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Guaranteed cost estimation and control for a class of nonlinear systems subject to actuator saturation

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ABSTRACT

The problems of guaranteed cost estimation (GCE) and guaranteed cost control (GCC) concern designing a state observer or a controller, respectively, such that some performance is maintained below an upper bound. This paper provides a matrix inequality-based observer/controller design procedure to perform GCE and GCC in a class of nonlinear systems affected by actuator saturation. In particular, this class of systems corresponds to those for which the origin of the state space is an equilibrium point when null inputs are considered, and the nonlinearity is differentiable with respect to the state and linear with respect to the saturated input. Simulation results obtained using a numerical example and a rotational single-arm inverted pendulum are used to illustrate the effectiveness of the proposed design procedure.

1. Introduction

The problem of optimal control consists in finding a control law for a given system so that some performance criterion is minimized. For a nonlinear generic system, this problem can be solved using Ponytryagin's maximum principle or solving the Hamilton-Jacobi-Bellman (HJB) equation [8,13,49]. However, in most of the cases, obtaining the analytical solution is a hard task, which motivated the search for alternative approaches to perform the above-mentioned minimization. A quite successful approach is the so-called guaranteed cost control (GCC), which was proposed first by Chang and Peng [3] as a way to guarantee the performance for uncertain systems by requiring it to be below some upper bound. Early results on GCC were obtained in the 1990s by Ian R. Petersen and his colleagues, who studied the design of robust state-feedback controllers that minimize an upper bound on a quadratic cost function [25,26]. A computationally efficient framework for finding the optimal guaranteed cost controller was provided by linear matrix inequalities (LMIs: see [31] for a tutorial), such that several approaches were developed, e.g. [6,43]. LMIs were also used in [19] to achieve GCC in bilateral teleoperation systems which included time-varying delays and model uncertainty. Although the initial attention of the research community was devoted to linear time invariant (LTI) systems, soon it was driven towards other classes of systems which could take into account variability in time, for example linear systems with varying parameters (LPV) [28,30] or fuzzy Takagi-Sugeno (TS) systems [9,41,42,46]. GCC is still a very active field of research, with several works appearing every year in the literature, mainly dealing with nonlinear systems, e.g. [11,15,33,39,45].

Strictly related to GCC is the problem of guaranteed cost estimation (GCE) in which instead of designing a controller, one wishes to design a state observer that minimizes some upper bound on a performance criterion, which is a function of the estimation error and the measurement noise. [25,26] showed that GCE extended the much celebrated Kalman filter to uncertain systems. A GCE-based approach was later developed by Petersen [24] using a class of state estimators that include copies of the globally Lipschitz system nonlinearities within them. More recently, Ishihara et al. [12] developed an approach based on regularization and penalty functions to solve the optimal filtering problem for discrete-time systems with norm–bounded parametric uncertainties.

Another topic that has attracted much attention by the control theory research community is how to deal with saturation nonlinearities. Saturation can be found everywhere in physical applications, since real-world actuators are constrained in the number of deliverable control actions. The control techniques that ignore these actuator limits can be affected by degraded performance or

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instability of the closed-loop system. Hence, the analysis and synthesis of control systems with saturating actuators has been investigated by several works, see e.g. [4,20,27,37,38]. The developed approaches can be divided into two main cathegories: the *two-step* paradigm (also referred to as *anti-windup compensation* [14,50]) ignores the saturation at the controller design step, and handles it by adding a compensator; on the other hand, the *one-step* paradigm (also called *direct control design* [7,36]) takes into account the saturation during the controller design phase. Among the most recent results concerning this topic, one may mention [48], where the problem of input saturation was solved by introducing an auxiliary design system, Shahri et al. [32], which employed the Lyapunov direct method for the stability analysis of fractional order linear systems subject to input saturation, and [29], which proposed a virtual actuator-based fault tolerant control strategy to deal with actuator saturations in unstable linear systems.

This work is motivated by the big importance held by the optimal design of observer and controller gains in automatic control systems. The literature review has shown that, although there exist a few results on GCC for systems with input saturation, e.g. [17,44,47], the problem of GCE for these systems has not been considered yet. Hence, this paper aims at developing a design procedure that addresses both the GCE and the GCC for a class of nonlinear saturating systems, while at the same time analysing the case in which the controller uses the estimate produced by the observer in order to update the control law.

More specifically, this paper proposes a matrix inequality-based guaranteed cost estimation and control design procedure for a class of discrete-time nonlinear systems subject to actuator saturation. This class of systems corresponds to those for which the origin of the state space is an equilibrium point when null inputs are considered, and the nonlinearity is differentiable with respect to the state and linear with respect to the saturated input. It is worth highlighting that these nonlinearities, which have been considered previously in the context of fault estimation by Zhu [2,40], encompass cases which cannot be dealt using the traditional bounding box method [35]. Hence, an alternative approach based on the application of the mean value theorem, as described by Lewis et al. [1,23], must be obtained.

The contributions of the paper can be resumed as follows:

1. a polytopic approach based on the application of the mean value theorem is described for the characterization of a class of discrete-time nonlinear systems subject to actuator saturation;
2. sufficient conditions for the synthesis of a state observer that achieves GCE and a state-feedback controller that achieves GCC for the above-mentioned class of systems are provided in the form of an LMI-based feasibility or optimization problem;
3. it is shown that for the above-mentioned class of systems, the celebrated separation principle holds only one-way in the sense that the observer can be designed independently from the controller, but the converse is not true. Hence, sufficient conditions for the design of an estimate-feedback guaranteed cost controller are obtained in the form of bilinear matrix inequalities (BMIs).

The paper is structured as follows. Section 2 describes the notation and some lemmas which are used in the proofs of the theoretical results. In Section 3, the class of considered nonlinear systems is defined, and the different design problems considered in this paper are formulated. Section 4 provides LMI-based sufficient conditions for the design of the state observer. Section 5 is devoted to providing LMI-based sufficient conditions for the design of the state-feedback controller. In Section 6, the state-feedback control law is replaced by an estimate-feedback control, and BMI-based conditions for the design of the controller gain are obtained. Section 7 summarizes the final procedure for designing and implementing the components of the control system that provide GCC and GCE. The theoretical results are illustrated by means of an illustrative example in Section 8, whereas an application to a nonlinear rotational single-arm inverted pendulum is given in Section 9. Finally, the main conclusions are outlined in Section 10.

2. Notation and preliminaries

For a real symmetric matrix $A \in \mathbb{R}^{n\times n}$, the notation $A \succ 0$ ($A \prec 0$) stands for a positive (negative) definite matrix and indicates that all the eigenvalues of A are positive (negative). Given a matrix $A \in \mathbb{R}^{n\times n}$ with $A \succ 0$, the symbol $\mathcal{E}_A$ denotes the ellipsoid:

$$\mathcal{E}_A = \left\{x \in \mathbb{R}^n : x^T A x \leq 1\right\}.$$

The symbol $\mathrm{Co}\{A_i, i = 1, \dots, N\}$ denotes the convex hull of a finite number of N vertex matrices:

$$\mathrm{Co}\{A_i, i = 1, \dots, N\} = \left\{\sum_{i=1}^{N} \mu_i A_i \middle| \sum_{i=1}^{N} \mu_i = 1, \mu_i \geq 0 \, \forall i = 1, \dots, N\right\}.$$

Given a vector $u \in \mathbb{R}^m$, the symbol $\sigma(\cdot)$ denotes the standard saturation function, such that $\sigma(u) = [\sigma(u_1), \sigma(u_2), \dots, \sigma(u_m)]^T$, where $\sigma(u_i) = \mathrm{sgn}(u_i)\min\{1, |u_i|\}$. Given a matrix $A \in \mathbb{R}^{m\times n}$, the symbol $\mathcal{L}_\sigma(A)$ denotes:

$$\mathcal{L}_\sigma(A) = \{x \in \mathbb{R}^n : \sigma(Ax) = Ax\}.$$

The following lemma are used throughout the paper.

Lemma 1. *Given two matrices $K, H \in \mathbb{R}^{m\times n}$ and a vector $x \in \mathbb{R}^n$, suppose that $|H_j x| \leq 1$ for $j = 1, 2, \dots, m$, where H_j denotes the jth row of H. Then:*

$$\sigma(Kx) \in \mathrm{Co}\left\{D_i K x + D_i^- H x, i = 1, \dots, 2^m\right\}, \tag{1}$$

where the matrices D_i are all the possible $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0, and $D_i^- = I - D_i$.

Proof. See [10]. □

Lemma 2. *Given the matrix $H \in \mathbb{R}^{m\times n}$, if the following inequality holds for $j = 1, \dots, m$:*

$$\begin{bmatrix} P & PH_j^T \\ H_j P & 1 \end{bmatrix} \succ 0, \tag{2}$$

then $|H_j x| \leq 1 \, \forall x \in \mathcal{E}_{P^{-1}}$.

Proof. See [21]. □

3. Problem formulation

Consider the following discrete-time nonlinear system:

$$x_{k+1} = A x_k + g(x_k, \sigma(u_k)), \tag{3}$$

$$y_k = C x_k, \tag{4}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^p$ is the output, A and C are constant matrices of appropriate dimensions, and the nonlinear function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to satisfy the following assumptions:

1. $g(x, \sigma(u))$ is affine in $\sigma(u)$, so that it can be rewritten as:

   $$g(x, \sigma(u)) = f(x) + F(x)\sigma(u), \tag{5}$$

   with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ appropriate functions such that $f(0) = 0$ and $F(0) \neq 0$. Note that a consequence of this fact is that $g(0, 0) = 0$;

2. $g(x, \sigma(u))$ is differentiable with respect to x with bounded partial derivatives:

$$\underline{a}_{i,j} \leq \frac{\partial g_i(x, \sigma(u))}{\partial x_j} \leq \bar{a}_{i,j}, \quad i = 1, \dots, n, \quad j = 1, \dots, n. \quad (6)$$

By applying the mean value theorem [5], the following relation holds:

$$g(a, \sigma(u)) - g(b, \sigma(u)) = M(a, b, \sigma(u))(a - b), \quad (7)$$

for some matrix $M(\cdot)$ obtained as follows:

$$M(a, b, \sigma(u)) = \begin{bmatrix} \frac{\partial g_1}{\partial x}(c_1, \sigma(u)) \\ \vdots \\ \frac{\partial g_n}{\partial x}(c_n, \sigma(u)) \end{bmatrix}, \quad (8)$$

with:

$$c_1, \dots, c_n \in \{c \in \mathbb{R}^n : c = \alpha a + \beta b, \alpha + \beta = 1, \alpha\beta \geq 0\}. \quad (9)$$

Taking into account lower and upper bounds $\underline{a}_{i,j}$, $\bar{a}_{i,j}$ and each possible permutation of these bounds, matrices $M_i \in \mathbb{R}^{n \times n}$, $i = 1, \dots, N$, can be obtained[1] such that:

$$M(a, b, \sigma(u)) \in \mathbb{M} \triangleq \text{Co}\{M_i, i = 1, \dots, N\}, \quad (10)$$

so that:

$$g(a, \sigma(u)) - g(b, \sigma(u)) \in \mathbb{M}(a - b). \quad (11)$$

Moreover, taking into account (5), the following holds:

$$g(x, \sigma(a)) - g(x, \sigma(b)) = F(x)(\sigma(a) - \sigma(b)). \quad (12)$$

Hereafter, the problems of observer and controller design are formulated.

3.1. State observer design

Let us consider a nonlinear discrete-time observer of the form:

$$\hat{x}_{k+1} = A\hat{x}_k + g\big(\hat{x}_k, \sigma(u_k)\big) + K_o(y_k - C\hat{x}_k), \quad (13)$$

where $\hat{x}_k$ denotes the estimate of the state x_k and K_o denotes the observer gain to be designed. Then, the dynamics of the estimation error $e_k = x_k - \hat{x}_k$ is given by:

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1} = \tilde{A}e_k + s_k, \quad (14)$$

where $\tilde{A} = A - K_oC$ and $s_k = g(x_k, \sigma(u_k)) - g\big(\hat{x}_k, \sigma(u_k)\big)$. Apart from the asymptotical convergence to zero of the estimation error, a bound on the following cost function:

$$J_o = \sum_{k=0}^{\infty} e_k^T Q_e e_k, \quad (15)$$

with given $Q_e \succ 0$, is considered as objective for the design of the observer. Hence, the GCE design problem can be formulated as follows.

Problem 1. (State observer design problem) Given $Q_e \succ 0$ and a scalar $\gamma_o > 1$, design K_o such that the dynamics of the estimation error (14) is asymptotically stable with:

$$J_o < \gamma_o e_0^T Q_e e_0. \quad (16)$$

[1] In general, $N = 2^{n^2}$ matrices are obtained using this approach, which is commonly named *bounding box method*, since it generates a hyperbox of matrices [35]. However, if the matrix function $M_{x,u}$ contains h constant elements which arise from the linearity of some function $g_i(x, \sigma(u))$ with respect to some state variable x_j, then a reduced number of matrices $N = 2^{(n-h)^2}$ is obtained.

3.2. State-feedback controller design

To design a robust controller, let us consider the following control law:

$$u_k = K_c x_k, \quad (17)$$

where K_c is the controller gain to be designed. Substituting the above equation into (3) gives the following closed-loop system:

$$x_{k+1} = Ax_k + g(x_k, \sigma(K_c x_k)). \quad (18)$$

Let us note that, since $g(0, 0) = 0$, then $g(x_k, \sigma(K_c x_k))$ can be rewritten as:

$$\begin{aligned} g(x_k, \sigma(K_c x_k)) &= g(x_k, \sigma(K_c x_k)) - g(0, \sigma(K_c x_k)) \\ &\quad + g(0, \sigma(K_c x_k)) - g(0, 0). \end{aligned} \quad (19)$$

Taking into account (11) and (12), the following is obtained:

$$g(x_k, \sigma(K_c x_k)) - g(0, \sigma(K_c x_k)) \in \mathbb{M}x_k, \quad (20)$$

$$g(0, \sigma(K_c x_k)) - g(0, 0) = F(0)\sigma(K_c x_k), \quad (21)$$

so that:

$$g(x_k, \sigma(K_c x_k)) \in \mathbb{M}x_k + F(0)\sigma(K_c x_k). \quad (22)$$

The following objectives are taken into account for the design of the controller: (i) asymptotical convergence to zero of x_k when x_0 belongs to the ellipsoid $\mathcal{E}_Q$ defined by a given matrix $Q \succ 0$; and (ii) bound on the following cost function:

$$J_c = \sum_{k=0}^{\infty} \big(x_k^T Q_x x_k + \sigma(u_k)^T Q_u \sigma(u_k)\big), \quad (23)$$

with given $Q_x \succ 0$ and $Q_u \succ 0$. Hence, the GCC design problem can be formulated as follows.

Problem 2. (State-feedback controller design problem) Given matrices $Q \succ 0$, $Q_x \succ 0$, $Q_u \succ 0$ and the scalar $\gamma_c > 1$, design K_c such that (18) is asymptotically stable and:

$$J_c < \gamma_c x_0^T Q_x x_0, \quad (24)$$

when $x_0 \in \mathcal{E}_Q$.

3.3. Estimate-feedback controller design

The controller design problem previously formulated (Problem 2) assumes that the real state x_k is available for feedback. However, a more realistic situation is the one in which the estimated state should be used instead, i.e. (17) changes into:

$$u_k = K_c \hat{x}_k, \quad (25)$$

where $\hat{x}_k$ is the estimated state given by the observer (13). In this case, the question about whether it is possible or not to design the observer and the controller separately arises.

Let us consider the interconnection of the system (3) and (4), the observer (13) and the control law (25) such that the overall system obeys (see Fig. 1 for a block diagram depicting their structures and interconnections):

$$e_{k+1} = \tilde{A}e_k + g\big(x_k, \sigma(K_c\hat{x}_k)\big) - g\big(\hat{x}_k, \sigma(K_c\hat{x}_k)\big), \quad (26)$$

$$x_{k+1} = Ax_k + g\big(x_k, \sigma(K_c\hat{x}_k)\big). \quad (27)$$

Let us note that:

$$\begin{aligned} g\big(x_k, \sigma(K_c\hat{x}_k)\big) &= g\big(x_k, \sigma(K_c\hat{x}_k)\big) - g\big(0, \sigma(K_c\hat{x}_k)\big) \\ &\quad + g\big(0, \sigma(K_c\hat{x}_k)\big) - g(0, 0). \end{aligned} \quad (28)$$

Taking into account (11):

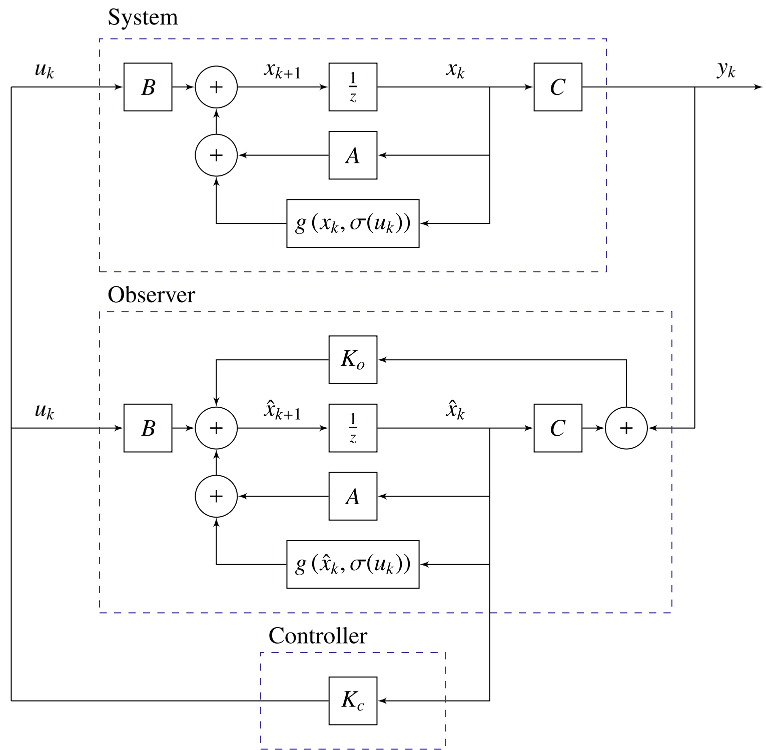

Fig. 1. Block diagram of the proposed control and estimation strategy.

$$g\big(x_k, \sigma(K_c\hat{x}_k)\big) - g\big(0, \sigma(K_c\hat{x}_k)\big) \in \mathbb{M}x_k, \tag{29}$$

$$g\big(x_k, \sigma(K_c\hat{x}_k)\big) - g\big(\hat{x}_k, \sigma(K_c\hat{x}_k)\big) \in \mathbb{M}\big(x_k - \hat{x}_k\big) = \mathbb{M}e_k. \tag{30}$$

Moreover, due to (12):

$$g\big(0, \sigma(K_c\hat{x}_k)\big) - g(0,0) = F(0)\sigma(K_c\hat{x}_k). \tag{31}$$

Hence:

$$g\big(x_k, \sigma(K_c\hat{x}_k)\big) \in \mathbb{M}x_k + F(0)\sigma\big(K_c\hat{x}_k\big), \tag{32}$$

which means that the overall system can be put in the equivalent form:

$$x_{k+1} \in (A + \mathbb{M})x_k + F(0)\sigma(K_c x_k - K_c e_k), \tag{33}$$

$$e_{k+1} \in (\tilde{A} + \mathbb{M})e_k, \tag{34}$$

where $\hat{x}_k = x_k - e_k$ has been used.

From (33) to (34) it can be seen that, due to the nonlinear term $\sigma(K_c x_k - K_c e_k)$, the separation principle holds only one-way, in the sense that, while the observer can be designed independently from the controller, this is not true for the controller, whose design procedure should be modified to take into account the effect of the evolution of e_k, driven by the specific choice of the observer gain K_o, on the nonlinearity $\sigma(\cdot)$. In order to deal with this situation, the requirements of Problem 2 are changed by requiring them to hold for $[x_0^T, e_0^T]^T \in \mathcal{E}_{[Q\ S\ ;\ S^T\ R]}$, where the ellipsoid $\mathcal{E}_{[Q\ S\ ;\ S^T\ R]}$ is defined by a given matrix:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ 0, \tag{35}$$

and that $J_c < \gamma_c\big(x_0^T Q_x x_0 + e_0^T Q_x e_0\big)$. Note that with this choice, in case $x_0 = 0$, then $J_c < \gamma_c \hat{x}_0^T Q_x \hat{x}_0$ is obtained, whereas if $e_0 = 0$, then the case $J_c < \gamma_c x_0^T Q_x x_0$ is recovered. Hence, the GCC design problem can be modified as follows:

Problem 3. (Estimate-feedback controller design problem) Given matrices $Q \succ 0$, S, $R \succ 0$, $Q_x \succ 0$, $Q_u \succ 0$, the observer gain K_o and the scalar $\gamma_c > 1$, design the controller gain K_c such that (33) and (34) is asymptotically stable and:

$$J_c < \gamma_c\big(x_0^T Q_x x_0 + e_0^T Q_x e_0\big), \tag{36}$$

when $[x_0^T, e_0^T]^T \in \mathcal{E}_{[Q\ S\ ;\ S^T\ R]}$.

4. Design of the state observer

The objective of this section is to solve Problem 1 by obtaining sufficient conditions for the synthesis of the state observer (13), which are given by the following theorem.

Theorem 1. *Let $P \succ 0$, $\gamma_o > 1$ and U be such that the following holds:*

$$P - \gamma_o Q_e \prec 0, \tag{37}$$

$$\begin{bmatrix} Q_e - P & * \\ P(A + M_i) - UC & -P \end{bmatrix} \prec 0, \qquad i = 1, \ldots, N. \tag{38}$$

Then, the nonlinear discrete–time observer given by (13), with gain calculated as $K_o = P^{-1}U$, is such that (14) is asymptotically stable and $J_o < \gamma_o e_0^T Q_e e_0$.

Proof. Let us consider the following inequality [30]:

$$\Delta V_k + e_k^T Q_e e_k < 0 \quad \text{for} \quad k = 0, \ldots, \infty, \tag{39}$$

where $\Delta V_k = V_{k+1} - V_k$, with $V_k = e_k^T P e_k$, $P \succ 0$. Since $e_k^T Q_e e_k > 0\, \forall e_k$, if inequality (39) holds then $\Delta V_k < 0$, which corresponds to the Lyapunov condition for asymptotic stability of (14). Then, by summing (39) from 0 to ∞, the following is obtained:

$$\sum_{k=0}^{\infty} e_k^T Q_e e_k = J_o < V_0 = e_0^T P e_0, \tag{40}$$

which, due to (37) ensuring that $V_0 < \gamma_o e_0^T Q_e e_0$ for any initial condition e_0, proves that $J_o < \gamma_o e_0^T Q_e e_0$ [22].

The remaining of the proof shows that inequality (39) follows from (38). In fact, taking into account (13), (39) can be rewritten as:

$$\Delta V_k + e_k^T Q_e e_k = e_k^T \left(\tilde{A}^T P \tilde{A} - P + Q_e\right) e_k + e_k^T \tilde{A}^T P s_k + s_k^T P \tilde{A} e_k + s_k^T P s_k < 0. \tag{41}$$

From (11), it follows that:

$$s_k = g(x_k, \sigma(u_k)) - g(\hat{x}_k, \sigma(u_k)) \in \mathbb{M}(x_k - \hat{x}_k) = \mathbb{M} e_k, \tag{42}$$

so that (41) is satisfied if:

$$\tilde{A}^T P A - P + Q_e + \tilde{A}^T P M + M^T P \tilde{A} + M^T P M \prec 0, \qquad \forall M \in \mathbb{M}, \tag{43}$$

which, using Schur complements, leads to:

$$\begin{bmatrix} Q_e - P & \left(\tilde{A}^T + M^T\right) P \\ P\left(\tilde{A} + M\right) & -P \end{bmatrix} \prec 0, \qquad \forall M \in \mathbb{M}. \tag{44}$$

Replacing:

$$P\tilde{A} = PA - PK_o C = PA - UC, \tag{45}$$

into (44), where $U = PK_o$, leads to:

$$\begin{bmatrix} Q_e - P & \star \\ P\left(\tilde{A} + M\right) - UC & -P \end{bmatrix} \prec 0, \qquad \forall M \in \mathbb{M}, \tag{46}$$

which is satisfied if (38) holds, thus completing the proof. □

Remark 1. The problem of determining the observer gain matrix described by *Theorem 1* can be treated as an optimization problem in which the cost performance index γ_o is minimized.

5. Design of the state-feedback controller

The objective of this section is to solve Problem 2 by obtaining sufficient conditions for the synthesis of the controller (17) for the system (3)-(4), which are given by the following theorem.

Theorem 2. *Let $P \succ 0$, $\gamma_c > 1$ and Γ, Z be such that the following holds:*

$$\begin{bmatrix} Q & I \\ I & P \end{bmatrix} \succeq 0, \tag{47}$$

$$\begin{bmatrix} P & Z_j^T \\ Z_j & 1 \end{bmatrix} \succeq 0, \qquad j = 1, \ldots, m, \tag{48}$$

$$\begin{bmatrix} -P & * & * & * \\ (A + M_l)P + F(0)\phi_i & -P & * & * \\ Q_x^{1/2} P & 0 & -I & * \\ \phi_i & 0 & 0 & -Q_u^{-1} \end{bmatrix} \prec 0, \qquad \begin{array}{c} i = 1, \ldots, 2^m \\ l = 1, \ldots, N \end{array}, \tag{49}$$

$$\begin{bmatrix} \gamma_c Q_x & I \\ I & P \end{bmatrix} \succ 0, \tag{50}$$

with $\phi_i = D_i \Gamma + D_i^- Z$, where Z_j denotes the jth row of the matrix Z, the matrices D_i are all the possible $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0, and $D_i^- = I - D_i$. Then the state-feedback control law (17), with gain calculated as $K_c = \Gamma P^{-1}$, is such that (18) is asymptotically stable and $J_c < \gamma_c x_0^T Q_x x_0$ when $x_0 \in \mathcal{E}_Q$.

Proof. In order to ensure that Problem 2 is solved, let us define the function $V_k = x_k^T P^{-1} x_k$, $P \succ 0$, and let us require that the ellipsoid $\mathcal{E}_Q$ is contained in $\mathcal{E}_{P^{-1}}$, which is equivalent to $Q - P^{-1} \succeq 0$ and, by Schur complements, to (47). Then, to ensure asymptotic stability for $x_0 \in \mathcal{E}_Q$, it is sufficient to ensure it for $x_0 \in \mathcal{E}_{P^{-1}}$, which together with the constraint on J_c, leads to the following constraints on V_k:

$$\Delta V_k + x_k^T Q_x x_k + \sigma(K_c x_k)^T Q_u \sigma(K_c x_k) < 0, \tag{51}$$

$$P^{-1} - \gamma_c Q_x \prec 0, \tag{52}$$

where $\Delta V_k = V_{k+1} - V_k$.

By defining:

$$\bar{x}_k^\sigma = [x_k^T, \ \sigma(K_c x_k)^T]^T, \tag{53}$$

and taking into account (22), the inequality (51) is satisfied if:

$$(\bar{x}_k^\sigma)^T \begin{bmatrix} \Lambda_{11}^\sigma & \Lambda_{12}^\sigma \\ * & \Lambda_{22}^\sigma \end{bmatrix} \bar{x}_k^\sigma = (\bar{x}_k^\sigma)^T \Lambda^\sigma \bar{x}_k^\sigma < 0, \qquad \forall M \in \mathbb{M}, \tag{54}$$

where:

$$\Lambda_{11}^\sigma = (A + M)^T P^{-1} (A + M) - P^{-1} + Q_x,$$
$$\Lambda_{12}^\sigma = (A + M)^T P^{-1} F(0),$$
$$\Lambda_{22}^\sigma = F(0)^T P^{-1} F(0) + Q_u.$$

According to Lemmas 1–2, by introducing an auxiliary feedback matrix H_c and the constraint (48), which enforces $\mathcal{E}_{P^{-1}} \subseteq \mathcal{L}_\sigma(H_c)$ and that is obtained from (2) through the change of variable $Z = H_c P$, then $\sigma(K_c x_k)$ can be placed into the convex hull of a group of linear feedbacks:

$$\sigma(K_c x_k) \in \mathrm{Co}\{D_i K_c x_k + D_i^- H_c x_k, i = 1 \ldots, 2^m\}. \tag{55}$$

Hence, from (55) and by convexity of the function V_k, it follows:

$$(\bar{x}_k^\sigma)^T \Lambda^\sigma \bar{x}_k^\sigma \leq \max_{i=1,\ldots,2^m} x_k^T \Lambda^i x_k, \tag{56}$$

where:

$$\Lambda^i = \Xi_i^T P^{-1} \Xi_i - P^{-1} + Q_x + \psi_i^T Q_u \psi_i, \tag{57}$$

with:

$$\Xi_i = A + M + F(0)\psi_i. \tag{58}$$

By requiring that $\Lambda^i \prec 0$ for $i = 1, \ldots, 2^m$, and applying Schur complements, with an appropriate congruence transformation, the following is obtained:

$$\begin{bmatrix} -P & P\Xi_i^T & PQ_x^{1/2} & P\psi_i^T \\ \Xi_i P & -P & 0 & 0 \\ Q_x^{1/2} P & 0 & -I & 0 \\ \psi_i P & 0 & 0 & -Q_u^{-1} \end{bmatrix} \prec 0, \qquad \forall M \in \mathbb{M}, \tag{59}$$

which, by replacing:

$$\begin{aligned} \Xi_i P &= (A + M)P + F(0)\psi_i P \\ &= (A + M)P + F(0) D_i K_c P + F(0) D_i^- H_c P \\ &= (A + M)P + F(0) D_i \Gamma + F(0) D_i^- Z, \end{aligned} \tag{60}$$

where $\Gamma = K_c P$, leads to:

$$\begin{bmatrix} -P & * & * & * \\ (A + M)P + F(0)\phi_i & -P & * & * \\ Q_x^{1/2} P & 0 & -I & * \\ \phi_i & 0 & 0 & -Q_u^{-1} \end{bmatrix} \prec 0, \qquad \begin{array}{c} i = 1, \ldots, 2^m \\ \forall M \in \mathbb{M} \end{array}, \tag{61}$$

which is satisfied if (49) holds. By applying Schur complements to (52), (50) is obtained, which completes the proof. □

Remark 2. Also in this case, the problem of determining the controller gain matrix described by Theorem 2 can be treated as an optimization problem in which the cost performance index γ_c is minimized.

6. Design of the estimate-feedback controller

The objective of this section is to solve Problem 3 by obtaining sufficient conditions for the synthesis of the controller (25) for the system (3) and (4) with state observer (13), which are given by the following theorem.

Theorem 3. *Given the observer gain K_o (hence, the matrix $\tilde{A}$), let $P \succ 0$, $\gamma_c > 1$ and the matrices K_c, H_c be such that:*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} - P \succeq 0, \tag{62}$$

$$P - \gamma_c \begin{bmatrix} Q_x & 0 \\ 0 & Q_x \end{bmatrix} \prec 0, \tag{63}$$

$$\begin{bmatrix} P & \begin{bmatrix} H_{c,j}^T \\ -H_{c,j}^T \end{bmatrix} \\ \begin{bmatrix} H_{c,j} & -H_{c,j} \end{bmatrix} & 1 \end{bmatrix} \succeq 0, \qquad j = 1, \ldots, m \tag{64}$$

$$\begin{bmatrix} P & P\begin{bmatrix} A + M_{l_1} + F(0)\phi_i & -F(0)\phi_i \\ 0 & \tilde{A} + M_{l_2} \end{bmatrix} \\ \star & P - \begin{bmatrix} Q_x + \phi_i^T Q_u \phi_i & -\phi_i^T Q_u \phi_i \\ -\phi_i^T Q_u \phi_i & \phi_i^T Q_u \phi_i \end{bmatrix} \end{bmatrix} \succ 0, \qquad \begin{matrix} i = 1, \ldots, 2^m \\ l_1 = 1, \ldots, N, \\ l_2 = 1, \ldots, N \end{matrix} \tag{65}$$

hold, where $\phi_i = D_i K_c + D_i^- H_c$, $H_{c,j}$ denotes the j-th row of the matrix H_c, and the matrices D_i are all the possible $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0, and $D_i^- = I - D_i$. Then the estimate-feedback control law (25) ensures that (18) is asymptotically stable when $[x_0^T, e_0^T]^T \in \mathcal{E}_{[Q\ S\ ;\ S^T\ R]}$ and $J_c < \gamma_c\left(x_0^T Q_x x_0 + e_0^T Q_x e_0\right)$.

Proof. Let us consider the function:

$$V_k = \begin{bmatrix} x_k \\ e_k \end{bmatrix}^T P \begin{bmatrix} x_k \\ e_k \end{bmatrix}, \tag{66}$$

with $P \succ 0$, and let us require that $\mathcal{E}_{[Q\ S\ ;\ S^T\ R]} \subseteq \mathcal{E}_P$, which is equivalent to (62). Then, to ensure asymptotic stability for $[x_0^T, e_0^T]^T \in \mathcal{E}_{[Q\ S\ ;\ S^T\ R]}$, it is sufficient to ensure it for $[x_0^T, e_0^T]^T \in \mathcal{E}_P$, which together with the constraint on J_c, leads to (63) and:

$$\Delta V_k + x_k^T Q_x x_k + \sigma(K_c\hat{x}_k)^T Q_u \sigma(K_c\hat{x}_k) < 0. \tag{67}$$

By performing a reasoning similar to the one in Theorem 2, using Lemma 2 the constraint (64) ensures that:

$$\mathcal{E}_P \subseteq \mathcal{L}\left(\begin{bmatrix} H_c & -H_c \end{bmatrix}\right), \tag{68}$$

and, according to Lemma 1:

$$\sigma(K_c\hat{x}_k) \in Co\left\{\tilde{\psi}_i \begin{bmatrix} x_k \\ e_k \end{bmatrix} = D_i \begin{bmatrix} K_c & -K_c \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} + D_i^- \begin{bmatrix} H_c & -H_c \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix}\right\}. \tag{69}$$

so that the following holds:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} \in \begin{bmatrix} A + \mathbb{M} + F(0)\phi_i & -F(0)\phi_i \\ 0 & \tilde{A} + \mathbb{M} \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix}, \tag{70}$$

which means that $\exists M_1, M_2 \in \mathbb{M}$ such that:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A + M_1 + F(0)\phi_i & -F(0)\phi_i \\ 0 & \tilde{A} + M_2 \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} = \bar{A} \begin{bmatrix} x_k \\ e_k \end{bmatrix}. \tag{71}$$

Then, (67) leads to:

$$\bar{A}^T P \bar{A} - P + \begin{bmatrix} Q_x & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \phi_i^T \\ -\phi_i^T \end{bmatrix} Q_u \begin{bmatrix} \phi_i & -\phi_i \end{bmatrix} \prec 0,$$

$$\forall M_1, M_2 \in \mathbb{M}, \tag{72}$$

which, by means of Schur complements, and an appropriate congruence transformation, leads to:

$$\begin{bmatrix} P & P\begin{bmatrix} A + M_1 + F(0)\phi_i & -F(0)\phi_i \\ 0 & \tilde{A} + M_2 \end{bmatrix} \\ \star & P - \begin{bmatrix} Q_x + \phi_i^T Q_u \phi_i & -\phi_i^T Q_u \phi_i \\ -\phi_i^T Q_u \phi_i & \phi_i^T Q_u \phi_i \end{bmatrix} \end{bmatrix} \succ 0, \qquad \begin{matrix} i = 1, \ldots, 2^m \\ \forall M_1, M_2 \in \mathbb{M} \end{matrix} \tag{73}$$

which is satisfied if (65) holds, thus completing the proof. □

Note that when applying Theorem 6, the matrix $\tilde{A}$ is considered to be known, since the observer gain K_o is assumed to be designed beforehand using Theorem 1. Taking the above considerations into account, similarly to previous cases, an optimization problem concerning minimization of the cost performance index γ_c can be defined. It is worth highlighting that the conditions provided by Theorem 6 are bilinear matrix inequalities (BMIs) due to the product between the unknown variables P and ϕ_i, hence their resolution suffers from being a non-convex problem.

7. Design and implementation procedure for the guaranteed cost estimation and control

The problem of determining the state observer and controller gain matrices is solved using the results given by Theorems 1–3. It is done by an optimization problem subject to minimization of the cost performance indexes γ_o and γ_c. The design and implementation procedure can be summarized as follows: *Off-line computation:*

1. Obtain a representation of the system of interest as in (3) and (4);
2. Calculate the Jacobians of the function $g(\cdot)$ with respect to state and input;
3. Compute lower and upper bounds for the elements of the Jacobians and use them to obtain (10) using the bounding box approach;
4. *(Observer design)* Obtain the observer gain K_o by solving the optimization problem:

   minimize γ_o

   subject to subject to (37)-(38)
5. *(Controller design)* Obtain the controller gain K_c by solving the optimization problem:

   minimize γ_c

   subject to subject to (47)-(50)

On-line computation:

1. Compute the state estimate using (13);
2. Compute the control action using (25).

Remark 3. The above procedure summarizes the necessary steps for the design of the state observer or the state-feedback controller. It could be applied to the case of the estimate-feedback controller, albeit some minor changes.

8. Numerical example

Let us consider the following system:

$$x_1(k+1) = a_{11}x_1(k) - 0.5x_2(k) + 0.1x_3(k) + \frac{\cos(x_1(k))\sigma_1(u_1(k))}{3x_1^2(k) + 2},$$

$$x_2(k+1) = -0.2x_1(k) + a_{22}x_2(k) + 0.1x_3(k) + \frac{\sigma_2(u_2(k))}{2 + x_2^2(k)},$$

$x_3(k+1) = 0.1x_1(k) - 0.1x_2(k) + a_{33}x_3(k),$

where the state variable $x_1(k)$ is assumed to be measured, which can be reshaped in the form (3) and (4) by considering:

$$A = \begin{bmatrix} a_{11} & -0.5 & 0.1 \\ -0.2 & a_{22} & 0.1 \\ 0.1 & -0.1 & a_{33}, \end{bmatrix}, \quad g(\cdot) = \begin{bmatrix} \frac{\cos(x_1)\sigma_1(u_1)}{3x_1^2+2} \\ \frac{\sigma_2(u_2)}{2+x_2^2} \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The nonlinear function $g(\cdot)$ is differentiable with respect to x and $\sigma(u)$:

$$\frac{\partial g(\cdot)}{\partial x} = \begin{bmatrix} -\frac{3\sin(x_1)x_1^2+6\cos(x_1)x_1+2\sin(x_1)}{(3x_1^2+2)^2}\sigma_1(u_1) & 0 & 0 \\ 0 & -\frac{2x_2}{(2+x_2^2)^2}\sigma_2(u_2) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\frac{\partial g(\cdot)}{\partial \sigma(u)} = \begin{bmatrix} \frac{\cos(x_1)}{3x_1^2(k)+2} & 0 \\ 0 & \frac{1}{2+x_2^2} \\ 0 & 0 \end{bmatrix} = F(x).$$

Note that the following holds:

$$-0.52 \le -\frac{3\sin(x_1)x_1^2+6\cos(x_1)x_1+2\sin(x_1)}{(3x_1^2+2)^2}\sigma_1(u_1) \le 0.52,$$

$$-0.23 \le -\frac{2x_2}{(2+x_2^2)^2}\sigma_2(u_2) \le 0.23.$$

Also, $g(0,0) = 0$ and:

$$F(0) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix}.$$

Taking into account the above computed bounds, it is possible to obtain the set defined in (10) as the convex combination of the following four matrices:

$$M_1 = \begin{bmatrix} -0.52 & 0 & 0 \\ 0 & -0.23 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -0.52 & 0 & 0 \\ 0 & 0.23 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0.52 & 0 & 0 \\ 0 & -0.23 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0.52 & 0 & 0 \\ 0 & 0.23 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8.1. Open-loop stable equilibrium

In this subsection, we will assume that $a_{11} = a_{22} = a_{33} = 0.6$, so that the origin of the state-space is an open-loop stable equilibrium point. Let us consider $Q_e = I$, and three different observer gains K_o^a, K_o^b and K_o^c, where K_o^a has been obtained through the minimization of γ_o using Theorem 1 [16,34], whereas K_o^b and K_o^c are observer gains designed by requiring only the stabilization of the estimation error dynamics. The observer gains are as follows:

$$K_o^a = \begin{bmatrix} 0.82 \\ -0.47 \\ 0.21 \end{bmatrix}, \quad K_o^b = \begin{bmatrix} 0.63 \\ -0.88 \\ -0.10 \end{bmatrix}, \quad K_o^c = \begin{bmatrix} 1.03 \\ -1.24 \\ 0.18 \end{bmatrix},$$

which deliver minimized values of γ_o as follows: $\gamma_o^a = 3.70$, $\gamma_o^b = 70.78$, $\gamma_o^c = 442.73$. In order to validate the proposed design technique, different simulations starting from initial conditions x_0 on the unit sphere $\mathcal{S}$, with $\hat{x}_0 = 0$, have been performed. Then, Fig. 2 shows the evolution of the following signals:

$$\tilde{J}(k|K_o^i) = \max_{x_0 \in \mathcal{S}} \sum_{t=0}^{k} e_t^T Q_e e_t | K_o = K_o^i, \tag{74}$$

with $i \in \{a, b, c\}$, that confirms that K_o^a is the best performing observer gain (see blue line). This can be seen also in Fig. 3, where the upper and lower envelopes of the estimation error trajectories are plotted, showing that K_o^a provides a faster convergence to zero of the estimation error.

Subsequently, selecting $Q = 100I$ (initial conditions in the sphere of radius 0.1, denoted in the following as $\tilde{\mathcal{S}}$) and:

$$Q_x^a = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_x^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q_x^c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \quad Q_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

three different controller gains have been designed, as follows:

$$K_c^a = \begin{bmatrix} -1.22 & 1.00 & -0.20 \\ 0.31 & -0.76 & -0.07 \end{bmatrix},$$

$$K_c^b = \begin{bmatrix} -1.04 & 1.39 & -0.26 \\ 0.41 & -1.20 & -0.18 \end{bmatrix},$$

$$K_c^c = \begin{bmatrix} -1.15 & 1.11 & -0.82 \\ 0.61 & -1.12 & 0.75 \end{bmatrix},$$

each one solving the minimization problem described in Section 5 [16,34], obtaining $\gamma_c^a = 3.34$, $\gamma_c^b = 3.71$ and $\gamma_c^c = 8.51$, respectively. Next, using:

$$\tilde{J}^i(k|K_c^i) = \max_{x_0 \in \tilde{\mathcal{S}}} \frac{\sum_{t=0}^{k} x_k^T Q_x^i x_k + u_k^T Q_u u_k}{x_0^T Q_x^i x_0} | K_c = K_c^i, \tag{75}$$

with $i \in \{a, b, c\}$, the validation of the proposed controller design technique can be performed, as shown in Fig. 4, where $\tilde{J}^i(k|K_c^i) < \gamma_c^i$ is satisfied in all simulations. As expected from the chosen values for Q_x^a, Q_x^b, Q_x^c, the controller gain K_c^a provides a faster convergence to zero of the state variable $x_1(k)$, K_c^b a faster convergence of $x_2(k)$, and K_c^c of $x_3(k)$, respectively, as shown in Fig. 5, where the upper and lower envelopes of the state trajectories for initial conditions on the frontier of $\tilde{\mathcal{S}}$ are plotted.

Finally, we have considered the design of the estimate-feedback controller using Theorem 3. To this end, it has been assumed that the estimated state is computed using the observer gain K_o^a, and that the region of possible initial conditions is described by matrices $Q = 100I$ and $R = 10^4 I$. At first, the performance of the previously designed controllers K_c^a, K_b^c, K_c^c has been evaluated using Theorem 3 as an analysis tool (hence, converting the BMIs into LMIs due to the decision variable K_c and H_c becoming known matrices), obtaining values of γ_c as follows: $\tilde{\gamma}_c^a = 379.2$, $\tilde{\gamma}_c^b = 41.2$ and $\tilde{\gamma}_c^c = 55.0$. Then, Theorem 3 has been employed as a design tool, with the solver PENLAB used to solve the BMIs. In this case, no solution was found for $Q_x = Q_x^a$ and $Q_x = Q_x^c$ (the solver got stuck indefinitely), whereas the following controller gain was obtained for $Q_x = Q_x^b$:

$$K_c^{b2} = \begin{bmatrix} -0.75 & 0.90 & -0.21 \\ 0.30 & -0.82 & 0.06 \end{bmatrix},$$

yielding the improved upper bounds $\tilde{\gamma}_c^{b2} = 19.8$.

8.2. Open-loop unstable equilibrium

In this subsection, we will assume that $a_{11} = 1.2$, $a_{22} = 1.2$ and $a_{33} = 0.7$, so that the origin of the state-space is an open-loop unstable equilibrium point. Let us consider $Q_e = I$, and three different observer gains K_o^d, K_o^e and K_o^f, where K_o^d has been obtained solving

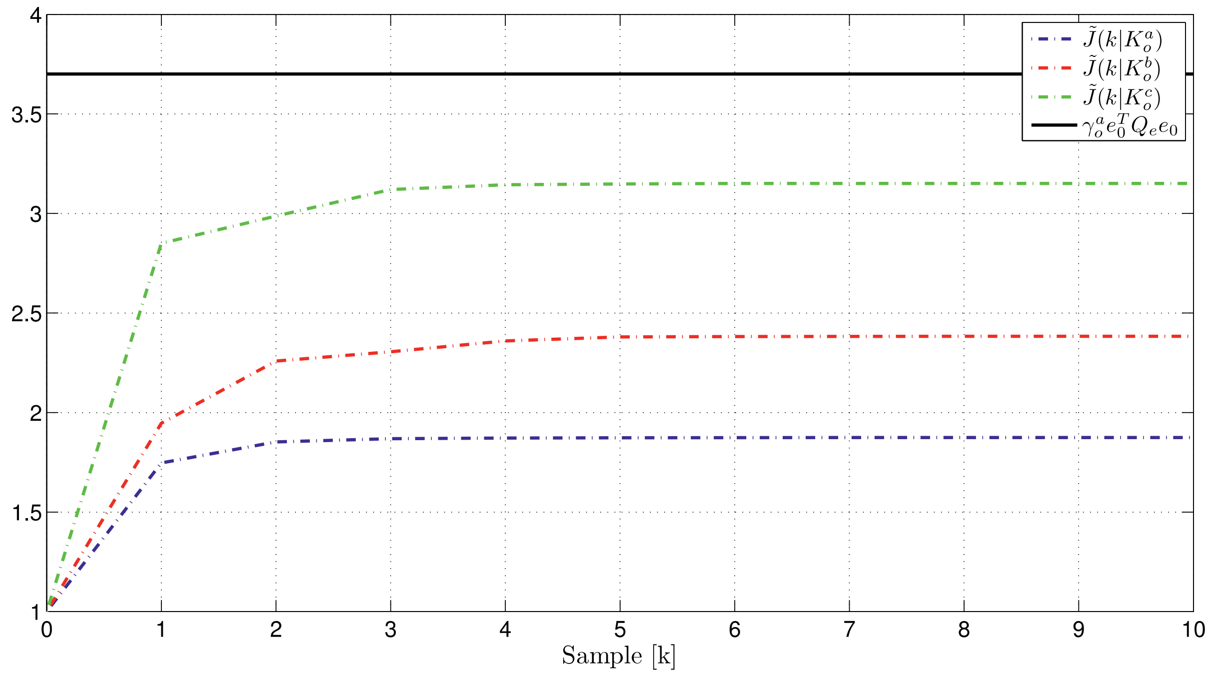

Fig. 2. Evolution of $\tilde{J}(k|K_o^i)$, $i \in \{a, b, c\}$, and upper bound $\gamma_o^a e_0^T Q_e e_0$ (open-loop stable).

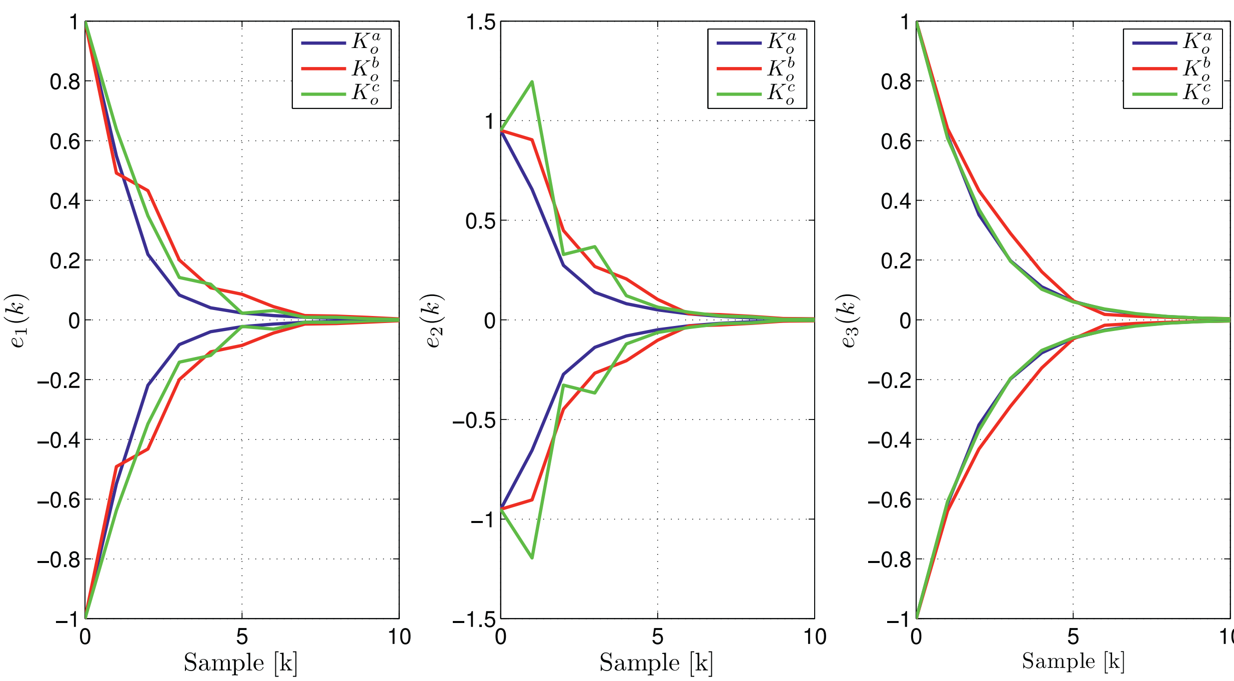

Fig. 3. Envelopes of the error $e(k)$ with observer gains K_o^a, K_o^b, K_o^c (open-loop stable).

the minimization of γ_o using Theorem 1, whereas K_o^e and K_o^f are observer gains designed by requiring only the stabilization of the estimation error dynamics. The observer gains are as follows:

$$K_o^d = \begin{bmatrix} 2.12 \\ -2.33 \\ 0.43 \end{bmatrix}, \quad K_o^e = \begin{bmatrix} 2.11 \\ -2.37 \\ 0.41 \end{bmatrix}, \quad K_o^f = \begin{bmatrix} 2.14 \\ -2.41 \\ 0.43 \end{bmatrix},$$

which deliver minimized values of γ_o as follows: $\gamma_o^d = 199.09$, $\gamma_o^e = 5528$, $\gamma_o^f = 1686.8$. Let us note that in this case, if compared to the open-loop stable one, very small changes in the elements of the observer gains cause big variations in the values of the computed upper bounds. Moreover, Fig. 6, which shows the evolution of the signal (74), $i \in \{d, e, f\}$, illustrates the increase in conservativeness of the proposed methodology when open-loop unstable plants are considered. For the sake of completeness in the presentation of the results, the upper and lower envelopes of the estimation error trajectories are shown in Fig. 7.

Subsequently, selecting $Q = 100I$, $Q_x^d = Q_x^a$, $Q_x^e = Q_x^b$, $Q_x^f = Q_x^c$ and $Q_u = I$, the following controller gains have been designed:

$$K_c^d = \begin{bmatrix} -2.38 & 1.02 & -0.20 \\ 0.29 & -1.85 & -0.15 \end{bmatrix},$$

$$K_c^e = \begin{bmatrix} -2.40 & 1.05 & -0.31 \\ 0.43 & -2.34 & -0.18 \end{bmatrix},$$

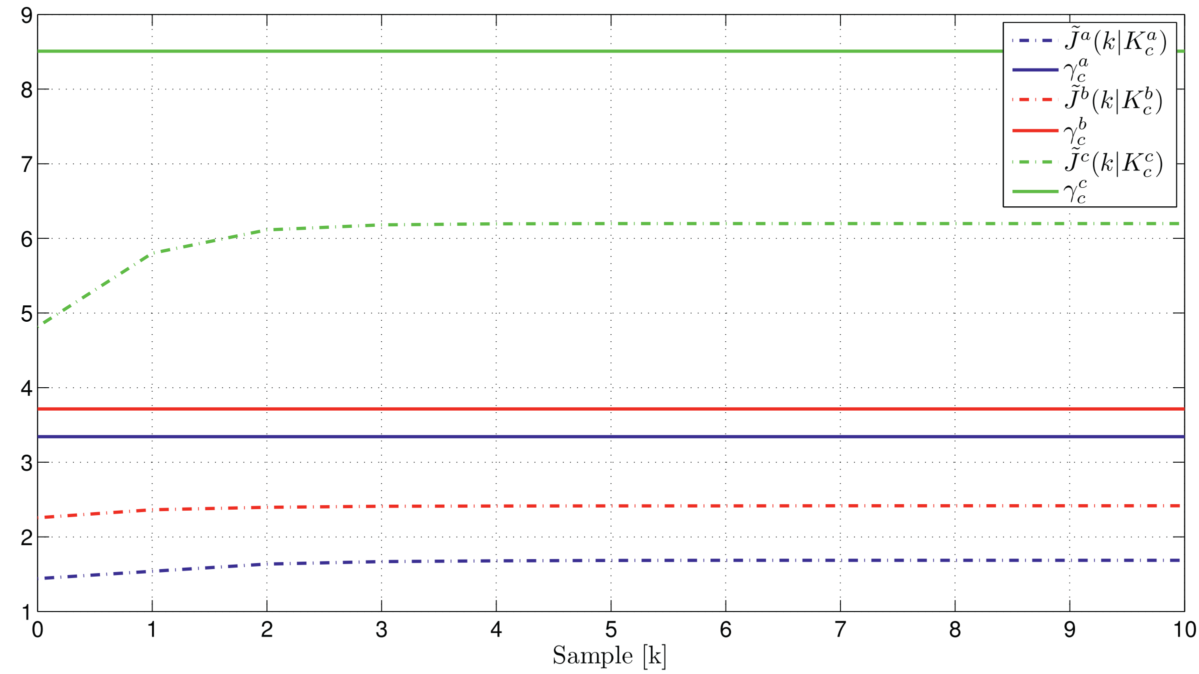

Fig. 4. Evolution of $\tilde{J}^i(k|K_c^i)$ and upper bounds γ_c^i, $i \in \{a, b, c\}$ (open-loop stable, state-feedback).

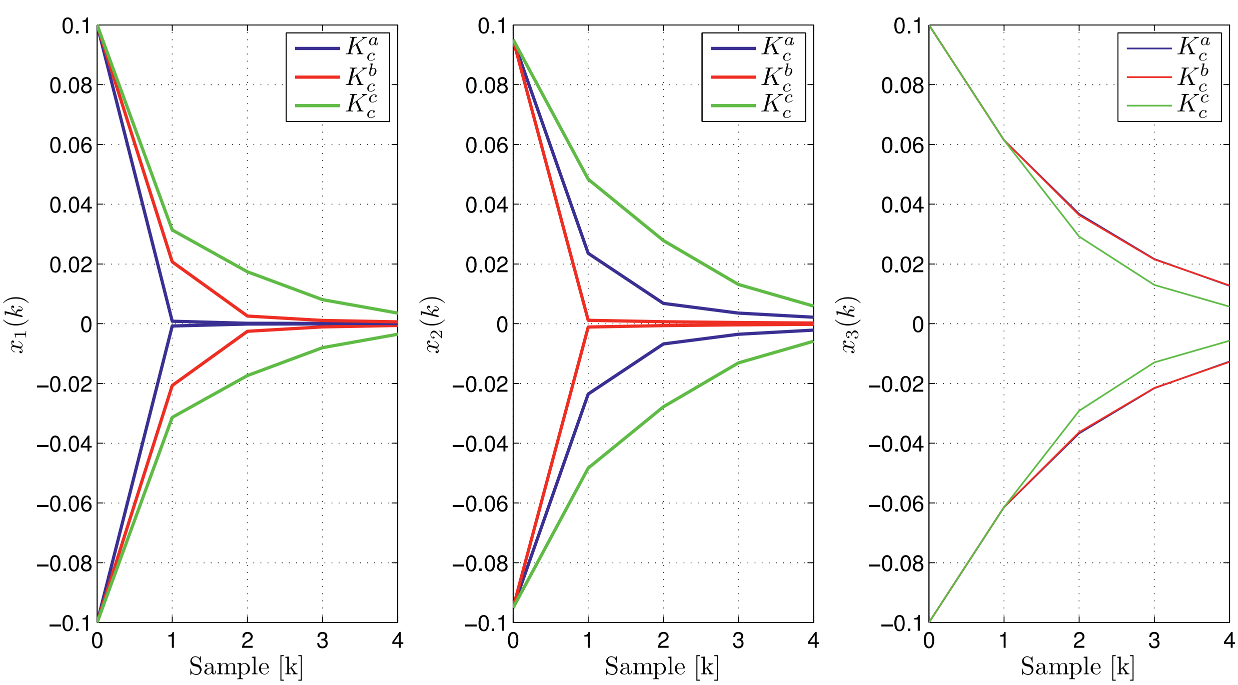

Fig. 5. Envelopes of $x(k)$ with controller gains K_c^a, K_c^b, K_c^c (open-loop stable, state-feedback).

$$K_c^f = \begin{bmatrix} -2.34 & 1.34 & -0.76 \\ 0.74 & -2.10 & 0.65 \end{bmatrix},$$

each one solving the minimization problem described in Section 5, obtaining $\gamma_c^d = 7.37$, $\gamma_c^e = 9.81$ and $\gamma_c^f = 16.40$, respectively. Fig. 8 shows the signal calculated using (75), $i \in \{d, e, f\}$, which demonstrates that $\tilde{J}^i(k|K_c^i) < \gamma_c^i$ is satisfied in all simulations. As in the previous case, the controller gain that provides a faster convergence to zero of the state variable $x_1(k)$ is K_c^d, whereas K_c^e and K_c^f provide a faster convergence of $x_2(k)$ and $x_3(k)$, respectively. For the sake of completeness, Fig. 9 shows the upper and lower envelopes of the state trajectories for initial conditions on the frontier of $\tilde{\mathcal{S}}$.

This can be seen also from Fig. 3, where the upper and lower envelopes of the estimation error trajectories are plotted, showing that K_o^a provides a faster convergence to zero of the estimation error.

9. Application to a rotational single-arm inverted pendulum

Let us consider the following nonlinear system describing the dynamics of a rotational single-arm inverted pendulum [18]:

$$x_1(k+1) = x_1(k) + T_s x_2(k),$$

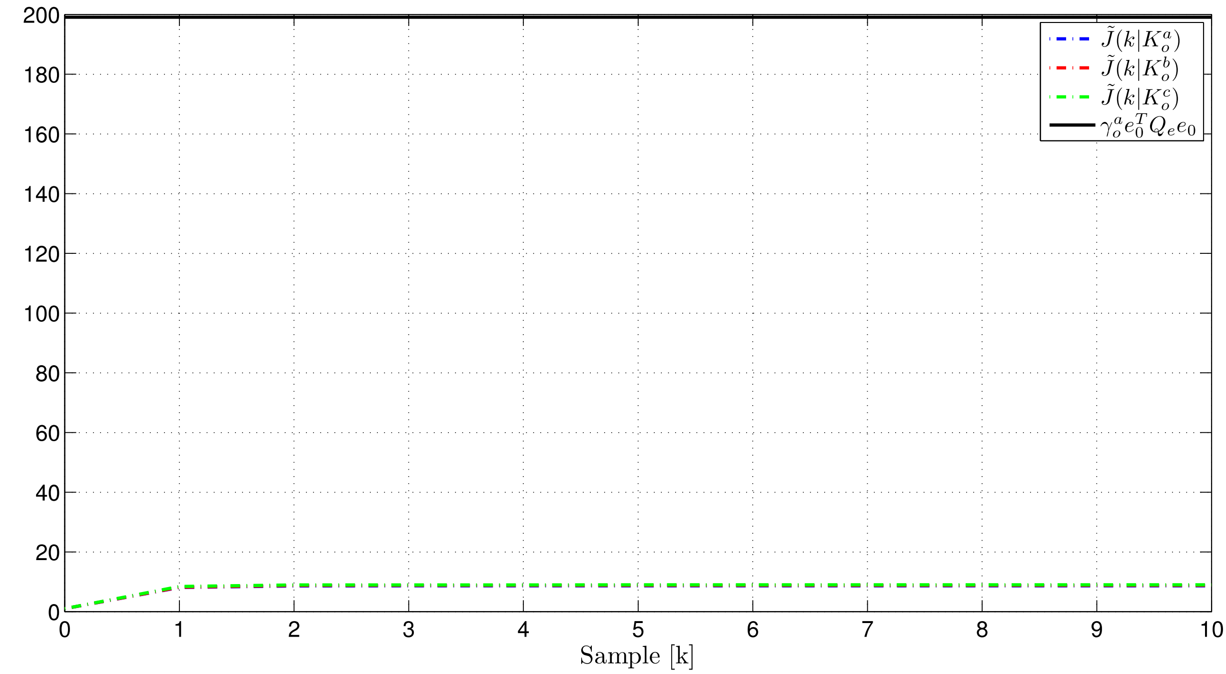

Fig. 6. Evolution of $\tilde{J}(k|K_o^i)$ and upper bounds $\gamma_o^d e_0^T Q_e e_0$, $i \in \{d, e, f\}$ (open-loop unstable).

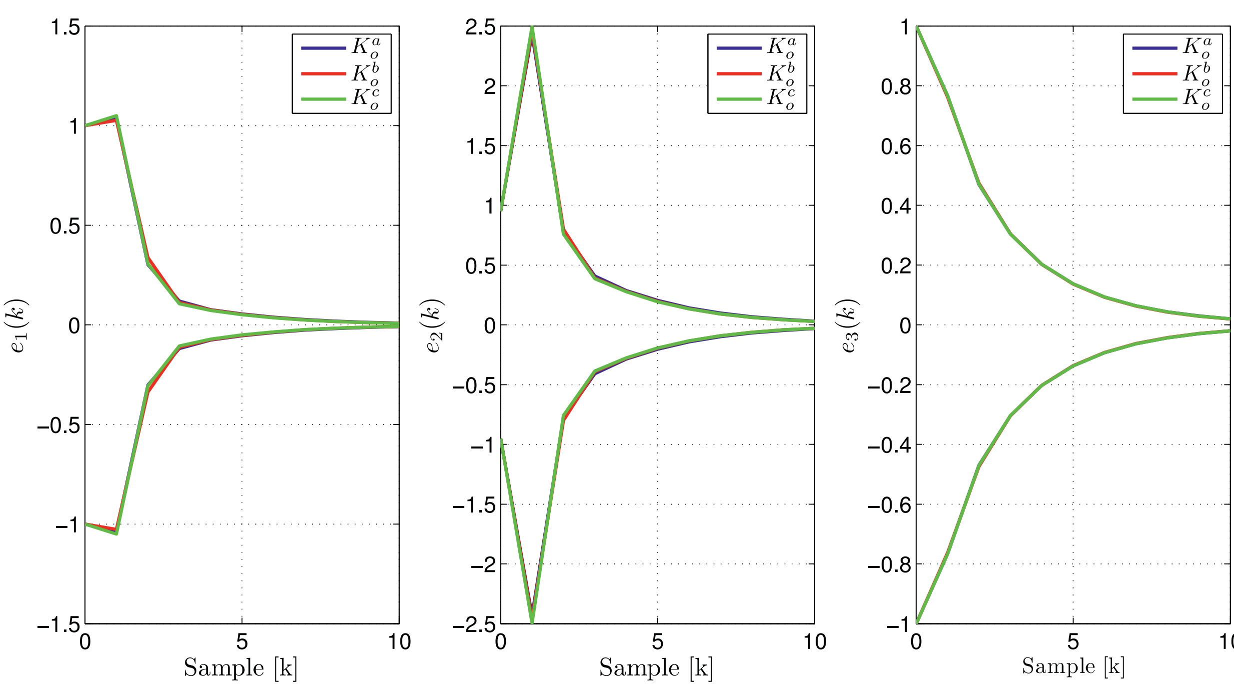

Fig. 7. Envelopes of the error $e(k)$ with observer gains K_o^d, K_o^e, K_o^f (open-loop unstable).

$$x_2(k+1) = T_s\frac{g}{l}\sin(x_1(k)) + (1 - T_s\frac{b}{ml^2})x_2(k) + \frac{T_s}{ml^2}\sigma(u_1(k)),$$

where $T_s = 0.01[s]$ is the sampling time, $m = 0.2[kg]$ is the mass of the pendulum, $l = 0.15[m]$ is the length of the pendulum, whereas $b = 0.0067[kgm^2s^{-1}]$ and $g = 9.81[ms^{-2}]$ are the friction coefficient and the gravitational acceleration, respectively. Assuming that the state variable $x_1(k)$ (angle of the pendulum) is measured, the pendulum model can be reshaped as:

$$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 - \frac{T_s b}{ml^2} \end{bmatrix}, \quad g(\cdot) = \begin{bmatrix} 0 \\ \frac{T_s g}{l}\sin(x_1(k)) + \frac{T_s}{ml^2}\sigma(u_1(k)) \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The nonlinear function $g(\cdot)$ is differentiable with respect to x and $\sigma(u)$:

$$\frac{\partial g(\cdot)}{\partial x} = \begin{bmatrix} 0 & 0 \\ T_s\frac{g}{l}\cos(x_1(k)) & 0 \end{bmatrix},$$

$$\frac{\partial g(\cdot)}{\partial \sigma(u)} = \begin{bmatrix} 0 \\ \frac{T_s}{ml^2} \end{bmatrix} = F(x).$$

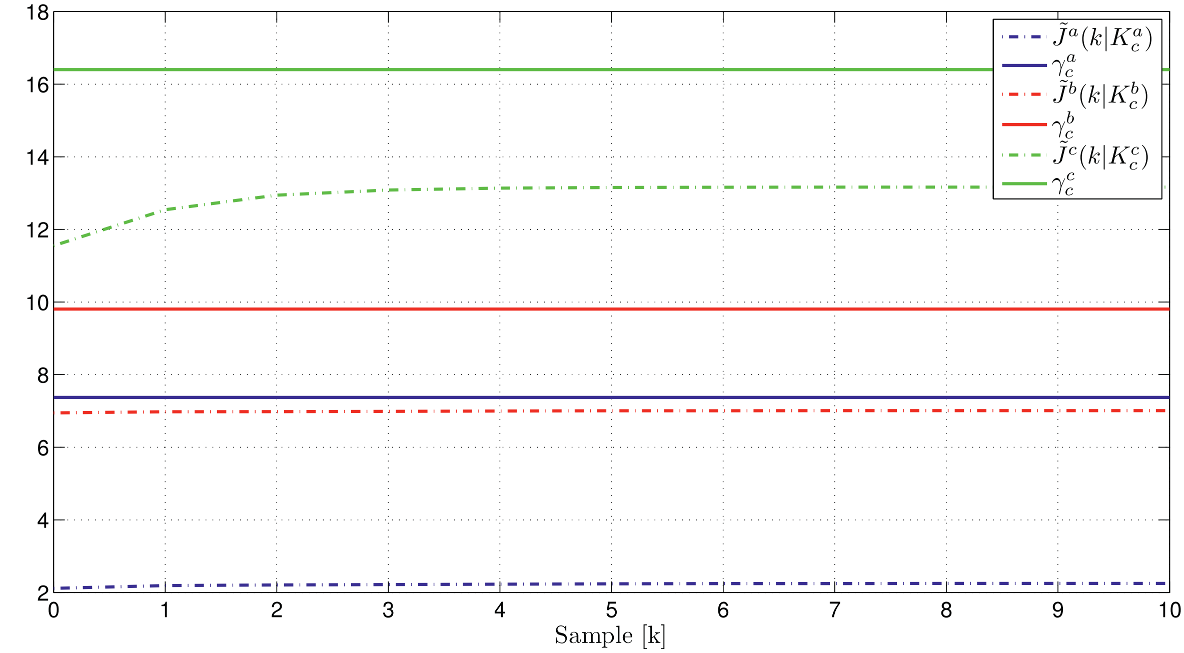

Fig. 8. Evolution of the signals $\tilde{J}^i(k|K_c^i)$ and upper bounds γ_c^i, $i \in \{d, e, f\}$ (open-loop unstable, state-feedback).

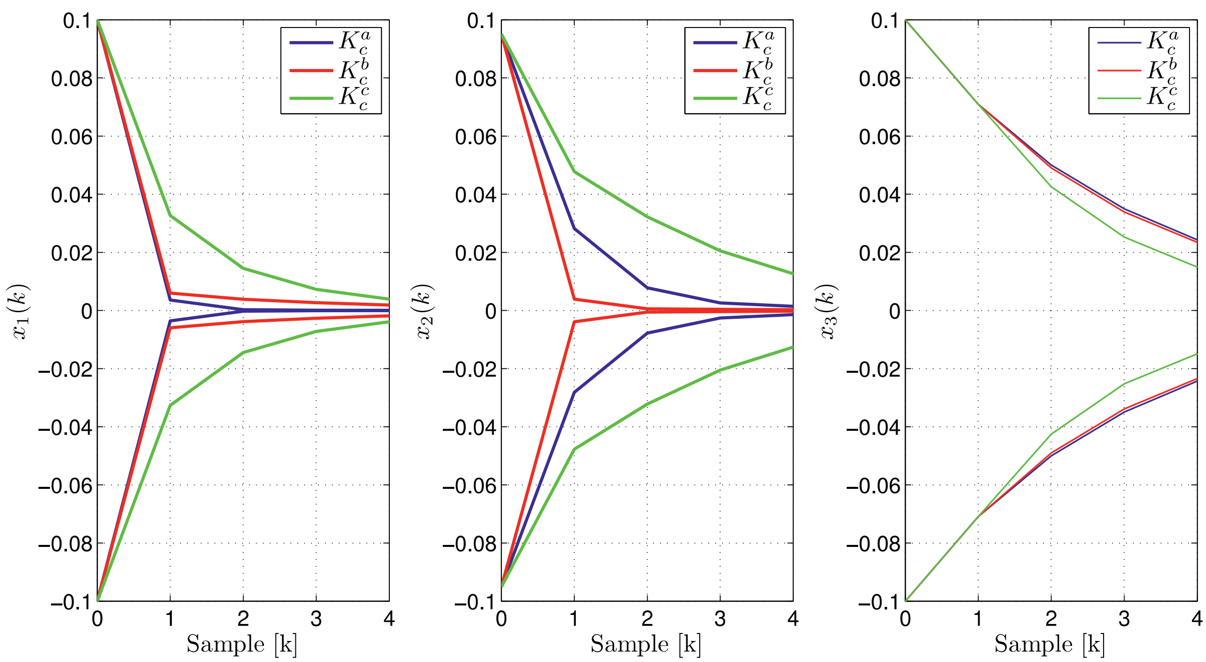

Fig. 9. Envelopes of the state trajectories $x(k)$ with different controller gains K_c^d, K_c^e, K_c^f.

Note that the following holds:

$$-0.6540 \le T_s \frac{g}{l}\cos(x_1(k)) \le 0.6540,$$

Also, $g(0,0) = 0$ and:

$$F(0) = \begin{bmatrix} 0 \\ 2.2222 \end{bmatrix}.$$

Taking into account the above computed bound, it is possible to obtain (10) as the convex combination of the following matrices:

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0.6540 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 \\ -0.6540 & 0 \end{bmatrix}$$

Taking into consideration $Q_e = I$, three different observer gain K_o^a, K_o^b and K_o^c have been obtained:

$$K_o^a = \begin{bmatrix} 1.0378 \\ 3.7197 \end{bmatrix}, \quad K_o^b = \begin{bmatrix} 0.4750 \\ 4.9875 \end{bmatrix}, \quad K_o^c = \begin{bmatrix} 0.5750 \\ 7.8375 \end{bmatrix},$$

where K_o^a has been obtained through the minimization of γ_o using Theorem 1 and whereas K_o^b and K_o^c provide only the stabilization of the estimation error dynamics. Using initial conditions x_0, Fig. 10, shows the evolution of (74) for $i \in \{a, b, c\}$. Also for this case, confirms that observer gain matrices K_o^a provide the best performance (see blue line). Moreover, Fig. 11 shows the upper and lower envelopes of the estimation error trajectories, confirms, that K_o^a pro-

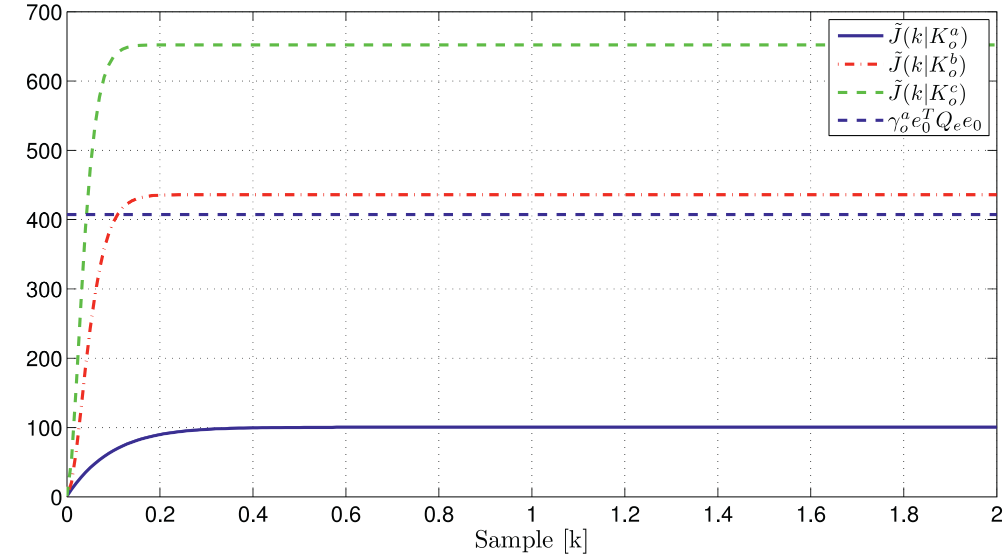

Fig. 10. Evolution of $\tilde{J}(k|K_o^i)$, $i \in \{a, b, c\}$, and upper bound $\gamma_o^a e_0^T Q_e e_0$.

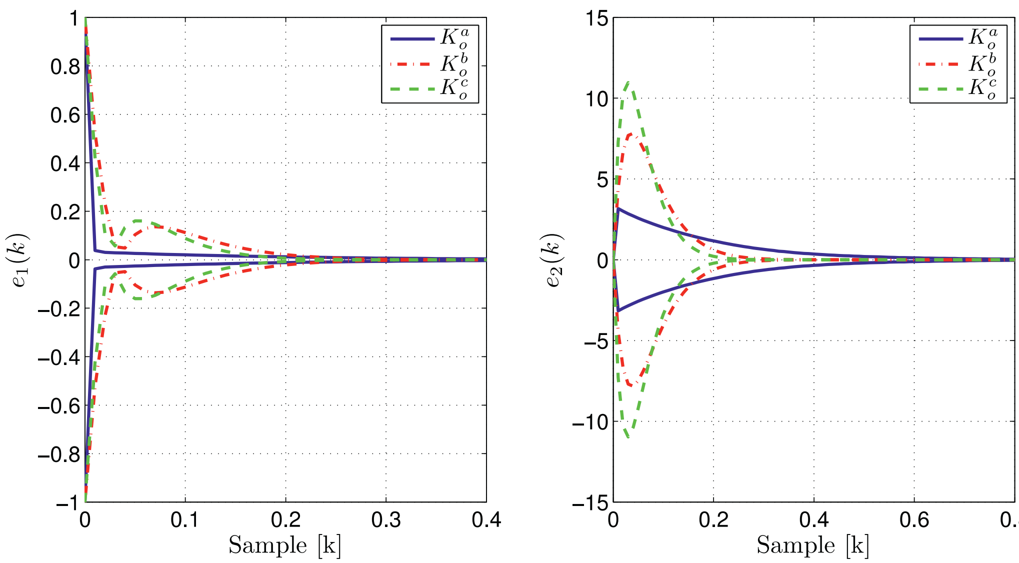

Fig. 11. Envelopes of the error $e(k)$ with observer gains K_o^a, K_o^b, K_o^c.

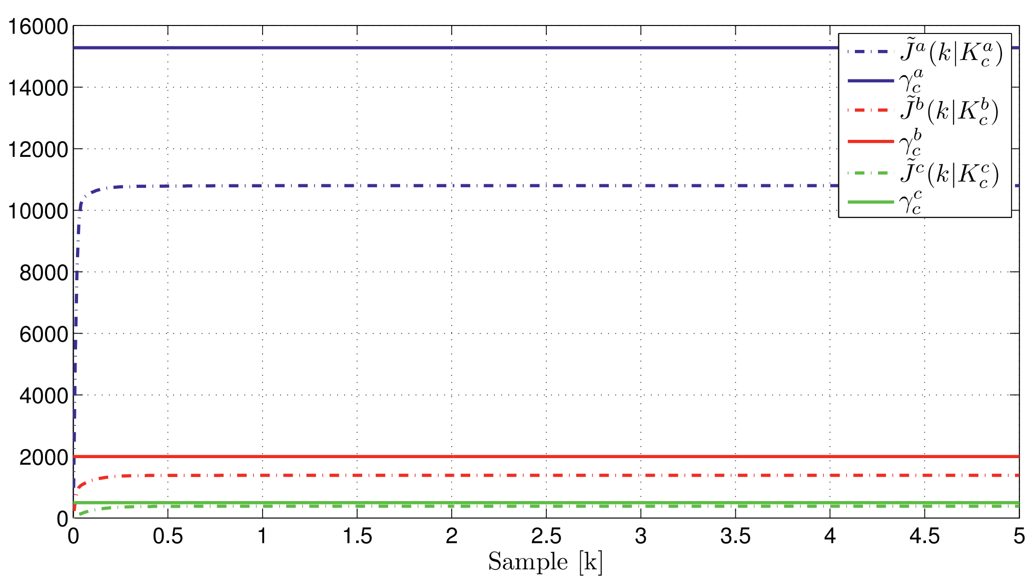

Fig. 12. Evolution of $\tilde{J}^i(k|K_c^i)$ and upper bounds γ_c^i, $i \in \{a, b, c\}$ (state-feedback).

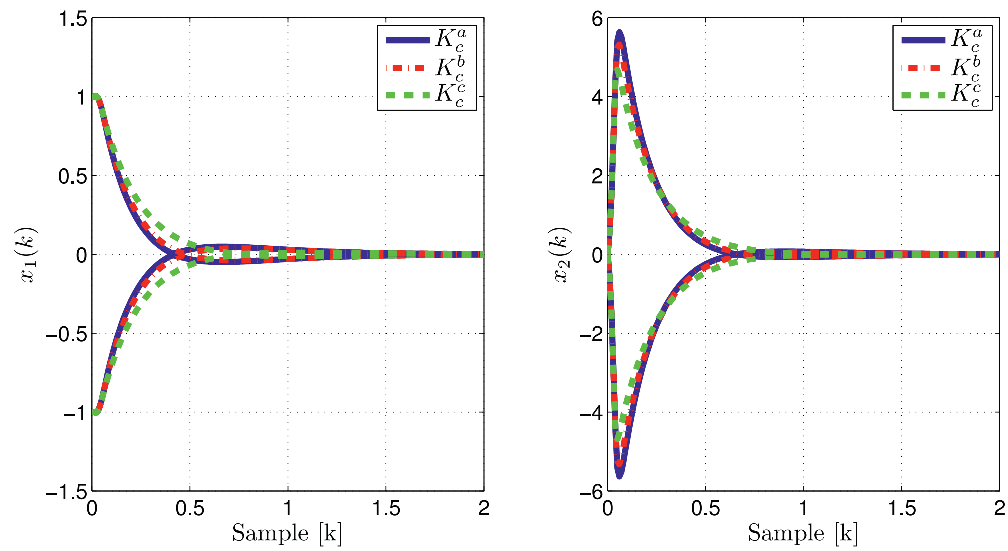

Fig. 13. Envelopes of $x(k)$ with controller gains K_c^a, K_c^b, K_c^c (state-feedback).

vides the smallest estimation error with respect to time and initial condition.

Subsequently, selecting $Q_e = \mathrm{diag}(10, 10000)$, $Q_x^b = \mathrm{diag}(0.0001, 0.0001)$, $Q_x^c = \mathrm{diag}(0.001, 0.001)$ and $Q_u^a = Q_u^b = Q_u^c = 0.01I$, the following controller gains have been designed:

$$K_c^a = \begin{bmatrix} -1.1967 & -0.2703 \end{bmatrix},$$

$$K_c^b = \begin{bmatrix} -1.1578 & -0.2968 \end{bmatrix},$$

$$K_c^c = \begin{bmatrix} -1.0410 & -0.3750 \end{bmatrix},$$

with $\gamma_c^a = 15279$, $\gamma_c^b = 19996$ and $\gamma_c^c = 501.74$, respectively. Fig. 12 shows the signal calculated using (75), $i \in \{a, b, c\}$, which demonstrates that $\tilde{J}^i(k|K_c^i) < \gamma_c^i$ is satisfied in all simulations. For this example, the controller gain that provides a faster convergence to zero of the state variable $x_1(k)$ and $x_2(k)$ is K_c^c. Finally, Fig. 13 shows the upper and lower envelopes of the state trajectories for initial conditions on the frontier of $\tilde{\mathcal{S}}$.

10. Conclusions

This paper has discussed the design of a state observer and a state-feedback controller that provide guaranteed cost estimation and guaranteed cost control, respectively, for a class of nonlinear systems affected by actuator saturations. The considered systems correspond to those for which the origin of the state space is an equilibrium point when null inputs are considered, and the nonlinearity is differentiable with respect to the state and linear with respect to the saturated input.

It has been shown that when both designs are considered separately, the procedure consists in solving LMIs, which is efficient to do using available solvers. The simulation results have shown the main characteristics of the proposed guaranteed cost design method, and the fact that less conservative solutions are found when the origin is an open-loop stable equilibrium.

On the other hand, it has been shown that in the more realistic situation in which a state estimate-feedback should be used, e.g., due to the lack of availability of some state variables for measurement, it is not possible to design the controller without taking into account the observer. In this case, the design procedure relies on bilinear matrix inequalities (BMIs). Some experiments using a BMI solver have shown that, although the proposed design procedure is viable in some cases, it suffers in returning a solution due to non-convexity issues.

In spite of the advantages of the proposed approach, the performance of the closed-loop system is affected by the conservativeness brought by the use of a quadratic Lyapunov function with constant Lyapunov matrix and constant observer/controller matrices. Future work will explore other types of Lyapunov functions which can decrease the conservativeness of the design procedure and the use of gain-scheduled (state-dependent) observer/controller gains. Moreover, other important directions for further research are the conversion of the BMIs obtained for computing the estimate-feedback controller gain into more computationally convenient LMIs, and the development of a procedure for the joint design of the observer and controller gain for estimate-feedback guaranteed cost estimation and control.

Declaration of Competing Interest

None.

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