The card game SET and its underlying mathematics

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Sections

1 Abstract

SET is a highly-awarded card game where each card has a number of objects with different shadings, colors and shapes. The goal is to find SETs of three cards following a certain matching rule. Despite the relatively simple rules of the game there are some advanced mathematical concepts hidden beneath the surface, and this thesis will explore some of these areas.
Using combinatorics we count different types of SETs. We introduce coordinates to each card and prove some surprising facts about the outcomes of each game, such as finding a card that was hidden at the start of the game. Finally we take a look at the connection between SET and finite geometry which resembles Euclidean geometry, except that there are only a finite number of lines and a finite number of points on each line.

2 Preface

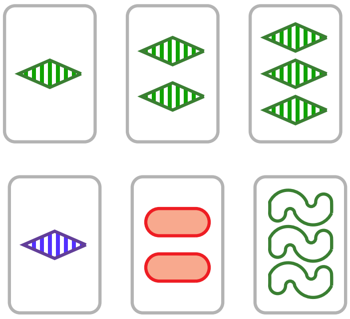
I would like to thank associate professor Tyson Ritter for guiding me through this project as supervisor. Although we could only met in person once, he has given me the opportunity to explore this exciting topic in a structured fashion and our regular meetings have been essential to the progression of this project. His swift and precise responses have made his dedication to this project obvious.

I would also like to thank my family and friends for keeping me company through what would otherwise be five lonely months of writing. Everyone I have spent time with both online and outside have helped me take some much needed time to relax.
A special thanks goes to my mother who listened to my semicoherent rants when I discovered something new or interesting relating to this thesis appeared in our games of SET. Every time you helped me collect my thoughts and showing genuine interest in further explanations to questions which otherwise might have gone unanswered. Finally a big than you to my brother for his help with references within the text, as well as some troubleshooting which saved me some headaches at the end of the project.

3 Introduction

The card game SET is a game of pattern recognition and has strong connections to different branches of mathematics on more than a visual level. The cards have a number of one, two or three objects, with different shadings, colors and shapes.
The game is played by placing 12 cards on the table, all the players look for SETs. If you think you spotted a SET you call out "SET!" and grab the three cards (it is always your turn, so be quick!). If you are correct you grab the SET and three new cards are placed on the table, this keeps going until there are no cards left, or there are no SETs left among the cards on the table. The winner is whoever collected the most SETs.

These are two of the many SETs you can find during a game.



The first parts of this thesis will cover some basic knowledge of the game and some simple (and some not so simple) counting. The later parts will look at some of the mathematics behind the game, including modular arithmetic and a branch called finite geometry.

SET was designed in 1974 by Marsha Falco and has been recognized as a great educational game and received numerous awards since its release through Set Enterprises in 1991, including MENSA select award in 1991 and teachers' choice learning awards in 2001 among the 37 awards listed on the company's website.

4 Definition of SET

SET is composed by cards of 4 properties, the number of objects, shade, color and shape. Each of these four properties can take 3 different values.

Notation	Number	Shade	Color	Shape
1	1	Empty	Red	Diamond
2	2	Striped	Green	Oval
0	3	Filled	Purple	Squiggle

Table 1: Definitions of properties and values

The naming used to describe the cards is chosen such that it is easy to learn the game. However another notation is useful to us as we explore the mathematics behind the game, $\{1, 2, 0\}$.

The deck consists of one card of every possible combination of these 4 properties with 3 possible values. All possible combinations of these four properties can be calculated by multiplying the number of possibilities, (3), for each property. i.e. $3 * 3 * 3 * 3 = 81$

If we want to look at a different version of the game where there is a different number of properties the number of cards would be 3^p where p is the number of properties. A new property might be the material of the cards, e.g. cards made of plastic, wood and metal. With five properties the number of cards in a deck would be $3^5 = 243$

We can describe a card $A = (a_1, a_2, a_3, a_4)$ where each a_i has a value of 1, 2 or 0. A SET is achieved with 3 cards, $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2, b_3, b_4)$ and $C = (c_1, c_2, c_3, c_4)$, each card having identical or unique values for each of the four properties. e.g.

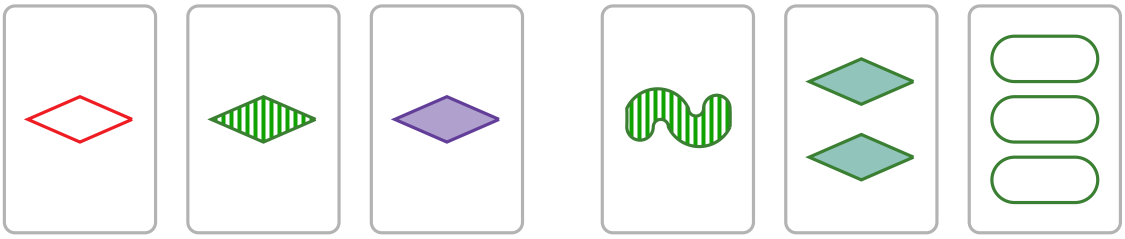
- **Number:** $a_1 = 1$, $b_1 = 1$, $c_1 = 1$
- **Shade:** $a_2 =$ empty, $b_2 =$ striped, $c_2 =$ filled
- **Color:** $a_3 =$ red, $b_3 =$ green, $c_3 =$ purple
- **Shape:** $a_4 =$ diamond, $b_4 =$ diamond, $c_4 =$ diamond

Definition 4.1. *Three cards form a SET if each variable, i, take either all different values,* $a_i \neq b_i$*,* $a_i \neq c_i$*,* $b_i \neq c_i$*, or all identical values,* $a_i = b_i = c_i$*.*

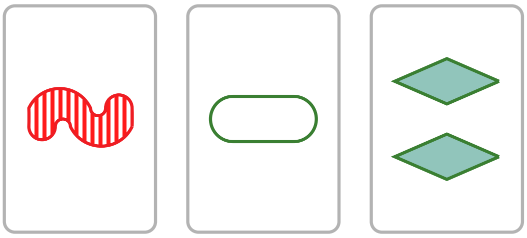
Table 2 gives an example of two complete SETs.

$$\begin{array}{ll} A = (1,1,1,1) & D = (1,2,2,0) \\ B = (1,2,2,1) & E = (2,0,2,1) \\ C = (1,0,0,1) & F = (0,1,2,2) \end{array}$$

Table 2: Complete SETs



A **B** **C** **D** **E** **F**



$(1,2,1,0)$ $(1,1,2,2)$ $(2,0,2,1)$

These three cards do not form a SET. Two of the cards have the same number of objects, but the third does not. Two of the cards are green, but the last one is red. Either of these cases by themselves would also ensure that the cards do not form a SET.

5 Fundamental theorem of SET

When you have played a couple of rounds of SET, some questions naturally come to mind. For example, once we choose two cards, is it always possible to complete a SET containing these two cards? And if it is possible, how many different ways are there? These questions are answered in the following theorem which is of such great importance throughout this thesis that we call it the fundamental theorem.

Theorem 1. *We only need two cards to determine the final card which completes the SET.*

Proof. Given two cards, $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ then a final unique card, $C = (c_1, c_2, c_3, c_4)$ is given by the condition that if the three cards are to form a SET we need each variable, i, to take either unique, $a_i \neq b_i$, $a_i \neq c_i$, $b_i \neq c_i$, or identical values, $a_i = b_i = c_i$ where a_i, b_i, c_i can take the values $\{1, 2, 0\}$. First of all, C must exist because the deck is made up of all possible combinations of property values. C can not be identical to A or B because each card only appears exactly once in a deck. If b_i is identical to a_i, this means that $a_i = b_i = c_i$ is the only possible combination that makes a SET and we now know c_i, if a_i is different from b_i then all values must be different and c_i has to take the final value among $\{1, 2, 0\}$. $\square$

This theorem holds for other versions of SET, by the same proof, where there are p properties each with three distinct values.

6 Counting

Many things are possible to count in SET. Counting the number of cards in the deck like we did in section 4 was an easy one.

In this section we will look at some things that are not as easy to count. You will come to find that the total number of SETs is not very difficult to find, but the number of SETs in a small number of cards can be more of a challenge.

6.1 Number of SETs

When talking about the number of SETs in the game we need to differentiate between ordered and unordered SETs. If a SET is made by the cards A, B and C we have six ordered SETs. (drawn in order: 1st, 2nd, 3rd): $(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B)$ and (C, B, A). These depend on the order you choose the cards. However if unordered, these six SETs are all the same. In calculations we can make sure we find the number of unordered SETs by dividing the number of ordered SETs by 3! which is the number of permutations we can arrange the three cards.

To calculate the total number of unordered SETs we have to choose the first card, A among the 81 cards in the deck. This first card can combine with any of the remaining 80 cards as card B to form a SET. After 2 cards are chosen the 3rd card, C, is uniquely determined by theorem 1. This gives us the calculation $81 * 80 * 1$, we get a total of 6480 ordered SETs. However, this counts each combination of 3 cards multiple times, in order to correct this we have to divide by the number of permutations in three cards, which is 3!
This gives us the total number of unordered SETs:

$$\frac{81 * 80 * 1}{3!} = 1080 \tag{1}$$

Theorem 2. *The total number of unordered SETs in a deck with p properties is given by*

$$\frac{3^p * (3^p - 1) * 1}{3!} = \frac{3^{2p} - 3^p}{3!} \tag{2}$$

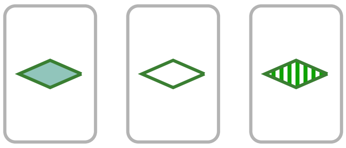
Proof. The total number of SETs is defined by choosing a card among all possible cards 3^p, and removing it, choosing one of the remaining $3^p - 1$ cards we now have two cards which, by the fundamental theorem, defines a unique SET. We complete the SET by choosing the unique card defined by the previous chosen cards (Theorem 1). This gives us the ordered number of SETs. However, we want to know the number of unordered SETs so we divide by 3!, which gives us the final result defined in Theorem 2. □

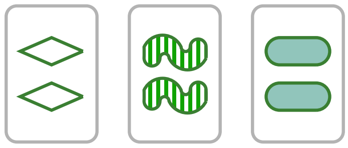
6.2 Types of SETs

We will now look at different types of SETs. A type is defined by how many of the properties are unique or identical. There are four different types in the regular version of the game which which will be presented in table 3.

Type	Identical properties	Different properties
x	x	$p - x$
0	0	4
1	1	3
2	2	2
3	3	1

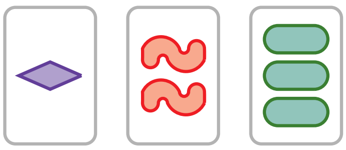
Table 3: Different types of SETs

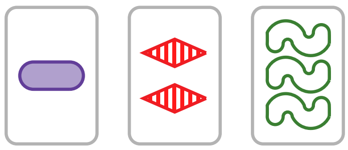
3 

2 

3 is a SET of type 3 where the number of objects, color and shape are identical.

2 is a SET of type 2 where the number of objects and color identical.

1 

0 

1 is a SET of type 1 where only the shading is identical.

0 is a SET of type 0 where none of the properties are identical.

You might wonder why there is no type 4. This is because of the arbitrary rules of the game where it was decided that only one copy of each card should be present in the deck. If you were to buy two additional decks and play with all three at the same time you could find a SET of type 4 where all cards are identical.

Now that we know how many SETs there are and what types of SETs exists it raises a new question. How many of the 1080 unordered SETs belong to each different type of SETs?

Theorem 3. *In a deck with p properties the number of unordered SETs of type x is given by:*

$$\frac{3^p * N * 1}{3!}, \qquad N = \binom{p}{x} * 2^{p-x} \tag{3}$$

x	N	T
0	$\binom{4}{0} * 2^4 = 16$	$\frac{81*16*1}{3!} = 216$
1	$\binom{4}{1} * 2^3 = 32$	$\frac{81*32*1}{3!} = 432$
2	$\binom{4}{2} * 2^2 = 24$	$\frac{81*24*1}{3!} = 324$
3	$\binom{4}{3} * 2^1 = 8$	$\frac{81*8*1}{3!} = 108$
Total	80	1080

Table 4: Number of unordered SETs in each type in the regular game.
The sums makes sense as 80 *is the number of possible second card choices when you do not care about which type you find and* 1080 *is the total number of SETs*

Proof. For a game with p properties we start by choosing card A among all the cards in the deck, of which there are 3^p. Then we have to choose card B among N cards such that we get a SET of type x. Type x has the condition that x among p properties are identical to the first card. This is done by $\binom{p}{x}$ to find how many of the properties can take the same values as the first card. After this we have to take a look at the properties that should be different on card B. There are 2 possible values for each property that are different from card A and we need $p - x$ of the properties to take one of these two values. The total number of possible combinations is therefore 2^{p-x}. Card C is uniquely defined by the fundamental theorem, given as theorem 1, and we want the number of unordered SETs so we divide by 3!. □

6.3 Maximum number of SETs among n cards

Continuing on we look at how many SETs can be found among n cards. We will ignore the obvious cases where there are no SETs, $n < 3$, as you need at least 3 cards to have a SET, and the case with exactly one SET, $n = 3$. Thus, we only care about cases where $n > 3$. When we look at the number of possible SETs among these n cards we can use a general formula.

$$\frac{n * (n-1) * 1}{3!} \tag{4}$$

This general formula, equation (4), is quite similar to equation (1), which gave us the total number of SETs when $n = 81$. Equation (4) gives us an upper limit to how many SETs are possible within any number of cards, however we have to look at the results a bit more closely as there might be more restrictions present that makes it impossible to obtain this upper limit that we find with the general formula.

The general formula, equation (4), can be found by first picking one of all available cards. Equation (1) had 81 available cards and now we only have a small $n < 81$. Then we pick one of the remaining cards, $n - 1$, and finally by the fundamental theorem, Theorem 1, the final card that completes the SET is now determined and there is only one "choice". As we choose to ignore the order we choose the cards, we need to divide by 3!.

Take the example of 4 cards, by equation (4) we have an upper bound of SETs for four cards given by $\frac{4*3*1}{3!} = 2$. To see if we can actually achieve two SETs we begin by labeling these four cards, A, B, C and D. We can assume that the cards A, B and C form a SET. We now suppose that there is a second SET within the four cards, which must therefore necessarily involve D. However the only available cards, A, B and C, already make up a SET and by the fundamental theorem, theorem 1, if you take any two of these cards there is only one unique card that completes the SET. Therefore we cannot find a SET which includes D.

Next we take 5 cards. The general formula, equation (4), tells us $\frac{5*4*1}{3!} = 3.3\overline{3}$ is the upper bound on five cards. This tell us that the upper bound is not 4 or more thus intuitively the upper bound is 3 SETs. To find the actual answer we have to look into the restrictions that comes after we find one SET among the five cards. We label five cards A, B, C, D and E where A, B and C form a SET. As we learned in the case of four cards, we need at least two cards that are not included in the SET $\{A, B, C\}$ to have a possible second SET. We assume that there is another SET which involves D and E, these two cards can by theorem 1 define a SET where by relabeling the cards in the first SET, C also completes the second SET, $\{C, D, E\}$, we have now found two SETs. Now suppose that there is a third SET among these five cards which include A and E. The final card in this third SET can either be B, C or D. However we know that if A is combined with B or C then by the fundamental theorem, theorem 1, the SET $\{A, B, C\}$ will be the only possibility. If we consider D as the third card then C is the only card which completes the SET with D and E. Thus there can only be two SETs among five cards.

7 Modular arithmetic

In section 4 we described a card as $A = (a_1, a_2, a_3, a_4)$. The strength of this description is that we can look at a card as a point in 4-dimensional space. In the fundamental theorem, theorem 1, we assigned $\{1, 2, 0\}$ as valid values for each property. We also said that if three cards form a SET then each property must take a given combination of values. Table 5 shows all possible combinations and which of these combinations make up a SET. All combinations that form a SET have a sum equivalent to 0 (mod 3).

SET Combinations	Sum	Non-SET Combinations	Sum
{0,0,0}	0	{0,0,1}	1
{1,1,1}	$3 \equiv 0$	{0,0,2}	2
{2,2,2}	$6 \equiv 0$	{0,1,1}	2
{0,1,2}	$3 \equiv 0$	{0,2,2}	$4 \equiv 1$
		{1,1,2}	$4 \equiv 1$
		{1,2,2}	$5 \equiv 2$

Table 5

Theorem 4. *Three cards $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2, b_3, b_4)$ and $C = (c_1, c_2, c_3, c_4)$ form a SET if and only if $A + B + C \equiv (0, 0, 0, 0)$ (mod 3)*

Proof. For the three cards to form a SET, each property, i, must be one of the four SET combinations from Table 5. These combinations all have a sum of 0 (mod 3).

For any non-SET combination of values the sum is different from 0 (mod 3). All combinations where the sum of the vectors is $(0, 0, 0, 0)$ must consequently be a SET. □

7.1 Game endings

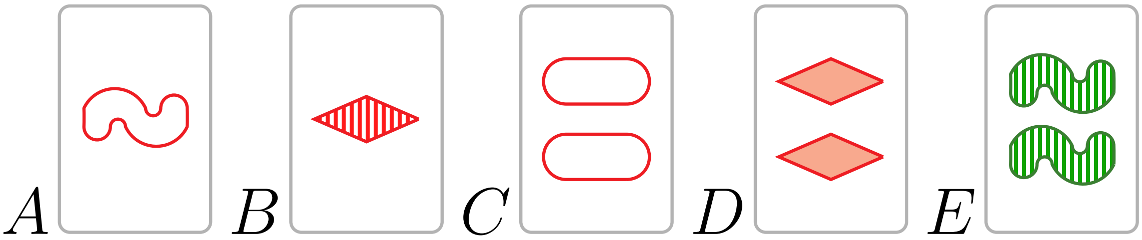
In Theorem 4 we learned that the sum of each SET is $(0, 0, 0, 0)$, as a consequence the sum of all the cards in the game is also $(0, 0, 0, 0)$. We can know this because it is possible to arrange all the cards in the game into SETs at the same time. This can easily be done by making only SETs of type 3 (table 4). e.g. Take all the red cards with diamonds with empty shading we now have three cards with one, two and three figures. Repeat this process while keeping only the number of objects different on the three cards and we will complete SETs with the entire deck.

This makes it impossible to end a game with only three cards left that do not form a SET. As all cards add up to $(0, 0, 0, 0)$ and we only remove SETs during the game, each SET have a sum of $(0, 0, 0, 0)$. Therefore all remaining cards at the end of the game can not have a different sum than the entire deck. Thus if there are only three cards left they must sum to $(0, 0, 0, 0)$ which is the definition of a SET and we can remove them from the table.

7.2 Hidden card

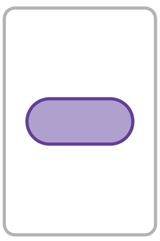
A fun trick you can do after reading about "endgames" is hide away a single card at the start of the game and not have anyone look at it before you play. The game is played as usual, but with $3n - 1$ cards left at the end. If there are only two cards left you can call out SET! and grab the remaining cards to get the final SET, this is because, assuming no mistakes have been made during the game, the final three cards have a sum of $(0, 0, 0, 0)$ and thus form a SET. However it is still possible to find the hidden card with more than two cards left at the end.

As explained earlier all cards left at the end must have the sum $(0, 0, 0, 0)$.
To find the last card we can take a look at each property in isolation and see what is missing such that the sum of this property is $0 \pmod 3$. A simple example will demonstrate the methods you can use to find the hidden card.



First we look at the color, there are 4 red cards they have color value 1, and there is 1 green card which has color value 2. We then add these and get $1 + 1 + 1 + 1 + 2 = 6 \equiv 0 \pmod 3$ we are missing 0 to get a multiple of 3. The final card must then have color value 0 so that the sum is unchanged, we now know that the card must be purple.
The numbers on the cards gives us $1 + 1 + 2 + 2 + 2 = 8 \equiv 2 \pmod 3$. Here we are missing 1 to get a multiple of 3 so the number on the hidden card must be 1.

We could repeat this for the remaining properties, but there is also a way to find the hidden properties without translating to numerical notation. We split the cards into groups of three where each property in isolation should follow the identical or all different rule from section 1. First the shape, there are two squiggly cards, two diamond cards and one oval card. We make one group with cards A, B, C, the second group now starts with two different shapes in D and E, the final shape in this group must also be different from the other two, which is oval.
Finally the shading uses the same argument as the shape. We start with the group B, C, D, we now have A, E in the second group with different shading. The final card has a full shading.

The hidden card is this: 

7.3 Lines and planes

Suppose you have two cards, $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, and want to find the unique card which completes the SET. This can be done with the following procedure. You find the vector from A to B. $\vec{AB} = B - A$. Then we add half of this direction vector to the first card to find the last card in the SET.

$$C = A + \frac{1}{2}\vec{AB} = (a_1, a_2, a_3, a_4) + (\frac{b_1 - a_1}{2}, \frac{b_2 - a_2}{2}, \frac{b_3 - a_3}{2}, \frac{b_4 - a_4}{2}) \tag{5}$$

$$C = (\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}, \frac{a_3 + b_3}{2}, \frac{a_4 + b_4}{2}) \tag{6}$$

Since we are working in modular arithmetic we need to know what $\frac{1}{2}$ is in mod 3, the definition of $\frac{1}{2}$ is something you multiply by 2 to get 1. In mod 3 we have $1 \equiv 4$ and $2 * 2 = 4 \equiv 1$ i.e. $\frac{1}{2} = 2 \bmod 3$.

This simple example uses all possible SET combinations (table 5) and serves as proof that this procedure works for all possible arrangements of SET combinations.

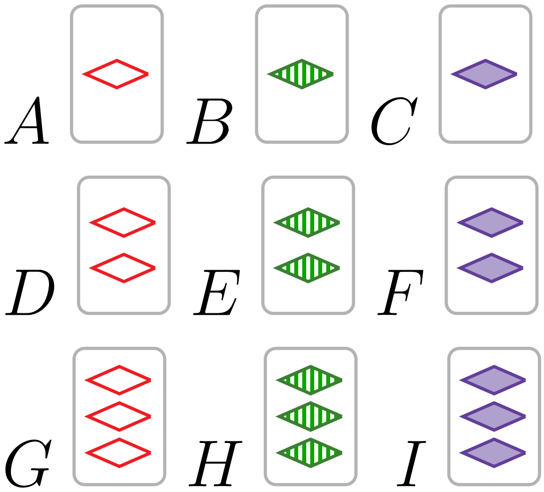
$$A = (0, 1, 2, 0), \quad B = (0, 1, 2, 1)$$

$$C = (\frac{0+0}{2}, \frac{1+1}{2}, \frac{2+2}{2}, \frac{0+1}{2}) = (0, 1, 2, \frac{1}{2}) \equiv (0, 1, 2, 2) \tag{7}$$

$$A + B + C = (0, 3, 6, 3) \equiv (0, 0, 0, 0) \tag{8}$$

This method might remind you of the method from Euclidean geometry for finding the midpoint on a line segment between the two points A and B. In this analogy we can imagine the SET being the line, and the cards would be points on the line. However unlike in Euclidean geometry there are not infinitely many points on the line, but only three. In the next section we will see that this is not just a superficial resemblance and we will discuss the deep connection between SET and something called finite geometry.

Continuing the analogy where we imagined SETs as lines, in a similar way we can try to imagine what a plane could look like in SET. A plane in SET is a combination of 9 cards which can be arranged to form 12 different SETs. Below is an example of a plane, P.



In each plane there are 12 SETs, some are easier to spot than others. There are the six more obvious SETs that are on horizontal or vertical lines. $\{A, B, C\}$, $\{D, E, F\}$, $\{G, H, I\}$, $\{A, D, G\}$, $\{B, E, H\}$ and $\{C, F, I\}$.
The six less obvious SETs are diagonal lines. $\{A, E, I\}$, $\{B, F, G\}$, $\{C, D, H\}$, $\{C, E, G\}$, $\{B, D, I\}$ and $\{A, F, H\}$

Like Euclidean geometry where three points not on the same line define a plane, in SET a plane is defined by three cards that do not form a SET.
Consider that we start with three cards that do not form a SET, it is then possible to build the plane by completing the SETs that are defined by the fundamental theorem (theorem 1). Below is one of the possible orders we can complete SETs to find the plane.

We know how many SETs there are, we know the number of cards in a plane and how many SETs are in a plane. What we do not know yet is how many planes are there in a deck of 81 cards. Earlier we learned that we need three cards that do not form a SET in order to define a plane. However each plane contains nine cards, this means that there are a lot of ways to define the same plane. A plane is defined by the cards it contains, not the position each card takes in the plane. If you compare the plane P (section 7.3) to P' (section 8.2) you will find the same cards, however P' (8.2) has the middle row of P (7.3) rotated to the right, and the bottom row rotated to the left. We define this to be the same plane.

In order to find the number of planes we need to know how many ways can we choose three cards that do not form a SET. As in section 6.1 we start by choosing two cards, where we have 81 and 80 choices for the first and second card respectively. However this time we want to choose a card that does not form a SET with these. By theorem 1 there is only one that completes the SET, thus we have 78 choices for the third card.

$$81 * 80 * 78 = 505440 \tag{9}$$

However, this counts each plane multiple times (as with the number of SETs). So exactly how many ways can we choose the same plane? In much the same way we choose first among all nine cards in the plane and secondly among the eight remaining there is only one card which completes this SET and we choose the third card among the other six that does not complete the SET. This means there are $9 * 8 * 6 = 432$ different ways to find each plane.

This gives us the total number of planes [1]

$$\frac{81 * 80 * 78}{9 * 8 * 6} = 1170 \tag{10}$$

8 Finite affine geometry

Finite affine geometry might be a bit of a head-scratcher when you are used to thinking in terms of Euclidean geometry, as the latter was built by Euclid in order to better understand the world we live in. There are four axioms for the Finite affine plane. [1]

- **Axiom 1.** There are at least 3 points that are not on the same line.
- **Axiom 2.** Every line contains at least 2 points.
- **Axiom 3.** Two points determine a unique line.
- **Axiom 4.** For any line, l, and point, Q, which is not on l there is exactly one line that contains Q and none of the points contained in l, this line through Q is said to be parallel to l.

The most simple case you can build with these axioms is what we call AG(2,2). This is the short form of affine geometry in 2 dimensions with 2 points on each line.

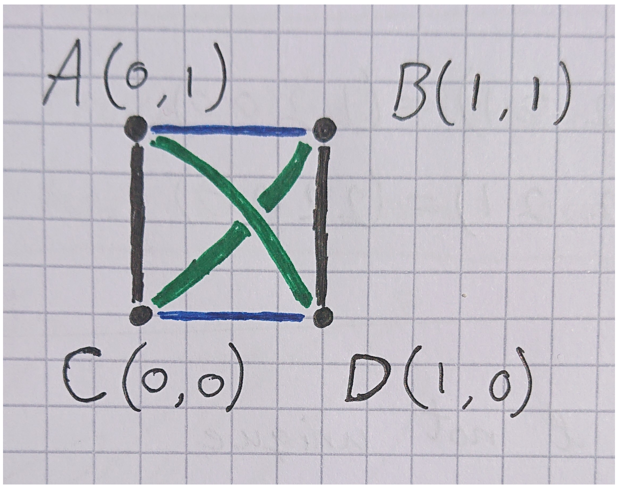


Figure 1: AG(2,2) plane

Note that even though it looks like the green lines intersect in the center of the figure, they actually do not. This is because lines only intersect in points on the line and the only points where lines can intersect are A, B, C or D.

The first three axioms are easy to see checks out, however the parallel line axiom is not intuitive for the lines AD and BC. What we can do to understand that these lines are in fact parallel here is assign coordinates to the points in mod 2. A way to see if lines are parallel is to compare the direction vectors to see if they are the same. to find the direction we just find the vector from A to D and from B to C. If the vectors are identical the lines are parallel.

$$\vec{AD} = (1,0) - (0,1) = (1,-1) \equiv (1,1) \quad \vec{BC} = (1,1) - (0,0) = (1,1)$$

8.1 Finite affine geometry in SET

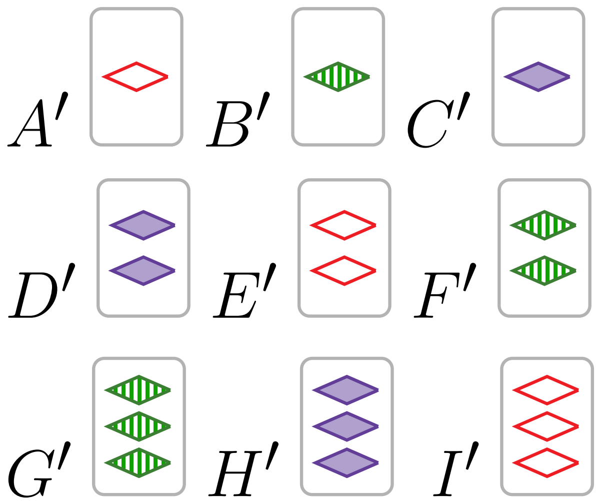
We will now revisit some of the themes discussed in the earlier sections from a geometric point of view.

Firstly we translate the axioms to SET terminology to see how SET is defined in finite geometry. [1]

- **Axiom 1.** There are at least 3 cards that are not in the same SET.
- **Axiom 2.** Every SET contains at least 2 cards.
- **Axiom 3.** Two cards determine a unique SET.
  - *This is what we called the fundamental theorem of SET in section 5*
- **Axiom 4.** For any SET, l, and any card, Q, which is not in l there is exactly one SET that contains Q and none of the cards contained in l, this SET containing Q is said to be parallel to l.
  - *What a parallel SET is and how it can be useful to us will be discussed further when we revisit lines and planes from section 7.3*

8.2 Parallel SETs and planes

Now we know what the axioms are in terms of the game, the next step is to figure out what parallel SETs are. To get an idea we again take a look at a plane, P'.



Parallel SETs have the same "direction" such as $\{A', B', C'\}$ and $\{D', E', F'\}$ or $\{A', D', G'\}$ and $\{C', F', I'\}$.

This raises another question, how do we determine the direction of a SET? The easiest way is to draw lines that we would call parallel in Euclidean geometry to see what we find. In section 7.3 we discussed the 12 SETs in a plane, and now I want to introduce the drawing of the plane AG(2,3) which will illustrate better where all of these SETs can be found.

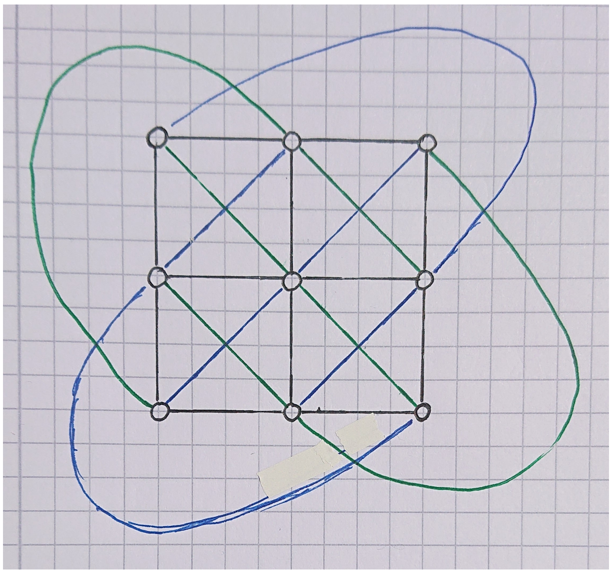
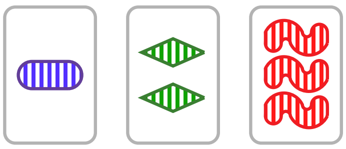


Figure 2: AG(2,3) Plane (2 dimensions, 3 points on each line)

There are four different directions in this picture illustrated by the lines between the points, the horizontal lines are parallel and the three vertical lines are parallel. It is less obvious that this is true for both the blue lines and the green lines as well. However if you take another look at the plane, P', earlier in section 8.2 draw the green lines from figure 2 in that plane you will find three SETs that are quite intuitively parallel.

One way to see if SETs are parallel is to take a look at the coordinates of the cards in the SET and find a vector between two of the cards. We used this in section 7.3 to find the final card in a SET. Another use of this vector is comparing it to other direction vectors. In Euclidean geometry two lines are parallel if the direction vectors are the same, and this is true for SET as well.

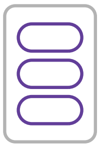
Another way to find a parallel SET and the direction is to look at the way each property "cycles". You get a cycle when you place anything in a circle and label them and then walk around the circle in a given direction and look at the order you find each label. e.g. $1 \to 2 \to 0 \to 1 \to 2 \to 0...$ we write this as the cycle $[1, 2, 0]$.

 Here we have a SET, we want to find a SET that is parallel.

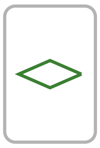
Firstly we look at the shape, in the first SET we have [oval, diamond, squiggles] or $[2, 1, 0]$. A parallel SET has to have this cycle for its shape values, either identical or an equivalent cycle ($[0, 2, 1]$ or $[1, 0, 2]$). These are equivalent because in order to get to the next value you subtract 1 from the value you start with, this is also the value of the shape coordinate in the direction vector.

The other properties have the following cycles: Number $[1, 2, 0]$ $(+1)$ (recall that we operate in mod 3, $2 + 1 = 3 \equiv 0$), shading $[2, 2, 2]$ $(+0)$, Color $[0, 2, 1]$ (-1). This gives us a direction vector of $(1, 0, -1, -1)$.

To find a parallel SET we need exactly one card which is not in our original SET (axiom 4). Suppose we start with this card:



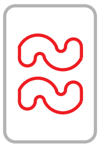
We take a look at our cycles in the original SET and see that after three objects comes one object, the shading is unchanged, after purple comes green and after oval comes diamond. This gives us the next card in the SET which has coordinates $(1, 1, 2, 1)$.

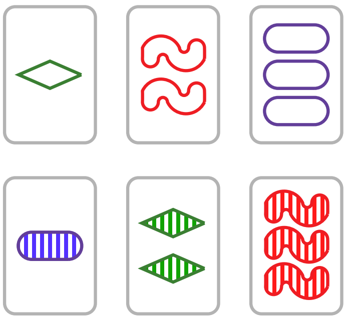


The final card can be found by adding the direction vector from the original SET to the second card.

$$(1, 1, 2, 1) + (1, 0, -1, -1) = (2, 1, 1, 0)$$

This gives us the final card and you can confirm that they indeed form a SET.





Now we have defined what parallel SETs are and we know how to construct a SET parallel to a given SET, containing a given card. How can we know for certain that these SETs do not intersect? For two SETs to intersect they have to share a card. As stated previously the two SETs have the same direction vector and by adding multiples of the direction vector to a card in the SET we generate the cards in the same SET. Thus, if one card is the same we would generate identical SETs, contradicting the fact that at least one card is different. With this we fulfilled all the axioms. A plane is isomorphic to AG(2,3) and the entire deck is isomorphic to AG(4,3) [1].

References

[1] Liz. McMahon et al. *The Joy of SET*. Princeton University Press, 2017, p. 307. ISBN: 9780691166148.