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RESEARCH CONTRIBUTION

Interpolation of Weighted Extremal Functions

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Abstract

An approach to interpolation of compact subsets of $\mathbb{C}^n$, including Brunn–Minkowski type inequalities for the capacities of the interpolating sets, was developed in [8] by means of plurisubharmonic geodesics between relative extremal functions of the given sets. Here we show that a much better control can be achieved by means of the geodesics between *weighted* relative extremal functions. In particular, we establish convexity properties of the capacities that are stronger than those given by the Brunn–Minkowski inequalities.

Keywords Plurisubharmonic geodesic · Coconvex sets · Brunn–Minkowski inequality · Relative extremal function

Mathematics Subject Classification 32U15 · 32U20 · 52A20 · 52A40

1 Introduction

Classical complex interpolation of Banach spaces, due to Calderón [5] (see [3] and, for more recent developments, [7]) is based on constructing holomorphic hulls generated by certain families of holomorphic mappings. A slightly different approach proposed in [8] rests on plurisubharmonic geodesics. The notion has been originally considered, starting from 1987, for metrics on compact Kähler manifolds (see [10] and the bibliography therein), while its local counterpart for plurisubharmonic functions on bounded hyperconvex domains of $\mathbb{C}^n$ was introduced more recently in [4] and [18], see also [1].

In the simplest case, the geodesics we need can be described as follows. Denote by $A = \{\zeta \in \mathbb{C} : \ 0 < \log|\zeta| < 1\}$ the annulus bounded by the circles $A_j = \{\zeta : \log|\zeta| = j\}$, $j = 0, 1$. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Given two plurisubharmonic functions u_0, u_1 in Ω, equal to zero on $\partial\Omega$, we consider the class

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W of all plurisubharmonic functions $u(z,\zeta)$ in $\Omega \times A$, such that

$$\limsup_{\zeta \to A_j} u(z,\zeta) \le u_j(z) \quad \forall z \in \Omega.$$

Its Perron envelope $\mathcal{P}_W(z,\zeta) = \sup\{u(z,\zeta) : u \in W\}$ belongs to the class and satisfies $\mathcal{P}_W(z,\zeta) = \mathcal{P}_W(z,|\zeta|)$, which gives rise to the functions

$$u_t(z) := \mathcal{P}_W(z, e^t), \quad 0 < t < 1,$$

the *geodesic* between u_0 and u_1. When the functions u_j are bounded, the geodesic u_t tends to u_j as $t \to j$, uniformly on Ω. One of the main properties of the geodesics is that they linearize the pluripotential energy functional

$$\mathcal{E}(u) = \int_\Omega u(dd^c u)^n,$$

which means

$$\mathcal{E}(u_t) = (1-t)\,\mathcal{E}(u_0) + t\,\mathcal{E}(u_1); \tag{1}$$

see the details in [4] and [18].

In [18], this was adapted to the case when the endpoints u_j are *relative extremal functions* ω_{K_j} of non-pluripolar compact sets $K_0, K_1 \subset \Omega$; we recall that

$$\omega_K(z) = \omega_{K,\Omega}(z) = \limsup_{y \to z} \mathcal{P}_{\mathcal{N}_K}(y),$$

where $\mathcal{N}_K$ is the collection of all negative plurisubharmonic functions u in Ω with $u|_K \le -1$, see [14]. Note that

$$\mathcal{E}(\omega_K) = -\mathrm{Cap}\,(K),$$

where

$$\mathrm{Cap}\,(K) = \mathrm{Cap}\,(K,\Omega) = (dd^c\omega_K)^n(\Omega) = (dd^c\omega_K)^n(K)$$

is the *Monge–Ampère capacity* of K relative to Ω.

If each K_j is polynomially convex (i.e., coincides with its polynomial hull), then the functions $u_j = -1$ exactly on K_j are continuous on $\overline{\Omega}$, and the geodesics $u_t \in C(\overline{\Omega} \times [0,1])$. Let

$$K_t = \{z \in \Omega : u_t(z) = -1\}, \quad 0 < t < 1; \tag{2}$$

then (1) implies:

$$\mathrm{Cap}\,(K_t) \le (1-t)\,\mathrm{Cap}\,(K_0) + t\,\mathrm{Cap}\,(K_1). \tag{3}$$

As was shown in [19], the functions u_t in general are different from the relative extremal functions of K_t. Moreover, if the sets K_j are Reinhardt (toric), then $u_t = \omega_{K_t}$

for some $t \in (0, 1)$ only if $K_0 = K_1$, so an equality in (3) is never possible unless the geodesic degenerates to a point.

Furthermore, in the toric case, the capacities (with respect to the unit polydisk $\mathbb{D}^n$) were proved in [8] to be not just convex functions of t, as is depicted in (3), but logarithmically convex:

$$\mathrm{Cap}\,(K_t, \mathbb{D}^n) \le \mathrm{Cap}\,(K_0, \mathbb{D}^n)^{1-t}\,\mathrm{Cap}\,(K_1, \mathbb{D}^n)^t. \tag{4}$$

This was done by representing the capacities, due to [2], as (co)volumes of certain sets in $\mathbb{R}^n$ and applying convex geometry methods to an operation of *copolar addition* introduced in [19]. Furthermore, the sets K_t in the toric situation were shown to be the geometric means (multiplicative combinations) of K_j, exactly as in the Calderón complex interpolation theory. And again, an equality in (4) is possible only if $K_0 = K_1$. It is worth mentioning that the *volumes* of K_t satisfy the opposite Brunn–Minkowski inequality [6]:

$$\mathrm{Vol}(K_t) \ge \mathrm{Vol}(K_0)^{1-t}\mathrm{Vol}(K_1)^t.$$

In this note, we apply the geodesic technique to *weighted relative extremal functions*

$$u_j^c = c_j\,\omega_{K_j}, \quad c_j > 0,$$

the sets K_t being replaced with the sets K_t^c where the functions u_t^c attain their minimal values, $-c_t$. The function $t \mapsto c_t$ turns out to be convex; moreover, it is actually linear, $c_t = (1-t)\,c_0 + t\,c_1$, provided $K_0 \cap K_1 \neq \emptyset$. With such an interpolation, one can have $u_t^c = c_t\,\omega_{K_t^c,\Omega}$ for a non-degenerate geodesic, in which case there is no loss in the transition from the functional $\mathcal{E}(u_t^c)$ to the capacity $\mathrm{Cap}\,(K_t^c)$. And in any case, we establish the weighted inequality

$$c_t^{n+1}\mathrm{Cap}\,(K_t^c) \le (1-t)\,c_0^{n+1}\mathrm{Cap}\,(K_0) + t\,c_1^{n+1}\mathrm{Cap}\,(K_1),$$

which, for a smart choice of the constants c_j, is stronger than (3) and even (in the toric case) than (4). In particular, it implies that the function

$$t \mapsto \left(\mathrm{Cap}\,(K_t^c)\right)^{-\frac{1}{n+1}}$$

is concave.

In the toric setting of Reinhardt sets K_j in the unit polydisk, we show that the interpolating sets K_t^c actually are the geometric means, so they do not depend on the weights c_j and coincide with the sets K_t in the non-weighted interpolation; we do not know if the latter is true in the general, non-toric setting.

Finally, we transfer the above results on the capacities of sets in $\mathbb{C}^n$ to the realm of convex geometry, developing thus the Brunn–Minkowski theory for volumes of (co)convex sets in $\mathbb{R}^n$ [8,12,19].

2 General Setting

Here, we consider the general case of $u_j^c = c_j\,\omega_{K_j}$ with $c_j > 0$ and K_j non-pluripolar, compact, polynomially convex subsets of a bounded hyperconvex domain Ω of $\mathbb{C}^n$. In this situation, the functions $u_j^c(z) = -c_j$ precisely on K_j and are continuous on $\overline{\Omega}$, the geodesics u_t converge to u_j, uniformly on Ω, as $t \to j$, and belong to $C(\overline{\Omega} \times [0, 1])$, as in the non-weighted case dealt with in [18] and [8].

Denote:

$$c_t = -\min\{u_t^c(z) : z \in \Omega\}$$

and

$$K_t^c = \{z \in \Omega : u_t^c(z) = -c_t\}, \quad 0 < t < 1, \tag{5}$$

the set where u_t^c attains its minimal value on Ω.

Theorem 1 *In the above setting, we have:*

(i) $c_t \le (1-t)\,c_0 + t\,c_1$, *with an equality if* $K_0 \cap K_1 \neq \emptyset$;
(ii) *the function* $t \mapsto c_t^{n+1}\mathrm{Cap}\,(K_t)$ *is convex:*

$$c_t^{n+1}\mathrm{Cap}\,(K_t^c) \le (1-t)\,c_0^{n+1}\mathrm{Cap}\,(K_0) + t\,c_1^{n+1}\mathrm{Cap}\,(K_1); \tag{6}$$

(iii) *if the weights* c_j *are chosen such that*

$$c_0^{n+1}\mathrm{Cap}\,(K_0) = c_1^{n+1}\mathrm{Cap}\,(K_1), \tag{7}$$

then the function

$$V(t) := \left(\mathrm{Cap}\,(K_t^c)\right)^{-\frac{1}{n+1}}$$

is concave and, consequently, the function

$$\rho(t) := V(t)^{-1} = \left(\mathrm{Cap}\,(K_t^c)\right)^{\frac{1}{n+1}}$$

is convex.

Proof (i) Consider $v_j = c_j\,\omega_K$ for $j = 0, 1$, where $K = K_0 \cup K_1$. The set K might be not polynomially convex, but ω_K is nevertheless a bounded plurisubharmonic function on Ω with zero boundary values, so the geodesic v_t^c is well defined and converge to v_j, uniformly on Ω, as $t \to j$ [18, Prop. 3.1]. Since $v_j \le u_j^c$, we have $v_t^c \le u_t^c$. Assume $c_0 \ge c_1$, then the corresponding geodesic $v_t^c = \max\{c_0\,\omega_K, -((1-t)\,c_0 + t\,c_1)\}$, because the right-hand side is maximal in $\Omega \times A$ and has the prescribed boundary values at $t = 0$ and $t = 1$. Therefore:

$$-c_t \ge \min\{v_t^c(z) : z \in \Omega\} \ge -((1-t)\,c_0 + t\,c_1),$$

which proves the convexity of c_t.

To finish the proof of (i), let $z^* \in K_0 \cap K_1 \neq \emptyset$, then $-c_t \le u_t^c(z^*)$. Since the convexity of the function $u_t^c(z^*)$ in t implies $u_t^c(z^*) \le -((1-t)\, c_0 + t\, c_1)$, we get $c_t \ge (1-t)\, c_0 + t\, c_1$ and thus the linearity.

(ii) Since $u_j^c = c_j\, \omega_{K_j}$, we have:

$$\mathcal{E}(u_j) = c_j^{n+1} \int_\Omega (dd^c \omega_{K_j})^n = -c_j^{n+1} \mathrm{Cap}\,(K_j), \quad j = 1, 2.$$

For any fixed t, the function $u_t^c = -c_t$ on K_t^c, so $u_t^c \le -c_t\, \omega_{K_t^c}$. By [18, Cor. 2.2] this implies

$$\mathcal{E}(u_t^c) \le \mathcal{E}(c_t\, \omega_{K_t^c}) = -c_t^{n+1} \mathrm{Cap}\,(K_t^c),$$

and (6) follows from the geodesic linearization property (1).

(iii) It suffices to prove the concavity of the function V. When the weights c_j satisfy (7), inequality (6) rewrites as

$$V(t) \ge \frac{c_t}{c_0} V(0)$$

and, since

$$c_1 = \frac{V(1)}{V(0)} c_0,$$

this gives us

$$V(t) \ge (1-t)\, V(0) + t\, V(1),$$

which completes the proof.

□

The convexity/concavity results in this theorem are stronger than inequality (3) obtained in [18] by the geodesic interpolation u_t of non-weighted extremal functions. In addition, the non-weighted geodesic u_t is very unlikely to be the extremal function of the set K_t (as shown in [19], this is never the case in the toric situation, unless $K_0 = K_1$), while this is perfectly possible in the weighted interpolation. For example, given $K_0 \Subset \Omega$, let

$$K_1 = \left\{ z \in \Omega : \omega_{K_0}(z) \le -\frac{1}{2} \right\},$$

then $\omega_{K_1,\Omega} = \max\{2\omega_{K_0,\Omega}, -1\}$. For $c_0 = 2$ and $c_1 = 1$, we get:

$$u_t^c = \max\{2\omega_{K_0}, -2 + t\} = (2-t)\, \omega_{K_t^c}$$

with

$$K_t^c = \{z \in \Omega : \omega_{K_0}(z) \le -1 + t/2\},$$

so

$$\operatorname{Cap}(K_t^c) = \left(1 - \frac{t}{2}\right)^{-1} |\mathcal{E}(u_t^c)| = \left(1 - \frac{t}{2}\right)^{-1-n} \operatorname{Cap}(K_0).$$

3 Toric Case

In this section, we assume $\Omega = \mathbb{D}^n$, the unit polydisk, and $K_0, K_1 \subset \mathbb{D}^n$ to be non-pluripolar, polynomially convex compact Reinhardt (multicircled, toric) sets. Polynomial convexity of such a set K means that its logarithmic image

$$\operatorname{Log} K = \{s \in \mathbb{R}^n_- : (e^{s_1}, \dots, e^{s_n}) \in K\}$$

is a *complete* convex subset of $\mathbb{R}^n_-$, i.e., $\operatorname{Log} K + \mathbb{R}^n_- \subset \log K$; we will also say that K is *complete logarithmically convex*. The functions $c_j\, \omega_{K_j}$ are toric, and so is their geodesic u_t^c. Note that since $K_0 \cap K_1 \ne \emptyset$, inequality (6) and the concavity/convexity statements of Theorem 1(iii) hold true.

It was shown in [8] that the sets K_t defined by (2) for the geodesic interpolation of non-weighted toric extremal functions ω_{K_j} are, as in the classical interpolation theory, the geometric means $K_t^\times$ of K_j. Here, we extend the result to the weighted interpolation, which shows, in particular, that the sets K_t^c do not depend on the weights c_j. The relation $K_t^\times \subset K_t^c$ is easy, while the reverse inclusion is more elaborate; we mimic the proof of the corresponding relation for the non-weighted case [8] that rests on a machinery developed in [19].

Any toric plurisubharmonic function $u(z)$ in $\mathbb{D}^n$ gives rise to a convex function

$$\check{u}(s) = u(e^{s_1}, \dots, e^{s_n}), \quad s \in \mathbb{R}^n_-, \tag{8}$$

and the geodesic u_t^c generates the function $\check{u}_t^c$, convex in $(s, t) \in \mathbb{R}^n_- \times (0, 1)$.

Given a convex function f on $\mathbb{R}^n_-$, we extend it to the whole $\mathbb{R}^n$ as a lower semicontinuous convex function on $\mathbb{R}^n$, equal to $+\infty$ on $\mathbb{R}^n \setminus \overline{\mathbb{R}^n_-}$, and we denote $\mathcal{L}[f]$ its *Legendre transform*:

$$\mathcal{L}[f](x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - f(y)\}.$$

Evidently, $\mathcal{L}[f](x) = +\infty$ if $x \notin \overline{\mathbb{R}^n_+}$, and the Legendre transform is an involutive duality between convex functions on $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$.

It was shown in [19] that for the relative extremal function $\omega_K = \omega_{K,\mathbb{D}^n}$

$$\mathcal{L}[\check{\omega}_K] = \max\{h_Q + 1, 0\},$$

where

$$h_Q(a) = \sup_{s\in Q}\langle a, s\rangle, \quad a \in \mathbb{R}^n_+$$

is the support function of the convex set $Q = \operatorname{Log} K \subset \mathbb{R}^n_-$. Therefore, for a weighted relative extremal function $u = c\,\omega_K$, we have:

$$\mathcal{L}[\check{u}](a) = c_j\,\mathcal{L}[\check{\omega}_{K_j}](c_j^{-1}a) = \max\{h_Q(a) + c_j, 0\}. \tag{9}$$

Theorem 2 *Given two non-pluripolar complete logarithmically convex compact Reinhardt sets $K_0, K_1 \subset \mathbb{D}^n$ and two positive numbers c_0 and c_1, let u_t^c be the geodesic connecting the functions $u_0 = c_0\,\omega_{K_0}$ and $u_1 = c_1\,\omega_{K_1}$. Then the interpolating sets K_t^c defined by (5) coincide with the geometric means:*

$$K_t^\times := K_0^{1-t}K_1^t = \{z:\ |z_l| = |\eta_l|^{1-t}|\xi_l|^t,\ 1\le l\le n,\ \eta\in K_0,\ \xi\in K_1\}.$$

Proof Since the sets $K_t^\times$ and K_t^c are complete logarithmically convex, it suffices to prove that $Q_t := \operatorname{Log} K_t^\times$ coincides with $Q_t^c := \operatorname{Log} K_t^c$.

The inclusion $Q_t \subset Q_t^c$ follows from convexity of the function $\check{u}_t^c(s)$ in $(s,t) \in \mathbb{R}^n_- \times (0,1)$: if $s \in Q_t$, then $s = (1-t)\,s_0 + t\,s_1$ for some $s_j \in Q_j$, so:

$$\check{u}_t(s) \le (1-t)\,\check{u}_0(s_0) + t\,\check{u}_1(s_1) = c_t,$$

while we have $\check{u}_t(s) \ge -c_t$ for all s. This gives us $s \in Q_t^c$.

To prove the reverse inclusion, take an arbitrary point $\xi \in \mathbb{R}^n_- \setminus Q_t$, then there exists $b \in \mathbb{R}^n_+$, such that

$$\langle b, \xi\rangle > h_{Q_t}(b) = (1-t)h_{Q_0}(b) + t\,h_{Q_1}(b). \tag{10}$$

By the homogeneity of the both sides, we can assume $h_{Q_0}(b) \ge -c_0$ and $h_{Q_1}(b) \ge -c_1$. Then, by (9) and (10), we have:

$$\begin{aligned}
\check{u}_t(\xi) &= \sup_a[\langle a, \xi\rangle - (1-t)\max\{h_{Q_0}(a) + c_0, 0\} - t\max\{h_{Q_1}(a) + c_1, 0\}]\\
&\ge \langle b, \xi\rangle - (1-t)\max\{h_{Q_0}(b) + c_0, 0\} - t\max\{h_{Q_1}(b) + c_1, 0\}\\
&> (1-t)[h_{Q_0}(b) - (h_{Q_0}(b) + 1)] + t[h_{Q_1}(b) - (h_{Q_1}(b) + 1)] = -1,
\end{aligned}$$

so $\xi \notin Q_t^c$. □

Now, the corresponding assertions of Theorem 1 can be stated as follows.

Theorem 3 *For non-pluripolar complete logarithmically convex compact Reinhardt sets $K_0, K_1 \subset \mathbb{D}^n$, the inequality*

$$c_t^{n+1}\operatorname{Cap}(K_t^\times, \mathbb{D}^n) \le (1-t)\,c_0^{n+1}\operatorname{Cap}(K_0, \mathbb{D}^n) + t\,c_1^{n+1}\operatorname{Cap}(K_1, \mathbb{D}^n) \tag{11}$$

holds true for any $c_0, c_1 > 0$ *and* $c_t = (1-t)\, c_0 + t\, c_1$.

In particular, the function

$$t \mapsto \left(\mathrm{Cap}\,(K_t^\times, \mathbb{D}^n)\right)^{-\frac{1}{n+1}}$$

is concave and consequently the function

$$t \mapsto \left(\mathrm{Cap}\,(K_t^\times, \mathbb{D}^n)\right)^{\frac{1}{n+1}}$$

is convex.

As we saw in the example in the previous section, sometimes one can have $u_t = \omega_{K_t^c}$ for $u_j = c_j\, \omega_{K_j}$, in which case (11) becomes an equality. Our next result determines when this is possible for the toric case.

Theorem 4 *In the conditions of Theorem 2, the geodesic* u_τ^c *equals* $c_\tau\, \omega_{K_\tau}$ *for some* $\tau \in (0, 1)$ *if and only if*

$$K_1^{c_0} = K_0^{c_1},$$

that is, $c_0 \,\mathrm{Log}\, K_1 = c_1 \,\mathrm{Log}\, K_0$.

Proof We will use the toric geodesic representation formula established in [19, Thm. 5.1]:

$$\check{u}_t = \mathcal{L}\left[(1-t)\mathcal{L}[\check{u}_0] + t\mathcal{L}[\check{u}_1]\right], \tag{12}$$

which is a local counterpart of Guan's result [9] for compact toric manifolds; here, $\check{u}$ is the convex image (8) of the toric plurisubharmonic function u.

Let $Q_t = \log K_t$, $0 \le t \le 1$. By (9), $u_\tau^c = c_\tau\, \omega_{K_\tau}$ means

$$(1-\tau)\max\{h_{Q_0}(a) + c_0, 0\} + \tau \max\{h_{Q_1}(a) + c_1, 0\} = \max\{h_{Q_\tau}(a) + c_\tau, 0\},$$

or, which is the same,

$$\max\{h_{(1-\tau)Q_0}(a) + (1-\tau)c_0, 0\} + \max\{h_{\tau Q_1}(a) + \tau c_1, 0\} = \max\{h_{Q_\tau}(a) + c_\tau, 0\}$$

for all $a \in \mathbb{R}^n_+$. Therefore, $h_{Q_0}(a) \le -c_0$ if and only if $h_{Q_1}(a) \le -c_1$, so $c_0\, Q_0^\circ = c_1\, Q_1^\circ$ and, since $(c\,Q)^\circ = c^{-1} Q^\circ$, we get $c_0\, Q_1 = c_1\, Q_0$. Here Q° is the *copolar* (14) to the set Q, see the beginning of the next section. □

4 Covolumes

In the toric case, the Monge–Ampère capacities with respect to the unit polydisk can be represented as volumes of certain sets [2,19]. Namely, if $K \Subset \mathbb{D}^n$ is complete and logarithmically convex, then $Q := \mathrm{Log}\, K \subset \mathbb{R}^n_-$ and

$$\mathrm{Cap}\,(K, \mathbb{D}^n) = n!\,\mathrm{Covol}(Q^\circ) := n!\,\mathrm{Vol}(\mathbb{R}^n_+ \setminus Q^\circ), \tag{13}$$

where the convex set $Q^\circ \subset \mathbb{R}^n_+$ defined by

$$Q^\circ = \{x \in \mathbb{R}^n : h_Q(x) \le -1\} = \{x \in \mathbb{R}^n : \langle x, y\rangle \le -1 \ \forall y \in Q\} \tag{14}$$

is, in the terminology of [19], the *copolar* to the set Q. In particular:

$$\mathrm{Cap}\,(K_t^\times, \mathbb{D}^n) = n!\,\mathrm{Covol}(Q_t^\circ)$$

for the copolar Q_t° of the set $Q_t = (1-t)Q_0 + t\,Q_1$; we would like to stress that $Q_t^\circ \ne (1-t)Q_0^\circ + t\,Q_1^\circ$.

Convex complete subsets P of $\mathbb{R}^n_+$ (i.e., $P + \mathbb{R}^n_+ \subset P$) appear in singularity theory and complex analysis (see, for example, [11–13,15–17]), their *covolumes* (the volumes of $\mathbb{R}^n_+ \setminus P$) being used for computation of the multiplicities of mappings, etc. Such a set P generates, by the same formula (14), its copolar $P^\circ \subset \mathbb{R}^n_-$, whose exponential image Exp P° (the closure of all points $(e^{s_1}, \dots, e^{s_n})$ with $s \in P^\circ$) is a complete logarithmically convex subset of $\mathbb{D}^n$. Since taking the copolar is an involution, the representation (13) translates coherently the inequalities on the capacities to those on the (co)volumes. Namely, $\mathrm{Cap}\,(Q_j)$ becomes replaced by $\mathrm{Covol}(P_j)$ with $P_j = Q_j^\circ \subset \mathbb{R}^n_+$ for $j = 0, 1$, while $\mathrm{Cap}\,(Q_t)$ has to be replaced with the covolume of the set whose copolar is Q_t, that is, with $\left((1-t)\,P_0^\circ + t\,P_1^\circ\right)^\circ$. The operation of *copolar addition*

$$P_0 \oplus P_1 := \left(P_0^\circ + P_1^\circ\right)^\circ$$

was introduced in [19]. In particular, it was shown there that the copolar sum of any pair of cosimplices in $\mathbb{R}^n_+$, unlike their Minkowski sum, is still a simplex.

Corollary 1 *Let P_0, P_1 be non-empty convex complete subsets of $\mathbb{R}^n_+$, and let the interpolating sets $P_t^\oplus$ be defined by*

$$P_t^\oplus = \left((1-t)P_0^\circ + t P_1^\circ\right)^\circ, \quad 0 < t < 1.$$

Then the inequality

$$c_t^{n+1}\mathrm{Covol}(P_t^\oplus) \le (1-t)\,c_0^{n+1}\mathrm{Covol}(P_0) + t\,c_1^{n+1}\mathrm{Covol}(P_1)$$

holds true for any $c_0, c_1 > 0$ and $c_t = (1-t)\,c_0 + t\,c_1$.

In particular, the function

$$V^\oplus[P](t) := \left(\mathrm{Covol}(P_t^\oplus)\right)^{-\frac{1}{n+1}}$$

is concave and, consequently, the function

$$\rho^\oplus[P](t) := \left(\mathrm{Covol}(P_t^\oplus)\right)^{\frac{1}{n+1}}$$

is convex.

Note that the convexity of $\rho^{\oplus}$ (following from the concavity of $V^{\oplus}$) implies that the function

$$\tilde{\rho}^{\oplus}[P](t) := \left(\mathrm{Covol}(P_t^{\oplus})\right)^{\frac{1}{n}}$$

is convex as well. Since $\tilde{\rho}^{\oplus}$ is a homogeneous function of P, that is,

$$\tilde{\rho}^{\oplus}[c\,P](t) = c\,\tilde{\rho}^{\oplus}[P](t)$$

for all $c > 0$ and $0 < t < 1$, its convexity is equivalent to the logarithmic convexity of the covolumes, established in [8] by convex geometry methods:

$$\mathrm{Covol}(P_t^{\oplus}) \le \mathrm{Covol}(P_0)^{1-t}\mathrm{Covol}(P_1)^t,$$

which is just another form of the Brunn–Minkowski type inequality (4). Therefore, the concavity of the function $V^{\oplus}$ is a stronger property than just the logarithmic convexity of the covolumes.

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References

1. Abja, S.: Geometry and topology of the space of plurisubharmonic functions. J. Geom. Anal. **29**(1), 510–541 (2019)
2. Aytuna, A., Rashkovskii, A., Zahariuta, V.: Widths asymptotics for a pair of Reinhardt domains. Ann. Polon. Math. **78**, 31–38 (2002)
3. Bergh, J., Löfström, J.: Interpolation Spaces. An Introduction. Springer, Berlin (1976)
4. Berman, R.J., Berndtsson, B.: Moser–Trudinger type inequalities for complex Monge–Ampère operators and Aubin's "hypothèse fondamentale". arXiv:1109.1263
5. Calderón, A.-P.: Intermediate spaces and interpolation, the complex method. Studia Math. **24**, 113–190 (1964)
6. Cordero-Erausquin, D.: Santaló's inequality on $\mathbb{C}^n$ by complex interpolation. C. R. Acad. Sci. Paris Ser. I **334**, 767–772 (2002)
7. Cordero-Erausquin, D., Klartag, B.: Interpolations, convexity and geometric inequalities. In: Geometric Aspects of Functional Analysis, Lecture Notes in Math., vol. 2050, pp. 151–168. Springer, Berlin (2012)
8. Cordero-Erausquin, D., Rashkovskii, A.: Plurisubharmonic geodesics and interpolating sets. Arch. Math. (Basel) **113**(1), 63–72 (2019)
9. Guan, D.: On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles. Math. Res. Lett. **6**(5–6), 547–555 (1999)

10. Guedj, V. (ed.): Complex Monge–Ampère Equations and Geodesics in the Space of Kähler Metrics. Lecture Notes in Math., vol. 2038. Springer, Berlin (2012)
11. Kaveh, K., Khovanskii, A.: Convex bodies and multiplicities of ideals. Proc. Steklov Inst. Math. **286**(1), 268–284 (2014)
12. Khovanskiĭ, A., Timorin, V.: On the theory of coconvex bodies. Discrete Comput. Geom. **52**(4), 806–823 (2014)
13. Kim, D., Rashkovskii, A.: Higher Lelong numbers and convex geometry. To appear in J. Geom. Anal.; arXiv:1803.07948
14. Klimek, M.: Pluripotential Theory. Oxford University Press, London (1991)
15. Kouchnirenko, A.G.: Polyèdres de Newton et nombres de Milnor. Invent. Math. **32**, 1–31 (1976)
16. Rashkovskii, A.: Newton numbers and residual measures of plurisubharmonic functions. Ann. Polon. Math. **75**(3), 213–231 (2000)
17. Rashkovskii, A.: Tropical analysis of plurisubharmonic singularities. In: Tropical and Idempotent Mathematics, Contemp. Math., vol. 495, pp. 305–315. Amer. Math. Soc., Providence (2009)
18. Rashkovskii, A.: Local geodesics for plurisubharmonic functions. Math. Z **287**, 73–83 (2017)
19. Rashkovskii, A.: Copolar convexity. Ann. Polon. Math. **120**(1), 83–95 (2017)

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