combining these results proves proposition (8.1).

8.2 Conformal Superalgebra

$$\begin{aligned}
\delta_\omega \phi &= \omega\pi \\
\delta_\omega \pi &= -\omega\phi \\
\delta_\omega \psi &= \frac{1}{2}\omega\gamma_5\psi \\
\delta_\omega \bar{\psi} &= \frac{1}{2}\omega\overline{\gamma_5\psi} \\
&= \frac{1}{2}\omega\bar{\psi}\gamma_5
\end{aligned} \tag{8.2}$$

$$\begin{aligned}
\delta_\omega S_{kin} &= \omega \int \pi\Box\phi - \phi\Box\pi - \frac{1}{2}\bar{\psi}\gamma_5\partial\!\!\!/\psi \\
&= -\frac{1}{2}\omega \int \psi^a(\gamma 5\gamma^\rho)_{ab}\partial_\rho\psi^b \\
&\underbrace{=}_{IBP} \frac{1}{2}\omega \int \partial_\rho\psi^a(\gamma_5\gamma^\rho)_{ab}\psi^b \\
&= -\frac{1}{2}\omega \int \psi^b(\gamma_5\gamma^\rho)_{ab}\partial_\rho\psi^a \\
&= \frac{1}{2}\omega \int \psi^b(\gamma_5\gamma^\rho)_{ba}\partial_\rho\psi^a \\
&= \frac{1}{2}\omega \int \psi^a(\gamma_5\gamma^\rho)_{ab}\partial_\rho\psi^b \\
&= 0
\end{aligned} \tag{8.3}$$

$$
\begin{aligned}
P_\mu &= \partial_\mu \\
M^\Phi_{\mu\nu} &= 2x_{[\mu}\partial_{\nu]} \\
M^\phi_{\mu\nu} &= M^\Phi_{\mu\nu} \\
M^{\psi/\pi}_{\mu\nu} &= M^\Phi_{\mu\nu} + \Sigma_{\mu\nu} \\
D^\Phi &= x^\mu \partial_\mu \\
D^{\phi/\pi} &= D^\Phi + 1 \\
D^\psi &= D^\Phi + 3/2 \\
K^\Phi_\mu &= 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu \\
K^{\phi/\pi}_\mu &= K^\Phi_\mu + 2x_\mu \\
K^\psi_\mu &= K^\Phi_\mu + 3x_\mu + 2x^\nu \Sigma_{\mu\nu}
\end{aligned}
\tag{8.4}
$$

$$
\begin{aligned}
[D,Q]\cdot\phi &= D^\psi \cdot \psi - Q\cdot(D^\phi\cdot\phi) \\
&= (D^\Phi + 3/2)\psi - (D^\Phi + 1)\psi \\
&= \frac{1}{2}\psi = \frac{1}{2}Q\cdot\phi \\
[D,Q]\cdot\pi &= D^\psi\gamma_5\cdot\psi - Q_a\cdot(D^\pi\cdot\pi) \\
&= (D^\Phi + 3/2)\gamma_5\psi - (D^\Phi + 1)\gamma_5\psi \\
&= \frac{1}{2}\gamma_5\psi = \frac{1}{2}Q\cdot\pi \\
[D,Q]\cdot\psi &= D^{\phi/\pi}\cdot\partial\!\!\!/(-\phi+\gamma_5\pi) - Q\cdot(D^\psi\cdot\psi) \\
&= (\partial\!\!\!/(D^\Phi+1) - (D^\Phi+3/2)\partial\!\!\!/)(-\phi+\gamma_5\pi) \\
&= (\gamma^\rho\delta^\mu_\rho\partial_\mu - \frac{1}{2}\partial\!\!\!/)(-\phi+\gamma_5\pi) \\
&= \frac{1}{2}\partial\!\!\!/(-\phi+\gamma_5\pi) = \frac{1}{2}Q\cdot\psi
\end{aligned}
\tag{8.5}
$$

Thus, $[D,Q] = \frac{1}{2}Q$.

$$
\begin{aligned}
[K_\mu, Q] \cdot \phi &= K_\mu^\psi \cdot \psi - Q \cdot (K_\mu^\phi \cdot \phi) \\
&= (K_\mu^\Phi + 3x_\mu + 2x^\nu \Sigma_{\mu\nu})\psi - (K_\mu^\Phi + 2x_\mu)\psi \\
&= (x_\mu + 2x^\nu \Sigma_{\mu\nu})\psi \\
&= (x_\mu + 2x^\nu \Sigma_{\mu\nu}) Q \cdot \phi \\
[K_\mu, Q] \cdot \pi &= K_\mu^\psi \gamma_5 \cdot \psi - Q_a \cdot (K_\mu^\pi \cdot \pi) \\
&= (K_\mu^\Phi + 3x_\mu + 2x^\nu \Sigma_{\mu\nu})\gamma_5\psi - (K_\mu^\Phi + 2x_\mu)\gamma_5\psi \\
&= (x_\mu + 2x^\nu \Sigma_{\mu\nu})\gamma_5\psi \\
&= (x_\mu + 2x^\nu \Sigma_{\mu\nu}) Q \cdot \phi \\
[K_\mu, Q_a] \cdot \psi_b &= K_\mu^{\phi/\pi} \cdot \partial\!\!\!/(-\phi + \gamma_5\pi) - Q \cdot (K_\mu^\psi \cdot \psi) \\
&= (\partial\!\!\!/(K_\mu^\Phi + 2x_\mu) - (K_\mu^\Phi + 3x_\mu + 2x^\nu \Sigma_{\mu\nu})\partial\!\!\!/)(-\phi + \gamma_5\pi) \\
&= (\gamma^\rho(2\eta_{\rho\mu}(x^\nu\partial_\nu + 1) + 2x_\mu\partial_\rho - 2x_\rho\partial_\mu - x_\mu\partial_\rho) - 2x^\nu\Sigma_{\mu\nu}\gamma^\rho\partial_\rho)(-\phi + \gamma_5\pi) \\
&= (2\eta_{\rho\mu}(x^\nu\partial_\nu + 1) + x_\mu\partial_\rho - 2x_\rho\partial_\mu)(-(\gamma^\rho)_{ab}\phi + (\gamma^\rho\gamma_5)_{ab}\pi) \\
&\quad - x^\nu\partial_\rho(-(\gamma_{\mu\nu}\gamma^\rho)_{ab}\phi + (\gamma_{\mu\nu}\gamma^\rho\gamma_5)_{ab}\pi)
\end{aligned}
\tag{8.6}
$$

The ψ-relation does not resolve into Q, and so a new spinorial generator S is needed. [3, Ch. 1] We define:

$$[K_\mu, Q_a] = (\gamma_\mu)_a{}^b S_b \tag{8.7}$$

To cancel γ_μ, we first note that $\frac{1}{4}(\gamma^\mu)_a{}^c(\gamma_\mu)_{cb} = \frac{1}{4}\delta^\mu_\mu(\mathbb{1})_{ab} = (\mathbb{1})_{ab} = C_{ab}$ and $(\gamma^\mu)_a{}^c(\gamma_{\mu\nu})_{cb}(= -\gamma^\mu\gamma_{\nu\mu}) = \delta^\mu_\mu\gamma_\nu - \delta^\mu_\nu\gamma_\mu = 3(\gamma_\nu)_{ab}$. We then note that $\frac{1}{4}(\gamma^\mu)_a{}^c(\gamma_\mu)_c{}^b S_b = S_a$. Furthermore, it is useful to note that $(\gamma_{\mu\nu}\gamma^\rho)_{ab} = (\gamma_{\mu\nu})_a{}^c(\gamma^\rho)_{cb}$ which after a transpose equals $(\gamma_{\mu\nu})_a{}^c(\gamma^\rho)_{bc} = (\gamma_{\mu\nu}\gamma^\rho)_{ba}$. So $(\gamma_{\mu\nu}\gamma^\rho) = (\gamma_{\mu\nu}\gamma^\rho)^t$. By a similar logic, $(\gamma_{\mu\nu}\gamma^\rho\gamma_5) = -(\gamma_{\mu\nu}\gamma^\rho\gamma_5)^t$. Finally, it is also useful to note that $\gamma_{\mu\nu}\gamma^\rho = \gamma_{\mu\nu}^\rho + \gamma_\mu\delta^\rho_\nu - \gamma_\nu\delta^\rho_\mu$

$$
\begin{aligned}
S_a \cdot \phi &= \frac{1}{4}(\gamma^\mu)_a^{\ c}[K_\mu, Q_c] \cdot \phi \\
&= \frac{1}{4}(\gamma^\mu)_a^{\ c}(x_\mu Q_c + 2x^\nu (\Sigma_{\mu\nu})_c^{\ b}) Q_b) \cdot \phi \\
&= \frac{1}{4}(\gamma^\mu x_\mu + 3x^\nu \gamma_\nu)\psi \\
&= x^\mu \gamma_\mu \psi_a \\
S_a \cdot \pi &= \frac{1}{4}\gamma^\mu [K_\mu, Q_a] \cdot \pi \\
&= x^\mu \gamma_\mu \gamma_5 \psi_a \\
S_a \cdot \psi_b &= \frac{1}{4}(\gamma^\mu)_a^{\ c}[K_\mu, Q_c] \cdot \psi_b \\
&= \frac{1}{4}(\gamma^\mu)_a^{\ c}\bigg((2\eta_{\rho\mu}(x^\nu \partial_\nu + 1) + x_\mu \partial_\rho - 2x_\rho \partial_\mu)(-(\gamma^\rho)_{cb}\phi + (\gamma^\rho \gamma_5)_{cb}\pi) \\
&\quad - x^\nu \partial_\rho (-(\gamma_{\mu\nu}\gamma^\rho)_{bc}\phi - (\gamma_{\mu\nu}\gamma^\rho\gamma_5)_{bc}\pi)\bigg) \\
&= \frac{1}{4}\bigg((2\eta_{\rho\mu}(x^\nu \partial_\nu + 1) + x_\mu \partial_\rho - 2x_\rho \partial_\mu)(-(\gamma^{\mu\rho} + \eta^{\mu\rho})_{ab}\phi + ((\gamma^{\mu\rho} + \eta^{\mu\rho})\gamma_5)_{ab}\pi) \\
&\quad - x^\nu \partial_\rho (-(\gamma^\mu \gamma_{\mu\nu}\gamma^\rho)_{ba}\phi - (\gamma^\mu\gamma_{\mu\nu}\gamma^\rho\gamma_5)_{ba}\pi)\bigg) \\
&= \frac{1}{4}\bigg((7x^\nu \partial_\nu + 8)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + 3x_\mu \partial_\rho (-(\gamma^{\mu\rho})_{ab}\phi + (\gamma^{\mu\rho}\gamma_5)_{ab}\pi) \\
&\quad - x_\nu \partial_\rho (-(\gamma^\mu(-\gamma_\mu^{\ \rho\nu} + \gamma_\mu \eta^{\rho\nu} - \gamma^\nu \delta_\mu^\rho))_{ba}\phi \\
&\quad - (\gamma^\mu(-\gamma_\mu^{\ \rho\nu} + \gamma_\mu \eta^{\rho\nu} - \gamma^\nu \delta_\mu^\rho)\gamma_5)_{ba}\pi)\bigg) \\
&= \frac{1}{4}\bigg((7x^\nu \partial_\nu + 8)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + 3x_\mu \partial_\rho (-(\gamma^{\mu\rho})_{ab}\phi + (\gamma^{\mu\rho}\gamma_5)_{ab}\pi) \\
&\quad + x_\nu \partial_\rho (-(\gamma^{\nu\rho} - 3\eta^{\rho\nu}))_{ab}\phi \\
&\quad + ((\gamma^{\nu\rho} - 3\eta^{\rho\nu})\gamma_5)_{ab}\pi)\bigg) \\
&= \frac{1}{4}\bigg((4x^\nu \partial_\nu + 8)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + 4x_\mu \partial_\rho (-(\gamma^{\mu\rho})_{ab}\phi + (\gamma^{\mu\rho}\gamma_5)_{ab}\pi)\bigg) \\
&= (x^\nu \partial_\nu + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + x_\mu \partial_\rho (-(\gamma^{\mu\rho})_{ab}\phi + (\gamma^{\mu\rho}\gamma_5)_{ab}\pi)
\end{aligned} \tag{8.8}
$$

In summary,

$$\begin{aligned} S_a \cdot \phi &= x^\mu \gamma_\mu \psi_a \\ S_a \cdot \pi &= x^\mu \gamma_\mu \gamma_5 \psi_a \\ S_a \cdot \psi_b &= (x^\nu \partial_\nu + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\ &\quad + x_\mu \partial_\rho(-(\gamma^{\mu\rho})_{ab}\phi + (\gamma^{\mu\rho}\gamma_5)_{ab}\pi) \end{aligned} \tag{8.9}$$

$$\begin{aligned} [P_\mu, S_a] \cdot \phi &= P_\mu \cdot x^\rho \gamma_\rho \psi_a - S_a \cdot P_\mu \phi \\ &= x^\rho \gamma_\rho \partial_\mu \psi_a - \partial_\mu (x^\rho \gamma_\rho \psi_a) \\ &= -\gamma_\mu \psi_a \\ &= -P_\mu Q_a \phi \end{aligned} \tag{8.10}$$

$$\begin{aligned} [P_\mu, S_a] \cdot \pi &= P_\mu \cdot x^\rho \gamma_\rho \gamma_5 \psi_a - S_a \cdot P_\mu \pi \\ &= x^\rho \gamma_\rho \gamma_5 \partial_\mu \psi_a - \partial_\mu (x^\rho \gamma_\rho \gamma_5 \psi_a) \\ &= -\gamma_\mu \gamma_5 \psi_a \\ &= -P_\mu Q_a \pi \end{aligned} \tag{8.11}$$

$$\begin{aligned} [P_\mu, S_a] \cdot \psi_b &= P_\mu \cdot (x^\nu \partial_\nu + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\ &\quad + x_\nu \partial_\rho(-(\gamma^{\nu\rho})_{ab}\phi + (\gamma^{\nu\rho}\gamma_5)_{ab}\pi) \\ &\quad - S_a \cdot P_\mu \psi_b \\ &= -\partial_\mu(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\ &\quad - \partial_\rho(-(\gamma_\mu{}^\rho)_{ab}\phi + (\gamma_\mu{}^\rho \gamma_5)_{ab}\pi) \\ &= -\partial_\mu(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\ &\quad - \partial_\rho(-(\gamma_\mu \gamma^\rho - \delta_\mu^\rho)_{ab}\phi + ((\gamma_\mu \gamma^\rho - \delta_\mu^\rho)\gamma_5)_{ab}\pi) \\ &= -\partial_\rho (\gamma_\mu)_a{}^c (-(\gamma^\rho)_{cb}\phi + (\gamma^\rho \gamma_5)_{cb}\pi) \\ &= -P_\mu Q_a \psi_b \\ \implies [P_\mu, S_a] &= -P_\mu Q_a \end{aligned} \tag{8.12}$$

First note that $\gamma_{\rho\mu\nu} = \gamma_{\mu\nu\rho}$ and $\gamma_\rho \gamma_{\mu\nu} = \gamma_{\mu\nu\rho} + \eta_{\rho\mu}\gamma_\nu - \eta_{\rho\nu}\gamma_\mu$. Then $\gamma_\rho \gamma_{\mu\nu} = \gamma_{\mu\nu}\gamma_\rho + 2\eta_{\rho\mu}\gamma_\nu - 2\eta_{\rho\nu}\gamma_\mu$. Similarly, $\gamma_5 \gamma_{\mu\nu} = \gamma_{\mu\nu}\gamma_5$

$$\begin{aligned} [M_{\mu\nu}, S_a] \cdot \phi &= M^\psi_{\mu\nu} \cdot x^\rho \gamma_\rho \psi_a - S_a \cdot M^\phi_{\mu\nu} \phi \\ &= x^\rho \gamma_\rho \Sigma_{\mu\nu} \psi_a - (2x_{[\mu}\partial_{\nu]} x^\rho)\gamma_\rho \psi_a \\ &= x^\rho \Sigma_{\mu\nu} \gamma_\rho \psi_a + 2x_{[\mu}\gamma_{\nu]}\psi_a - 2x_{[\mu}\gamma_{\nu]}\psi_a \\ &= \Sigma_{\mu\nu} x^\rho \gamma_\rho \psi_a \\ &= (\Sigma_{\mu\nu})_a{}^b S_b \cdot \phi \end{aligned} \tag{8.13}$$

$$
\begin{aligned}
[M_{\mu\nu}, S_a]\cdot\pi &= M^{\psi}_{\mu\nu}\cdot x^\rho\gamma_\rho\gamma_5\psi_a - S_a\cdot M^{\pi}_{\mu\nu}\pi \\
&= x^\rho\gamma_\rho\gamma_5\Sigma_{\mu\nu}\psi_a - (2x_{[\mu}\partial_{\nu]}x^\rho)\gamma_\rho\gamma_5\psi_a \\
&= x^\rho\Sigma_{\mu\nu}\gamma_\rho\gamma_5\psi_a + 2x_{[\mu}\gamma_{\nu]}\psi_a - 2x_{[\mu}\gamma_{\nu]}\psi_a \\
&= \Sigma_{\mu\nu}x^\rho\gamma_\rho\gamma_5\psi_a \\
&= (\Sigma_{\mu\nu})_a{}^b S_b\cdot\pi
\end{aligned}
\tag{8.14}
$$

It will now be useful to recall from (3.75) that $(\gamma_{\mu\nu}\gamma^{\rho\sigma})_{ab} = 2(\delta^\sigma_{[\mu}\gamma^{\ \rho}_{\nu]} - \delta^\rho_{[\mu}\gamma^{\ \sigma}_{\nu]})_{ab}$. Therefore, $(\Sigma_{\mu\nu}\gamma^{\rho\sigma})_{ab} = (-\Sigma_{\mu\nu}\gamma^{\rho\sigma} + 2\delta^\sigma_{[\mu}\gamma^{\ \rho}_{\nu]} - 2\delta^\rho_{[\mu}\gamma^{\ \sigma}_{\nu]})_{ab}$.

$$
\begin{aligned}
[M_{\mu\nu}, S_a]\cdot\psi_b &= M^{\phi/\pi}_{\mu\nu}\cdot(x^\rho\partial_\rho + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + x_\rho\partial_\sigma(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) - S_a\cdot M^{\psi}_{\mu\nu}\psi_b \\
&= (x^\rho\partial_\rho 2x_{[\mu}\partial_{\nu]})(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + (x_\rho\partial_\sigma 2x_{[\mu}\partial_{\nu]})(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&\quad - (2x_{[\mu}\partial_{\nu]}x^\rho\partial_\rho)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad - (2x_{[\mu}\partial_{\nu]}x_\rho\partial_\sigma)(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&\quad - \Sigma_{\mu\nu}S_a\psi_b \\
&= (2x_{[\mu}\partial_{\nu]})(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + (x_\rho 2\eta_{\sigma[\mu}\partial_{\nu]})(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&\quad - (2x_{[\mu}\partial_{\nu]})(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad - (2x_{[\mu}\eta_{\nu]\rho}\partial_\sigma)(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&\quad - (\Sigma_{\mu\nu})_a{}^c(x^\rho\partial_\rho + 2)(-C_{cb}\phi + (\gamma_5)_{cb}\pi) \\
&\quad - x_\rho\partial_\sigma(-(\Sigma_{\mu\nu}\gamma^{\rho\sigma})_{ab}\phi + (\Sigma_{\mu\nu}\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&= (-2x_\rho\partial_{[\mu}\eta_{\nu]\sigma})((\gamma^{\sigma\rho})_{ab}\phi - (\gamma^{\sigma\rho}\gamma_5)_{ab}\pi) \\
&\quad - (2x_{[\mu}\eta_{\nu]\rho}\partial_\sigma)(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&\quad + (\Sigma_{\mu\nu})_a{}^c(x^\rho\partial_\rho + 2)(-C_{bc}\phi + (\gamma_5)_{bc}\pi) \\
&\quad - x_\rho\partial_\sigma(-(-\Sigma_{\mu\nu}\gamma^{\rho\sigma} + 2\delta^\sigma_{[\mu}\gamma^{\ \rho}_{\nu]} - 2\delta^\rho_{[\mu}\gamma^{\ \sigma}_{\nu]})_{ab}\phi \\
&\quad + (-\Sigma_{\mu\nu}\gamma^{\rho\sigma}\gamma_5 + 2\delta^\sigma_{[\mu}\gamma^{\ \rho}_{\nu]}\gamma_5 - 2\delta^\rho_{[\mu}\gamma^{\ \sigma}_{\nu]}\gamma_5)_{ab}\pi) \\
&= (x^\rho\partial_\rho + 2)(-(\Sigma_{\mu\nu})_{ba}\phi + (\Sigma_{\mu\nu}\gamma_5)_{ba}\pi) \\
&\quad + x_\rho\partial_\sigma((-\Sigma_{\mu\nu}\gamma^{\rho\sigma})_{ab}\phi + (\Sigma_{\mu\nu}\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&= (x^\rho\partial_\rho + 2)(-(\Sigma_{\mu\nu})_{ab}\phi + (\Sigma_{\mu\nu}\gamma_5)_{ab}\pi) \\
&\quad + x_\rho\partial_\sigma((-\Sigma_{\mu\nu}\gamma^{\rho\sigma})_{ab}\phi + (\Sigma_{\mu\nu}\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&= (\Sigma_{\mu\nu})_a{}^c S_c\cdot\psi_b \\
\implies [M_{\mu\nu}, S_a] &= (\Sigma_{\mu\nu})_a{}^b S_b
\end{aligned}
\tag{8.15}
$$

$$
\begin{aligned}
[D, S_a] \cdot \phi &= D^{\psi} \cdot x^{\rho}\gamma_{\rho}\psi_a - S_a \cdot D^{\phi}\phi \\
&= \frac{1}{2}x^{\rho}\gamma_{\rho}\psi_a - (x^{\mu}\partial_{\mu}x^{\rho})\gamma_{\rho}\psi_a \\
&= \frac{1}{2}x^{\mu}\gamma_{\mu}\psi_a - x^{\mu}\gamma_{\mu}\psi_a \\
&= -\frac{1}{2}x^{\mu}\gamma_{\mu}\psi_a \\
&= -\frac{1}{2}S_a \cdot \phi
\end{aligned} \tag{8.16}
$$

$$
\begin{aligned}
[D, S_a] \cdot \pi &= D^{\psi} \cdot x^{\rho}\gamma_{\rho}\gamma_5\psi_a - S_a \cdot D^{\pi}\pi \\
&= \frac{1}{2}x^{\rho}\gamma_{\rho}\gamma_5\psi_a - (x^{\mu}\partial_{\mu}x^{\rho})\gamma_{\rho}\gamma_5\psi_a \\
&= -\frac{1}{2}x^{\mu}\gamma_{\mu}\gamma_5\psi_a \\
&= -\frac{1}{2}S_a \cdot \pi
\end{aligned} \tag{8.17}
$$

$$
\begin{aligned}
[D, S_a] \cdot \psi_b &= D^{\phi/\pi} \cdot (x^{\nu}\partial_{\nu} + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + x_{\rho}\partial_{\sigma}(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) - S_a \cdot D^{\psi}\psi_b \\
&= (x^{\nu}\partial_{\nu}x^{\mu}\partial_{\mu})(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + (x_{\rho}\partial_{\sigma}x^{\mu}\partial_{\mu})(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&\quad - (x^{\mu}\partial_{\mu}x^{\nu}\partial_{\nu})(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad - (x^{\mu}\partial_{\mu}x_{\rho}\partial_{\sigma})(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi) \\
&\quad - \frac{1}{2}S_a \cdot \psi_b \\
&= -\frac{1}{2}S_a \cdot \psi_b \\
\implies [D, S_a] &= -\frac{1}{2}S_a
\end{aligned} \tag{8.18}
$$

One useful observation is that $x^{\nu}x^{\rho}$ is symmetric in the indices $\nu \leftrightarrow \rho$ while $\gamma_{\rho\nu\mu}$ is antisymmetric, and therefore $x^{\nu}x^{\rho}\gamma_{\rho\nu\mu}$ vanishes. Also, note from (3.54) that $\gamma_5\gamma_{\mu\nu} = \gamma_{\mu\nu}\gamma_5$.

$$
\begin{aligned}
[K_\mu, S_a]\cdot\phi &= K^\psi_\mu \cdot x^\rho\gamma_\rho\psi_a - S_a\cdot K^\phi\phi \\
&= -(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)x^\rho\gamma_\rho\psi_a + (x_\mu + 2x^\nu\Sigma_{\mu\nu})S_a\cdot\phi \\
&= -(2x_\mu x^\nu\gamma_\nu - x^2\gamma_\mu)\psi_a + x^\rho\gamma_\rho(x_\mu + 2x^\nu\Sigma_{\mu\nu})\psi_a \\
&= (x^2\gamma_\mu - x_\mu x^\nu\gamma_\nu - x^\nu x^\rho\gamma_\rho\gamma_{\nu\mu})\psi_a \\
&= (x^2\gamma_\mu - x_\mu x^\nu\gamma_\nu - x^\nu x^\rho(\gamma_{\rho\nu\mu} + \eta_{\rho\nu}\gamma_\mu - \eta_{\rho\mu}\gamma_\nu))\psi_a \\
&= (x^2\gamma_\mu - x_\mu x^\nu\gamma_\nu - x^\nu x_\nu\gamma_\mu + x^\nu x_\mu\gamma_\nu))\psi_a \\
&= 0 \qquad (8.19) \\
[K_\mu, S_a]\cdot\pi &= K^\psi_\mu \cdot x^\rho\gamma_\rho\gamma_5\psi_a - S_a\cdot K^\pi\pi \\
&= -(2x_\mu x^\nu\gamma_\nu - x^2\gamma_\mu)\gamma_5\psi_a + x^\rho\gamma_\rho\gamma_5(x_\mu + 2x^\nu\Sigma_{\mu\nu})\psi_a \\
&= (x^2\gamma_\mu - x_\mu x^\nu\gamma_\nu - x^\nu x^\rho\gamma_\rho\gamma_{\nu\mu})\gamma_5\psi_a \\
&= (x^2\gamma_\mu - x_\mu x^\nu\gamma_\nu - x^\nu x^\rho(\gamma_{\rho\nu\mu} + \eta_{\rho\nu}\gamma_\mu - \eta_{\rho\mu}\gamma_\nu))\psi_a \\
&= (x^2\gamma_\mu - x_\mu x^\nu\gamma_\nu - x^\nu x_\nu\gamma_\mu + x^\nu x_\mu\gamma_\nu))\psi_a \\
&= 0
\end{aligned}
$$

Using (3.74) we can rewrite $(2\Sigma_{\mu\nu})_a^{\ c}(-C_{cb}+(\gamma_5)_{cb})$ as $(\Sigma_{\mu\nu})_a^{\ c}(-C_{cb}+(\gamma_5)_{cb})+(\Sigma_{\mu\nu})_a^{\ c}(C_{bc}-(\gamma_5)_{bc})$. This equals $(-\Sigma_{\mu\nu})_{ab}+(\Sigma_{\mu\nu})_{ba}+(\Sigma_{\mu\nu}\gamma_5)_{ab}-(\Sigma_{\mu\nu}\gamma_5)_{ba} = 0$. Furthermore, for this calculation it is an advantage to rewrite $(\gamma_{\mu\nu}\gamma^{\rho\sigma})_{ab} = 2(\delta^\sigma_{[\mu}\gamma^{\ \rho}_{\nu]} - \delta^\rho_{[\mu}\gamma^{\ \sigma}_{\nu]})_{ab}$ as $2(\delta^\sigma_{[\mu}\gamma_{\nu]}\gamma^\rho - \delta^\rho_{[\mu}\gamma_{\nu]}\gamma^\sigma)_{ab}$ -$2(\delta^\sigma_{[\mu}\delta^{\ \rho}_{\nu]} - \delta^\rho_{[\mu}\delta^{\ \sigma}_{\nu]})_{ab}$.

$$
\begin{aligned}
[K_\mu, S_a] \cdot \psi_b &= K_\mu^{\phi/\pi} \cdot ((x^\rho \partial_\rho + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + x_\rho \partial_\sigma(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi)) - S_a \cdot K^\psi \psi_b \\
&= ((x^\rho \partial_\rho)(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu + 2x_\mu) \\
&\quad - (2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)(x^\rho \partial_\rho))(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + ((x_\rho \partial_\sigma)(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu + 2x_\mu) \\
&\quad - (2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)(x_\rho \partial_\sigma))(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi)) \\
&\quad - (x_\mu + 2x^\nu \Sigma_{\mu\nu})((x^\rho \partial_\rho + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + x_\rho \partial_\sigma(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi)) \\
&= (4x_\mu x^\nu \partial_\nu - 2x^2 \partial_\mu + 2x_\mu \\
&\quad - 2x_\mu x^\nu \partial_\nu + x^2 \partial_\mu)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + (2(\eta_{\sigma\mu} x^\nu + \delta_\sigma^\nu x_\mu) x_\rho \partial_\nu - 2x_\rho x_\sigma \partial_\mu + 2x_\rho \eta_{\sigma\mu} \\
&\quad - (2x_\mu x_\rho \partial_\sigma - x^2 \eta_{\mu\rho} \partial_\sigma + x_\mu x_\rho \partial_\sigma))(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi)) \\
&\quad - (x_\mu x^\rho \partial_\rho + 2x_\mu)((-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + x^\nu x_\rho \partial_\sigma(-(\gamma_{\mu\nu}\gamma^{\rho\sigma})_{ab}\phi + (\gamma_{\mu\nu}\gamma^{\rho\sigma}\gamma_5)_{ab}\pi)) \\
&= (x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + (2\eta_{\sigma\mu} x^\nu x_\rho \partial_\nu + x_\mu x_\rho \partial_\sigma + 2x_\rho \eta_{\sigma\mu} \\
&\quad + x^2 \eta_{\mu\rho} \partial_\sigma - x_\mu x_\rho \partial_\sigma)(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi)) \\
&\quad - (x^\nu x_\rho \partial_\sigma(-2\delta^\sigma_{[\mu}\gamma_{\nu]}\gamma^\rho + 2\delta^\rho_{[\mu}\gamma_{\nu]}\gamma^\sigma + 2\delta^\sigma_{[\mu}\delta_{\nu]}^{\ \rho} - 2\delta^\rho_{[\mu}\delta_{\nu]}^{\ \sigma})_{ab}\phi \\
&\quad + ((2\delta^\sigma_{[\mu}\gamma_{\nu]}\gamma^\rho - 2\delta^\rho_{[\mu}\gamma_{\nu]}\gamma^\sigma - 2\delta^\sigma_{[\mu}\delta_{\nu]}^{\ \rho} + 2\delta^\rho_{[\mu}\delta_{\nu]}^{\ \sigma})\gamma_5)_{ab}\pi)) \\
&= (2x^\nu x_{[\mu}\partial_{\nu]})(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + (2\eta_{\sigma\mu} x^\nu x_\rho \partial_\nu + 2x_\rho \eta_{\sigma\mu} \\
&\quad + x^2 \eta_{\mu\rho} \partial_\sigma)(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi)) \\
&\quad - (x^\nu x_\rho \partial_\sigma(-2(\delta^\sigma_{[\mu}\gamma_{\nu]}^{\ \rho} - \delta^\rho_{[\mu}\gamma_{\nu]}^{\ \sigma})_{ab}\phi + 2((\delta^\sigma_{[\mu}\gamma_{\nu]}^{\ \rho} - \delta^\rho_{[\mu}\gamma_{\nu]}^{\ \sigma})\gamma_5)_{ab}\pi)) \\
&= 0 \\
\implies [K_\mu, S_a] &= 0
\end{aligned}
\tag{8.20}
$$

$$
\begin{aligned}
S_a \cdot \phi &= x^\mu \gamma_\mu \psi_a \\
S_a \cdot \pi &= x^\mu \gamma_\mu \gamma_5 \psi_a \\
S_a \cdot \psi_b &= (x^\nu \partial_\nu + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + x_\rho \partial_\sigma(-(\gamma^{\rho\sigma})_{ab}\phi + (\gamma^{\rho\sigma}\gamma_5)_{ab}\pi)
\end{aligned}
\tag{8.21}
$$

For the next calculation it is again useful to employ (3.74), this time for the term $(\gamma_\mu \gamma^{\rho\sigma}\gamma_5)_{ab}$.

$$\begin{aligned}(\gamma_\mu \gamma^{\rho\sigma}\gamma_5)_{ab} &= (\gamma_\mu\gamma_5\gamma^{\rho\sigma})_{ab} = (\gamma_\mu\gamma_5)_{ac}(\gamma^{\rho\sigma})_b^{\ c} \\ &= -(\gamma_\mu\gamma_5)_{ca}(\gamma^{\rho\sigma})_b^{\ c} = -(\gamma_\mu\gamma_5\gamma^{\rho\sigma})_{ba} \\ &= -(\gamma_\mu\gamma^{\rho\sigma}\gamma_5)_{ba}\end{aligned} \tag{8.22}$$

So the term is antisymmetric. Since S_a is odd, we will need to use the anticommutator.

$$\begin{aligned}[S_a, S_b]\cdot\phi &= 2S_{(a}\cdot x^\mu\gamma_\mu\psi_{b)} \\ &= 2((x^\mu)(x^\nu\partial_\nu+2))((\gamma_\mu)_{(ba)}\phi + (\gamma_\mu\gamma_5)_{(ab)}\pi) \\ &\quad + 2((x^\mu)x_\rho\partial_\sigma)(-(\gamma_\mu\gamma^{\rho\sigma})_{(ab)}\phi + (\gamma_\mu\gamma^{\rho\sigma}\gamma_5)_{(ab)}\pi) \\ &= 2(x^\mu(x^\nu\partial_\nu+2))((\gamma_\mu)_{ab}\phi) \\ &\quad + 2(x^\mu x_\rho\partial_\sigma)(-(\gamma_\mu^{\ \rho\sigma} + \delta_\mu^\rho\gamma^\sigma - \delta_\mu^\sigma\gamma^\rho)_{(ab)}\phi) \\ &= 2(x_\mu x^\nu\partial_\nu + 2x_\mu)(\gamma^\mu)_{ab}\phi \\ &\quad + 2(-x^\nu x_\nu\partial_\mu + x_\mu x^\nu\partial_\nu)(\gamma^\mu)_{ab}\phi \\ &= 2(2x_\mu x^\nu\partial_\nu - x^\nu x_\nu\partial_\mu + 2x_\mu)(\gamma^\mu)_{ab}\phi \\ &= 2(\gamma^\mu)_{ab}K_\mu\cdot\phi\end{aligned} \tag{8.23}$$

$$\begin{aligned}[S_a, S_b]\cdot\pi &= 2S_{(a}\cdot x^\mu\gamma_\mu\gamma_5\psi_{b)} \\ &= 2((x^\mu)(x^\nu\partial_\nu+2))(-(\gamma_\mu\gamma_5)_{(ab)}\phi - (\gamma_\mu\gamma_5\gamma_5)_{(ba)}\pi) \\ &\quad + 2((x^\mu)x_\rho\partial_\sigma)(-(\gamma_\mu\gamma_5\gamma^{\rho\sigma})_{(ab)}\phi + (\gamma_\mu\gamma_5\gamma^{\rho\sigma}\gamma_5)_{(ab)}\pi) \\ &= 2((x^\mu)(x^\nu\partial_\nu+2))(\gamma_\mu)_{ab}\pi \\ &\quad + 2((x^\mu)x_\rho\partial_\sigma)(-(\gamma_\mu\gamma^{\rho\sigma})_{(ab)}\pi) \\ &= 2(\gamma^\mu)_{ab}K_\mu\cdot\pi\end{aligned} \tag{8.24}$$

$$\begin{aligned}\delta_\zeta\phi &= \bar\zeta x^\rho\gamma_\rho\psi \\ \delta_\zeta\pi &= \bar\zeta x^\mu\gamma_\mu\gamma_5\psi \\ \delta_\zeta\psi &= -\not{\partial}(\phi+\pi\gamma_5)x^\mu\gamma_\mu\zeta - 2(\phi-\pi\gamma_5)\zeta\end{aligned} \tag{8.25}$$

$$
\begin{aligned}
[\delta_{\zeta_1}, \delta_{\zeta_2}] \cdot \psi &= \delta_{\zeta_1} \cdot (-\partial\!\!\!/(\phi + \pi\gamma_5)x^\mu\gamma_\mu\zeta_2 - 2(\phi - \pi\gamma_5)\zeta_2) \\
&- \delta_{\zeta_2} \cdot (-\partial\!\!\!/(\phi + \pi\gamma_5)x^\mu\gamma_\mu\zeta_1 - 2(\phi - \pi\gamma_5)\zeta_1) \\
&= (-\partial\!\!\!/(\bar{\zeta}_1 x^\rho\gamma_\rho\psi + \bar{\zeta}_1 x^\rho\gamma_\rho\gamma_5\psi\gamma_5)x^\mu\gamma_\mu\zeta_2 - 2(\bar{\zeta}_1 x^\rho\gamma_\rho\psi - \bar{\zeta}_1 x^\rho\gamma_\rho\gamma_5\psi\gamma_5)\zeta_2) \\
&- (-\partial\!\!\!/(\bar{\zeta}_2 x^\rho\gamma_\rho\psi + \bar{\zeta}_2 x^\rho\gamma_\rho\gamma_5\psi\gamma_5)x^\mu\gamma_\mu\zeta_1 - 2(\bar{\zeta}_2 x^\rho\gamma_\rho\psi - \bar{\zeta}_2 x^\rho\gamma_\rho\gamma_5\psi\gamma_5)\zeta_1) \\
&= (-\gamma^\nu\partial_\nu((\zeta_2\bar{\zeta}_1 - \zeta_1\bar{\zeta}_2)x^\rho\gamma_\rho\psi + \gamma_5(\zeta_2\bar{\zeta}_1 - \zeta_1\bar{\zeta}_2)x^\rho\gamma_\rho\gamma_5\psi)x^\mu\gamma_\mu \\
&- 2((\zeta_2\bar{\zeta}_1 - \zeta_1\bar{\zeta}_2)x^\rho\gamma_\rho\psi - \gamma_5(\zeta_2\bar{\zeta}_1 - \zeta_1\bar{\zeta}_2)x^\rho\gamma_\rho\gamma_5\psi)) \\
&= -\gamma^\nu\partial_\nu(\bar{\zeta}_2\gamma^\kappa\zeta_1\gamma_\kappa)x^\rho\gamma_\rho\psi x^\mu\gamma_\mu - 2\bar{\zeta}_2\gamma^\kappa\zeta_1\gamma_\kappa)x^\rho\gamma_\rho\psi \\
&= -(\bar{\zeta}_2\gamma^\kappa\zeta_1)(\gamma^\nu\gamma_\kappa\partial_\nu x^\rho x^\mu\gamma_\rho\psi\gamma_\mu + 2\gamma_\kappa x^\rho\gamma_\rho\psi) \\
&= \zeta_1\zeta_2\gamma^\kappa(\gamma^\nu\gamma_\kappa\partial_\nu x^\rho x^\mu\gamma_\rho\psi\gamma_\mu + 2\gamma_\kappa x^\rho\gamma_\rho\psi) \\
&= -\zeta_1\zeta_2 2\gamma^\nu(\partial_\nu x^\rho x^\mu\gamma_\rho)\psi\gamma_\mu + 8x^\rho\gamma_\rho\psi)
\end{aligned}
\tag{8.26}
$$

$$
\begin{aligned}
Q_a \cdot \phi &= \psi_a \\
Q_a \cdot \pi &= (\gamma_5)^b{}_a\psi_b = \gamma_5\psi_a \\
Q_a \cdot \psi_b &= -(\gamma^\rho)_{ab}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ab}\partial_\rho\pi
\end{aligned}
\tag{8.27}
$$

$$
\begin{aligned}
[Q_a, S_b] \cdot \phi &= Q_a \cdot x^\nu\gamma_\nu\psi_b + S_b\psi_a \\
&= x^\nu\gamma_\nu(-(\gamma^\mu)_{ba}\partial_\mu\phi - (\gamma^\mu\gamma_5)_{ba}\partial_\mu\pi) \\
&\quad + (x^\nu\partial_\nu + 2)(-C_{ba}\phi + (\gamma_5)_{ba}\pi) \\
&\quad + x_\mu\partial_\nu(-(\gamma^{\mu\nu})_{ba}\phi + (\gamma^{\mu\nu}\gamma_5)_{ba}\pi) \\
&= x_\nu\partial_\mu(-(\gamma^{\nu\mu} + \eta^{\nu\mu})_{ba}\phi - (\eta^{\nu\mu}\gamma_5)_{ba}\pi)) \\
&\quad + (x^\nu\partial_\nu + 2)(C_{ab}\phi - (\gamma_5)_{ab}\pi) \\
&\quad + x_\mu\partial_\nu(-(\gamma^{\mu\nu})_{ab}\phi) \\
&= x_\nu\partial_\mu((\gamma^{\mu\nu} + \eta^{\nu\mu})_{ab}\phi) + x^\mu\partial_\mu(\gamma_5)_{ab}\phi) \\
&\quad + (x^\nu\partial_\nu + 2)(C_{ab}\phi - (\gamma_5)_{ab}\pi) \\
&\quad + x_\mu\partial_\nu(-(\gamma^{\mu\nu})_{ab}\phi) \\
&= -(x_\mu\partial_\nu - x_\nu\partial_\mu)((\gamma^{\mu\nu})_{ab}\phi) \\
&\quad + 2(x^\nu\partial_\nu + 1)C_{ab}\phi - 2(\gamma_5)_{ab}\pi \\
&= -M^\phi_{\mu\nu}(\gamma^{\mu\nu})_{ab}\phi + 2D^\phi C_{ab}\phi - 2(\gamma_5)_{ab}R^\phi
\end{aligned}
\tag{8.28}
$$

To achieve algebraic closure we must define another generator R such that $R \cdot \phi = \pi$.

$$
\begin{aligned}
[Q_a, S_b] \cdot \pi &= Q_a \cdot x^\nu \gamma_\nu \gamma_5 \psi_b + S_b \gamma_5 \psi_a \\
&= x^\nu \gamma_\nu \gamma_5 (-(\gamma^\mu)_{ba} \partial_\mu \phi + (\gamma_5 \gamma^\mu)_{ba} \partial_\mu \pi) \\
&\quad + \gamma_5 (x^\nu \partial_\nu + 2)(-C_{ba} \phi + (\gamma_5)_{ba} \pi) \\
&\quad + \gamma_5 x_\mu \partial_\nu (-(\gamma^{\mu\nu})_{ba} \phi + (\gamma_5 \gamma^{\mu\nu})_{ba} \pi) \\
&= x_\nu \partial_\mu ((\eta^{\nu\mu} \gamma_5)_{ba} \phi - (\eta^{\nu\mu} + \gamma^{\nu\mu})_{ba} \pi)) \\
&\quad + (x^\nu \partial_\nu + 2)((\gamma_5)_{ab} \phi + C_{ab} \pi) \\
&\quad + x_\mu \partial_\nu (\gamma_5 \gamma_5 \gamma^{\mu\nu})_{ab} \pi) \\
&= -x^\mu \partial_\mu (\gamma_5)_{ab} \phi + x_\nu \partial_\mu (\eta^{\nu\mu} + \gamma^{\mu\nu})_{ab} \pi \\
&\quad + (x^\nu \partial_\nu + 2)((\gamma_5)_{ab} \phi + C_{ab} \pi) \\
&\quad + x_\mu \partial_\nu (-(\gamma^{\mu\nu})_{ab} \pi) \\
&= -(x_\mu \partial_\nu - x_\nu \partial_\mu)((\gamma^{\mu\nu})_{ab} \pi) \\
&\quad + 2(x^\nu \partial_\nu + 1) C_{ab} \pi + 2(\gamma_5)_{ab} \phi \\
&= -M^\pi_{\mu\nu} (\gamma^{\mu\nu})_{ab} \pi + 2 D^\pi C_{ab} \pi - 2(\gamma_5)_{ab} R^\pi
\end{aligned} \tag{8.29}
$$

To achieve algebraic closure we must define R^π such that $R \cdot \pi = -\phi$.

$$
\begin{aligned}
R \cdot \phi &= \pi \\
R \cdot \pi &= -\phi \\
R \cdot \psi &= \frac{1}{2} \gamma_5 \psi
\end{aligned} \tag{8.30}
$$

We need to check the brackets between R and the other generators to see if they are closed. Since R maps spin $0/\frac{1}{2}$ fields to spin $0/\frac{1}{2}$ fields respectively, R must therefore have spin 0, and belong to $\mathfrak{g}_0$ which means that only commutators will be needed in the algebra involving R.

$$
\begin{aligned}
[P_\mu, R] \cdot \phi &= P_\mu \cdot \pi - R \cdot P_\mu \phi \\
&= 0 \\
[P_\mu, R] \cdot \pi &= -P_\mu \cdot \phi - R \cdot P_\mu \pi \\
&= 0 \\
[P_\mu, R] \cdot \psi &= P_\mu \cdot \frac{1}{2} \gamma_5 \psi - R \cdot P_\mu \psi \\
&= 0 \\
\implies [P_\mu, R] &= 0
\end{aligned} \tag{8.31}
$$

$$\begin{aligned}
[M_{\mu\nu}, R] \cdot \phi &= M^{\pi}_{\mu\nu} \cdot \pi - R \cdot M^{\phi}_{\mu\nu}\phi \\
&= 0 \\
[M_{\mu\nu}, R] \cdot \pi &= -M^{\phi}_{\mu\nu} \cdot \phi - R \cdot M^{\pi}_{\mu\nu}\pi \\
&= 0 \\
[M_{\mu\nu}, R] \cdot \psi &= M^{\psi}_{\mu\nu} \cdot \frac{1}{2}\gamma_5\psi - R \cdot M^{\psi}_{\mu\nu}\psi \\
&= 0 \\
\implies [M_{\mu\nu}, R] &= 0
\end{aligned} \tag{8.32}$$

$$\begin{aligned}
[D, R] \cdot \phi &= D^{\pi} \cdot \pi - R \cdot D^{\phi}\phi \\
&= 0 \\
[D, R] \cdot \pi &= -D^{\phi} \cdot \phi - R \cdot D^{\pi}\pi \\
&= 0 \\
[D, R] \cdot \psi &= D^{\psi} \cdot \frac{1}{2}\gamma_5\psi - R \cdot M^{\psi}_{\mu\nu}\psi \\
&= 0 \\
\implies [D, R] &= 0
\end{aligned} \tag{8.33}$$

$$\begin{aligned}
[K_{\mu}, R] \cdot \phi &= K^{\pi}_{\mu} \cdot \pi - R \cdot K^{\phi}_{\mu}\phi \\
&= 0 \\
[K_{\mu}, R] \cdot \pi &= -K^{\phi}_{\mu} \cdot \phi - R \cdot K^{\pi}_{\mu}\pi \\
&= 0 \\
[K_{\mu}, R] \cdot \psi &= K^{\psi}_{\mu} \cdot \frac{1}{2}\gamma_5\psi - R \cdot M^{\psi}_{\mu\nu}\psi \\
&= 0 \\
\implies [K_{\mu}, R] &= 0
\end{aligned} \tag{8.34}$$

$$\begin{aligned}
[R, R] \cdot \phi &= R^{\pi} \cdot \pi - R \cdot R^{\phi}\phi \\
&= 0 \\
[R, R] \cdot \pi &= -R^{\phi} \cdot \phi - R \cdot R^{\pi}\pi \\
&= 0 \\
[R, R] \cdot \psi &= R^{\psi} \cdot \frac{1}{2}\gamma_5\psi - R \cdot R^{\psi}\psi \\
&= 0 \\
\implies [R, R] &= 0
\end{aligned} \tag{8.35}$$

With all the bosonic generators G, $G^{\phi} = G^{\pi}$ and there are no spacetime dependencies in any of the products of R. Therefore all brackets trivially equal zero, including $[R, R]$. Things only get interesting with the fermionic supercharges.

$$\begin{aligned}
[R,Q]\cdot\phi &= R^{\psi}\cdot\psi - Q\cdot R^{\phi}\phi \\
&= \frac{1}{2}\gamma_5\psi - \gamma_5\psi = -\frac{1}{2}\gamma_5\psi \\
&= -\frac{1}{2}\gamma_5 Q\cdot\phi \\
[R,Q]\cdot\pi &= R^{\psi}\cdot\gamma_5\psi - Q\cdot R^{\pi}\pi \\
&= \frac{1}{2}\gamma_5\gamma_5\psi + \psi = \frac{1}{2}\psi \\
&= -\frac{1}{2}\gamma_5 Q\cdot\pi \\
[R,Q]\cdot\psi &= R^{\phi/\pi}\cdot\partial\!\!\!/(-\phi+\gamma_5\pi) - R\cdot R^{\psi}\psi \\
&= \partial\!\!\!/(-\pi-\gamma_5\phi) - Q\cdot\frac{1}{2}\gamma_5\psi \\
&= \partial\!\!\!/(-\pi-\gamma_5\phi) - \frac{1}{2}\partial\!\!\!/(-\gamma_5\phi+\gamma_5\gamma_5\pi) \\
&= \frac{1}{2}\partial\!\!\!/(-\gamma_5\phi-\pi) \\
&= -\frac{1}{2}\gamma_5 Q\cdot\psi \\
\implies [R,Q] &= -\frac{1}{2}\gamma_5 Q
\end{aligned} \tag{8.36}$$

$$\begin{aligned}
[R,S_a]\cdot\phi &= R^{\psi}\cdot x^{\mu}\gamma_\mu\psi_a - S_a\cdot R^{\phi}\phi \\
&= x^{\mu}\gamma_\mu\frac{1}{2}\gamma_5\psi_a - x^{\mu}\gamma_\mu\gamma_5\psi_a \\
&= -\frac{1}{2}x^{\mu}\gamma_\mu\gamma_5\psi_a \\
&= \frac{1}{2}\gamma_5 x^{\mu}\gamma_\mu\psi_a \\
&= \frac{1}{2}\gamma_5 S_a\cdot\phi
\end{aligned} \tag{8.37}$$

$$
\begin{aligned}
[R, S_a] \cdot \pi &= R^{\psi} \cdot x^{\mu}\gamma_{\mu}\gamma_5\psi_a - S_a \cdot R^{\pi}\pi \\
&= x^{\mu}\gamma_{\mu}\gamma_5\frac{1}{2}\gamma_5\psi_a + x^{\mu}\gamma_{\mu}\psi_a \\
&= -x^{\mu}\gamma_5\gamma_{\mu}\frac{1}{2}\gamma_5\psi_a + x^{\mu}\gamma_5\gamma_{\mu}\gamma_5\psi_a \\
&= \frac{1}{2}\gamma_5 x^{\mu}\gamma_{\mu}\gamma_5\psi_a \\
&= \frac{1}{2}\gamma_5 S_a \cdot \pi
\end{aligned}
\tag{8.38}
$$

$$
\begin{aligned}
[R, S_a] \cdot \psi_b &= R^{\phi/\pi} \cdot ((x^{\nu}\partial_{\nu} + 2)(-C_{ab}\phi + (\gamma_5)_{ab}\pi) \\
&\quad + x_{\mu}\partial_{\rho}(-(\gamma^{\mu\rho})_{ab}\phi + (\gamma^{\mu\rho}\gamma_5)_{ab}\pi)) - S_a \cdot R^{\psi}\psi_b \\
&= (x^{\nu}\partial_{\nu} + 2)(-C_{ab}\pi - (\gamma_5)_{ab}\phi) \\
&\quad + x_{\mu}\partial_{\rho}(-(\gamma^{\mu\rho})_{ab}\pi - (\gamma^{\mu\rho}\gamma_5)_{ab}\phi) \\
&\quad - \frac{1}{2}((x^{\nu}\partial_{\nu} + 2)(-(\gamma_5)_{ab}\phi - C_{ab}\pi) \\
&\quad + x_{\mu}\partial_{\rho}(-(\gamma^{\mu\rho}\gamma_5)_{ab}\phi - (\gamma^{\mu\rho})_{ab}\pi)) \\
&= \frac{1}{2}((x^{\nu}\partial_{\nu} + 2)(-(\gamma_5)_{ab}\phi + (\gamma_5\gamma_5)_{ab}\pi) \\
&\quad + x_{\mu}\partial_{\rho}(-(\gamma_5\gamma^{\mu\rho})_{ab}\phi + (\gamma_5\gamma^{\mu\rho}\gamma_5)_{ab}\pi)) \\
&= \frac{1}{2}\gamma_5 S_a \cdot \psi_b \\
\implies [R, S_a] &= \frac{1}{2}\gamma_5 S_a
\end{aligned}
\tag{8.39}
$$

The super-Jacobi identity

$$
\begin{aligned}
&[P_{\mu}, [D, Q_a]] - [[P_{\mu}, D], Q_a] - [D, [P_{\mu}, Q_a]] \\
&= [P_{\mu}, \propto Q_a] - [P_{\mu}, Q_a] - 0 \\
&= 0
\end{aligned}
\tag{8.40}
$$

$$
\begin{aligned}
&[M_{\mu\nu}, [D, Q_a]] - [[M_{\mu\nu}, D], Q_a] - [D, [M_{\mu\nu}, Q_a]] \\
&= [M_{\mu\nu}, \frac{1}{2}Q_a] - 0 - [D, \Sigma_{\mu\nu}Q_a] \\
&= \frac{1}{2}\Sigma_{\mu\nu}Q_a - \Sigma_{\mu\nu}\frac{1}{2}Q_a \\
&= 0
\end{aligned}
\tag{8.41}
$$

$$
\begin{aligned}
&[D, [D, Q_a]] - [[D, D], Q_a] - [D, [D, Q_a]] \\
&= [D, \frac{1}{2}Q_a] - 0 - [D, \frac{1}{2}Q_a] \\
&= 0
\end{aligned}
\tag{8.42}
$$

$$
\begin{aligned}
&[D,[Q_a,Q_b]]-[[D,Q_a],Q_b]-[Q_a,[D,Q_b]]\\
&=[D,-2(\gamma^\rho)_{ab}P_\rho]-[\frac{1}{2}Q_a,Q_b]-[Q_a,\frac{1}{2}Q_b]\\
&=2(\gamma^\rho)_{ab}P_\rho-2(\gamma^\rho)_{ab}P_\rho\\
&=0
\end{aligned}
\tag{8.43}
$$

$$
\begin{aligned}
&[P_\mu,[K_\nu,Q_a]]-[[P_\mu,K_\nu],Q_a]-[K_\nu,[P_\mu,Q_a]]\\
&=[P_\mu,-\gamma_\nu S_a]-[-2(\eta_{\mu\nu}D-M_{\mu\nu}),Q_a]-0\\
&=-\gamma_\nu\gamma_\mu Q_a+\eta_{\mu\nu}Q_a-\gamma_{\mu\nu}Q_a\\
&=0
\end{aligned}
\tag{8.44}
$$

$$
\begin{aligned}
&[M_{\mu\nu},[K_\rho,Q_a]]-[[M_{\mu\nu},K_\rho],Q_a]-[[K_\rho,[M_{\mu\nu},Q_a]]\\
&=[M_{\mu\nu},-\gamma_\rho S_a]-[-(\eta_{\rho\nu}K_\mu-\eta_{\rho\mu}K_\nu),Q_a]-[K_\rho,\Sigma_{\mu\nu}Q_a]\\
&=-\gamma_\rho\Sigma_{\mu\nu}S_a-(\eta_{\rho\nu}\gamma_\mu-\eta_{\rho\mu}\gamma_\nu)S_a+\Sigma_{\mu\nu}\gamma_\rho S_a\\
&=(\eta_{\rho\nu}\gamma_\mu-\eta_{\rho\mu}\gamma_\nu)S_a-(\eta_{\rho\nu}\gamma_\mu-\eta_{\rho\mu}\gamma_\nu)S_a\\
&=0
\end{aligned}
\tag{8.45}
$$

$$
\begin{aligned}
&[D,[K_\rho,Q_a]]-[[D,K_\rho],Q_a]-[K_\rho,[D,Q_a]]\\
&=[D,-\gamma_\rho S_a]-[-K_\rho,Q_a]-[K_\rho,\frac{1}{2}Q_a]\\
&=\frac{1}{2}\gamma_\rho S_a-\gamma_\rho S_a+\frac{1}{2}\gamma_\rho S_a\\
&=0
\end{aligned}
\tag{8.46}
$$

$$
\begin{aligned}
&[K_\sigma,[K_\rho,Q_a]]-[[K_\sigma,K_\rho],Q_a]-[K_\rho,[K_\sigma,Q_a]]\\
&=[K_\sigma,\propto S_a]-0-[K_\rho,\propto S_a]\\
&=0-0\\
&=0
\end{aligned}
\tag{8.47}
$$

$$
\begin{aligned}
&[K_\rho,[Q_a,Q_b]]-[[K_\rho,Q_a],Q_b]-[Q_a,[K_\rho,Q_b]]\\
&=[K_\rho,-2(\gamma^\sigma)_{ab}P_\sigma]-[-\gamma_\sigma S_a,Q_b]-[Q_a,-\gamma_\sigma S_b]\\
&=2(\gamma^\rho)_{ab}(2\eta_{\rho\sigma}D-2M_{\rho\sigma})\\
&-(2(\gamma_\sigma)_{ba}D-2(\gamma_\sigma\gamma_5)_{ba}R+(\gamma_\sigma\gamma^{\mu\nu})_{ba}M_{\mu\nu})\\
&-(2(\gamma_\sigma)_{ab}D-2(\gamma_\sigma\gamma_5)_{ab}R+(\gamma_\sigma\gamma^{\mu\nu})_{ab}M_{\mu\nu})\\
&=4(\gamma_\sigma)_{ab}D-4(\gamma^\rho)_{ab}M_{\rho\sigma}\\
&-4(\gamma_\sigma)_{ab}D+(\delta^\mu_\sigma\gamma^\nu-\delta^\nu_\sigma\gamma^\mu)_{ab}M_{\mu\nu}\\
&+(\delta^\mu_\sigma\gamma^\nu-\delta^\nu_\sigma\gamma^\mu)_{ab}M_{\mu\nu}\\
&=-4(\gamma^\rho)_{ab}M_{\rho\sigma}\\
&+2(\gamma^\nu)_{ab}M_{\sigma\nu}+2(\gamma^\mu)_{ab}M_{\sigma\mu}\\
&=0
\end{aligned}
\tag{8.48}
$$

$$
\begin{aligned}
&[P_\mu,[P_\nu,S_a]]-[[P_\mu,P_\nu],S_a]-[P_\nu,[P_\mu,S_a]]\\
&=[P_\mu,\gamma_\nu Q_a]-0-[P_\nu,\gamma_\mu Q_a]\\
&=0
\end{aligned}
\tag{8.49}
$$

The next is solved like (8.48).

$$
\begin{aligned}
&[P_\mu,[S_a,S_b]]-[[P_\mu,S_a],S_b]-[S_a,[P_\mu,S_b]]\\
&=[P_\mu,2\gamma^\nu_{ab}K_\nu]-[\gamma_\mu Q_a,S_b]-[S_a,\gamma_\mu Q_b]\\
&=-4\gamma^\nu_{ab}(\eta_{\mu\nu}D-2M_{\mu\nu})-[\gamma_\mu Q_a,S_b]-[S_a,\gamma_\mu Q_b]\\
&=0
\end{aligned}
\tag{8.50}
$$

$$
\begin{aligned}
&[S_c,[S_a,S_b]]-[[S_c,S_a],S_b]+[S_a,[S_c,S_b]]\\
&=[S_c,2\gamma^\nu_{ab}K_\nu]-[2\gamma^\nu_{ca}K_\nu,S_b]+[S_a,2\gamma^\nu_{cb}K_\nu]\\
&=0
\end{aligned}
\tag{8.51}
$$

This one is solved similarly to (8.45).

$$
\begin{aligned}
&[P_\rho,[M_{\mu\nu},S_a]]-[[P_\rho,M_{\mu\nu}],S_a]-[M_{\mu\nu},[P_\rho,S_a]]\\
&=[P_\rho,\Sigma_{\mu\nu}S_a]-[\eta_{\rho\nu}P_\mu-\eta_{\rho\mu}P_\nu,S_a]-[M_{\mu\nu},\gamma_\rho Q_a]\\
&=\Sigma_{\mu\nu}\gamma_\rho Q_a-\eta_{\rho\nu}\gamma_\mu Q_a+\eta_{\rho\mu}\gamma_\nu Q_a-\gamma_\rho\Sigma_{\mu\nu}Q_a\\
&=0
\end{aligned}
\tag{8.52}
$$

$$
\begin{aligned}
&[M_{\rho\sigma},[M_{\mu\nu},S_a]]-[[M_{\rho\sigma},M_{\mu\nu}],S_a]-[M_{\mu\nu},[M_{\rho\sigma},S_a]]\\
&=[M_{\rho\sigma},\Sigma_{\mu\nu}S_a]-[\Sigma_{\rho\sigma},\Sigma_{\mu\nu}]S_a-[M_{\mu\nu},\Sigma_{\rho\sigma},S_a]\\
&=\Sigma_{\mu\nu}\Sigma_{\rho\sigma}S_a]-[\Sigma_{\rho\sigma},\Sigma_{\mu\nu}]S_a-\Sigma_{\rho\sigma}\Sigma_{\mu\nu}S_a\\
&=0
\end{aligned}
\tag{8.53}
$$

Analogous to (5.24):

$$\begin{aligned}
&[M_{\rho\sigma},[S_a,S_b]]-[[M_{\rho\sigma},S_a],S_b]-[S_a,[M_{\rho\sigma},S_b]]\\
&=[M_{\rho\sigma},2\gamma^\mu K_\mu]-[\Sigma_{\rho\sigma}S_a,S_b]-[S_a,\Sigma_{\rho\sigma}S_b]\\
&=[M_{\rho\sigma},2\gamma^\mu K_\mu]-\Sigma_{\rho\sigma}2\gamma^\mu K_\mu-\Sigma_{\rho\sigma}2\gamma^\mu K_\mu\\
&=2\gamma^\mu(\eta_{\rho\mu}K_\sigma-\eta_{\sigma\mu}K_\rho)-2\gamma_{\rho\sigma}\gamma^\mu K_\mu\\
&=0
\end{aligned}\tag{8.54}$$

$$\begin{aligned}
&[P_\rho,[D,S_a]]-[[P_\rho,D],S_a]-[D,[P_\rho,S_a]]\\
&=[P_\rho,-\frac{1}{2}S_a]-[-P_\mu,S_a]-[D,\gamma_\rho Q_a]\\
&=-\frac{1}{2}\gamma_\rho Q_a+\gamma_\rho Q_a-\frac{1}{2}\gamma_\rho Q_a\\
&=0
\end{aligned}\tag{8.55}$$

$$\begin{aligned}
&[M_{\mu\nu},[D,S_a]]-[[M_{\mu\nu},D],S_a]-[D,[M_{\mu\nu},S_a]]\\
&=[M_{\mu\nu},-\frac{1}{2}S_a]-0-[D,\Sigma_{\mu\nu}S_a]\\
&=-\frac{1}{2}\Sigma_{\mu\nu}S_a+\frac{1}{2}\Sigma_{\mu\nu}S_a\\
&=0
\end{aligned}\tag{8.56}$$

$$\begin{aligned}
&[D,[D,S_a]]-[[D,D],S_a]-[D,[D,S_a]]\\
&=[D,-\frac{1}{2}S_a]-0-[D,-\frac{1}{2}S_a]\\
&=0
\end{aligned}\tag{8.57}$$

$$\begin{aligned}
&[D,[S_a,S_b]]-[[D,S_a],S_b]-[S_a,[D,S_b]]\\
&=[D,2\gamma^\mu K_\mu]-[-\frac{1}{2}S_a,S_b]-[S_a,-\frac{1}{2}S_b]\\
&=-2\gamma^\mu K_\mu+\gamma^\mu K_\mu+\gamma^\mu K_\mu\\
&=0
\end{aligned}\tag{8.58}$$

$$\begin{aligned}
&[P_\mu,[K_\nu,S_a]]-[[P_\mu,K_\nu],S_a]-[K_\nu,[P_\mu,S_a]]\\
&=0-[-2(\eta_{\mu\nu}D-2M_{\mu\nu}),S_a]-[K_\nu,\gamma_\mu Q_a]\\
&=-\eta_{\mu\nu}S_a-\gamma_{\mu\nu}S_a+\gamma_\mu\gamma_\nu S_a\\
&=0
\end{aligned}\tag{8.59}$$

$$\begin{aligned}
&[M_{\rho\sigma},[K_\nu,S_a]]-[[M_{\rho\sigma},K_\nu],S_a]-[K_\nu,[M_{\rho\sigma},S_a]]\\
&=0-[\propto K_\rho,S_a]-[K_\nu,\propto S_a]\\
&=0
\end{aligned}\tag{8.60}$$

$$\begin{aligned}&[D,[K_\nu,S_a]]-[[D,K_\nu],S_a]-[K_\nu,[D,S_a]]\\&=0-[\propto K_\rho,S_a]-[K_\nu,\propto S_a]\\&=0\end{aligned}\tag{8.61}$$

$$\begin{aligned}&[K_\mu,[K_\nu,S_a]]-[[K_\mu,K_\nu],S_a]-[K_\nu,[K_\mu,S_a]]\\&=0-0-0\\&=0\end{aligned}\tag{8.62}$$

$$\begin{aligned}&[K_\mu,[S_a,S_b]]-[[K_\mu,S_a],S_b]-[S_a,[K_\mu,S_a]]\\&=[K_\rho,\propto K_\rho]-0-0\\&=0\end{aligned}\tag{8.63}$$

$$\begin{aligned}&[P_\rho,[Q_a,S_b]]-[[P_\rho,Q_a],S_b]-[Q_a,[P_\rho,S_b]]\\&=[P_\rho,-(2C_{ab}D-2(\gamma_5)_{ab}R+(\gamma^{\mu\nu})_{ab}M_{\mu\nu})]-0-[Q_a,\gamma_\rho Q_b]\\&=2C_{ab}P_\rho-(\gamma^\mu_{\ \rho})_{ab}P_\mu+(\gamma_\rho^{\ \nu})_{ab}P_\nu+2(\gamma^\mu\gamma_\rho)_{ba}P_\mu\\&=2C_{ab}P_\rho-2(\gamma^\mu_{\ \rho})_{ab}P_\mu+2(\gamma^\mu_{\ \rho})_{ab}P_\mu-2C_{ab}P_\rho\\&=0\end{aligned}\tag{8.64}$$

$$\begin{aligned}&[M_{\rho\sigma},[Q_a,S_b]]-[[M_{\rho\sigma},Q_a],S_b]-[Q_a,[M_{\rho\sigma},S_b]]\\&=[M_{\rho\sigma},-(2C_{ab}D-2(\gamma_5)_{ab}R+(\gamma^{\mu\nu})_{ab}M_{\mu\nu})]\\&-[\Sigma_{\rho\sigma}Q_a,S_b]-[Q_a,\Sigma_{\rho\sigma}S_b]\\&=-(\gamma^{\mu\nu})_{ab}[M_{\rho\sigma},M_{\mu\nu}]+\gamma_{\rho\sigma}(\gamma^{\mu\nu})_{ab}M_{\mu\nu}\\&=-(2\gamma_\rho^{\ \mu}M_{\mu\sigma}-2\gamma_\sigma^{\ \nu}M_{\nu\rho})+(2\gamma_\rho^{\ \mu}M_{\mu\sigma}-2\gamma_\sigma^{\ \nu}M_{\nu\rho})\\&=0\end{aligned}\tag{8.65}$$

$$\begin{aligned}&[D,[Q_a,S_b]]-[[D,Q_a],S_b]-[Q_a,[D,S_b]]\\&=[D,-(2C_{ab}D-2(\gamma_5)_{ab}R+(\gamma^{\mu\nu})_{ab}M_{\mu\nu})]\\&-[\frac{1}{2}Q_a,S_b]-[Q_a,-\frac{1}{2}S_b]\\&=-(\gamma^{\mu\nu})_{ab}[M_{\rho\sigma},M_{\mu\nu}]+\gamma_{\rho\sigma}(\gamma^{\mu\nu})_{ab}M_{\mu\nu}\\&=-(2\gamma_\rho^{\ \mu}M_{\mu\sigma}-2\gamma_\sigma^{\ \nu}M_{\nu\rho})+(2\gamma_\rho^{\ \mu}M_{\mu\sigma}-2\gamma_\sigma^{\ \nu}M_{\nu\rho})\\&=0\end{aligned}\tag{8.66}$$

$$
\begin{aligned}
&[K_\rho,[Q_a,S_b]] - [[K_\rho,Q_a],S_b] - [Q_a,[K_\rho,S_b]] \\
&= [K_\rho, -(2C_{ab}D - 2(\gamma_5)_{ab}R + (\gamma^{\mu\nu})_{ab}M_{\mu\nu})] \\
&- [-\gamma_\rho S_a, S_b] - 0 \\
&= 2C_{ab}K_\rho - (\gamma^{\mu\nu})_{ab}(\eta_{\rho\nu}K_\mu - \eta_{\rho\mu}K_\nu) + 2(\gamma^\mu\gamma_\rho)_{ba}K_\mu \\
&= 0 \\
&= 2C_{ab}K_\rho - 2(\gamma^\mu_{\ \rho})_{ab}K_\mu + 2(\gamma^\mu_{\ \rho})_{ab}K_\mu - 2C_{ab}K_\rho \\
&= 0
\end{aligned} \tag{8.67}
$$

$$
\begin{aligned}
&[Q_c,[Q_a,S_b]] - [[Q_c,Q_a],S_b] + [Q_a,[Q_c,S_b]] \\
&= [Q_c, -(2C_{ab}D - 2(\gamma_5)_{ab}R + (\gamma^{\mu\nu})_{ab}M_{\mu\nu})] \\
&- [-2(\gamma^\rho)_{ca}P_\rho, S_b] \\
&+ [Q_a, -(2C_{cb}D - 2(\gamma_5)_{cb}R + (\gamma^{\mu\nu})_{cb}M_{\mu\nu})] \\
&= C_{ab}Q_c + (\gamma_5)_{ab}(\gamma_5)Q_c + (\gamma^{\mu\nu})_{ab}\Sigma_{\mu\nu})Q_c \\
&+ 2(\gamma^\rho\gamma_\rho)_{ca}Q_b \\
&+ C_{cb}Q_a + (\gamma_5)_{cb}(\gamma_5)Q_a + (\gamma^{\mu\nu})_{cb}\Sigma_{\mu\nu})Q_a \\
&= (\gamma^{\mu\nu})_{ab}\Sigma_{\mu\nu}Q_c + (\gamma^{\mu\nu})_{cb}\Sigma_{\mu\nu}Q_a \\
&= (\delta^\nu_{[\mu}\gamma^{\ \mu}_{\nu]} - \delta^\mu_{[\mu}\gamma^{\ \nu}_{\nu]})_{ab}Q_c + (\delta^\nu_{[\mu}\gamma^{\ \mu}_{\nu]} - \delta^\mu_{[\mu}\gamma^{\ \nu}_{\nu]})_{cb}Q_a \\
&= 0
\end{aligned} \tag{8.68}
$$

$$
\begin{aligned}
&[Q_c,[S_a,S_b]] - [[Q_c,S_a],S_b] + [S_a,[Q_c,S_b]] \\
&= [Q_c, (\gamma^\mu)_{ab}K_\mu] \\
&- [-(2C_{ca}D - 2(\gamma_5)_{ca}R + (\gamma^{\mu\nu})_{ca}M_{\mu\nu}), S_b] \\
&+ [S_a, -(2C_{cb}D - 2(\gamma_5)_{cb}R + (\gamma^{\mu\nu})_{cb}M_{\mu\nu})] \\
&= (\gamma^\mu)_{ab}\gamma_\mu S_c \\
&- C_{ca}S_b - (\gamma_5)_{ca}\gamma_5 S_b + (\gamma^{\mu\nu})_{ca}\Sigma_{\mu\nu}S_b \\
&- C_{cb}S_a - (\gamma_5)_{cb}\gamma_5 S_a + (\gamma^{\mu\nu})_{cb}\Sigma_{\mu\nu}S_a \\
&= (\gamma^{\mu\nu})_{ca}\Sigma_{\mu\nu}S_b + (\gamma^{\mu\nu})_{cb}\Sigma_{\mu\nu}S_a \\
&= (\delta^\nu_{[\mu}\gamma^{\ \mu}_{\nu]} - \delta^\mu_{[\mu}\gamma^{\ \nu}_{\nu]})_{ab}S_c + (\delta^\nu_{[\mu}\gamma^{\ \mu}_{\nu]} - \delta^\mu_{[\mu}\gamma^{\ \nu}_{\nu]})_{cb}S_a \\
&= 0
\end{aligned} \tag{8.69}
$$

$$
\begin{aligned}
&[P_\mu,[P_\nu,R]] - [[P_\mu,P_\nu],R] - [P_\nu,[P_\mu,R]] \\
&= 0 - 0 - 0 \\
&= 0
\end{aligned} \tag{8.70}
$$

$$
\begin{aligned}
&[P_\mu,[R,R]] - [[P_\mu,R],R] - [R,[P_\mu,R]] \\
&= 0 - 0 - 0 \\
&= 0
\end{aligned} \tag{8.71}
$$

$$
\begin{aligned}
&[R,[R,R]]-[[R,R],R]-[R,[R,R]]\\
&=0-0-0\\
&=0
\end{aligned}
\tag{8.72}
$$

$$
\begin{aligned}
&[P_\mu,[M_{\rho\nu},R]]-[[P_\mu,M_{\rho\nu}],R]-[M_{\rho\nu},[P_\mu,R]]\\
&=0-[\propto P_\mu,R]-0\\
&=0
\end{aligned}
\tag{8.73}
$$

$$
\begin{aligned}
&[M_{\mu\sigma},[M_{\rho\nu},R]]-[[M_{\mu\sigma},M_{\rho\nu}],R]-[M_{\rho\nu},[M_{\mu\sigma},R]]\\
&=0-[\propto M_\mu,R]-0\\
&=0
\end{aligned}
\tag{8.74}
$$

$$
\begin{aligned}
&[M_{\mu\sigma},[R,R]]-[[M_{\mu\sigma},R],R]-[R,[M_{\mu\sigma},R]]\\
&=0-0-0\\
&=0
\end{aligned}
\tag{8.75}
$$

$$
\begin{aligned}
&[P_\mu,[D,R]]-[[P_\mu,D],R]-[D,[P_\mu,R]]\\
&=0-[\propto P,R]-0\\
&=0
\end{aligned}
\tag{8.76}
$$

$$
\begin{aligned}
&[M_{\mu\nu},[D,R]]-[[M_{\mu\nu},D],R]-[D,[M_{\mu\nu},R]]\\
&=0-0-0\\
&=0
\end{aligned}
\tag{8.77}
$$

$$
\begin{aligned}
&[D,[D,R]]-[[D,D],R]-[D,[D,R]]\\
&=0-0-0\\
&=0
\end{aligned}
\tag{8.78}
$$

$$
\begin{aligned}
&[D,[R,R]]-[[D,R],R]-[D,[R,R]]\\
&=0-0-0\\
&=0
\end{aligned}
\tag{8.79}
$$

$$
\begin{aligned}
&[P_\mu,[K_\nu,R]]-[[P_\mu,K_\nu],R]-[K_\nu,[P_\mu,R]]\\
&=0-[\propto D+\propto M,R]-0\\
&=0
\end{aligned}
\tag{8.80}
$$

$$
\begin{aligned}
&[M_{\mu\nu}, [K_\rho, R]] - [[M_{\mu\nu}, K_\rho], R] - [K_\rho, [M_{\mu\nu}, R]] \\
&= 0 - [\propto K, R] - 0 \\
&= 0
\end{aligned} \tag{8.81}
$$

$$
\begin{aligned}
&[D, [K_\rho, R]] - [[D, K_\rho], R] - [K_\rho, [D, R]] \\
&= 0 - [\propto K, R] - 0 \\
&= 0
\end{aligned} \tag{8.82}
$$

$$
\begin{aligned}
&[K_\sigma, [K_\rho, R]] - [[K_\sigma, K_\rho], R] - [K_\rho, [K_\sigma, R]] \\
&= 0 - 0 - 0 \\
&= 0
\end{aligned} \tag{8.83}
$$

$$
\begin{aligned}
&[K_\sigma, [R, R]] - [[K_\sigma, R], R] - [R, [K_\sigma, R]] \\
&= 0 - 0 - 0 \\
&= 0
\end{aligned} \tag{8.84}
$$

$$
\begin{aligned}
&[P_\mu, [Q_a, R]] - [[P_\mu, Q_a], R] - [Q_a, [P_\mu, R]] \\
&= [P_\mu, \propto Q_a]] - 0 - 0 \\
&= 0
\end{aligned} \tag{8.85}
$$

$$
\begin{aligned}
&[M_{\mu\nu}, [Q_a, R]] - [[M_{\mu\nu}, Q_a], R] - [Q_a, [M_{\mu\nu}, R]] \\
&= [M_{\mu\nu}, \frac{1}{2}\gamma_5 Q_a] - [\Sigma_{\mu\nu} Q_a, R] - 0 \\
&= \frac{1}{2}\gamma_5 \Sigma_{\mu\nu} Q_a - \Sigma_{\mu\nu} \frac{1}{2}\gamma_5 Q_a \\
&= 0
\end{aligned} \tag{8.86}
$$

$$
\begin{aligned}
&[D, [Q_a, R]] - [[D, Q_a], R] - [Q_a, [D, R]] \\
&= [D, \frac{1}{2}\gamma_5 Q_a] - [\frac{1}{2} Q_a, R] - 0 \\
&= \frac{1}{4}\gamma_5 Q_a - \frac{1}{4}\gamma_5 Q_a \\
&= 0
\end{aligned} \tag{8.87}
$$

$$
\begin{aligned}
&[K_\mu, [Q_a, R]] - [[K_\mu, Q_a], R] - [Q_a, [K_\mu, R]] \\
&= [K_\mu, \frac{1}{2}\gamma_5 Q_a] - [-\gamma_\mu S_a, R] - 0 \\
&= -\frac{1}{2}\gamma_5 \gamma_\mu S_a - \gamma_\mu \frac{1}{2}\gamma_5 S_a \\
&= \frac{1}{2}\gamma_\mu \gamma_5 S_a - \frac{1}{2}\gamma_\mu \gamma_5 S_a \\
&= 0
\end{aligned} \tag{8.88}
$$

$$\begin{aligned}
&[Q_a,[Q_b,R]]-[[Q_a,Q_b],R]+[Q_b,[Q_a,R]]\\
&=[Q_a,\frac{1}{2}\gamma_5 Q_b]-[\propto P_\mu,R]+[Q_b,\frac{1}{2}\gamma_5 Q_a]\\
&=\frac{1}{2}(\gamma_5\gamma^\mu)_{ab}P_\mu+\frac{1}{2}(\gamma_5\gamma^\mu)_{ba}P_\mu\\
&=0
\end{aligned}\tag{8.89}$$

$$\begin{aligned}
&[Q_a,[R,R]]-[[Q_a,R],R]-[R,[Q_a,R]]\\
&=0-[\frac{1}{2}\gamma_5 Q_a,R]-[R,\frac{1}{2}\gamma_5 Q_a]\\
&=-\frac{1}{4}\gamma_5\gamma_5 Q_a+\frac{1}{4}\gamma_5\gamma_5 Q_a\\
&=0
\end{aligned}\tag{8.90}$$

$$\begin{aligned}
&[P_\mu,[S_a,R]]-[[P_\mu,S_a],R]-[S_a,[P_\mu,R]]\\
&=[P_\mu,-\frac{1}{2}\gamma_5 S_a]]-[\gamma_\mu Q_a,R]-0\\
&=-\frac{1}{2}\gamma_5\gamma_\mu Q_a-\frac{1}{2}\gamma_\mu\gamma_5 S_a\\
&=\frac{1}{2}\gamma_\mu\gamma_5 Q_a-\frac{1}{2}\gamma_\mu\gamma_5 Q_a\\
&=0
\end{aligned}\tag{8.91}$$

$$\begin{aligned}
&[M_{\mu\nu},[S_a,R]]-[[M_{\mu\nu},S_a],R]-[S_a,[M_{\mu\nu},R]]\\
&=[M_{\mu\nu},-\frac{1}{2}\gamma_5 S_a]]-[\Sigma_{\mu\nu}S_a,R]-0\\
&=-\frac{1}{2}\gamma_5\Sigma_{\mu\nu}S_a+\frac{1}{2}\Sigma_{\mu\nu}\gamma_5 S_a\\
&=0
\end{aligned}\tag{8.92}$$

$$\begin{aligned}
&[D,[S_a,R]]-[[D,S_a],R]-[S_a,[D,R]]\\
&=[D,-\frac{1}{2}\gamma_5 S_a]]-[-\frac{1}{2}S_a,R]-0\\
&=\frac{1}{2}\gamma_5 S_a-\frac{1}{2}\gamma_5 S_a\\
&=0
\end{aligned}\tag{8.93}$$

$$\begin{aligned}
&[K_\mu,[S_a,R]]-[[K_\mu,S_a],R]-[S_a,[K_\mu,R]]\\
&=[K_\mu,\propto S_a]]-0-0\\
&=0
\end{aligned}\tag{8.94}$$

$$
\begin{aligned}
&[Q_a, [S_b, R]] - [[Q_a, S_b], R] + [S_b, [Q_a, R]] \\
&= [Q_a, -\frac{1}{2}\gamma_5 S_b]] - [-(2C_{ab}D - 2(\gamma_5)_{ab}R + (\gamma^{\mu\nu})_{ab}M_{\mu\nu}), R] + [S_b, \frac{1}{2}\gamma_5 Q_a] \\
&= \frac{1}{2}\gamma_5(2C_{ab}D - 2(\gamma_5)_{ab}R + (\gamma^{\mu\nu})_{ab}M_{\mu\nu}) - 0 \\
&+ \frac{1}{2}\gamma_5(2C_{ba}D - 2(\gamma_5)_{ba}R) + (\gamma^{\mu\nu})_{ba}M_{\mu\nu}) \\
&= 0
\end{aligned}
\tag{8.95}
$$

$$
\begin{aligned}
&[S_a, [S_b, R]] - [[S_a, S_b], R] + [S_b, [S_a, R]] \\
&= [S_a, -\frac{1}{2}\gamma_5 S_b]] - [2(\gamma^\mu)_{ab}K_\mu, R] + [S_b, \frac{1}{2}\gamma_5 S_a] \\
&= -\frac{1}{2}(\gamma_5\gamma^\mu)_{ba}K_\mu - [2\gamma^\mu K_\mu, R] + \frac{1}{2}(\gamma_5\gamma^\mu)_{ba}K_\mu \\
&= 0
\end{aligned}
\tag{8.96}
$$

$$
\begin{aligned}
&[S_a, [R, R]] - [[S_a, R], R] - [R, [S_a, R]] \\
&= 0 - [-\frac{1}{2}\gamma_5 S_a, R] - [R, -\frac{1}{2}\gamma_5 S_a] \\
&= -\frac{1}{4}\gamma_5\gamma_5 S_a + \frac{1}{4}\gamma_5\gamma_5 S_a
\end{aligned}
\tag{8.97}
$$

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