University of Stavanger

FACULTY OF SCIENCE AND TECHNOLOGY

MASTER'S THESIS

Study programme/specialisation: Mathematics and Physics - MSc	Spring/ Autumn semester, 20.20..... Open / Confidential
Author: Styrbjørn Juhl Markestad	
Programme coordinator: Anders Tranberg Supervisor(s): Sigbjørn Hervik	
Title of master's thesis: Left-Invariant Pseudo-Riemannian Metrics on Lie Groups	
Credits: 60	
Keywords: Lie groups, Lie algebras, differential geometry, pseudo-Riemannian metrics, invariant theory	Number of pages: 51………………… + supplemental material/other: ………… Stavanger, 10/08/2020……….. date/year

Left-Invariant Pseudo-Riemannian Metrics on Lie Groups

Styrbjørn Juhl Markestad

10/08/2020

Abstract

In differential geometry and mathematical physics, there is interest in left-invariant pseudo-Riemannian metrics on Lie groups. We learn and review Lie theory, representation theory, geometric invariant theory, and differential geometry. We apply this theory to find pseudo-Riemannian metrics for certain Lie groups such that all polynomial curvature invariants are identically zero. We find that the six nilpotent Lie algebras of dimension five can be equipped with pseudo-Riemannian metrics with non-zero curvature such that the Ricci tensor is zero, and all polynomial curvature invariants are identically zero as well. We also find that a class of Lie groups G that can be realized as products or semi-direct products of a Lie group H and $\mathbb{R}^n$ in a certain way can be equipped with pseudo-Riemannian metrics in such a way that all polynomial curvature invariants are identically zero.

Contents

Chapter 1

Introduction

In mathematics, notions of continuous, differentiable symmetry are encoded in mathematical objects called Lie groups. These are smooth manifolds that also carry a group structure compatible with the smooth structure. Any other mathematical object acted upon by the Lie group such that the object is unchanged is said to have a symmetry of the Lie group. For instance, a simple cylinder $\mathbb{R} \times S^1$ carries a circle symmetry by the circle acting on itself in the usual way.

Of key interest in differential geometry and mathematical physics are pseudo-Riemannian manifolds with smooth symmetries. For instance, flat Minkowski space is invariant under translations and Lorentz transformations. As it happens, translations and Lorentz transformations each form Lie groups $\mathbb{R}^4$ and $O(1,3)$, and under semi-direct product, these two groups together form the affine group for Minkowski space, the Poincaré group $\mathrm{ISO}(1,3)$.

If a Lie group acts transitively and faithfully on a set, then the group is in bijection with the set itself. As such, Lie groups encode their own symmetries.

Consider a Lie group G and a topologically closed subgroup H of G. If we form a new topological space from the left cosets gH, $g \in G$, under the quotient topology, then this new space will turn out to be a smooth manifold with a left G-action (although it will not in general be a new Lie group). In other words, the resulting manifold is invariant under the action of G.

Now consider a pseudo-Riemannian space M with metric ρ. Then the set of metric preserving diffeomorphisms,

$$\mathrm{Isom}(M) := \{\varphi\colon M \to M \mid \varphi^*\rho = \rho\},$$

forms a Lie group (results in Ch. IV in [**Pal57**]). If the group acts transitively on the manifold M, then $M \simeq \mathrm{Isom}(M)/H$, where H is the stabilizer subgroup of $\mathrm{Isom}(M)$ for any arbitrary point $p \in M$. For instance, Minkowski space is the homogeneous space $\mathrm{ISO}(1,3)/O(1,3)$. In the special case that $\mathrm{Isom}(M)$ has a Lie subgroup K that acts transitively and faithfully on M, possibly equal to $\mathrm{Isom}(M)$ itself, then K is referred to as simply transitive [**Her10**], respectively, and we are left to study invariant metrics on K itself.

In either case, homogeneous pseudo-Riemannian spaces are model spaces for the local symmetries of more general pseudo-Riemannian spaces, and their study plays a role in topics such as cosmology and quantum gravity. An entry point into this body of mathematical theory is to understand the simply transitive spaces and their metrics, which is the topic of this thesis.

In Chapter 2, we introduce the basic theory of Lie groups in a manner that emphasizes those aspects of the theory that we need in subsequent chapters. In particular, we spend some time discussing left-invariant tensors on Lie groups, and the Maurer-Cartan equation.

In Chapter 3, we go on to introduce basic notions of representation theory and structure theory for Lie groups and Lie algebras, and subsequently give a brief account of some key results from geometric invariant theory that are essential to this thesis.

In Chapter 4, we review the theory of pseudo-Riemannian geometry, discussing the key results relating to curvature.

Finally, in Chapter 5, we put all of this theory to use. To be more specific, we will search for left-invariant pseudo-Riemannian metrics on Lie groups with the special property that all polynomial curvature invariants are identically zero. This stringent condition means that we cannot expect all Lie groups to admit such a metric, and must therefore perform a more targeted search, using results from geometric invariant theory as well as Lie group theory and differential geometry. We obtain some new results that to my knowledge have so far not appeared in the literature. These are theorems 27, 28, and 29.

Chapter 2

Basic Lie Theory

We begin with the basic theory of Lie groups that we need. Any book on Lie groups should contain these results, but here we have used the books [**Kir08**, **Lee13**, **Tu11**, **Tu17**] in particular, as these emphasize a primarily geometric point of view.

2.1 Lie Groups

Definition 1. A Lie group is a smooth (real or complex) manifold with a group structure such that both the multiplication map $\mu\colon G \times G \to G$ and the inverse map $i\colon G \to G$ are smooth maps.

Definition 2. Define left and right translation by a fixed element $g \in G$ to be the maps $l_g\colon G \to G$, $h \mapsto gh$ and $r_g\colon G \to G$, $h \mapsto hg$, respectively.

From the above definitions, it follows that left and right translation by fixed elements of a Lie group G are globally defined diffeomorphisms.

Examples of Lie groups are: $\mathbb{R}^n$ under addition, $\mathbb{R}\backslash\{0\}$ and $S^1 \subset \mathbb{C}$ under multiplication, and — perhaps the canonical example of a Lie group — $GL(V)$, the group of invertible linear transformations of a vector space V (real or complex). Fix a basis and regard $GL(V)$ as a set of invertible matrices. Then $GL(V)$ may be regarded as an open subset of $\mathrm{End}(V) = \mathbb{R}^{n\times n}$, where $n = \mathrm{dim}V$, via the pullback of the determinant: $GL(V) = \det^{-1}(\mathbb{R}\backslash\{0\})$. Here, the determinant function $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous, and in fact smooth, as it is a polynomial of the matrix elements.

Because any finite-dimensional vector space V is isomorphic to either $\mathbb{R}^n$ or $\mathbb{C}^n$ as vector spaces, we often write $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ instead of $GL(V)$, especially if we do not have any particular vector space in mind and wish to explore the group in the abstract.

Other notions from group theory carry over to the Lie group case as well:

Definition 3. A Lie group homomorphism is a group homomorphism $\phi\colon G \to H$ between Lie groups G and H that is also a smooth map.

Definition 4 (Def. 15.8, [**Tu11**])**.** A Lie subgroup of a Lie group G is a subgroup H of G such that H is an immersed manifold under the inclusion map with a Lie group structure inherited from the group operations of G.

This last definition is very general; a more restrictive definition, considering only subgroups which are *embedded* submanifolds — in which case it follows almost immediately that they are Lie groups (see [**Lee13**]) — would be too restrictive, for reasons relating to the correspondence between Lie groups and Lie algebras, to be described later.

The group $GL(V)$ has many interesting subgroups, depending on what kind of vector space V is. A few examples:

- The orthogonal group $O(V)$, the group of invertible linear transformations preserving the euclidean inner product, where V is a real vector space.
- The special linear group $SL(V)$, the group of invertible linear transformations of determinant one, i.e. volume preserving transformations, where V is a real or complex vector space
- The special orthogonal group $SO(V) \coloneqq O(V) \cap SL(V)$, the space of rigid "rotations" of a real vector space V.
- The unitary group $U(V)$, the group of invertible linear transformations preserving the hermitian inner product, where V is a complex vector space.
- The special unitary group $SU(V) \coloneqq U(V) \cap SL(V)$, the space of rigid complex "rotations" of a complex vector space V.

In this thesis we shall be particularly interested in the Lie groups $O(p, q)$, the groups preserving an inner product (or metric) of signature (p, q), where p and q are integers. We look at this Lie group and its associated Lie algebra in Section 4.2.

Theorem 1 (Thm. 20.12, [**Lee13**])**.** *If $H \subset G$ is an abstract subgroup of G which is topologically closed in G, then H is a Lie subgroup of G.*

If $H = \ker \phi$ for some Lie group homomorphism ϕ, then H is a closed subgroup of G, hence a Lie subgroup of G.

If a Lie group has already been given to us, we may investigate its covering spaces. From topology we know that any connected and locally simply connected topological space has a universal covering space [**Lee11**]. In particular, a connected (smooth) manifold M fits this description, so therefore has a universal covering space $\tilde{M}$. Furthermore, this covering space $\tilde{M}$ is itself a smooth manifold:

Proposition 1 (Prop. 4.40, [**Lee13**])**.** *If M is a connected smooth manifold and $q \colon \tilde{M} \to M$ is a topological covering map, then there is a unique smooth structure on $\tilde{M}$ such that $\tilde{M}$ is a smooth manifold and q is a smooth covering map.*

This construction gives rise to new and interesting examples of Lie groups. For example, the universal covering space of $SO(3)$ is $SU(2)$, a double cover of $SO(3)$. There is a one to one correspondence between simply connected Lie groups and Lie algebras, to be discussed in due course.

2.2 Lie Algebras

The fact that left and right translations are diffeomorphisms has interesting consequences. For instance, l_g has inverse $l_{g^{-1}}$, and maps the tangent space T_hG isomorphically to $T_{gh}G$ for any $h \in G$. Thus, the tangent space over any point on G is isomorphic to the tangent space over any other point in G. In particular, the tangent space over the identity element e, T_eG, is isomorphic to T_gG for any $g \in G$. It follows that Lie groups are parallelizable as manifolds, meaning the tangent bundle is a trivial bundle, i.e. $TG = G \times T_eG \simeq G \times \mathbb{R}^n$.

Definition 5. A vector field $X \in \mathfrak{X}(G)$ is said to be left-invariant if it is l_g-related to itself for any $g \in G$, i.e. if $l_{g*}X_h = X_{gh}$.

Note that if $X \in \mathfrak{X}(G)$ is left-invariant, then $X_e = l_{g^{-1}*}X_g$, which means that the left invariant vector fields of G form a vector space over $\mathbb{R}$ (or $\mathbb{C}$) isomorphic to T_eG via the map $\phi\colon X_g \mapsto X_e = l_{g^{-1}*}X_g$. We will adopt the convention of [**Lee13**] and denote this vector space of left-invariant vector fields by the notation $\mathrm{Lie}(G)$.

If if $e_1, \dots, e_n$ is a fixed basis for $\mathfrak{g} \simeq \mathrm{Lie}(G)$, then any vector field on G may be written as a linear combination of these basis fields, i.e. $X \in \mathfrak{X}(G)$ may be written $X_g = X^i(g)e_i$.

Proposition 2 (Prop. 16.9, [**Tu11**]; Prop. 8.33, [**Lee13**])**.** *$[X, Y]$ is a left-invariant vector field on G whenever X and Y are left-invariant vector fields on G.*

By taking the usual Lie bracket of vector fields and restricting to $\mathfrak{g} \simeq \mathrm{Lie}(G)$, regarded as a vector space over $\mathbb{R}$ (or $\mathbb{C}$), we obtain a bilinear product with the same properties as the regular vector bracket defined for any manifold, namely anti-symmetry and the Jacobi identity.

Definition 6. A Lie algebra is a finite-dimensional real (or complex) vector space endowed with an anti-symmetric bilinear product, denoted by $[\cdot, \cdot]$ and referred to as the "bracket product", that satisfies the Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$$

or, equivalently,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

for all $X, Y, Z \in \mathfrak{g}$.

Example 1. If we think of $GL(V)$ as a subspace of $\mathrm{End}(V)$, we may exploit the vector space structure of $\mathrm{End}(V)$ to find $\mathfrak{gl}(V) \coloneqq T_eGL(V)$. The fact that that $\mathrm{End}(V)$ is a vector space means that we may identify the tangent space over any point of $\mathrm{End}(V)$ with itself. Thus, $\mathfrak{gl}(V) = \mathrm{End}(V)$ as a vector space. Let $\gamma\colon \mathbb{R} \to GL(V)$ be a smooth curve in $GL(V)$, with $\gamma(0) = Id$ and $\gamma'(0) = X_{Id}$, where $X \in \mathrm{End}(V)$. We then find that for $A \in GL(V)$, $X_A = (l_A)_{*,Id}X_{Id} = \frac{d}{dt}\Big|_{t=0} A \cdot \gamma(t) = AX_{Id}$. We now wish to compute the bracket operation in $\mathfrak{gl}(V)$, i.e. we wish to find $[X, Y]_{Id}$ for $X, Y \in \mathrm{Lie}(GL(V))$. To do so, we will compute $([X, Y]_{Id})x_{ij} = ([X, Y]x_{ij})_{Id}$, where (x_{ij}) are the standard coordinates for $GL(V) \subset \mathrm{End}(V)$. We do not use Einstein notation for this example. We observe that for $X, Y \in \mathrm{Lie}(GL(V))$,

$$Xx_{ij} = dx_{ij}(X)) = dx_{ij}(AX_{Id}) = \sum A_{ik}(X_{Id})_{kj} = X_{ij},$$

and

$$\begin{aligned} YXx_{ij} = YX_{ij} &= \sum B_{qr}(Y_{Id})_{rs} \frac{\partial}{\partial x_{qr}}\Big|_{Id}(A_{ik}(X_{Id})_{kj}) \\ &= \sum B_{qr}(Y_{Id})_{rs}\delta_{qi}\delta_{sk}(X_{Id})_{kj} \\ &= \sum B_{ir}(Y_{Id})_{rk}(X_{Id})_{kj}. \end{aligned}$$

Thus,

$$\begin{aligned} ([X,Y]_{Id})x_{ij} &= ([X,Y]x_{ij})_{Id} \\ &= ((XY - YX)x_{ij})_{Id} \\ &= \sum (A_{ir}(X_{Id})_{rk}(Y_{Id})_{kj})_{Id} - (B_{ir}(Y_{Id})_{rk}(X_{Id})_{kj})_{Id}, \\ &= \sum \delta_{ir}(X_{Id})_{rk}(Y_{Id})_{kj} - \delta_{ir}(Y_{Id})_{rk}(X_{Id})_{kj} \\ &= \sum (X_{Id})_{ik}(Y_{Id})_{kj} - (Y_{Id})_{ik}(X_{Id})_{kj} \end{aligned}$$

and we see that in $\mathfrak{gl}(V)$, $[X,Y] = XY - YX$, where $X, Y \in \mathfrak{gl}(V)$ and XY and YX denotes matrix multiplication.

Definition 7. A vector subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a Lie subalgebra of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A vector subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an ideal of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

Note that if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra. This need not be the case if $\mathfrak{h}$ was merely a Lie subalgebra.

Definition 8. A vector space homomorphism $T\colon \mathfrak{g} \to \mathfrak{h}$ is called a Lie algebra homomorphism if $T[X,Y] = [TX, TY]$ for all $XY \in \mathfrak{g}$.

Proposition 3 (Theorem 8.44, [**Lee13**])**.** *If $F\colon G \to H$ is a Lie group homomorphism, then — identifying $\mathfrak{g}$ with Lie(G), and $\mathfrak{h}$ with Lie(H) — $F_*\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. Since F is a Lie group homomorphism,

$$F \circ l_g(h) = F(gh) = F(g)F(h) = l_{F(g)} \circ F(h),$$

for arbitrary $h \in G$, hence $F \circ l_g = l_{F(g)} \circ F$ and

$$F_* \circ (l_g)_* = (l_{F(g)})_* \circ F_*.$$

Thus, keeping in mind that $F(e) = e' \in H$, we have

$$F_{*,g}(X_g) = F_{*,e} \circ (l_g)_{*,e}(X_e) = (l_{F(g)})_{*,e'} \circ F_{*,e}(X_e) = (l_{F(g)})_{*,e'}(Z_{e'}) = Z_{F(g)}.$$

Identifying $\mathfrak{g}$ with Lie(G) as before, and with a mild abuse of notation, we see that $X \in \mathfrak{g}$ is F-related to $F_*(X) = Z \in \mathfrak{h}$. From here, $F_*[X,Y] = [F_*X, F_*Y]$ follows from the naturality of the Lie bracket for vector fields.

$\square$

Theorem 2. *If the kernel* $\ker\Phi$ *of a Lie group homomorphism* $\Phi\colon G \to H$ *is discrete, then* $\Phi_*\colon \mathfrak{g} \to \mathfrak{h}$ *is an injective Lie algebra homomorphism.*

Proof. Since the kernel of a Lie group homomorphism $\Phi\colon G \to H$ is a closed subgroup of the original Lie group G, it is a Lie subgroup $\ker\Phi \subset G$. If the kernel is discrete, then by definition of a closed set, it is closed if and only if it has no limit points in G. We can therefore find a neighbourhood U of $e \in G$ such that e is the only point in U that is also in the kernel of Φ, and a further restricted neighbourhood $V \subset U$ of e such that $gg'^{-1} \in U$ for any $g, g' \in V$. We see that if $g, g' \in V$ then $\Phi(g) = \Phi(g')$ implies $\Phi(gg'^{-1}) = e'$, which in turn implies that $g = g'$ since V contains no element in the kernel of Φ other than e. Thus, Φ is injective restricted to V, hence $\Phi_*\colon \mathfrak{g} \to \mathfrak{k}$ is also injective. $\square$

Theorem 3 (Prop. 21.28; [**Lee13**])**.** *Any discrete subgroup* H *of a Lie group* G *is a closed subgroup.*

Proposition 3 implies that if $H \subset G$ is a Lie subgroup, then $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, via the inclusion map $\iota\colon H \hookrightarrow G$. Conversely, we have the following theorem:

Theorem 4 (Theorem 19.26, [**Lee13**])**.** *For a Lie group* G *with Lie algebra* $\mathfrak{g}$ *and subalgebra* $\mathfrak{h} \subset \mathfrak{g}$*, there is a unique connected Lie subgroup* $H \subset G$ *whose Lie algebra is* $\mathfrak{h}$*.*

Note. The proof of Theorem 4 makes use of the theory of distributions: Pushing $\mathfrak{h}$ around on G defines an involutive distribution of TG, which may be integrated to obtain a foliation of G. The leaf of the foliation containing $e \in G$ will be the Lie subgroup H; H is an integral manifold of $\mathfrak{h}$. This is the reason why the definition for Lie subgroups encompasses immersed submanifolds: The leaves of a foliation are not in general embedded submanifolds, although they are weakly embedded (see [**Lee13**]).

The above theory tells us that the Lie subgroups of $GL(V)$ have Lie subalgebras in $\mathfrak{gl}(V)$.

Theorem 5 (Ado's Theorem; Thm. E.4, p. 501 [**FH04**])**.** *Every Lie algebra* $\mathfrak{g}$ *has an injective (faithful) representation* $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ *for some finite-dimensional vector space* V*.*

Combining the results at the end of Section 2.1 with Ado's Theorem (5) and Theorem (4), we see that every Lie algebra $\mathfrak{g}$ has associated to it a universal covering group: Simply integrate $\mathfrak{g} \subset \mathfrak{gl}(V)$ to obtain a group $G \subset GL(V)$, and find the universal covering group $\tilde{G}$ of G.

A smooth manifold M is called a homogeneous space if there is a Lie group G acting smoothly and transitively on M.

Theorem 6 (Thm. 21.17, 21.26; [**Lee13**])**.** *Let* G *be a Lie group with a closed subgroup* H*. Then* G/H *is a homogeneous smooth manifold, with a left action by* G *on* G/H *given by* $g' \cdot gH = (g'g) \cdot H$*, and the quotient map* $\pi\colon G \to G/H$ *is a smooth submersion. If additionally* H *is normal as a subgroup, then* G/H *is a Lie group, and the quotient map* $\pi\colon G \to G/H$ *is a surjective Lie group homomorphism with kernel* H*.*

Theorem 7 (Thm. 21.18; [**Lee13**])**.** *Let* G *be a Lie group acting on a homogeneous space* M*. Then the stabilizer group* G_p *for any point* p *is a closed subgroup of* G*, and is isomorphic to the stabilizer group* $G_{p'}$ *of any other point* p'*. Moreover,* $M \simeq G/G_p$ *via the map* $gG_p \mapsto g \cdot p$*.*

2.3 Left-Invariant Tensor Fields and Maurer-Cartan

In the same way that TG is parallelizable, so too is T^*G for the same reason: $T^*_gG \simeq T^*_eG$ via the map $l^*_g\colon T^*_gG \to T^*_eG$, therefore $T^*G = G \times T^*_eG \simeq G \times \mathbb{R}^{n*} \simeq G \times \mathbb{R}^n$.

Let $\varphi \in \Gamma(T^*G)$ be a 1-form on G. We say that φ is left-invariant if $l^*_g\varphi = \varphi$ for all $g \in G$. Note that for $X \in \mathfrak{X}(G)$, $\varphi_g(X_g) = (l^*_{g^{-1}}\varphi_e)(X_g) = \varphi_e(l_{g^{-1}*}X_g)$, and if $X \in \mathrm{Lie}(G) \subset \mathfrak{X}(X)$, then $\varphi_e(l_{g*}X_g) = \varphi_e(X_e)$ is constant over G. Therefore, the $\mathbb{R}$-vector space (or $\mathbb{C}$-vector space) of left-invariant 1-forms is the dual space to $\mathrm{Lie}(G) \simeq \mathfrak{g}$, *regarded as a vector space.* Denote this space by $\mathrm{Lie}(G)^* \simeq \mathfrak{g}^*$. Just as any vector field on G may be written $X = X^i(g)e_i$, where $\{e_1, \ldots, e_n\}$ is a basis of left-invariant vector fields on G, any covector field on G may be written $\varphi = \varphi_i(g)e^i$, where $\{e^1, \ldots, e^n\}$ is the dual basis of left-invariant covector fields on G associated to $\{e_1, \ldots, e_n\}$.

More generally, $T^{(k,l)}_gG \simeq T^{(k,l)}_eG$ via the map $l^*_g\colon T^{(k,l)}_gG \to T^{(k,l)}_eG$ defined by

$$T^{i_1\ldots i_k}_{j_1\ldots j_l}(g)e_{i_1}\otimes\cdots\otimes e_{i_k}\otimes e^{j_1}\otimes\cdots\otimes e^{j_l} \mapsto T^{i_1\ldots i_k}_{j_1\ldots j_l}(e)(l_{g^{-1}*}e_{i_1})\otimes\cdots\otimes(l_{g^{-1}*}e_{i_k})\otimes(l^*_ge^{j_1})\otimes\cdots\otimes(l^*_ge^{j_l}),$$

for an arbitrary local frame and corresponding coframe, and so we may define left-invariant tensor fields by the formula $l^*_gT = T$. By the same calculations performed in the case of vector and covector fields, we likewise see that left-invariant tensor fields are determined by their value at the origin, and that the coefficients of a left-invariant tensor field T are constant with respect to a left-invariant basis. From this, we see that we may conceive of left-invariant tensors as elements in tensor products of $\mathfrak{g}$,

$$\mathfrak{g}^{(k,l)} := \overbrace{\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}}^{k}\otimes\overbrace{\mathfrak{g}^*\otimes\cdots\otimes\mathfrak{g}}^{l}.$$

We may conceive of the bracket product on a Lie algebra $\mathfrak{g}$ as a tensor field on any Lie group G that has $\mathfrak{g}$ as its Lie algebra. Fix a basis $e_1, \ldots, e_n$ with dual basis $e^1, \ldots, e^n$ for $\mathfrak{g} \simeq \mathrm{Lie}(G)$ and let $C^k_{ij} = e^k([e_i, e_j])$. As previously discussed, any vector field on G may be written $X = X^i(g)e_i$, so we may define a "bracket tensor" $C \in \Gamma(T^{(1,2)}G)$ by the formula $C(X,Y) = X^i(g)Y^j(g)C^k_{ij}e_k$. For any e_i, e_j, e_k in a left-invariant basis, the Jacobi identity then becomes $C^q_{il}C^l_{jk} + C^q_{jl}C^l_{ki} + C^q_{kl}C^l_{ij} = 0$. This construction only works because of the existence of left-invariant vector fields: For arbitrary vector fields $X, Y \in \mathfrak{X}(G)$, the bracket of vector fields is not $C^\infty(G)$-linear.

Given a left-invariant covector φ, its exterior derivative $d\varphi$ will also be left-invariant. We need the following result:

Proposition 4 (Prop. 20.13, [**Tu11**]). *For any smooth 1-form φ and smooth $X, Y \in \mathfrak{X}(G)$, the formula*

$$d\varphi(X,Y) = X\varphi(Y) - Y\varphi(X) - \varphi([X,Y]) \tag{2.1}$$

holds on any smooth manifold.

This will give us the following result:

Proposition 5 (Prop. 7.2, p. 137, [**Hel78**])**.** *If φ is a left-invariant covector field on a Lie group G, then $d\varphi$ is a left-invariant 2-form determined by the Maurer-Cartan equation*

$$d\varphi(X,Y) = -\varphi([X,Y]). \tag{2.2}$$

In particular, if $\{e_1,\ldots,e_n\}$ is basis of left-invariant vector fields and $\{e^1,\ldots,e^n\}$ its cobasis, then

$$de^i = -\frac{1}{2}C^i_{jk}e^j \wedge e^k, \tag{2.3}$$

where C^i_{jk} are defined by the formula $C^k_{ij} = e^k([e_i,e_j])$.

Proof. $d\varphi$ is a tensor, and is therefore determined pointwise. Thus, it suffices to compute $d\varphi_p(X_p,Y_p))$ for arbitrary points p in the manifold. Since φ is left-invariant on a Lie group G, we may compute $d\varphi_p(X_p,Y_p)$ by extending X_p,Y_p to left-invariant vector fields on all of G, and then using formula (2.1). Then the terms $X\varphi(Y)$ and $Y\varphi(X)$ in (2.1) vanish, and we are left with the equation $d\varphi(X,Y) = -\varphi([X,Y])$.

Furthermore, with respect to a basis $\{e_1,\ldots,e_n\}$ and cobasis $\{e^1,\ldots,e^n\}$ for $\mathfrak{g}$,

$$\begin{aligned}(d\varphi)_{jk} &= d\varphi(e_j,e_k)\\ &= -\varphi([e_j,e_k])\\ &= -\varphi(C^i_{jk}e_i)\\ &= -C^i_{jk}\varphi_i,\end{aligned} \tag{2.4}$$

hence $d\varphi = (d\varphi)_{jk}e^j \otimes e^k = -\frac{1}{2}C^i_{jk}\varphi_i e^j \wedge e^k$. In particular, $de^i = -\frac{1}{2}C^i_{jk}e^j \wedge e^k$. Since $d\varphi$ has constant components relative to a left-invariant cobasis, $d\varphi$ is also left-invariant. □

2.4 The Exponential Map

The isomorphism $\phi\colon \mathfrak{g} \to \mathrm{Lie}(G)$ implies that there is a map between $\mathfrak{g}$ and G. Let $\varphi_X : \mathbb{R} \times G \to G$ be the flow of $X \in \mathrm{Lie}(G)$. Then there is a well defined map $X_e \mapsto \varphi_{\phi(X_e)}(1,e) = \exp(X_e)$ from $\mathfrak{g}$ to G, called the exponential map.

Proposition 6 (Prop. 15.9, [**Tu17**])**.**

1. *The integral curve of $X \in Lie(G)$ starting at $g \in G$ is $g\exp(tX_e)$.*

2. *For a fixed $X_e \in \mathfrak{g}$, the map $t \mapsto \exp(tX_e)$ is a Lie group homomorphism.*

3. $\exp\colon \mathfrak{g} \to G$ *is C^∞.*

4. *The pushforward of* exp *at* 0, $\exp_{*,0}\colon T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$ *is the identity map.*

5. *For $X \in \mathfrak{gl}(V)$,*

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

By the inverse function theorem, item 4 in Proposition 6 in particular means that there exists a neighbourhood U of 0 in $\mathfrak{g}$ which maps diffeomorphically onto a neighbourhood $\exp(U)$ of e in G. Item 5 in Proposition 6 explains why the exponential map has the name that it has.

Theorem 8 (Thm. 15.12, [**Tu17**])**.** *If $F\colon G \to H$ is a Lie group homomorphism, then $F \circ \exp = \exp \circ F_*$.*

Proof. Fix $X \in \mathfrak{g}$ and let $\gamma(t) := t \mapsto F \circ \exp(tX)$ for $X \in \mathfrak{g}$. Then

$$\gamma'(t) = F_* \circ (l_{\gamma(t)})_*(X) = (l_{F(\gamma(t))})_* \circ F_*(X) = (l_{F(\gamma(t))})_* \circ Y,$$

for $Y = F_*X \in \mathfrak{h}$. Therefore, $\gamma(t)$ is an integral curve of the left-invariant vector field that is F-related to X, with $\gamma(0) = e'$, and so by the uniqueness of integral curves is equal to $\exp(tY) = \exp(tF_*(X))$. Setting $t = 1$, we obtain our result.

□

Theorem 9 (Thm. 1.6.1; [**DK00**])**.** *For some neighbourhood $U \subset \mathfrak{g}$ of 0, the expression*

$$\exp(X)\exp(Y) = \exp(\mu(X,Y))$$

holds for $X, Y \in U$, where $\mu\colon U \times U \to \mathfrak{g}$ is a real or complex analytic function whenever $\mathfrak{g}$ is a real or complex Lie algebra, respectively.

Theorem 9 ultimately means that real Lie groups are in fact analytic (Thm. 1.6.3; [**DK00**]). Complex Lie groups are by definition holomorphic (complex analytic), so Theorem 9 merely reaffirms this. The function μ has a series expansion known as the Campbell-Baker-Hausdorff formula, or the Dynkin formula, depending on source.

Chapter 3

Representation Theory

We account for some of the basic theory of representations of Lie groups and Lie algebras. General theory can be found in the books [**GW10**, **Kir08**, **Kna05**, **Lee13**, **Sep10**, **Wal18**], which we have used here in particular.

3.1 Representations and Actions

Although actions and representations can be defined for any group, we only need the theory for Lie groups and algebras, and finite-dimensional vector spaces.

Definition 9 ([**Lee13**]). A left Lie group action on a smooth manifold M is a smooth mapping $\varphi\colon G \times M \to M$, $G \times M \ni (g,p) \mapsto g \cdot p \in M$, such that $g \cdot (h \cdot p) = (gh) \cdot p$ and $e \cdot p = p$.

A right Lie group action on a smooth manifold M is a smooth mapping $\varphi\colon M \times G \to M$, $M \times G \ni (p,g) \mapsto p \cdot g \in M$, such that $(p \cdot h) \cdot g = p \cdot (hg)$ and $p \cdot e = p$.

Any left Lie group action gives rise to a right Lie group action by setting $p \cdot g \coloneqq g^{-1} \cdot p$, and any right Lie group action gives rise to a left Lie group action by setting $g \cdot p \coloneqq p \cdot g^{-1}$.

Definition 10. We define the orbit of a point $p \in M$ in a manifold under the action of a Lie group G to be the set

$$G \cdot p \coloneqq \{p' \in M \mid p' = g \cdot p \text{ for some } g \in G\},$$

and the stabilizer group of a point $p \in M$ to be the set

$$G_p \coloneqq \{g \in G \mid g \cdot p = p\}. \tag{3.1}$$

It is easy to see that the orbits of a manifold M under the action of a Lie group G define equivalence classes.

The stabilizer group is clearly an abstract subgroup, and it is also topologically closed in its parent Lie group, as it is equal to the set $G_p = F_p^{-1}(p)$, where F_p is the map defined by $F_p(g) = g \cdot p$. By the closed subgroup theorem (Theorem 1), G_p is a Lie subgroup of G if the action of G on M is transitive.

Theorem 10 (Thm. 20.15 and 20.18, [**Lee13**])**.** *Any left or right Lie group action φ on a manifold M gives rise to a map $\hat{\varphi}\colon \mathfrak{g} \to \mathfrak{X}(M)$ by the formula*

$$\hat{X}_p = \left.\frac{d}{dt}\right|_{t=0} \exp tX \cdot p \tag{3.2}$$

for a left action, and the formula

$$\hat{X}_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp tX \tag{3.3}$$

for a right action, such that $\hat{\varphi}$ is a Lie algebra anti-homomorphism or Lie algebra homomorphism, for left and right actions respectively.

Definition 11. A Lie group representation is a Lie group homomorphism $\Phi\colon G \to GL(V)$ for some finite-dimensional vector space V over either the real or the complex numbers.

Similarly, a Lie algebra representation is a Lie algebra homomorphism $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ for some finite-dimensional vector space V over either the real or the complex numbers.

We see that a Lie group representation is a special case of a Lie group action, where the space acted upon is a vector space (of finite dimension in our case) and the action itself is linear.

By Proposition 3, to any Lie group representation there is an associated Lie algebra representation. This is obtained by taking the pushforward (or differential) of the Lie group homomorphism from G to $GL(V)$, as per Proposition 3.

3.2 Linear Actions

The Lie group $GL(V)$ and its Lie subgroups have canonical, or defining, representations, and any other representations of these groups may be obtained by conjugation with an element of $GL(V)$, which may be regarded as a change of basis for the underlying vector space.

We may use the universal property of tensor products to find the canonical actions of $GL(V)$ and $\mathfrak{gl}(V)$ on symmetric and alternating products of vector spaces and their dual spaces.

Proposition 7. *If G acts on a vector space V through the Lie group representation $\Phi\colon G \to GL(V)$, and on W through the Lie group representation $\Psi\colon G \to GL(W)$, then:*

- *G acts on V^* through the Lie group representation $\Phi^*\colon G \to GL(V^*)$ defined by $g \mapsto \Phi(g^{-1})^*$.*
- *G acts on $V \oplus W$ through the Lie group representation $\Phi \oplus \Psi\colon G \to GL(V \oplus W)$ defined by $g \mapsto \Phi(g) \oplus \Psi(g)$.*
- *G acts on $V \otimes W$ through the Lie group representation $\Phi \otimes \Psi\colon G \to GL(V \otimes W)$ defined by $g \mapsto \Phi(g) \otimes \Psi(g)$.*

- *G acts on $V \odot W$ through through the Lie group representation $\Phi \odot \Psi \colon G \to GL(V \odot W)$ defined by $g \mapsto \Phi(g) \odot \Psi(g)$.*

- *G acts on $V \wedge W$ through through the Lie group representation $\Phi \wedge \Psi \colon G \to GL(V \wedge W)$ defined by $g \mapsto \Phi(g) \wedge \Psi(g)$.*

Proposition 8. *If $\mathfrak{g}$ acts on a vector space V through the Lie algebra representation $\phi \colon \mathfrak{g} \to \mathfrak{gl}(V)$, and on W through the Lie algebra representation $\psi \colon \mathfrak{g} \to \mathfrak{gl}(W)$, then:*

- *$\mathfrak{g}$ acts on V^* through the Lie algebra representation $\phi^* \colon \mathfrak{g} \to \mathfrak{gl}(V^*)$ defined by $X \mapsto -\phi(X)^*$.*

- *$\mathfrak{g}$ acts on $V \oplus W$ through the Lie algebra representation $\phi \oplus \psi \colon \mathfrak{g} \to \mathfrak{gl}(V \oplus W)$ defined by $X \mapsto \phi(X) \oplus \psi(X)$.*

- *$\mathfrak{g}$ acts on $V \otimes W$ through the Lie algebra representation $\phi \otimes \psi \colon \mathfrak{g} \to \mathfrak{gl}(V \otimes W)$ defined by $X \mapsto \phi(X) \otimes Id(X) + Id(X) \otimes \psi(X)$.*

- *$\mathfrak{g}$ acts on $V \odot W$ through through the Lie algebra representation $\phi \odot \psi \colon \mathfrak{g} \to \mathfrak{gl}(V \odot W)$ defined by $X \mapsto \phi(X) \odot Id(X) + Id(X) \odot \psi(X)$.*

- *$\mathfrak{g}$ acts on $V \wedge W$ through through the Lie algebra representation $\phi \wedge \psi \colon \mathfrak{g} \to \mathfrak{gl}(V \wedge W)$ defined by $X \mapsto \phi(X) \wedge Id(X) + Id(X) \wedge \psi(X)$.*

Definition 12. A representation V of a Lie group G or Lie algebra $\mathfrak{g}$ is said to have a subrepresentation $W \subset V$ if $G \cdot W \subset W$ or $\mathfrak{g} \cdot W \subset W$, respectively. If a representation has a subrepresentation, it is said to be reducible.

Definition 13. A representation V of a Lie group G or Lie algebra $\mathfrak{g}$ is said to be irreducible if it has no subrepresentations other than 0 or V itself. A reducible representation V of a Lie group G or Lie algebra $\mathfrak{g}$ is said to be completely reducible if it can be decomposed into a finite direct sum $V = \bigoplus_{k=1}^{n} V_k$ of irreducible representations V_k.

For $GL(V)$ or $\mathfrak{gl}(V)$ acting on V, we may either regard the action as being a left action on the coefficients of any vector $v \in V$ expressed in a fixed basis, or we may regard the action as being a right action on the basis in which a vector is expressed leaving the coefficients fixed. By the results above, this notion may be extended to V^* and arbitrary tensor products of V and V^*.

We may look at the action of $GL(V)$ on $V \otimes V^* \otimes V^*$, and see that if we restrict our attention to those elements $C \in V \otimes V^* \otimes V^*$ that satisfy the Jacobi identity (see Section 2.3) then the orbits of these C in $V \otimes V^* \otimes V^*$ under the action of $GL(V)$ are equivalence classes of Lie algebras. This motivates the following definition.

Definition 14. If V is a vector space over a field $\mathbb{K}$, real or complex, of dimension n, we define the set of Lie algebra structure coefficients of dimension n to be the set

$$\mathfrak{G}(n, \mathbb{K}) \coloneqq \{C \in V \otimes V^* \otimes V^* \mid C \text{ satisfies the Jacobi identity}\}.$$

The set $\mathfrak{G}(n, \mathbb{K})/GL(n, \mathbb{K})$ is then the moduli space of Lie algebras.

We may define the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ to be the subgroup

$$\mathrm{Aut}(\mathfrak{g}) \coloneqq \{g \in GL(\mathfrak{g}) \mid g \cdot [X,Y] = [g \cdot X, g \cdot Y] \text{ for any } X, Y \in \mathfrak{g}\} \tag{3.4}$$

of $GL(\mathfrak{g})$, which, if $\dim \mathfrak{g} = n$, is equivalent to the stabilizer subgroup $GL(n,\mathbb{R})_C \subset GL(n,\mathbb{R})$ of $C \in \mathfrak{G}$. As it is a stabilizer subgroup, it is a closed Lie group.

3.3 Adjoint Representation

There is one representation that is of central importance to the study of Lie groups and Lie algebras, and that is the adjoint representation. Let $c_g\colon G \to G$ denote the conjugation map $h \mapsto ghg^{-1}$, for any $g \in G$. Then c_g is a Lie group automorphism that maps the identity element e to itself. The adjoint map assigns to the element $g \in G$ the pushforward (or differential) of C_g at the identity:

$$\mathrm{Ad}(g) = (c_g)_{*,e}\colon \mathfrak{g} \to \mathfrak{g}.$$

The map $(c_g)_*$ preserves Lie brackets. We have that $\mathrm{Ad}(gh) = (c_{gh})_{*,e} = (c_g \circ c_h)_{*,e} = (c_g)_{*,e} \circ (c_h)_{*,e} = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$, and moreover that the adjoint map is smooth (Prop. 15.14, [**Tu17**]), so the adjoint map is a Lie group homomorphism $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$, called the adjoint representation of the Lie group G.

We may now take the pushforward of the adjoint representation at the identity to obtain the adjoint representation of the Lie algebra $\mathrm{ad} \coloneqq \mathrm{Ad}_{*,e}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$, where $\mathrm{Der}(\mathfrak{g}) = \{A \in \mathfrak{gl}(\mathfrak{g}) \mid A \cdot [X,Y] = [A \cdot X, Y] + [X, A \cdot Y],\ X, Y \in \mathfrak{g}\}$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$. We write $\mathrm{ad}(X)(Y)$ or $\mathrm{ad}_X Y$ to indicate $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ acting on $Y \in \mathfrak{g}$.

Proposition 9 (Prop. 15.15, [**Tu17**])**.** *$ad_X Y = [X,Y]$ for $X, Y \in \mathfrak{g}$.*

We have that $\ker \mathrm{Ad} = \{g \in G \mid (c_g)_{*,e}(X) = \mathrm{Id}\}$, with $Z(G) = \{g \in G \mid c_g(h) = ghg^{-1} = h\} \subset \ker \mathrm{Ad}$ and $Z(G) = \ker \mathrm{Ad}$ if G is connected. $Z(G)$ is a closed, normal Lie subgroup of G. This in turn tells us that the Lie algebra of $Z(G)$ is $\ker \mathrm{ad} = \{X \in \mathfrak{g} \mid [X,Y] = 0,\ \forall Y \in \mathfrak{g}\}$, by Proposition 9. We denote this algebra $\mathfrak{z}(\mathfrak{g}) \coloneqq \ker \mathrm{ad}$. It may happen that $Z(G)$ is discrete, in which case $\mathfrak{z}(\mathfrak{g}) = 0$.

3.4 Structure Theory

Definition 15. A Lie algebra $\mathfrak{g}$ is called abelian, or commutative, if $[\mathfrak{g},\mathfrak{g}] = 0$.

Definition 16. For a Lie algebra $\mathfrak{g}$, define $\mathfrak{g}^0 = \mathfrak{g}_0 \coloneqq \mathfrak{g}$. Define recursively the derived series by

$$\mathfrak{g}^{i+1} \coloneqq [\mathfrak{g}^i, \mathfrak{g}^i],$$

and the lower central series by

$$\mathfrak{g}_{i+1} \coloneqq [\mathfrak{g}, \mathfrak{g}_i].$$

Note that an abelian Lie algebra is also nilpotent.

Definition 17. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called an ideal if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

The kernel of any Lie algebra homomorphism $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ is an ideal: if $X \in \mathfrak{g}$ and $Y \in \ker \phi$, then $\phi([X, Y]) = [\phi(X), \phi(Y)] = [\phi(X), 0] = 0$, and so $[X, Y] \in \ker \phi$.

Proposition 10. *The Lie bracket of two ideals $\mathfrak{k}$ and $\mathfrak{l}$ of $\mathfrak{g}$ is again an ideal. Likewise, $\mathfrak{k}+\mathfrak{l}$ and $\mathfrak{k} \cap \mathfrak{l}$ are ideals.*

Proof. Suppose $\mathfrak{k}$ and $\mathfrak{l}$ are ideals in $\mathfrak{g}$.

Let $X \in \mathfrak{g}$, $Y \in \mathfrak{k}$, and $Z \in \mathfrak{l}$. Then $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$. $[Y, X] \in \mathfrak{k}$ and $[X, Z] \in \mathfrak{l}$, hence $[[X, Y], Z] \in [\mathfrak{k}, \mathfrak{l}]$ and $[Y, [X, Z]] \in [\mathfrak{k}, \mathfrak{l}]$.

Let $X \in \mathfrak{g}$ and $Y \in (\mathfrak{k} + \mathfrak{l})$, meaning Y is either in $\mathfrak{k}$ or in $\mathfrak{l}$. Then $[X, Y]$ is in $\mathfrak{k}$ or in $\mathfrak{l}$, and so is in $\mathfrak{k} + \mathfrak{l}$.

Let $X \in \mathfrak{g}$ and $Y \in \mathfrak{k} \cap \mathfrak{l}$. Then $[X, Y]$ is in $\mathfrak{k}$ and in $\mathfrak{l}$, and so is in $\mathfrak{k} \cap \mathfrak{l}$. □

Definition 18. A Lie algebra $\mathfrak{g}$ is called solvable if the derived series terminates, and nilpotent if the lower central series terminates.

Proposition 11. *A nilpotent Lie algebra $\mathfrak{g}$ is solvable.*

Proof. Suppose $\mathfrak{g}$ is nilpotent. Then $\mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0] = [\mathfrak{g}, \mathfrak{g}^0] = [\mathfrak{g}, \mathfrak{g}_0] = \mathfrak{g}_1$, such that $\mathfrak{g}^0 \subset \mathfrak{g}_0$ trivially. We proceed by induction and assume that $\mathfrak{g}^i \subset \mathfrak{g}_i$ for all $i \leq k$. Then $\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k] \subset [\mathfrak{g}, \mathfrak{g}^k] \subset [\mathfrak{g}, \mathfrak{g}_k] = \mathfrak{g}_{k+1}$. Since $\mathfrak{g}$ is nilpotent, there exists some $k \leq 0$ such that $\mathfrak{g}^k \subset \mathfrak{g}_k = 0$, hence $\mathfrak{g}$ is solvable. □

Definition 19. A Lie algebra $\mathfrak{g}$ is called semisimple if it does not contain any nonzero solvable ideals, and simple if it does not contain any other ideals than $\mathfrak{g}$ itself and 0.

For a simple Lie algebra $\mathfrak{g}$, it must be the case that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and of course $[0, 0] = 0$, so $\mathfrak{g}$ is the only nonzero ideal. Clearly, the derived series does not terminate, so $\mathfrak{g}$ is semisimple.

Definition 20. A bilinear form m on a vector space V is said to be symmetric if $m(u, v) = m(v, u)$ for any $u, v \in V$.

Definition 21. A bilinear form m on a vector space V is said to be nondegenerate if, for a fixed $u \in V$ and all $v \in V$, $m(u, v) = 0$ implies that $u = 0$.

Definition 22. Let V be a vector space equipped with a nondegenerate bilinear form m and $W \subset V$ be a subspace. Then the vector space

$$W^{\perp} := \{u \in V \mid m(u, v) = 0 \text{ for all } v \in W\} \tag{3.5}$$

is called the m-orthogonal (or just orthogonal) complement to W.

Proposition 12. *For a finite-dimensional vector space V, nondegeneracy of a bilinear form m implies that the map $m^\flat\colon V \to V^*$ defined by $u \mapsto m(u, -)$ is an isomorphism. We denote the inverse to $m^\flat$ by $m^\sharp$.*

Proof. We examine $\ker m^\flat = V^\perp = \{u \in V \mid m(u,v) = 0 \text{ for all } v \in V\}$. By nondegeneracy, the only $u \in V$ satisfying the condition $m(u,v) = 0$ for all $v \in V$ is $u = 0$. Thus, the map $m^\flat$ is injective. Since $\dim V = \dim V^*$, $m^\flat$ is an isomorphism. □

A nondegenerate bilinear form m thus has an inverse (or dual) bilinear form $\tilde{m}$ associated to it, defined by the condition $\tilde{m}(\alpha, \beta) \coloneqq m(m^\sharp(\alpha), m^\sharp(\beta))$.

Definition 23. The Killing form of a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form $K_\mathfrak{g}$ defined by

$$K_\mathfrak{g}(X, Y) \coloneqq \operatorname{tr}(\mathrm{ad}_X \circ \mathrm{ad}_X),$$

for $X, Y \in \mathfrak{g}$.

Proposition 13. *Let $\mathfrak{g}$ be a Lie algebra, $K_\mathfrak{g}$ its Killing form, and $\mathfrak{h} \subset \mathfrak{g}$ any ideal in $\mathfrak{g}$. Then the orthogonal complement with respect to $K_\mathfrak{g}$, $\mathfrak{h}^\perp$, is also an ideal in $\mathfrak{g}$.*

Proof. By the Jacobi identity,

$$\mathrm{ad}_{[X,Y]}Z = [[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]] = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X = [\mathrm{ad}_X, \mathrm{ad}_Y].$$

Hence,

$$\begin{aligned} K_\mathfrak{g}([X,Y],Z) &= \operatorname{tr}(\mathrm{ad}_{[X,Y]} \circ \mathrm{ad}_Z) = \operatorname{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \circ \mathrm{ad}_Z - \mathrm{ad}_Y \circ \mathrm{ad}_X \circ \circ \mathrm{ad}_Z) \\ &= \operatorname{tr}(\mathrm{ad}_Y \circ \circ \mathrm{ad}_Z \circ \mathrm{ad}_X - \mathrm{ad}_Y \circ \mathrm{ad}_X \circ \circ \mathrm{ad}_Z) \\ &= -\operatorname{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_{[X,Z]}) \\ &= -K_\mathfrak{g}(Y, [X,Z]). \end{aligned}$$

Thus, for $[\mathfrak{g}, \mathfrak{h}^\perp] \subset \mathfrak{h}^\perp$ to be true, we must have $K_\mathfrak{g}([X,Y],Z) = 0$ for any $X \in \mathfrak{g}$, $Y \in \mathfrak{h}^\perp$, and $Z \in \mathfrak{h}$. But $K_\mathfrak{g}([X,Y],Z) = -K_\mathfrak{g}(Y,[X,Z])$, and $[X,Z] \in \mathfrak{h}$ since $\mathfrak{h}$ is an ideal, hence $K_\mathfrak{g}(Y,[X,Z]) = 0$ by definition of $\mathfrak{h}^\perp$.

□

Theorem 11 (Thm. 5.34, [**Kir08**])**.** *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if for every $X \in \mathfrak{g}$, ad_X is a nilpotent operator.*

Theorem 12 (Cartan's criterion for solvability; Thm. 5.52, [**Kir08**])**.** *A Lie algebra $\mathfrak{g}$ is solvable if and only if $K_\mathfrak{g}([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) \equiv 0$.*

Theorem 13 (Cartan's criterion for semisimplicity; Thm. 5.53, [**Kir08**])**.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if $K_\mathfrak{g}$ is nondegenerate.*

We shall also refer to Lie groups as nilpotent, solvable, and semisimple if their Lie algebras are nilpotent, solvable, or semisimple, respectively.

3.5 Root and Weight Decompositions

Proposition 14 (Cor. 6.4, 6.5; [**Kir08**]). *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it decomposes as a direct sum $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ of simple Lie algebras $\mathfrak{g}_i$. Any ideal in $\mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus_{j \in J \subset I} \mathfrak{g}_j$.*

This means in particular that if $\mathfrak{g} = \bigoplus_{k=1}^{n} \mathfrak{g}_k$, then $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ whenever $i \neq j$, and that $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ for all i. Consequently, if $\mathfrak{g}$ is semisimple, then assuming that $\mathfrak{z}(\mathfrak{g}) \neq 0$,

$$\mathfrak{z}(\mathfrak{g}) = \ker \operatorname{ad} = \{X \in \mathfrak{g} \mid [X, Y] = 0,\ \forall Y \in \mathfrak{g}\} = \bigoplus_{i \in I} \mathfrak{g}_i,$$

for $\mathfrak{g}_i$ simple ideals. Then $[\mathfrak{z}(\mathfrak{g}), \mathfrak{g}] \neq 0$, a contradiction. Thus, $\mathfrak{z}(\mathfrak{g}) = \ker \operatorname{ad} = 0$, and $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an injective embedding.

Any Lie algebra $\mathfrak{g}$ may be complexified by taking the tensor product $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and extending the bracket operation by linearity. All Lie algebra homomorphisms then also extend by linearity.

Proposition 15 (Prop. 3.4.7; [**Wal18**]). *If $\mathfrak{g}$ is a complex semisimple Lie algebra, $X \in \mathfrak{g}$, and $ad_X = S + N$ is the unique decomposition of ad_X into a sum of semisimple operator S and nilpotent operator N, then there exist elements X_s and X_n such that $S = ad_{X_s}$ and $N = ad_{X_n}$. The elements X_s and X_n are then described as semisimple and nilpotent, respectively, themselves.*

Proposition 15 facilitates the analysis of Lie algebras to such an extent that for the rest of this chapter, we shall consider only complex Lie algebras, even when those are the complexifications of real Lie algebras, and hope to be able to split the complex Lie algebras into direct sums of real and complex parts in such a way as to preserve the root or weight decompositions, to be described shortly.

Definition 24. A Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$, such that every element of $\mathfrak{h}$ is semisimple.

By standard results in linear algebra, a semisimple operator on a finite-dimensional vector space may be diagonalized and the underlying vector space split into a direct sum of eigenspaces, and commuting operators may be simultaneously diagonalized.

For a Lie algebra representation V, $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$, we have that $\phi([X, Y]) = [\phi(X), \phi(Y)]$, hence the operators $\phi(\mathfrak{h}) \subset \mathfrak{gl}(V)$ all commute. In the particular case when $V = \mathfrak{g}$, we have the following definition:

Definition 25. For a Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a root decomposition of $\mathfrak{g}$ is the direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda,$$

where the set $\Delta \subset \mathfrak{h}^*$ is the finite collection of nonzero, generalized eigenfunctionals on $\mathfrak{h}$ such that the sets

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid (\mathrm{ad}_H - \lambda(H))^n X = 0,\ \forall H \in \mathfrak{h},\ n \text{ large enough}\}$$

are the nonzero generalized eigenspaces of $\mathfrak{h}$, i.e. $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$. The set Δ is called the root system of $\mathfrak{g}$.

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ plays a key role in the representation theory of semisimple Lie algebras. $\mathfrak{sl}(2,\mathbb{C})$ is a three-dimensional Lie algebra spanned by the elements e, f, h with bracket relations

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

.

Proposition 16 (Thm. 4.52; [**Kir08**])**.** *Any finite-dimensional representation of* $\mathfrak{sl}(2,\mathbb{C})$ *is reducible into a direct sum of irreducible representations.*

Theorem 14 (Thm. 1.66, p. 62; [**Kna05**])**.** *For each n, there exists an n-dimensional, irreducible, complex representation* $\pi\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V_n)$*, unique up to isomorphism, where* $V_n := \mathit{span}\{v_0, \ldots, v_n\}$*, such that*

$$\pi(h)v_k = (\lambda - 2k)v_k, \quad \pi(f)v_k = (k+1)v_{k+1}, \quad \pi(e)v_k = (\lambda - (k-1))v_{k-1}.$$

If $K_\mathfrak{g}$ is the Killing form of the complex semisimple Lie algebra $\mathfrak{g}$, then the map $K_\mathfrak{g}^\flat\colon \mathfrak{h} \to \mathfrak{h}^*$, as defined in Proposition 12, is an isomorphism by the same proposition. Let for $\alpha \in \mathfrak{h}^*$, $\alpha^* \in \mathfrak{h}$ denote the element isomorphic to α. Let also $\tilde{K}_\mathfrak{g}$ denote the inverse Killing form. We borrow the notation of inner products, and let $\langle \cdot, \cdot \rangle$ denote both $K_\mathfrak{g}(\cdot,\cdot)$ and $\tilde{K}_\mathfrak{g}(\cdot,\cdot)$.

Theorem 15 (p.117-124,132; [**Kir08**])**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda$ its root decomposition. Then:*

1. $\dim \mathfrak{g}_\alpha = 1$ *if $\alpha \in \Delta$, 0 otherwise, and* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ *if $\alpha + \beta \in \Delta$, and 0 otherwise.*
2. *For $\alpha, \beta \in \Delta$, $K_\mathfrak{g}$ is a degenerate pairing on $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta$ if $\alpha + \beta \neq 0$, and positive definite on $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta$ if $\alpha + \beta = 0$. Moreover, $K_\mathfrak{g}$ is nondegenerate, positive definite on $\mathfrak{h}$.*
3. *If $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$, then $[e,f] = \langle e, f\rangle \alpha^* \in \mathfrak{h}$.*
4. *If $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$, with $\langle e, f \rangle = 2/\langle \alpha, \alpha\rangle$ and $\alpha^\vee := 2\alpha^*/\langle \alpha, \alpha \rangle$, then $e, f, \alpha^\vee$ satisfy the commutation relations of $\mathfrak{sl}(2,\mathbb{C})$. We denote this embedding of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{g}$ by $\mathfrak{sl}(2,\mathbb{C})_\alpha$.*
5. *For $\alpha, \beta \in \Delta$, $\beta \neq \pm\alpha$, the subspace $W := \bigotimes_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$ is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$.*

The root system Δ has the following properties:

1. *Δ generates $\mathfrak{h}^*$ as a vector space.*
2. *For any $\alpha, \beta \in \Delta$, the number $n_{\beta\alpha} := \alpha^\vee(\beta) = 2\langle \alpha, \beta\rangle/\langle \alpha, \alpha \rangle$ is an integer.*

3. *If $\alpha, \beta \in \Delta$, then $\beta - n_{\beta\alpha}\alpha = \beta - (2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle)\alpha$ is also in Δ.*

4. *If $\alpha \in \Delta$, then α and $-\alpha$ are the only multiples of α in Δ.*

We may note that since $\dim \mathfrak{g}_\alpha = 1$, the generalized eigenspaces $\mathfrak{g}_\alpha$ are in fact true eigenspaces.

Definition 26. Let Δ be a root system for the complex semisimple Lie algebra $\mathfrak{g}$. We define the weight lattice to be the set $P \coloneqq \{\lambda \in \mathfrak{h}^* \mid \alpha^\vee(\lambda) \in \mathbb{Z},\ \forall \alpha \in \Delta\} \subset \mathfrak{h}$.

Definition 27. For a complex representation $\phi \coloneqq \mathfrak{g} \to \mathfrak{gl}(V)$ of a complex semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the weight decomposition of V is the direct sum decomposition

$$V = \bigoplus_{\lambda \in P(V)} V_\lambda,$$

where the set $P(V)$ is the finite collection of nonzero, eigenfunctionals on V such that the sets

$$V_\lambda \coloneqq \{v \in V \mid (\phi(H) - \lambda(H))v = 0,\ \forall H \in \mathfrak{h}\}$$

are the nonzero eigenspaces of V, i.e. $P(V) = \{\lambda \in \mathfrak{h}^* \setminus \{0\} \mid V_\lambda \neq 0\}$. The set $P(V)$ is called the weight space of V.

Theorem 16 (Thm. 8.2; [**Kir08**])**.** *Every finite-dimensional representation of a complex semisimple Lie algebra $\mathfrak{g}$ has a weight decomposition, and the weight space $P(V)$ is a subset of the weight lattice P.*

Proposition 17 (Lemma 8.3; [**Kir08**])**.** *For $X \in \mathfrak{g}_\alpha$, $\phi(X)V_\lambda \subset V_{\lambda+\alpha}$.*

Proof. Suppose $v \in V_\lambda$, $X \in \mathfrak{g}_\alpha$, and $H \in \mathfrak{h}$. Then

$$\begin{aligned}
\phi(H)\phi(X)v &= [\phi(H), \phi(X)]v + \phi(X)\phi(H)v \\
&= \phi([H, X])v + \lambda(H)\phi(X)v \\
&= \alpha(H)\phi(X)v + \lambda(H)\phi(X)v \\
&= (\alpha(H) + \lambda(H))\phi(X)v,
\end{aligned}$$

which means that $\phi(X)V_\lambda \subset V_{\lambda+\alpha}$ since v, H, X were all arbitrary.

□

3.6 Real Forms and Cartan Decomposition

Definition 28. A real subalgebra $\mathfrak{g}_0$ of a complex Lie algebra $\mathfrak{g}$ is called a real form if $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$.

Definition 29. A Lie algebra $\mathfrak{g}$ is said to be compact if there exists a compact Lie group for which $\mathfrak{g}$ is the Lie algebra.

Theorem 17 (Thm. 3.7.5; [**Wal18**]). *If $\mathfrak{g}$ is a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, then there exists a compact real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $i\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, where $\mathfrak{h}_0 \coloneqq \mathbb{R} \cdot \{H \in \mathfrak{h} \mid H = \alpha^*,\ \alpha \in \Delta_{\mathfrak{g}}\}$.*

Definition 30. If $\mathfrak{g}$ is a real semisimple Lie algebra, then a Cartan involution of $\mathfrak{g}$ is an involutive automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g})$ — involutive meaning $\sigma^2 = Id$ — such that if $\mathfrak{k} = \{X \in \mathfrak{g} \mid \sigma X = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \sigma X = -X\}$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $K_{\mathfrak{g}}|_{\mathfrak{k}}$ negative definite and $K_{\mathfrak{g}}|_{\mathfrak{p}}$ positive definite. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the Cartan decomposition of $\mathfrak{g}$.

Note that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ follows from the definition of Cartan decomposition.

For $\mathfrak{gl}(V)$, the map $\mathrm{Aut}(\mathfrak{g}) \ni \sigma \colon \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ given by $X \mapsto -X^T$ is a Cartan involution:

$$X^T Y^T - Y^T X^T = (YX)^T - (XY)^T = -(XY - YX)^T.$$

This then holds for all Lie subalgebras $\mathfrak{g}$ of $\mathfrak{gl}(V)$ also. This means that for any semisimple Lie subgroup $G \subset GL(V)$, $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{o}(V) \subset \mathfrak{gl}(V) \cap \mathfrak{o}(V) = \mathfrak{o}(V)$ and $\mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(V) \subset \mathfrak{gl}(V) \cap \mathrm{Sym}(V) = \mathrm{Sym}(V)$ with respect to the standard euclidean inner product on V, where $\mathrm{Sym}(V)$ is the space of symmetric endomorphisms of V.

Theorem 18 (Thm. 3.7.9; [**Wal18**]). *If $\mathfrak{g}$ is a real semisimple Lie algebra, then $\mathfrak{g}$ has a Cartan involution and corresponding Cartan decomposition.*

Theorem 19 (Thm. 6.31; [**Kna05**]). *Let G be a semisimple real Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition due to the Cartan involution σ. Let K be the Lie subgroup of G such that $Lie(K) = \mathfrak{k}$, and let Z be the center group of G. Then*

1. *There exists a Lie group involution Σ of G such that $\Sigma_* = \sigma$.*
2. *$\Sigma(K) = K$.*
3. *G is diffeomorphic to $K \times \exp(\mathfrak{p})$.*
4. *K is closed and $Z \subset K$, moreover K is compact if and only if Z is finite, in which case K is a maximal compact Lie subgroup of G.*

In the case of a semisimple Lie subgroup G of $GL(V)$, K is a Lie subgroup of $O(V)$. The center subgroup of $GL(V)$ is the group of non-zero scalar multiples of $Id_V \in GL(V)$, which means that the center Z of G is $\mathbb{Z}_2$, as the only scalar multiples of $Id \in GL(V)$ that lie in $O(V)$ are Id and $-Id$.

Thus, for the semisimple Lie subgroups of $GL(V)$, K is compact and maximal. Furthermore, it is possible to prove the following:

Proposition 18 (Prop. A.1, Cor. A.2, [**BL17**]; Lem. 7.38, Thm. 7.39, [**Kna05**]). *Let G be a real semisimple Lie subgroup of $GL(V)$, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to the Cartan involution $\sigma(X) = -X^T$. If $\mathfrak{t}_1, \mathfrak{t}_2 \subset \mathfrak{p}$ are any two maximal abelian*

Lie subalgebras of $\mathfrak{p}$*, then there exists an element* $k \in K$ *such that* $k\mathfrak{t}_1k^{-1} = \mathfrak{t}_2$*, and for any maximal abelian subalgebra* $\mathfrak{t} \subset \mathfrak{p}$*,*

$$\mathfrak{p} = \cup_{k\in K} k\mathfrak{t}k^{-1}.$$

Consequently, for any maximal abelian Lie subalgebra $\mathfrak{t} \subset \mathfrak{p}$*,* $G = KTK$*, where* $T = \exp(\mathfrak{t})$*.*

3.7 Polynomial Invariants

Let G be a Lie group acting on a vector space V, and let $\mathbb{K}$ be either the real or the complex numbers. Let $e_1, \ldots, e_n$ be a basis for V, and let $e^1, \ldots, e^n$ be the dual basis. Let $\mathbb{K}[V] \coloneqq \mathbb{K}[e^1, \ldots, e^n] = \mathbb{K}[x_1, \ldots, x_n]$ be the ring of polynomials in the coordinate functions of V. The ring $\mathbb{K}[V]$ is often called the coordinate ring of V or the ring of regular functions on V. In fact, $\mathbb{K}[V]$ is an algebra graded by the degree of the polynomials, as $\mathbb{K}[V]$ is a vector space over $\mathbb{K}$ and two polynomials p, q of degree k, l, respectively, multiply to a polynomial r of degree $k + l$. The action of G on V induces a natural action of G on $\mathbb{K}[V]$ via the mapping $g \cdot p(x) \coloneqq p(g^{-1} \cdot x)$ for $p(x) \in \mathbb{K}[V]$ and $g \in G$. This action is linear over $\mathbb{K}$, and respects the grading of the algebra. A polynomial function $p \in \mathbb{K}[V]$ is said to be invariant if $g \cdot p = p$. The set $\mathbb{K}[V]^G$ of all invariant polynomial functions is a subalgebra of $\mathbb{K}[V]$, and is called the ring of invariants. We refer to [**GW10**, **Pro07**] for these basic results.

Definition 31 ([**BL17**])**.** Let G be a real semisimple Lie group, and $\rho\colon G \to GL(V)$ be a faithful representation on V real and finite-dimensional, with $\rho(G)$ closed in $GL(V)$. Then G is said to be real reductive if there exists an inner product on V compatible with the Cartan decomposition of $\mathfrak{g}$ in such a way that

$$G = K \cdot \exp(\mathfrak{k}),$$

with $K = G \cap O(V)$, $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{o}(V)$ and $\mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(V)$, where $\mathrm{Sym}(V)$ is the space of symmetric endomorphisms of V.

All real semisimple Lie groups G with finitely many connected components are real reductive [**BL17**, **Mos55**].

We may complexify the vector space V by taking the tensor product of V with $\mathbb{C}$ over $\mathbb{R}$: $V^{\mathbb{C}} \coloneqq V \otimes_{\mathbb{R}} \mathbb{C}$. By natural inclusion, $GL(V) \subset GL(V^{\mathbb{C}})$, hence $G \subset GL(V^{\mathbb{C}})$ also. We may then take the complexification of the Lie algebra $\mathfrak{g}$ of G also, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, such that $\mathfrak{g} \subset \mathfrak{gl}(V) \subset \mathfrak{gl}(V^{\mathbb{C}})$. For a semisimple group G with Lie algebra $\mathfrak{g}$, a complex analytic group $G^{\mathbb{C}}$ is said to be a complexification of G if G is an analytic subgroup of $G^{\mathbb{C}}$ and the Lie algebra of $G^{\mathbb{C}}$ is $\mathfrak{g}^{\mathbb{C}}$.

From geometric invariant theory generally, and the papers [**BL17**, **RS90**] in particular, we have the following results:

Theorem 20 ([**BL17**, **Pro07**, **RS90**])**.** *Let* G *be a real reductive Lie group faithfully embedded in* $GL(V)$*,* V *real and finite-dimensional. Let* $G^{\mathbb{C}} \subset GL(V^{\mathbb{C}})$ *be a complexification of* G*. Then we have the following:*

1. $\mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}} \subset \mathbb{C}[V^{\mathbb{C}}]$ *is finitely generated.*

2. *The polynomials of* $\mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}}$ *separate the closed orbits of the action of* $G^{\mathbb{C}}$ *on* $V^{\mathbb{C}}$*, meaning that for any two closed orbits we may find a polynomial in* $\mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}}$ *such that the polynomial takes distinct values on each of the closed orbits.*

3. *Regarding* V *as a subset of* $V^{\mathbb{C}}$*, any closed orbit of the action of* G *acting on* V *is contained in a closed orbit of the action of* $G^{\mathbb{C}}$ *on* $V^{\mathbb{C}}$*, and the closed orbits of the action of* $G^{\mathbb{C}}$ *on* $V^{\mathbb{C}}$ *only ever contain finitely many closed orbits of the action of* G *acting on* V*.*

4. *If an orbit of the action of* G *acting on* V *contains a minimal vector* $v \in V$ *in the compatible norm, then the orbit is closed.*

5. *If for some* $v \in V$ *the orbit* $G \cdot v$ *is not closed, then there exists an element* $X \in \mathfrak{p}$ *such that the limit* $w = \lim_{t\to\infty} \exp(tX)$ *exists, and* $G \cdot w$ *is a closed orbit.*

6. *The closure of an orbit contains only one closed orbit.*

Chapter 4

Basic Pseudo-Riemannian Geometry

We recount the basic theory of pseudo-Riemannian geometry. We primarily follow the books by [**Jos11**, **Lee97**, **O'N83**, **Tu17**].

4.1 Pseudo-Riemannian Metrics

Proposition 19 (Adapted from Prop. 1.3 in [**Ber01**])**.** *By an appropriate choice of basis for a finite-dimensional vector space V, a nondegenerate, symmetric bilinear form m may be brought to the form*

$$(m_{ij}) = \mathit{diag}\{\epsilon_1, \ldots, \epsilon_n\},$$

where $\epsilon_i = -1$ for $i = 1, \ldots, p$ and $\epsilon_i = 1$ for $i = p+1, \ldots, n$, where $p \leq n$ and $n = \dim V$.

Proof. Obviously, for m to be nondegenerate it must be nonzero. Thus, there exist vectors $u, v \in V$ such that $m(u, v) \neq 0$. Since m is symmetric,

$$m(u, v) = \frac{1}{4}\Big(m(u+v, u+v) - m(u-v, u-v)\Big),$$

hence if $m(u, v)$ is nonzero, then either $m(u+v, u+v)$ or $m(u-v, u-v)$ is nonzero. Thus, there exists a vector $e_1 \in V$ such that $m(e_1, e_1) \neq 0$. By rescaling e_1 if necessary, we may assume that $m(e_1, e_1) = \epsilon_1$, which is equal to either -1 or 1.

Let $V_1 \coloneqq \operatorname{span}\{e_1\}$ and let $V_2 \coloneqq V_1^{\perp} = \{u \in V \mid m(u, v) = 0 \text{ for all } v \in V_1\}$. We claim that $V_1 \cap V_2 = \{0\}$ and $V = V_1 + V_2$, and that therefore $V = V_1 \oplus V_2$. First, if $u \in V_1$, then $m(u, u) \neq 0$, and so $u \notin V_2$. Secondly, if $u \in V$, then $u - \epsilon_1 m(u, e_1)e_1$ lies in V_2:

$$\begin{aligned}
m(u - \epsilon_1 m(u, e_1)e_1, e_1) &= m(u, e_1) - m(\epsilon_1 m(u, e_1)e_1, e_1) \\
&= m(u, e_1) - \epsilon_1 m(u, e_1)m(e_1, e_1) \\
&= m(u, e_1) - (\epsilon_1)^2 m(u, e_1) \\
&= m(u, e_1) - m(u, e_1) \\
&= 0.
\end{aligned}$$

We now need to check that $m|_{V_2}$ is nondegenerate. We know that m is nondegenerate on all of V. Suppose $u \in V_2$ is is a nonzero vector such that $m(u, v) = 0$ for all $v \in V_2$, i.e. $u \in \ker(m|_{V_2})^\flat$. Then, by nondegeneracy, it must be the case that $m(u, e_1)$; however, this contradicts the assumption that $v \in V_2 = V_1^\perp$. Thus, it must be the case that $u = 0$. Therefore, $\ker(m|_{V_2})^\flat = \{0\}$, and $m|_{V_2}$ is nondegenerate.

The result then follows by induction, and simple rearrangement of the basis vectors.

□

Definition 32. A nondegenerate, symmetric bilinear form m on a finite-dimensional vector space V is said to be of index p or signature (p, q) if it can be brought to the form described in Proposition 19 with p entries on the diagonal equal to -1 and q entries on the diagonal equal to 1.

The form m on $\mathbb{R}^n$ such that $(m_{ij}) = \operatorname{diag}\{\epsilon_1, \ldots, \epsilon_n\}$ in the standard basis shall be labelled η or $\eta(p, q)$ as per Proposition 19 and Definition 32, and a basis on a vector space V with nondegenerate, symmetric bilinear form m such that m takes the form η in that basis shall be called orthonormal.

Definition 33. A pseudo-Riemannian metric m on a smooth manifold m is a smoothly varying symmetric, nondegenerate $(0, 2)$-tensor field of constant index.

If $\{e_1, \ldots, e_n\}$ is a local frame for some patch of a manifold M, with coframe $\{e^1, \ldots, e^n\}$ then we write

$$\eta(p, q) = -(e^1e^1 + \cdots + e^pe^p) + (e^{p+1}e^{p+1} + \cdots + e^{p+q}e^{p+q}),$$

where $e^ie^j := (1/2)(e^i \otimes e^j + e^j \otimes e^i)$ is the symmetric product. If we perform a change of basis from $\{e_1, \ldots, e_n\}$ to $f_1, \ldots, f_n$ given by

$$\begin{aligned}
&f_1, \ldots, f_n = \\
&\{(1/\sqrt{2})(e_1 - e_{1+p}), (1/\sqrt{2})(e_2 - e_{2+p}), \ldots, \\
&(1/\sqrt{2})(e_p - e_{2p}), (1/\sqrt{2})(e_1 + e_{1+p}), (1/\sqrt{2})(e_2 + e_{2+p}), \ldots, \\
&(1/\sqrt{2})(e_p + e_{2p}), e_{n-2p+1}, \ldots, e_n\},
\end{aligned}$$

then $\{e^1, \ldots, e^n\}$ goes to

$$\begin{aligned}
&f^1, \ldots, f^n = \\
&\{(1/\sqrt{2})(e^1 - e^{1+p}), (1/\sqrt{2})(e^2 - e_{2+p}), \ldots, \\
&(1/\sqrt{2})(e^p - e^{2p}), (1/\sqrt{2})(e^1 + e^{1+p}), (1/\sqrt{2})(e^2 + e^{2+p}), \ldots, \\
&(1/\sqrt{2})(e^p + e^{2p}), e^{n-2p+1}, \ldots, e^n\}.
\end{aligned}$$

Correspondingly, $\eta(p, q)$ goes to

$$\rho = 2(f^1f^{1+p} + \cdots + f^pf^{2p}) + (f^{2p+1}f^{2p+1} + \cdots + f^{p+q}f^{p+q}).$$

This form for the metric is often more convenient for computation, because pure boosts are diagonal operators relative to this form.

4.2 Isometry Group of a Pseudo-Riemannian Metric

The group $O(p, q)$ of isometric automorphisms of a real vector space V equipped with a pseudo-Riemannian tensor $\eta(p, q)$ is a semisimple Lie subgroup of the group $GL(V)$ of linear isomorphisms. Because any pseudo-Riemannian tensor η may be brought to the form $\eta(p, q)$ by a change of basis, we will start by describing the group in this basis first. Subsequently, we will discuss a different basis that facilitates certain computations.

The group $O(p, q)$ is the group of elements $G \in GL(V) \mid \eta_{p,q}(Gu, Gv) = \eta_{p,q}(u, v)\ \forall u, v \in V$. Consequently, if $X \in \mathfrak{o}(p, q)$, then

$$\eta_{p,q}(\exp(tX)u, \exp(tX)v) = \eta_{p,q}(u, v)$$

and

$$d/dt|_{t=0}\Big(\eta_{p,q}(\exp(tX)u, \exp(tX)v)\Big) = \eta_{p,q}(Xu, v) + \eta_{p,q}(u, Xv) = 0. \tag{4.1}$$

Thinking of $\eta(p, q)$ as a diagonal matrix, equation (4.1) is saying that $X^T\eta(p, q)+\eta(p, q)X = 0$ for all $X \in \mathfrak{o}(p, q)$. Writing X as a matrix

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

of submatrices X_1, X_2, X_3, and X_4 of dimensions $p \times p$, $p \times q$, $q \times p$, and $q \times q$, respectively, we get the equation

$$\begin{bmatrix} X_1^T & X_3^T \\ X_2^T & X_4^T \end{bmatrix}\begin{bmatrix} -Id_p & 0 \\ 0 & Id_q \end{bmatrix} + \begin{bmatrix} -Id_p & 0 \\ 0 & Id_q \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = 0.$$

Thus, we obtain the conditions $X_1 = -X_1^T$, $X_4 = -X_4^T$, and $X_3 = X_2^T$. X may be split into parts

$$X = Y + Z = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix},$$

where $A = -A^T$, $B = -B^T$, and C is arbitrary. We see that this accords with the canonical Cartan decomposition for Lie subgroups of $GL(n, \mathbb{R})$ given by the map $\sigma(X) = -X^T$, with

$$\mathfrak{k} = \{X \in M_{p+q}(\mathbb{R}) \mid X = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},\ A = A^T,\ B = B^T,\ A \in M_p(\mathbb{R}),\ B \in M_q(\mathbb{R})\}$$

and

$$\mathfrak{p} = \{X \in M_{p+q}(\mathbb{R}) \mid X = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix},\ C \in M_{p,q}(\mathbb{R})\}.$$

The group K corresponding to $\mathfrak{k}$ is the group $O(p, \mathbb{R}) \oplus O(q, \mathbb{R})$. We wish to employ Proposition 18, and so therefore wish to discover a maximal abelian Lie subgroup $\mathfrak{t} \subset \mathfrak{p}$. To that end, we observe that if $X, Y \in \mathfrak{p}$, then

$$[X,Y] = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} 0 & D \\ D^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & D \\ D^T & 0 \end{bmatrix} \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix} = \begin{bmatrix} CD^T - (CD^T)^T & 0 \\ 0 & C^T D - (C^T D)^T \end{bmatrix},$$

hence we wish to find conditions on C, D such that the terms $CD^T - (CD^T)^T$ and $C^T D - (C^T D)^T$ are zero, and $X, Y \in \mathfrak{t}$. If we let C and D take the forms

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix},$$

with C_1 and D_1 being $p \times p$ matrices, and C_2 and D_2 being $p \times (q-p)$ matrices, then

$$CD^T - (CD^T)^T = C_1 D_1^T - (C_1 D_1^T)^T + C_2 D_2^T - (C_2 D_2^T)^T$$

is a $p \times p$ matrix and

$$C^T D - (C^T D)^T = \begin{bmatrix} C_1^T D_1 - (C_1^T D_1)^T & C_1^T D_2 - (C_2^T D_1)^T \\ C_2^T D_1 - (C_1^T D_2)^T & C_2^T D_2 - (C_2^T D_2)^T \end{bmatrix}$$

is a $q \times q$ matrix. For the terms $CD^T - (CD^T)^T$ and $C^T D - (C^T D)^T$ to be zero, we see from the above that we need $C_1 D_1^T + C_2 D_2^T$, $C_1^T D_1$, and $C_2^T D_2$ to be symmetric, and $C_1^T D_2 = (C_2^T D_1)^T$.

Proposition 20. *$\mathfrak{t} \subset \mathfrak{p}$ given by*

$$\mathfrak{t} = \{X \in M_{p+q}(\mathbb{R}) \mid X = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix},\ C = \begin{bmatrix} C_1 & 0 \end{bmatrix},\ C_1 \in M_p(\mathbb{R}) \textit{ diagonal}\}$$

is a maximal abelian Lie subalgebra contained in $\mathfrak{p}$.

Proof. Suppose we try to expand $\mathfrak{t}$ by letting the matrix C_2 be non-zero. Let $X \in \mathfrak{p}$ be such that C takes the form $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$, with C_1 diagonal and C_2 arbitrary, and similarly for $Y \in \mathfrak{p}$ with D taking the form $\begin{bmatrix} D_1 & D_2 \end{bmatrix}$, with D_1 diagonal and D_2 arbitrary. We must have $C_1^T D_2$ symmetric, i.e. $C_1^T D_2 - (C_2^T D_1)^T = C_1 D_2 - D_1 C_2 = 0$, since C_1 and D_1 are diagonal. Suppose in particular that $C_2 = D_2$, which is admissible if we were trying to expand $\mathfrak{t}$ by a new basis element in the form of a matrix C_2. Then this condition reads $(C_1 - D_1)C_2 = 0$, which is only possible if $C_1 \equiv D_1$, against assumption. Therefore, we are precluded from extending the abelian algebra $\mathfrak{t}$ by any such matrix as a new basis element.

Suppose now that we try to make use of more general matrices C_1, D_1. By essentially the same Cartan decomposition σ as before, only for $p \times p$ matrices, we may split C_1 into a symmetric and anti-symmetric part, $C_1 = M + N$ where $M = (1/2)(C_1 + C_1^T)$ and $N = (1/2)(C_1 - C_1^T)$. Let $M' + N'$ be a corresponding split for $D_1 = M' + N'$. Then the condition that $C_1 D_1^T - (C_1 D_1^T)^T = 0$ reads

$$
\begin{aligned}
C_1 D_1^T - (C_1 D_1^T)^T &= (M+N)(M'-N') - (M'+N')(M-N) \\
&= MM' - NN' - MN' + NM' - M'M + N'N - N'M + M'N \\
&= (MM' - M'M) - (NN' - N'N) - (MN' + N'M) + (NM' - M'N) \\
&= (MM' - (MM')^T) + (NM' + (NM')^T) \\
&- \Big((NN' - (NN')^T) + (MN' - (MN')^T)\Big) \\
&= 0.
\end{aligned}
$$

Here, we have split $C_1 D_1^T - (C_1 D_1^T)^T$ into its symmetric and anti-symmetric parts, which each have to be zero. Thus, we have to have $(MM' - (MM')^T) + (NM' + (NM')^T) = 0$ and $(NN' - (NN')^T) + (MN' - (MN')^T) = 0$, independently of each other.

Suppose first that we try to expand $\mathfrak{t}$ by an antisymmetric basis element, N. M, M' are diagonal, so our conditions now read $NM' + (NM')^T = 0$ and $MN - (MN)^T = 0$. Neither condition is true for general diagonal matrices M, M'. So we cannot extend $\mathfrak{t}$ by any anti-symmetric matrix as a new basis element.

We are left to consider C_1, D_1 as general symmetric matrices, as would happen if we tried to expand $\mathfrak{t}$ by adding a non-diagonal symmetric matrix as a basis element. Then the condition $C_1 D_1^T - (C_1 D_1^T)^T = 0$ reads as $C_1 D_1 - D_1 C_1 = 0$, i.e. the matrices C_1, D_1 have to commute. This is in general only possible if both matrices are diagonal, thus no such expansion of $\mathfrak{t}$ is possible.

Finally, it is easy to see that for any fixed $X \in \mathfrak{k}$ there is a $Y \in \mathfrak{t}$ such that $[X, Y] \neq 0$, thus we cannot hope to expand $\mathfrak{t}$ by any element $X \in \mathfrak{k}$. Thus we have shown that $\mathfrak{t}$ is a maximal abelian Lie subalgebra of $\mathfrak{p}$.

□

The elements of $\mathfrak{t} \subset \mathfrak{gl}(V)$ may be simultaneously diagonalized by a change of basis from the standard $\{e_1, \ldots, e_n\}$ to the basis

$$
\begin{aligned}
&f_1, \ldots, f_n = \\
&\{(1/\sqrt{2})(e_1 - e_{1+p}), (1/\sqrt{2})(e_2 - e_{2+p}), \ldots, \\
&(1/\sqrt{2})(e_p - e_{2p}), (1/\sqrt{2})(e_1 + e_{1+p}), (1/\sqrt{2})(e_2 + e_{2+p}), \ldots, \\
&(1/\sqrt{2})(e_p + e_{2p}), e_{n-2p+1}, \ldots, e_n\},
\end{aligned}
$$

where $n = p + q$ and $p < q$ such that $2p < n$, as described in Section 4.2. In this case, the eigenfunctionals of $\mathfrak{t}$ are real, and for an operator $H \in \mathfrak{t}$ such that

$$
H = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix}, \; C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \; C_1 = \mathrm{diag}(\ldots, \lambda_i, \ldots),
$$

the elements $(1/\sqrt{2})(e_i - e_{i+p})$ have eigenvalue $-\lambda_i$ and the elements $(1/\sqrt{2})(e_i + e_{i+p})$ have eigenvalue λ_i, for $i = 1, \ldots, p$, and the remaining e_j have eigenvalue 0.

Proposition 21. *The abelian subalgebra $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra in the complexified Lie algebra $\mathfrak{g}$.*

Proof. We need to show that $\mathrm{ad}(H)$ is semisimple for any $H \in \mathfrak{t}^{\mathbb{C}}$. If H is as described above, then H is a diagonal matrix:

$$H = \mathrm{diag}(\rho_1, \dots, \rho_{p+q})$$

such that $\rho_{2i-1} = -\lambda_i$ and $\rho_{2i} = \lambda_i$ for $i = 1, \dots, p$, and $\rho_j = 0$ for $j = 2p+1, \dots, p+q$. Let E_{ij} be the matrix with a 1 in the (i,j)-th position. Then

$$\mathrm{ad}(H)(E_{ij}) = HE_{ij} - E_{ij}H = (\rho_i - \rho_j)E_{ij}.$$

Thus, $\mathrm{ad}(H)$ acts as a diagonal matrix on elements of $\mathfrak{g} \subset \mathfrak{gl}(V)$, and thus is semisimple. $\square$

We have thus, conveniently, discovered that there is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$. Then the root space decomposition of $\mathfrak{g}$ splits nicely into a real and a complex part such that the real part is the original $\mathfrak{o}(p,q)$ that we started with. Now, all the results for root and weight space decompositions apply to $\mathfrak{o}(p,q)$ as a real Lie algebra.

4.3 Connections on Vector Bundles

Definition 34 ([**Jos11**, **Tu17**]). A connection on a vector bundle $\pi \colon E \to M$ is a map

$$\nabla \colon \Gamma(E) \to \Gamma(E \otimes T^*M)$$

such that ∇ is $\mathbb{R}$-linear and ∇ satisfies the Leibniz rule

$$\nabla(f\sigma) = \sigma \otimes df + f \cdot \nabla\sigma \tag{4.2}$$

Note that by the Leibniz rule (4.2), on any local trivialization (U, φ) of E with local frame $\{e_1, \dots, e_n\}$ and smooth section $Y \in \Gamma(E)$,

$$\begin{aligned} \nabla Y &= \nabla(Y^j e_j) \\ &= e_j \otimes d(Y^j) + Y^j \nabla e_j \\ &= e_k \otimes \frac{\partial Y^k}{\partial x^i} dx^i + Y^j \Gamma^k_{ij} e_k \otimes dx^i \\ &= \Big(\frac{\partial Y^k}{\partial x^i} + Y^j \Gamma^k_{ij}\Big) e_k \otimes dx^i, \end{aligned} \tag{4.3}$$

where $\varphi = (x^1, \dots, x^n)$ are local coordinates and we have introduced the Christoffel symbol $\Gamma^k_{ij} e_k = \nabla_{e_i} e_j$ to express the components of ∇Y in a local frame.

Alternatively, describing ∇ as a map

$$\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

we have for $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$,

$$\begin{aligned}\nabla_X Y &= \nabla_X Y^j e_j \\ &= X(Y^k)e_k + Y^j \nabla_X e_j \\ &= dY^k(X)e_k + Y^j \omega_j^k(X)e_k\Big) \\ &= \Big(dY^k(X) + Y^j\omega_j^k(X)\Big)e_k \\ &= dY(X) + \omega(X)\cdot Y,\end{aligned} \tag{4.4}$$

such that

$$\nabla Y = dY + \omega \cdot Y$$

may be regarded as a multilinear map from $E \times T^*M$ to $\mathbb{R}$ (and therefore defines a tensor field), where $\nabla_X e_j = \omega_j^k(X)e_k$ and $\nabla e_j = \omega_j^k e_k$, $dY(X) = d(Y^k)(X)e_k$ and $dY = d(Y^k)e_k$, and $\omega(X)\cdot Y$ and $\omega \cdot Y$ are both given by the standard action of a linear operator acting on a vector. Note that $\Gamma_{ij}^k e_k = \omega_j^k(e_i)e_k$.

The following proposition has a form that holds for more general vector bundles, but we state it only for tensor bundles (see also Section 4.1, [**Jos11**]):

Proposition 22 (Lemmas 4.6, 4.7, and 4.8, [**Lee97**]; Prop. 22.7, Thm. 22.8, [**Tu17**])**.** *A connection ∇ on the tangent bundle $\pi\colon TM \to M$ may be uniquely extended to a connection $\tilde{\nabla}$ on any tensor bundle*

$$\pi\colon \overbrace{TM \otimes \cdots \otimes TM}^{k \text{ copies}} \otimes \overbrace{T^*M \otimes \cdots \otimes T^*M}^{l \text{ copies}} \to M$$

of $\pi\colon TM \to M$, for arbitrary k, l, by demanding the following:

1. *$\tilde{\nabla}$ agrees with ∇ on TM.*
2. *On $TM^{(0,0)} \simeq \mathfrak{X}(M)$, the space of smooth functions on M, $\tilde{\nabla}$ acts on functions $f\colon M \to \mathbb{R}$ by $\tilde{\nabla} f = df$.*
3. *$\tilde{\nabla}$ obeys the product rule with respect to tensor products, i.e. if $Z = X \otimes Y$, then*

$$\tilde{\nabla} Z = \tilde{\nabla}(X \otimes Y) = (\tilde{\nabla} X) \otimes Y + X \otimes (\tilde{\nabla} Y)$$

4. *$\tilde{\nabla}$ commutes with contractions of indices, meaning if tr: $T^{(k,l)}M \to T^{(k-1,l-1)}M$ denotes a contraction or "trace" operator over any pair of indices, then $\tilde{\nabla}$ commutes with tr, i.e.*

$$\tilde{\nabla}(tr(X)) = tr(\tilde{\nabla} X)$$

for any smooth tensor field $X \in \Gamma(T^{(k,l)}M)$.

The above then implies that

1. *For $\varphi \in \Gamma(T^*M)$, $\tilde{\nabla}\varphi = d\varphi + \omega^* \cdot \varphi$, where $d\varphi = d(\varphi_k)e^k$ and $\omega^* \cdot \varphi = -\omega^T \cdot \varphi = -\omega^k_j \varphi_k$, or alternatively*

$$\tilde{\nabla}\varphi = \Big(\frac{\partial \varphi_k}{\partial x^j} - \varphi_i \Gamma^i_{kj}\Big) dx^k \otimes dx^j \tag{4.5}$$

in local coordinates $\phi = (x^1, \dots, x^n)$, and the same Christoffel symbols as in the covariant case.

2. *For $T \in \Gamma(T^{(k,l)}M)$, $\tilde{\nabla}T = dT + \omega \cdot T = dT + \omega \cdot_i T^{(\dots,i,\dots)} + \omega^* \cdot_j T_{(\dots,j,\dots)}$, where $\omega \cdot T$ should be understood to mean that ω acts on the tensor T by acting on each component of T separately and then summing the contributions, i.e. $\omega \cdot_i T^{(\dots,i,\dots)}$ and $\omega^* \cdot_j T_{(\dots,j,\dots)}$ should be understood to mean that ω acts on the i-th contravariant component and j-th covariant component of the tensor field T, respectively, in accordance with (4.4) and (4.5), respectively. Alternatively, in local coordinates $\phi = (x^1, \dots, x^n)$ and Christoffel symbols as before,*

$$\tilde{\nabla}T = \left(\frac{\partial T^{i_1\dots i_k}_{j_1\dots j_l}}{\partial x^q} + \sum_{s=1}^{k} T^{i_1\dots r\dots i_k}_{j_1\dots j_l}\Gamma^{i_s}_{rq} - \sum_{t=1}^{l} T^{i_1\dots i_k}_{j_1\dots r\dots j_l}\Gamma^{r}_{j_t q}\right)\partial_{i_1}\otimes\cdots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}\otimes dx^q. \tag{4.6}$$

4.4 The Levi-Civita Connection

Definition 35. A connection ∇ on the tangent bundle $\pi\colon TM \to M$ is said to be torsion free if the torsion tensor τ, defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{4.7}$$

vanishes identically, where $[-,-]$ is the Lie bracket of vector fields on M. In other words, if $\tau \equiv 0$.

Theorem 21 (Thm. 11.7, [**Tu17**]). *Relative to a local frame $\{e^1, \dots, e^n\}$ and coframe $\{e^1, \dots, e^n\}$, the torsion tensor satisfies the first structural equation,*

$$\tau^i = de^i + \omega^i_j e^j. \tag{4.8}$$

Definition 36. A connection ∇ on the tangent bundle $\pi\colon TM \to M$ is said to be compatible with a metric m on M if $\nabla m \equiv 0$, i.e. if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{4.9}$$

for smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$, where we have used the formula (4.6) to obtain (4.9).

Theorem 22 (Thm. 6.6, [**Tu17**]; Thm. 11, Ch. 3, [**O'N83**]). *On a pseudo-Riemannanian manifold, there is a unique affine connection that is torsion free and compatible with the pseudo-Riemannian metric, defined by the Koszul formula*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &- \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle, \end{aligned} \tag{4.10}$$

where X, Y, Z are smooth vector fields on the manifold.

Proposition 23 (Prop. 11.4, [**Tu17**]). *For a metric compatible connection ∇ on the tangent bundle $\pi\colon TM \to M$, we have that*

$$X\langle e_i, e_j\rangle = \langle \nabla_X e_i, e_j\rangle + \langle e_i, \nabla_X e_j\rangle, \tag{4.11}$$

or

$$dm_{ij} = \omega_i^k m_{kj} + m_{ik}\omega_j^k, \tag{4.12}$$

for a local frame $\{e_1, \ldots, e_n\}$. In the special case that the coefficients of the metric are constant relative to a local frame we have

$$0 = \omega_i^k m_{kj} + m_{ik}\omega_j^k, \tag{4.13}$$

and in the special case that the frame is orthonormal, i.e. $\langle e_i, e_j\rangle = \eta_{ij}$,

$$0 = \omega_i^k \eta_{kj} + \eta_{ik}\omega_j^k \tag{4.14}$$

Thus, in a local frame for which the components of the metric tensor are constant, then if we regard ω as an endomorphism of the tangent space, the adjoint of ω is equal to the negative of its transpose, that is to say, $\omega^* = -\omega^T$. We may therefore regard ω as a representative element for a representation $\rho\colon \mathfrak{o}(p, q) \to \mathfrak{gl}(n, \mathbb{R})$.

By Theorem 22, the torsion free, metric compatible connection on the tangent bundle is unique. Working in a local frame for which the components of the metric tensor are constant, we may thus use the first structural equation (4.8), with $\tau \equiv 0$, as well as the expression (4.13), to uniquely specify the connection form ω in terms of that local frame.

4.5 Curvature

Definition 37. A connection on a vector bundle $\pi\colon E \to M$ gives rise to a curvature tensor, also called the Riemann tensor, defined by

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{4.15}$$

for smooth vector fields $X, Y \in \mathfrak{X}(M)$ and smooth section $Z \in \Gamma(E)$.

A simple calculation shows that R is in fact linear in each of its arguments. Relative to some local frame, we may write $R = R^i_{jkl} e_i \otimes e^j \otimes e^k \otimes e^l$, with $R^i_{jkl} = e^i(R(e_j, e_k, e_l))$

or $R^i_{jkl} = e^i(R(e_k, e_l, e_j))$, depending on convention. We shall adopt the second of these notational conventions henceforth.

We may also regard R as a linear map $\Omega\colon TM \otimes TM \to E \otimes E^*$ by defining Ω by the formula $R(X,Y,Z) = \Omega(X,Y) \cdot Z$, where Ω acts on $Z \in \Gamma(E)$ by the standard action of a linear endmorphism on a vector.

Theorem 23 (Thm. 11.1, [**Tu17**])**.** *For a connection ∇ on a vector bundle $\pi\colon E \to M$, the curvature form Ω satisfies the second structural equation,*

$$\Omega = d\omega + \omega \wedge \omega, \tag{4.16}$$

or in terms of its components,

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j. \tag{4.17}$$

Proposition 24. *The Levi-Civita connection on a tangent bundle $\pi\colon TM \to M$ satisfies the condition*

$$0 = \Omega^k_i m_{kj} + m_{ik}\Omega^k_j, \tag{4.18}$$

when expressed in a local frame for which the coefficients of the metric tensor are constant.

Proof. We perform a simple calculation making use of condition (4.13):

$$\begin{aligned}
\Omega^k_i m_{kj} + m_{ik}\Omega^k_j &= (d\omega^k_i + \omega^k_l \wedge \omega^l_i)m_{kj} + m_{ik}(d\omega^k_j + \omega^k_l \wedge \omega^l_j) \\
&= (d\omega^k_i m_{kj} + m_{ik} d\omega^k_j) + ((\omega^k_l \wedge \omega^l_i)m_{kj} + m_{ik}(\omega^k_l \wedge \omega^l_j)) \\
&= d(\omega^k_i m_{kj} + m_{ik}\omega^k_j) + ((m_{kj}\omega^k_l) \wedge \omega^l_i + (m_{ik}\omega^k_l) \wedge \omega^l_j) \\
&= (m_{kj}\omega^k_l) \wedge \omega^l_i + (m_{ik}\omega^k_l) \wedge \omega^l_j \\
&= (m_{kj}\omega^k_l) \wedge \omega^l_i + (-m_{lk}\omega^k_i) \wedge \omega^l_j \\
&= (m_{kj}\omega^k_l) \wedge \omega^l_i + (m_{lk}\omega^l_j) \wedge \omega^k_i \\
&= (m_{kj}\omega^k_l) \wedge \omega^l_i + (m_{kl}\omega^k_j) \wedge \omega^l_i \\
&= (m_{kj}\omega^k_l + m_{kl}\omega^k_j)) \wedge \omega^l_i \\
&= 0
\end{aligned}$$

□

Other tensors that contain information about curvature can be computed from the Riemann tensor. For instance the Ricci curvature:

Definition 38. The Ricci tensor (Ric or R_{ij}) is the contraction (or trace) of the map

$$Z \mapsto R(Z, X, Y).$$

Writing $R^k_{ilj} = e_k(R(e_l, e_j, e_i))$, the coefficients of the Ricci tensor are

$$R_{ij} \coloneqq R^k_{ikj}.$$

A further contraction of the Ricci tensor produces the scalar curvature:

Definition 39. The scalar curvature S is the contraction of the Ricci curvature, such that

$$S \coloneqq m^{ij} R_{ij},$$

where m^{ij} are the components of the inverse metric.

We may take arbitrary covariant derivatives of the Riemann tensor, arbitrary sums of these, and contract any indices of the resulting tensors that we wish. Any scalar quantity obtained by fully contracting all indices is a **polynomial curvature invariant**, independent of choice of local frame.

Chapter 5

Left-Invariant Metrics

The main topic of this thesis is the study of left-invariant, pseudo-Riemannian metrics on Lie groups. As discussed in Section 2.3, such a metric m may be regarded as an element in $\Sigma^2(\mathfrak{g}^*)$, the space of symmetric, covariant 2-tensors over $\mathfrak{g}$.

The metric may also be be right-invariant, i.e. if $r_g^* m = m$, in which case the metric is said to be bi-invariant.

5.1 Levi-Civita Connection and Curvature

The metric being left-invariant has implications for the Levi-Civita connection. The connection forms ω_j^k are defined using the formula $\nabla_X e_j = \omega_j^k(X) e_k$, for some frame $\{e_1, \dots, e_n\}$. On a Lie group, which has a global frame of left-invariant vector fields, it suffices to compute $\omega_j^k(X)$ for a left-invariant basis $\{e_1, \dots, e_n\} \in \mathrm{Lie}(G) \simeq \mathfrak{g}$. Since $X = X^i(g) e_i$ for any $X \in \mathfrak{X}(G)$, it further suffices to compute $\omega_j^k(e_i)$.

Proposition 25. *For a left-invariant, pseudo-Riemannian metric on a Lie group, the connection coefficients ω_j^k of the associated Levi-Civita connection are determined by the formula*

$$\omega_j^k(e_i) = \frac{1}{2} m^{kq} \Big(- C_{ijq} + C_{jqi} + C_{qij} \Big), \tag{5.1}$$

where the m^{kq} are the components of the inverse metric, and $C_{ijk} = m_{il} C_{jk}^l$ are the lowered structure coefficients.

Proof. By the Koszul formula (4.10), letting $\nabla_{e_i} e_j = \omega_j^r(e_i) e_r$, we have that

$$
\begin{aligned}
2\omega^k_j(e_i) &= 2m^{kq}\omega_{qj}(e_i)\\
&= 2m^{kq}\langle \omega^r_j(e_i)e_r, e_q\rangle\\
&= 2m^{kq}\langle \nabla_{e_i}e_j, e_q\rangle\\
&= m^{kq}\Big(e_i\langle e_j, e_q\rangle + e_j\langle e_q, e_i\rangle - e_q\langle e_i, e_j\rangle\\
&- \langle e_i, [e_j, e_q]\rangle + \langle e_j, [e_q, e_i]\rangle + \langle e_q, [e_i, e_j]\rangle\Big)\\
&= m^{kq}\Big(- \langle e_i, [e_j, e_q]\rangle + \langle e_j, [e_q, e_i]\rangle + \langle e_q, [e_i, e_j]\rangle\Big)\\
&= m^{kq}\Big(- \langle e_i, C^l_{jq}e_l\rangle + \langle e_j, C^l_{qi}e_l\rangle + \langle e_q, C^l_{ij}e_l\rangle\Big)\\
&= m^{kq}\Big(- m_{il}C^l_{jq} + m_{jl}C^l_{qi} + m_{ql}C^l_{ij}\Big)\\
&= m^{kq}\Big(- C_{ijq} + C_{jqi} + C_{qij}\Big).
\end{aligned}
$$

□

Corollary 1. *The connection form ω and the curvature form Ω are constant as functions of $g \in G$ — in other words, they are left-invariant — and their components are polynomial functions of the components of the metric and the structure coefficients.*

Proof. From inspection of (5.1), the connection form is seen to be independent of $g \in G$. The curvature form is determined by the second structural equation (4.17). Letting $\omega^k_j = \Gamma^k_{ij}e^i$, and using the Maurer-Cartan formula (2.3), we get

$$
\begin{aligned}
\Omega^i_j &= d\omega^i_j + \omega^i_k \wedge \omega^k_j\\
&= -\tfrac{1}{2}\Gamma^i_{kj}C^k_{qr}e_i \otimes e^j \otimes (e^q \wedge e^r) + \Gamma^i_{qk}\Gamma^k_{rj}e_i \otimes e^j \otimes (e^q \wedge e^r)\\
&= \Big(- \Gamma^i_{kj}C^k_{qr} + \Gamma^i_{qk}\Gamma^k_{rj} - \Gamma^i_{rk}\Gamma^k_{qj}\Big)e_i \otimes e^j \otimes e^q \otimes e^r,
\end{aligned}
\tag{5.2}
$$

from which we conclude that the curvature form is also constant as a function of $g \in G$, since the structure coefficients and Christoffel symbols are constant.

From (5.1) and (5.2), it is also clear that the coefficients of both the connection form and the curvature form are polynomials in the coefficients of the metric and the structure coefficients. □

The above means that we may disregard the underlying Lie group G that we started with and solely consider the Lie algebra $\mathfrak{g}$. Thus, we may derive results for entire equivalence classes of Lie groups all at once by investigating their Lie algebras.

Theorem 24 ([**Her10**]). *The Ricci tensor takes the form*

$$R_{ij} = Q_{ij} - \tfrac{1}{2}K_{ij} - Z_{ij},$$

where K_{ij} is the Killing form of $\mathfrak{g}$, with $K_{ij} = C^k_{\ il}C^l_{\ jk}$, and Q_{ij} and Z_{ij} are defined by

$$Q_{ij} = \tfrac{1}{4}C_{ikl}C_j^{\ kl} - \tfrac{1}{2}C_{kli}C^{kl}_{\ \ j}$$

and

$$Z_{ij} = \tfrac{1}{2}C^k_{\ kl}C_{ij}^{\ \ l} + \tfrac{1}{2}C^k_{\ kl}C_{ji}^{\ \ l},$$

respectively.

Proof. From equation (5.2) and the definition of the Ricci tensor, the coefficients R_{ij} of the Ricci tensor must be

$$R_{ij} = R^k_{\ ikj} = -\Gamma^k_{\ li}C^l_{\ kj} + \Gamma^k_{\ kl}\Gamma^l_{\ ji} - \Gamma^k_{\ jl}\Gamma^l_{\ ki}. \tag{5.3}$$

From here, it is a matter of carefully expanding out the Christoffel symbols.

The first term in (5.3) becomes, after expanding and re-indexing:

$$-\Gamma^k_{\ li}C^l_{\ kj} = -\tfrac{1}{2}C_{kli}C^{kl}_{\ \ j} + \tfrac{1}{2}C_{ikl}C^{kl}_{\ \ j} - \tfrac{1}{2}C^k_{\ il}C^l_{\ jk}.$$

The second term in (5.3) becomes, after expanding, re-indexing, utilizing the Jacobi identity, and knowing that the Killing form is symmetric:

$$\begin{aligned}
\Gamma^k_{\ kl}\Gamma^l_{\ ji} &= \tfrac{1}{4}m^{kq}m^{lr}\Big(C_{klq}C_{jir} - C_{klq}C_{irj} - C_{klq}C_{rji} \\
&\quad - C_{lqk}C_{jir} + C_{lqk}C_{irj} + C_{lqk}C_{rji} \\
&\quad - C_{qkl}C_{jir} + C_{qkl}C_{irj} + C_{qkl}C_{rji}\Big) \\
&= -\tfrac{1}{2}C^k_{\ kl}\Big(C_{ij}^{\ \ l} + C_{ji}^{\ \ l}\Big) - \tfrac{1}{2}C^k_{\ kl}C^l_{\ ij} + \tfrac{1}{4}C_{lk}^{\ \ k}\Big(-C_{ji}^{\ \ l} - C_{ij}^{\ \ l} + C^l_{\ ji}\Big) \\
&= -\tfrac{1}{2}C^k_{\ kl}\Big(C_{ij}^{\ \ l} + C_{ji}^{\ \ l}\Big) - \tfrac{1}{2}\Big(-C^k_{\ il}C^l_{\ jk} + C^k_{\ jl}C^l_{\ ik}\Big) + \tfrac{1}{4}C_{lk}^{\ \ k}\Big(-C_{ji}^{\ \ l} - C_{ij}^{\ \ l} + C^l_{\ ji}\Big) \\
&= -\tfrac{1}{2}C^k_{\ kl}\Big(C_{ij}^{\ \ l} + C_{ji}^{\ \ l}\Big).
\end{aligned}$$

The third term in (5.3) becomes, after expanding and re-indexing slightly:

$$\begin{aligned}
-\Gamma^k_{\ jl}\Gamma^l_{\ ki} &= -\tfrac{1}{4}m^{kq}m^{lr}\Big(C_{jlq}C_{kir} - C_{jlq}C_{irk} - C_{jlq}C_{rki} \\
&\quad - C_{lqj}C_{kir} + C_{lqj}C_{irk} + C_{lqj}C_{rki} \\
&\quad - C_{qjl}C_{kir} + C_{qjl}C_{irk} + C_{qjl}C_{rki}\Big) \\
&= \tfrac{1}{4}C_{ikl}C_j^{\ kl} - \tfrac{1}{2}C_{ikl}C^{kl}_{\ \ j}.
\end{aligned}$$

Summing these terms, we obtain our result.

□

The tensor Z is identically zero for the unimodular Lie groups (a class of Lie groups which we have not discussed in this thesis, but which include the compact Lie groups, and therefore the semisimple Lie groups), and both Z and K are identically zero for nilpotent Lie groups ([**Her10**]). Unsurprisingly, the Lie algebra structure has consequences for the possible geometries that can arise from a left-invariant metric. For instance, a semisimple Lie group will always have a non-zero contribution to its Ricci tensor from the Killing form.

5.2 Group Actions on $\mathfrak{g}$

The canonical action of $GL(p+q,\mathbb{R})$ on $\mathfrak{g} \simeq \mathbb{R}^{p+q}$ (as vector spaces) induces a canonical action on any arbitrary tensor product of copies of $\mathbb{R}^{p+q}$ and $(\mathbb{R}^{p+q})^*$ (see Section 3.2). As such, there is in particular a canonical action of $GL(p+q,\mathbb{R})$ on

$$(\mathbb{R}^{p+q})^* \odot (\mathbb{R}^{p+q})^* := \mathrm{Sym}((\mathbb{R}^{p+q})^* \otimes (\mathbb{R}^{p+q})^*),$$

which is equivalent to an action of $GL(p+q,\mathbb{R})$ on the space of pseudo-Riemannian metrics on $\mathbb{R}^{p+q}$ defined by

$$(g \cdot m)(X,Y) = m(g^{-1} \cdot X, g^{-1} \cdot Y).$$

Moreover, the $GL(p+q,\mathbb{R})$ acts on $\mathbb{R}^{p+q}$ induces an action on the space of structure coefficients $\mathfrak{G}(p+q,\mathbb{R})$, as described in Section 3.2. For a given Lie algebra $\mathfrak{g}$ with corresponding structure coefficients given by $C \in \mathfrak{G}(p+q,\mathbb{R})$, an arbitrary pseudo-Riemannian metric m' on $\mathbb{R}^{p+q}$ becomes a left-invariant metric for any Lie group G which has as its Lie algebra $\mathfrak{g}$. By the results of Section 4.1, there exists $g \in GL(p+q,\mathbb{R})$ such that $g \cdot m' = m$, where m is a metric of a form of our choosing. This same element g simultaneously acts on C to produce a new structure tensor $C' = g \cdot C$.

The Lie group $GL(p+q,\mathbb{R})$ acts transitively on the space of pseudo-Riemannian metrics of signature (p,q). Since $O(p,q)$ leaves the form of any metric invariant, this space may be parametrized as $GL(p+q,\mathbb{R})/O(p,q)$, a homogeneous space since $O(p,q)$ is closed. As noted in Section 3.2, the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ fixes the structure coefficients, hence the space of structure coefficients for a particular Lie algebra $\mathfrak{g}$ is parametrized by the homogeneous space $GL(p+q,\mathbb{R})/\mathrm{Aut}(\mathfrak{g})$. By Proposition 25 and equation (5.1), the combined action of $O(p,q)$ and $\mathrm{Aut}(\mathfrak{g})$ fixes the connection coefficients of the Levi-Civita connection. In other words, the group $H := O(p,q) \cap \mathrm{Aut}(\mathfrak{g})$ leaves invariant any polynomial or tensor obtained algebraically from the connection coefficients. The group H, being the intersection of two closed subgroups, is a closed subgroup of $GL(p+q,\mathbb{R})$, and is thus a Lie subgroup of $GL(p+q,\mathbb{R})$ by Theorem 1. Hence the homogeneous space $GL(p+q,\mathbb{R})/H$ parametrizes the possible connection coefficients of a Lie algebra $\mathfrak{g}$, and consequently the possible curvature properties of $\mathfrak{g}$. An appropriate choice of representative element in $GL(p+q,\mathbb{R})$ for each equivalence class in $GL(p+q,\mathbb{R})/H$ can facilitate computation of various geometric properties.

In summary, by Proposition 25 and equation (5.1), we may investigate all possible left-invariant metrics on $\mathfrak{g}$ and their associated curvature properties by fixing a particular metric form m and letting the structure coefficients of the Levi-Civita connection relative to the Lie algebra vary under action of $GL(p+q,\mathbb{R})$ on $\mathfrak{g}$.

5.3 Group Actions on $\mathfrak{G}(p+q,\mathbb{R})$

We may look for specific Lie algebras with particular geometric properties. Fixing p and q, we may let the group $O(p,q)$ act on $\mathfrak{G}(p+q,\mathbb{R})$ through the natural action obtained by taking tensor products of the the natural representation $\Phi\colon O(p,q) \to GL(p+q,\mathbb{R})$. In other words, for a fixed pseudo-Riemannian metric m on $\mathfrak{g}$, the group $O(p,q)$ preserves m but does alter the connection and curvature coefficients ω_{ij} and Ω_{ij}, respectively, through its action on $\mathfrak{G}(p+q,\mathbb{R}) \subset V \otimes V^* \otimes V^*$ by $\Phi \otimes \Phi^* \otimes \Phi^*$, where we regard $V \simeq \mathfrak{g}$ solely as a vector space.

Any bilinear form μ on a vector space V gives rise to a bilinear form μ^* on V^*, and we may take the tensor product of two bilinear forms $\mu\colon V \otimes V \to \mathbb{K}$ and $\nu\colon W \otimes W \to \mathbb{K}$ to obtain a new bilinear form on $V \otimes W$ by defining the multilinear map

$$mu \times \nu)(v \otimes w, v' \otimes w') = \mu(v \otimes v')\nu(w \otimes w'), \tag{5.4}$$

which descends to the unique tensor product $\mu \otimes \nu$ by the universal mapping property. These constructions may be extended to arbitrary tensor products $V_1 \otimes \cdots \otimes V_n \otimes V_1^* \otimes \cdots \otimes V_m^*$.

It is relatively simple to show that $\mu \otimes \nu$ is nondegenerate if μ and ν are each nondegenerate: $\mu \otimes \nu$ would be nondegenerate if for a fixed $v \otimes w$ and all $v' \otimes w'$, $\mu \otimes \nu(v \otimes w, v' \otimes w') = 0$ implies that $v \otimes w = 0$. Since $\mu \otimes \nu((v \otimes w) \otimes (v' \otimes w')) = \mu(v \otimes v')\nu(w \otimes w')$, it must be the case that either $\mu(v \otimes v') = 0$ for all v' or $\nu(w \otimes w') = 0$ for all w'. By nondegeneracy of μ and ν, either v or w must be 0. Additionally, equation (5.4) also implies that if μ and ν are symmetric, then so is $\mu \otimes \nu$.

From this, we can also see that the classical Lie subgroups of $GL(V)$ embed into

$$GL(V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*)$$

such that the extensions of the defining forms (euclidean, pseudo-Riemannian, symplectic) for the classical groups to the tensor product $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$. Consequently, the group $O(p,q)$ with representation $\Phi\colon O(p,q) \to GL(V \otimes V^* \otimes V^*)$ as described in Section 4.2 is a real reductive group in accordance with Definition 31, as the Cartan decomposition $O(p,q) = O(p) \times O(q) \times \exp(\mathfrak{p})$ is compatible with the euclidean inner product on $V \otimes V^* \otimes V^*$ inherited from the euclidean inner product on V.

The real weight decomposition of $O(p,q)$ also extends to the arbitrary tensor product $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$. Let the weights relative to $\mathfrak{t}$ be $\lambda_1, \ldots, \lambda_{2p}$ for the standard representation on V^* (as per Section 4.2). There are p positive and p negative weights, as seen from the analysis in Section 4.2. Then $Hv_{\lambda_i} = \lambda_i(H)v_{\lambda_i}$ for any $v_{\lambda_i} \in V_{\lambda_i}$, and $\exp(H)v_\lambda = \exp(\lambda(H))v_\lambda$. For the tensor element $v_{\lambda_{i_1}} \otimes \cdots \otimes v_{\lambda_{i_n}}$,

$$H(v_{\lambda_{i_1}} \otimes \cdots \otimes v_{\lambda_{i_n}}) = \lambda_{i_1}(H) \ldots \lambda_{i_n}(H) v_{\lambda_{i_1}} \otimes \cdots \otimes v_{\lambda_{i_n}},$$

and

$$\begin{aligned} \exp(H)(v_{\lambda_{i_1}} \otimes \cdots \otimes v_{\lambda_{i_n}}) &= \exp(\lambda_{i_1}(H)) \ldots \exp(\lambda_{i_n}(H))(v_{\lambda_{i_1}} \otimes \cdots \otimes v_{\lambda_{i_n}}) \\ &= \exp(\lambda_{i_1}(H) + \cdots + \lambda_{i_n}(H))(v_{\lambda_{i_1}} \otimes \cdots \otimes v_{\lambda_{i_n}}). \end{aligned}$$

If we account for duplicates among the λ_{i_j}, then we may write $\lambda_{i_1} + \cdots + \lambda_{i_n} = \mathbf{b} \cdot \lambda$, where $(b_1, \ldots, b_p) = \mathbf{b} \in \mathbb{Z}^p$ and $\lambda = (\lambda_1, \ldots, \lambda_p)$ is a vector of the positive weights of the standard representation on V. Acting on copies of V^* merely flips the sign of any given weight. $b_1 + \cdots + b_p \leq n$, where n is the number of copies of V and V^* tensored together. Thus,

$$V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^* = \bigoplus_{\mathbf{b}\cdot\lambda} V_{\mathbf{b}\cdot\lambda}$$

is a weight decomposition, where we sum over all admissible $\mathbf{b} \in \mathbb{Z}^p$.

Let

$$T = \sum_{\mathbf{b}\in B} T_{\mathbf{b}} v_{\mathbf{b}},$$

where $T_{\mathbf{b}} v_{\mathbf{b}}$ means

$$T_{\mathbf{b}} v_{\mathbf{b}} = T^{i_1 \ldots i_k}_{j_1 \ldots j_l} v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v^{j_1} \otimes \cdots \otimes v^{j_l},$$

such that $(\lambda_{i_1}, \ldots, \lambda_{i_k}, \lambda_{j_1}, \ldots, \lambda_{j_l}) = \mathbf{b} \cdot \lambda$, where v_i is a basis for $\bigoplus_{\mathbf{b}\cdot\lambda} V_{\mathbf{b}\cdot\lambda}$, and B is the set of admissible $\mathbf{b} \in \mathbb{Z}^p$. If $\mathbf{b} \cdot \lambda < 0$ for all $\mathbf{b}$, then for any $H \in \mathfrak{t}$,

$$\lim_{t\to\infty} \exp(tH) \cdot T = 0.$$

Clearly, $O(p, q) \cdot 0 = 0$, with 0 a closed set in $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$. Therefore, by Theorem 20, 0 is the unique closed orbit in the closure of the orbit of $O(p, q)$ acting on T. Consequently, all orbits with 0 as a limit point are uniquely determined by the values the invariant polynomials take at the limit point 0. For any polynomials without constant term, that value is necessarily 0.

If we now apply this theory to $\mathfrak{G}(p + q, \mathbb{R}) \subset V \otimes V^* \otimes V^*$, we see from Proposition 25 and equation (5.1) that the group $O(p, q)$ will preserve the metric and act on the structure coefficients, and that if $\mathbf{b} \cdot \lambda < 0$ then all polynomial invariants constructed from the connection coefficients will be 0. Thus, we have a means of searching for Lie algebras with the special property that all curvature invariants (polynomials constructed from the curvature coefficients that are invariant under action of $O(p, q)$ on the tensor bundle) are 0. We could also have demanded that $\mathbf{b} \cdot \lambda > 0$ for all $\mathbf{b}$ for the same effect.

All the above leads to the following results:

Theorem 25 (Thm. 2.1, 2.2, [**Her12**])**.** *A tensor*

$$T = \sum_{\mathbf{b}\in B} T_{\mathbf{b}} v_{\mathbf{b}},$$

has vanishing polynomial curvature invariants if and only if either $\mathbf{b} \cdot \lambda < 0$ *or* $\mathbf{b} \cdot \lambda > 0$ *for all* $\mathbf{b} \in B$.

Theorem 26. *A Lie algebra* $\mathfrak{g}$ *with metric tensor* ρ *of the form*

$$\rho = 2(e^1 e^{p+1} + \cdots + e^p e^{2p}) + (e^{2p+1} e^{2p+1} + \cdots + e^{p+q} e^{p+q}).$$

has polynomial curvature invariants identically equal to 0 *if and only if* $\mathbf{b} \cdot \lambda < 0$ *or* $\mathbf{b} \cdot \lambda > 0$ *for all* $\mathbf{b}$ *such that* $V_{\mathbf{b}\cdot\lambda}$ *is nonzero in the weight decomposition*

$$V \otimes V^* \otimes V^* = \bigoplus_{\mathbf{b}\cdot\lambda} V_{\mathbf{b}\cdot\lambda}.$$

Theorem 26 follows from Theorem 25.

5.4 Example Applications and New Results

We now wish to apply some of this theory, and find examples.

We might try to start a search among nilpotent Lie algebras of low dimension, and look for any that admit pseudo-Riemannian metrics with desireable properties. The nilpotent Lie algebras are good candidates since they are of a comparatively simple structure, which one may hope will more easily fit the pattern described in Theorem 26. In five dimensions, there are a total of six nilpotent Lie algebras [**ŠW14**]. We obtain the following result:

Theorem 27. *For each of the six nilpotent Lie algebras of dimension five, there exists a pseudo-Riemannian metric such that the curvature tensor is non-zero, but the Ricci tensor and all polynomial curvature invariants are identically zero.*

Note. This result for the Lie algebra $\mathfrak{n}_{5,5}$ was know to my advisor Sigbjørn Hervik, the remaining five algebras were then easy to work out using the techniques outlined in this thesis.

Proof. We shall fix a metric

$$\rho = 2(e^1e^3 + e^2e^4) + e^5e^5,$$

of signature $(2,3)$, and look for five-dimensional, nilpotent Lie algebras such that $\mathbf{b}\cdot\lambda$ and Theorem 26 applies. The hope is to realize the nilpotent Lie algebras of five dimensions, as listed in [**ŠW14**], such that we maintain the metric ρ and the Lie algebra structure coefficients are of a form such that $\mathbf{b}\cdot\lambda < 0$ in the weight decomposition.

In [**ŠW14**], the six nilpotent Lie algebras of dimension five are listed as $\mathfrak{n}_{5,1-6}$. In five dimensions, the structure coefficients such that $\mathbf{b}\cdot\lambda < 0$ for the metric ρ are:

- C^1_{12}
- C^2_{12}
- C^3_{12}, C^3_{13}, C^3_{14}, C^3_{15}, C^3_{23}, C^3_{24}, and C^3_{25}
- C^4_{12}, C^4_{13}, C^4_{14}, C^4_{15}, C^4_{23}, C^4_{24}, and C^4_{25}
- C^5_{12}, C^5_{15}, and C^5_{25}

Here, indices $1,2$ give a positive contribution in upper position, and a negative contribution in lower position, while the indices $3,4$ give a negative contribution in upper position and a positive contribution in lower position. Amongst the above structure coefficients, we can realize the six Lie algebras with curvature tensors R as follows:

1. $\mathfrak{n}_{5,1}$: $C^3_{15} = 1$ and $C^4_{25} = 1$.
$$R = -f_3 \otimes f^2 \otimes f^1 \otimes f^2 + f_3 \otimes f^2 \otimes f^2 \otimes f^1 + f_4 \otimes f^1 \otimes f^1 \otimes f^2 - f_4 \otimes f^1 \otimes f^2 \otimes f^1$$
2. $\mathfrak{n}_{5,2}$: $C^3_{51} = 1$, $C^4_{52} = 1$, and $C^5_{12} = 1$
$$R = -\frac{7}{4} f_3 \otimes f^2 \otimes f^1 \otimes f^2 + \frac{7}{4} f_3 \otimes f^2 \otimes f^2 \otimes f^1 + \frac{7}{4} f_4 \otimes f^1 \otimes f^1 \otimes f^2 - \frac{7}{4} f_4 \otimes f^1 \otimes f^2 \otimes f^1$$

3. $\mathfrak{n}_{5,3}$: $C^3_{14} = 1$ and $C^3_{25} = 1$

$$R = \frac{1}{4} f_3 \otimes f^2 \otimes f^1 \otimes f^2 - \frac{1}{4} f_3 \otimes f^2 \otimes f^2 \otimes f^1 - \frac{1}{4} f_4 \otimes f^1 \otimes f^1 \otimes f^2 + \frac{1}{4} f_4 \otimes f^1 \otimes f^2 \otimes f^1$$

4. $\mathfrak{n}_{5,4}$: $C^3_{12} = 1$, $C^4_{51} = 1$, and $C^4_{32} = 1$

$$R = \frac{1}{4} f_3 \otimes f^2 \otimes f^1 \otimes f^2 - \frac{1}{4} f_3 \otimes f^2 \otimes f^2 \otimes f^1 - \frac{1}{4} f_4 \otimes f^1 \otimes f^1 \otimes f^2 + \frac{1}{4} f_4 \otimes f^1 \otimes f^2 \otimes f^1$$

5. $\mathfrak{n}_{5,5}$: $C^3_{52} = 1$, $C^4_{32} = 1$, and $C^5_{12} = 1$

$$R = -f_3 \otimes f^2 \otimes f^1 \otimes f^2 + f_3 \otimes f^2 \otimes f^2 \otimes f^1 + f_4 \otimes f^1 \otimes f^1 \otimes f^2 - f_4 \otimes f^1 \otimes f^2 \otimes f^1$$

6. $\mathfrak{n}_{5,6}$: $C^3_{52} = 1$, $C^4_{32} = 1$, $C^4_{51} = 1$, and $C^5_{12} = 1$

$$R = \frac{1}{4} f_3 \otimes f^2 \otimes f^1 \otimes f^2 - \frac{1}{4} f_3 \otimes f^2 \otimes f^2 \otimes f^1 - \frac{1}{4} f_4 \otimes f^1 \otimes f^1 \otimes f^2 + \frac{1}{4} f_4 \otimes f^1 \otimes f^2 \otimes f^1$$

These all have non-zero curvature tensors relative to ρ, but all polynomial curvature invariants are zero by construction. The curvature tensors are all scalar multiples of each other. Incidentally, they all also have Ricci tensors identically zero, which from Theorem 24 means that the tensor $Q \equiv 0$. □

Further examples of Lie groups with zero polynomial curvature invariants can be found amongst product groups and certain semi-direct product groups, such as the affine Lie subgroups.

Theorem 28. *Any product Lie group $G^p \times \mathbb{R}^q$, with $p \leq q$ indicating the dimension of the Lie group G^p, can be equipped with a pseudo-Riemannian metric such that all polynomial curvature invariants are identically zero, and the Ricci tensor R is equal to $-\frac{1}{2}K$, where K is the Killing form that $G^p \times \mathbb{R}^q$ inherits from G^p.*

Proof. Any product group $G^p \times \mathbb{R}^q$ — with G^p indicating that the Lie group G is of dimension p — under the metric

$$\rho = 2(f^1 f^{1+p} + \cdots + f^p f^{2p}) + (f^{2p+1} f^{2p+1} + \cdots + f^{p+q} f^{p+q})$$

has its non-zero structure coefficients all satisfying the condition $\mathbf{b} \cdot \lambda < 0$ for the weight decomposition of $O(p,q)$ acting on tensor products of $\mathfrak{g}$. That is to say, we can realize the Lie algebra in terms of the structure coefficients C^k_{ij}, $i, j, k \in \{1, \ldots, p\}$, relative to a basis $e_1, \ldots, e_{p+q}$ for the Lie algebra, and these coefficients all have strictly negative weight. Consequently, these groups all have zero polynomial curvature invariants.

From Theorem 5.3, we have that the components of the Ricci tensor are given by terms Q, Z, and the Killing form K. Of these, the terms Q and Z must be identically zero.

We first examine Q. Recall:

$$Q_{ij} = \tfrac{1}{4} C_{ikl} C_j{}^{kl} - \tfrac{1}{2} C_{kli} C^{kl}{}_j.$$

If $C_{ikl}C_j$ and C_{kli} are to be non-zero, then l must be less than or equal to p. But then, under the metric ρ, $C_j{}^{kl}$ and $C^{kl}{}_j$ must be identically zero, as the raised index $l \leq p$ in the second and third position must correspond to a lowered index $l' > p$ in the second or third position, for which the structure coefficients are zero.

Similarly, for Z:

$$Z_{ij} = \frac{1}{2}C^k{}_{kl}C_{ij}{}^l + \frac{1}{2}C^k{}_{kl}C_{ji}{}^l.$$

If $C^k{}_{kl}$ is to be non-zero, then l must be less than or equal to p in the lowered position, but that leads to $C_{ij}{}^l$ and $C_{ij}{}^l$ having $l \leq p$ in the raised position, corresponding to $l > p$ in the lowered position such that $C_{ij}{}^l$ and $C_{ij}{}^l$ must be zero.

This leaves just the Killing form K:

$$K_{ij} = C^k{}_{il}C^l{}_{jk}.$$

Here, there are no conflicts as long as all indices are less than or equal to p, which also means that the Killing form of $G^p \times \mathbb{R}^q$ is just the embedded Killing form of G^p.

$\square$

Theorem 29. *If G^p is a Lie group acting faithfully and linearly on $\mathbb{R}^p$ — with G^p indicating that the Lie group is of dimension p — then the affine semi-direct product $\mathbb{R}^p \ltimes G^p$ can be equipped with a pseudo-Riemannian metric such that all polynomial curvature invariants are identically zero.*

Proof. We can represent $\mathbb{R}^p \ltimes G^p$ in $GL(p+1, \mathbb{R})$ as the set of matrices of the form

$$A = \begin{bmatrix} \Phi(g) & x \\ 0 & 1 \end{bmatrix},$$

where $\Phi(g) \in GL(p, \mathbb{R})$ is the representative matrix element of $g \in G^p$, and $x \in \mathbb{R}^p$. Then the Lie algebra of $\mathbb{R}^p \ltimes G^p$, $\mathbb{R}^p \ltimes \mathfrak{g}^p$, can be represented by matrices

$$Q = \begin{bmatrix} \phi(X) & u \\ 0 & 0 \end{bmatrix},$$

where $\phi(X) \in \mathfrak{gl}(p, \mathbb{R})$ is the representative matrix element of $X \in \mathfrak{g}^p$, and $u \in \mathbb{R}^p$. If we simplify this expression to $Q = (X, u)$, then the bracket operation is

$$[(X, u), (Y, v)]_{\mathbb{R}^p \ltimes \mathfrak{g}^p} = ([X, Y]_{\mathfrak{g}^p}, X \cdot v - Y \cdot u).$$

From this, we can see that if we let $e_1, \ldots, e_p$ be the basis elements for $\mathfrak{g}^p$, and $e_{p+1}, \ldots, e_{2p}$ be the basis elements for $\mathbb{R}^p$, then the structure coefficients C^k_{ij} come in two categories: The first is when a basis element e_i, $i \leq p$, interacts with another basis element e_j, $j \leq p$, in which case, the bracket product is expressible in basis elements $e_{i_1}, \ldots, e_{i_k}$ such that $i_l \leq p$, and the structure coefficient C^k_{ij} when $i, j \leq p$ is only non-zero when $k \leq p$ also. The second is when a basis element e_i, $i \leq p$, interacts with another basis element e_j, $j > p$, in which case, the bracket product is expressible in basis elements $e_{i_1}, \ldots, e_{i_k}$ such that $i_l > p$, and the structure coefficient C^k_{ij} when $i \leq p$ and $j > p$ is only non-zero when $k > p$ also. All other coefficients are zero. Relative to the metric ρ,

$$\rho = 2(f^1 f^{1+p} + \cdots + f^p f^{2p}),$$

this means that all weights are strictly less than 0, i.e. that $\mathbf{b} \cdot \lambda < 0$. Thus, the polynomial curvature invariants are all identically zero. $\square$

Note that we are no longer guaranteed to have $R = -\frac{1}{2}K$ for the Ricci tensor; the tensors Q and Z may be non zero for the semi-product group $\mathbb{R}^p \ltimes G^p$. Nor is it the case that the Killing form of the group $\mathbb{R}^p \ltimes G^p$ is necessarily the embedded Killing form of G^p.

To give a few concrete examples, consider the affine groups $\mathbb{R}^3 \ltimes O(3)$ and $\mathbb{R}^3 \ltimes SL(2,\mathbb{R})$, with metric

$$\rho = 2(e^1 e^4 + e^2 e^5 + e^3 e^6).$$

The structure coefficients for $\mathbb{R}^3 \ltimes O(3)$ are then:

- For $[\mathfrak{o}(3), \mathfrak{o}(3)]$: $C^1_{23} = 1$, $C^2_{31} = 1$, and $C^3_{12} = 1$.
- For $[\mathfrak{o}(3), \mathbb{R}^3]$: $C^5_{14} = -1$, $C^4_{15} = 1$, $C^6_{24} = 1$, $C^4_{26} = -1$, $C^6_{35} = -1$, and $C^6_{36} = 1$.

And for $\mathbb{R}^3 \ltimes SL(2,\mathbb{R})$:

- For $[\mathfrak{sl}(2,\mathbb{R}), \mathfrak{sl}(2,\mathbb{R})]$: $C^1_{12} = 2$, $C^2_{13} = -1$, and $C^3_{23} = 2$.
- For $[\mathfrak{sl}(2,\mathbb{R}), \mathbb{R}^3]$: $C^4_{15} = 2$, $C^4_{24} = -2$, $C^5_{16} = -1$, $C^5_{34} = 1$, $C^6_{26} = 2$, and $C^6_{35} = -2$.

These groups then have non-zero curvature tensor and Ricci tensor, but all polynomial curvature invariants are identically zero.

Chapter 6

Summary

We have explored results in Lie theory, representation theory, geometric invariant theory, and differential geometry with the goal of understanding left-invariant pseudo-Riemannian metrics on Lie groups. Through the use of representation theory and geometric invariant theory, we have seen how the algebraic structure of a Lie algebra $\mathfrak{g}$ interacts with a metric on $\mathfrak{g}$ to determine the curvature properties of any Lie group with $\mathfrak{g}$ as its Lie algebra. Finally, we have discovered new results, in the form of theorems 27, 28, and 29, that to my knowledge have not previously appeared in publication.

Appendix A

Nilpotent Lie algebras

The nilpotent Lie algebras of dimension five given in [**ŠW14**] are as follows:

1. $\mathfrak{n}_{5,1}$:

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	0
e_3			0	0	e_1
e_4				0	e_2

2. $\mathfrak{n}_{5,2}$:

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	0
e_3			0	e_2	e_1
e_4				0	e_3

3. $\mathfrak{n}_{5,3}$:

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	e_1	0
e_3			0	0	e_1
e_4				0	0

4. $\mathfrak{n}_{5,4}$:

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	e_1
e_3			0	e_1	0
e_4				0	e_2

5. $\mathfrak{n}_{5,5}$:

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	e_1
e_3			0	0	e_2
e_4				0	e_3

6. $\mathfrak{n}_{5,6}$:

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	e_1
e_3			0	e_1	e_2
e_4				0	e_3

Bibliography

[Ber01] Rolf Berndt. *An Introduction to Symplectic Geometry.* American Mathematical Society, Providence, Rhode Island, 2001.

[BL17] Christoph Böhm and Ramiro A. Lafuente. Real geometric invariant theory. *arXiv e-prints*, page arXiv:1701.00643, January 2017.

[DK00] J. J. Duistermaat and J. A. C. Kolk. *Lie Groups.* Springer-Verlag, Berlin, 2000.

[FH04] William Fulton and Joe Harris. *Representation Theory.* Springer Science+Business Media, New York, 2004.

[GW10] Roe Goodman and Nolan R. Wallach. *Symmetry, Representations, and Invariants.* Springer Science+Business Media, New York, 2010.

[Hel78] Sigurdur Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces.* American Mathematical Society, Providence, Rhode Island, 1978.

[Her10] Sigbjorn Hervik. Negatively curved left-invariant metrics on Lie groups. In *CMS/CAIMS Summer 2004 Meeting*, 2 2010.

[Her12] Sigbjørn Hervik. Pseudo-riemannian VSI spaces II. *Classical and Quantum Gravity*, 29(9):095011, apr 2012.

[Jos11] Jürgen Jost. *Riemannian Geometry and Geometric Analysis.* Springer-Verlag, Berlin, 2011.

[Kir08] Alexander Kirillov, Jr. *An Introduction to Lie Groups and Lie Algebras.* Cambridge University Press, Cambridge, 2008.

[Kna05] Anthony W. Knapp. *Lie Groups Beyond an Introduction.* Birkhäuser Boston, Boston, Massachusetts, 2005.

[Lee97] John M. Lee. *Riemannian Manifolds.* Springer Science+Business Media, New York, 1997.

[Lee11] John M. Lee. *Introduction to Topological Manifolds.* Springer Science+Business Media, New York, 2011.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds.* Springer Science+Business Media, New York, second edition, 2013.

[Mos55] G. D. Mostow. Self-adjoint groups. *Annals of Mathematics*, 62(1):44–55, 1955.

[O'N83] Barrett O'Neill. *Semi-Riemannian Geometry*. Academic Press, San Diego, 1983.

[Pal57] Richard Palais. A global formulation of the lie theory of transformation groups. *Memoirs of the American Mathematical Society*, 0:0–0, 01 1957.

[Pro07] Claudio Procesi. *Lie Groups*. Springer Science+Business Media, New York, 2007.

[RS90] R. W. Richardson and P. J. Slodowy. Minimum vectors for real reductive algebraic groups. *Journal of the London Mathematical Society*, s2-42(3):409–429, 1990.

[Sep10] Mark R. Sepanski. *Compact Lie Groups*. Springer Science+Business Media, New York, 2010.

[ŠW14] Libor Šnobl and Pavel Winternitz. *Classification and Identification of Lie Algebras*. American Mathematical Society, Providence, Rhode Island, 2014.

[Tu11] Loring Tu. *An Introduction to Manifolds*. Springer Science+Business Media, New York, 2011.

[Tu17] Loring Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes*. Springer Nature, Switzerland, 2017.

[Wal18] Nolan R. Wallach. *Harmonic Analysis on Homogeneous Spaces*. Dover Publications, Mineola, New York, 1973, 2018.