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Forside for bacheloroppgaven Det
Teknisk-naturvitenskapelige fakultet

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1 Introduction

1.1 Outline

1.1.1 General Problem

In this paper I firstly seek to show that the laws of Kepler can be derived more fundamentally from Newtons laws of motion. Secondly that the orbits of the planets are described by harmonic oscillation, and that they are possible to model as such through numerical solutions.

1.1.2 What Have I Done

In this paper I have briefly described the historical context of the development of celestial orbital mechanics from Tycho Brahe, to Johannes Kepler and finishing with Isaac Newton.
After presenting Kepler's Laws of planetary movement, and Newton's Laws of motion, I used Newton's laws to arrive at Kepler's results.
Finally using mathematica to solve second order differential equations of motion, combined with Newton's and Kepler's Laws I modeled celestial orbits of our solar system, both real, and imagined.

1.1.3 Tools Used

For this paper I have used the computational program Wolfram Mathematica v12.2

1.1.4 Preview

In this paper I have shown that the empirical results of Johannes Kepler, descriptive models of our solar system based on the observational data of Tycho Brahe, can be derived from the laws of motion later postulated by Newton. This derivation which is based on mathematical models, allows for newer insights which were obscured by the inaccuracies in Brahe's data.
I have also found that the numerical solutions of Wolfram Mathematica, while producing some errors, are accurate enough to model the orbits of planets for up to at least the time of their orbital period.

1.2 Historical Introduction

The era of modern astronomy stared with the observations of Danish astronomer Tycho Brahe (1546-1601), who in 1572 observed a supernova. This observation was in stark contradiction with the current dogma of the church, that the skies were unchanging. The observation was of such significance that it prompted King Frederick II to construct and gift Brahe with the observatory Uraniborg at the island of Hveen. From this observatory, Brahe went on to compile his extensive notes on the movement of the planets through the heavens over several years. Despite the lack of optical equipment, Brahe was able to measure positions with an accuracy of at least 4' (four arc minutes), or 1/8 the angular diameter of a full moon. While Brahe was able to show through his observations a number of things - such as comets - changing outside of the atmosphere, he was unable to find evidence of Earths own movement through the sky. Thus he was forced to discard the Heliocentric model, and keep the Ptolemaic model of Geo-centrism.

While Brahe was an astute observer, and able to draw several conclusions from his observations, it was not until German mathematician Johannes Kepler (1571-1630) was invited to work with him that the data would be thoroughly analysed.

Johannes Kepler wished to use Brahe's data to formulate a mathematical model consistent with a Heliocentric world view. Kepler inherited Brahe's data upon his death, allowing him to continue his work.
At first held back by the Ptolemaic presumptions of planetary motion in concentric circles, Kepler tried to match the data to perfect circular orbits. This would however lead to a discrepancy of 8', or twice that of Brahe's accuracy in the data.

Sir Isaac Newton (1642-1727) would later expand upon Kepler's ideas, by incorporating his own theory of gravity and the mathematical tool of calculus. By applying his laws, which is outlined in a following section, Newton was able to arrive at Kepler's laws theoretically, rather than by relying on observational data alone, as Kepler had done. This would allow Newton to gleam new insight into the properties of the system which were hidden from Kepler, due to the inherent margin of error in the observations he based his laws on.
The content of this section is based on (Carol and Ostlie, 2018, chapter 2: Celestial Mechanics)

2 Theory

2.1 Kepler's Law's of Planetary Motion

1st: All planets move in elliptical orbits with the Sun at one focus
2nd: The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals

$$\frac{dA}{dt} = \frac{L}{2M} \tag{1}$$

3rd: The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

$$P^2 = ka^3 \tag{2}$$

2.2 Newton's Law's of Motion

1st: An object at rest will remain at rest and and object in motion will remain in motion at a constant velocity untess acted upon by an external force.
2nd: The sum of all forces acting upon an object is equal to its mass times acceleration.
3rd: For every action there is an equal and opposide reaction.

2.3 Mathematics of Conic Sections

There are some mathematical properties of conic sections which will be used in the following derivations that we will take for granted. These properties will be presented in this section.

In any ellipse, there is a relation between the semi-minor axis, b, the semi-major axis, a, and the eccentricity, e

$$b^2 = a^2(1 - e^2) \tag{3}$$

The distance from the principal focus to the edge of the ellipse is given by

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}, \qquad (0 \leq e < 1) \tag{4}$$

The eccentricity of an ellipse is given by

$$e = \frac{r_1 - r_0}{r_1 + r_0} \tag{5}$$

where r_1 is the distance at perihelion, the point on the ellipse closest to the principal focus, and r_0 is the distance at aphelion, the point on the ellipse farthest from the principal focus.

The distance at perihelion and aphelion is also related to the semi-major axis

$$a = \frac{r_1 + r_0}{2} \tag{6}$$

2.4 Coordinate Systems

While analyzing systems of orbiting bodies, it is relevant to consider where to choose to place our reference frame. It can be shown that a system of N bodies orbiting around a mass M can be reduced to an equivalent system of a reduced mass μ orbiting around M (Caroll and Ostlie, 2018, p. 47-50)

$$\mu = \frac{m_1 * m_2 * ... * m_N}{m_1 + m_2 + ... + m_N} \tag{7}$$

We may also place the mass M at the center of our coordinate system, as this will be an inertial frame of reference (Caroll and Ostlie, 2018, p. 49).

Finally as a simplification of the problem, we shall consider only cases where the body M is much more massive than the imagined body of reduced mass μ. This will allow us to consider one body orbiting another, rather than two bodies orbiting around a common center of attraction.

3 Derivation

The following derivations of the gravitational constant, and Kepler's Laws are based off the derivations from "An Introduction to Modern Astrophysics" (Caroll and Ostlie, 2018, chapter 2: Celestial Mechanics)

3.1 The Universal Gravitational Constant

By starting with Kepler's 3rd law of planetary motion, and assuming a circular orbit we can set the semi-major axis, a, as equal to the radius, r, of the circle. We also know that the orbital period will be given by the circumference of the circle traced by the orbit, divided by the speed of the satellite

$$P^2 = kr^3, \qquad P = \frac{2\pi r}{v} \tag{8}$$

In putting these two expressions together we get

$$\frac{4\pi^2 r^2}{v^2} = kr^3 \tag{9}$$

By multiplying the previous expression by the mass, m, and a little bit of algebra, we arrive at the following expression

$$\frac{mv^2}{r} = \frac{4\pi^2 m}{kr^2}, \qquad \frac{v^2}{r} = a \quad \text{in circular motion} \tag{10}$$

By Newton's 2nd law, we can recognize the expression on the LHS of the equation (10) as the force of the mass m, on another mass M. Given that $M >> m$ this will be the force keeping m in orbit around M.
However, by Newton's 3rd law, we know that the force of m on M will be the same magnitude as the force of M on m.

$$\begin{aligned} F &= \frac{4\pi^2 m}{kr^2} \\ F &= \frac{4\pi^2 M}{k' r^2} \end{aligned} \tag{11}$$

By this symmetry of the system, we can relate the force of the two objects orbiting each other by the proportionality constant k in the following way

$$F = \frac{4\pi^2 Mm}{k'' r^2}, \qquad k = \frac{k''}{M}, \qquad k' = \frac{k''}{m} \tag{12}$$

Finally we group all the constants together to arrive at the universal gravitational constant, G

$$G = \frac{4\pi^2}{k''} \approx 6.673 * 10^{-11} \quad Nm^2 kg^{-2} \tag{13}$$

3.2 Kepler's Laws of Planetary Motion from Newton

3.2.1 Kepler's 1.st Law

We wish to show that the orbital pattern of celestial objects whose motion is described by Newtons laws are elliptical. From Newton we know that acceleration due to M of a system with reduced mass is expressed in vector form as

$$\vec{a} = -\frac{GM}{r^2}\hat{r} \tag{14}$$

The angular momentum of the system is defined as the vector product of the distance vector, $\vec{r}$, and the momentum. The distance vector is here the distance between M and the reduced mass μ

$$\vec{L} = \mu\vec{r} \times \vec{v} = \vec{r} \times \vec{p}, \qquad \vec{p} \equiv \mu\vec{v} \tag{15}$$

By taking the time derivative of equation (15) and applying Newtons 2nd Law, we can analyze how the angular momentum changes over time. Since both $\vec{r}$ and $\vec{p}$ are functions of time, we apply the product rule for differentiation. Also, we can see that the change in distance with respect to time is simply the velocity, and by Newton's 2nd Law, the change in momentum with respect to time is simply the force.

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F} \tag{16}$$

From equation (16) we can now conclude that the angular momentum does not change with respect to time (17). This is because for central force, both the velocity and momentum vector, and the distance and force vector are parallel to each other respectively. This results in both cross products being zero.

$$\frac{d\vec{L}}{dt} = 0 \qquad \text{for central force} \tag{17}$$

Taking a step back, we can see that the angular momentum in equation (15) can also be expressed in terms of its unit vectors. By doing this, expanding the time derivative, and simplifying the cross product, we end up with the final step of equation (18), to get an expression for the angular momentum

$$\begin{aligned} \vec{L} &= \mu r\hat{r} \times \frac{d}{dt}(r\hat{r}) \\ &= \mu r\hat{r} \times (\frac{dr}{dt}\hat{r} + r\frac{d}{dt}\hat{r}), \qquad \hat{r} \times \hat{r} = 0 \\ &= \mu r^2\hat{r} \times \frac{d}{dt}\hat{r} \end{aligned} \tag{18}$$

Now that we have expressed both the acceleration and angular momentum by its unit vectors, we can take the cross product of the two. The vector identity $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$ helps us simplify the answer, and we get

$$\begin{aligned} \vec{a} \times \vec{L} &= -\frac{GM}{r^2}\hat{r} \times (\mu r^2\hat{r} \times \frac{d}{dt}\hat{r}) \\ &= -GM\mu[(\hat{r} \cdot \frac{d}{dt}\hat{r})\hat{r} - (\hat{r} \cdot \hat{r})\frac{d}{dt}\hat{r}] \\ &= GM\mu\frac{d}{dt}\hat{r} \end{aligned} \tag{19}$$

In equation (19) we can rearrange the RHS and take advantage of the fact that acceleration is the time derivative of velocity to arrive at an expression which we can easily integrate with respect to time.

$$\frac{d}{dt}(\vec{v} \times \vec{L}) = \frac{d}{dt}(GM\mu\hat{r}) \tag{20}$$

Since derivation and integration are inverse operations we are simply left with the same expression up to a constant D, after integrating. While we can not specify the magnitude of D, we can see that its orientation will lie in the orbital plane. This follows necessarily from the fact that both $\vec{v} \times \vec{L}$ and $\hat{r}$ does.

$$\vec{v} \times \vec{L} = GM\mu\hat{r} + \vec{D} \tag{21}$$

Now recalling that our goal is to demonstrate an elliptical orbit, and that mathematically ellipses can be described as a function of the distance r, we see that we can achieve this by taking equation (21) and applying the dot product of $\vec{r}$ to it.
In the following manipulation for the first step we apply the dot product.
For the second step we simply recall the identity $A \cdot (B \times C) = (A \times B) \cdot C$, and the fact that any two unit vectors are parallel, and the resulting dot product will therefore be a scalar value of one.
Finally it is worth noting that the angle θ introduced by the dot product of $\vec{r}$ and $\vec{D}$ refers to the angle of the reduced mass of the system measured from the direction to perihelion.

$$\begin{aligned} \hat{r} \cdot (\vec{v} \times \vec{L}) &= GM\mu r\hat{r} \cdot \hat{r} + \vec{r} \cdot \vec{D} \\ (\vec{r} \times \vec{v}) \cdot \vec{L} &= GM\mu r + rD\cos\theta \\ \frac{L^2}{\mu} &= GM\mu r(1 + \frac{D\cos\theta}{GM\mu}) \end{aligned} \tag{22}$$

Finally after some algebraic manipulation to make the scalar magnitude, r, our subject, we arrive at an expression for the distance, r, that is analogous to that of a conic section. By comparison we can define the eccentricity, e, of our system. From this we may infer that it, and by extension the shape of our conic section will depend on the masses involved in our celestial system.

$$r = \frac{\frac{L^2}{\mu^2}}{GM(1 + e\cos\theta)}, \qquad e \equiv \frac{D}{GM\mu} \tag{23}$$

3.2.2 Kepler's 2.nd Law

Kepler's 2nd law is a statement about how an area changes over time. A change in area of an orbital plane can be described by the following differential

$$dA = r\,dr\,d\theta = \frac{1}{2}r^2 d\theta \tag{24}$$

Moreover the way the area changes with respect to time, is related to how the angle θ changes with respect to time

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \tag{25}$$

In the case we are considering, in the presence of a central force, we can split the velocity into a radial and angular component. This means we can express the velocity components as their respective time derivatives, multiplied by a unit vector in the spacial direction we require them to face

$$\vec{v} = \vec{v_r} + \vec{v_\theta} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} \tag{26}$$

From equation (26) we can observe that the angular velocity is already a part of our expression for how the area changes over time, as is shown in the following expression

$$\frac{dA}{dt} = \frac{1}{2}rv_\theta \tag{27}$$

Next we notice that the product of the radius and the angular momentum is equal to the magnitude of the cross product $\vec{r}$ and $\vec{v}$. This is true because $\vec{r}$ and $\vec{v_\theta}$ are perpendicular, and $\vec{r}$ and $\vec{v_r}$ are parallel.

$$rv_\theta = |\vec{r} \times \vec{v}| = \left|\frac{\vec{L}}{\mu}\right| = \frac{L}{\mu} \tag{28}$$

In combining equations (27) and (28), we get

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu} \tag{29}$$

This tells us that the change in area with respect to time is proportional to the angular momentum. However, from our previous derivation of Kepler's first law, we showed that the angular momentum is constant. Since this is our only parameter, it must mean that the change in area is constant as well. This is exactly what Kepler's 2nd law tells us.

3.2.3 Kepler's 3.rd Law

In order to derive Kepler's 3rd law, we wish to relate the period of an orbiting object to the semi-major axis of its orbit. We can obtain an expression that lets us do this by starting with our final result in the previous section, equation (29), and integrating both sides over one orbital period, P. This will give us

$$A = \frac{1}{2}\frac{L}{\mu}P \tag{30}$$

Since we know the shape of the orbit will be elliptical from Kepler's 1st law, we can substitute in the area of an ellipse, πab. We then change the subject to the period, and square both sides, since we are looking for the period squared

$$P^2 = \frac{4\pi^2 a^2 b^2 \mu^2}{L^2} \tag{31}$$

By combining our physical interpretation of r that we found in equation (23) and our mathematical interpretation that we have from equation (4), we can solve for angular momentum to get an expression that holds for closed orbits

$$L = \mu\sqrt{aGM(1-e^2)} \tag{32}$$

If we substitute in equations (3) and (32) into (31) we get

$$P^2 = \frac{4\pi^2\mu}{GM}a^3 \tag{33}$$

4 Harmonic Oscillator

4.1 Harmonic motion in 1 dimension

The orbiting motion of a celestial object can be described as an oscillating system. In order to describe and analyze such systems to compare and test the theories of Kepler and Newton, we will first consider the simplest of such systems - the harmonic oscillator in one dimension. By considering this case we will be able to solve both analytically and numerically with mathematica, and compare the two results. This will serve as a proof of concept that the numerical solutions of mathematica provides reliable results.

The harmonic oscillator is described as

$$\frac{d^2x}{dt^2} = -kx \tag{34}$$

where x is the displacement of oscillation, and k is the spring constant. The negative sign implies a restorative force.
The solution to this second order differential equation is then

$$x(t) = A\cos\omega t + B\sin\omega t, \qquad \omega = \sqrt{k} \tag{35}$$

We get the case of simple harmonic motion on one dimension by imposing the initial conditions of some displacement X_0 at time t = 0, and initial velocity $V_0 = 0$ also at time t = 0.
The choice of initial displacement corresponds to the variable A in our analytic solution, and the choice of initial velocity corresponds to the variable B. Since the initial velocity is set to zero, the last term of the solution will vanish and we are left with an expression for the position as a function of time (Serway and Jewett, 2016, p. 420)

$$x(t) = A\cos\omega t \tag{36}$$

The graphs in figures 1 through 6 show the motion of the harmonic oscillator as a function of time. In figure 1, 3 and 5 the solution is found analytically by hand, and graphed in mathematica, while in figure 2, 4 and 6 the solution is found numerically through mathematicas built in differential equation functionality.

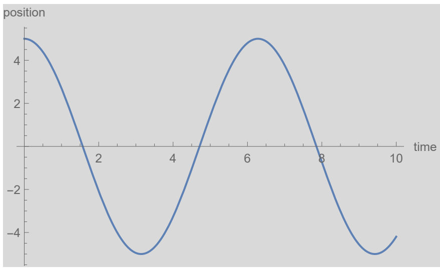

Figure 1: The position of an oscillating system as a function of time, found analytically

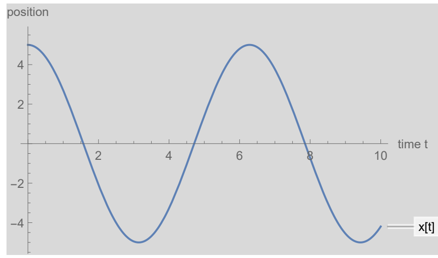

Figure 2: The position of an oscillating system as a function of time, found numerically

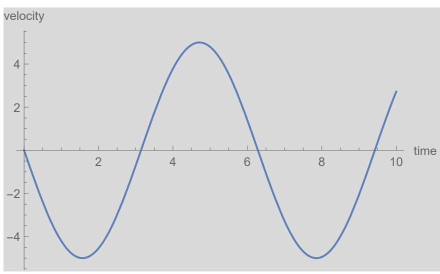

Figure 3: The velocity of an oscillating system as a function of time, found analytically

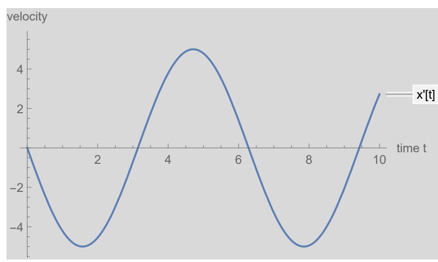

Figure 4: The velocity of an oscillating system as a function of time, found numerically

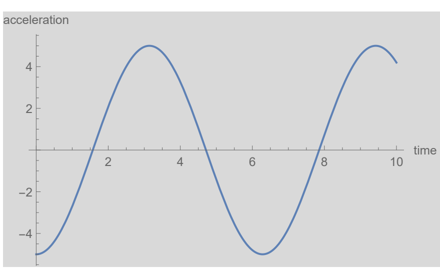

Figure 5: The acceleration of an oscillating system as a function of time, found analytically

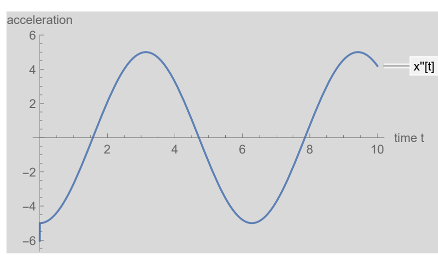

Figure 6: The acceleration of an oscillating system as a function of time, found numerically

The two approaches to the solution appear to yield a similar result. To see how close the two are, I plot the difference of the respective graphs. This difference can be seen in figures 7, 8 and 9, for the position, velocity and acceleration respectively. From the graphs we can see that as time progress, the margin of error increases. The scale at which the error propagates however is so small, 10^{-6}, that for the purpose of modeling orbiting systems of one body the two solutions are practically the same.

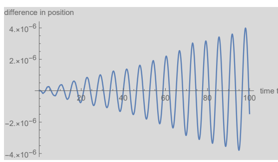

Figure 7: The difference in position between an analytic and numerical approach

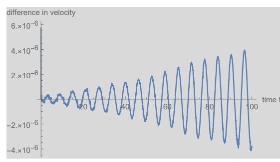

Figure 8: The difference in velocity between an analytic and numerical approach

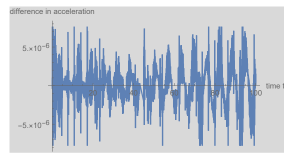

Figure 9: The difference in acceleration between an analytic and numerical approach

4.2 Harmonic motion in 2 dimensions

4.2.1 By explicit variables

In order to plot the orbital motion of celestial objects, we need to translate the one dimensional periodic motion into two dimensions. This is accomplished by considering a second separate second order differential equation to represent the orthogonal direction to our already established oscillator.
This will give us one expression for how our x-coordinate changes over time, and another for how our y-coordinate changes over time.

$$\begin{aligned} \frac{d^2x}{dt^2} &= -ax \\ \frac{d^2y}{dt^2} &= -by \end{aligned} \tag{37}$$

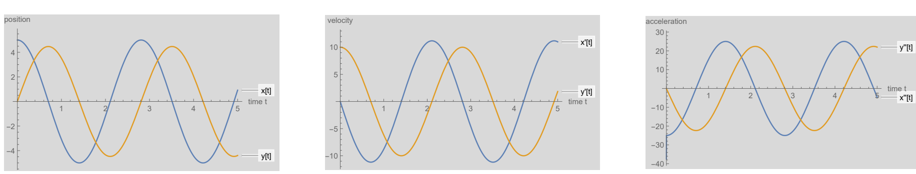

Figure 10: Numeric approach. The initial conditions have been set to illustrate the modelling of a circular path

These two solutions to the harmonic oscillators with their own unique initial conditions, can be used to generate a set of ordered pairs of values as a function of time, or directly plotted against each other to graph a parametric curve.

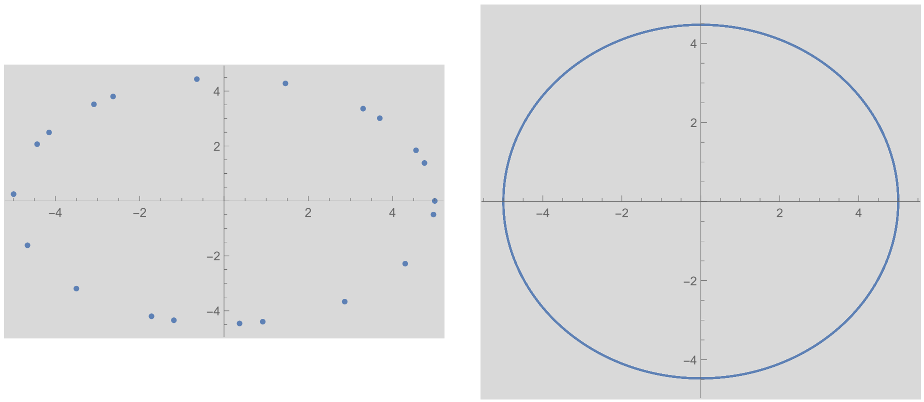

Figure 11: The two solutions graphed and plotted together with initial conditions describing circular motion

By adjusting the initial conditions of our differential equations, we can generate periodic motion of varying degree of eccentricity, periodicity and amplitude.

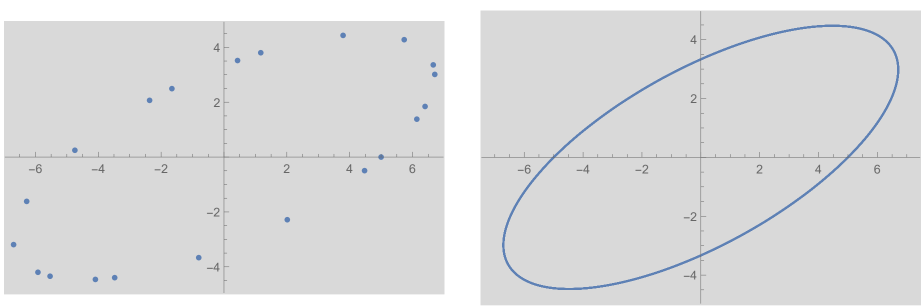

Figure 12: The two solutions graphed and plotted together with initial conditions describing elliptical motion

4.2.2 By implicit variables

If we wish to take it one step further and graph real world systems, we have to take into account that our two differential equations are implicitly described by each other. To see why this is true we consider what we already know from equation (14), that the centripetal acceleration due to a mass M of a system is inversely proportional to the square of distance between the objects. If we wish to impose Cartesian coordinates on this system, and we place M at the center, the distance between the objects at any time is described by

$$r[t]^2 = x[t]^2 + y[t]^2 \tag{38}$$

In this frame of reference, the centripetal acceleration is a linear combination of our basis vectors $\hat{x}$ and $\hat{y}$. The decomposed acceleration is then

$$\begin{aligned} a_x &= -\frac{GM}{r^2}\cos\theta, \qquad \cos\theta = \frac{x[t]}{r} \\ a_y &= -\frac{GM}{r^2}\sin\theta, \qquad \cos\theta = \frac{y[t]}{r} \end{aligned} \tag{39}$$

Putting equations (38) and (39) together, and expressing the acceleration in terms of the second derivative of position, we arrive at the two differential equations that will allow us to express orbital motion

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -\frac{GMm}{(x[t]^2 + y[t]^2)^{3/2}}x[t] \\ m\frac{d^2y}{dt^2} &= -\frac{GMm}{(x[t]^2 + y[t]^2)^{3/2}}y[t] \end{aligned} \tag{40}$$

For brevity I shall be using the notation x'[t] and x"[t] interchangeably with $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ respectively throughout the rest of the text, while discussing the choice of initial conditions of our systems.

4.2.3 Orbit of the Moon around the Earth

To test the differential equations we will consider the Moons orbit around Earth. If we approximate the orbit to be circular, we can find the values we need for our initial conditions.
In our choice of coordinates we will start at time t = 0 on the x-axis. This means that we set x[0] = r and y[0] = 0.
The orbital velocity can be expressed in terms of the orbital period P

$$v = \frac{2\pi r}{P} \tag{41}$$

Since the orbital velocity is always perpendicular to r in circular motion, the full magnitude of the velocity will be projected in the y direction at t = 0. Therefore we get our two last initial conditions x'[t] = 0 and y'[t] = v.
Figure (13) shows the parameterisation of the orbit with and orbital period P = 27.3 days and a distance $r = 3.88401 * 10^8$ m.

The centripetal acceleration for circular motion is given by

$$a_c = \frac{v^2}{r} \tag{42}$$

Using equation (41) for v, this gives a value for $a_c = 0.00272 \quad m/s^2$. We can compare this to the evaluation of our differential x"[t] at a given time, for

example x"[t] = -0.00269, and see that we get a similar value. The difference in sign is accounted for by the fact that a_c is a scalar magnitude with implied direction towards the systems center, while x"[0] is constrained to our coordinate system, and that acceleration at t = 0 is pointing in the negative x direction. Theoretically, we should expect to get the same value for x"[t] for any choice of t, since acceleration is constant for circular motion. By examining for instance x"[P/2] = 0.00266, we see this is not the case. This is explained by the tendency for errors to propagate with time as shown in section (4.1).
We can also compare our graphs eccentricity with the expected value of 0, by finding the distance at perihelion, $\left|x[0]\right|$, and distance at aphelion, $\left|x[P/2]\right|$. Recalling equation (5) we get e = 0.00305. This is not quite zero, again explained by the results of section (4.1), but very close to a circular orbit.

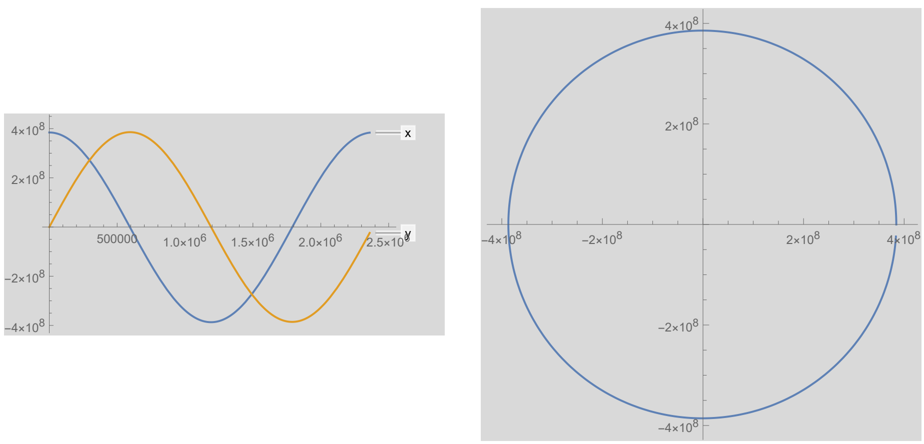

Figure 13: The orbit of the Moon around the Earth for one orbital period

4.2.4 Orbit of the Earth around the Sun

To model the earths motion around the sun, we will use a similar approach to our model of the moon. Instead of assuming circular motion we will use table values of earths orbit. x[0] will again be the distance between the Earth and Sun at perihelion, and y[0] will be zero. Because of our choice in initial position, we can again set the x'[0] to zero, and y'[0] can be calculated by the vis-viva equation (*Orbits, p. 6*)

$$v = \sqrt{GM(\frac{2}{r} - \frac{1}{a})} \tag{43}$$

With the mass of the Sun at $M = 1.989 * 10^{30}$ kg, the distance at perihelion $r = 1.471 * 10^{11}$ m and semi-major axis $a = 1.496 * 10^{11}$ m, we get $v = 30281$ m/s
With this information we can now graph the orbit.
Using the same method as in the previous section to find our graphs eccentricity we get a value e = 0.0162, compared to the expected value of e = 0.0167.

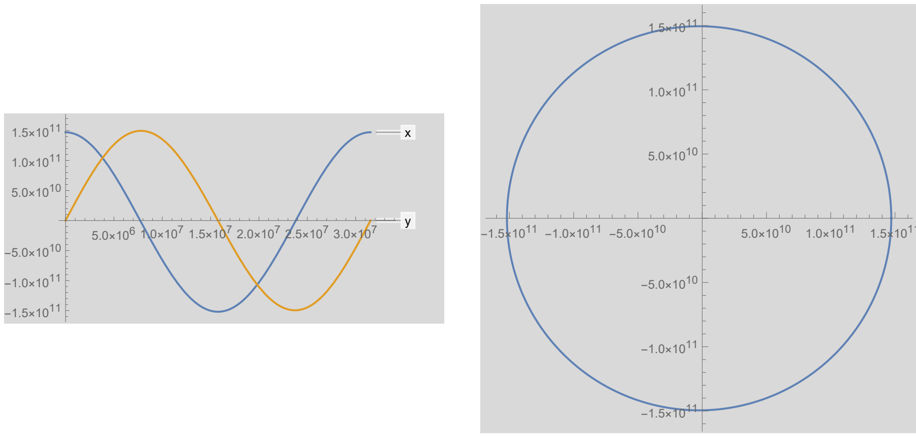

Figure 14: The orbit of the Earth around the Sun for one orbital period

4.2.5 Eccentric orbits and Escape velocity

In continuing to work with the orbit of the earth, one way to increase the eccentricity of the orbit, is to increase the velocity at perihelion. By increasing the velocity, the orbit and orbital period will increase, as can be seen from figure (15).

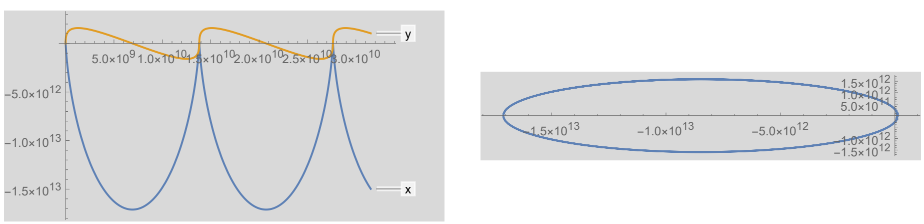

Figure 15: The orbit of the Earth around the Sun with a velocity at perihelion of 42300 m/s. Modelled at 1000 times the orbital period of the earth to complete one orbit.

Should the velocity at this point exceed a certain value, the planet will have enough kinetic energy to escape the gravitational pull of the Sun, and assume a hyperbolic path. This escape velocity can be found by considering the energy of the system. The initial energy of the system is given by

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r} \tag{44}$$

By convention, the potential energy goes to zero at a point infinitely far away. Assuming that the velocity will be zero at this point also, the total energy is zero. By conservation of energy the total energy must then be zero at all times

(Caroll and Ostlie, 2018, p. 47)

$$\frac{1}{2}mv^2 = G\frac{Mm}{r} \tag{45}$$

And so we finally get

$$v_e = \sqrt{\frac{2GM}{r_0}} \tag{46}$$

By using numbers from our scenario of the Earth orbiting the Sun, we find an escape velocity $v_e = 42482$, and from our graphs we can see how this velocity results in a trajectory away from the Sun.

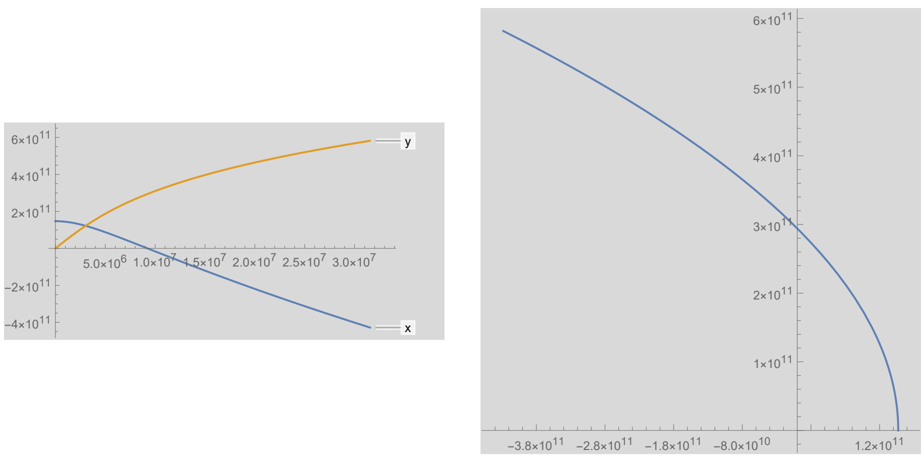

Figure 16: The Earth starting at perihelion, with enough velocity in the y direction to escape the Suns gravitational field

5 References

Bradley W. Caroll and Dale A. Ostlie (2018). *An Introdution to Modern Astrophysics 2nd Edition*. Cambridge University Press.
Raymond A. Serway and John W. Jewett, Jr. (2017). *FYS100 Fysikk, Custom Edition*. Cengage Learning EMEA
Florida International University. *Orbits*. Avaliable at: https://faculty.fiu.edu/ vanhamme/ast3213/orbits.pdf. (read 15.05.2021)

6 Appendix

In the appendix is enclosed the code used to arrive at the various graphs, and comparative values.

Bachelor Simulation Code

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1 dimension

In[11]:=

```
(* Parameters for analytic solution to harmonic oscillator *)
Clear[r, dr, ddr, k, m, ω, A]
Dynamic[A] (* Start posisjon *)
Slider[Dynamic[A], {0, 5}]
k = 1;(* Fjærkonstant *)
m = 2;(* masse *)
Dynamic[ω]
Slider[Dynamic[ω]]

r[t_, A_, ω_] := A * Cos[ω * t]
dr[t_, A_, ω_] := -ω * A * Sin[ω * t]
ddr[t_, A_, ω_] := -ω^2 * A * Cos[ω * t]
```

In[21]:=

```
(* Plot of harmonic oscillator in 1d solved analytically *)
Clear[tf];
tf = 10;
Dynamic[Plot[r[t, A, ω], {t, 0, tf}, AxesLabel → {"time t", "position"}]]
Dynamic[Plot[dr[t, A, ω], {t, 0, tf}, AxesLabel → {"time t", "velocity"}]]
Dynamic[Plot[ddr[t, A, ω], {t, 0, tf}, AxesLabel → {"time t", "acceleration"}]]
```

In[26]:=

```
(* Plot of harmonic oscillator 1d solved numerically *)
Clear[num, A, ω, tf];
A = 5;
ω = 1;
tf = 10;
num = NDSolveValue[{x''[t] + ω^2 * x[t] == 0, x[0] == A, x'[0] == 0}, x, {t, 0, tf}];
Plot[num[t], {t, 0, tf}, AxesLabel → {"time t", "position"}, PlotLabels → {"x[t]"}]
Plot[num'[t], {t, 0, tf},
 AxesLabel → {"time t", "velocity"}, PlotLabels → {"x'[t]"}]
Plot[num''[t], {t, 0, tf}, AxesLabel → {"time t", "acceleration"},
 PlotLabels → {"x''[t]"}]
```

```
(* Difference/margin of error of analytic and numerical solution *)
Clear[tf];
tf = 100;
Plot[{r[t, A, ω] - num[t]}, {t, 0, tf},
 AxesLabel → {"time t", "difference in position"}]
Plot[{dr[t, A, ω] - num'[t]}, {t, 0, tf},
 AxesLabel → {"time t", "difference in velocity"}]
Plot[{ddr[t, A, ω] - num''[t]}, {t, 0, tf},
 AxesLabel → {"time t", "difference in acceleration"}]
```

2 dimensions

In[76]:=

```
(* Numerical solution to harmonic
 oscillation of two explicit parameters x and y *)
Clear[a, b, X0, Y0, V0x, V0y, xsol, ysol];
a = 5;
b = a;
X0 = 5;
Y0 = 0;
V0x = 10;
V0y = 10;
xsol = NDSolveValue[{x''[t] == -a * x[t], x[0] == X0, x'[0] == V0x}, x, {t, 0, 20}];
ysol = NDSolveValue[{y''[t] == -b * y[t], y[0] == Y0, y'[0] == V0y}, y, {t, 0, 20}];
xsol[0];
```

In[ ]:=

```
Plot[{xsol[t], ysol[t]}, {t, 0, 5},
 AxesLabel → {"time t", "position"}, PlotLabels → {"x[t]", "y[t]"}]
Plot[{xsol'[t], ysol'[t]}, {t, 0, 5}, AxesLabel → {"time t", "velocity"},
 PlotLabels → {"x'[t]", "y'[t]"}]
Plot[{xsol''[t], ysol''[t]}, {t, 0, 5}, AxesLabel → {"time t", "acceleration"},
 PlotLabels → {"x''[t]", "y''[t]"}]
```

In[72]:=

```
(* Parametric representation of the two parameters combined to form an orbit →
 V0x:0 and V0y:10*)
Clear[table1];
table1 = Table[{xsol[t], ysol[t]}, {t, 0, 20}];
ListPlot[table1]
ParametricPlot[{xsol[t], ysol[t]}, {t, 0, 20}]
```

In[86]:=

```
(* Parametric representation of the two parameters combined to form an orbit →
 V0x:10 and V0y:10*)
Clear[table2];
table2 = Table[{xsol[t], ysol[t]}, {t, 0, 20}];
ListPlot[table2]
ParametricPlot[{xsol[t], ysol[t]}, {t, 0, 20}]
```

In[90]:=

```
(* Earth orbit around Sun *)
Clear[M, G, X0, Y0, V0x, V0y, P, sol1x, r0, r1, e];
M = 1.989 * 10^30; (* mass Sun *)
G = 6.673 * 10^-11; (* Grav constant *)
X0 = 147 091 144 000; (* distance from earth to sun at perihelion *)
Y0 = 0;
V0x = 0;
V0y = 30 281; (* velocity at perihelion *)
P = 365 * 24 * 3600;
sol1x = NDSolve[
  {x''[t] == -M * G * x[t] / (x[t]^2 + y[t]^2)^(3/2), y''[t] == -M * G * y[t] / (x[t]^2 + y[t]^2)^(3/2),
   x[0] == X0, y[0] == Y0, x'[0] == V0x, y'[0] == V0y}, {x, y}, {t, 0, P}];

Plot[Evaluate[{{x[t], y[t]} /. sol1x}],
 {t, 0, P}, PlotRange → All, PlotLabels → {"x", "y"}]
ParametricPlot[Evaluate[{x[t], y[t]} /. sol1x], {t, 0, P}]
(*{X,Y}=sol1x[[1,All,2]];*)
r0 = Abs[Evaluate[x[0] /. sol1x]] (* r0: distance to perihelion *);
r1 = Abs[Evaluate[x[P / 2] /. sol1x]] (* r1: distance to aphelion *);
e = (r1 - r0) / (r1 + r0) (* eccentricity of earths orbit around Sun *)
```

In[104]:=

```
(* Moon orbit around Earth *)
Clear[M, G, X0, Y0, V0x, V0y, P, sol1x, r0, r1, e];
M = 5.9736 * 10^24; (* mass earth *)
G = 6.673 * 10^-11; (* Grav constant *)
X0 = 3.84401 * 10^8; (* distance from earth to sun at perihelion *)
Y0 = 0;
V0x = 0;
V0y = 1020; (* velocity at perihelion *)
P = 2.36 * 10^6;
sol1x = NDSolve[
  {x''[t] == -M * G * x[t] / (x[t]^2 + y[t]^2)^(3/2), y''[t] == -M * G * y[t] / (x[t]^2 + y[t]^2)^(3/2),
   x[0] == X0, y[0] == Y0, x'[0] == V0x, y'[0] == V0y}, {x, y}, {t, 0, P}];

Plot[Evaluate[{{x[t], y[t]} /. sol1x}],
 {t, 0, P}, PlotRange → All, PlotLabels → {"x", "y"}]
ParametricPlot[Evaluate[{x[t], y[t]} /. sol1x], {t, 0, P}]
Evaluate[{x''[0], y''[0]} /. sol1x] (* centripetal acceleration *)
Evaluate[{x''[P / 2], y''[P / 2]} /. sol1x] (* centripetal acceleration *)
r0 = Abs[Evaluate[x[0] /. sol1x]] (* r0: distance to perihelion *);
r1 = Abs[Evaluate[x[P / 2] /. sol1x]] (* r1: distance to aphelion *);
e = (r1 - r0) / (r1 + r0) (* eccentricity of earths orbit around Sun *)
```

In[120]:=

```
(* Earth orbit around Sun alternate eccentricity*)
Clear[M, G, X0, Y0, V0x, V0y, P, sol1x];
M = 1.989 * 10^30; (* mass Sun *)
G = 6.673 * 10^-11; (* Grav constant *)
X0 = 147 091 144 000; (* distance from earth to sun at perihelion *)
Y0 = 0;
V0x = 0;
V0y = 42 300;
P = 365 * 24 * 3600;
sol1x = NDSolve[
  {x''[t] == -M * G * x[t]/(x[t]^2 + y[t]^2)^(3/2), y''[t] == -M * G * y[t]/(x[t]^2 + y[t]^2)^(3/2),
   x[0] == X0, y[0] == Y0, x'[0] == V0x, y'[0] == V0y}, {x, y}, {t, 0, 1000 P}];

Plot[Evaluate[{{x[t], y[t]} /. sol1x}],
 {t, 0, 1000 P}, PlotRange -> All, PlotLabels -> {"x", "y"}]
ParametricPlot[Evaluate[{x[t], y[t]} /. sol1x], {t, 0, 1000 P}]
```

In[131]:=

```
(* Earth orbit around Sun at escape velocity*)
Clear[M, G, X0, Y0, V0x, V0y, P, sol1x];
M = 1.989 * 10^30; (* mass Sun *)
G = 6.673 * 10^-11; (* Grav constant *)
X0 = 147 091 144 000; (* distance from earth to sun at perihelion *)
Y0 = 0;
V0x = 0;
V0y = Ve; (* velocity at perihelion *)
Ve = Sqrt[2 G * M / X0] (* Escape velocity at perihelion *)
P = 365 * 24 * 3600;
sol1x = NDSolve[
  {x''[t] == -M * G * x[t]/(x[t]^2 + y[t]^2)^(3/2), y''[t] == -M * G * y[t]/(x[t]^2 + y[t]^2)^(3/2),
   x[0] == X0, y[0] == Y0, x'[0] == V0x, y'[0] == V0y}, {x, y}, {t, 0, P}];

Plot[Evaluate[{{x[t], y[t]} /. sol1x}],
 {t, 0, P}, PlotRange -> All, PlotLabels -> {"x", "y"}]
ParametricPlot[Evaluate[{x[t], y[t]} /. sol1x], {t, 0, P}]
(*{X,Y}=sol1x[[1,All,2]];*)
```