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Abstract

This thesis is an introduction to graph theory with its core concepts and formulas, including a look at the mostly untapped classroom potential of this field of mathematics. Included are in-depth explanations and proofs of the Euler characteristic, platonic solids, the Kuratowski planarity criterion, and the 5- colour theorem, in addition to an introduction to the 4-colour theorem. For platonic solids, the Kuratowski planarity criterion, and the 4-colour theorem, there are suggestions for how to implement these concepts in the classroom, some with variations based on age group. After completing some of the lesson plans suggested in this thesis with students ranging from 4th to 9th grade, I have reason to believe that there is huge potential in introducing more concepts from graph theory at an earlier age and in teaching maths more visually.

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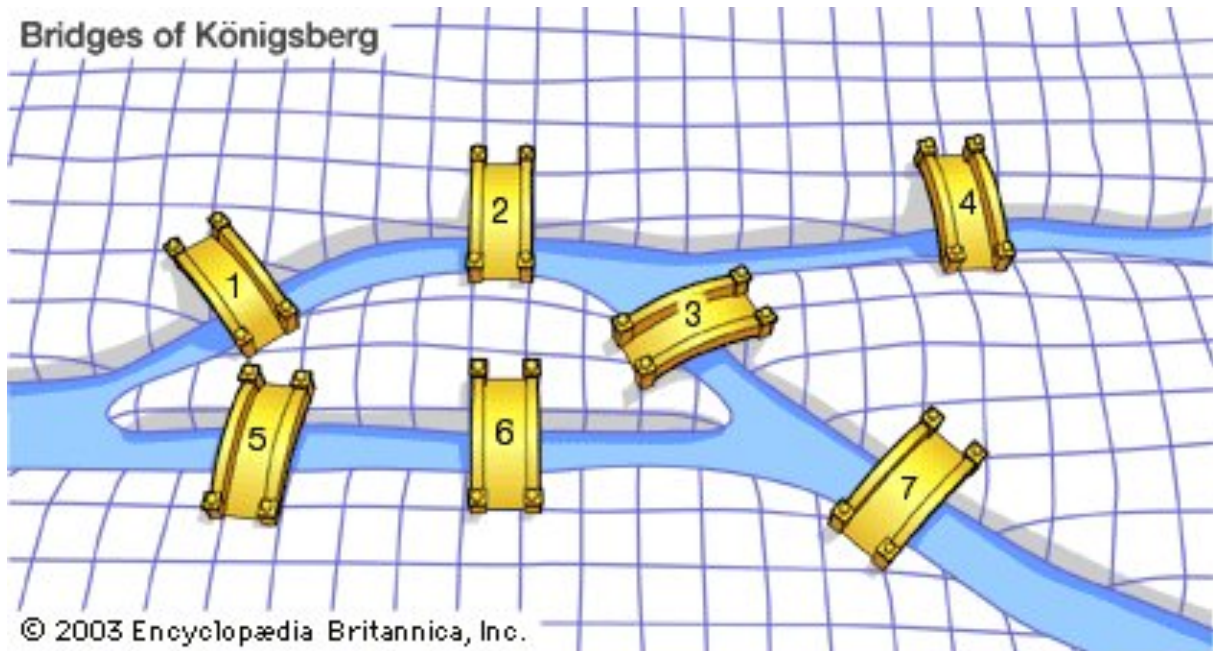
Introduction

Graph theory is part of discrete mathematics and has gained wide notoriety due to it being so easily applicable in many fields of study, especially in the digital age. A search for ‘graph theory’ on Google will yield more than 400 million results, and this field of mathematics is expanding more and more as time passes. By using planar graphs to reduce the scope of the problem down to the objects and the relationship between those objects, the problem becomes much more manageable. With computer use going mainstream in the 70’s and 80’s, the interest in graph theory was also on the rise. Problems such as the 4-colour theorem, which had first been posed in the 1800s, could be proven mathematically for the first time. Though the concepts within graph theory are quite easy to grab, it isn’t properly introduced in schools before university level courses. This means that there possibly exists a huge resource for students that remains largely untapped, when there are students who could really benefit from it being introduced at an earlier age. Students who struggle with math because it’s ‘just numbers’ could more easily visualize the problem using the concepts of graph theory. There are articles such as John Niman’s “graph theory in the elementary school” that go in-depth into the vast potential for graph theory in schools, but still it has yet to be implemented into the curriculum. In this thesis, I will include some basic definitions, an introduction to platonic solids, definitions of the Euler characteristic and the Kuratowski planarity criterion, a complete proof of the 5 colour theorem, and an introduction of the 4 colour theorem, with the overbearing topic in mind being introduction and execution for the concepts in a classroom setting. I was in the fortunate position of being able to test some of the concepts out classes ranging from 4th to 9th grade, my observations and suggestions for fine-tuning the lesson plans are to be found throughout the thesis.

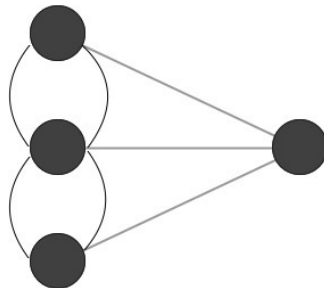
History

Graph theory goes all the way back to the old Greeks, who studied platonic solids, but the Konigsberg Bridge problem is widely regarded as the proper start of graph theory, after being solved by Euler in 1736. The famous math problem was based on the town plan of

Konigsberg, which is illustrated below.

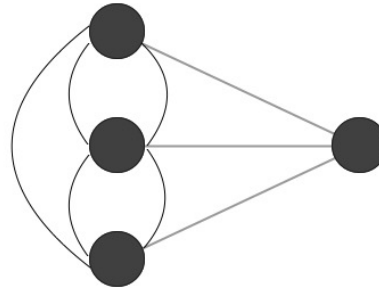


The problem was to find a path that utilized all 7 bridges only once, the path ending in the same place it started. A path of a graph that visits each edge only once is today known as a 'Eulerian path'. Euler simplified the problem by creating a planar graph with 7 edges representing the 7 bridges, and 4 vertices representing the islands and banks. He could then conclude that the problem had no solution (Niman, 1975, p. 353). Euler's rendition of the problem as a planar graph is depicted in the figure below.



There are many similar mathematical problems to the Königsberg Bridge problem, that actually have been used in schools under the guise of 'riddles', 'puzzles' or 'brain teasers'. Using one of these would be an excellent way of introducing graph theory. There are many to choose from, but why not start from the beginning? Students of all ages would be able to attempt to solve a problem similar to the Königsberg Bridge problem. The reason why the Königsberg Bridge problem has no solution is due to the vertex degrees (see definitions) of the vertices. A Eulerian path, which is what we want, exists in a graph if the number of vertices with an odd vertex degree is either zero or two. Therefore, one could start with

introducing the Königsberg bridge problem, where there are 4 vertices with an odd vertex degree and consequently not Eulerian, and then add a new bridge like in the graph below, making the number of vertices with an odd vertex degree 2, meaning there is a Eulerian path. Taking a different route, one could start with showing the graph where there is a solution, and then removing one of the bridges to reveal the Königsberg bridge problem.

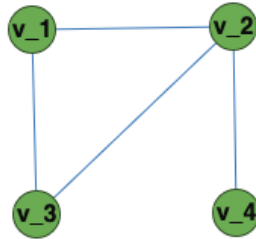


Some of these ‘brain teasers’ require a unique blend of good visual representation and creative problem-solving. And to think there is an entire mathematical field designed to make mathematical problem solving easier! Such a field of mathematics would be tremendously welcome in the school system, as the subject is regarded as boring and senseless by many students. If some aspects of math could prove to be fun, sensible and visually pleasing for the students, it could encourage them to not be so quick to dismiss other aspects of it. For students who struggle with envisioning a math problem, graph theory could prove to be extremely useful. There are many resources out there for teachers to use in order to engage the students in mathematical problem-solving using graph theory. The website for the Freudenthal Institute for Science and Mathematics Education have listed many assignments where using graph theory is an option, for example the ‘Evacuation’ problem, where the students must find the most efficient way to escape the building during a fire alarm. This is part of their Mathematics Olympiad, catering mostly to older students, but showing that there has long been a push for more inclusion of graph theory in the curriculum.

Definitions

The following are a few important definitions of terms that will be frequently referred to in this thesis. These definitions are standard, and most can be found in any book about graph theory, including “*Graph theory with applications*” by J. A. Bondy and U. S. R. Murty which has been the reference point for the definitions in this thesis (Bondy & Murty, 1976).

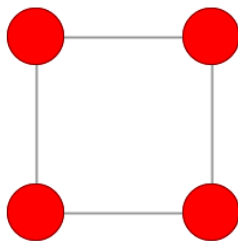
A **graph** G is a pair (V,E) where V is a set of vertices and E is a set of 2-element subsets of V . If we have $V=\{1,2,3,4\}$ and $E=\{\{1,3\},\{1,2\},\{2,4\},\{2,3\}\}$, the resulting **graph** is illustrated below.



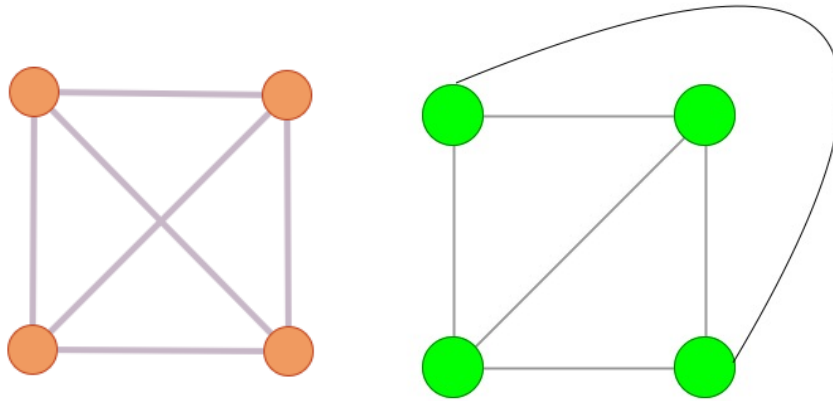
In this thesis, our focus will be on finite graphs, meaning graphs that have a finite number of vertices.

The **vertex degree** of a vertex v is the number of edges that are connected to v . Often, the vertex degree is indicated by a number within the vertex itself, while other times the number refers to v_1 and so forth. In this thesis, if a lone number is stated within the vertex, it will refer to the vertex degree. See fig. 1 for example.

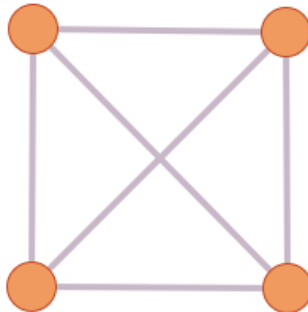
A **regular graph** is a graph where all vertices have the same vertex degree, such as the graph below, where all the vertices have a degree of 2.



A **planar graph** is a graph G that can be drawn in the plane without the edges crossing, for example on a flat piece of paper. Most graphs are drawn using straight edges as it looks neater and more visually pleasing, so seeing a graph with edges crossing over each other is not uncommon. This does not mean, however, that the graph is inherently nonplanar. A K_4 graph, for example, is typically drawn like in the first figure below, but by moving one of the edges we see that the graph is clearly planar.



A **complete graph K_n** is a graph with n vertices in which each pair of graph vertices is connected by an edge. An example is the complete K_4 graph, as seen below.



A **complete bipartite graph $K_{n,m}$** is a graph with a set of vertices that have been divided into two disjoint sets U and V , each with n and m vertices, respectively. Edges connect each vertex in U to each vertex in V , but not to the vertices of the same set. See figure 1 below, which shows a $K_{4,3}$ graph.

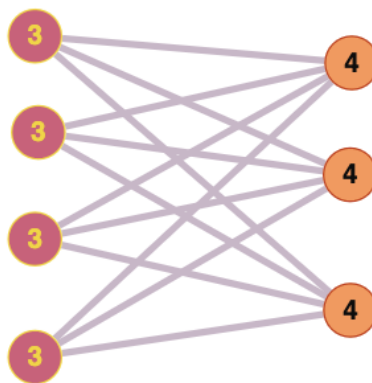
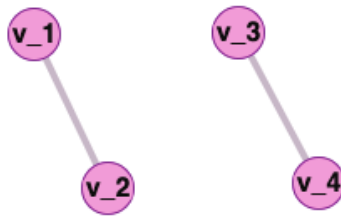


Figure 1

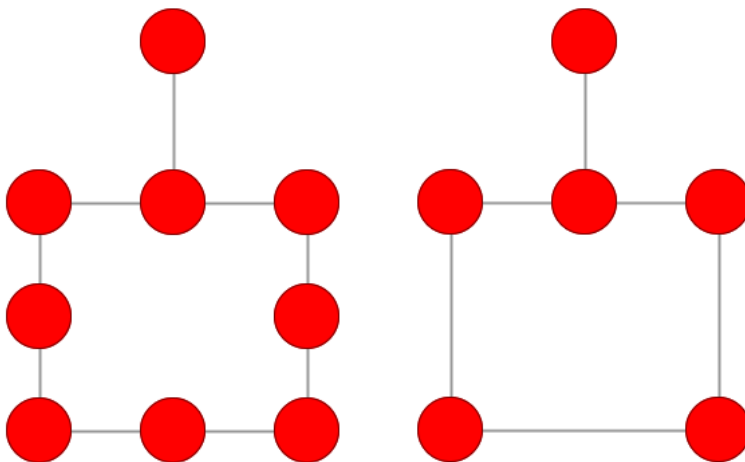
A **disconnected graph** is a graph where at least two vertices are not connected by a path, such as in the graph below.



The **utility graph** is another name for the $K_{3,3}$ graph. It is called the utility graph because the original problem the graph originates from is called the utility problem, where three houses are connected to three different utility companies by water pipes, gas lines and power lines.

The **pentatope graph** is another name for the K_5 graph, as it is isomorphic to the skeleton of the pentatope figure.

A **graph subdivision** is a graph that can be obtained from a given graph G by breaking up each edge into segments (one or more) by inserting intermediate vertices between its two ends. The first graph below, for example, is a graph subdivision of the second graph.



Definitions explained for children

For some of the lesson plans in this thesis, it is important to define some of the terms beforehand, or during the conclusion. Most of the definitions in the section above are worded in a way that would be difficult for younger students to comprehend, therefore I have included

a short section with explanations of the terms aimed at children. These definitions are how I chose to word myself when testing the various lesson plans mentioned in this thesis.

A **graph** is something we can draw that shows us the relationship between objects. Examples of things that can easily be turned into a graph includes maps, bus schedules, family trees, etc.

A **planar graph** is a graph that can be drawn on a flat piece of paper where we have the opportunity to draw the graph without crossing any edges over each other.

A **regular graph** is a graph where each circle has the same number of circle neighbours, and therefore the same number of lines coming out of each circle.

A **complete graph** is a graph where each circle is connected to all the other circles. A graph could also be complete if there are two sets of circles, using two different colours, where each circle with colour *A* is neighbours with each circle with colour *B*, while not being neighbours with the circles that have the same colouring.

The Euler characteristic

The Euler characteristic, C , is a number “that is a topological characteristic of various classes of geometric figure” based on the relationship between the number of the figure’s vertices (v), edges (e) and faces (f) (Britannica, 2017). In this context, a face of the graph is a region bounded by a set of edges and vertices in the embedding, including what is known as the *infinite face*, which is the unbounded area surrounding the graph, though many like to think of it as a sort of ‘backdrop’ or ‘surface’ on which our graph is placed.

The theorem for the Euler characteristic, henceforth referred to as Euler’s theorem in this thesis, is stated as such:

Theorem: Euler characteristic of planar graphs

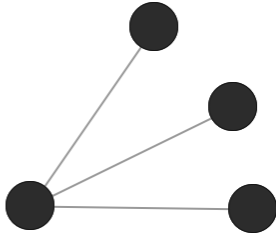
If G is a connected, planar graph with v vertices, e edges and f faces, then

$$v - e + f = 2$$

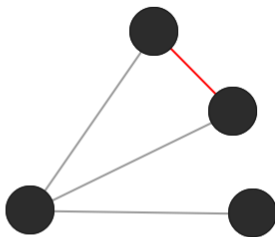
Proof by induction:

- 1) For our proof, we will make an induction of the number of edges e . We start with a connected graph G on v vertices, v being a fixed number of vertices, with a minimal number of edges. This makes the number of edges $e=v-1$, giving us a tree graph. The formula holds for all such graphs:

$$v - (v - 1) + 1 = v - v + 1 + 1 = 2$$



- 2) So we make our induction step. If the formula holds for e , it must also hold for $e+1$. We assume the graph is planar and connected, with v vertices, e edges and f faces such that Euler's formula holds. We make a new graph G' out of G by adding one edge to G , such that G' is still planar. Then G' has $v'=v$ vertices, $e'=e+1$ edges, and $f'=f+1$ faces.



The following calculation proves that the graph G' still yields a Euler characteristic of 2.

$$c' = v' - e' + f' = v - (e + 1) + (f + 1) = v - e + f = 2$$

This proves the claim.

Example:

Taking the tetrahedron as an example, $f=4$, $v=4$ and $e=6$, giving us $4 - 6 + 4 = 2$. If we imagine a 10-sided dice with each face being the shape of a congruent kite, we would have $f=10$, $v=12$ and $e=20$, which leads to the formula for Euler's characteristic $12 - 20 + 10 = 2$ being proven true again. If we were to remove one of the faces from the 10-sided dice, the number of vertices and edges would stay the same, but the formula for the Euler characteristic would yield $c=1$ instead of 2, which would correspond to a polygon (two-dimensional figure).

Moving onwards, there is also an important inequality that will be essential in further proof within this thesis. By definition, each face of a polyhedron must consist of at least 3 sides, meaning the least amount of sides in the resulting polyhedron must be three times the number of faces. Also, by definition, an edge is where two faces meet, meaning that the total number of sides consist of exactly two times the number of edges.

Lemma 1:
$$3f \leq 2e$$

Going further, if we combine Euler's formula with this inequality, we can substitute the f with $f=2-v+e$, such that $3(2 - v + e) \leq 2e = 6 - 3v + 3e \leq 2e = 3e - 2e \leq 3v - 6$ and we end up with:

Lemma 2:
$$e \leq 3v - 6$$

Lemma 3: Every planar graph has a vertex of degree 5 or less.

Proof of lemma 3 by contradiction: Suppose there is a planar graph where the vertex degrees are all 6 or higher. We know that the sum of vertex degrees is equal to twice the number of edges, as each edge has one vertex at both ends, namely:

$$6v \leq \sum d_v = 2e$$

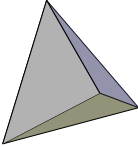
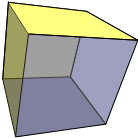
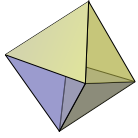
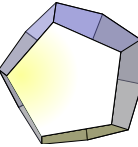
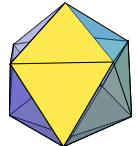
By lemma 2, $e \leq 3v - 6$, so we can put these inequalities together and end up with:

$$3v \leq e \leq 3v - 6$$

This is a contradiction, proving that every graph has a vertex with a vertex degree of 5 or less.

Platonic solids

Platonic solids were a source of great fascination for the ancient Greeks, particularly Plato, to whom they owe their name. A platonic solid is a geometric solid "whose faces are all identical, regular polygons meeting at the same three-dimensional angles" (Heilbron, 2013) of which there exists five. The most familiar platonic solid is the cube, and additionally there is the tetrahedron, the octahedron, the dodecahedron and the icosahedron. The chart below shows the name of each platonic solid, accompanied by an illustration and the total number of nodes, vertices and edges.

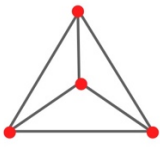
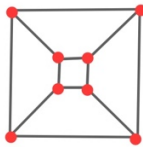
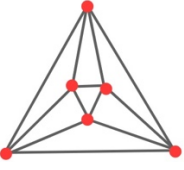
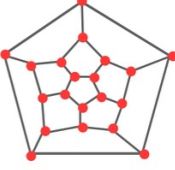
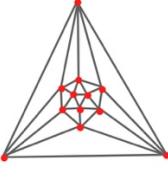
Type of platonic solid	Illustration	No. of faces (f)	No. of vertices (v)	No. of edges (e)
Tetrahedron		4	4	6
Cube		6	8	12
Octahedron		8	6	12
Dodecahedron		12	20	30
Icosahedron		20	12	30

All 5 platonic solids and polyhedrons that aren't regular (as long as they do not contain holes and do not intersect themselves) adhere to Euler's formula. If we take the icosahedron, for example, Euler's formula becomes $C = 12 - 30 + 20$, making $C=2$, as expected.

Platonic solids are particularly easy to graph, as their planar representations are reminiscent of their true three-dimensional shape. This also makes it much easier to explain to young students, as they might struggle with understanding how one can simply turn a three-dimensional shape into a planar graph to be drawn on a flat piece of paper, even more so if the graph looks nothing like the original shape at all. Showing a class a three-dimensional shape,

for example a cube, and then demonstrating how one can represent such a figure planarly (see chart below), would be an easy task, and a smooth transition into the world of graph theory.

The following chart shows each platonic solid and their planar representation.

Type of platonic solid	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Planar representation					

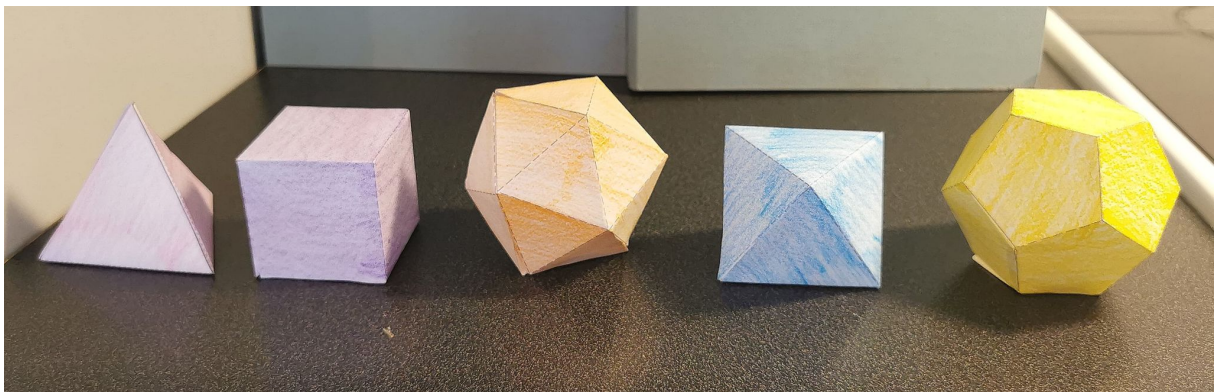
Platonic solids: Classroom potential

I was given the opportunity by one of my colleagues to ‘borrow’ her 4th grade class for a math lesson, so that I could better prepare myself and assess the difficulty level of the students beforehand. Prior to, I have accompanied the class on weekly field trips, which meant I was quite familiar with them. This is a clear advantage, as classroom management is much easier when you know the students’ names and have established a good relationship with them. Over the winter break, I started to work on a few variations of lesson plans in graph theory. I decided to introduce the lesson with explaining, in simple terms, what a graph is. Loosely translated, I explained that a graph is something we can draw that shows a number of objects and how those objects relate to each other. I then placed some appropriately sized 3D figures on a desk, big enough for everyone to see from where they were sitting. I asked them which of the three shapes (octahedron, hexagonal prism and rectangular prism) was special, a picture of the models I used can be seen below.



They made several good points but couldn’t come to a conclusion, so I prompted that one them is special because it has the same shape on each face, and each of the shapes have the

same angles. They then decided that the blue one, the octahedron, was special. I could then go on to tell them about platonic solids and what that entails and could then ask them if they were familiar with other 3D figures that satisfy our demands for a platonic solid. Some suggested a pyramid, and I replied that a pyramid with three sides and a bottom is indeed a platonic solid. I expected them to suggest a cube as a platonic solid, but they didn't until I prompted them to by pointing at some dice that were in the classroom. In preparation for this lesson I had made all the platonic solids into 3D figures using thick paper and showed the class all 5 platonic solids, as seen below. I showed the students how we could draw the octahedron and the cube as planar graphs, while holding the 3D figures, and pointing out which face of the figure corresponded to which face on the planar graph. If I were to make a small improvement here, I would probably colour each face a different colour so that I could demonstrate this more easily. The lesson went by quite fast, meaning I couldn't go on any further. Had the students been older, I may have been tempted to introduce them to the Euler characteristic, but I felt the level of this lesson was appropriate for this particular age group.



Kuratowski planarity criterion

The Kuratowski planarity criterion is a great tool in graph theory, as it helps in determining whether a graph is planar or not. Since the main focus for this thesis is planar graphs, stating Kuratowski's planarity criterion is absolutely essential. For complete proof of the theorem, see p. 153 of Bondy & Murty's "Graph theory with applications".

Theorem: Kuratowski's theorem (1930):

A graph is planar if and only if it contains no subdivisions of $K_{3,3}$ or K_5

K_5 , known as the pentatope graph, is represented in fig. 2 and $K_{3,3}$, known as the utility graph, is represented in fig. 3.

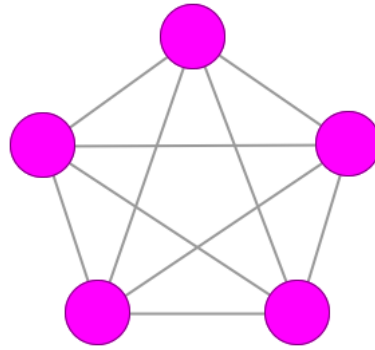


Figure 2

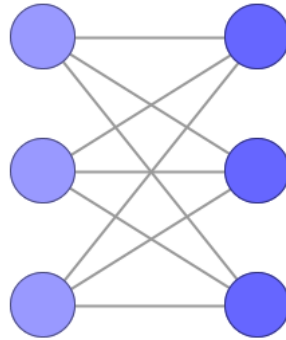
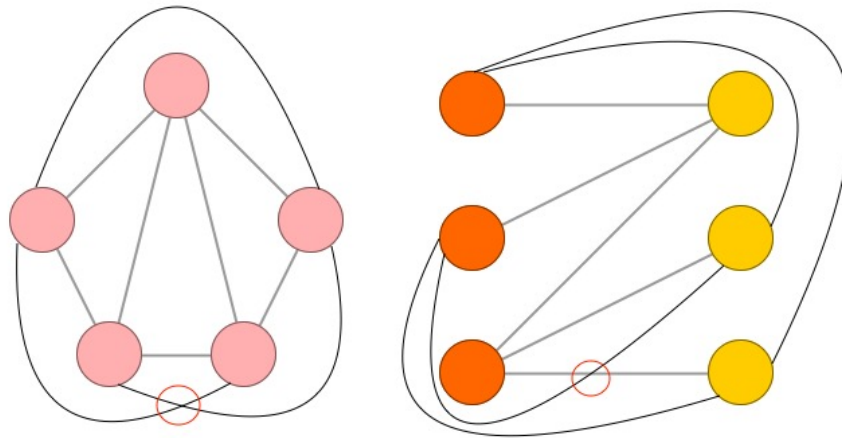


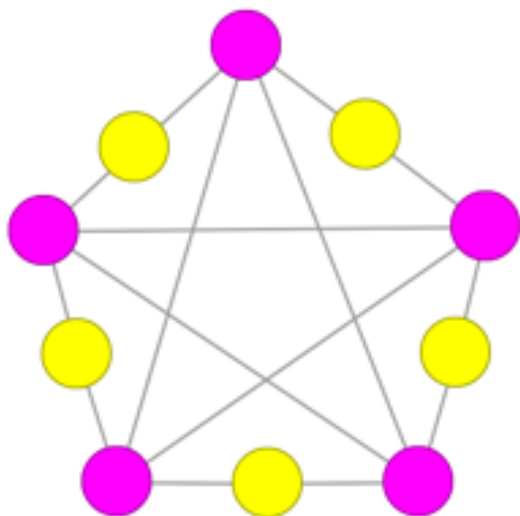
Figure 3

Using the formula for Euler's characteristic, it can be shown mathematically that the $K_{3,3}$ or K_5 graphs are not planar. Euler's characteristic is stated as $c = v - e + f$. Taking the K_5 first, we must first (wrongly) assume that the graph is planar, meaning that $c=2$. We know by definition that $v=5$, counting the number of edges we see that $e=10$, and so, by Euler's formula, K_5 must have 7 faces. Already we can dispute this by showing that it does not satisfy the Euler inequality, since $21 \not\leq 20$. For the $K_{3,3}$ graph, however, $v=6$ and $e=9$, meaning that $f=5$ to satisfy the Euler characteristic formula. This graph does in fact satisfy the Euler inequality above, so to prove its non-planarity, we must utilize the fact that the $K_{3,3}$ graph is bipartite, meaning that every face has an even number of sides. This transforms our previous inequality into $4f \leq 2e$, as 4 is the minimal number of sides a bipartite graph can have. Inputting our numbers into the updated inequality, we see that $20 \not\leq 18$, thus we have a contradiction. For both the $K_{3,3}$ and K_5 graph, it is also evident graphically that they are not

planar, as they cannot be connected in a way that does not include at least one crossover. See example below.

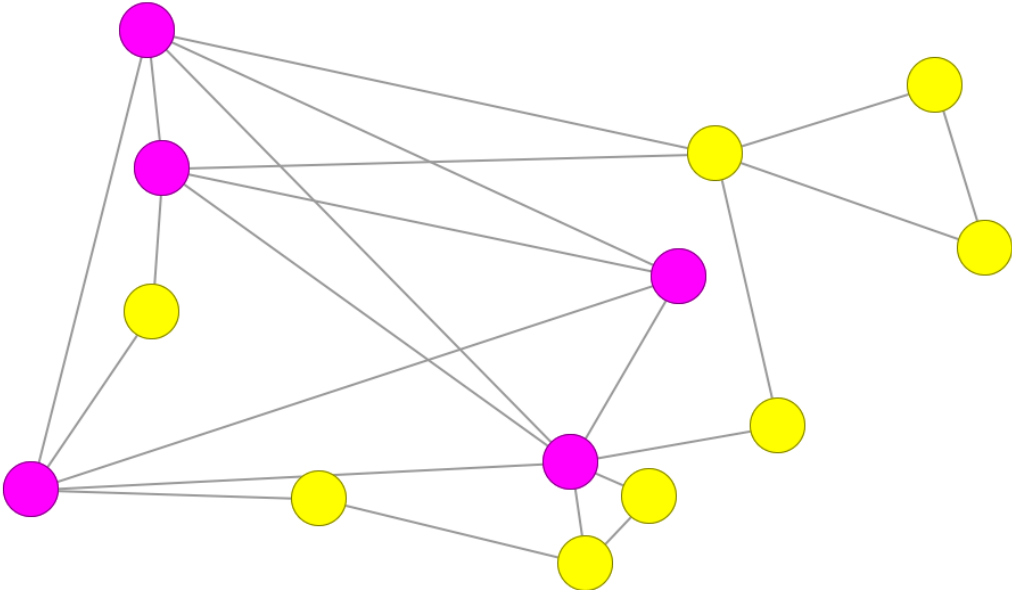
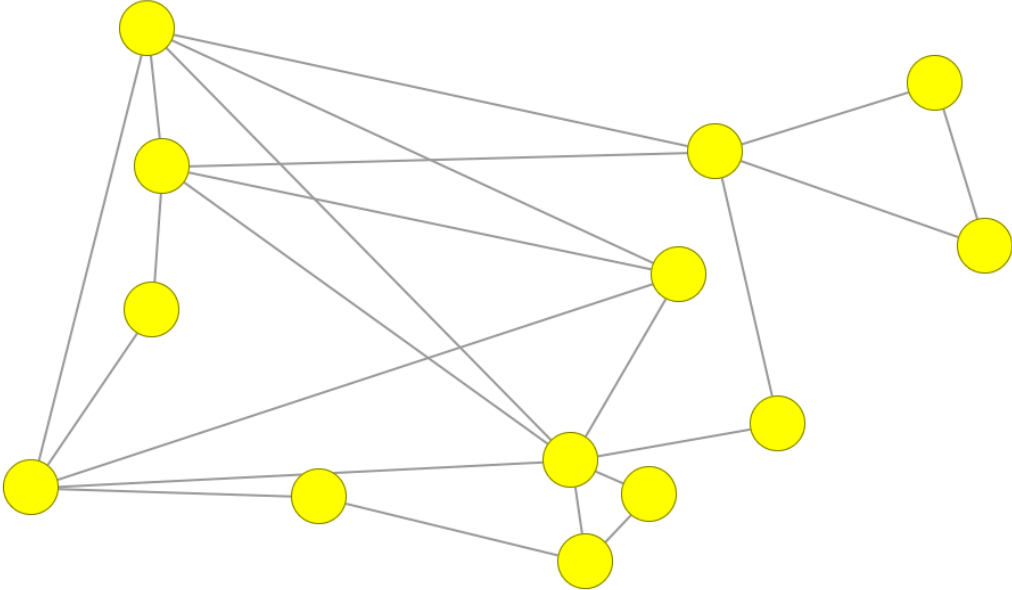


Having shown that $K_{3,3}$ and K_5 graphs are non-planar themselves, it must be emphasized yet again that a graph with a **subdivision** of $K_{3,3}$ or K_5 is per the Kuratowski planarity criterion not considered planar. Below is an example of a graph where edges have been removed, new vertices have been added, and new edges have been added to connect them.

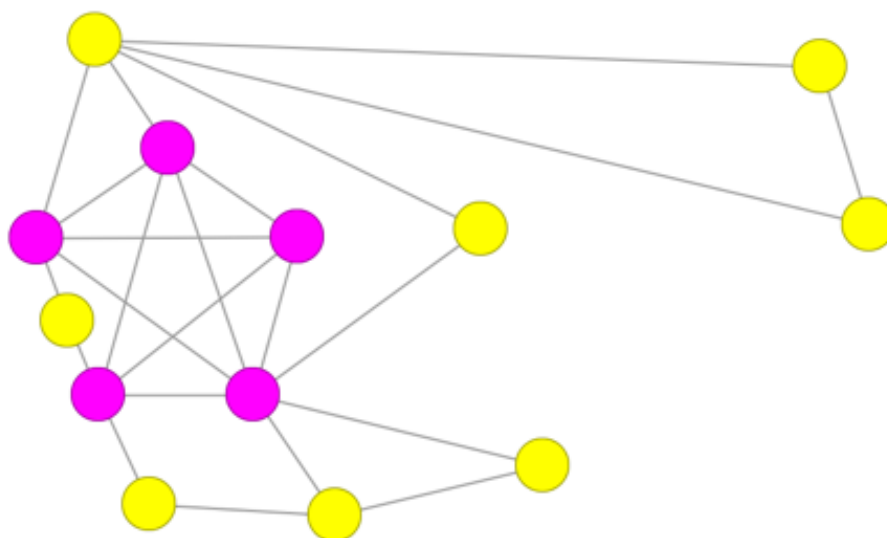


This example of a subdivision is fairly obvious, but it isn't always so. A graph could have added vertices and edges that aren't arranged neatly along the original edges of the pentatope or utility graph. Since our last example used a K_5 graph subdivision, our next example will showcase a more realistic example using a K_5 graph subdivision. In the first graph below we see that there are overlapping edges. This does not inherently prove that the graph is nonplanar. However, if we were to attempt a planar representation of the graph, it would prove impossible. The graph has a K_5 graph subdivision, which is found by sectioning off the

graph like in the second graph below, where the purple vertices represent the K_5 graph subdivision.



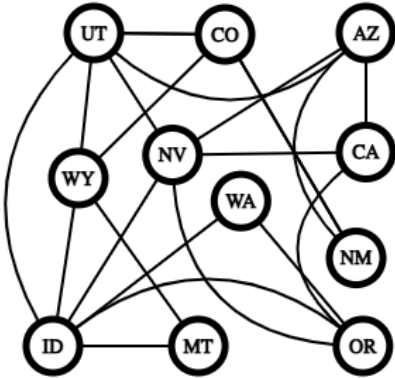
The following graph is showing how we could easily see that the graph is indeed a K_5 subgraph had the vertices been arranged differently.



Kuratowski's planarity criterion is relatively simple to understand, though one must first be familiar with the terms used. Visualizing it makes it even easier to comprehend, particularly for younger students. There are a number of ways to demonstrate the Kuratowski planarity criterion in a classroom, or even outside on the pavement using chalk, for instance. Each vertex could be represented by a bucket, a paper plate, a student, etc., and the edges are either drawn in chalk or represented by jump rope or yarn, and the students are given the task of connecting the vertices to each other using the edges, without any of the edges crossing over one another.

I decided to plan a hypothetical lesson involving this activity. The ideal age group for this type of activity would probably be 5th grade and up. Middle schoolers are, in my experience, the most likely to actually be excited about something like this, while some of the students in 8th grade and up are generally less enthusiastic. I wanted to start the lesson off with explaining what a planar graph is and what it could be used for, and then moving on to the activity for the remainder of the lesson. It would probably be smart to divide the class into teams of 4-6 people, depending on the size of the class. I would then set up the graphs on the floor using coloured paper plates and yarn, replicating a $K_{3,2}$ graph, telling the students to use the pieces of yarn to connect each red plate to each green plate and vice versa, but not connecting the red plates to the other red plates, and not connecting the green plates to the other green plates. If they see fit, they may move the plates. Depending on how quick they are, I may include a K_4 graph as well, using only one colour of plates, and this time connecting each blue plate to all the other blue plates. Moving on, they would need to tackle

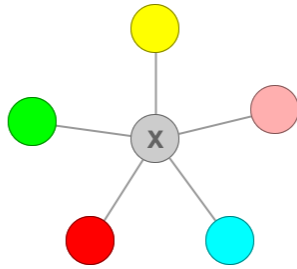
borders between them as the edges of a planar graph, we restrict ourselves to using only the truly relevant information. As an example, the US map seen above can easily be represented by a planar graph. The figure below shows such a graph with the states Washington, Oregon, California, Nevada, Arizona, Idaho, Montana, Utah, Wyoming, Colorado and New Mexico as vertices and their borders represented as edges. For the 5-colour theorem to be true, it must mean that any planar graphs vertices can be coloured using 5 colours, given that any vertex that is connected to another vertex by an edge cannot be coloured in the same colour.



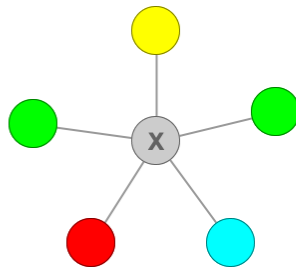
According to the five-colour theorem, any map is 5-colorable, and therefore any simple planar graph's vertices are also 5-colorable. To reiterate, this means that any map or simple planar graph can be coloured using **at most** 5 colours. For any simple graph with 5 or less vertices, the five-colour theorem is trivially true. We will prove that the theorem also holds for graphs with more than 5 vertices.

5-colour theorem: Proof by induction:

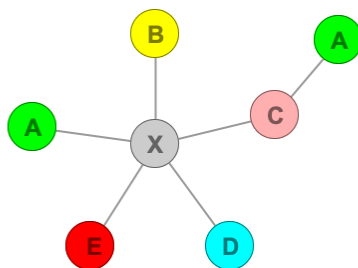
- 1) For our proof, we will do an induction on the number v of vertices. If $v < 6$, the statement is trivially true, as each vertex can have a different colouring regardless of their connection to the other vertices.
- 2) We will assume that all planar graphs of v vertices are colourable by 5 colours or less. Meaning, we have a graph G with v vertices to which we can assign a vertex colouring of 5 colours or less.
- 3) Let G be a planar graph with $v+1$ vertices. By lemma 3, G has a vertex X with a vertex degree of <6 .
- 4) If we remove X from G , we obtain a planar graph with v vertices which, by the induction hypothesis, is 5-colourable. The figure below is showing a part of the graph G containing the non-coloured vertex X and its neighbours, but is not a complete representation of the whole graph G .



If X has vertex degree <5 , there is a colour available for X to achieve a 5-colouring of G .
 If X has vertex degree exactly 5, and the neighbouring vertices are coloured by <5 colours, there is still a colour available for X .

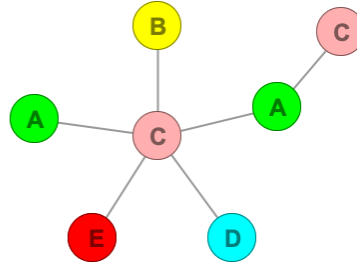


5) We therefore assume that all 5 colours occur around X , meaning all 5 neighbours have different colouring. These 5 colours are named A (green), B (yellow), C (pink), D (turquoise), and E (red), ordered clockwise around X .

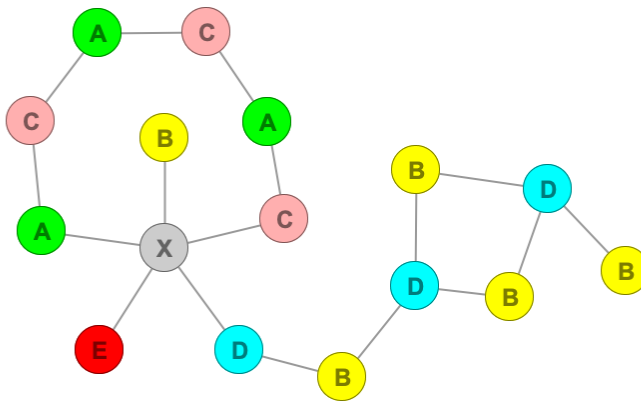


6) Let G_1 be the subgraph of all vertices with vertex colour A and C. If the neighbours A and C to X are **disconnected** from each other, we can revert the colours A and C in one component of the subgraph G_1 , freeing a colour for X . In the figure below, C is attached to an A, but is disconnected to the neighbouring A to X.

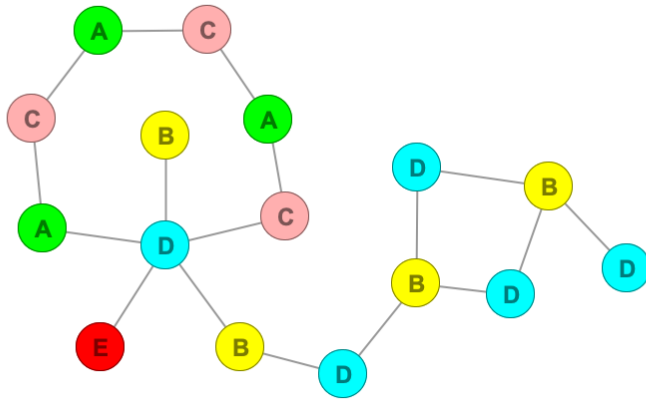
- 7) Since the neighbours A and C to X are disconnected in G_1 , we can swap C for A , which means the attached A to C is now reverted like in the following figure. Now X can be coloured using colour C .



- 8) If the neighbours A and C to X **are** connected in G_1 , there is a vertex chain that keeps us from reverting the colours. This means we must turn our attention to a new subgraph, G_2 , with vertex colour B and D . Since there is a vertex chain connecting the neighbours A and C to X , the neighbours B and D to X **must** be disconnected in G_2 in order for the graph G to be planar, otherwise the two chains would cross over one another. In the figure below, there **is** a chain connected to the neighbour D to X in G_2 , but this chain is disconnected from the neighbour B to X .



- 9) Since the neighbours B and D to X are disconnected in G_2 , we are free to swap the vertex colours B and D . This means all the colours in the chain going out from the neighbour D to X needs to be reverted as in the graph below, such that X can be coloured using colour D .



Now all the vertices are coloured, and we have proved that 5 colours are sufficient to colour a graph with any number of vertices, thus confirming the 5-colour theorem.

4-colour theorem: Classroom potential

The remaining question now has to be why the four-colour theorem is even necessary? The answer is that it isn't, really, other than making maps look more aesthetically pleasing and cohesive, maybe you save on ink, or maybe the world just hasn't found a proper use for it yet, but anyhow it makes for a very interesting theoretical result. Oftentimes, mathematical problems are hard to visualize and require huge amounts of prior knowledge of the terms and concepts, but the four-colour theorem is relatively easy to introduce to younger students and can gradually be adjusted according to the age and knowledge. For the youngest students, the teacher can hand out printed-out maps of the world or maps of the regions in their own country, and ask the pupils to colour in the maps so that no two regions that are touching are coloured the same colour, and trying to use as few colours as necessary. As a teacher you can either hand out only 4 colours, or you can ask the pupils at the end how many colours they used. A good idea could also be to discuss their colouring strategies, whether they coloured as many non-bordering colours with colour A first, then colour B, etc. or if they had a completely different strategy. A different exercise that is appropriate for all levels is asking the students to draw a map of their own **made-up** country with regions or their own continent with new countries. This exercise has many variations, maybe you can collect them and hand each student an arbitrary map made by a fellow pupil and ask them to colour it with as few colours as possible, maybe you can scan them to a computer and show the class a few examples up on the board, or the pupils can colour their own map. For older pupils, the teacher can start by showing an ordinary map and maybe handing them out to colour, and

then reveal that the maps can be represented by graphs while still maintaining the borders. Depending on the students, I imagine an 8th grade class would have the capacity to comprehend some of the more complicated concepts regarding the four-colour theorem and graph theory in general.

I had the opportunity to test these exercises out on pupils ranging from 5th to 9th grade due to my employment as a substitute teacher at a local elementary school. I decided that the best way to introduce the theorem to the pupils would be if they were to, in part, discover it themselves, and then I could reveal that there is a theorem called the 4-colour theorem. As preparation, I printed out some maps of Europe and various coloured pencils. I was able to test the lesson plan three different times, with three different age groups; 5th grade, 8th grade and 9th grade. Although I slightly varied the lesson plans according to the age group, the younger students were massively more intrigued and excited by the reveal of the theorem, showing tremendous excitement in trying to prove me wrong. It seemed that many students in 8th and 9th grade felt it was silly and juvenile to spend a whole lesson colouring maps. It was harder to engage these students from the start, and although some of them were intrigued by the four-colour theorem, it didn't have the "wow"-effect that it had on the younger students. This was probably in part due to the fact that I came in as an unknown substitute teacher, therefore not having established their trust in my teaching abilities. Seeing as the main critique from the students was that they felt it to be too juvenile, I decided to try to make a new lesson plan that would cater better to their abilities. I unfortunately wasn't able to test it out due to time constraints, but I believe it would meet better reception. The modifications I would make would be to ask them to colour a map using their computer. In a previous section, I used the website mapchart.net, which makes colouring the maps a smooth transaction, and it saves time so we can move on to the more advanced concepts. I believe that the act of using actual crayons to colour the maps made the process seem much more juvenile than it is. Going further I would show the students how we can represent the map we chose as a planar graph while maintaining the integrity of the map. I would then draw a graph showing different bus stops as the vertices, and bus routes as the edges, and we could do some problem solving where a person needs to get from A to B and they want to take the shortest route, or they need to make a stop at another location C before going to B. Then we could find out if the graph has any cycles, and if so, which is the shortest, how many choices of routes we have at any given bus stop, what is the longest route you could possible take to get from one bus stop to another, etc. Essentially, what we are then determining as a class is the width, girth and diameter of this graph. Though there are travel apps that can already do this for

them, it can prove useful nonetheless, and given its relevance, it is probably a much more appropriate introduction to graph theory for this age group.

In 5th grade, I first handed out the maps of Europe and asked the students to colour it in, following some rules. The rules were to 1) use as few colours as possible, 2) two neighbouring countries cannot have the same colouring, although they can if they only touch at the tip, and 3) don't worry so much about the islands. The students began carefully choosing what colours they wanted and started the assignment. Many began asking why they were doing this in math class, to which I replied that it is math! Some did a more systematic, particular job of staying inside the lines and making it look neat, while others just drew a coloured x or a line over each country, resulting in wildly varying finishing times. When most of them were finishing up, I asked how many colours they had to use. Though some claimed to have used 3 (which I later disproved) and others used 5, most of the students agreed that 4 were the fewest number of colours you could use. I then handed out a blank sheet of paper and asked them to try to draw a map of their own continent where you **had** to use 5 colours to draw it in, and then to switch maps with the person next to them so they could colour it in, again, using as few colours as possible. For many of the students, the realization hit quite quickly that it actually was possible to use only four colours, again. At that point, I told them that it's true that you only need four colours to colour in a map, any map, and that we call it the four-colour theorem. I had planned to go a little bit more into the theorem, but there were only 2 minutes left of the lesson and we had hit a natural stopping point. This was by far the most successful rendition of the lesson plan based on the theorem, and judging by the engagement level of the students, it seemed they too were happy with the lesson, hopefully gaining a more positive outlook on what math can be like.

Conclusion

Graph theory deserves a place in Norwegian classrooms. The visualisation of mathematical concepts is often lost and downgraded in favour of the more traditional means of teaching maths. Many swear by the traditional, ritualistic lesson structure of presenting a mathematical concept on the blackboard, and then having the students use this new knowledge to solve math problems for the remainder of the lesson. For many students, this is effective. However, for other students, who lack drive, inspiration and motivation to focus, it pushes their existing perspective that math is a boring subject, because they can't make sense of it. By making an effort to visualize the concepts for students, and making it a priority, one could really be

saving time in the long run. The goal of an educator shouldn't be to just 'get through' the curriculum in time, it should be to teach independence, problem solving skills, and the art of visualising the problem, the path and the solution. Graph theory is an excellent way to engage students, often without them fully realising they are solving a mathematical problem at all. The lesson plans in this thesis that I had the opportunity to try out were well-received, the students enjoyed a break from the traditional setup of a math lesson. The concepts were easy to demonstrate visually and were therefore easy to understand. There are also many, many other uses for graph theory in the classroom that aren't included in this thesis. The potential for this field of mathematics in schools is boundless, hopefully this realisation will be shared by other educators that will make the choice to deepen their students' understanding of mathematical concepts.

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