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Semisimple Lie Group Theory with application in Real Geometric Invariant Theory

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Summary

The goal of this thesis was to learn about the theory of Lie algebras, Lie groups and some real geometric invariant theory, with emphasis on semisimple Lie group theory. The material in this thesis was not known to me beforehand, so the thesis represents the material which I have learned from scratch. The main motivation for studying the material in this thesis was to get insight into a problem in pseudo-Riemannian geometry intersecting real geometric invariant theory. This problem consists of classifying tensors under the action of a semisimple Lie group. An important case is where the Lie group is the orthogonal group $O(n, \mathbb{C})$ and the real forms are $O(p, q)$ where $p + q = n$. We investigate some aspects of this problem at the end of the thesis, where we consider $O(n)$ (the compact real form) and $O(p, q)$ an arbitrary real form acting on their Lie algebras via the adjoint action.

Consider a real semisimple matrix group G with real Lie algebra $\mathfrak{g}$. Let $G^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ be the complexifications, i.e G is a real form of $G^{\mathbb{C}}$ and similarly $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. We let $\tilde{G}$ be another real form of $G^{\mathbb{C}}$, i.e it's Lie algebra $\tilde{\mathfrak{g}}$ is another real form of $\mathfrak{g}^{\mathbb{C}}$. Suppose $G^{\mathbb{C}}$ acts on a complex vector space $V^{\mathbb{C}}$, and that the action restricts to actions of the real forms $G, \tilde{G}$ on real forms $V, \tilde{V}$ of $V^{\mathbb{C}}$. An example of this is the adjoint action where the Lie groups act on their Lie algebras. We can ask questions about the relationship between the real orbits contained in a complex orbit.

Suppose $Gx \subset V$ and $\tilde{G}\tilde{x} \subset \tilde{V}$ are two real orbits contained in the complex orbit $G^{\mathbb{C}}x \subset V^{\mathbb{C}}$. Do they intersect in general? What if one of the real orbits is closed, do they intersect? If so is there a relationship between the minimal vectors of one orbit to the other?

The adjoint action can be extended to an action on the vector space of endomorphisms of the Lie algebra. Suppose $\tilde{G}$ is an arbitrary real form of $G^{\mathbb{C}}$ and G is the compact real form chosen w.r.t a Cartan involution $\tilde{\theta}$ of $\tilde{G}$. We prove that if two real orbits $G\mathcal{R}$ and $\tilde{G}\tilde{\mathcal{R}}$ are conjugate, then the symmetric/antisymmetric parts of $\tilde{\mathcal{R}}$ w.r.t to the Killing form $\kappa(-,-)$ must coincide with the symmetric/antisymmetric parts w.r.t the inner product $\kappa_{\tilde{\theta}}(-,-)$.

Contents

CHAPTER 1

The structure of a complex semisimple Lie algebra

1. Preliminaries

Most of what is written in this chapter is based on material from [**1**] and [**3**].

Definition 1.1. A Lie algebra L over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space equipped with a Lie bracket $[-,-] : L \times L \to L$ satisfying the following conditions:

(1) The Lie bracket $[-,-]$ is bilinear.
(2) For all $x \in L$ we have $[x, x] = 0$.
(3) The Jacobi identity holds, i.e $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

We assume throughout the chapter that our Lie algebra is finite dimensional and we will always work over the fields $\mathbb{R}$ or $\mathbb{C}$. Here is a list of some standard examples of matrix Lie algebras equipped with the commutator bracket: $[X, Y] = XY - YX$ for square matrices X, Y.

Example 1.1.

- $\mathfrak{gl}(n, \mathbb{C}) = M(n, \mathbb{C}) = \{\ n \times n \textit{ square matrices over } \mathbb{C}\}$.
- $\mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | tr(X) = 0\}$, this is called the special linear Lie algebra.
- $\mathfrak{b}(k, \mathbb{C}) = \{X \in \mathfrak{gl}(k, \mathbb{C}) | X \textit{ is upper triangular}\}$.
- $\mathfrak{n}(k, \mathbb{C}) = \{X \in \mathfrak{gl}(k, \mathbb{C}) | X \textit{ is strictly upper triangular}\}$.
- $\mathfrak{o}(p, q) = \{X \in \mathfrak{gl}(n, \mathbb{R}) | X^t I_{p,q} = -I_{p,q} X\}$ where $p + q = n$, this is called the orthogonal Lie algebra.

In general if $X \in \mathfrak{gl}(n, \mathbb{K})$ then the set $\mathfrak{gl}_X(n, \mathbb{K}) = \{x \in \mathfrak{gl}(n, \mathbb{K}) | x^t X = -Xx\}$ forms a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{K})$, here are some important examples.

Definition 1.2. [**Classical Lie algebras**]. Let $n \geq 1$ and S_1 be the $2n \times 2n$ matrix in block form $S_1 = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$, we define the symplectic Lie algebra

$$\mathfrak{sp}(2n, \mathbb{C}) = \{x \in \mathfrak{gl}(2n, \mathbb{C}) | x^t S_1 = -S_1 x\} = \mathfrak{gl}_{S_1}(2n, \mathbb{C}).$$

Similarly let $S_2 = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}$ be the $2n \times 2n$ matrix then we define the special orthogonal Lie algebra to be:

$$\mathfrak{so}(2n, \mathbb{C}) = \mathfrak{gl}_{S_2}(n, \mathbb{C}).$$

Finally let $S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$ be the $2n+1 \times 2n+1$ matrix then we define

$$\mathfrak{so}(2n+1, \mathbb{C}) = \mathfrak{gl}_{S_3}(n, \mathbb{C}).$$

The classical Lie algebras are defined to be $\mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sp}(2n, \mathbb{C})$.

Definition 1.3. Let $L, \tilde{L}$ be Lie algebras over $\mathbb{K}$ then a Lie homomorphism $\phi : L \to \tilde{L}$ is a linear map such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L$. We say it is a Lie isomorphism if the map is also bijective.

For later notation we will refer to a surjective homomorphism of Lie algebras as an **epimorphism** and an injective one as a **monomorphism**. We write $Aut(L)$ for the group of all automorphisms of L, i.e the group of all Lie isomorphisms $L \to L$.

Now since our Lie algebra L is a finite dimensional vector space then given a basis $\{e_j\}_j$ of L we can write $[e_i, e_j] = \sum_l C_{ij}^l e_l$ for suitable constants C_{ij}^l in our field. These are called the **structure constants** of L w.r.t $\{e_j\}_j$. It is not difficult to see that in order to have a Lie isomorphism $L \to \tilde{L}$ for two Lie algebras, then there must be a basis of L and a basis of $\tilde{L}$ such that their structure constants are the same.

Definition 1.4. A Lie algebra L is said to be semisimple if it has no non-zero abelian ideals. Moreover we say that L is simple if L is non-abelian and has no non-trivial proper ideals.

Let $L, \tilde{L}$ be Lie algebras then we define $[L, \tilde{L}]$ to be the Lie algebra generated by the set $\{[x, y] | x \in L, y \in \tilde{L}\}$. Define now $L^k = [L, L^k]$ for $k \geq 1$ and $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$ for $k \geq 1$ where $L^{(0)} = L^1 = L$ and set $L' = L^{(1)}$. We say that L is **nilpotent** if $L^k = 0$ for some $k \geq 1$, similarly we say that L is **solvable** if $L^{(k)} = 0$ for some $k \geq 0$. An example of a nilpotent Lie algebra is $\mathfrak{n}(k, \mathbb{C})$, and a solvable one is $\mathfrak{b}(k, \mathbb{C})$.

We also have the notion of an **abelian** Lie algebra L. This is one which the Lie bracket is zero, i.e $[x, y] = 0$ for all $x, y \in L$. We note that the following inclusions hold within Lie algebras which is analogous to groups:

$$\{\textbf{abelian}\} \subset \{\textbf{nilpotent}\} \subset \{\textbf{solvable}\}.$$

Consider now a Lie algebra L then there is an ideal of L containing every solvable ideal of L, denote by $Rad(L)$ called the **radical** of L. To see this we note that if L does not contain any non-trivial solvable ideals then clearly we can set $Rad(L) = 0$. So assume $0 \neq I \lhd L$ is solvable and let $Rad(L)$ be the maximal solvable ideal containing I. Now if $\tilde{I}$ is any other solvable ideal of L then obviously $I + \tilde{I}$ is also solvable and an ideal of L. So $I + \tilde{I} = Rad(L)$ and $\tilde{I} \subset Rad(L)$. This shows the existence of such an ideal.

Proposition 1.1. *Let L be a Lie algebra then the following are equivalent:*

(1) $Rad(L) = 0$ where $Rad(L)$ is the radical of L.
(2) L is semisimple.
(3) L contain no non-trivial solvable ideals.

Recall that a linear map $f : V \to V$ is said to be **nilpotent** if $f^k = 0$ for a suitable $k \geq 1$. Now if V is complex then we can choose an eigenvalue of f and by an argument using induction, we can show that f being nilpotent is equivalent to saying that there is a basis of V in which the matrix representing f is strictly upper triangular.

The following theorem is important and generalizes nilpotency of maps to subalgebras of maps in $\mathfrak{gl}(n, \mathbb{C})$. It will be used throughout this chapter.

Theorem 1.1. ***[Engel's Theorem]*** *Let $L \leq \mathfrak{gl}(V)$ be a Lie subalgebra of $\mathfrak{gl}(V)$ with V a finite complex vector space. Assume that every element of L can be represented by a strictly upper triangular matrix. Then there exist a basis of V such that every element can be simultaneously represented by a strictly upper triangular matrix.*

PROOF. For proof see [**3**], section 6.1. □

Theorem 1.2. ***[Lie's Theorem]*** *Let $L \leq \mathfrak{gl}(V)$ be a solvable Lie subalgebra where V is a finite dimensional complex vector space. Then there exist a basis of V such that every element of L can simultaneously be represented by a upper triangular matrix.*

PROOF. For proof see [**3**], section 6.4. □

So we see that if $L \leq \mathfrak{gl}(V)$ satisfies the conditions in Engel's theorem then L lives in $\mathfrak{n}(k, \mathbb{C})$, i.e there is an obvious monomorphism $L \hookrightarrow \mathfrak{n}(k, \mathbb{C})$ for $k = Dim(V)$. This shows that L is nilpotent since $\mathfrak{n}(k, \mathbb{C})$ is nilpotent. Similarly if L satisfies the conditions in Lie's theorem then there is a copy of L in $\mathfrak{b}(k, \mathbb{C})$, so L is solvable.

If L is a Lie algebra then we define $ad(L) \leq \mathfrak{gl}(L)$ to be the **adjoint** Lie algebra consisting of all maps of the form $ad(x) : L \to L$ for a suitable $x \in L$ where

$$ad(x)(y) = [x, y]$$

for all $y \in L$. By using the Jacobi identity we see that these maps are in fact Lie homomorphisms of L. In fact there is a natural homomorphism $Ad : L \to \mathfrak{gl}(L)$ defined by $Ad(x) = ad(x)$ and we see that $Z(L) = ker(Ad)$. This homomorphism is called the **adjoint representation**. In particular if L is semisimple then the center $Z(L)$ must be trivial, i.e Ad is an isomorphism from L to $ad(L)$ in this case.

Lemma 1.1. *Let $\psi : L \to \hat{L}$ be an epimorphism of Lie algebras then L is nilpotent if and only if $\hat{L}$ is nilpotent. In particular L is nilpotent if and only if $ad(L)$ is nilpotent.*

Proof. For $k \geq 0$ let $L^{k+1} = [L, L^k]$, we proceed by induction on k to show that $\psi(L^k) = \hat{L}^k$. We note that since ψ is linear then it is enough to consider elements of the form $[x, y]$ for $x, y \in L$. For $k = 0$ then $L^1 = L'$ and so if $[x, y] \in L'$ we have $\psi([x, y]) = [\psi(x), \psi(y)] \in \hat{L}'$, so $\psi(L') \subset \hat{L}'$. Moreover since ψ is an epimorphism then any $[\hat{x}, \hat{y}] \in \hat{L}'$ has the form $[\psi(x), \psi(y)]$ for suitable $x, y \in L$, hence $\psi([x, y]) = [\hat{x}, \hat{y}]$ so $\hat{L}' \subset \psi(L')$ as required. Now if the statement holds for all $k \geq 0$ then we need to show that $\psi(L^{k+1+1}) = \hat{L}^{k+1+1}$. Indeed we have

$$\psi(L^{k+1+1}) = \psi([L, L^{k+1}]) = [\hat{L}, \psi(L^{k+1})] = [\hat{L}, \hat{L}^{k+1}] = \hat{L}^{k+1+1}$$

as required. Now the second statement follows from the fact that the adjoint representation $L \to ad(L)$ is an epimorphism. $\square$

We note that the previous lemma also works for solvable Lie algebras as well. The proof follows in a similar way.

Corollary 1.1. *Let L be a nilpotent complex Lie algebra then every element of $ad(L)'$ is nilpotent. In particular L' is nilpotent.*

Proof. Since $ad(L)$ is solvable then applying Lie's theorem we can find a basis such that every element of $ad(L)$ can simultaneously be represented by an upper triangular matrix. In particular for any $x, y \in L$ the map $[ad(x), ad(y)]$ is represented by a strictly upper triangular matrix with respect to this basis, hence it is nilpotent. Now apply Engel's theorem to $ad(L)'$ to show that there is a monomorphism

$$ad(L)' \hookrightarrow \mathfrak{n}(Dim(L), \mathbb{C}).$$

But $\mathfrak{n}(Dim(L), \mathbb{C})$ is nilpotent so therefore so is $ad(L)'$. This shows that L' is nilpotent as required. $\square$

2. The Killing form

Definition 1.5. Let L be a Lie algebra then the Killing form κ of L is defined to be the map $\kappa : L \times L \to \mathbb{K}$ given by $\kappa(x, y) = tr(ad(x) \circ ad(y))$ for all $x, y \in L$.

Occasionally we will write κ_L when referring to the Killing form of a Lie algebra L. We also denote the orthogonal space $L^\perp = \{x \in L | \kappa(x, y) = 0, \forall y \in L\}$ of L w.r.t κ. This space is in fact an ideal of L, which is a consequence of the associative property of κ, we will come back to this later.

We note that the Killing form κ of a Lie algebra L does not depend on a basis chosen. To see this let $x, y \in L$ and A be the matrix representing the linear map $ad(x) \circ ad(y) : L \to L$. Then a change of basis leads to similarity of matrices. So if $\tilde{A}$ is the resulting matrix of $ad(x) \circ ad(y)$ after a basis change then $\tilde{A} = S^{-1}AS$ for a suitable invertible matrix S. Hence

$$tr(\tilde{A}) = tr(S^{-1}AS) = tr(ASS^{-1}) = tr(A)$$

by symmetry of the trace.

Proposition 1.2. *Given a Lie algebra L the Killing form κ is a symmetric associative bilinear form on L.*

Proof. We show associativity of κ. So let $x, y, z \in L$ then

$$\begin{aligned}\kappa(x, [y, z]) = tr(ad(x) \circ ad[y, z]) = tr(ad(x) \circ (ad(y) \circ ad(z) - ad(z) \circ ad(y))) = \\ = tr(ad(x) \circ ad(y) \circ ad(z)) - tr(ad(x) \circ ad(z) \circ ad(y)) = \\ = tr(ad(x) \circ ad(y) \circ ad(z)) - tr(ad(y) \circ ad(x) \circ ad(z)) = \\ = tr(ad([x, y]) \circ ad(z))) = \kappa([x, y], z).\end{aligned}$$

All by the symmetry and linearity of the trace map, and since the adjoint representation ad is a Lie homomorphism. $\square$

Proposition 1.3. *If L is a Lie algebra and $\psi \in Aut(L)$ then $\kappa(\psi(x), \psi(y)) = \kappa(x, y)$ for all $x, y \in L$. More generally if $L \cong \hat{L}$ with isomorphism $L \to_\psi \hat{L}$ then $\kappa_L(x, y) = \kappa_{\hat{L}}(\psi(x), \psi(y))$ for all $x, y \in L$.*

Proof. Assume $ad(x)$ and $ad(y)$ are represented by the matrices A, B respectively with respect to a basis $x_1, x_2, \ldots, x_n$ of L. We observe that $\psi(x_1), \ldots \psi(x_n)$ is a basis for L as well because ψ is invertible. In particular with respect to this basis $ad(\psi(x))$ and $ad(\psi(y))$ also have the matrices A, B respectively as ψ preserves the Lie bracket, and so the first statement follows. A similar argument holds for the second statement. $\square$

In particular if L and $\tilde{L}$ are two Lie algebras such that κ_L is negative semi definite but $\kappa_{\tilde{L}}$ is not then L and $\tilde{L}$ cannot be isomorphic. The following is an example of this.

Example 1.2. Consider $\mathfrak{sl}(2,\mathbb{R})$ and $\mathbb{R}^3_\wedge$ which is the Lie algebra $\mathbb{R}^3$ with the Lie bracket given by the cross product. They are clearly isomorphic as vector spaces since they both have dimension 3, but they are not as Lie algebras. To see this an easy calculation shows that the Killing form of $\mathbb{R}^3_\wedge$ is negative definite, which is not the case for $\mathfrak{sl}(2,\mathbb{R})$. In fact $\mathfrak{sl}(2,\mathbb{R})$ with the standard basis

$$\{h = e_{11} - e_{22}, e = e_{12}, f = e_{21}\}$$

have Killing form matrix given by: $\begin{pmatrix} 0 & 4 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$. So in particular $\kappa(h,h) > 0$. While $\mathbb{R}^3_\wedge$ has Killing form matrix given by: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ w.r.t the standard basis $\{i, j, k\}$.

It turns out however that over $\mathbb{C}$ the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathbb{C}^3_\wedge$ are in fact isomorphic.

If $L, \tilde{L}$ have the same Killing form matrix then they need not be isomorphic, indeed consider L any 1-dimensional Lie algebra and $\tilde{L}$ be trivial then their Killing form matrices are both zero but $L \ncong \tilde{L}$.

It is clear that an abelian Lie algebra must have Killing form which is identically zero, more generally we have.

Proposition 1.4. *The Killing form of a nilpotent complex Lie algebra L is identically zero. In particular $L^\perp = L$.*

PROOF. Let $\beta, \alpha \in L$ and suppose L has nilpotency class $k \geq 1$, i.e $L^k = 0$ where k is minimal. Consider the linear map $f = ad(\beta) \circ ad(\alpha)$ then we begin by noting that $f^2(x) = [\beta, [\alpha, [\beta, [\alpha, x]]]] \in L^4$ for $x \in L$. So by induction we claim that $f^n(L) \subset L^{2n}$. Indeed if $n = 1$ then $f(L) = [\beta, [\alpha, L]] \subset L^2$ as $[\alpha, L] \subset L^1$. Now if $f^n(L) \subset L^{2n}$ holds for all $n \geq 1$ then

$$f^{n+1}(L) = ad(\beta) \circ ad(\alpha) \circ f^n(L) = [\beta, [\alpha, f^n(L)]] \subset [\beta, [\alpha, L^{2n}]] \subset [\beta, L^{2n+1}] \subset L^{2(n+1)}$$

as required. This shows that $f^k = 0$ hence f is nilpotent and so can be represented by a strictly upper triangular matrix, in particular

$$tr(ad(\beta) \circ ad(\alpha)) = \kappa(\beta, \alpha) = 0.$$

Since this holds for any $\alpha, \beta \in L$ the proof is complete. $\square$

Note that the previous proof also works over algebraically closed fields, since then a nilpotent map can be represented by a strictly upper triangular matrix.

3. Cartan's criteria

We assume now that our Lie algebra is over $\mathbb{C}$. In this section we will see how the Killing form is related to the semisimplicity of a Lie algebra. We will also see that in order to classify the semisimple Lie algebras over $\mathbb{C}$ it is enough to classify the simple ones.

Lemma 1.2. *If $L \subset \mathfrak{gl}(V)$ is a Lie algebra such that $tr(x \circ y) = 0$ for all $x, y \in L$ then L is solvable.*

PROOF. See for example [**3**], Proposition 9.3. $\square$

The following lemma shows that the Killing form of L restricted to an ideal of L coincides with Killing form of the ideal. If $I \leq L$ is a Lie subalgebra we will denote the orthogonal subspace to I w.r.t κ by $I^\perp = \{x \in L | \kappa(x, y) = 0, \forall y \in I\}$.

Lemma 1.3. *Let L be a Lie algebra and $I \trianglelefteq L$ then $\kappa_I(I, I) = \kappa_L(I, I)$. Moreover we have $I^\perp \trianglelefteq L$, where $I^\perp$ is the orthogonal subspace to I with respect to the Killing form of L.*

PROOF. Let $x, y \in I \trianglelefteq L$ be given and suppose $x_1, \ldots x_n$ is a basis for I then we can extend to a basis for L say $x_1, \ldots x_n, y_{n+1}, \ldots y_m$. So denote A, B for the matrices which represents $ad_I(x)$ and $ad_I(y)$ w.r.t this basis respectively. Then $ad(x)(x_j) = ad_I(x)(x_j)$ for all j and $ad(x)(y_j) = \sum_l \lambda_l x_l$ for some $\lambda_j \in \mathbb{C}$, similarly for $ad(y)$. So that the matrix of $ad(x)$ have the form $\begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix}$ and similarly $ad(y)$ has matrix $\begin{pmatrix} B & \tilde{S} \\ 0 & 0 \end{pmatrix}$ for some $n \times n$ matrices $S, \tilde{S}$. This shows that $ad(x) \circ ad(y)$ has matrix

$$\begin{pmatrix} AB & A\tilde{S} \\ 0 & 0 \end{pmatrix}.$$

So we get

$$\kappa_L(x,y) = tr(ad(x) \circ ad(y)) = tr(AB) = tr(ad_I(x) \circ ad_I(y)) = \kappa_I(x,y)$$

as required. For the second statement we observe that if $x \in I^\perp$ with $y \in L$ and $\gamma \in I$ then $\kappa([y,x],\gamma) = \kappa(y,[x,\gamma]) = 0$ by associativity of κ so that $[x,y] \in I^\perp$, this shows that $I^\perp \trianglelefteq L$. $\square$

Recall that if X, Y are upper triangular matrices then the Lie bracket $[X,Y]$ is strictly upper triangular. Also we note that a product of a upper triangular matrix with a strictly upper triangular matrix gives a strictly upper triangular matrix.

Theorem 1.3. ***[Cartan's first criterion]***. *Let L be a Lie algebra with Killing form κ then L is solvable if and only if $(\forall x \in L)(\forall y \in L')(\kappa(x,y) = 0)$.*

PROOF. ($\Rightarrow$). Suppose L is solvable then so is $ad(L)$ hence by Lie's theorem we may choose a basis of L such that the matrix of $ad(x)$ is upper triangular for every choice of $x \in L$. So we note that if $\alpha, \beta \in ad(L)$ then with respect to this basis the matrix of $[\alpha, \beta]$ is strictly upper triangular. In particular for $x \in L$ and $y \in L'$ the linear map $ad(x) \circ ad(y)$ is represented by a strictly upper triangular matrix as well. This shows that $\kappa(x,y) = 0$ for any $x \in L$ and $y \in L'$.

($\Leftarrow$). Conversely suppose $\kappa(x,y) = 0$ for all $x \in L$ and $y \in L'$ then in particular the the Killing form on L' satisfies

$$\kappa_{L'}(x,y) = 0 = tr(ad(x) \circ ad(y))$$

for all $x, y \in L'$, since L' is an ideal of L. So it follows that $ad(L') \subset gl(L)$ is solvable, in particular since $ad(L)' = ad(L')$ then $ad(L)$ is solvable, hence so is L as required. The theorem is proved. $\square$

We now obtain a criterion for when a complex Lie algebra is semisimple, this is strongly related to the Killing form in the following way.

Theorem 1.4. ***[Cartan's second criterion]***. *A Lie algebra L is semisimple if and only if κ is non-degenerate (i.e $L^\perp = 0$).*

PROOF. ($\Rightarrow$). Assume L is semisimple but κ is degenerate, then we can choose $x \in L$ such that $\kappa(x,y) = 0$ for any $y \in L$. In particular $0 \neq L^\perp \trianglelefteq L$ is a proper non-zero ideal of L. Indeed if $L^\perp = L$ then $\kappa(L, L') = 0$ so L would be solvable and so not semisimple. Moreover since $\kappa(L^\perp, L) = 0$ then in particular

$$\kappa(L^\perp, L^{\perp\prime}) = \kappa_{L^\perp}(L^\perp, L^{\perp\prime}) = 0$$

since $L^{\perp} \trianglelefteq L$. So that $L^{\perp}$ is a solvable non-zero proper ideal of L, contradicting the fact that L is semisimple.

($\Leftarrow$). Conversely suppose κ is non-degenerate but L is not semisimple. So we may choose a non-zero abelian ideal $\tilde{L} \trianglelefteq L$. So given any $0 \neq x \in \tilde{L}$ and any choice of $z \in L$ we have $ad(x) \circ ad(z) \circ ad(x) \circ ad(z) = 0$ so that the map $f = ad(x) \circ ad(z)$ is nilpotent, hence $tr(ad(x) \circ ad(z)) = 0 = \kappa(x, z)$ so that $L^{\perp} \neq 0$ a contradiction. The theorem is proved. $\square$

We see that in the proof of Cartan's second criterion that for Lie algebras over general algebraically closed fields, the direction ($\Leftarrow$) will still hold, i.e if κ is non-degenerate then L is semisimple. In fact we will see later when we discuss real forms, that Cartan's second criterion in fact holds for semisimple real Lie algebras as well.

The following corollary is immediate.

Corollary 1.2. *A Lie algebra L is semisimple if and only if the matrix of κ is invertible.* $\square$

The following is a standard result in linear algebra, we omit the proof.

Lemma 1.4. *Let V be a vector space with non-degenerate bilinear form $\boldsymbol{b}$ then for a subspace $\tilde{V} \leq V$ we have $V = \tilde{V} \oplus \tilde{V}^{\perp}$.*

In particular when L is a semisimple Lie algebra and $I \lhd L$ then we can always decompose $L = I \oplus I^{\perp}$ with respect to the Killing form κ on L. In fact a Lie algebra L which has such a property is said to be **reductive**, i.e whenever I is an ideal of L then L can be decomposed as $L = I \oplus \tilde{I}$ for some ideal $\tilde{I} \trianglelefteq L$.

We recall how the Lie bracket is defined on a quotient. Suppose we have a Lie algebra L with an ideal I, then the quotient is also a natural Lie algebra with Lie bracket given by $[x + I, y + I] = [x, y] + I$ for all $x, y \in L$. In particular the quotient map $p : L \to L/I$ is naturally an epimorphism.

Corollary 1.3. *Any proper ideal I of a semisimple Lie algebra L is again semisimple. Moreover any quotient of L is also semisimple. In particular $L' = L$.*

Proof. If $I \lhd L$ is a proper ideal of L which is not semisimple then the Killing form on I is degenerate hence choose $x \in I$ such that $\kappa_I(x, y) = 0$ for any $y \in I$. Now because $I \lhd L$ we have $\kappa_I(x, y) = \kappa_L(x, y) = 0$. We claim that we can choose $0 \neq y \in I$ such that $[x, y] \neq 0$. Indeed if not then the center, $Z(I)$ is non-trivial and by the Jacobi identity it is an abelian ideal of L. This contradicts our assumptions. So choose any $\alpha \in L$ then

$$0 = \kappa_I([\alpha, y], x) = \kappa([\alpha, y], x) = \kappa(\alpha, [y, x])$$

so $0 \neq [x, y] \in L^\perp$, a contradiction. For the second statement let $I \lhd L$ be non-zero and proper then we know that $L = I^\perp \oplus I$ and we claim that $L/I \cong I^\perp$. Indeed define a map $\psi : I^\perp \to L/I$ by

$$\psi(x) = x + I$$

then ψ is clearly linear and it is injective, as $I \cap I^\perp = 0$. Now since $Dim(I^\perp) = Dim(L) - Dim(I)$ then ψ is also surjective. Finally

$$\psi([x, y]) = [x, y] + I = [x + I, y + I] = [\psi(x), \psi(y)]$$

for all $x, y \in L$ by definition of the Lie bracket on a quotient, so ψ is a Lie isomorphism. Now since $I^\perp \lhd L$ is proper then it is semisimple by the first part, but then so is L/I as required. The last statement follows immediately from the fact that L/L' is abelian. $\square$

Finally we derive the theorem which describes the structure of a semisimple complex Lie algebra, and the meaning of the word "semisimple" becomes clear.

Theorem 1.5. *A Lie algebra L is semisimple if and only if it is a direct sum of simple ideals.*

Proof. ($\Rightarrow$). First if L does not contain any proper non-zero ideals then L is simple as L is non-abelian hence we are done. So we may assume that L contain a non-zero proper ideal $I \lhd L$. In particular we know that $L = I \oplus I^\perp$ where $I^\perp$ is the orthogonal space to I with respect to the Killing form κ of L. But as $I \lhd L$ and $I^\perp \lhd L$ then they are both semisimple, so if they are both simple we are done. If not we can extract non-zero proper ideals $I_1 \lhd I$ and $I_2 \lhd I^\perp$ so that

$$L = I_1 \oplus I_1^\perp \oplus I_2 \oplus I_2^\perp$$

where $Dim(I_1) < Dim(I)$ and $Dim(I_2) < Dim(I^\perp)$. In particular I_1, I_2 are semisimple, so we can repeat the procedure and eventually the process stops and L is a direct sum of simple ideals.

($\Leftarrow$). Assume that L is the direct sum $\oplus_j^n L_j$ of simple ideals $L_j \lhd L$, and suppose by contradiction that L is not semisimple. Then we can choose an abelian non-zero ideal $\hat{L} \lhd L$. We note that

$$[\hat{L}, L] = \oplus_j^n [\hat{L}, L_j]$$

since the L_j are ideals of L. We also have $[\hat{L}, L_j] \lhd L_j$ for all j by the Jacobi identity. Now suppose $[\hat{L}, L_j]$ are all zero, then $[\hat{L}, L] = 0$ hence $\hat{L} \leq Z(L)$. However we have $Z(L) \subset \oplus_j^n Z(L_j)$, but this sum is trivial since the L_j are all simple. So that $\hat{L} = 0$, this contradicts our assumptions. So there is some j such that $[\hat{L}, L_j] \neq 0$, but since $[\hat{L}, L_j] \lhd L_j$ then we have $\hat{L} = L_j$, so that L_j is abelian, which is again a contradiction. This completes the proof. □

Corollary 1.4. *If L is semisimple with decomposition $\oplus_{1\leq j\leq n} L_j$ of simple ideals $l_j \lhd L$ and $I \lhd L$ is any simple ideal then $I = L_i$ for some $1 \leq i \leq n$. In particular a decomposition of a semisimple Lie algebra into simple ideals is unique up to rearrangement.*

PROOF. Note that $[L_i, I] \unlhd L_i$ and $[L_i, I] \unlhd I$ together with $[L_i, I] \lhd L$ all by the Jacobi identity. So if $[L_j, I] = 0$ for all j then $[L, I] = 0$ so I would be abelian hence not simple. So as L_j and I are simple for all j then it must be the case that $[L_i, I] = L_i = I$ for some i as required. □

4. Trace forms

In this section we will prove that any associative symmetric non-degenerate bilinear form on a simple complex Lie algebra is just proportional to the Killing form. We will use some elementary representation theory of Lie algebras (see appendix B for details).

Let L be a Lie algebra and suppose we have a finite representation $\psi : L \to \mathfrak{gl}(V)$. Then we can naturally define a map $\mathbf{b} : L \times L \to \mathbb{C}$ by

$$\mathbf{b}(x, y) = tr(\psi(x) \circ \psi(y))$$

for all $x, y \in L$. Analogous to the Killing form one can show that the trace form $\mathbf{b}$ is a symmetric associative bilinear form on L. We call this form for a **trace** form on L.

Proposition 1.5. *Let L be a complex semisimple Lie algebra with trace form $\boldsymbol{b}$ with respect to a faithful representation $L \to_\psi gl(V)$ then $\boldsymbol{b}$ is non-degenerate.*

PROOF. Consider the space $L^\perp = \{x \in L | \forall y \in L : \mathbf{b}(x, y) = 0\}$ of elements perpendicular to L w.r.t $\mathbf{b}$. Then this is an ideal of L by associativity of $\mathbf{b}$, moreover it is clear that

$$\mathbf{b}(x, y) = 0 = tr(\psi(x) \circ \psi(y))$$

for all $x, y \in L^\perp$. So it follows that $\psi(L^\perp)$ must be a solvable ideal in $\mathfrak{gl}(V)$. Clearly it is a proper ideal of $\mathfrak{gl}(V)$ since L is semisimple and ψ is faithful. However $\psi(L)$ is also semisimple so we must have $\psi(L^{\perp}) = 0$ which gives $L^{\perp} = 0$ as required. This proves the result. $\square$

We see that if we take the adjoint representation, $Ad : L \to \mathfrak{gl}(L)$ which is faithful on a semisimple Lie algebra L, then we recover the Killing form.

Suppose now that V is an L-module, $\phi : L \times V \to V$ and write $x \cdot v = \psi(x, v)$ for this action. We can consider the dual space V^* and a new map $\psi : L \times V^* \to V^*$ defined by

$$(x \cdot \alpha)(v) = -\alpha(x \cdot v)$$

for all $x \in L, v \in V$ and $\alpha \in V^*$. It follows that V^* is also an L-module defined in this way.

Lemma 1.5. *V^* is an L-module.*

PROOF. We check the axioms for V^* to be an L-module. If $x, y \in L$ and $\alpha \in V^*$ then

$$([x,y]\cdot\alpha)(\star) = -\alpha([x,y]\cdot\star) = -\alpha(x(y\cdot\star)-y(x\cdot\star)) = -\alpha(x(y\cdot\star))+\alpha(y(x\cdot\star)) = x\cdot(y\cdot\alpha)-y\cdot(x\cdot\alpha)$$

using that α is linear and V is an L-module. Now given $\lambda_1, \lambda_2 \in \mathbb{K}$ and $x, y \in L$ with $\alpha \in V^*$ then

$$((\lambda_1 x + \lambda_2 y) \cdot \alpha)(\star) = -\alpha((\lambda_1 x + \lambda_2 y) \cdot \star)$$

so linearity of ψ follows from the fact that α is linear and V is an L-module. Similarly

$$(x \cdot (\lambda_1\alpha + \lambda_2\beta))(\star) = -\lambda_1\alpha(x \cdot \star) - \lambda_1\beta(x \cdot \star) = \lambda_1(x \cdot \alpha) + \lambda_2(x \cdot \beta)$$

for every $\alpha, \beta \in V^*$. We deduce that V^* is an L-module as required. $\square$

Theorem 1.6. *Let L be a simple complex Lie algebra together with a symmetric associative non-degenerate bilinear form $\boldsymbol{b}$ then $\boldsymbol{b} = \lambda\kappa$ for a suitable $\lambda \in \mathbb{C}$.*

PROOF. Let $L \to_{\mathbf{B}} L^*$ be the linear map $\mathbf{B}(x) = \mathbf{b}(x, -)$. We can view L as the adjoint L-module given by $x \cdot y = [x, y]$ for $x, y \in L$. So consider L^* as the L-module described above, i.e it is given by $(x \cdot \alpha)(\star) = -\alpha(x \cdot \star) = -\alpha([x, \star])$ for all $x \in L$ and $\alpha \in L^*$. We claim that $\mathbf{B}$ is a homomorphism of L-modules. Indeed $\mathbf{B}$ is linear and for $x, y \in L$ we have

$$\mathbf{B}(x \cdot y)(\star) = \mathbf{B}([x, y])(\star) = \mathbf{b}([x, y], \star) = -\mathbf{b}([y, x], \star) = -\mathbf{b}(y, [x, \star]) = x \cdot \mathbf{b}(y, \star).$$

This shows that $\mathbf{B}$ is an homomorphism of L-modules as required. Now since the kernel of $\mathbf{B}$ is $L^{\perp}$ and $\mathbf{b}$ is non-degenerate then $\mathbf{B}$ is an isomorphism of L-modules. Also since L is simple then it is semisimple and so κ is non-degenerate as well. Hence there is also an isomorphism $L \to_{\mathbf{K}} L^*$ of L-modules where $\mathbf{K}(x) = \kappa(x, -)$, this follows by the same argument as for $\mathbf{B}$. Moreover since L is simple then the adjoint module is irreducible (since all L-submodules are ideals of L). So the composition

$$L \to_{\mathbf{B}} L^* \to_{\mathbf{K}^{-1}} L$$

is an isomorphism from the adjoint module to itself. By Schur's lemma there is an $\lambda \in \mathbb{C}$ such that

$$\kappa = \lambda \mathbf{b}$$

as required. □

We will see later when we discuss real forms that the previous result also hold for semisimple real Lie algebras.

Corollary 1.5. *If L is a complex simple Lie algebra with Killing form κ then there exist a trace form $\boldsymbol{b}$ and some $\lambda \in \mathbb{C}$ such that $\boldsymbol{b} = \lambda\kappa$*

Proof. Combine the previous results with Ado's Theorem (appendix B). □

These results show that if $L \subset \mathfrak{gl}(n, \mathbb{C})$ is a simple matrix Lie algebra then the Killing from κ is given by $\kappa(X, Y) = \lambda tr(XY)$ for all $X, Y \in L$, for a suitable $\lambda \in \mathbb{C}$. This is because there is a natural faithful representation $\psi : L \hookrightarrow \mathfrak{gl}(V)$ where V is an n-dimensional complex vector space, given by sending X to the linear map $V \to V$ represented by X. So we have a trace form $\mathbf{b}(X, Y) = tr(\psi(X) \circ \psi(Y)) = tr(XY)$.

Example 1.3. One can show by an easy argument that $\mathfrak{sl}(2, \mathbb{C})$ is simple. Consider the standard basis $\{e, f, h\}$ of $\mathfrak{sl}(2, \mathbb{C})$. Now an easy calculation shows that $\kappa(e, f) = 4$ and $tr(ef) = 1$, so it follows that $\kappa(X, Y) = 4tr(XY)$ for all $X, Y \in \mathfrak{sl}(2, \mathbb{C})$.

From the results we have the following is an observation: A semisimple Lie algebra L can be decomposed into a direct sum of ideals say $L = \oplus_j L_j$. Now since $[L_i, L_j] = 0$ for all $i \neq j$ and $[L_i, L_i] = L_i$, then clearly $[L_i, [L_j, L_s]] = 0$ whenever $i \neq j$. This means that $\kappa(L_i, L_j) = 0$ whenever $i \neq j$. So if $x = \sum_s x_s$ and $y = \sum_l x'_l$ for $x'_l \in L_l$ and $x_s \in L_s$ then we obtain that

$$\kappa(x, y) = \sum_k \kappa(x_k, x'_k).$$

Hence if $\mathbf{b}$ is any symmetric, associative, non-degenerate bilinear form on L then

$$\mathbf{b}(x, y) = \sum_k \lambda_k \kappa(x_k, x'_k)$$

for suitable $\lambda_k \in \mathbb{C}$.

5. The root space decomposition

In this chapter we always assume L is a Lie algebra over $\mathbb{C}$ unless otherwise stated.

6. Cartan subalgebras

Definition 1.6. An element $x \in L$ is said to have an abstract Jordan decomposition if it is possible to decompose $x = d + n$ for elements $d, n \in L$ such that $ad(d)$ diagonalisable and $ad(n)$ nilpotent. Moreover an element $x \in L$ is said to be semisimple if $n = 0$ in the decomposition.

We say that a Lie homomorphism $\phi : L \to L$ is a **derivation** if $\phi([x, y]) = [x, \phi(y)] + [\phi(x), y]$ for all $x, y \in L$. One may show that the set of all derivations, $Der(L)$ is a Lie algebra with the commutator bracket. In particular it is clear that $ad(L) \leq Der(L)$.

Theorem 1.7. *Let L be a semisimple Lie algebra then $ad(L) = Der(L)$.*

PROOF. We first claim that $ad(L) \trianglelefteq Der(L)$. Indeed if $ad(x) \in Ad(L)$ and $\delta \in Der(L)$ then we have

$$[\delta, ad(x)] = \delta \circ ad(x) - ad(x) \circ \delta = \delta([x, -]) - [x, \delta(-)] = [\delta(x), -] = ad(\delta(x)) \in ad(L).$$

Now since L is semisimple then $L \cong ad(L)$ and so $ad(L)$ is also semisimple. Now we can write $Der(L) = ad(L) \oplus ad(L)^{\perp}$ where $ad(L)^{\perp}$ is the orthogonal space to $ad(L)$ in $Der(L)$ with respect to the Killing form κ on $Der(L)$. Our aim is to show that $ad(L)^{\perp} = 0$. Suppose not then there exist $0 \neq \delta \in Der(L)$ such that $\kappa(\delta, \gamma) = 0$ for every choice of $\gamma \in ad(L)$. Note that if we can find $\alpha \in ad(L)$ such that $[\delta, \alpha] \neq 0$ then we would have

$$\kappa(\delta, [\alpha, \gamma]) = 0 = \kappa([\delta, \alpha], \gamma) = \kappa_{ad(L)}([\delta, \alpha], \gamma).$$

So we would contradict the fact that $ad(L)$ is semisimple. We claim that we can find such an element α. If not then $[\delta, ad(x)] = ad(\delta(x)) = 0$ for all $x \in L$. But because the adjoint representation, $Ad : L \hookrightarrow ad(L)$ is an isomorphism, we have

$$(\forall x \in L)(ad(\delta(x)) = 0) \Leftrightarrow (\forall x \in L)(\delta(x) = 0) \Leftrightarrow \delta = 0,$$

which contradicts our assumptions as $\delta \neq 0$. The theorem is proved. $\square$

We will see that this theorem also hold for semisimple real Lie algebras, in fact one can mimic the proof above.

Lemma 1.6. *Let L be a Lie algebra and suppose $x \in Der(L)$ can be expressed as the sum $x = z + y$ such that $z \in \mathfrak{gl}(L)$ is diagonalisable and $y \in \mathfrak{gl}(L)$ is nilpotent, then $z, y \in Der(L)$.*

Proof. For proof see for example [**3**], chapter 9, Proposition 9.14. □

In turns out that every element of a semisimple Lie algebra has an abstract Jordan decomposition. In fact it is not difficult to see that if a linear map $V \to V \in \mathfrak{gl}(V)$ have an abstract Jordan decomposition, then it must coincide with the usual Jordan decomposition by uniqueness.

Theorem 1.8. *Every element x of a semisimple Lie algebra L has a unique abstract Jordan decomposition, $x = d + n$ with $[d, n] = 0$.*

Proof. Let $x \in L$ be given then we can decompose $ad(x)$ into it's Jordan decomposition: $ad(x) = D + N$ for which $D, N \in \mathfrak{gl}(L)$ are diagonalisable and nilpotent respectively. In particular $[D, N] = 0$ so it follows that D, N are both derivations of L. So because $ad(L) = Der(L)$ we must have $ad(x) = ad(d) + ad(n)$ for suitable $d, n \in L$. Now since the adjoint representation, $Ad : L \to ad(L)$ is a monomorphism then result follows. □

Definition 1.7. A Cartan subalgebra H of a Lie algebra L is a Lie subalgebra of L such that the following are satisfied:

(1) Every element in H is semisimple.
(2) H is abelian.
(3) H is maximal with properties (1) and (2).

Suppose we have a Cartan subalgebra $H \leq L$. Now since H is abelian then $ad(H) \subset \mathfrak{gl}(L)$ is also abelian. Moreover since every element in H is semisimple, then it is possible to choose a basis $\{e_j\}_j$ of H such that every map $ad(x)$ is represented by a diagonal matrix w.r.t $\{e_j\}_j$. We say the maps in $ad(H)$ are simultaneously diagonalisable. This is a standard result in linear algebra and we omit the proof.

Lemma 1.7. *Let V be a complex vector space and $\psi_1, \dots, \psi_n$ be linear diagonalisable maps $V \to V$. Then these maps commute pairwise if and only if there exist a basis in which all the ψ_j are represented by a diagonal matrix.*

Having a Cartan subalgebra $H \leq L$ we can choose a linear functional of H, say $\alpha \in H^*$, and define the following subspace of L:

$$L_\alpha = \{x \in L | [h, x] = \alpha(h)x, \ (\forall h \in H)\}.$$

If $L_\alpha \neq 0$ we call α a **root** of L, and L_α a **root space** of L. Consider the set $\Omega = \{0 \neq \alpha \in H^* | L_\alpha \neq 0\}$, then we observe that $L = L_0 + \sum_{\alpha \in \Omega} L_\alpha$. Indeed we can choose a basis $\{e_j\}_j$ of L such that every map in $ad(H)$ is represented by a diagonal matrix, so in particular if $h \in H$ then $ad(h)(e_j) = \lambda_h e_j$ for suitable $\lambda_h \in \mathbb{C}$. So define a map $\alpha_j : H \to \mathbb{C}$ taking $h \to \lambda_h$. It is easy to check that this map is linear. So we have $e_j \in L_{\alpha_j}$.

We also claim that the sum is direct. To see this suppose that we have $\sum_{\alpha \in \Omega} x_\alpha = 0$ for a choice of $x_\alpha \in L_\alpha$. Then pick any $h \in H$. A vector x_α is contained in the eigenspace of $ad(h)$ corresponding to the eigenvalue $\alpha(h)$. So we must have $x_\alpha = 0$, since a finite set of eigenvectors corresponding to distinct eigenvalues are linearly independent.

Proposition 1.6. *Any semisimple Lie algebra L contain a Cartan subalgebra H.*

Proof. Since L is semisimple then every element of L has an abstract Jordan decomposition. So given $x \in L$ we can write $x = d + n$ with $ad(d)$ diagonalisable and $ad(n)$ nilpotent. Now suppose the diagonal part is alway zero for every $x \in L$ then $ad(x)$ is nilpotent, so by Engel's theorem there is a monomorphism $L \hookrightarrow \mathfrak{n}(k, \mathbb{C})$ with $k = Dim(L)$ where $\mathfrak{n}(k, \mathbb{C})$ is the Lie algebra of strictly upper triangular matrices over $\mathbb{C}$. This shows that L is nilpotent, as $\mathfrak{n}(k, \mathbb{C})$ is nilpotent. But this can not happen as then L would not be semisimple. So L must contain semisimple elements, in particular we can choose H of largest possible dimension such that it is abelian and contain only semisimple elements. $\square$

The previous results show that we can always decompose a semisimple Lie algebra L into a direct sum:

$$L = L_0 \oplus_{\alpha \in \Omega} L_\alpha$$

for a choice of Cartan subalgebra $H \leq L$, where L_α are the root spaces of L w.r.t H. We call such a decomposition a **root decomposition** of L.

We note that H is always contained in L_0, in fact L_0 coincides with H.

Theorem 1.9. *Let L be a semisimple Lie algebra with Cartan subalgebra H and denote L_0 the zero root space, then $H = L_0 = C_L(H)$.*

Proof. For proof see for example [**3**], section 10.2. $\square$

Example 1.4. One can easily show that the Lie algebra $L = \mathfrak{sl}(n, \mathbb{C})$ with basis $S = \{e_{ij} | i \neq j\} \cup \{e_{ii} - e_{nn} | 1 \leq i \leq n\}$ has root space decomposition

$$\mathfrak{sl}(n, \mathbb{C}) = H \oplus_{i \neq j} L_{h_{ii} - h_{jj}},$$

where $H = \langle e_{ii} - e_{nn} | 1 \leq i \leq n-1 \rangle$ and the roots have have the form $\alpha_{ij}(h) = h_{ii} - h_{jj}$ for each $h \in H$ and $i \neq j$. We show that H is a Cartan subalgebra of L. It clear that H is abelian and all elements are clearly semisimple, since we can show that $[h, e_{ij}] = (h_{ii} - h_{jj})e_{ij}$ $(i \neq j)$ for all $h \in H$. We show that it is maximal abelian in L. Suppose there is an element $x \notin H$ such that $[h, x] = 0$ for every $h \in H$. In particular there is some matrix entry $x_{ij} \neq 0$ for $i \neq j$, and an easy calculation shows that $[e_{ii}, x]_{ij} = x_{ij}$ and $[e_{jj}, x]_{ij} = -x_{ij}$ for $i \neq j$. Now $h = e_{ii} - e_{jj} \in H$ and $[h, x]_{ij} = 2x_{ij} \neq 0$ hence $[h, x] \neq 0$, this contradicts our assumptions. So H is a Cartan subalgebra of L as required.

We will investigate in the next section some properties which the root spaces must have when L is semisimple. In fact if one can find a suitable root decomposition for a Lie algebra L, such that the root spaces satisfy some of these properties, then this is enough to determine the semisimplicity of the Lie algebra. So we can either use the Killing form or find a root decomposition in order to determine the semisimplicity of a Lie algebra.

7. The structure of the root spaces

Let L be a semisimple Lie algebra and H be a Cartan subalgebra and $L = H \oplus_{\alpha \in \Omega} L_\alpha$ be the corresponding root decomposition. In this section we study properties of the root spaces.

Proposition 1.7. *Let $\alpha, \beta \in \Omega$ then the following hold:*

(1) If $\alpha + \beta \neq 0$ then $\kappa(L_\alpha, L_\beta) = 0$.
(2) $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.

PROOF. For (1) let $\alpha, \beta \in \Omega$ be roots such that $\alpha + \beta \neq 0$, then given $x \in L_\alpha$ and $y \in L_\beta$ we have for any $h \in H$ such that $\alpha(h) + \alpha(h) \neq 0$:

$$(\alpha(h) + \beta(h))\kappa(x, y) = \kappa([h, x], y) + \kappa(x, [h, y]) =$$

$$= \kappa([h, x], y) + \kappa([x, h], y) = \kappa([h, x], y) - \kappa([h, x], y) = 0.$$

by associativity of κ. This proves (1). Now for (2) suppose $x \in L_\alpha$ and $y \in L_\beta$ then for $h \in H$ by the Jacobi identity:

$$[h, [x, y]] = -[x, [y, h]] - [y, [h, x]] = -[x, -\beta(h)y] - [y, \alpha(h)x] = (\beta(h) + \alpha(h))[x, y],$$

showing that $[x, y] \in L_{\alpha+\beta}$ as required. $\square$

We note if α is a root then so is $-\alpha$. Indeed we can decompose any element $z \in L$ as

$$z = h + \sum_{\beta} x_\beta,$$

so if $x_\alpha \in L_\alpha$ is non-trivial then by the previous proposition $\kappa(x_\alpha, z) = 0$ unless there exist $\gamma \in \Omega$ such that $\alpha + \gamma = 0$. The same argument also shows that there is no element $0 \neq x_\alpha \in L_\alpha$ such that $\kappa(x_\alpha, x_{-\alpha}) = 0$ for all $x_{-\alpha} \in L_{-\alpha}$. Hence the map $L_\alpha \to L_{-\alpha}^*$ given by $x_\alpha \to \kappa(x_\alpha, -)$ is an injective linear map.

Proposition 1.8. *The Killing form κ on L restricted to the Cartan subalgebra H is non-degenerate and positive definite.*

Proof. Suppose for a contradiction that there exist $x \in H$ such that for all $y \in H$ we have $\kappa(x, y) = 0$. For any $z \in L$ we can write

$$z = h_0 + \sum_{\gamma \in \Omega} x_\gamma$$

for $h_0 \in H$. So $\kappa(x, z) = \kappa(x, h) + \sum_\gamma \kappa(x, x_\gamma) = 0$ as $\kappa(H, L_\gamma) = 0$ because $\gamma \neq 0$. This shows that κ is degenerate, a contradiction as L is semisimple. Moreover if $h, \tilde{h} \in H$ and $x \in L_\alpha$ then $[h, [\tilde{h}, x]] = \alpha(h)\alpha(\tilde{h})x$, i.e

$$\kappa(h, \tilde{h}) = \sum_{\alpha \in \Omega} \alpha(h)\alpha(\tilde{h}), \ \forall h, \tilde{h} \in H.$$

This shows that κ is positive definite when restricted to H. The proposition is proved. □

The proposition shows that the linear map $H \to H^*$ given by $h \to \kappa(h, -)$ is an isomorphism. Indeed the kernel is given by $\{h \in H | \kappa(h, \tilde{h}) = 0, \forall \tilde{h} \in H\}$, which was shown to be trivial. In particular for every root $\alpha \in \Omega$ there is a unique element $h \in H$ such that $\alpha(-) = \kappa(h, -)$. For future references we will refer to this unique element as t_α for a root $\alpha \in \Omega$.

Corollary 1.6. *Let $\alpha \in \Omega$ be any root. If $x \in L_\alpha$ and $y \in L_{-\alpha}$ then $[x, y] = \kappa(x, y)t_\alpha$.*

Proof. Given an arbitrary $h \in H$ we have by associativity of κ:

$$\kappa([x, y], h) = \kappa(x, [y, h]) = -\alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y)$$

in particular this shows that

$$\kappa([x, y], h) - \kappa(t_\alpha \kappa(x, y), h) = 0 = \kappa([x, y] - t_\alpha \kappa(x, y), h).$$

Note that $[x, y] - t_\alpha \kappa(x, y) \in H$. But the restricted Killing form of L on H is non-degenerate, so that $[x, y] = t_\alpha \kappa(x, y)$ as required. □

Lemma 1.8. *Let α be a root in Ω and suppose $x \in L_\alpha$ and $y \in L_{-\alpha}$ is such that $0 \neq [x, y] = h \in L_0$. Then $\alpha(h) \neq 0$.*

Proof. Let $\beta, \alpha \in \Omega$ be roots and consider the sum

$$\tilde{L} = \sum_n L_{\beta+n\alpha}$$

where $n \in \mathbb{Z}$ runs over all possibilities such that $\beta + n\alpha \in \Omega$. This is a non-zero subspace of L and it is clear that $[H, \tilde{L}] \subset \tilde{L}$. So if $x \in L_\alpha$ and $y \in L_{-\alpha}$ are non-zero such that $h \neq 0$ then by considering the trace of the map $ad(h)$ restricted to $\tilde{L}$ we obtain:

$$tr(ad(h)) = tr([ad(x), ad(y)]) = tr(ad(x) \circ ad(y) - ad(y) \circ ad(x)) = 0$$

but this is also equal to

$$\sum_n (\beta(h) + n\alpha(h))Dim(L_{\beta+n\alpha})$$

so we get the equality

$$\beta(h) \sum_n Dim(L_{\beta+n\alpha}) = -\alpha(h) \sum_n nDim(L_{\beta+n\alpha}).$$

Now since β can be any root then $\alpha(h) \neq 0$, indeed if this is the case then for any root β we have $\beta(h) = 0$, because $Dim(L_{\beta+n\alpha}) \geq 1$, and so $h \in Z(L) = 0$ as L is semisimple, a contradiction. □

We observe the following. Choose $x \in L_\alpha$ and $y \in L_{-\alpha}$ such that $[x, y] \neq 0$ with $\alpha([x, y]) \neq 0$. Consider the subspace $\tilde{L} = \langle x, y, [x, y]\rangle$ of L. It is clear that this forms a 3-dimensional Lie subalgebra of L. Moreover by comparing the structure constants to that of $\mathfrak{sl}(2, \mathbb{C})$ w.r.t the standard basis $\{e, f, h\}$, we see that $\tilde{L} \cong \mathfrak{sl}(2, \mathbb{C})$. So for each root $\alpha \in \Omega$ we can find a copy of $\mathfrak{sl}(2, \mathbb{C})$ in L. We will denote this copy by $\mathfrak{sl}(\alpha)$, and we will set $\{e_\alpha, f_\alpha, h_\alpha\}$ for a basis of $\mathfrak{sl}(\alpha)$ chosen such that

$$h_\alpha = [e_\alpha, f_\alpha], \quad [h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha, \quad [h_\alpha, f_\alpha] = -\alpha(h_\alpha)f_\alpha$$

with $\alpha(h_\alpha) = 2$.

8. The root system of a semisimple Lie algebra

Our aim in this section is to show that the set of roots Ω of a semisimple Lie algebra L with respect to a Cartan subalgebra H form a root system on a subspace of H^* (see appendix A for a definition). We begin by introducing a root string which is strongly related to the root decomposition of L. So as before we let $L = H \oplus_{\alpha\in\Omega} L_\alpha$ be a root decomposition of L throughout this section.

Given a root $\alpha \in \Omega$ and $\beta \in \Omega \cup \{0\}$ we define the α-**root string** through β to be the subspace

$$\oplus_n L_{\beta+n\alpha} \subset L$$

where $n \in \mathbb{Z}$ runs over all possibilities where: $\beta+n\alpha$ is a root of L, i.e $\beta+n\alpha \in \Omega\cup\{0\}$. Suppose a root string $\oplus_n L_{\beta+n\alpha}$ has the form $\oplus_{a\leq n\leq b} L_{\beta+n\alpha}$ for a suitable choice of integers $a, b \in \mathbb{Z}$ such that $\beta+(b+1)\alpha$ and $\beta+(a-1)\alpha$ are not roots. Then we say that the α-root string through β is a **maximal root string**.

Clearly any root string can be decomposed into a sum of maximal root strings. This follows because the set of roots Ω is finite, so we can start by choosing a minimal $a \in \mathbb{Z}$ such that $\beta + a\alpha$ is a root. Let $a \leq b$ be the maximal integer such that $\beta + b\alpha$ is a root. Choose $a \leq a_1 \leq b$ to be the largest integer such that $\beta + k\alpha$ are roots for all $a \leq k \leq a_1$. This gives the first maximal root string. Continue to choose the minimal integer $a_1 + 1 < a_2 \leq b$ such that $\beta + a_2\beta$ is a root. So taking the largest integer $a_1 \leq a_3 \leq b$ such that $\beta + k\alpha$ are roots for all $a_1 \leq k \leq a_3$ we get a second maximal root string. We continue until we reach b. This gives a decomposition into maximal root strings.

Recall now the basis $\{e_\alpha, f_\alpha, h_\alpha\}$ chosen for $\mathfrak{sl}(\alpha)$ where α is a root of L. We now show that any root space L_α must have dimension 1.

Lemma 1.9. *$Dim(L_\alpha) = 1$ for every $\alpha \in \Omega$.*

PROOF. Consider the basis element $h_\alpha = [e_\alpha, f_\alpha] \in \mathfrak{sl}(\alpha)$ and the subspace

$$0 \neq \tilde{L} = \oplus_{n\geq 0} L_{n\alpha}$$

of the α-root string through 0 where $n\alpha$ is a root of L. We know $\tilde{L}$ has the form $\oplus_{0\leq n\leq k} L_{n\alpha}$ for some natural number $k \geq 1$. Next consider the subspace

$$\tilde{\tilde{L}} = \langle f_\alpha \rangle \oplus \tilde{L} \leq L$$

then it is easy to check that $[h_\alpha, \tilde{\tilde{L}}] \subset \tilde{\tilde{L}}$. So we can consider the linear map $ad(h_\alpha) : \tilde{\tilde{L}} \to \tilde{\tilde{L}}$, and computing the trace of $ad(h_\alpha)$ restricted to $\tilde{\tilde{L}}$ we get:

$$tr(ad(h_\alpha)) = tr(ad([e_\alpha, f_\alpha])) = 0$$

since the maps $ad(e_\alpha) \circ ad(f_\alpha)$ and $ad(f_\alpha) \circ ad(e_\alpha)$ are also linear maps on $\tilde{\tilde{L}}$. But the trace of $ad(h_\alpha)$ is also equal to

$$\sum_{0 \leq n \leq k} Dim(L_{n\alpha}) n\alpha(h_\alpha) - \alpha(h_\alpha)$$

hence

$$1 = \sum_{0 \leq n \leq k} Dim(L_{n\alpha}) n$$

as $\alpha(h_\alpha) \neq 0$. So if $Dim(L_{n\alpha}) > 1$ for all $k \geq n \geq 0$ then $1 > \frac{k(k+1)}{2} \geq 1$ as $k \geq 1$, not possible. So in particular $Dim(L_\alpha) = 1$ as required. $\square$

Following the previous proof we observe that if $k > 1$ then $1 = \frac{k(k+1)}{2} \geq 3$ i.e we must have $k = 1$. This proves that the only integers k such that $k\alpha$ is a root are $k = \pm 1$. In fact if $\lambda \in \mathbb{C}$ and $\lambda\alpha$ is a root then using the space $\tilde{\tilde{L}}$ as in the proof but with $\lambda\alpha$ instead of α we get that $\lambda = \frac{1}{N}$ for a natural number N. So $N(\lambda\alpha) = \alpha$ is a root, but then $N = \pm 1$, hence $\lambda = \pm 1$. In particular we have proved:

Corollary 1.7. *Let α be a root in Ω then the following hold.*

(1) $\lambda\alpha \in \Omega$ for some $\lambda \in \mathbb{C}$ if and only $\lambda \in \{\pm 1\}$.
(2) $\mathfrak{sl}(\alpha) = L_\alpha \oplus L_{-\alpha} \oplus [L_\alpha, L_{-\alpha}]$. $\square$

Recall the linear injective map $L_\alpha \to L^*_{-\alpha}$ given by $x \to \kappa(x, -)$. Since all root spaces have dimension 1, it follows that this map is an isomorphism. We will use this in the following corollary.

Corollary 1.8. *We can find $x \in L_\alpha$ and $y \in L_{-\alpha}$ such that $[x, y] \neq 0$ and $[x, y] = t_\alpha$ where $\kappa(t_\alpha, -) = \alpha(-)$.*

PROOF. Consider the linear functional $\gamma : L_{-\alpha} \to L_{-\alpha}$ given by $\gamma(x_{-\alpha}) = 1$ where $\{x_{-\alpha}\}$ is a basis for $L_{-\alpha}$. Now there must be some $0 \neq x_\alpha \in L_\alpha$ with $\kappa(x_\alpha, -) = \gamma(-)$ as $\gamma \neq 0$, hence $\kappa(x_\alpha, x_{-\alpha}) = \gamma(x_{-\alpha}) = 1$. Therefore $[x_\alpha, x_{-\alpha}] = \kappa(x_\alpha, x_{-\alpha})t_\alpha = t_\alpha$ for the unique $0 \neq t_\alpha \in H$ as required. $\square$

It follows from our previous results that the α-root string through 0 is maximal, it has the form $L_{-\alpha} \oplus H \oplus L_\alpha$. In fact it turns out that any α-root string $\tilde{L}$ through a root β is a maximal root string, i.e: $\tilde{L} = \oplus_{a \leq n \leq b} L_{\beta + n\alpha}$. Moreover the integers $a, b \in \mathbb{Z}$ are strongly related to the basis element $h_\alpha \in \mathfrak{sl}(\alpha)$. We have the following.

Corollary 1.9. *For any choice of a root $\alpha \in \Omega$ and $\beta \in \Omega \cup \{0\}$ the α-root string through β is a maximal root string, i.e $\oplus L_{\beta+n\alpha} = \oplus_{a \leq n \leq b} L_{\beta+n\alpha}$ for suitable $a, b \in \mathbb{Z}$.*

In fact we have $a+b=-\beta(h_\alpha)$ where h_α is the basis element $[e_\alpha, f_\alpha]$ of $\mathfrak{sl}(\alpha)$ for which $\alpha(h_\alpha)=2$.

PROOF. Let H be the Cartan subalgebra of L associated with the roots Ω. Let $\tilde{L}$ be an α-root string through β of L written as a direct sum of maximal root strings $\{\tilde{L}_k\}_k$. Then we note that $ad(H)(\tilde{L}_k) \subset \tilde{L}_k$ since $[H, L_{\beta+n\alpha}] \subset L_{\beta+n\alpha}$ for any root $\beta+n\alpha \in \Omega$. Now we know that there exist $0 \neq h = [x,y]$ for $x \in L_\alpha$ and $y \in L_{-\alpha}$ with $\alpha(h) \neq 0$. So we can consider the trace of the matrix of the map $ad(h)$ restricted to a maximal string root string of $\tilde{L}$, say $\tilde{L}_k$. Write $\tilde{L}_k = \oplus_{a \leq n \leq b} L_{\beta+n\alpha}$. Denote $ad_k(h) : \tilde{L}_k \to \tilde{L}_k$ for the restriction. Hence we have

$$tr(ad_k(h)) = \sum_{a \leq n \leq b} (\beta(h) + n\alpha(h)) Dim(L_{\beta+n\alpha}) = \sum_{a \leq n \leq b} (\beta(h) + n\alpha(h)),$$

as $Dim(L_{\beta+n\alpha}) = 1$ for $\beta+n\alpha \in \Omega$. But we also have

$$tr(ad_k(h)) = tr(ad_k([x,y])) = tr(ad(x) \circ ad(y) - ad(y) \circ ad(x)) = 0$$

so we deduce that

$$0 = \beta(h)(b-a+1) + \alpha(h) \sum_{k=0}^{b-a} (n+a) = (b-a+1)\beta(h) + \alpha(h)(a(b-a+1) + \frac{(b-a)(b-a+1)}{2})$$

hence

$$0 = 2\beta(h) + \alpha(h)(a+b)$$

which gives

$$-2\frac{\beta(h)}{\alpha(h)} = a+b.$$

This shows that the α-root string through β is a maximal root string. In particular if we set $h = h_\alpha$ then as $\alpha(h_\alpha) = 2$ we get the last statement. $\square$

Recall that we can choose $x \in L_\alpha$ and $y \in L_{-\alpha}$ such that $[x,y] = t_\alpha$ where $\kappa(t_\alpha, -) = \alpha(-)$. Let $\gamma \in \Omega$ then we can write $-2\frac{\gamma(t_\alpha)}{\alpha(t_\alpha)} = a_\gamma + b_\gamma$ where $a_\gamma, b_\gamma \in \mathbb{Z}$. In particular we have the following.

Corollary 1.10. *$\beta(t_\alpha) \in \mathbb{Q}$ for all $\beta \in \Omega$.*

PROOF. We note that

$$\kappa(t_\alpha, t_\alpha) = \sum_{\gamma \in \Omega} \gamma(t_\alpha)^2 = \alpha(t_\alpha)$$

which is equivalent to

$$\alpha(t_\alpha) = \sum_\gamma \alpha(t_\alpha)^2 (-\frac{1}{2}(a_\gamma + b_\gamma))^2 = \frac{1}{4}\alpha(t_\alpha)^2 \sum_\gamma (a_\gamma + b_\gamma)^2.$$

Now since $\alpha(t_\alpha) \neq 0$ this shows that $\alpha(t_\alpha) \in \mathbb{Q}$ and is > 0. In particular $\beta(t_\alpha)$ must be in $\mathbb{Q}$ for every $\beta \in \Omega$ as well. The proof is complete. $\square$

Consider now a vector space V and $W \leq V$ then we define the **annihilator** of W to be the subspace

$$W^\circ = \{\alpha \in V^* | \alpha(W) = 0\} \leq V^*.$$

We can define a linear map $W^\circ \to (V/W)^*$ by sending $\alpha \to \alpha + W$ where $(\alpha + W)(x + W) = \alpha(x)$ for all $x \in V$. This is clearly an isomorphism, and we get the result $Dim(W^\circ) = Dim(V) - Dim(W)$. So in particular we can consider a subspace $\tilde{W} \leq V^*$ and the annihilator $\tilde{W}^\circ \leq V^{**}$. Consider the isomorphism $\epsilon : V \to V^{**}$ given by the evaluation map, i.e $\epsilon(v)(\alpha) = \alpha(v)$ for all $v \in V$ and $\alpha \in V^*$. Then the annihilator $\tilde{W}^\circ$ under ϵ^{-1} is identified with the set $\{v \in V | \alpha(v) = 0, \ (\forall \alpha \in \tilde{W})\}$. We will use this result in the next proposition.

Proposition 1.9. *Ω spans H^*, i.e $\langle \Omega \rangle = H^*$. In particular $|\Omega| \geq Dim(H)$.*

PROOF. Let $\langle \Omega \rangle^\circ \cong \{h \in H | \alpha(h) = 0, \forall \alpha \in \langle \Omega \rangle\} \leq H$ be the annihilator of $\langle \Omega \rangle$ then $\langle \Omega \rangle^\circ = 0$. Indeed if not then for some $0 \neq h$ we have $\alpha(h) = 0$ for all $\alpha \in \Omega$. Hence given any $z \in L$ we can write $[h, z] = \sum_{\alpha \in \Omega} \alpha(h) x_\alpha = 0$ where $x_\alpha \in L_\alpha$, so that $h \in Z(L)$. But since L is semisimple then $Z(L) = 0$, a contradiction. In particular

$$Dim(\langle \Omega \rangle^\circ) + Dim(\langle \Omega \rangle) = Dim(H^*).$$

This shows that $Dim(H^*) = Dim(\langle \Omega \rangle)$, hence Ω spans H^* as required. $\square$

Now since Ω spans H^* we can choose a basis $S \subset \Omega$ of H^* and consider the real span of S, namely the real subspace $V = \langle S \rangle \leq H^*$. Moreover we can define an inner product $\mathbf{b}$ on V as follows: Given $\alpha, \beta \in \Omega$ set

$$\mathbf{b}(\alpha, \beta) = \kappa(t_\alpha, t_\beta) = \alpha(t_\beta) \in \mathbb{R}.$$

Here $t_\beta, t_\alpha \in H$ are the unique elements such that $\kappa(t_\alpha, -) = \alpha(-)$ and $\kappa(t_\beta, -) = \beta(-)$. This is justified since the Killing form restricted to H is positive definite.

Lemma 1.10. *Let $\alpha, \beta \in \Omega$ then the following hold.*

(1) $\beta(h_\alpha) = 2\frac{\mathbf{b}(\beta,\alpha)}{\mathbf{b}(\alpha,\alpha)} \in \mathbb{Z}$.
(2) $\beta - \beta(h_\alpha)\alpha \in \Omega$ where $h_\alpha = [e_\alpha, f_\alpha]$ is the basis element of $\mathfrak{sl}(\alpha)$.
(3) If $\alpha + \beta \in \Omega$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

PROOF. We note that

$$\beta(h_\alpha) = -(a + b) = 2\frac{\beta(t_\alpha)}{\alpha(t_\alpha)} = 2\frac{\mathbf{b}(\beta, \alpha)}{\mathbf{b}(\alpha, \alpha)} \in \mathbb{Z}$$

where a, b are the integers corresponding to the α-root string through β namely, $\oplus L_{\beta+n\alpha} = \oplus_{a\leq n\leq b} L_{\beta+n\alpha}$. This shows case (1). Next we claim that $a \leq a+b \leq b$ as then (2) would follow. Note first that $b \geq 0$ as β is a root of L, similarly $a \leq 0$. If $a \leq -1$ then clearly it holds, since $a \leq a+b \leq b-1 < b$. Now if $a = 0$ then it also holds, since $a \leq a+b = b \leq b$. This proves (2). For the last case, we already know that $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ so it is enough to show that the space is non-zero, as $Dim(L_{\alpha+\beta}) = 1$. Now consider the subspace

$$\hat{L} = \langle f_\alpha \rangle \oplus [L_\alpha, L_\beta] \subset L$$

where β, α are non-zero. If one of them is zero then the result clearly follows. We have that $[h_\alpha, \hat{L}] \subset \hat{L}$. So computing the trace of the map $ad(h_\alpha)$ when restricted to $\hat{L}$ we get

$$tr(ad(h_\alpha)) = 0 = -\alpha(h_\alpha) + Dim([L_\alpha, L_\beta])(\alpha(h_\alpha) + \beta(h_\alpha)).$$

This shows that $[L_\alpha, L_\beta] \neq 0$, since $\alpha(h_\alpha) \neq 0$. The proof of the lemma is complete. □

Now let $V \subset H^*$ be a subspace for which Ω spans V, equipped with the inner product **b**. Then part (2) of the previous lemma shows for each $\alpha \in \Omega$, that the image of the reflection, $s_\alpha : V \to V$, in the hyperplane normal to α when restricted to Ω is contained in Ω. Moreover it is clear that it is a bijection since $L_\alpha \neq L_\beta$ for all roots $\alpha \neq \beta \in \Omega$. So finally we derive our theorem that Ω forms a root system on V.

Theorem 1.10. *Ω forms a root system on $V \leq H^*$ where V is equipped with the inner product $\boldsymbol{b} : V \times V \to \mathbb{R}$ given by*

$$\boldsymbol{b}(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$$

for all $\alpha, \beta \in \Omega$. □

Before we end this chapter we introduce a useful basis for L. We will use this basis when we introduce real forms, in particular using the root decomposition of L, it will reveal the existence of a compact and a split real form of L. These are special real Lie algebras contained in L of the same dimension.

So consider now $S \subset \Omega$, then the **hull** of S, $\hat{S}$, is defined to be the set of all $\pm\alpha$ and $\pm(\alpha+\beta)$ such that $\alpha, \beta \in \Omega$. Moreover let $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$ be chosen such that $\kappa(x_\alpha, x_{-\alpha}) = 1$ for each $\alpha \in \Omega$. We define a function $\lambda : \hat{S} \times \hat{S} \to \mathbb{C}$ by $\lambda(\alpha, \beta) = 0$ if $\alpha + \beta \notin \Omega$ while $[x_\alpha, x_\beta] = \lambda(\alpha, \beta) x_{\alpha+\beta}$ if $\alpha + \beta \in \hat{S}$.

For a complete proof of the following theorem we refer to [**1**], chapter 3, section 5.

Theorem 1.11. *There exist for each root $\alpha \in \Omega$ elements $x_\alpha \in L_\alpha$ satisfying the following conditions.*
(1) $[x_\alpha, x_{-\alpha}] = t_\alpha$ for the unique $\kappa(t_\alpha, -) = \alpha(t_\alpha)$, so $\kappa(x_\alpha, x_{-\alpha}) = 1$.
(2) $[x_\alpha, x_\beta] = \lambda(\alpha, \beta)x_{\alpha+\beta}$.
(3) $\lambda(\alpha, \beta) = -\lambda(-\alpha, -\beta)$.
(4) $\lambda(\alpha, \beta) \in \mathbb{R}$.

We can choose a basis $S = \{t_\alpha | \alpha \in \Omega, \alpha \neq -\alpha\} \cup \{x_\alpha | \alpha \in \Omega\}$ for L with the properties of the previous theorem. This is often called a **Cartan Weyl basis**.

Consider now the real subalgebra $H_\mathbb{R} \leq H$ spanned by the set $\{t_\alpha | \alpha \in \Omega, \alpha \neq -\alpha\}$. Since the inner product $\mathbf{b}$ is defined by $\mathbf{b}(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$ for roots $\alpha, \beta \in \Omega$ then it follows that κ restricted to $H_\mathbb{R}$ is clearly an inner product. In fact it turns out that every element in the Cartan subalgebra H can be written uniquely as the sum $h + i\tilde{h}$ where $h, \tilde{h} \in H_\mathbb{R}$. We say H is the **complexification** of $H_\mathbb{R}$, this is written

$$H = H_\mathbb{R}^\mathbb{C} = \{h + i\tilde{h} | h, \tilde{h} \in H_\mathbb{R}\}.$$

To see that this is true we note that if we consider $H_\mathbb{R}^\mathbb{C}$ as a real Lie algebra then it is a direct sum $H_\mathbb{R} \oplus iH_\mathbb{R}$. This follows because κ is real on $H_\mathbb{R}$. We call $H_\mathbb{R} \oplus iH_\mathbb{R}$ the **realification** of H. It is therefore enough to show that H is equal to the set $H_\mathbb{R}^\mathbb{C}$. But this is clear since we have seen that H^* is spanned by the set of roots Ω. In particular since the map $H \to H^*$ given by $t_\alpha \to \kappa(t_\alpha, -) = \alpha(-)$ is an isomorphism, then H is spanned by $\{t_\alpha | \alpha \in \Omega, \alpha \neq -\alpha\}$.

So we have proved that:

Proposition 1.10. $H_\mathbb{R}^\mathbb{C} = H$. $\square$

We end the classification theory of semisimple complex Lie algebras by stating the classification theorem of simple complex Lie algebras. This theorem is a consequence of the root system constructed for semisimple complex Lie algebras. For a partial proof of this theorem we refer to [**3**], chapters 11-14.

Theorem 1.12. *The complete list of simple complex Lie algebras are:*

***(1)[Classical Lie algebras]**: $\mathfrak{sl}(n, \mathbb{C})$ for $n > 1$, $\mathfrak{so}(2n, \mathbb{C})$ for $n \neq 1, 2$, $\mathfrak{so}(2n+1, \mathbb{C})$ for $n \geq 1$ and $\mathfrak{sp}(2n, \mathbb{C})$ for $n \geq 1$.*
***(2)[Exceptional Lie algebras]**: e_6, e_7, e_8, f_4 and g_2.*

CHAPTER 2

Matrix groups

1. A matrix group

Most of this chapter is based on [**4**] and [**1**].

Let $\mathbb{K}$ be either the real or the complex numbers $\mathbb{R}, \mathbb{C}$. Throughout this chapter we always assume $M(n, \mathbb{K})$ is equipped with the usual norm metric $||,||$ where $||A|| = sup\{|Ax||x \in \mathbb{K}^n, |x| = 1\}$ for $A \in M(n, \mathbb{K})$. So that we have a natural topology on $M(n, \mathbb{K})$.

In particular we note that a sequence $(a_n)_{n\in\mathbb{N}}$ in $M(n, \mathbb{C})$ converges to a matrix A if and only if the matrix entries of $(a_n)_{n\in\mathbb{N}}$ converges to the matrix entries of A. One also see that the norm $||,||$ on $M(n, \mathbb{C})$ extends the norm on $M(n, \mathbb{R})$. Indeed let $A \in M(n, \mathbb{R})$ then clearly $||A||_{\mathbb{R}} \leq ||A||_{\mathbb{C}}$. But consider the standard basis $\{e_j\}_j$ of $\mathbb{C}^n$. Then for $v \in \mathbb{C}^n$ with norm 1, we can write $v = \sum_j \lambda_j e_j$ for $|\lambda_j| \leq 1$. So $|Av|_{\mathbb{C}} \leq |Ae_j|_{\mathbb{R}} \leq ||A||_{\mathbb{R}}$. This shows that $||A||_{\mathbb{C}} \leq ||A||_{\mathbb{R}}$.

Definition 2.1. A subgroup $G \leq GL(n, \mathbb{K})$ for some $n \geq 1$ is said to be a **matrix group** if G is closed with respect to the induced subspace topology.

Definition 2.2. Let G be a matrix group then a matrix subgroup H is a subgroup $H \leq G$ which is closed with respect to the induced subspace topology.

Not every subgroup H of a matrix group G is a matrix subgroup. For instance we can identify $\mathbb{R}$ as the matrix group consisting of diagonal entries in $GL(2, \mathbb{R})$. Now the subgroup $\mathbb{Q}$ of rational diagonal entries is clearly not closed in $\mathbb{R}$. Take for instance the sequence $(a_n)_{n\in\mathbb{N}} = \begin{pmatrix} \frac{[n\sqrt{5}]}{n} & 0 \\ 0 & \frac{[n\sqrt{5}]}{n} \end{pmatrix}$ in $\mathbb{Q}$ where $[,]$ is the floor function. Then $a_n \to_{n\to\infty} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \notin \mathbb{Q}$.

The following is immediate.

Proposition 2.1. *If G is a matrix group and H is a matrix subgroup then H is also a matrix group.* $\square$

Here are some examples of matrix groups:

Example 2.1.

- The general linear group: $GL(n, \mathbb{R})$.
- The special linear group: $SL(n, \mathbb{R}) = \{X \in GL(n, \mathbb{R}) | det(X) = 1\}$.
- The orthogonal group: $O(n, \mathbb{R}) = \{X \in GL(n, \mathbb{R}) | XX^t = I\}$.
- The special orthogonal group: $SO(n, \mathbb{R}) = \{X \in O(n, \mathbb{R}) | det(X) = 1\}$.

Since we have the notion of convergence of matrices w.r.t to the norm $||,||$, then we can consider power series of matrices $s(X) = \sum_{0 \le n < \infty} z_n X^n$ where $(z_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{C}$ and $X \in M(n, \mathbb{C})$. In particular we may define the **exponential** of a matrix $X \in M(n, \mathbb{C})$ to be the power series

$$e^X = \sum_{0 \le n < \infty} \frac{1}{n!} X^n.$$

To see that it is well-defined, let $A_N = \sum_{0 \le n \le N} \frac{1}{n!} X^n$ then:

$$||A_N - A_M|| \le |a_N - a_M|$$

where $a_N = \sum_{0 \le n \le N} \frac{1}{n!} ||X||^n$. So because $a_N \to e^{||X||}$ as $N \to \infty$ then A_N is a Cauchy sequence. The exponential function turns out to be a very important tool in the theory of matrix groups, as we will see it will relate a matrix group G to a matrix Lie algebra $Lie(G)$ via the exponential map.

Proposition 2.2.

(1) If $X, Y \in M(n, \mathbb{C})$ commute i.e $[X, Y] = 0$ then $e^{X+Y} = e^X e^Y$.
(2) The exponential map $exp : M(n, \mathbb{C}) \to M(n, \mathbb{C})$ by $exp(X) = e^X$ is continuous.

If $[X, Y] \neq 0$ then case (1) may fail, as the following example illustrates:
Take $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then $X^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n > 0$ and so $e^X = \begin{pmatrix} 1+e & 1+e \\ 1 & 1+e \end{pmatrix}$. Also let $Y = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$ then $Y^{2n} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $Y^{2n+1} = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$ for $n > 0$, so $e^Y = I$.
Now $[X, Y] \neq 0$ and $e^{X+Y} = \begin{pmatrix} 1+e & 0 \\ 0 & 0 \end{pmatrix}$ but $e^X e^Y = e^X \neq e^{X+Y}$.

We see that if X is any matrix in $M(n, \mathbb{C})$ then X commutes with $-X$ so that $I = e^X e^{-X}$, hence the image under the exponential map is contained in $GL(n, \mathbb{C})$.

2. Differentiable curves and one parameter subgroups

Definition 2.3. Let G be a matrix group and $0 \in (a,b)$ be an open interval. Then a differential curve in G is a map $\gamma : (a,b) \to G$ which is differentiable, i.e the limit

$$\gamma'(t_0) = lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists in $M(n, \mathbb{K})$ for all $t_0 \in (a,b)$. A curve γ in G at X is a curve with $\gamma(0) = X$.

If $X \in GL(n, \mathbb{C})$ then the map $t \to e^{tX}$ is a curve in $GL(n, \mathbb{C})$. We show that the map is differentiable at $t = 0$. By definition we have,

$$(e^{tX})'(0) = lim_{t \to 0} \frac{e^{tX} - I}{t} = X + lim_{t \to 0} \sum_{1 \leq n < \infty} \frac{(tX)^n}{(n+1)!}$$

but $|| \sum_{1 \leq n < \infty} \frac{(tX)^n}{(n+1)!} || \leq \sum_{1 \leq n < \infty} \frac{t^n ||X||^n}{(n+1)!} \to 0$ as $t \to 0$. This follows because the power series $S(t) = \sum_{1 \leq n < \infty} \frac{||X||^n}{(n+1)!} t^n$ in $\mathbb{R}$ has radius of converge $R = \infty$, and so is continuous with $S(0) = 0$. We deduce that $(e^{tX})'(0) = X$.

It is worth noting that if γ is a map $(a,b) \to G$ then writing $\gamma(t) = (\gamma_{ij}(t))_{ij}$ in matrix form, γ is differentiable at t if and only if $\gamma_{ij}(t)$ are differentiable at t as maps $(a,b) \to \mathbb{K}$. Also if $\gamma'(t)$ exists then $\gamma'(t) = (\gamma'_{ij}(t))_{ij}$.

Lemma 2.1. *Let G be a matrix group and $\gamma : (a,b) \to G$ a map in G. Then γ is a differentiable curve in G if and only if the following is satisfied: For any $t_0 \in (a,b)$ we can find $Y \in M(n, \mathbb{K})$ and a continuous map $\epsilon : (a,b) \to M(n, \mathbb{K})$ such that $lim_{t \to t_0} \epsilon(t) = 0$ together with*

$$\gamma(t) = \gamma(t_0) + (t - t_0)Y + (t - t_0)\epsilon(t)$$

for all $t \in (a,b)$.

Proof. Suppose $\gamma : (a,b) \to G$ is a curve in G. Define the function $\epsilon : (a,b) \to M(n, \mathbb{K})$ by setting $\epsilon(t) = \frac{\gamma(t) - \gamma(t_0)}{t - t_0} - Y$ where $Y = \gamma'(t_0) \in M(n, \mathbb{K})$. So the statement follows. The converse is clear. □

Corollary 2.1. *Let G be a matrix group and $\alpha, \beta : (a,b) \to G$ be differentiable curves in G then the following hold.*

(1) $(\alpha + \beta)'(0) = \alpha'(0) + \beta'(0)$ (Sum rule).
(2) $(\alpha\beta)'(t) = \alpha'(t)\beta(t) + \alpha(t)\beta'(t)$ (Product rule).
(3) $(\alpha^{-1})'(t) = -\alpha(t)^{-1}\alpha'(t)\alpha(t)^{-1}$ (Quotient rule).

Proof. Since α, β are differentiable curves in G then given $t_0 \in (a,b)$ we can write

$$\alpha(t) = \alpha(t_0) + (t - t_0)Y + (t - t_0)\epsilon(t)$$

and similarly

$$\beta(t) = \beta(t_0) + (t - t_0)\tilde{Y} + (t - t_0)\tilde{\epsilon(t)}$$

for suitable functions $\epsilon, \tilde{\epsilon} : (a,b) \to M(n, \mathbb{K})$ which both tend to 0 as $t \to 0$ and $Y, \tilde{Y} \in M(n, \mathbb{K})$. By taking the product and the sum of these equalities, cases (1) and (2) follows immediately. Case (3) follows from case (2) by noting that $(\alpha(t)\alpha(t)^{-1})'(0) = 0 = \alpha'(t)\alpha(t)^{-1} + \alpha(t)(\alpha^{-1})'(t)$. $\square$

Definition 2.4. Let G be a matrix group and γ be a continuous homomorphism $\mathbb{R} \to_\gamma G$, differentiable at $t = 0$. Then γ is said to be a **one parameter subgroup** of G. Moreover if $\mathbb{R}$ is replaced by an interval of the form $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ and $\gamma(t+s) = \gamma(t)\gamma(s)$ whenever $|t + s| < \epsilon$ then γ is said to be a **one parameter semi-subgroup** of G.

Note that any one parameter subgroup is differentiable with derivative $\gamma'(t) = \gamma(t)\gamma'(0)$.

As an example the exponential curve $\mathbb{R} \to GL(n, \mathbb{K})$ by $t \to e^{tX}$ for any choice of $X \in GL(n, \mathbb{K})$ is a one parameter subgroup of $GL(n, \mathbb{K})$. This is in fact the only one.

Theorem 2.1. *Let G be a matrix group then any one parameter subgroup $\mathbb{R} \to_\gamma G$ has the form $\gamma(t) = e^{At}$ for some $A \in G$.*

Proof. Suppose γ has initial value $\gamma'(0) = A$ then it satisfies the differential equation $\gamma'(t) = \gamma'(0)\gamma(t)$, and so does the exponential curve $\beta(t) = e^{tA}$. We claim that γ must be unique. To see this we note that $\frac{d}{dt}_{|t=t_0} e^{-At}\alpha(t) = 0$ at any choice of $t_0 \in \mathbb{R}$. Hence $e^{-At}\alpha(t) = C$ for some $C \in M(n, \mathbb{K})$, so as $\alpha(0) = I$ we get $C = I$ and uniqueness follows. This proves the result. $\square$

Definition 2.5. Let $G, H \subseteq GL(n, \mathbb{K})$ be matrix groups and $G \to_\psi H$ be a continuous group homomorphism. Then we say ψ is a **Lie homomorphism** or just **smooth** if the following two conditions are satisfied:

(1) If $I \to_\gamma G$ is a differentiable curve in G then the map $\psi \circ \gamma$ is a differentiable curve in H.
(2) Given any two differentiable curves γ, α at $X \in G$ such that $\gamma'(0) = \alpha'(0)$ then $(\psi \circ \gamma)'(0) = (\psi \circ \alpha)'(0)$.

Moreover if ψ is in addition bijective and ψ^{-1} is also a Lie homomorphism then we say that ψ is a Lie isomorphism. We write in this case $G \cong H$, and say they are **Lie isomorphic**.

Consider the maps $R_g, L_g, C_g : G \to G$ given by $R_g(h) = hg$ and $L_g(h) = gh$ together with $C_g(h) = g^{-1}hg$. These are important examples of Lie isomorphisms of G. If γ is a differentiable curve at $1 \in G$ then obviously we have:

$$(R_g \circ \gamma)'(0) = \gamma'(0)g, \ (L_g \circ \gamma)'(0) = g\gamma'(0), \ and \ (C_g \circ \gamma)'(0) = g^{-1}\gamma'(0)g.$$

Similarly the inverse map $I : G \to G$ by $g \to g^{-1}$ is also an example of a Lie isomorphism of G with $(I \circ \gamma)'(0) = -\gamma'(0)$.

The following is immediate.

Corollary 2.2. *Let $G, H \subseteq GL(n, \mathbb{K})$ be matrix groups and $G \to_\psi H$ be a continuous group homomorphism. Then ψ is smooth if and only if for every differentiable curve $(a, b) \to_\gamma G$ the following is satisfied: For every $a < t_0 < b$ we can find $Y \in M(n, \mathbb{K})$ and a continuous map $\epsilon : (a, b) \to M(n, \mathbb{K})$ such that $\epsilon(t) \to 0$ as $t \to t_0$ where*

$$\psi(\gamma(t)) = \psi(\gamma(t_0)) + (t - t_0)Y + (t - t_0)\epsilon(t)$$

for all $t \in (a, b)$. $\square$

Corollary 2.3. *If $G, H \subseteq GL(n, \mathbb{K})$ are matrix groups and $G \to_\psi H$ is a smooth map and $\gamma, \alpha, \beta : (a, b) \to G$ are differentiable curves then the following is true.*

(1) If $\tau(t) = \psi(\gamma(\lambda t))$ where $\lambda t \in (a, b)$ then $\tau'(t) = \lambda(\psi \circ \gamma)'(t)$.
(2) If $\tau(t) = \psi(\alpha(t)\beta(t))$ then $\tau'(0) = (\psi \circ \alpha)'(0) + (\psi \circ \beta)'(0)$.

Now given matrix groups $G, H \leq GL(n, \mathbb{K})$ then $G \times H$ is also easily seen to be a matrix group. Indeed let $\psi : G \to GL(n, \mathbb{K})$ and $\tilde{\psi} : H \to GL(n, \mathbb{K})$ be continuous monomorphisms of matrix groups. Then we can take a pair $(g, h) \in G \times H$ and send it to the matrix $\begin{pmatrix} \psi(g) & 0 \\ 0 & \tilde{\psi}(h) \end{pmatrix} \in GL(2n, \mathbb{K})$. This defines a group monomorphism

$$\psi \times \tilde{\psi} : G \times H \to GL(2n, \mathbb{K}),$$

and if we equip the image of $\psi \times \tilde{\psi}$ under $G \times H$ with the induced topology from $GL(2n, \mathbb{K})$ then the image is clearly closed. In particular G, H can be considered as matrix subgroups of $G \times H$.

Another observation is that the identity component G_0 of a matrix group G is a normal matrix subgroup of G. To see this we can make use of the Lie isomorphism $L_g : G \to G$ for $g \in G$. So by continuity $L_g(G_0) = gG_0$ is a connected component of G. In particular if $h, g \in G_0$ we have $g^{-1}G_0 = G_0 = h^{-1}G_0$ and so $(gh)G_0 = gG_0 = G_0$. This shows that G_0 is a subgroup of G. Now since G_0 is a connected component of G then it closed. Moreover if $g \in G$ then using the conjugation map $g^{-1}G_0g$ is a connected component of G. Moreover $g^{-1}G_0g = G_0$ since $1 \in G_0$ which proves that $G_0 \trianglelefteq G$. In particular we see that every connected component of G has the form gG_0 for some $g \in G_0$.

So we have shown that:

Proposition 2.3. *The identity component G_0 of a matrix group G is a normal matrix subgroup of G.* $\square$

3. The Lie algebra of a matrix group

In this section we show that to every matrix group there is a Lie algebra attached to it.

Definition 2.6. Let $G \subseteq GL(n, \mathbb{K})$ be a matrix group and suppose $x \in G$. Then the **tangent space** at x is defined to be the set

$$T_xG = \{\gamma'(0) | I \to_\gamma G \; is \; a \; differentiable \; curve \; at \; x\} \subset M(n, \mathbb{K}),$$

where $0 \in I$ is some open interval in $\mathbb{R}$.

Proposition 2.4. *Let G be a matrix group of $GL(n, \mathbb{K})$ and suppose $x \in G$ then the tangent space T_xG at x forms a real vector subspace of $M(n, \mathbb{K})$.*

Proof. Let $\alpha : I_1 \to G, \beta : I_2 \to G$ be differentiable curves at x in G with $\alpha'(0) = a$ and $\beta'(0) = b$. Now consider $\gamma : I \to G$ to be the differentiable curve

$$\gamma(t) = \alpha(t)\beta(0)^{-1}\beta(t)$$

in G at x, where I is some interval contained in $I_1 \cap I_2$ with $o \in I$. Then by the product rule

$$\gamma'(0) = \alpha'(0) + \beta'(0) = a + b.$$

So that if $a, b \in T_xG$ then so is $a + b$. Let now $\lambda \in \mathbb{R}$ be given, and let (a, b) be an interval such that $\lambda t \in I_1$ for all $t \in (a, b)$. Define a new differentiable curve $\tau : (a, b) \to G$ by $\tau(t) = \alpha(\lambda t)$ at x. Then $\tau'(0) = \lambda\gamma'(0) = \lambda a$. Hence $\lambda a \in T_xG$. Now the constant curve at x in G is clearly differentiable with derivative 0 at $t = 0$, i.e $0 \in T_xG$. This shows that $T_xG \leq M(n, \mathbb{K})$. $\square$

Note that when G is a complex matrix group then the proof only show that T_xG is a real vector subspace of $M(n, \mathbb{C})$. Since in the proof the curve $\tau(t) = \gamma(\lambda t)$ for which $\lambda \in \mathbb{C}$ does not make sense. There are in fact cases where T_xG is only a real vector space and not a complex one, we will see examples of this when we introduce real forms.

Lemma 2.2. *Any vector subspace of a normed* $\mathbb{K}$*-vector space is closed.*

Proof. See for example [**4**], section 1.8. □

Proposition 2.5. *Let* $G \subseteq GL(n,\mathbb{R})$ *be a matrix group then the tangent space* T_1G *at the identity forms a real Lie subalgebra of* $\mathfrak{gl}(n,\mathbb{R})$*. Moreover if* $G \subseteq GL(n,\mathbb{C})$ *is a complex matrix group and* T_1G *is a complex subspace of* $M(n,\mathbb{C})$ *then* T_1G *is a complex Lie subalgebra of* $\mathfrak{gl}(n,\mathbb{C})$*.*

Proof. Suppose $I_1 \to_\alpha G$ and $I_2 \to_\beta G$ are curves at 1 in G with $\alpha'(0) = a$ and $\beta'(0) = b$. Now define the curve $\tau = \alpha(t)b\alpha(t)^{-1}$ in $M(n,\mathbb{K})$ at b on I_1. Then τ has derivative $[a,b]$ at $t = 0$. Now fix any $t \in (-\epsilon,\epsilon) \subset I_1 \cap I_2$ for some $\epsilon > 0$. Then the curve

$$\gamma_t(s) = \alpha(t)\beta(s)\alpha(t)^{-1}$$

at 1 in G has derivative $(\gamma_t)'(0) = \alpha(t)\beta'(0)\alpha(t)^{-1}$. Hence $\frac{\alpha(t)\beta'(0)\alpha(t)^{-1}-\beta'(0)}{t} \in T_1G$ for any $t \in (-\epsilon,\epsilon)$ since $T_1G \leq M(n,\mathbb{K})$. So it follows that

$$lim_{t\to 0}\frac{\alpha(t)\beta'(0)\alpha(t)^{-1} - \beta'(0)}{t} = lim_{t\to 0}\frac{\tau(t) - \tau(0)}{t} = \tau'(0) = [a,b].$$

But because T_1G is closed in $M(n,\mathbb{K})$ then $[a,b] \in T_1G$ as required. This shows that T_1G is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{K})$. □

Observe even though the tangent space at the identity of a complex matrix group is not necessarily complex, it will always be a real Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. So we can always think of a Lie algebra attached to a complex matrix group as well. We have the following definition.

Definition 2.7. Let G be a matrix group then the tangent space T_1G at $1 \in G$ is defined to be the Lie algebra of the matrix group G, denoted $Lie(G)$.

Example 2.2. The Lie algebra of $GL(n,\mathbb{K})$ is just $\mathfrak{gl}(n,\mathbb{K})$. Clearly $Lie(GL(n,\mathbb{K})) \subseteq \mathfrak{gl}(n,\mathbb{K})$ by definition. Moreover if $X \in \mathfrak{gl}(n,\mathbb{K})$ then consider the exponential curve $t \to e^{tX}$ in $GL(n,\mathbb{K})$. It has derivative X at $t = 0$. Hence $\mathfrak{gl}(n,\mathbb{K}) \subseteq Lie(GL(n,\mathbb{K})$.

Following the proof of the previous proposition we see that if our matrix group G is abelian then so is the Lie algebra T_1G. Indeed if α,β are curves at $1 \in G$ with $\alpha'(0) = x$ and $\beta'(0) = y \in Lie(G)$ then the curve $\tau_s(t) = \alpha(s)\beta(t)\alpha^{-1}(s)$ has derivative $\tau_s'(0) = \alpha(s)y\alpha(s)^{-1} \in Lie(G)$. But $\tau_s'(0)$ has derivative $[x,y]$ at $s = 0$. So because G is abelian then $\tau_s(t) = \beta(t)$ and $\tau_s'(0) = y$ therefore $[x,y] = 0$ as required. We state this as a corollary.

Corollary 2.4. *If $G \subseteq GL(n, \mathbb{K})$ is an abelian matrix group with Lie algebra $Lie(G)$ then $Lie(G)$ is abelian.* $\square$

Consider the matrix group $G \times H \leq GL(2n, \mathbb{K})$ described earlier. It is straight forward to show that $Lie(G \times H) \cong Lie(G) \times Lie(H)$, here the Lie bracket on the direct product is defined by: $[(x, y), (a, b)] = ([x, a], [y, b])$ for all $x, a \in Lie(G)$ and $y, b \in Lie(H)$.

Definition 2.8. [Differential/pushforward]. Let $G, H \subseteq GL(n, \mathbb{K})$ be matrix groups and $G \to_{\psi} H$ be a Lie homomorphism then we define the differential of ψ at $g \in G$ to be the map

$$d\psi_g : T_g G \to T_{\psi(g)} H$$

by $d\psi_g(\gamma'(0)) = (\psi \circ \gamma)'(0)$. In the case of $g = 1$ we just write $d\psi$ for the differential.

Note that by definition of a Lie homomorphism the differential is well defined. It turns out that the differential is $\mathbb{R}$-linear. Indeed if γ is a differentiable curve at g in G, then we can adjust the interval such that $\tau = \gamma(\lambda t)$ for $\lambda \in \mathbb{R}$ is a curve in G at g. We have seen that $\psi \circ \tau$ has derivative $\lambda((\psi \circ \gamma)'(0))$, this shows that $\lambda d\psi_g(\gamma'(0)) = d\psi_g(\lambda\gamma'(0))$. Similarly if β is another differentiable curve at g in G then

$$\psi(\gamma'(0) + \beta'(0)) = (\psi \circ \gamma(t) g \beta(t)^{-1})'(0) =$$
$$= (\psi \circ \gamma)'(0)\psi(g)\psi(\beta(0))^{-1} + \psi(\gamma(0))\psi(g)(\psi \circ \beta)'(0) = (\psi \circ \gamma)'(0) + (\psi \circ \beta)'(0).$$

This shows that $d\psi_g$ is linear. So we have proved the following proposition.

Proposition 2.6. *If $\psi : G \to H$ is a Lie homomorphism then $d\psi_g$ is a $\mathbb{R}$-linear map for each $g \in G$.* $\square$

Lemma 2.3. *Let G, H, Z be matrix groups and $G \to_{\psi_2} H \to_{\psi_1} Z$ be a composition of Lie homomorphisms then the composition $\psi_1 \circ \psi_2$ is also smooth.*

Proof. Indeed if α is a curve in G then $\psi_2 \circ \alpha$ is a curve in H since ψ_2 is smooth, but ψ_1 is also smooth hence $\psi_1(\psi_2(\alpha))'(0)$ exist. Moreover if β is another curve in G such that $\beta(0) = \alpha(0)$ and $\alpha'(0) = \beta'(0)$ then $\psi_2(\alpha(0)) = \psi_2(\beta(0))$ and $(\psi_2 \circ \alpha)'(0) = (\psi_2 \circ \beta)'(0)$ since ψ_2 is smooth. Finally because ψ_1 is smooth $(\psi_1 \circ \psi_2 \circ \alpha)'(0) = (\psi_1 \circ \psi_2 \circ \beta)'(0)$ as required. $\square$

Proposition 2.7. ***[Chain rule]****. Let G, H, Z be matrix groups and $G \to_{\tilde{\psi}} H \to_{\psi} Z$ be a composition of smooth maps. Then given any $g \in G$ we have*

$$d(\psi \circ \tilde{\psi})_g = d\psi_{\tilde{\psi}(g)} \circ d\tilde{\psi}_g.$$

PROOF. Let γ be a differentiable curve in G at g with $\gamma'(0) = a$ then

$$d(\psi \circ \tilde{\psi})_g(a) = (\psi \circ (\tilde{\psi} \circ \gamma))'(0) = d\psi_{\tilde{\psi}(g)} \circ d\tilde{\psi}_g(a).$$

□

The differential at the Lie algebra of a matrix group is not only linear but in fact also a Lie homomorphism.

Theorem 2.2. *Suppose $G, H \subseteq GL(n, \mathbb{K})$ are matrix groups then given a Lie homomorphism $G \to_\psi H$ the differential $d\psi : Lie(G) \to Lie(H)$ is a real Lie homomorphism.*

PROOF. It remains to show that $d\psi([x, y]) = [d\psi(x), d\psi(y)]$ for all $x, y \in Lie(G)$. Let $\alpha, \beta : (a, b) \to G$ be differentiable curves at $1 \in G$ with $\alpha'(0) = x$ and $\beta'(0) = y$. Then the differentiable curve $\tau_s(t) = \alpha(s)\beta(t)\alpha(s)^{-1}$ has derivative $\alpha(s)y\alpha(s)^{-1} \in Lie(G)$ at $t = 0$ for all $s \in (a, b)$. We have seen that the derivative of $\tau_s'(0)$ at $s = 0$ is just $[x, y]$. Now

$$d\psi(\tau_s'(0)) = (\psi \circ \alpha)(s) \cdot d\psi(y) \cdot (\psi \circ \alpha)^{-1}(s)$$

and so the derivative of the curve $d\psi(\tau_s'(0))$ at $s = 0$ is just $[d\psi(x), d\psi(y)]$. But as $d\psi$ is linear then it is also continuous, so the result follows since

$$lim_{s\to 0}\frac{d\psi(\tau_s'(0)) - d\psi(\tau_0'(0))}{s} = d\psi(lim_{s\to 0}\frac{\tau_s'(0) - \tau_0'(0)}{s}) = d\psi([x, y]).$$

□

Observe that if G, H are complex matrix groups with complex Lie algebras $Lie(G)$ and $Lie(H)$, then a Lie homomorphism $\psi : G \to H$ gives rise to a Lie homomorphism

$$d\psi : Lie(G)^{\mathbb{R}} \to Lie(H)^{\mathbb{R}}.$$

We define the **dimension** of a matrix group to be the dimension of the Lie algebra as a vector space.

Corollary 2.5. *Let $G, H \subseteq GL(n, \mathbb{K})$ be matrix groups and $Lie(G), Lie(H)$ be their Lie algebras and suppose G is Lie isomorphic to H then $Lie(G) \cong Lie(H)$. In particular $Dim(G) = Dim(H)$.*

PROOF. Choose a Lie isomorphism $G \to_\psi H$, i.e ψ is smooth with smooth inverse ψ^{-1}. So by the chain rule $d(\psi \circ \psi^{-1}) = 1_{Lie(G)} = d\psi \circ d\psi^{-1}$ and $d(\psi^{-1} \circ \psi) = 1_{Lie(H)} = d\psi^{-1} \circ d\psi$, hence the differential $d\psi$ is an isomorphism of Lie algebras as required. □

The property for two matrix groups to be Lie isomorphic is a very strong property, indeed if $G \cong_{Lie\ isomorphic} H$ then:

$$G \cong_{groups} H \ , \ G \cong_{topology} H \ and \ Lie(G) \cong_{Lie\ algebra} Lie(H).$$

Example 2.3. Here is a nice application of the Lie algebra. Consider the special orthogonal group $SO(3,\mathbb{R})$ and the special linear group $SL(2,\mathbb{R})$. One can show that their Lie algebras are $\cong \mathbb{R}^3_\wedge$(cross product on $\mathbb{R}^3$) and $\mathfrak{sl}(2,\mathbb{R}) = \{x \in \mathfrak{gl}(2,\mathbb{R}) | tr(x) = 0\}$ respectively. Now if the matrix groups are Lie isomorphic then we can choose a Lie isomorphism $\mathfrak{so}(3,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R})$. However this is impossible because the Killing form is negative definite on $\mathfrak{so}(3)$ while is not on $\mathfrak{sl}(2,\mathbb{R})$. Hence $SO(3,\mathbb{R}) \ncong SL(2,\mathbb{R})$. In fact one can also show that $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C})$ by uniqueness of semisimple complex Lie algebras of dimension 3. However in this case $SL(2,\mathbb{C}) \ncong SO(3,\mathbb{C})$ (they have non-isomorphic fundamental groups). This shows that the Lie algebra does not encode all the information about the matrix group.

The identity component of a matrix group G has the same Lie algebra as G. To see this let $L = Lie(G)$ then we note that if $x \in L$ the map $t \to e^{tx}$ is a path in G from $1 \to e^x$, hence $e^x \in G_0$ by definition. From this it is clear that $L \subseteq Lie(G_0)$, i.e $L = Lie(G)$.

4. The exponential map

We mention here that the theory of matrix groups is just a special case of a more general construction, known as Lie groups. A Lie group G is a smooth manifold which is also a group such that the group operations are smooth maps (see for instance [**1**] for details). It can be shown that all matrix groups are Lie groups, and to any Lie group one can attach a Lie algebra just as with matrix groups. One can also define the exponential map as a map $exp : Lie(G) \to G$. The advantage of starting from the Lie group point of view is that the image of the exponential is by definition contained in G. However from the matrix group point of view this takes a bit of work to prove. We refer to [**4**], section 7.6 for details regarding the proof of the next theorem.

Theorem 2.3. *If $G \subseteq GL(n,\mathbb{K})$ is a matrix group with Lie algebra $Lie(G)$ and $x \in Lie(G)$ then the exponential map $exp : Lie(G) \to GL(n,\mathbb{K})$ has image contained in G.*

Theorem 2.4. *If G is a real matrix group and $L \leq Lie(G)$ is a Lie subalgebra. Then the following hold.*

(1) There is a unique connected matrix subgroup $H \leq G$ with Lie algebra L.
(2) H is generated by the image of L under the exponential map.

PROOF. For proof see for example [**1**], chapter 2, section 2, Theorem 2.1. □

Proposition 2.8. *Let G, H be matrix groups and $G \to_{\psi} H$ be a smooth map then*

$$\psi(e^x) = e^{d\psi(x)}$$

for every $x \in Lie(G)$.

PROOF. Let $x \in Lie(G)$ and consider the exponential one parameter subgroup of G: $\gamma : \mathbb{R} \to G$ given by $\gamma(t) = e^{tx}$. Then

$$\psi(\gamma(t+s)) = \psi(\gamma(t)\gamma(s)) = \psi(\gamma(t))\psi(\gamma(s))$$

for all $t, s \in \mathbb{R}$. Hence $\psi \circ \gamma$ is a one parameter subgroup of H, but by uniqueness $\psi \circ \gamma = e^{yt}$ for a suitable $y \in H$. Now $(\psi \circ \gamma)'(0) = d\psi(\gamma'(0)) = d\psi(x) = y$, i.e

$$\psi(e^{tx}) = e^{td\psi(x)}$$

for all $t \in \mathbb{R}$. This proves the result. □

Let $G \subseteq GL(n, \mathbb{K})$ be a matrix group with Lie algebra $Lie(G) \subseteq \mathfrak{gl}(n, \mathbb{K})$. We now show that if $x \in Lie(G)$ then the determinant of e^x is given by $det(e^x) = e^{tr(x)}$ where $tr(x)$ is the trace of x.

Lemma 2.4. *Given a differentiable curve $\gamma : (a, b) \to G$ at $1 \in G$ then $det(\gamma)'(0) = tr(\gamma'(0))$.*

PROOF. Write $\gamma(t) = (a_{ij}(t))_{ij}$ for the matrix entries of γ. Then by the cofactor expansion along the first column we may write

$$det(\gamma(t)) = \sum_l (-1)^{1+l} a_{l1}(t) det(m_{l1}(t))$$

so that

$$det(\gamma)'(0) = \sum_l (-1)^{1+l} a'_{l1}(0) det(m_{l1}(0)) + \sum_l (-1)^{l+1} a_{l1}(0) det(m_{l1}(t))'(0)$$

here $det(m_{l1}(t))$ is the $(l, 1)$-minor of the matrix $\gamma(t)$. Now since $\gamma(0) = I_n$ it follows that $a_{l1}(0) = 0$ if $l \neq 1$ and $a_{11}(0) = 1$. Also note that $det(m_{l1}(0)) = 0$ whenever $l \neq 1$ since there is a row of zeroes, and we have $det(m_{11}(0)) = 1$. Hence

$$det(\gamma)'(0) = a'_{11}(0) + det(m_{11}(t))'(0),$$

but $m_{11}(0) = I_{n-1}$ so we may proceed by induction over n. The case $n = 1$ is clear, so assume now that it holds for all $1 \le k \le n-1$ over all curves in $M(k, \mathbb{K})$ with $\gamma(0) = I_k$. So by the induction hypothesis

$$det(\gamma)'(0) = a'_{11}(0) + tr(m'_{11}(0)) = a'_{11}(0) + a'_{22}(0) + \cdots + a'_{nn}(0) = tr(\gamma'(0))$$

as required. $\square$

Now by identifying $x \in \mathbb{K}^\times$ with the diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ then $\mathbb{K}^\times$ is a matrix group and we can define the determinant map: $Det : G \to \mathbb{K}^\times$ by $Det(g) = \begin{pmatrix} det(g) & 0 \\ 0 & det(g) \end{pmatrix}$. By the argument above this is a Lie homomorphism and the differential is given by:

$$\gamma'(0) \to \begin{pmatrix} tr(\gamma'(0)) & 0 \\ 0 & tr(\gamma'(0)) \end{pmatrix}$$

so that

$$Det(e^x) = \begin{pmatrix} e^{tr(\gamma'(0))} & 0 \\ 0 & e^{tr(\gamma'(0))} \end{pmatrix}.$$

So finally we get,

Proposition 2.9. *Let $G \subseteq GL(n, \mathbb{K})$ be a matrix group and $x \in Lie(G)$ then $det(e^x) = e^{tr(x)}$.* $\square$

This result can be used to reveal the Lie algebras of many matrix groups. Indeed to see the power, consider for instance $SL(n, \mathbb{R})$ and a differentiable curve γ at 1. Then by the result above $det(e^{\gamma'(0)}) = 1 = e^{tr(\gamma'(0))}$ giving $tr(\gamma'(0)) = 0$. Hence $Lie(SL(n, \mathbb{R})) \subseteq \mathfrak{sl}(n, \mathbb{R})$. Similarly if X has trace zero then again by the result above $t \to e^{tX}$ is a curve in $SL(n, \mathbb{R})$. Hence it follows that $SL(n, \mathbb{R})$ has Lie algebra $\mathfrak{sl}(n, \mathbb{R})$.

Here is a list of some matrix groups and their Lie algebras.

Example 2.4.

- $SL(n, \mathbb{R})$ with Lie algebra $\mathfrak{sl}(n, \mathbb{R})$.
- $O(n, \mathbb{R})$ with Lie algebra $\mathfrak{o}(n, \mathbb{R})$.
- $H_3(\mathbb{R}) = \{X \in \mathfrak{gl}(3, \mathbb{R}) | X = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\}$ (Heisenberg group) with Heisenberg Lie algebra $\mathfrak{n}(3, \mathbb{R})$ (3×3 strictly upper triangular matrices).

• The circle S^1 identified with $SO(2,\mathbb{R})$ has Lie algebra $\cong \mathbb{R}$.
• The 3-sphere S^3 identified with $SU(2,\mathbb{C}) = \{X \in GL(2,\mathbb{C}) | X\bar{X}^t = I, det(X) = 1\}$ has Lie algebra $\mathfrak{su}(2,\mathbb{C}) = \{X \in \mathfrak{gl}(2,\mathbb{C}) | \bar{X}^t = -X, tr(X) = 0\}$.

Given a matrix group with a certain group property one may ask how this transfers to a property of the Lie algebra. For instance if we have an abelian matrix group G then we have seen that $Lie(G)$ must also be abelian. Here are some other properties.

Proposition 2.10. *Let G be a matrix group with Lie algebra $Lie(G)$ and suppose $N \lhd G$ is a matrix subgroup of G then $Lie(N) \trianglelefteq Lie(G)$.*

PROOF. First it is clear that if $N \leq G$ then $Lie(N) \leq Lie(G)$ since a differentiable curve in N at 1 is also a differentiable curve in G at 1. Moreover suppose $x \in Lie(N)$ and $y \in Lie(G)$ and let α, β be differentiable curves at 1 in G, N respectively with $\alpha'(0) = y$ and $\beta'(0) = x$. We observe that $\tau(t) = \alpha(t)\beta'(0)\alpha(t)^{-1} \in Lie(N)$ for all t, this is seen by considering the differentiable curve $\tau_t(s) = \alpha(t)\beta(s)\alpha(t)^{-1}$ in N, since $N \lhd G$. Now we have seen that $\tau'(0) = [x, y]$ and since $Lie(H)$ is closed in $\mathfrak{gl}(n,\mathbb{K})$ then it follows that $[x, y] \in Lie(N)$. This shows that $Lie(N)$ is an ideal of $Lie(G)$ as required. ☐

Proposition 2.11. *If $\psi : G \to H$ is a Lie homomorphism of real matrix groups then the following hold.*

(1) $Lie(ker(\psi)) = ker(d\psi)$.
(2) If G is connected and $\psi(G)$ is closed in H then $Lie(\psi(G)) = d\psi(Lie(G))$.

PROOF. First since ψ is a Lie homomorphism and $\psi(G)$ is closed in H then we know that $ker(\psi)$ and $\psi(G)$ are matrix subgroups of G, H respectively, and so $Lie(ker(\psi)) \leq Lie(G)$ and $Lie(\psi(G)) \leq Lie(H)$. Now for case (1), if γ is a curve at $1 \in G$ in $ker(\psi)$ with $\gamma'(0) = x$ then we have

$$d\psi(x) = (\psi \circ \gamma)'(0) = (I_n)'(0) = 0$$

so that $Lie(ker(\psi)) \subseteq ker(d\psi)$. Conversely if $x \in ker(d\psi)$ then $tx \in ker(d\psi)$ for any $t \in \mathbb{R}$, so by considering the exponential map we note that $\psi(e^{tx}) = 1$ hence $e^{tx} \in ker(\psi)$. So $t \to e^{tx}$ is a curve in $ker(\psi)$ with derivative $x \in Lie(ker(\psi))$. Now for the second case let $\tilde{G} \leq H$ be the unique matrix subgroup with Lie algebra $d\psi(Lie(G))$. It is enough to show that $\tilde{G} = \psi(G)$. Now because G is connected then G is generated by $\{e^z | z \in Lie(G)\}$, in particular $\psi(G)$ is generated by $\{e^z | z \in d\psi(Lie(G))\}$. But this is also the case for $\tilde{G}$, i.e $\tilde{G} = \psi(G)$, so that $d\psi(Lie(G)) = Lie(\psi(G))$ as required. ☐

5. The inner automorphism group

Now we consider a Lie algebra L and the the automorphism group $G = Aut(L)$. We claim that this is a matrix subgroup of $GL(n, \mathbb{K})$ where $n = Dim(L)$. Indeed first it is clear that we can embed $\psi : G \hookrightarrow GL(n, \mathbb{K})$ via a group homomorphism. Since we can fix a basis $\{e_j\}_j$ of L and represent an automorphism $\phi : L \to L$ by it's matrix, X, so set $\psi(\phi) = X$. Now we can equip $\psi(G)$ with the induced topology from $GL(n, \mathbb{K})$. It remains to check that $\psi(G)$ is closed in $GL(n, \mathbb{K})$. To see this we let $[e_i, e_j] = \sum_k C_{ij}^k e_k$ where C_{ij}^k are the structure constants of L in $\mathbb{K}$. Consider now a convergent sequence of automorphisms $(\phi_n)_{n\in\mathbb{N}} \to \phi$ and denote $(\phi_n)_{ij} = (\phi_{ij}(n))_{ij}$ for the matrix w.r.t $\{e_j\}_j$. Since ϕ_n are automorphisms of L then the following equality must be satisfied:

$$\sum_t \phi_{ti}(n) \sum_s \phi_{sj}(n) C_{ts}^l = \sum_k C_{ij}^k \phi_{lk}(n)$$

for every l, i, j and $n \in \mathbb{N}$. This means that since the entries of ϕ_n converges to the entries of ϕ then ϕ must be an automorphism as well. Since we can take the limit as $n \to \infty$. We conclude that $\psi(G)$ must be closed in $GL(n, \mathbb{K})$.

Proposition 2.12. *Let L be a Lie algebra then $Lie(Aut(L)) = Der(L)$.*

PROOF. We show the case $Lie(Aut(L)) \subseteq Der(L)$. Let $Aut(L)$ be embedded in $GL(n, \mathbb{K})$ as in the argument above w.r.t a fixed basis $\{e_j\}_j$ of L. Suppose that $D : L \to L$ is a derivation of L, i.e $D([a, b]) = [D(a), b] + [a, D(b)]$ for all $a, b \in L$. Let X be the matrix which represents D w.r.t $\{e_j\}_j$. Now we will identify D with it's matrix $X \in \mathfrak{gl}(n, \mathbb{K})$ w.r.t $\{e_j\}_j$ so that $Der(L)$ is embedded in $\mathfrak{gl}(n, \mathbb{K})$. In particular a matrix $X = (x_{ij})_{ij}$ is a matrix of a derivation if and only if it satisfies the equality:

$$C_{ij}^t = \sum_l x_{li} C_{lj}^t + \sum_l x_{lj} C_{il}^t$$

for all t, i, j where C_{ij}^k denotes the structure constants of L w.r.t $\{e_j\}_j$. Suppose now that $\alpha : I \to Aut(L)$ is a curve at 1 with $\alpha'(0) = X$. Write $(\alpha(t))_{ij} = \alpha_{ij}(t)$ for the matrix entries w.r.t the basis $\{e_j\}_j$. Then because $\alpha(t)$ is an automorphism of L we know that the following equality must hold:

$$\sum_k \alpha_{ki}(t) \sum_s \alpha_{sj}(t) C_{ks}^l = \sum_k C_{ij}^k \alpha_{lk}(t)$$

for every l, i, j. Now differentiating both sides of this equation we obtain the property of X being a derivation. This shows that $Lie(Aut(L)) \subseteq Der(L)$. $\square$

Definition 2.9. If G is a matrix group we define the **adjoint group**

$$Ad(G) = \{Ad_g | g \in G\} \leq Aut(Lie(G))$$

where $Ad_g : Lie(G) \to Lie(G)$ is the automorphism of $Lie(G)$ defined by $Ad_g(x) = gxg^{-1}$ for all $x \in Lie(G)$.

Analogous to $Aut(Lie(G))$ one may show that $Ad(G)$ is a matrix subgroup of $Aut(Lie(G)) \subseteq GL(n, \mathbb{K})$ by identifying an automorphism in $Ad(G)$ with it's matrix $X \in Aut(Lie(G))$ w.r.t to the fixed basis $\{e_j\}$ of $Lie(G)$. In this way a matrix X is a matrix of an automorphism in $Ad(G)$ if and only if there is some $g \in G$ such that

$$ge_ig^{-1} = \sum_l x_{li}e_l$$

for all i. Similarly we identify the adjoint Lie algebra $ad(Lie(G)) \subseteq Der(Lie(G)) \subseteq \mathfrak{gl}(n, \mathbb{K})$. So that a matrix X is a matrix of a map $ad(z) : Lie(G) \to Lie(G)$ if and only if

$$[z, e_i] = \sum_l x_{li}e_l$$

for all i. In particular if $\alpha : I \to G$ is a curve in G at 1 with matrix entries $(\alpha_{ij}(t))_{ij}$ and $X(t)$ denotes the matrix of the map $Ad_{\alpha(t)}$ then

$$\alpha(t)e_i\alpha(t)^{-1} = \sum_l X_{li}(t)e_l$$

for all i. By differentiating both sides we get

$$[\alpha'(0), e_i] = \sum_l X'_{lj}(0)e_l.$$

So we deduce that $ad(Lie(G)) \subseteq Lie(Ad(G))$. In fact it can be shown that the inclusion is an equality.

The homomorphism $Ad : G \to Ad(G)$ given by $g \to Ad_g$ is called the **adjoint representation** of G. By the previous argument Ad is smooth with differential $ad : Lie(G) \to ad(Lie(G))$. We immediately deduce that

$$Ad(e^X) = e^{ad(X)}$$

for all $X \in Lie(G)$.

Definition 2.10. Given a matrix group G with Lie algebra $Lie(G)$ we define the matrix group of **inner automorphisms** $Int(Lie(G))$ of $Lie(G)$ to be the identity component of $Ad(G)$ namely, $Ad(G)_0$. An automorphism $Lie(G) \to_\psi Lie(G)$ is said to be an **inner automorphism** if $\psi \in Int(Lie(G))$.

Example 2.5. If G is abelian then $Ad_g : Lie(G) \to Lie(G)$ is the trivial homomorphism for any $g \in G$, i.e $Ad(G)$ and $Int(Lie(G))$ are the trivial groups.

It follows that $Int(Lie(G))$ is generated by the set $\{e^{ad(x)} | x \in Lie(G)\}$. This is because $Int(Lie(G))$ is connected so it follows that $Int(Lie(G))$ is generated by the image under the exponential map. We also note that if $Lie(G)$ is semisimple then we must have $Aut(Lie(G))_0 = Int(Lie(G))$. This follows because $Der(Lie(G)) = ad(Lie(G))$, so as $Int(Lie(G))$ is connected then by uniqueness $Aut(Lie(G))_0 = Int(Lie(G))$. In particular the inner automorphism group of a semisimple Lie algebra is defined naturally without specifying any matrix group.

CHAPTER 3

Semisimple Lie algebras over $\mathbb{R}$

1. Complexification and realification of Lie algebras

Most of this chapter is based on [**1**] and [**5**].

Definition 3.1. Let L_0 be a real Lie algebra and suppose $J : L_0 \to L_0$ is a linear map such that $J^2 = -1_{L_0}$. Then J is said to be a complex structure on L if in addition $[x, J(y)] = J([x, y])$ for all $x, y \in L$.

We note that if J is a complex structure on a real Lie algebra L_0 then so is $-J$ (since $(-J)^2 = J^2 = -1_{L_0}$). The dimension of L_0 must be even if a complex structure were to exist, this is because J is an isomorphism of vector spaces so $det(J)^2 = (-1)^{Dim(L_0)}$. The point with a complex structure is that we can construct a complex Lie algebra using L_0 as follows. Define for each $x \in L_0$ the scalar multiplication of a complex number by,

$$(a + ib)x = ax + bJ(x)$$

for any $a, b \in \mathbb{R}$. It can be easily verified that this defines a complex Lie algebra with the same Lie bracket $[-, -]$ inherited from L_0. Denote this complex Lie algebra by $\bar{L_0}$. We note that if $\{x_j\}_j$ is a basis for $\bar{L_0}$ then $\{x_j\}_j \cup \{J(x_j)\}_j$ is a basis of L_0. Consequently $Dim_{\mathbb{C}}(\bar{L_0}) = \frac{1}{2}Dim_{\mathbb{R}}(L_0)$. The following argument shows this.

Proposition 3.1. *$Dim_{\mathbb{C}}(\bar{L_0}) = \frac{1}{2}Dim_{\mathbb{R}}(L_0)$.*

Proof. Given a basis for $\bar{L}$ say $\{x_j\}_j$ then we claim that $S = \{x_j\}_j \cup \{J(x_j)\}_j$ is a basis for L_0. Indeed if $x \in \bar{L}$ then we can write

$$x = \sum_j (re(\lambda_j) + iIm(\lambda_j))x_j$$

for suitable complex numbers $\lambda_j \in \mathbb{C}$. In particular $x = \sum_j re(\lambda_j)x_j + Im(\lambda_j)J(x_j)$ so that S spans L_0. Moreover if

$$\sum_j \beta_j x_j + \sum_j \alpha_j J(x_j) = 0$$

for $\beta_j, \alpha_j \in \mathbb{R}$ then $\sum_j(\alpha_j + i\beta_j)x_j = 0$, hence it follows that α_j, β_j are all zero since $\{x_j\}_j$ is a basis for L. □

Definition 3.2. Given a complex Lie algebra L we define the **realification** of L denoted $L^{\mathbb{R}}$ to be the Lie algebra L over $\mathbb{R}$ with the Lie bracket inherited from L over $\mathbb{C}$.

Observe that given a complex Lie algebra L then $L^{\mathbb{R}}$ has a complex structure J given by $J(x) = ix \in L^{\mathbb{R}}$, as $[x, J(y)] = [x, iy] = i[x, y]$ since L is a complex Lie algebra. In particular the corresponding complex Lie algebra constructed by defining

$$(a + ib)x = ax + bJ(x) = ax + ibx,$$

for all $x, y \in L^{\mathbb{R}}$ and $a, b \in \mathbb{R}$ is just the original Lie algebra L over $\mathbb{C}$. In particular by our argument above it follows that $Dim_{\mathbb{R}}(L^{\mathbb{R}}) = 2Dim_{\mathbb{C}}(L)$.

Recall the construction of $\mathbb{C}$ from $\mathbb{R}$ using $\mathbb{R} \times \mathbb{R}$. We now extend this construction to real vector spaces V, i.e we will construct a complex vector space denoted $V^{\mathbb{C}}$ using a natural complex structure on $V \times V$. The complex vector space $V^{\mathbb{C}}$ is defined to be the **complexification** of V. This construction will also work for Lie algebras.

Let now L be a real vector space and consider the endomorphism map $J : L \times L \to L \times L$ given by $J(x, y) = (-y, x)$ for all $x, y \in L$. Define now a scalar multiplication of complex numbers by:

$$(a + ib)(x, y) = (ax, ay) + bJ(x, y) = (ax - by, ay + bx)$$

for all $a, b \in \mathbb{R}$ and $x, y \in L$. Set $L^{\mathbb{C}} = \{x + iy | x, y \in L\}$ where we identify a pair $(x, y) = (x, 0) + J((y, 0))$ with $x + iy$ for $(x, y) \in L \times L$. This is a complex vector space. Moreover if L is a Lie algebra with bracket $[-, -]$, then we can define a Lie bracket on $L^{\mathbb{C}}$ defined by:

$$[x + iy, z + it] = [x, z] - [t, x] + i([y, z] + [x, t])$$

for all $x, y, z, t \in L$. So $L^{\mathbb{C}}$ becomes a complex Lie algebra. We immediately see that $Dim_{\mathbb{R}}(L) = Dim_{\mathbb{C}}(L^{\mathbb{C}})$.

2. Real forms

Definition 3.3. Let L be a complex Lie algebra and $L_0 \leq L^{\mathbb{R}}$ be a Lie subalgebra of $L^{\mathbb{R}}$. We say L_0 is a real form of L if

$$L^{\mathbb{R}} = L_0 \oplus J(L_0),$$

here $L^{\mathbb{R}} \to_J L^{\mathbb{R}}$ is the complex structure on $L^{\mathbb{R}}$ given by $ix = J(x)$ for every $x \in L$. We will usually just write i instead of J.

It follows immediately that if L_0 is a real form of a complex Lie algebra L then $L_0^{\mathbb{C}} = L$.

Example 3.1. The special linear Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is a real form of $\mathfrak{sl}(2,\mathbb{C})$. In fact consider $\{e, f, h\}$ the standard basis of $\mathfrak{sl}(2,\mathbb{C})$ then the subspace C spanned by $\{ih, e-f, i(e+f)\}$ is also a real form of $\mathfrak{sl}(2,\mathbb{C})$. Indeed we calculate the following brackets:

$$[ih, e-f] = 2i(e+f),\ [ih, i(e+f)] = -2(e-f),\ [e-f, i(e+f)] = 2ih.$$

We deduce that the Lie algebra is real, so it is a Lie subalgebra of $\mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$. It can also be easily checked that $\mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}} = C \oplus iC$. It turns out that $C \cong \mathbb{R}^3_\wedge$ (cross product). The two real forms we found are in fact all the real forms of $\mathfrak{sl}(2,\mathbb{C})$.

We say that a map $\sigma : L \to L$ is a antilinear map if σ satisfies $\sigma(x+y) = \sigma(x)+\sigma(y)$ and $\sigma(\lambda x) = \bar{\lambda} x$ for every $x \in L$ and $\lambda \in \mathbb{C}$. Now since I is a real form of L then it gives rise to an involutive antilinear map $\sigma : L \to L$ given by conjugation $\sigma(x+iy) = x - iy$. Conversely if we have an involutive antilinear map $\sigma : L \to L$ which is a homomorphism of Lie algebras, then we claim that the subspace fixed by σ must be a real form of L. Indeed note that σ is an involution when restricted on $L^{\mathbb{R}}$, so we can write $L^{\mathbb{R}} = L_+ \oplus L_-$. We clearly have $L_- = iL_+$ by definition. Moreover L_+ is clearly a Lie subalgebra of $L^{\mathbb{R}}$, since $\sigma([x,y]) = [x,y]$ whenever $x, y \in L_+$. So L_+ is a real form of L.

Definition 3.4. A real structure σ on a complex Lie algebra L is an involutive antilinear homomorphism $\sigma : L \to L$.

Proposition 3.2. *Let L be a complex Lie algebra with real forms L_0, L_1 and conjugation maps τ_0, τ_1 respectively. Then the following hold:*
(1) τ_0 restricted to $L^{\mathbb{R}}$ is an automorphism of $L^{\mathbb{R}}$, i.e $\tau_0 \in Aut(L)$.
(2) $\tau_0 \circ \tau_1 \in Aut(L)$.
(3) $\tau_1 \circ \tau_0 = \tau_0 \circ \tau_1$ if and only if $\tau_0(L_1) = L_1$ and $\tau_1(L_0) = L_0$.
(4) If (3) holds then $L_0 = (L_0 \cap L_1) \oplus (L_0 \cap iL_1)$ and $L_1 = (L_0 \cap L_1) \oplus (L_1 \oplus iL_0)$.
(5) $(\forall x, y \in L)(\kappa(\tau_0(x), \tau_0(y)) = \overline{\kappa(x,y)})$.

Lemma 3.1. *Let L_0 be a real Lie algebra and denote κ_{L_0} for the Killing form. Denote $L_0^{\mathbb{C}} = L$ and $\hat{L} = L^{\mathbb{R}}$. Then the following is true.*
(1) $(\forall x, y \in L_0)(\kappa_{L_0}(x,y) = \kappa_L(x,y))$.
(2) $(\forall x, y \in \hat{L})(\kappa_{\hat{L}}(x,y) = 2Re(\kappa_L(x,y)))$ where Re is the real part.

PROOF. Case (1) is clear as the Lie bracket of L_0 is an extension to the bracket on L, moreover a choice of bases for L_0 is also a basis for L. For (2) if $S = \{x_j\}_j$ is a basis for L then we know that $S \cup J(S) = \hat{S}$ is a basis for $\hat{L}$, here J is the complex structure on $\hat{L}$ given by $J(x) = ix$. So given $x \in \hat{L}$ we have

$$ad_{\hat{L}}(x)(x_j) = [x, x_j] = ad_L(x)(x_j) = \sum_l (Re(a_{lj}) + iIm(a_{lj}))x_l =$$

$$= \sum_l Re(a_{lj})x_l + \sum_l Im(a_{lj})J(x_l).$$

Here $(a_{ij})_{ij}$ is the matrix for the map $ad_L(x)$. Moreover we have since J is a complex structure:

$$ad_{\hat{L}}(x)(J(x_j)) = [x, J(x_j)] = J([x, x_j]) = J(ad_{\hat{L}}(x)(x_j)) =$$

$$= \sum_l -Im(a_{lj})x_l + \sum_l Re(a_{lj})J(x_l).$$

This shows that the matrix of $ad_{\hat{L}}(x)$ has the block form

$$\begin{pmatrix} Re(A) & -Im(A) \\ Im(A) & Re(A) \end{pmatrix}.$$

This shows that if B is the matrix for $ad_L(y)$ for $y \in \hat{L}$ then $ad_{\hat{L}}(x) \circ ad_{\hat{L}}(y)$ has matrix

$$\begin{pmatrix} Re(A)Re(B) - Im(A)Im(B) & \star \\ \star & Re(A)Re(B) - Im(A)Im(B) \end{pmatrix} = \begin{pmatrix} Re(AB) & \star \\ \star & Re(AB) \end{pmatrix}.$$

Now because the real part $Re(-)$ is linear then $\kappa_{\hat{L}}(x, y) = 2tr(Re(AB)) = 2Re(tr(AB))) = 2Re(\kappa_L(x, y))$ as required. $\square$

It follows immediately that the property of a complex Lie algebra of being semisimple is conserved within its real forms.

Corollary 3.1. *Let L_0 be a real Lie algebra then following are equivalent.*

(1) L_0 is semisimple.
(2) $L_0^{\mathbb{C}}$ is semisimple.
(3) $(L_0^{\mathbb{C}})^{\mathbb{R}}$ is semisimple.

In particular if L is a complex Lie algebra with real form L_0 then L is semisimple if and only if L_0 is semisimple. $\square$

Having the notion of complexification we see that Cartan's second criterion also hold for real semisimple Lie algebras. In particular it follows that $ad(L_0) = Der(L_0)$ also for semisimple real Lie algebras L_0, we can mimic the proof we used for complex semisimple Lie algebras. Another observation is that if $L_0^{\mathbb{C}}$ is simple then so is L_0. Since if we have an ideal $I \trianglelefteq L_0$ then $I^{\mathbb{C}}$ is an ideal of $L_0^{\mathbb{C}}$. So it follows that any bilinear form of L_0 which is associative, symmetric and non-degenerate must be proportional to the Killing form, κ, on L_0. This follows because we can extend the bilinear form to a bilinear form on $L_0^{\mathbb{C}}$ with the same properties.

3. A compact real form

Definition 3.5. [**Compact real form**]. Let L be a complex Lie algebra. We say that a real form L_0 on L is compact if the Killing form κ on L is negative definite when restricted to L_0, i.e $\kappa(x,x) < 0$ for all $x \neq 0$, and $\kappa(x,x) = 0$ if and only if $x = 0$.

We will now show that every semsimple complex Lie algebra L has a compact real form, this follows from the fact that L has a root decomposition. So choose a Cartan subalgebra of L say H and write a root decomposition:

$$L = H \oplus_{\alpha \in \Omega} L_\alpha.$$

Recall now the definition of the Cartan-Weyl basis and the function λ defined on the hull of a subset $S \subset \Omega$. We will assume that L is equipped with this basis

$$\{t_\alpha | \alpha \in \Omega, \alpha \neq -\alpha\} \cup \{x_\alpha | \alpha \in \Omega\}.$$

Consider now the real subspace

$$C = iH_{\mathbb{R}} \oplus_{\alpha \neq -\alpha} \langle i(x_\alpha + x_{-\alpha})\rangle \oplus_{\alpha \neq -\alpha} \langle x_\alpha - x_{-\alpha}\rangle \subset L$$

where $H_{\mathbb{R}}$ is the real span of $\{t_\alpha | \alpha \in \Omega, \alpha \neq -\alpha\}$. Recall that $\kappa(t_\alpha, -) = \alpha(-)$. We will show that this is the required compact real form of L.

Lemma 3.2. *The real subspace C of L is a real Lie subalgebra of L. Moreover if we restrict the Killing form κ on L to C then κ is negative definite.*

PROOF. Write for elements $z, \hat{z} \in C$,

$$z = \sum_\alpha \lambda_\alpha i t_\alpha + \sum_\alpha c_\alpha i(x_\alpha + x_{-\alpha}) + \sum_\alpha b_\alpha(x_\alpha - x_{-\alpha})$$

and

$$\hat{z} = \sum_\beta \hat{\lambda}_\beta i t_\beta + \sum_\beta \hat{c}_\beta i(x_\beta + x_{-\beta}) + \sum_\beta \hat{b}_\beta(x_\beta - x_{-\beta}).$$

Then an easy calculation shows that

$$\kappa(z, \hat{z}) = -\sum_{\alpha,\beta} \lambda_\alpha \hat{\lambda}_\beta \alpha(t_\beta) - \sum_\alpha c_\alpha(2\hat{c}_\alpha) + \sum_\alpha b_\alpha(-2\hat{b}_\alpha) \in \mathbb{R},$$

using that $\kappa(x_\alpha, x_{-\alpha}) = 1$ for all $\alpha \in \Omega$ and $\kappa(L_\alpha, L_\beta) = 0$ if $\alpha + \beta \neq 0$. Denote $h = \sum_\alpha \lambda_\alpha i t_\alpha$ then

$$\kappa(z,z) = -\kappa(h,h) - 2\sum_\alpha c_\alpha^2 - 2\sum_\alpha b_\alpha^2 < 0$$

when $z \neq 0$ as κ is positive definite on the Cartan subalgebra H. Replacing κ with the Lie bracket $[-,-]$ one shows similarly that $[z, \hat{z}] \in C$ for every $z, \hat{z} \in C$ by using the fact $\lambda(\alpha,\beta) = -\lambda(-\alpha,-\beta)$ for all $\alpha, \beta \in \Omega$ and that $\lambda(\alpha,\beta)$ is real. The lemma is proved. □

Consider now the real Lie subalgebra $C \leq L^{\mathbb{R}}$ as above. We claim that this is a real form of L. Indeed recall that $H_{\mathbb{R}}$ is a real form of H, so any element $z \in L$ can be decomposed as follows:

$$z = h_1 + ih_2 - \frac{i}{2}\sum_\alpha \lambda_\alpha i(x_\alpha + x_{-\alpha}) + \frac{1}{2}\sum_\alpha \lambda_\alpha(x_\alpha - x_{-\alpha})$$

for some $h_1, h_2 \in H_{\mathbb{R}}$ and $x_\alpha \in L_\alpha$. This shows that $L^{\mathbb{R}} = C + iC$ and it is clear that $C \cap iC = 0$, since κ is real on C.

Thus we have proved the following theorem.

Theorem 3.1. *Every complex semisimple Lie algebra L has a compact real form.* □

For later use, when we refer to the compact real form C associated to a root decomposition of L, then we will mean the one we just found previously. It is immediate that C satisfies the following.

Proposition 3.3.

(1) $H \cap C = iH_{\mathbb{R}}$ is maximal abelian in C.
(2) $H_{\mathbb{R}} \subset iC$.

Theorem 3.2. *Let L_0 be a real semisimple Lie algebra with complexification L, i.e L_0 is a real form on L. Now let C be a compact real form on L and denote σ for the conjugation map of L_0. There exist $\phi \in Aut(L)$ such that $\sigma(\phi(C)) \subset \phi(C)$.*

Proof. Let $\tau : L^{\mathbb{R}} \to L^{\mathbb{R}}$ denote the conjugation map w.r.t the compact real form C of L. Then the composition $\theta = \sigma \circ \tau$ is an involution $\theta : L \to L$. Since C is compact then the bilinear form κ_τ defined by:

$$\kappa_\tau(x,y) = -\kappa(x, \tau(y))$$

for all $x, y \in L$ forms an inner product on L. In particular since κ is invariant under automorphisms of L and σ, τ are involutions of $L^{\mathbb{R}}$, then θ is symmetric w.r.t κ_τ. So w.r.t some orthonormal basis we can assume θ is represented by a diagonal matrix with non-zero entries. In particular $\tilde{\theta} = \theta^2$ has positive diagonal entries say, $\lambda_j > 0$. Consider now $\tilde{\theta}^t$ for any real number $t \in \mathbb{R}$. We claim that $\tilde{\theta}^t$ is an automorphism of L. Indeed this is true if and only if

$$\lambda_i^t \lambda_j^t C_{ij}^k = C_{ij}^k \lambda_k^t$$

for every k, i, j, t, where C_{ij}^k are the structure constants. However we already know that it is true in the case of $t = 1$, so it easily extends to $\mathbb{R}$. Similarly we see that $\tau \circ \theta = \theta^{-1} \circ \tau$ so this extends to $\tau \circ \tilde{\theta}^t = \tilde{\theta}^{-t} \circ \tau$ for any $t \in \mathbb{R}$. In particular if we set $\psi_t = \tilde{\theta}^t \circ \tau \circ \tilde{\theta}^{-t}$ then using the previous inequality we get

$$\sigma \circ \psi_t = \theta \circ \tilde{\theta}^{-2t}$$

and

$$\psi_t \circ \sigma = \theta^{-1} \circ \tilde{\theta}^{2t}.$$

So in particular when $t = \frac{1}{4}$ and since $\tilde{\theta}^{\frac{1}{2}} = \theta$ it follows that σ commutes with $\psi_{\frac{1}{4}}$. This means that if we choose the automorphism $\phi = \tilde{\theta}^{\frac{1}{4}} = \theta^{\frac{1}{2}}$ of L we get the required result. □

From the previous proof it follows immediately that two compact real forms $C, \tilde{C} \leq L^{\mathbb{R}}$ of L are isomorphic. Since if τ_C denotes the conjugation map w.r.t C then we can find a one parameter subgroup $\mathbb{R} \to Aut(L)$: $\psi^t : L \to L$, such that $\tau_C(\tilde{\psi}(\tilde{C})) \subset \tilde{\psi}(\tilde{C})$ where $\tilde{\psi} = \psi^{\frac{1}{4}}$ i.e

$$\tilde{\psi}(\tilde{C}) = (\tilde{\psi}(\tilde{C}) \cap C) \oplus (\tilde{\psi}(\tilde{C}) \cap iC).$$

But since C and $\tilde{\psi}(\tilde{C})$ are both compact it follows that $\tilde{\psi}(\tilde{C}) = C$. In particular we know by the theory of matrix groups that $\psi^t = e^{tad(x)}$ for a suitable $x \in L$, hence ψ^t are inner automorphisms of L. In particular every two compact real forms of L are related by a one parameter subgroup of $Aut(L)$. So it follows that if $G \subset GL(n, \mathbb{C})$ is a matrix group with Lie algebra L then there is an element $g \in G$ such that $g\tilde{C}g^{-1} = C$.

Definition 3.6. **[Conjugacy]**. Let $H_1, H_2 \leq L$ where L is a real Lie algebra then we say H_1 is conjugate to H_2 if we can find an inner automorphism $\psi \in Int(L)$ such that $\psi(H_1) = H_2$. Moreover if θ_1, θ_2 are two automorphisms of L then we say that they are conjugate if we can find an inner automorphism ψ such that $\psi^{-1} \circ \theta_1 \circ \psi = \theta_2$.

4. A Cartan decomposition

Definition 3.7. [Cartan decomposition]. Let L_0 be a real Lie algebra with complexification L. We say that L_0 has a Cartan decomposition if

$$L_0 = (L_0 \cap C) \oplus (L_0 \cap iC)$$

for some compact real form C of L with $\sigma(C) \subset C$.

Corollary 3.2. *Every semisimple real Lie algebra L_0 has a Cartan decomposition.*

Proof. Let L be the complexification of L_0 and C any compact real form of L, also let σ be the conjugation map of L_0. Then we know that there exist a automorphism $\psi \in Aut(L)$ such that σ fixes $\psi(C) = \tilde{C}$ i.e $\sigma(\tilde{C}) \subset \tilde{C}$, in particular $\tilde{C}$ is also a compact real form of L. So if τ denotes the conjugation map of $\tilde{C}$ then σ commutes with τ, and so τ fixes L_0. This shows that

$$L_0 = (L_0 \cap \tilde{C}) \oplus (L_0 \cap i\tilde{C})$$

which is by definition a Cartan decomposition of L_0 as required. $\square$

Consider now any involution $\theta : L_0 \to L_0$ and the Killing form κ of L_0. Then we can construct a new symmetric associative bilinear form, κ_θ on L_0 by defining $\kappa_\theta(-,-) := -\kappa(-,\theta(-))$. We are interested in which involutions θ this bilinear form is positive definite, we will see that this is strongly related to the Cartan decomposition of L_0. So we have the following definition.

Definition 3.8. [Cartan involution]. Let L_0 be a real Lie algebra and suppose there is an involution $\theta : L_0 \to L_0$ such that κ_θ is positive definite. Then we say that θ is a Cartan involution of L_0.

From this definition we immediately see that if C is a compact real form of $L = C^{\mathbb{C}}$ then the identity $1_C = \theta$ serves as Cartan involution, since the Killing form is negative definite on C. Note that if $T_0(+) \oplus P_0(-)$ is the eigenspace decomposition of a Cartan involution θ, then κ restricted to T_0 must be negative definite while κ restricted to P_0 must be positive definite. Also if $t_0 \in T_0$ and $p_0 \in P_0$ then $\kappa(t_0, \theta(p_0)) = \kappa(t_0, -p_0) = \kappa(\theta(t_0), p_0) = \kappa(t_0, p_0)$, i.e $\kappa_\theta(t_0, p_0) = 0$.

Example 3.2. Consider $\mathfrak{sl}(2,\mathbb{R})$ with standard basis $\{e, f, h\}$. Then the involution given by $x \to -x^t$ is a Cartan involution of $\mathfrak{sl}(2,\mathbb{R})$. Indeed let $T_0 \oplus P_0$ be the eigenspace decomposition. Then clearly $T_0 = \langle e - f\rangle$ and $P_0 = \langle h, e + f\rangle$. We have seen that the real subspace C spanned by $\{e - f, ih, i(e + f)\}$ is a compact real form of $\mathfrak{sl}(2,\mathbb{C})$. But since $T_0 \oplus iP_0 = C$ then κ is negative definite on T_0 and positive definite on P_0 as required.

Proposition 3.4. *Let L_0 be a real semisimple Lie algebra with Cartan involution $\theta \in Aut(L_0)$ then the following hold.*

(1) If L is the complexification of L_0 then the real subspace $T_0 \oplus iP_0 \subseteq L^{\mathbb{R}}$ is a compact real form of L.
(2) The eigenspace decomposition $L_0 = T_0 \oplus P_0$ with respect to θ is a Cartan decomposition.

It follows that every semisimple real Lie algebra L_0 has a Cartan involution $\theta : L_0 \to L_0$ which is restricted from an involution $\theta^{\mathbb{C}} : L \to L$ where $L = L_0^{\mathbb{C}}$. In fact L_0 has a Cartan decomposition if and only if L_0 has a Cartan involution.

Proposition 3.5. *Every real semisimple Lie algebra L_0 has a Cartan involution. Moreover every Cartan involution of L_0 can be extended to an involutive automorphism of $L = L_0^{\mathbb{C}}$.*

PROOF. Choose a Cartan decomposition of L_0 say $L_0 = T_0 \oplus P_0$ then we claim that the map $L_0 \to_\theta L_0$ given by $\theta(x+y) = x - y$ is a Cartan involution. To see this first observe that $T_0 = L_0 \cap C$ and $P_0 = L_0 \cap iC$ for some compact form C of L, by definition of a Cartan decomposition. So we must have that κ is negative definite on T_0, while κ is positive definite on P_0. This shows that for $x \in T_0$ and $y \in P_0$,

$$\kappa_\theta(x+y, x+y) = -\kappa(x+y, x-y) = -\kappa(x,x) + \kappa(y,y) > 0$$

for $x + y \neq 0$, while it is zero if and only if $x + y = 0$. So κ_θ is positive definite. Now θ is clearly a linear isomorphism and $\theta^2 = 1$. To show that it is an automorphism in $Aut(L_0)$ we first consider the automorphism

$$\theta^{\mathbb{C}} = \sigma \circ \tau \in Aut(L)$$

where σ, τ are the conjugation maps for L_0, C respectively. Now since σ and τ commute then obviously $\theta^{\mathbb{C}}$ is an involution. It is clear that $\theta^{\mathbb{C}}$ extends θ. In particular θ must be an automorphism as well. This proves the proposition. □

We now follow closely the idea in the proof of Theorem 3.2.

Consider now a Cartan involution θ of L_0 with Cartan decomposition $L_0 = T_0 \oplus P_0$ and set $C = T_0 \oplus iP_0$ for the corresponding compact real form of $L = L_0^{\mathbb{C}}$. Denote $\theta^{\mathbb{C}} : L \to L$ for the extension to L then $\kappa_{\theta^{\mathbb{C}}}$ forms an Hermitian form on L. Now it follows that if we define $\widetilde{\theta^{\mathbb{C}}} = (\theta^{\mathbb{C}})^2$ then the automorphism $\widetilde{\theta^{\mathbb{C}}}^{\,t}$ of L restricts to an involution $L_0 \to L_0$ for all $t \in \mathbb{R}$, since this holds for $\theta^{\mathbb{C}}$. In particular we have a one parameter subgroup $\mathbb{R} \to Aut(L_0)$, so we know that $\tilde{\theta^{\mathbb{C}}}^t = e^{tad(x)}$ for a suitable $x \in L_0$. But then $\tilde{\theta^{\mathbb{C}}}^t \in Int(L_0)$ for every $t \in \mathbb{R}$ and we in particular have

$$\theta = \tilde{\theta^{\mathbb{C}}}^{\frac{1}{2}} \in Int(L_0).$$

This means that if we have a real matrix group G with semisimple Lie algebra L_0, then any Cartan involution has the form $\theta = Ad_g$ for a suitable $g \in G$.

Theorem 3.3. *Suppose L_0 is a semisimple real Lie algebra and*

$$L_0 = T_0 \oplus P_0 = T_1 \oplus P_1$$

are two Cartan decompositions of L_0. Then we can find an inner automorphism $\psi \in Int(L_0)$ such that $\psi(T_0) = T_1$ and $\psi(P_0) = P_1$. In particular $T_0 \cong T_1$ and $P_1 \cong P_0$.

Proof. Denote $T_0 \oplus iP_0 = C$ and $T_1 \oplus iP_1 = \tilde{C}$ for the compact real forms of $L = L_0^{\mathbb{C}}$ w.r.t to the two Cartan decompositions of L_0. Now let $\sigma, \tau, \tilde{\tau}$ be the conjugation maps of $L_0, C, \tilde{C}$ respectively and set $\theta = \tau\tilde{\tau}$ with $\tilde{\theta} = \theta^2$. Then it follows that $\phi(C) = \tilde{C}$ for which $\phi = \tilde{\theta}^{\frac{1}{4}}$ (see Theorem 3.2). We claim that $\phi(T_0) = \tilde{T_0}$ and $\phi(P_0) = \tilde{P_0}$. Indeed $\tilde{\theta}$ restricts to an involution $L_0 \to L_0$ since $\tilde{\tau}(L_0) \subset L_0$ and $\tau(L_0) \subset L_0$ so in particular $\tilde{\theta}(L_0) \subset L_0$. But then this easily extends to $\tilde{\theta}^t$ for any $t \in \mathbb{R}$. So in particular ϕ is an inner automorphism of L_0. Now the theorem follows since σ commutes with both τ and $\tilde{\tau}$. □

It follows from the previous results that any two Cartan involutions $\theta, \tilde{\theta} : L_0 \to L_0$ of L_0 must be conjugate. In particular any Cartan involution θ of a compact real form C of $C^{\mathbb{C}}$ is unique, i.e $\theta = 1_C$.

Let L_0 be semisimple and consider now any involution $\rho : L_0 \to L_0$ of L_0 with eigenspace decomposition $T \oplus P$. Define $\rho^{\mathbb{C}}$ for the extended involution to $L_0^{\mathbb{C}}$ by $\rho^{\mathbb{C}}(x + iy) = \rho(x) + i\rho(y)$ for all $x, y \in L_0$. The following proposition shows when and how it is related to a Cartan involution of L_0.

Proposition 3.6. *ρ is a Cartan involution if and only if $T \oplus iP$ is a compact real form of $L_0^{\mathbb{C}}$. Moreover if $\tilde{\theta}$ is a Cartan involution of L_0 then there exist an inner automorphism $\psi \in Int(L_0)$ such that $\psi \circ \tilde{\theta} \circ \psi^{-1}$ commutes with ρ.*

Proof. If $C = T \oplus iP$ is a compact real form of $L_0^{\mathbb{C}}$ then since $T \subset C$ we must have that κ is negative definite on T. Similarly since $iP \subset C$ then κ restricted P must be positive definite. This shows that ρ is a Cartan involution as required.

Conversely suppose that ρ is a Cartan involution. Then by definition $T \oplus P$ is a Cartan decomposition, hence $T \oplus iP$ is a compact real form of $L_0^{\mathbb{C}}$. The second statement follows from Theorem 3.2 by replacing the following components in the proof: replace τ with $\tilde{\theta}^{\mathbb{C}}$ and σ with $\rho^{\mathbb{C}}$, and set $\theta = \rho^{\mathbb{C}} \circ \tilde{\theta}^{\mathbb{C}}$ then the proof is still valid. So following the proof there is a one parameter subgroup of $Aut(L_0^{\mathbb{C}})$ of the form $\phi^t = (\theta^2)^t$ such that when $t = \frac{1}{4}$ then $\phi^{-t} \circ \theta^{\mathbb{C}} \circ \phi^t$ commutes with $\rho^{\mathbb{C}}$. In particular ϕ^t is a one parameter subgroup of $Aut(L_0)$, hence restricting to L_0 we get the result with $\psi = \phi^{\frac{1}{4}}$. The proposition is proved. $\square$

We now investigate Cartan involutions of $L^{\mathbb{R}}$ where L is a semisimple complex Lie algebra.

Since L is semisimple then $L^{\mathbb{R}}$ is also semisimple. Let $C \leq L^{\mathbb{R}}$ be a compact real form of L. Then the decomposition

$$L^{\mathbb{R}} = C \oplus iC$$

is a Cartan decomposition of $L^{\mathbb{R}}$. To see this we claim that the conjugation map $\theta : L^{\mathbb{R}} \to L^{\mathbb{R}}$ given by $\theta(x + iy) = x - iy$ is a Cartan involution. It is clearly an involutive Lie homomorphism. Now we have seen that the Killing form on $L^{\mathbb{R}}$ is just given by

$$\kappa(x, y) = 2Re(\kappa_L(x, y))$$

for all $x, y \in L^{\mathbb{R}}$. In particular $\kappa(x, x) = 2\kappa_L(x, x)$ for all $x \in L^{\mathbb{R}}$, and so clearly κ is negative definite on C. Similarly we see that $\kappa(ix, ix) = -2\kappa_L(x, x)$ so it positive definite on iC. This shows that θ is a Cartan involution of $L^{\mathbb{R}}$.

Conversely suppose θ is any Cartan involution of $L^{\mathbb{R}}$ with Cartan decomposition $T_0 \oplus P_0 = L^{\mathbb{R}}$. We can choose a compact real form of L, say C. Then we know that there is an inner automorphism $\psi \in Int(L^{\mathbb{R}})$ such that $\psi(T_0) = C$ and $\psi(P_0) = iC$. So in particular T_0 must be a compact real form of L, since κ is negative definite on T_0 and so is positive definite on iT_0. Hence $T_0 \cap iT_0 = 0$. So we have proved the following theorem.

Theorem 3.4. *Let L be a complex semisimple Lie algebra then there is a bijection:*

$$\{Cartan\ involutions\ L^{\mathbb{R}} \to L^{\mathbb{R}}\} \leftrightarrow \{\ Compact\ real\ forms\ of\ L\}.$$

5. Cartan subalgebras of real Lie algebras

Definition 3.9. Let L_0 be real Lie algebra with complexification L then we say that a Lie subalgebra $H \leq L_0$ is a Cartan subalgebra of L_0 if $H^{\mathbb{C}}$ is a Cartan subalgebra of L.

We define the **rank** of L_0 to be the dimension of a Cartan subalgebra of L_0. The rank is well-defined since it can be shown that every Cartan subalgebra of a complex Lie algebra L is conjugate. This is however not the case over the reals, for instance it can be shown that the Cartan subalgebras of $\mathfrak{sl}(2,\mathbb{R})$ can be divided into two conjugacy classes. In fact any semisimple real Lie algebra L_0 has a finite number of conjugate classes, and in the case where L_0 is compact then every Cartan subalgebra is in fact conjugate.

Example 3.3. Consider L a semisimple complex Lie algebra and denote C for the compact form associated to a root decomposition $L = H \oplus_{\alpha\in\Omega} L_\alpha$ where H is a Cartan subalgebra of L. Recall the real subalgebra $H_{\mathbb{R}} \subset H$. It follows that $iH_{\mathbb{R}}$ is an abelian subalgebra of C, moreover we know that $(iH_{\mathbb{R}})^{\mathbb{C}}$ is just H, i.e it is a Cartan subalgebra of C. To illustrate let $L = \mathfrak{sl}(2,\mathbb{C})$ with standard basis $\{e, f, h\}$, then we can take H to be the span of $\{h\}$. Here $iH_{\mathbb{R}}$ is the real span of $\{ih\}$, where C is the real span of $\{ih, i(e+f), e-f\}$.

A useful remark is the following. Consider now a real Lie algebra L_0 with Cartan involution θ with the inner product κ_θ. Write $L_0 = T_0 \oplus P_0$ for the Cartan decomposition w.r.t θ. Then we observe that if $x, y, z \in L_0$ we have

$$\kappa_\theta(x, ad(\theta(z))(y)) = -\kappa(x, \theta([\theta(z), y])) = \kappa([z, x], \theta(y)) = -\kappa_\theta(ad(z)(x), y).$$

So given $x = t_0 + p_0$ for $t_0 \in T_0$ and $p_0 \in P_0$ then $ad(t_0)$ and $ad(p_0)$ are the antisymmetric/symmetric parts of $ad(x)$ w.r.t κ_θ.

Using this observation we can prove that every semisimple real Lie algebra is a matrix Lie algebra. This is a special case of Ado's theorem (see appendix B).

Theorem 3.5. *Let L_0 be a real semisimple Lie algebra the the following is true.*

(1) There is a monomorphism

$$L_0 \hookrightarrow_\psi \mathfrak{gl}(Dim(L_0), \mathbb{R})$$

such that $\psi(L_0)$ is closed under taking transpose.
(2) If θ denotes a Cartan involution of L_0 then the corresponding Cartan involution of $\psi(L_0)$ is given by $A \to -A^t$.

Proof. Denote θ for a Cartan involution of L_0, with corresponding Cartan decomposition $T_0 \oplus P_0$. Now $ad(L_0) \cong L_0$ so we clearly have the following embedding:

$$L_0 \hookrightarrow_{Ad} ad(L_0) \hookrightarrow_\phi \mathfrak{gl}(n, \mathbb{R})$$

where $\phi(ad(x))$ is sent to the matrix which it is represented by w.r.t a orthonormal basis chosen w.r.t the inner product κ_θ. But then $ad(T_0)$ is sent to antisymmetric matrices and $ad(P_0)$ is sent to symmetric matrices. So (1) follows immediately. If we denote composition above by ψ then $\tilde{\theta} = \psi \circ \theta \circ \psi^{-1}$ restricted to $\psi(L_0)$ is a Cartan involution of $\psi(L_0)$. This follows because the Killing form is preserved under the embedding, i.e

$$\kappa_{\psi(L_0)}(\psi(x), \psi(y)) = \kappa_{L_0}(x, y)$$

for all $x, y \in L_0$. In particular

$$-\kappa_{\psi(L_0)}(\psi(x), \tilde{\theta}(\psi(y))) = -\kappa_{L_0}(x, (\theta \circ \psi^{-1})(y))$$

for all $x, y \in L_0$. This shows that it is a Cartan involution of $\psi(L_0)$ with decomposition $\psi(T_0) \oplus \psi(P_0)$ as required. Moreover it is clear that $\tilde{\theta}(A) = -A^t$ for all $A \in \psi(L_0)$. The theorem is proved. $\square$

We immediately see that any semisimple compact Lie algebra C is embedded into the special orthogonal Lie algebra $\mathfrak{so}(n, \mathbb{R})$ for $n = Dim(C)$. Now the dimension of $\mathfrak{so}(n, \mathbb{R})$ can be seen to be $\binom{n}{2}$. So for example $\mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R})$ is semisimple of dimension 6. Hence it has a compact form embedded in $\mathfrak{so}(6, \mathbb{R})$ which has dimension 15. Note that the dimension $n = Dim(C)$ is the best possible, indeed it is easily verified that $\mathfrak{sl}(2, \mathbb{C})$ has a compact real form isomorphic to $\mathfrak{so}(3, \mathbb{R})$.

Recall now the definition of a reductive Lie algebra.

Proposition 3.7. *The following is true.*

(1) A reductive Lie algebra L has the form $L = [L, L] \oplus Z(L)$ where $[L, L]$ is semisimple.
(2) A real Lie algebra $L_0 \subset \mathfrak{gl}(n, \mathbb{R})$ which is closed under taking transpose, i.e $x^t \in L_0$ for all $x \in L_0$ is reductive.

PROOF. For proof of (1) see for example [**5**], chapter 1, section 7, Corollary 1.53. A proof of (2) can be found in [**5**], section 8, Proposition 1.56. $\square$

Let L_0 be a real Lie algebra and H be a Lie subalgebra of L_0. Suppose ψ is an automorphism of L_0. Then we say that H is ψ-**stable** if $\psi(H) \subset H$.

It is now immediate that any θ-stable Lie subalgebra of a semisimple real Lie algebra L_0 must be reductive. Since embed L_0 into $\mathfrak{gl}(n, \mathbb{R})$ and denote this copy by $\tilde{L_0}$, by our results above we can assume $\tilde{L_0}$ is closed under taking transpose. Now the Cartan involution is now identified with $\tilde{\theta}$ given by $\tilde{\theta}(x) = -x^t$ for all $x \in \tilde{L_0}$, so it follows that the copy of H in $\tilde{L_0}$ must be $\tilde{\theta}-$stable. Hence it is reductive, in particular H is reductive.

Having a Cartan decomposition gives rise to Cartan subalgebras in the following way.

Proposition 3.8. *Let L_0 be a real semisimple Lie algebra with Cartan involution θ and Cartan decomposition $L_0 = T_0 \oplus P_0$ then the following hold:*

(1) Let $t_0 \subset T_0$ be maximal abelian then the centralizer $H = C_{L_0}(t_0)$ is a θ-stable Cartan subalgebra of L_0.
(2) Let $p_0 \subset P_0$ be maximal abelian and $t_0 \subset C_{T_0}(p_0)$ be maximal abelian then $t_0 \oplus p_0 = H$ is a Cartan subalgebra of L_0.

Proof. For case (1) it is clear that the subalgebra $H = C_{L_0}(t_0) \subset L_0$ is θ-stable. So in particular we know that H is reductive and has the form $H = T_0 \cap H \oplus P_0 \cap H$. It follows that H is abelian, since $[H, H]$ is semisimple and abelian so must be trivial, i.e $H = Z(H)$. Now H is clearly maximal abelian since if $H \subseteq \tilde{H} \subset L_0$ where $\tilde{H}$ is another abelian subalgebra of L_0, then by definition of H we have $\tilde{H} \subseteq H$. It remains to show that every $h \in H^{\mathbb{C}}$ is semisimple in $L = L_0^{\mathbb{C}}$. If $\theta^{\mathbb{C}}$ denotes the extension of θ, then $\kappa_{\theta^{\mathbb{C}}}$ is an inner product on L. Now every element h of $H^{\mathbb{C}}$ can written as

$$h = -iit_0 + p_0 + i(\tilde{t_0} + \tilde{p_0})$$

for suitable $t_0, \tilde{t_0} \in T_0$ and $p_0, \tilde{p_0} \in P_0$. Choose an orthonormal basis for L w.r.t $\kappa_{\theta^{\mathbb{C}}}$. Now w.r.t this basis it follows that $ad(p_0)$ and $ad(\tilde{p_0})$ are Hermitian, while $ad(\tilde{t_0})$ and $ad(t_0)$ are antisymmetric. In particular $ad(i\tilde{t_0})$ is also Hermitian. But we also have $ad(-iit_0) = -iad(it_0)$, so that this is also Hermitian w.r.t this basis. This proves the result. Case (2) follows in a similar way. □

In particular we have proved the following.

Theorem 3.6. *Every real semisimple Lie algebra L_0 with Cartan involution θ has a θ-stable Cartan subalgebra.* □

6. θ-stable Cartan subalgebras

Although two Cartan subalgebras of a semisimple real Lie algebra are not necessarily conjugate, we prove in this section that every Cartan subalgebra is conjugate to a θ-stable Cartan subalgebra. This is trivially true when the Lie algebra is compact.

Consider now a real semisimple Lie algebra L_0 with complexification L, and suppose H is a Cartan subalgebra of L_0. So we can write a root decomposition of L,

$$L = H^{\mathbb{C}} \oplus_{\alpha \in \Omega} L_\alpha.$$

Recall that there is a compact real form of L associated to every root decomposition. Set C for this compact real form. Now we also recall that $H^{\mathbb{C}} = H_{\mathbb{R}} \oplus iH_{\mathbb{R}}$ for which $H_{\mathbb{R}} \subset iC$ is a real Lie subalgebra of $H^{\mathbb{C}}$. Here $H^{\mathbb{C}} \cap C = iH_{\mathbb{R}}$ is maximal abelian in C. We will use this setup in the following theorem:

Theorem 3.7. *There exist a θ-stable Cartan subalgebra $\tilde{H} \subset L_0$ such that H and $\tilde{H}$ are conjugate.*

Proof. Denote σ, τ_C for the conjugation maps of L_0 and C respectively. Now we know that there is an automorphism $\psi \in Aut(L)$ for which

$$\sigma(\psi(C)) \subset \psi(C).$$

So set $U = \psi(C)$, which is another compact real form of L. We claim that $\psi(H^{\mathbb{C}}) = H^{\mathbb{C}}$. Indeed since $\sigma(H^{\mathbb{C}}) \subset H^{\mathbb{C}}$ and $\tau_C(H^{\mathbb{C}}) \subset H^{\mathbb{C}}$ then obviously this extends to ψ. This follows because we can choose ψ of the form $(\sigma \circ \tau_C)^{\frac{1}{2}}$ (See Theorem 3.2 for details). Also observe that $\sigma(h) = h$ for all $h \in H$. Now we may write a Cartan decomposition

$$L_0 = L_0 \cap U \oplus L_0 \cap iU$$

of L_0 with Cartan involution $\tilde{\theta} = \sigma \circ \tau_U$. There exist an inner automorphism $\gamma \in Int(L_0)$ such that

$$\gamma^{-1}\theta\gamma = \tilde{\theta}.$$

So set $\tilde{H} = \gamma(H)$ as our Cartan subalgebra of L_0, then obviously $\theta(\tilde{H}) = \gamma(\tilde{\theta}(H))$. It is therefore enough to show that $\tilde{\theta}(H) = H$. First note that $\psi(H^{\mathbb{C}} \cap C) = H^{\mathbb{C}} \cap U$ is maximal abelian in U since $H^{\mathbb{C}} \cap C$ is maximal abelian in C. We claim that we can decompose:

$$H^{\mathbb{C}} = H^{\mathbb{C}} \cap U \oplus H^{\mathbb{C}} \cap iU.$$

Indeed suppose there is an $0 \neq h \in H^{\mathbb{C}}$ such that $h \notin H^{\mathbb{C}} \cap U \oplus H^{\mathbb{C}} \cap iU$. We can write $h = a + ib$ for $a, b \in U$ since U is a real form of L, we can assume w.l.g that $a \notin H^{\mathbb{C}} \cap U$. So

$$[H^{\mathbb{C}} \cap U, h] = [H^{\mathbb{C}} \cap U, a] + i[H^{\mathbb{C}} \cap U, b] = 0$$

and therefore we must have $[H^{\mathbb{C}} \cap U, a] = 0$ and $[H^{\mathbb{C}} \cap U, b] = 0$. So we can extend $H^{\mathbb{C}} \cap U$ to a larger abelian Lie subalgebra of U, this contradicts our assumptions.

Hence $\tau_U(H^{\mathbb{C}}) = H^{\mathbb{C}}$. Now since σ commutes with τ_U then for $h \in H$ we have

$$\tilde{\theta}(h) = \sigma(\tau(h)) = \sigma(\tilde{h} + i\tilde{h}') = \tilde{h} - i\tilde{h}' = \tau_U(h) = \tilde{h} + i\tilde{h}'$$

for suitable $\tilde{h}, \tilde{h}' \in H$, i.e $\tilde{h}' = 0$. The theorem is proved. $\square$

Now we consider a real semisimple Lie algebra L_0 with Cartan involution θ and with corresponding Cartan decomposition $L_0 = T_0 \oplus P_0$. Recall that a Lie subalgebra H which is θ-stable has the form $H = T_0 \cap H \oplus P_0 \cap H$.

Definition 3.10. A θ-stable Cartan subalgebra $H \leq L$ of the form $H = t_0 \oplus p_0$ for $t_0 \subset T_0$ and $p_0 \subset P_0$ is said to be **maximally compact** if $t_0 \subset T_0$ is maximal. By maximal we mean that if $t_0 \subset t_1 \subset T_0$ and $t_1 \oplus a_1$ is a θ-stable Cartan subalgebra for some $a_1 \subset P_0$ then $t_1 = t_0$. Similarly we say that H is **maximally non-compact** if $p_0 \subset P_0$ is maximal.

Clearly any Cartan subalgebra of a compact semisimple Lie algebra is maximally compact. In fact the other extrema also exist, i.e we will see that any semisimple complex Lie algebra has a real form with Cartan subalgebra contained in P_0. This real form is known as a **split real form**.

Proposition 3.9. *A θ-stable Cartan subalgebra $H = t_0 \oplus p_0$ of L_0 is maximally compact if and only if t_0 is maximal abelian in T_0. Similarly it is maximally non-compact if and only if p_0 is maximal abelian in P_0.*

PROOF. Suppose $H = t_0 \oplus p_0$ is a θ-stable Cartan subalgebra of L_0. If $t_0 \subset T_0$ is maximal then obviously H is maximal compact. Conversely suppose H is maximally compact. Assume $t_0 \subseteq \tilde{t_0} \subset T_0$ for a maximal abelian subalgebra $\tilde{t_0} \subset T_0$ containing t_0. Then we know that the centralizer $\tilde{H} = C_{L_0}(\tilde{t_0})$ is a θ-stable Cartan subalgebra of L_0. In particular $\tilde{H} = \tilde{t_0} \oplus \tilde{p_0}$ for some $\tilde{p_0} \subset P_0$, so we must have $\tilde{t_0} = t_0$ since H is maximally compact. Hence t_0 is maximal abelian in T_0. Now if p_0 is maximal in P_0 then obviously H is maximally non-compact. Conversely if H is maximally non-compact then assume $p_0 \subset \tilde{p_0} \subset P_0$ is maximal abelian in P_0 containing p_0. Then we can choose a θ-stable Cartan subalgebra of the form $\tilde{H} = \tilde{t_0} \oplus \tilde{p_0}$ where $\tilde{t_0} \subset C_{T_0}(\tilde{p_0})$ is maximal abelian. In particular $p_0 = \tilde{p_0}$, so must be maximal abelian as required. $\square$

7. The split real form

Definition 3.11. [**Split real form**]. Let L be a semisimple complex Lie algebra and L_0 a real form of L. We say L_0 is a split real form if for every Cartan decomposition $L_0 = T_0 \oplus P_0$ we can find a maximal abelian subalgebra $H_0 \leq L_0$ such that $H_0 \subseteq P_0$.

Clearly H_0 must be a Cartan subalgebra of L_0. Indeed we have seen that $H = T \oplus H_0$ is a Cartan subalgebra for which T is maximal abelian in $C_{T_0}(H_0)$. However H is also maximal abelian in L_0, in particular $H_0 = H$, i.e $T = 0$. Note also that if the conditions are satisfied in the definition for one Cartan decomposition then clearly it holds for every Cartan decomposition. This is because Cartan involutions are conjugate.

Example 3.4. $\mathfrak{sl}(2,\mathbb{R})$ is a split real form of $\mathfrak{sl}(2,\mathbb{C})$. Since if $\{e, f, h\}$ is the standard basis of $\mathfrak{sl}(2,\mathbb{C})$ then the span of $\{h\}$ is a Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$. But $h \in \mathfrak{sl}(2,\mathbb{R})$ and $h \in P_0$ where P_0 consists of all symmetric matrices of $\mathfrak{sl}(2,\mathbb{R})$. So we can take H_0 equal to the real span of h.

Having a split real form allows us to consider a real version of a root decomposition of it's complexification. Often this is called the **restricted root decomposition**. To see this let L_0 be a semisimple split real form of $L = L_0^{\mathbb{C}}$ and $L_0 = T_0 \oplus P_0$ be a Cartan decomposition of L_0. Choose $H_0 \subset P_0$ to be a maximal subalgebra of L_0. Set $H = H_0^{\mathbb{C}}$. Now we can choose a basis $\{x_j\}_j$ of L_0 such that every linear map in $ad(H_0)$ is represented by a diagonal matrix. So we clearly have a real root decomposition

$$L_0 = L_o \oplus_{\lambda \in \sum} G_\lambda$$

where $G_\lambda = \{x \in L_0 | [h_0, x] = \lambda(h_0)x, \forall h_0 \in H_0\}$ and $\sum = \{0 \neq \lambda \in H_0^* | G_\lambda \neq 0\}$. Clearly $L_o = H_0$ since H_0 is maximal abelian in L_0.

Now since every element h in H has the form $h = h_0 + ih_0'$ for $h_0, h_0' \in H_0$ then clearly $ad(h)$ is diagonal in L w.r.t the same basis as for L_0. Let $H \oplus_{\alpha \in \Omega} L_\alpha$ be a root decomposition of L w.r.t H. Now given $0 \neq x \in L_\alpha$ we can write $x = \sum_j a_j x_j$ for unique $a_j \in \mathbb{R}$. So given $h_0 \in H_0$ we note that

$$[h_0, x] = \sum_j a_j \lambda_j(h_0) x_j = \sum_j a_j \alpha(h_0) x_j$$

where $x_j \in G_{\lambda_j}$. In particular there is at least one $a_i \neq 0$ so that $\lambda_i(h_0)a_i = \alpha(h_0)a_i$, i.e $\alpha(h_0) = \lambda_i(h_0) \in \mathbb{R}$. This shows that every root in Ω has the form $\alpha = \lambda^{\mathbb{C}}$ for some $\lambda \in \sum$. Here $\lambda^{\mathbb{C}}(h_0 + ih_0') = \lambda(h_0) + i\lambda(h_0')$ for all $h_0, h_0' \in H_0$. So we deduce that $G_\lambda = L_{\lambda^{\mathbb{C}}} \cap L_0$. So our real root decomposition has the form

$$L_0 = H_0 \oplus_{\alpha \in \Omega} (L_\alpha \cap L_0).$$

We end our discussion of semisimple real Lie algebras by showing that every semisimple complex Lie algebra has a split real form. This we do by exploiting the fact that L has a root decomposition, we will see that this split real form is strongly related to the compact real form associated to the root decomposition.

Let L be a complex semisimple Lie algebra with root decomposition

$$L = H \oplus_{\alpha\in\Omega} L_\alpha.$$

Also denote the real Lie algebra $H_{\mathbb{R}} = \oplus_{\alpha\neq -\alpha}\langle t_\alpha\rangle \leq H$ for which $t_\alpha \in H$ are the unique elements satisfying $\kappa(t_\alpha, -) = \alpha(-)$. Equip L with a Cartan-Weyl basis:

$$\{t_\alpha|\alpha\in\Omega\}\cup\{x_\alpha|\alpha\in\Omega\}.$$

We claim that the following subspace of L:

$$L_0 = H_{\mathbb{R}} \oplus_{\alpha\in\Omega} \langle x_\alpha\rangle$$

is a real form of L. First it is clear that L_0 is a real Lie algebra because $[x_\alpha, x_\beta] = \lambda(\alpha,\beta)x_{\alpha+\beta}$ with $\lambda(\alpha,\beta)\in\mathbb{R}$ for every $\alpha,\beta\in\Omega$. Similarly $[t_\alpha, x_\beta] = \beta(t_\alpha)x_\beta$ for every $\alpha,\beta\in\Omega$, where $\beta(t_\alpha)\in\mathbb{R}$. This is all by definition of a Cartan-Weyl basis. Moreover since $H_{\mathbb{R}}^{\mathbb{C}} = H$ and $L_\alpha = \langle x_\alpha\rangle^{\mathbb{C}}$ then clearly $L_0 \oplus iL_0 = L^{\mathbb{R}}$. Now let C be the compact real form of L associated with the root decomposition of L:

$$C = iH_{\mathbb{R}} \oplus_{\alpha\neq-\alpha} \langle i(x_\alpha + x_{-\alpha})\rangle \oplus_{\alpha\neq-\alpha} \langle x_\alpha - x_{-\alpha}\rangle.$$

Then we claim that

$$L_0 = L_0\cap C \oplus L_0 \cap iC.$$

Indeed if τ denotes the conjugation map of C then as $H_{\mathbb{R}}\subset iC$ we have $\tau(H_{\mathbb{R}})\subset H_{\mathbb{R}}$. Also we note that

$$x_\alpha = \frac{1}{2}(x_\alpha - x_{-\alpha}) - \frac{i}{2}i(x_\alpha + x_{-\alpha})$$

for each $\alpha\in\Omega$. So $\tau(x_\alpha) = -x_{-\alpha}$ hence τ fixes $\oplus_{\alpha\in\Omega}\langle x_\alpha\rangle$ as well. So τ fixes L_0 and so the decomposition follows, and in particular if σ denotes the conjugation map of L_0 then $\sigma(C)\subset C$ as the conjugation maps commute. This shows that it is a Cartan decomposition of L_0. Hence the automorphism $\theta = \sigma\circ\tau$ of L is a Cartan involution of L_0 when restricted to L_0. Now $\theta(h) = -h$ for all $h\in H_{\mathbb{R}}$ and $H_{\mathbb{R}}$ is clearly maximal abelian in L_0. Indeed if $h\in L_0$ but $h\notin H_{\mathbb{R}}$ and $[H_{\mathbb{R}}, h] = 0$ then write $h = h_1 + \sum_\alpha \lambda_\alpha x_\alpha$ for $\lambda_\alpha\in\mathbb{R}$, where $\lambda_\beta\neq 0$ say. So that $[t_\beta, h] = 0 = \sum_\alpha \lambda_\alpha\alpha(t_\beta)x_\alpha$, however this is impossible as $\lambda_\beta\beta(t_\beta)\neq 0$. Thus L_0 is a split real form of L, and we have proved the following theorem.

Theorem 3.8. *Every semisimple complex Lie algebra L has a split real form. Moreover every complex semisimple Lie algebra contain at least two non-isomorphic real forms.* $\square$

CHAPTER 4

Real forms and semisimple matrix groups

1. Complexification and realification of matrix groups

Most of what is written in this chapter is based on material from [**11**], [**6**] and [**9**].

Let G be a real matrix group $G \subseteq GL(n,\mathbb{R})$ then the Lie algebra $Lie(G)$ is naturally contained in $\mathfrak{gl}(n,\mathbb{R}) \subseteq \mathfrak{gl}(n,\mathbb{C})$. So therefore so is the complexification, i.e $Lie(G)^{\mathbb{C}} \subseteq \mathfrak{gl}(n,\mathbb{C})$. Now there is a unique connected matrix subgroup $\tilde{G}$ in $GL(n,\mathbb{C})$ with Lie algebra $Lie(G)^{\mathbb{C}}$. So set $G^{\mathbb{C}} = G \cdot \tilde{G}$ then it can be shown that this is a complex matrix group with Lie algebra $Lie(G)^{\mathbb{C}}$. Analogous to the complexification of a Lie algebra we define $G^{\mathbb{C}}$ to be the **complexification** of G.

Proposition 4.1. *$G^{\mathbb{C}}$ is a complex matrix group with complex Lie algebra $Lie(G^{\mathbb{C}}) = Lie(G)^{\mathbb{C}}$.*

It turns out that any element in $G^{\mathbb{C}}$ can be uniquely written as the product ge^{ix} for a suitable $g \in G$ and $x \in Lie(G)$. So that we can write $G^{\mathbb{C}} = Ge^{iLie(G)} = \{ge^{ix} | x \in Lie(G), g \in G\}$. Here the identity component of $G^{\mathbb{C}}$ is generated by the set $\{e^{ix} | x \in Lie(G)\}$.

Here are some standard examples:

Example 4.1.

- $SO(n)^{\mathbb{C}} = SO(n,\mathbb{C})$.
- $SL(n,\mathbb{R})^{\mathbb{C}} = SL(n,\mathbb{C})$.
- $O(p,q)^{\mathbb{C}} = O(p,q,\mathbb{C})$.

We want to define the realification of a complex matrix group G which is analogous to the notion of realification of Lie algebras. Suppose G has complex Lie algebra $Lie(G)$. It is clear that we can embed $GL(n,\mathbb{C})$ into $GL(2n,\mathbb{R})$ via a continuous monomorphism Ψ of groups. Indeed consider the map $\tilde{\Psi} : \mathbb{C} \to M(2,\mathbb{R})$ given by

$$a + ib \to \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

this is clearly a continuous monomorphism. Now we can extend this map to a map $\Psi : M(n, \mathbb{C}) \to M(2n, \mathbb{R})$ given by sending a matrix $(a_{ij})_{ij}$ to the $2n \times 2n$ matrix having block form consisting of 2×2 blocks: $\tilde{\psi}(a_{ij})$. This defines a continuous injective ring homomorphism, in fact it is also easily seen to be $\mathbb{R}$-linear. In particular the differential is well-defined and is just the map itself: $\Psi : \mathfrak{gl}(n, \mathbb{C})^{\mathbb{R}} \to \mathfrak{gl}(2n, \mathbb{R})$. So it is a Lie monomorphism. We define $\Psi(G) = G^{\mathbb{R}}$ to be the **realification** of G. So it is immediate that $Lie(G)^{\mathbb{R}} \cong Lie(G^{\mathbb{R}})$. For later use we will refer to this map as Ψ.

Similarly to Lie algebras we can define real and complex structures on matrix groups, this lead to real forms of complex matrix groups.

Definition 4.1. Let G be a real matrix group then we say that an involution $G \to_J G$ is a complex structure if the differential $Lie(G) \to_{dJ} Lie(G)$ is a **complex structure** on $Lie(G)$. Similarly if G is complex then an involution $G \to_\psi G$ is said to be a real structure on G, if the differential $d\psi$ is a **real structure** on $Lie(G)$, i.e $d\psi : Lie(G) \to Lie(G)$ is an involutory antilinear automorphism.

Definition 4.2. [**Real form**]. Let $G \subset GL(n, \mathbb{C})$ be a complex matrix group with complex Lie algebra $Lie(G)$. A subgroup $H \leq G$ is said to be a real form of G if there is a real structure $\sigma : G \to G$ such that $G^\sigma = H$ (the fix group of σ).

Note that a real form H of G is naturally a matrix subgroup of G. Moreover since there is a real structure σ on G with fix group H, then the real Lie algebra fixed by the differential $d\sigma$ clearly coincides with the tangent space T_1H of H. We note that T_1H is not a complex vector space but rather a real vector space, $Lie(H)$ is a real form of $Lie(G)$.

In particular if G is a real matrix group then it is a real form of $G^{\mathbb{C}}$. Indeed we can consider the complex conjugation $\sigma : G^{\mathbb{C}} \to G^{\mathbb{C}}$ as our real structure. It is clear that $G \subset G^{\mathbb{C}} \cap GL(n, \mathbb{R}) = (G^{\mathbb{C}})^\sigma$. Now suppose there is some element $g \in GL(n, \mathbb{R}) \cap G^{\mathbb{C}}$ which is not in G. Then by removing g from $G^{\mathbb{C}}$ we can make a smaller complex group containing G, this contradicts the definition of $G^{\mathbb{C}}$.

Example 4.2. Consider the complex orthogonal group $O(n, \mathbb{C})$ then the complex conjugation map $g \to \bar{g}$ clearly gives rise to the real form $O(n, \mathbb{R})$. Moreover if we consider the map σ of $O(n, \mathbb{C})$ given by $g \to I_{p,q}\bar{g}I_{p,q}$ then this defines an involution with differential $d\sigma$ given by $X \to I_{p,q}\bar{X}I_{p,q}$. In particular σ is a real structure on $O(n, \mathbb{C})$. So the fix group

$$G_{p,q} = \{g \in O(n, \mathbb{C}) | I_{p,q}\bar{g}I_{p,q} = g\}$$

is a real form of $O(n, \mathbb{C})$ for every p, q such that $p+q=n$. In fact one can show that $G_{p,q}$ can be identified with the orthogonal group $O(p,q)$. So we can think of $O(p,q)$ as a real form of $O(n, \mathbb{C})$.

Definition 4.3. Given a complex matrix group G we say a real form H of G is compact if $Lie(H)$ is a compact real form of $Lie(G)$.

2. Semisimple matrix groups

Definition 4.4. A matrix group G is said to be semisimple if G has a semisimple Lie algebra $Lie(G)$.

So in particular if G is a complex semisimple matrix group then obviously any real form of G is semisimple. Similarly $G^{\mathbb{R}}$ is also semisimple.

Example 4.3. Here are some examples of semisimple matrix groups:

• $O(p,q) = \{X \in GL(n, \mathbb{R}) | X^t I_{p,q} X = I_{p,q}\}$ with semisimple Lie algebra $\mathfrak{o}(p,q)$ where $p+q=n$.
• $SL(n, \mathbb{R}) = \{X \in GL(n, \mathbb{R}) | det(X) = 1\}$ with semisimple Lie algebra $\mathfrak{sl}(n, \mathbb{R})$.
• $SO(n, \mathbb{R}) = \{X \in O(n, \mathbb{R}) | det(X) = 1\}$ with semisimple Lie algebra $\mathfrak{so}(n, \mathbb{R})$.
• $SL(2, \mathbb{R}) \times SO(3, \mathbb{R})$ with semisimple Lie algebra $\cong \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{so}(3, \mathbb{R})$.

We make the following observation: We know that any semisimple Lie algebra L is isomorphic to the adjoint Lie algebra $ad(L)$. In particular L isomorphic to a matrix Lie algebra. So L is isomorphic to a Lie algebra of some semisimple matrix group. This follows because we can take the matrix group of inner automorphisms, $Int(L)$ in this case. So there is a map:

$$\{\textit{semisimple Lie algebras}\} \to \{\textit{ semisimple matrix groups}\}.$$

The following two proceeding theorems show that there is a strong connection between a semisimple matrix group and it's Lie algebra.

Theorem 4.1. *A semisimple real Lie algebra L is compact if and only if there exist a compact matrix group G with Lie(G) $\cong$ L.*

PROOF. For proof see for example [**1**], chapter 2, section 6. □

In particular a compact semisimple matrix group G must have a compact semisimple Lie algebra.

Theorem 4.2. ***[Cartan decomposition]***. *Let G be a semisimple real matrix group and θ be a Cartan involution of $Lie(G)$ and suppose $Lie(G) = T_0 \oplus P_0$ is the corresponding Cartan decomposition. Then the following is true.*

(1) The subgroup $K = \{g \in G | Ad_g \circ \theta = \theta \circ Ad_g\}$ is a matrix subgroup of G with Lie algebra T_0.
(2) There exist an involution $\Theta : G \to G$ such that $d\Theta = \theta$ and $K = G^{\Theta}$.
(3) We can decompose $g \in G$ uniquely as the product $g = ke^{p_0}$ for $p_0 \in P_0$ and $k \in K = G^{\Theta}$, this is known as a ***Cartan decomposition*** *of G w.r.t Θ.*
(4) If the center $Z(G^0)$ is finite and G has finitely many components then K is maximally compact.

PROOF. For proof see for example [**6**], chapter 4, Theorem 3.2. □

In fact one may prove that the involution Θ is unique, in the sense that if $\tilde{\Theta}$ is another involution of G lifting θ and fixing K then $\Theta = \tilde{\Theta}$. So we have the following definition.

Definition 4.5. Let G be a real matrix group then we say an involution $\Theta : G \to G$ is a **Cartan involution** of G if the following is satisfied:

(1) There exist a Cartan involution $\theta : Lie(G) \to Lie(G)$ such that the differential of Θ is θ.
(2) If $K = \{g \in G | Ad_g \circ \theta = \theta \circ Ad_g\}$ then $\Theta(g) = g$ for all $g \in K$.

Here are two examples.

Example 4.4. Consider the semisimple matrix group $SL(2, \mathbb{R})$. Then we can use the Cartan involution $\theta : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R})$ given by $\theta(x) = -x^t$. In particular a Cartan decomposition of $\mathfrak{sl}(2, \mathbb{R})$ is given by $\mathfrak{so}(2, \mathbb{R}) \oplus P_0$. It is clear that $SO(2, \mathbb{R}) \subset K$. Now if $X \in K$ then $(X^t X)A(X^t X)^{-1} = A$ for all $A \in SL(2, \mathbb{R})$. So by a straight forward calculation one can show that $XX^t = I$, or one can simply use the fact that $SO(2, \mathbb{R})$ is connected. So there is a unique matrix subgroup in $SL(2, \mathbb{R})$ with Lie algebra $\mathfrak{so}(2, \mathbb{R})$. We conclude that $SO(2, \mathbb{R}) = K$. A Cartan decomposition of $SL(2, \mathbb{R})$ is therefore given by:

$$SL(2, \mathbb{R}) = SO(2, \mathbb{R})e^{P_0}.$$

Note that Θ in this case is just the involution given by $A \to A^{-t}$ for all $A \in SL(2, \mathbb{R})$.

Example 4.5. Consider the orthogonal group $O(p, q)$ where $p + q = n$ with Lie algebra $\mathfrak{o}(p, q)$ equipped with the Cartan involution $\theta : \mathfrak{o}(p, q) \to \mathfrak{o}(p, q)$ given by $\theta(x) = I_{p,q} x I_{p,q}$. Then a Cartan decomposition of $\mathfrak{o}(p, q)$ is given by $\mathfrak{o}(p, q) = T_0 \oplus P_0$

where T_0 consists of matrices of the form $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ where X, Y are antisymmetric $p \times p$ and $q \times q$ matrices respectively. Now the corresponding Cartan involution Θ of $O(p,q)$ can be shown to be Ad_h where $h = I_{p,q}$. So we see that the fix group of Θ is given by $K = \{g \in G | gh = hg\}$, and it follows that an element in K has the form $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ where X, Y are orthogonal i.e $XX^t = I_p$ and $YY^t = I_q$. This is a maximal compact subgroup of $O(p,q)$ and we see that $K \cong O(p,0) \times O(q,0)$.

In the case where G is compact and semisimple our Lie algebra $Lie(G)$ will have Cartan involution given by the identity $1_{Lie(G)}$. So in the Cartan decomposition theorem $P_0 = 0$ and so our fix group of Θ is just G itself, i.e $K = G$. Hence Θ is also the identity, 1_G.

Corollary 4.1. *Any Cartan involution of a semisimple compact real matrix group G is just the identity $1_G : G \to G$.* $\square$

Given a semisimple matrix group G with Lie algebra L then we know that two Cartan involutions $\theta, \tilde{\theta}$ of L are conjugate, this is also true for two Cartan involutions $\Theta, \tilde{\Theta}$ of G.

Proposition 4.2. *Two Cartan involutions $\Theta, \tilde{\Theta} : G \to G$ of a semisimple matrix group G are conjugate, i.e there is some $g \in G$ such that $g^{-1}\Theta g = \tilde{\Theta}$.*

Given a Cartan involution $\theta : L \to L$, it can be extended to a Cartan involution $\theta^{\mathbb{R}} : (L^{\mathbb{C}})^{\mathbb{R}} \to (L^{\mathbb{C}})^{\mathbb{R}}$. This is a real structure on $L^{\mathbb{C}}$. We now show that this is also possible for semisimple matrix groups, i.e given a Cartan involution $\Theta : G \to G$ lifting θ, we can extend to an involution $\Theta^{\mathbb{C}} : G^{\mathbb{C}} \to G^{\mathbb{C}}$ lifting $\theta^{\mathbb{R}}$. Moreover we show that there is a Cartan decomposition of $G^{\mathbb{C}}$ w.r.t $\Theta^{\mathbb{C}}$.

To see this let $G^{\mathbb{C}} = \tilde{G}$ and consider the realification $\Psi(\tilde{G}) = \tilde{G}^{\mathbb{R}} \subset GL(2n, \mathbb{R})$. Choose a Cartan involution for G say Θ, write a Cartan decomposition $Lie(G) = T_0 \oplus P_0$ w.r.t $d\Theta$. Denote also K for the fix group of Θ. So via Ψ we obtain a new Cartan involution of $\Psi(G)$:

$$\tilde{\Theta} = \Psi \circ \Theta \circ \Psi^{-1}$$

with fix group $\Psi(K)$. Now the realification of $\tilde{G}$ is also semisimple, and we can write a Cartan decomposition $Lie(\Psi(G)) = \Psi(T_0) \oplus \Psi(P_0)$ for $\Psi(G)$ w.r.t the differential $d\tilde{\Theta}$. Thus we can extend the Cartan involution $d\tilde{\Theta}$ to a Cartan involution of $Lie(\tilde{G}^{\mathbb{R}})$, say $d\tilde{\Theta}^{\mathbb{R}}$. This follows because if

$$C = \Psi(T_0) \oplus i\Psi(P_0)$$

then $C \oplus iC$ is a Cartan decomposition of $Lie(\tilde{G}^{\mathbb{R}})$. Denote $\tilde{\Theta}^{\mathbb{R}}$ for the corresponding Cartan involution on $\tilde{G}^{\mathbb{R}}$. Now the fix group $\tilde{U}$ of $\tilde{\Theta}^{\mathbb{R}}$ has the form

$$\tilde{U} = \{g \in \tilde{G}^{\mathbb{R}} | Ad_g \circ d\tilde{\Theta}^{\mathbb{R}} = d\tilde{\Theta}^{\mathbb{R}} \circ Ad_g\}$$

and so we clearly see that the Cartan involution of $\tilde{G}^{\mathbb{R}}$ is just an extension of $\tilde{\Theta}$ on $\Psi(G)$. This shows that $\Psi(K) = \Psi(G) \cap \tilde{U}$.

Finally transferring all this back to $\tilde{G} = G^{\mathbb{C}}$ via Ψ^{-1} we see that there is an involution $\Theta^{\mathbb{C}} : G^{\mathbb{C}} \to G^{\mathbb{C}}$ extending $\Theta : G \to G$. Moreover using the Cartan decomposition theorem on $\tilde{G}^{\mathbb{R}}$, then every element g in $G^{\mathbb{C}}$ can be written uniquely as the product $g = ue^p$ for $u \in U$ (the fix group of $\Theta^{\mathbb{C}}$ which is $U = \Psi^{-1}(\tilde{U})$) and $p \in iT_0 \oplus P_0$. In particular $K = U \cap G$.

Note that U is a real form of $G^{\mathbb{C}}$ since $\Theta^{\mathbb{C}}$ is a real structure with differential $d\Theta^{\mathbb{R}}$. If G happens to be compact then the groups G, K, U must all coincide. Indeed since G is compact then we know that $Lie(G)$ is compact so $K = G$ since $P_0 = 0$. However $\Theta^{\mathbb{C}}$ is now the just the conjugation map, i.e $U = G$ as required.

We have proved the following theorem.

Theorem 4.3. *Every Cartan involution Θ of G extends to a real structure $\Theta^{\mathbb{C}}$ of $G^{\mathbb{C}}$. Moreover every element in $G^{\mathbb{C}}$ can be written uniquely as the product $g = ue^p$ where $u \in U$ (the fix group of $\Theta^{\mathbb{C}}$) and $p \in iT_0 \oplus P_0$ (where $T_0 \oplus P_0$ is the Cartan decomposition of $Lie(G)$).* $\square$

We will use this result in the next chapter.

CHAPTER 5

Real orbits of semisimple matrix groups

1. Preliminaries

In this chapter we assume that a vector space V is finite dimensional over $\mathbb{R}$ or $\mathbb{C}$, and all topological properties will be w.r.t the classical topology on V, inherited from $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively. If we speak of an inner product $\langle -,-\rangle$ on V then we will define $||v|| = \langle v,v\rangle^{\frac{1}{2}}$.

Fix a basis for V. Let $End(V)$ be equipped with the usual norm metric $||,||$ inherited from $M(n,\mathbb{K})$, i.e if $f : V \to V$ is a linear map represented by a matrix A w.r.t this fixed basis then: $||f|| = ||A||$. In this way $GL(V)$ is isomorphic to $GL(n,\mathbb{K})$. We can define analogously the exponential of a linear map

$$exp(f) = \sum_{0\le n<\infty} \frac{1}{n!} f^n.$$

Similarly we can define a curve $\gamma : (a,b) \to GL(V)$ to be differentiable if and only if the curve $(a,b) \to_\gamma GL(V) \to_\phi GL(n,\mathbb{K})$ is differentiable. Here ϕ sends an invertible map to it's matrix. If the curve is differentiable we set $\gamma'(t_0)$ to be the linear map given by the matrix $(\phi\circ\gamma)'(t_0)$. In this way the tangent space of $GL(V)$ coincides with $\mathfrak{gl}(V)$. In particular if $f \in \mathfrak{gl}(V)$ then $exp(f) \in GL(V)$.

Definition 5.1. Let G be a matrix group and V be a vector space. Then a continuous group homomorphism $G \to GL(V)$ is said to be a **representation** of G or a **continuous linear group action** on V. We will often just say G acts on a vector space V.

A representation $\psi : G \to GL(V)$ is smooth if and only if $G \to GL(n,\mathbb{K})$ is smooth factoring through $GL(V)$ via ϕ. Hence the differential $Lie(G) \to \mathfrak{gl}(n,\mathbb{K})$ of $\phi\circ\psi$ factors through $\mathfrak{gl}(V)$ via the Lie isomorphism $\mathfrak{gl}(V) \to \mathfrak{gl}(n,\mathbb{K})$, which is given by sending a linear map to it's matrix. Thus the differential $Lie(G) \to \mathfrak{gl}(V)$ of ψ is a well-defined Lie homomorphism.

We will not prove this here but it turns out that any continuous homomorphism $G \to H$ is a Lie homomorphism. In particular there is a well-defined differential attached to any representation which is in particular a representation itself.

Proposition 5.1. *Let G, H be matrix groups then any continuous homomorphism $G \to H$ is a Lie homomorphism.*

PROOF. See for example [**11**], section 1.3.6, Proposition 1.3.14. □

Lemma 5.1. *Any representation $\psi : G \to GL(V)$ extends to a complex representation $\psi^{\mathbb{C}} : G^{\mathbb{C}} \to GL(V^{\mathbb{C}})$. Moreover the differential $d\psi^{\mathbb{C}}$ is a complex representation.*

We will refer to the following example throughout this chapter as an illustration of the results.

Example 5.1. Consider a semisimple real matrix group G with Lie algebra $\mathfrak{g}$. Then the adjoint action $Ad : G \to GL(\mathfrak{g})$ is a representation, and the differential is just the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. We see that the adjoint representation $Ad^{\mathbb{C}}$ of $G^{\mathbb{C}}$ acting on $\mathfrak{g}^{\mathbb{C}}$ extends Ad. Similarly if $\tilde{G}$ is another real form of $G^{\mathbb{C}}$ with Lie algebra $\tilde{\mathfrak{g}}$ then $Ad^{\mathbb{C}}$ extends the adjoint action of $\tilde{G}$ acting on $\tilde{\mathfrak{g}}$ as well.

Definition 5.2. Let G be a group which acts on an inner product space V then the **orbit space** at $v \in V$ is defined to be the induced subspace $Gv = \{g \cdot v | g \in G\} \subseteq V$ of V.

Definition 5.3. Let $G \to GL(V)$ be a group action on an inner product space V. Then we say that a vector $v \in V$ is **minimal** if for all $g \in G$:

$$||g \cdot v|| \geq ||v||.$$

We denote the set of minimal vectors by $\mathcal{M}(G, V)$.

Given a representation $\psi : G \to GL(V)$ and any $v \in V$ it is straightforward to check that the stabilizer/isotropy subgroup $G_v = \{g \in G | g \cdot v = v\}$ is a matrix subgroup of G. So we can define the **isotropy** Lie algebra $Lie(G_v) \leq Lie(G)$. It is easy to check that $Lie(G_v)$ consists of elements x of $Lie(G)$ such that $d\psi(x)(v) = 0$.

2. Minimal vectors and closure of semisimple real orbits

In this section we will always assume that $G \subseteq GL(n, \mathbb{R})$ is a real semisimple matrix group. We will follow closely what is written in [**2**]. Our setup is as follows:

Suppose we have a representation $\psi : G \to GL(V)$ where V is a real finite dimensional vector space. Now we will assume that there is a Cartan involution $\Theta : G \to G$ of G with differential θ, and an inner product $\langle -, - \rangle_\theta$ on V which is $K = G^\Theta-$invariant, i.e

$$\langle k \cdot v, k \cdot \tilde{v} \rangle_\theta = \langle v, \tilde{v} \rangle_\theta$$

for all $v, \tilde{v} \in V$ and $k \in K$. Let $Lie(G) = T_0 \oplus P_0$ be the corresponding Cartan decomposition w.r.t $\theta : Lie(G) \to Lie(G)$. We will assume that the differential $d\psi$ of ψ has the property such that if $p_0 \in P_0$ then $d\psi(p_0)$ is symmetric w.r.t $\langle -, - \rangle_\theta$, and similarly if $t_0 \in T_0$ then $d\psi(t_0)$ is antisymmetric w.r.t $\langle -, - \rangle_\theta$. Denote the Cartan decomposition of G: $G = Ke^{P_0}$ w.r.t Θ.

Example 5.2. Consider the adjoint action $Ad : G \to GL(\mathfrak{g})$. Then there is a natural inner product on $\mathfrak{g}$ which satisfies the criteria above. Indeed consider the Killing form of $\mathfrak{g}$ together with a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$. We know that the bilinear form

$$\langle -, - \rangle_\theta = \lambda \kappa_\theta(-, -) = -\lambda \kappa(-, \theta(-))$$

for any $\lambda > 0$ is an inner product on $\mathfrak{g}$. If we denote Θ for the Cartan involution of G lifting θ, then the Killing form is clearly invariant under automorphisms Ad_g for $g \in G$. Moreover by definition of K the automorphisms Ad_k commute with θ, i.e the inner product is invariant under K. We have also seen that if $x \in \mathfrak{g}$ is decomposed as $x = t_0 + p_0$ w.r.t θ, then $ad(x)$ has symmetric/antisymmetric parts $ad(p_0)$ and $ad(t_0)$ w.r.t $\langle -, - \rangle_\theta$ respectively.

In fact an inner product with the properties above always exist on V when G is a semisimple matrix group. The image of a semisimple Lie algebra under a Lie homomorphism is also semisimple, so the identity component $\psi(G)_0$ of $\psi(G)$ is also a semisimple matrix group. In particular if θ is a Cartan involution of $Lie(G)$ then it can be shown that the involution: $d\psi(x) \to d\psi(\theta(x))$ is a Cartan involution of $d\psi(Lie(G))$. So there is a Cartan involution Θ' of $\psi(G)_0$ with Cartan decomposition

$$\psi(G)_0 = \psi(K)_0 e^{d\psi(P_0)}.$$

The condition $\psi \circ \Theta = \Theta' \circ \psi$ ensures that such an inner product exists.

We state this result as a lemma.

Lemma 5.2. *Let $G \to GL(V)$ be a finite real representation of a real semisimple matrix group G. Then for any choice of Cartan involution θ of $Lie(G)$ there exist an inner product $\langle -, - \rangle_\theta$ with the properties above.*

PROOF. For details about the proof see for example [**12**], Proposition 13.5. □

Having the setup described above, our aim in this section is to show the relationship between minimal vectors, the Lie algebra $Lie(G)$ and the closure of a real orbit. We start with some technical lemma's.

Lemma 5.3. *Let $v \in V$ and $\psi \in \mathfrak{gl}(V)$ be a self adjoint map w.r.t $\langle -,-\rangle_\theta$. Also let $f : \mathbb{R} \to \mathbb{R}$ be the map given by $f(t) = ||e^{t\psi}(v)||^2$ for all $t \in \mathbb{R}$. Then if $\psi(v) \neq 0$ we have $f''(t) > 0$ for all $t \in \mathbb{R}$. Moreover f is smooth.*

Proof. Since ψ is self adjoint then all eigenvalues are real and we may write $V = \oplus_\lambda V_\lambda$ as a decomposition into the eigenspaces of ψ. In particular we may choose a basis of eigenvectors $\{v_j\}_j$ for V, and clearly $\langle v_i, v_j\rangle_\theta = 0$ for all $i \neq j$. So with respect to this basis ψ has a diagonal matrix with diagonal entries say, λ_j. Write $v = \sum_j a_j v_j$ for $a_j \in \mathbb{R}$. Then

$$f(t) = \sum_j e^{2t\lambda_j}||v_j||^2 a_j^2$$

since $e^{t\psi}$ is diagonal with diagonal entries $e^{t\lambda_j}$. So

$$f'(t) = \sum_j 2\lambda_j e^{2t\lambda_j}||v_j||^2 a_j^2$$

and therefore

$$f''(t) = \sum_j 4\lambda_j^2 ||v_j||^2 a_j^2 e^{2t\lambda_j} \geq 0.$$

Now since $\psi(v) \neq 0$ then obviously there is some $\lambda_i a_i \neq 0$ so that $f''(t) > 0$ for all $t \in \mathbb{R}$ as required. □

The lemma shows that if $p_0 \in P_0$ is such that $d\psi(p_0)(v) \neq 0$ (i.e $p_0 \notin Lie(G_v)$) then the function $f(t) = ||e^{tp_0} \cdot v||^2$ has the property: $f''(t) > 0$ for all $t \in \mathbb{R}$. Indeed this follows because

$$e^{tp_0} \cdot v =^{def} \psi(e^{tp_0})(v) = e^{td\psi(p_0)}(v),$$

and $d\psi(p_0) \in \mathfrak{gl}(V)$ is assumed to be self-adjoint w.r.t $\langle -,-\rangle_\theta$. We will use this map in the upcoming proofs.

Lemma 5.4. *If $\mathbb{R} \to_f \mathbb{R}$ is a smooth function satisfying the conditions:*

(1) For all $t \in \mathbb{R}$ we have $f''(t) > 0$.
(2) There exist $t_0 \in \mathbb{R}$ such that $f'(t_0) = 0$.

Then $f(t) > f(t_0)$ for all $t_0 \neq t \in \mathbb{R}$ and we have $lim_{t\to\pm\infty} f(t) = \infty$.

We now define a function $F_v : G \to \mathbb{R}^\times$ for a vector $0 \neq v \in V$ given by $F_v(g) = ||g \cdot v||^2 = \langle g \cdot v, g \cdot v\rangle_\theta$ for all $g \in G$. One can show that there is a well-defined differential $dF_v : Lie(G) \to \mathbb{R}$ given by

$$dF_v(x) = 2\langle x \cdot v, v\rangle_\theta$$

for all $x \in Lie(G)$. Here $x \cdot v = d\psi(x)(v)$, where $d\psi$ denotes the differential of the group representation $G \to_\psi GL(V)$. To see that this is true consider γ a curve at $1 \in G$ and let $\{e_j\}_j$ be an orthonormal basis for V. Also set $x = \gamma'(0) \in Lie(G)$ and suppose $\psi(\gamma(t))$ has matrix $(a_{ij}(t))_{ij}$ w.r.t to this basis. So $d\psi(\gamma'(0))$ has matrix $(a'_{ij}(0))_{ij}$. Write $v = \sum_j \lambda_j e_j$ for $\lambda_j \in \mathbb{R}$. Then we have

$$(\langle \gamma(t) \cdot v, \gamma(t) \cdot v \rangle_\theta)'(0) = 2 \sum_{l,j} \lambda_l \lambda_j a'_{jl}(0) = 2\langle x \cdot v, v \rangle_\theta.$$

This is clearly a Lie homomorphism since $\mathbb{R}$ is abelian.

So in the case where $f(t) = ||e^{tx} \cdot v||^2$ for $x \in Lie(G)$ we get

$$f'(0) = 2\langle x \cdot v, v \rangle_\theta = dF_v(x).$$

We will say that $1 \in G$ is a **critical point** for F_v if the differential $dF_v : Lie(G) \to \mathbb{R}$ is not surjective, which is the case if and only if dF_v is the zero map, since $\mathbb{R}$ is 1-dimensional.

Corollary 5.1. *If $1 \in G$ is a critical point of the function F_v for some $v \in V$ then $||e^x \cdot v|| \geq ||v||$ for every $x \in P_0$. Moreover if $x \in P_0$ then $||e^x \cdot v|| = ||v||$ if and only if $x \in Lie(G_v) \cap P_0 = P_v$ where G_v is the isotropy subgroup of G.*

Proof. Suppose $0 \neq x \in P_0$ and $x \notin Lie(G_v)$, so in particular $d\psi(x)(v) \neq 0$. Therefore we can consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(t) = ||e^{tx} \cdot v||^2$$

for all $t \in \mathbb{R}$. We know that $f''(t) > 0$ for all $t \in \mathbb{R}$. Moreover since $1 \in G$ is a critical point for F_v then $dF_v(x) = 0$ for all $x \in Lie(G)$. But we recall that $dF_v(x) = f'(0)$ so $f'(0) = 0$. Therefore we know that the function f also satisfies: $f(t) > f(0) = ||v||^2$ for all $t \neq 0$, and so $f(1) = ||e^x \cdot v||^2 > f(0) = ||v||^2$ as required. For the last statement, if $x \in Lie(G_v)$ then $e^x \in G_v$ so by definition $e^x \cdot v = v$, showing that $f(1) = f(0) = ||v||^2$. Conversely if $||e^x \cdot v|| = ||v||$ for $x \in P_0$ and $x \notin Lie(G_v)$ then by the argument above we get a contradiction, i.e $x \in P_v$. The corollary is proved. $\square$

The corollary shows the following: Write $g \in G$ of the form $g = ke^{p_0}$ where $k \in K$ and $p_0 \in P_0$. Then we have

$$||g \cdot v|| = ||ke^{p_0} \cdot v|| = ||e^{p_0} \cdot v||$$

for all $v \in V$ by the K-invariance of $\langle -, - \rangle_\theta$. In particular if $1 \in G$ is a critical point of F_v for some $v \in V$ then

$$||g \cdot v|| \geq ||v||,$$

hence v is minimal, i.e $v \in \mathcal{M}(G, V)$.

The following theorem show the full relationship between minimal vectors and the non-compact part P_0 of the Cartan involution θ.

Theorem 5.1. *Let $v \in V$ then the following are equivalent.*

(1) $v \in \mathcal{M}(G, V)$.
(2) $1 \in G$ is a critical point for the function F_v.
(3) For all $x \in P_0$ we have $dF_v(x) = 0$.

PROOF. Suppose v is a minimal vector in V then $F_v(g) \geq F_v(1)$ for all $g \in G$. Let $x \in Lie(G)$. Now since $dF_v(x) = f'(0)$ for which $f(t) = ||e^{tx} \cdot v||$, then obviously f has a minimum at $t = 0$, hence $dF_v(x) = 0$. So $1 \in G$ is a critical point for F_v. This proves $[(1) \Rightarrow (2)]$. Case $[(2) \Rightarrow (3)]$ is clear. The case $[(3) \Rightarrow (1)]$ has already been shown. This proves the theorem. $\square$

Consider as an example the adjoint action $Ad : G \to GL(\mathfrak{g})$ where $Lie(G) = \mathfrak{g}$. The previous theorem says that $x \in \mathfrak{g}$ is minimal if and only if $\kappa([p_0, x], \theta(x)) = 0$ for all $p_0 \in P_0$. The next proposition show what this means in terms of the properties of the Lie algebra $\mathfrak{g}$.

Proposition 5.2. *Let $x \in \mathfrak{g}$ then the following are equivalent.*

(1) $x \in \mathcal{M}(G, \mathfrak{g})$ is a minimal vector.
(2) $[x, \theta(x)] = 0$.
(3) There exist a maximally compact Cartan subalgebra H of $\mathfrak{g}$ containing x which is θ-stable, i.e $\theta(H) = H$.

In particular we see that $T_0 \cup P_0 \subseteq \mathcal{M}(G, \mathfrak{g})$.

PROOF. $[(1) \Leftrightarrow (2)]$. By definition $x \in \mathfrak{g}$ is minimal if and only if

$$-\kappa([p_0, x], \theta(x)) = 0 = -\kappa(p_0, [x, \theta(x)])$$

for all $p_0 \in P_0$. However if $[x, \theta(x)] \neq 0$ then κ would be degenerate, which contradicts the semisimplicity of $\mathfrak{g}$. $[(3) \Rightarrow (1)]$. If x is contained in a θ-stable Cartan subalgebra $H \subseteq \mathfrak{g}$ then obviously we have $H = T_0 \cap H \oplus P_0 \cap H$. So if $x = t_0 + p_0$ for $t_0 \in T_0 \cap H$ and $p_0 \in P_0 \cap H$, then $[t_0, p_0] = 0$ and by the previous equivalence x is minimal in $\mathfrak{g}$ as required. $[(2) \Rightarrow (3)]$. Suppose that $[x, \theta(x)] = 0$ for some $x \in \mathfrak{g}$ written as $t_0 + p_0 = x$. We can choose a maximal abelian subalgebra $H_0 \subset T_0$ containing t_0. So that x is contained in the centralizer, $x \in C_{\mathfrak{g}}(H_0)$ since $[t_0, p_0] = 0$. It follows that $H = C_{\mathfrak{g}}(H_0)$ is a Cartan subalgebra of $\mathfrak{g}$ and is clearly θ-stable, since if $h \in T_0$ and

$\tilde{h} \in H$ then $\theta([h, \tilde{h}]) = 0 = [h, \theta(\tilde{h})]$. We see that H is maximally compact since H_0 is chosen maximal abelian in T_0. $\square$

Example 5.3. As an example consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, the split real form of $\mathfrak{sl}(2, \mathbb{C})$. Choose the Cartan involution given by $X \to -X^t$. Then one easy calculate that $T_0 \cup P_0 = \mathcal{M}(G, \mathfrak{g})$.

Coming back to the more general case, the following corollary describes the set of minimal vectors in a real orbit.

Corollary 5.2. *Suppose $v \in \mathcal{M}(G, V)$ then the following is true.*

(1) $Gv \cap \mathcal{M}(G, V) = Kv$.
(2) $G_v = K_v e^{P_v}$ where $K_v = \{g \in K | g \cdot v = v\}$ and $P_v = Lie(G_v) \cap P_0$.

Proof. Suppose $v \in \mathcal{M}(G, V)$ then clearly $Kv \subseteq \mathcal{M}(G, V)$, since if $k \in K$ then obviously $||gk \cdot v|| \geq ||v|| = ||k \cdot v||$ for all $g \in G$, as v is minimal. Conversely suppose that $\beta = g \cdot v \in \mathcal{M}(G, V)$ and write $g = ke^p$ for $k \in K$ and $p \in P_0$. Now we have,

$$||g^{-1} \cdot \beta|| = ||v|| \geq ||\beta||,$$

but since v is also minimal then $||\beta|| \geq ||v||$ so $||g \cdot v|| = ||e^p \cdot v|| = ||v||$ hence $p \in P_v$. This shows that $g \cdot v = k \cdot v$, and so (1) is proved. Now suppose that $g = ke^p$ for $k \in K$ and $p \in P_0$, assume $g \in G_v$. Then similarly

$$||e^p \cdot v|| = ||g \cdot v|| = ||v||$$

so that $p \in P_v$ since v is minimal. In particular $g \cdot v = k \cdot v = v$ which shows that $k \in K_v$. Now the inclusion $K_v e^{P_v} \subseteq G_v$ is clear, this shows case (2). $\square$

Note in the case where G is also compact then G coincides with K, and clearly $\mathcal{M}(G, V) = V$. So every orbit is closed.

Lemma 5.5. *Let $v \in V$ and suppose the orbit space Gv is not closed. Then there exist $x \in P_0$ such that $lim_{t \to -\infty} e^{tx} \cdot v \in V$ exist w.r.t to the classical topology on V. Moreover if this limit is $\alpha \in V$ then the real orbit $G\alpha$ is closed in V.*

Proof. For proof see [**2**], Lemma 3.3. $\square$

We note that the function $f : \mathbb{R} \to \mathbb{R}$ given by $t \to ||e^{tx} \cdot v||^2$ is clearly continuous for any $v \in V$ and $x \in Lie(G)$. Moreover if $e^{tx} \cdot v \to \beta$ for $t \to -\infty$ then as $\big| ||e^{tx} \cdot v|| - ||\beta|| \big| \leq ||e^{tx} \cdot v - \beta||$ we have $||e^{tx} \cdot v|| \to ||\beta||$ as $t \to -\infty$. So in particular if the limit of $f(t)$ as $t \to -\infty$ does not exist, then the limit of $e^{tx} \cdot v$ does not exist either. We will use this simple observation in the next theorem.

The following theorem shows the connection between a minimal vector and the closure of a real orbit.

Theorem 5.2. *Let $v \in V$ then the orbit space Gv is closed if and only if Gv contains a minimal vector $\alpha \in \mathcal{M}(G,V)$.*

Proof. We show the direction [($\Leftarrow$)]. If Gv is not closed but there is some minimal vector $\alpha \in Gv$ then we can find $x \in P_0$ such that $e^{tx} \cdot \alpha \to \beta$ as $t \to -\infty$ for a suitable $\beta \in V$. Moreover $G\beta$ is closed. We claim that $x \cdot \alpha \neq \alpha$. Indeed if this was the case then for any $t \in \mathbb{R}$ we have $tx \in Lie(G_\alpha)$. So that $e^{tx} \cdot \alpha = \alpha$ for all $t \in \mathbb{R}$, and consequently $\beta = \alpha$ so that $G\beta = Gv$ and Gv would also be closed, this contradicts our assumption. Now $1 \in G$ is a critical point of the function F_α. So by considering the smooth function $f(t) = ||e^{tx} \cdot \alpha||^2$ for $t \in \mathbb{R}$, we conclude by the previous results that $f(t) \to \infty$ as $t \to -\infty$. This is impossible. $\square$

Corollary 5.3. *Let $v \in V$ and Gv be an orbit. Then the closure $cl(Gv)$ of Gv contains a minimal vector in $\mathcal{M}(G,V)$.*

Proof. If Gv is closed we are done. Assume Gv is not closed then we can choose $x \in P_0$ such that the sequence $(a_n)_{n\in\mathbb{N}} = e^{-nx} \cdot v$ is a sequence in Gv which converges to a vector $\gamma \in V$ such that $G\gamma$ is closed in V. But clearly given $g \in G$ the sequence ga_n in Gv converges to $g\gamma \in G\gamma$. So that $G\gamma \subseteq cl(G\gamma)$. Hence we can find $\beta \in G\gamma \subseteq cl(G\gamma)$ such that $\beta \in \mathcal{M}(G,V)$ as required. $\square$

3. Complex versus the real case

In this section we will explore the connection between real and complex orbits under the actions of a complex semisimple matrix group $G^{\mathbb{C}} \subseteq GL(n,\mathbb{C})$ with corresponding real form $G \subseteq GL(n,\mathbb{R})$. We will follow the same setup for our real semisimple matrix group G as in the previous section. To relate G and $G^{\mathbb{C}}$ we will do via their Cartan decompositions (see the end of the previous chapter). So our setup is as follows.

Let $\psi^{\mathbb{C}} : G^{\mathbb{C}} \to GL(V^{\mathbb{C}})$ be an extended representation of G and denote $d\psi^{\mathbb{C}}$ for the differential. Let

$$\Theta^{\mathbb{C}} : G^{\mathbb{C}} \to G^{\mathbb{C}}$$

be the extension of Θ with differential $\theta^{\mathbb{R}}$ (Cartan involution of $Lie(G^{\mathbb{C}})^{\mathbb{R}}$). So we have Cartan decompositions:

$$Lie(G) = T_0 \oplus P_0,\ Lie(G^{\mathbb{C}})^{\mathbb{R}} = (T_0 \oplus iP_0) \oplus (iT_0 \oplus P_0) = C \oplus iC$$

w.r.t θ and $\theta^{\mathbb{R}}$. Similarly for G and $G^{\mathbb{C}}$ we have Cartan decompositions $G^{\mathbb{C}} = Ue^{iC}$ and $G = Ke^{P_0}$ w.r.t Θ and $\Theta^{\mathbb{C}}$. Here U has compact Lie algebra $T_0 \oplus iP_0 = C$. In fact the following is true for U.

Lemma 5.6. *Every element in U can be written uniquely as the product ke^{ip_0} for $k \in K$ and $p_0 \in P_0$.*

Write as before $\langle -, - \rangle_\theta$ for the inner product on V as in the previous section with norm $||v||^2 = \langle v, v \rangle_\theta$. We will now consider an inner product on the realification of $V^{\mathbb{C}}$, defined by

$$\langle x + iy, \tilde{x} + i\tilde{y} \rangle_{\theta^{\mathbb{R}}} = \langle x, \tilde{x} \rangle_\theta + \langle y, \tilde{y} \rangle_\theta$$

for all $x, \tilde{x}, y, \tilde{y} \in V$. This clearly extends the inner product on V. Now since we have the notion of the realification of $G^{\mathbb{C}}$ which is a real semisimple matrix group in $GL(2n, \mathbb{R})$, then we can study the closure and minimal vectors of a complex orbit of $\psi^{\mathbb{C}}$ via the realification $(G^{\mathbb{C}})^{\mathbb{R}}$. Moreover since we have an inner product which extends $\langle -, - \rangle_\theta$, then we can relate minimal vectors of real orbits of ψ to minimal vectors of complex orbits of $\psi^{\mathbb{C}}$. We show this connection in the next results.

Lemma 5.7. *The inner product $\langle -, - \rangle_{\theta^{\mathbb{R}}}$ on the realification of $V^{\mathbb{C}}$ is U-invariant.*

PROOF. Since every element in U can be written uniquely as the product ke^{ip_0} for $k \in K$ and $p_0 \in P_0$, then it is enough to show that the linear map $d\psi^{\mathbb{C}}(ip_0)$ is antisymmetric w.r.t $\langle -, - \rangle_{\theta^{\mathbb{R}}}$. This however follows by an easy calculation since $d\psi^{\mathbb{C}}$ is a complex representation extending $d\psi$, and $d\psi(p_0)$ is symmetric w.r.t $\langle -, - \rangle_\theta$. $\square$

Example 5.4. Consider $Ad^{\mathbb{C}} : G^{\mathbb{C}} \to GL(\mathfrak{g}^{\mathbb{C}})$ to be the adjoint action extending $Ad : G \to GL(\mathfrak{g})$. Denote $\langle -, - \rangle_\theta = \lambda\kappa_\theta(-, -)$ $(\lambda > 0)$ for the usual inner product on $\mathfrak{g}$. Then we can take our inner product $\langle -, - \rangle_{\theta^{\mathbb{R}}}$ on the realification of $\mathfrak{g}^{\mathbb{C}}$ to be

$$\lambda\kappa_{\theta^{\mathbb{R}}}(-, -) = \langle -, - \rangle_{\theta^{\mathbb{R}}},$$

noting that $\theta^{\mathbb{R}}$ is just the conjugation map of $C = T_0 \oplus iP_0$. We see directly that it is U-invariant, since by definition of U:

$$U = \{g \in G^{\mathbb{C}} | Ad_g \circ \theta^{\mathbb{R}} = \theta^{\mathbb{R}} \circ Ad_g\}.$$

Recall the map $\Psi : G^{\mathbb{C}} \to (G^{\mathbb{C}})^{\mathbb{R}}$ between $G^{\mathbb{C}}$ and the realification, with real differential Ψ itself (see beginning of the previous chapter). Define for $v \in V^{\mathbb{C}}$ as in the real case the map $F_v^{\mathbb{C}} : G^{\mathbb{C}} \to \mathbb{R}$ by

$$F_v^{\mathbb{C}}(g) = \langle g \cdot v, g \cdot v \rangle_{\theta^{\mathbb{R}}}$$

for all $g \in G^{\mathbb{C}}$. Then analogous to the real case we have:

Theorem 5.3. ***[Complex case]***. *The following are equivalent statements.*

(1) $v \in \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}})$.
(2) $1 \in G^{\mathbb{C}}$ *is a critical point for the function* $F_v^{\mathbb{C}}$.
(3) For all $x \in iC$ *we have* $2\langle x \cdot v, v\rangle_{\theta^{\mathbb{R}}} = dF_v^{\mathbb{C}}(x) = 0$.

Moreover if $v \in V^{\mathbb{C}}$ *is minimal then* $G^{\mathbb{C}}v \cap M(G^{\mathbb{C}}, V^{\mathbb{C}}) = Uv$.

PROOF. Since the realification of $G^{\mathbb{C}}$ is semisimple, then one can apply all the results in the previous section to the representation

$$\psi^{\mathbb{R}} : (G^{\mathbb{C}})^{\mathbb{R}} \to GL((V^{\mathbb{C}})^{\mathbb{R}})$$

given by $\psi^{\mathbb{R}} = 1 \circ \psi^{\mathbb{C}} \circ \Psi^{-1}$. The differential becomes $d\psi^{\mathbb{R}} = 1 \circ d\psi^{\mathbb{C}} \circ \Psi^{-1}$. Here the Lie algebra of the realification have Cartan decomposition $\Psi(C) \oplus \Psi(iC)$ with corresponding decomposition $(G^{\mathbb{C}})^{\mathbb{R}} = \Psi(U)e^{\Psi(iC)}$. This real representation preserves the complex orbits of $\psi^{\mathbb{C}}$. Indeed if $\tilde{g} \in G^{\mathbb{R}}$ then $\tilde{g} = \Psi(g)$ for a unique $g \in G^{\mathbb{C}}$ so that

$$\psi^{\mathbb{R}}(\tilde{g})(v) = \psi^{\mathbb{C}}(g)(v) = g \cdot v$$

for all $v \in V^{\mathbb{C}}$, i.e $G^{\mathbb{R}}v = G^{\mathbb{C}}v$. $\square$

In particular a complex orbit $G^{\mathbb{C}}v$ is closed if and only it intersects $\mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}})$. The following proposition relates the minimal vectors of V to minimal vectors of $V^{\mathbb{C}}$.

Proposition 5.3. *We have* $\mathcal{M}(G, V) = \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}) \cap V$.

PROOF. The inclusion $\mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}) \cap V \subseteq \mathcal{M}(G, V)$ is clear by definition since $\langle -, -\rangle_{\theta^{\mathbb{R}}}$ extends $\langle -, -\rangle_{\theta}$ and $P_0 \subseteq iC$. Now given $X \in iC$ write $X = x + iy$ for $x \in P_0$ and $y \in T_0$ then

$$dF_v^{\mathbb{C}}(X) = 2\langle X \cdot v, v\rangle_{\theta^{\mathbb{R}}} = 2\langle x \cdot v + iy \cdot v, v\rangle_{\theta^{\mathbb{R}}} = dF_v(x) + \langle iy \cdot v, v\rangle_{\theta^{\mathbb{R}}} = \langle iy \cdot v, v\rangle_{\theta^{\mathbb{R}}} = 0,$$

since $v \in V$ and $y \cdot v \in V$. Hence v is minimal in $\mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}})$ as required. The proposition is proved. $\square$

The proposition shows that if we have a real orbit $Gv \subseteq V$ which is closed then the complex orbit $G^{\mathbb{C}}v \subseteq V^{\mathbb{C}}$ must also be closed. The converse is also true but is harder to prove.

Theorem 5.4. *Suppose* $v \in V$ *and the complex orbit* $G^{\mathbb{C}}v$ *is closed in* $V^{\mathbb{C}}$ *then so is the real orbit* Gv *in* V.

PROOF. For proof see [**10**], Lemma 2.2. $\square$

Corollary 5.4. *If G is a real semisimple compact matrix group and $v, \mu \in V$ then*

(1) $G^{\mathbb{C}}v \cap M(G^{\mathbb{C}}, V^{\mathbb{C}}) = Gv = G^{\mathbb{C}}v \cap V$.
(2) If $G^{\mathbb{C}}v = G^{\mathbb{C}}\mu$ then $Gv = G\mu$.

PROOF. We note first that $\mathcal{M}(G, V) = V$ since G is compact. So given $v \in V$ then v is minimal in the complex orbit $G^{\mathbb{C}}v$. Hence given any $\alpha \in G^{\mathbb{C}}v$ which is minimal then $\alpha \in U \cdot v$, but since G is compact then $G = U$ and so $\alpha \in G \cdot v \subseteq V$. This proves (1). Now case (2) follows immediately from case (1).

The previous corollary is actually a special case of a more general result for semisimple matrix groups G. The result states that if $v \in V$ then $G^{\mathbb{C}}v \cap V$ is a finite disjoint union of real orbits Gv_j for $v_j \in V$. In particular if $G^{\mathbb{C}}v$ is closed then so are all the real orbits Gv_j. So if $\alpha_j \in Gv_j$ are minimal then we deduce that $Uv \cap V$ is a finite disjoint union of K-orbits, $K\alpha_j$.

Remark. Although we have assumed in this section that G is a real matrix group with complexification $G^{\mathbb{C}}$, we could also have worked with an arbitrary complex semisimple matrix group $G^{\mathbb{C}}$ and a real form G (not necessarily a real matrix group). This follows because one can always embed the real form G into the realification of $G^{\mathbb{C}}$, so that G becomes a real matrix group inside $GL(2n, \mathbb{R})$.

4. Intersection of semisimple real orbits

The following section is my own work.

In this section we continue with the notation from the previous section, except now we let $Lie(G) = \mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}} = Lie(G^{\mathbb{C}})$. We now consider another real form of $G^{\mathbb{C}}$ say $\tilde{G}$ and let $\tilde{V}$ be a real form of $V^{\mathbb{C}}$. We suppose $\tilde{\psi} : \tilde{G} \to GL(\tilde{V})$ is another representation which restricts from $\psi^{\mathbb{C}}$, define $d\tilde{\psi}$ for the real differential. Similarly to G, we can equip $\tilde{V}$ with an inner product with the usual properties, determined by a Cartan involution $\tilde{\theta} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$, with corresponding Cartan decomposition

$$\tilde{\mathfrak{g}} = \tilde{T}_0 \oplus \tilde{P}_0.$$

We denote this inner product similarly by $\langle -, - \rangle_{\tilde{\theta}}$. Note that a Cartan decomposition: $\tilde{G} = \tilde{K}e^{\tilde{P}_0}$ is now complex, i.e $\tilde{K}$ and $e^{\tilde{P}_0}$ are not necessarily subsets of real matrices. We find a Cartan decomposition of $\tilde{G}$ by first embedding it in the realification of $G^{\mathbb{C}}$ (i.e in $GL(2n, \mathbb{R})$) and then transferring the information back to $\tilde{G}$ via the usual realification map Ψ. So the inner product $\langle -, - \rangle_{\tilde{\theta}}$ is here $\tilde{K}$-invariant. Now we can extend to an inner product on the realification of $V^{\mathbb{C}}$ as we did in the previous section for $\langle -, - \rangle_{\theta}$, denote this similarly by $\langle -, - \rangle_{\tilde{\theta}^{\mathbb{R}}}$.

We get as in the previous section a decomposition $G^{\mathbb{C}} = \tilde{U}e^{i\tilde{C}}$ where $C = \tilde{T}_0 \oplus i\tilde{P}_0$, here $\tilde{\theta}^{\mathbb{R}}$ denotes the Cartan involution of $\mathfrak{g}^{\mathbb{C}}$ with decomposition: $\tilde{C} \oplus i\tilde{C}$. Now analogous to ψ all the results in the previous two sections also hold for $\tilde{\psi}$, these results are proved by using the realification map Ψ. The matrix groups U and $\tilde{U}$ are compact real forms of $G^{\mathbb{C}}$.

It can be assumed that G is also an arbitrary real form of $G^{\mathbb{C}}$ (not necessarily contained in $GL(n, \mathbb{R})$). Since we may as well assume that $G^{\mathbb{C}}$ is an arbitrary complex semisimple matrix group.

We will say that two real orbits $\tilde{G}\tilde{x} \subseteq \tilde{V}$ and $Gx \subseteq V$ are **conjugate** if $G^{\mathbb{C}}x = G^{\mathbb{C}}\tilde{x}$.

For the rest of this section we assume $\psi, \tilde{\psi}, \psi^{\mathbb{C}}$ are the adjoint representations, i.e $\mathfrak{g} = V$, $\tilde{\mathfrak{g}} = \tilde{V}$ and $V^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$. Let $\kappa_\theta(-,-)$ and $\kappa_{\tilde{\theta}}(-,-)$ be the usual inner products on $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ respectively, defined via the Killing form κ. We also equip the realification of $\mathfrak{g}^{\mathbb{C}}$ with the inner products $\kappa_{\theta^{\mathbb{R}}}(-,-)$ and $\kappa_{\tilde{\theta}^{\mathbb{R}}}(-,-)$. Here $\kappa_{\theta^{\mathbb{R}}}(-,-) = 2\kappa_\theta(-,-)$ on $\mathfrak{g}$.

The following problem is explored in this section:

Problem. Consider two conjugate real orbits $Gx \subseteq V$ and $\tilde{G}\tilde{x} \subseteq \tilde{V}$.

(1) Do they intersect in general?
(2) If one of the orbits is closed do they intersect? If so is there a relationship between the minimal vectors of one orbit to the other?

We prove that if $G = \tilde{U}$ is the compact real form of $G^{\mathbb{C}}$ and $\tilde{G}$ is an arbitrary real form of $G^{\mathbb{C}}$ then two conjugate real orbits must intersect in a minimal vector. The adjoint action can also be extended to an action on the vector space of endomorphisms: $\mathfrak{gl}(\mathfrak{g})$. We consider this extended action and prove a result concerning the symmetric/antisymmetric parts of an endomorphism $\mathcal{R}$ w.r.t the bilinear forms: $\kappa(-,-)$ and $\kappa_\theta(-,-)$.

Lemma 5.8. *If $\tilde{G}\tilde{x}$ is conjugate to Gx then Gx is closed $\Leftrightarrow$ $\tilde{G}\tilde{x}$ is closed.*

PROOF. If $\tilde{G}\tilde{x}$ is closed then the complex orbit $G^{\mathbb{C}}x$ is closed hence so is Gx. Similarly if Gx is closed then $G^{\mathbb{C}}x$ is closed hence so is $\tilde{G}\tilde{x}$. $\square$

Consider the orthogonal group $G^{\mathbb{C}} = O(n, \mathbb{C})$ with real forms $G = O(n)$ and $\tilde{G} = O(p,q) = \{X \in O(n, \mathbb{C}) | I_{p,q}\bar{X}I_{p,q} = X\}$ where $p + q = n$. Here $\mathfrak{g} = \mathfrak{o}(n)$ and $\tilde{\mathfrak{g}} = \mathfrak{o}(p,q)$ are real forms of $\mathfrak{o}(n, \mathbb{C})$. We claim that any two conjugate real orbits intersect, this is proved in the next proposition.

Proposition 5.4. *Let $\tilde{G} = O(p,q)$ and $G = O(n)$. Suppose $Gx \subseteq \mathfrak{g}$ and $\tilde{G}\tilde{x} \subseteq \tilde{\mathfrak{g}}$ are conjugate real orbits. Then they intersect in a minimal vector.*

PROOF. Take $\tilde{\theta} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ to be the Cartan involution given by $X \to \bar{X}$. We note that a Cartan involution $\theta^{\mathbb{R}}$ of the realification of $\mathfrak{g}^{\mathbb{C}}$ is also given by this map. This follows because $\theta^{\mathbb{R}}$ is an antilinear Lie homomorphism with fix point set $\mathfrak{g}$, i.e when restricting to $\mathfrak{g}$ we obtain the identity $1_{\mathfrak{g}}$, which is of course the Cartan involution of $\mathfrak{g}$. Now the real orbit Gx is closed since the Cartan involution on $\mathfrak{g}$ is the identity, so therefore $\tilde{G}\tilde{x}$ is also closed. This means that we can choose a minimal vector X in $\tilde{G}\tilde{x}$. But X is also minimal in $G^{\mathbb{C}}x$, i.e

$$X \in G^{\mathbb{C}}x \cap \mathcal{M}(G^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) = Gx.$$

Hence Gx intersect with $\tilde{G}\tilde{x}$ as claimed. □

The reason why the previous proposition works is because $O(n)$ coincides with $\tilde{U}$, since $\mathfrak{o}(n) = T_0 \oplus iP_0 = C$ w.r.t the Cartan involution $X \to \bar{X}$ of $\mathfrak{o}(p,q)$. Indeed when we extend our Cartan involution θ to a Cartan involution $\theta^{\mathbb{R}}$ on the realification of $\mathfrak{g}^{\mathbb{C}}$, then $\theta^{\mathbb{R}}$ also extends the identity on the compact real form C. But this is the Lie algebra of the compact real form U. Hence we have the following trivial observation.

Proposition 5.5. *Set $\tilde{G} = U$ (the compact real form). Let Gx and Ux' be conjugate real orbits. Then Gx intersect Ux' in a minimal vector.*

PROOF. Since $Ux' \subseteq C$ is closed then so is $Gx \subseteq \mathfrak{g}$, i.e we can choose a minimal vector $X \in Gx$. But we know that X is also minimal in $G^{\mathbb{C}}x$ and is therefore contained in Ux' by construction. □

We now show that there is an embedding of the real form $\tilde{\mathfrak{g}}$ into $\mathfrak{g}^{\mathbb{C}}$ such that the compact form $\tilde{C}$ coincides with C. In this way U coincides with $\tilde{U}$ and $K, \tilde{K} \leq U$.

This we can do via an inner automorphism of $\mathfrak{g}^{\mathbb{C}}$. Indeed we know that any two compact real forms are conjugate via an element $g \in G^{\mathbb{C}}$, so we can use $Ad_g : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ to embed $\tilde{\mathfrak{g}}$ in $\mathfrak{g}^{\mathbb{C}}$. Denote this new copy also by $\tilde{\mathfrak{g}}$. It follows that we can choose a real structure

$$\theta^{\mathbb{R}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$$

which restricts to new Cartan involutions $\theta^{\mathbb{R}}_{|_{\mathfrak{g}}} = \theta$ and $\theta^{\mathbb{R}}_{|_{\tilde{\mathfrak{g}}}} = \tilde{\theta}$ for $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ respectively. To see this we only need to note that having a Cartan decomposition $T_0 \oplus P_0$ of $\mathfrak{g}$ extends to a Cartan decomposition of the realification of $\mathfrak{g}^{\mathbb{C}}$, i.e

$$(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}} = C \oplus iC.$$

So θ extends to $\theta^{\mathbb{R}}$. Moreover take $Ad_g \circ \tilde{\theta} \circ Ad_{g^{-1}}$ to be the new Cartan involution $\tilde{\theta}$ for $\tilde{\mathfrak{g}}$. Then the extended Cartan involution to the realification is also $\theta^{\mathbb{R}}$ because the compact real forms coincide, i.e

$$C = T_0 \oplus iP_0 = \tilde{T_0} \oplus i\tilde{P_0} = \tilde{C}.$$

In particular if $\mathfrak{g} \cap \tilde{\mathfrak{g}} \neq 0$, then it follows that

$$\mathfrak{g} \cap \tilde{\mathfrak{g}} = (T_0 \cap \tilde{T_0}) \oplus (P_0 \cap \tilde{P_0}).$$

In this way $\tilde{G}$ is embedded in $G^{\mathbb{C}}$ also via Ad_g.

Now recall what was done in the end of chapter 4, where we gave a Cartan decomposition of $G^{\mathbb{C}}$ and the definition of U. By definition U and $\tilde{U}$ are those elements in $G^{\mathbb{C}}$ such that Ad_g commute with the extension $\theta^{\mathbb{R}}$, i.e $U = \tilde{U}$ and so $K, \tilde{K} \leq U$.

We have thus proved the following lemma:

Lemma 5.9. *There is an embedding of $\tilde{\mathfrak{g}}$ in $\mathfrak{g}^{\mathbb{C}}$ such that the following properties hold.*

(1) $C = \tilde{T_0} \oplus i\tilde{P_0} = T_0 \oplus iP_0 = \tilde{C}$. In particular when $\mathfrak{g}$ is compact we get $\tilde{T_0} \oplus i\tilde{P_0} = \mathfrak{g}$.
(2) θ and $\tilde{\theta}$ are both restricted from the Cartan involution $\theta^{\mathbb{R}}$ of $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}}$ with Cartan decomposition $C \oplus iC$.
(3) $U = \tilde{U}$ and $K, \tilde{K} \leq U$.
(4) The inner products on $\mathfrak{g}, \tilde{\mathfrak{g}}$ both extend to the inner product $\langle -, - \rangle_{\theta^{\mathbb{R}}} = \frac{1}{2}\kappa_{\theta^{\mathbb{R}}}(-, -)$.

Example 5.5. The real forms $\mathfrak{o}(p, q)$ and $\mathfrak{o}(\tilde{p}, \tilde{q})$ of $\mathfrak{o}(n, \mathbb{C})$ satisfy the previous lemma naturally. Indeed they both share the Cartan involution: $X \to \bar{X}$, and so $C = \tilde{C} = \mathfrak{o}(n)$. So in this case $U = \tilde{U} = O(n)$.

Having property (4) in the previous lemma means that we can speak of a common set of minimal vectors $\mathcal{M}(G^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ for $\mathfrak{g}^{\mathbb{C}}$. In particular we see that if $v \in \tilde{\mathfrak{g}} \cap \mathfrak{g}$ is minimal in $\mathfrak{g}$ then it must also be minimal in $\tilde{\mathfrak{g}}$. This follows since $\langle X \cdot v, v \rangle_{\theta^{\mathbb{R}}} = 0$ for all $X \in iT_0 + P_0 = i\tilde{T_0} + \tilde{P_0}$ so by restricting to $\tilde{P_0}$ we see that X is also minimal in $\tilde{\mathfrak{g}}$.

We will occasionally write $\tilde{V} = \tilde{\mathfrak{g}}$ and $V = \mathfrak{g}$.

One can also extend to an obvious action of G on the vector space of endomorphisms of V: $End(V) = \mathcal{V}$. Indeed define the action of G by:

$$(g \cdot f)(v) = g \cdot (f(v)),\ f \in \mathcal{V},\ v \in V.$$

It is easy to check that the differential in this case is also given by:

$$(x \cdot f)(v) = x \cdot (f(v)),\ x \in \mathfrak{g},\ f \in \mathcal{V}.$$

Now if we fix an orthonormal basis $\{e_j\}_j$ of V w.r.t $\langle -, - \rangle_\theta$ then we can define an inner product $\langle\langle -, - \rangle\rangle_\theta$ on $\mathcal{V}$ by:

$$\langle\langle f, g \rangle\rangle_\theta = \sum_j \langle f(e_j), g(e_j) \rangle_\theta, \quad f, g \in \mathcal{V}.$$

It is easy to check that it has the required properties similar to that of $\langle -, - \rangle_\theta$. We can do this similarly for $\tilde{\mathcal{V}} = End(\tilde{V})$ and $\mathcal{V}, \tilde{\mathcal{V}}$ are real forms of $\mathcal{V}^{\mathbb{C}} = End(V^{\mathbb{C}})$. This follows by identifying a linear map $\mathcal{R} : V \to V$ with the extension $\mathcal{R}^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$, defined by

$$\mathcal{R}^{\mathbb{C}}(x + iy) = \mathcal{R}(x) + i\mathcal{R}(y), \quad x, y \in V.$$

Lemma 5.10. *Let $\mathcal{M}$ be the minimal vectors of $\mathfrak{g}$ w.r.t the inner product $\kappa_\theta(-, -)$. Suppose θ' is a another Cartan involution of $\mathfrak{g}$ of the form $\gamma\theta\gamma^{-1}$ for any choice of $\gamma \in Aut(\mathfrak{g})$. Then the minimal vectors $\mathcal{M}'$ w.r.t the inner product $\kappa_{\theta'}(-, -)$ is just $\gamma(\mathcal{M})$.*

Proof. Let α be minimal in $\mathfrak{g}$ w.r.t $\kappa_\theta(-, -)$ then $[\alpha, \theta(\alpha)] = 0$. But since the new Cartan involution of $\mathfrak{g}$ has the form $\theta' = \gamma\theta\gamma^{-1}$ we get:

$$[\gamma(\alpha), \theta'(\gamma(\alpha))] = [\gamma(\alpha), \gamma(\theta(\alpha))] = \gamma([\alpha, \theta(\alpha)]) = \gamma(0) = 0.$$

This shows that $\gamma(\mathcal{M}) = \mathcal{M}'$. $\square$

The lemma says that by conjugating the Cartan involution of our Lie algebra we conjugate the set of minimal vectors. So consider a change of Cartan involution of $\mathfrak{g}$ by a conjugation of an automorphism γ in $Ad(G)$. Conjugate the groups G and $\tilde{G}$ in $G^{\mathbb{C}}$ by γ and similarly the Lie algebras. Now since we are using the adjoint representation then the orbits also become conjugated by γ. In particular we loose no information about the orbits. A real orbit Gx become $G\gamma(x) = Gx$ and a real orbit $\tilde{G}\tilde{x}$ become $\gamma(G)\gamma(\tilde{x})$. So when $G = \tilde{U}$ is the compact real form of $G^{\mathbb{C}}$ then we get the following corollary.

Corollary 5.5. *Suppose $G = \tilde{U}$ is the compact real form of $G^{\mathbb{C}}$. Let $Gx \subseteq \mathfrak{g}$ and $\tilde{G}\tilde{x} \subseteq \tilde{\mathfrak{g}}$ be two real conjugate orbits. Then we may assume w.l.o.g that $\tilde{x}$ is minimal.*

Proof. Since $G = \tilde{U}$ is the compact real form then we know that Gx and $\tilde{G}\tilde{x}$ intersect in a minimal vector α. Suppose $\alpha = \tilde{g} \cdot \tilde{x}$ for a suitable $\tilde{g} \in \tilde{G}$. Now we can conjugate our setup by $\tilde{g}^{-1}$ via the isomorphism $\gamma = Ad_{\tilde{g}^{-1}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$. In this way our Cartan involutions become conjugated by γ. Denote $\tilde{\theta}'$ for the new Cartan involution of $\tilde{\mathfrak{g}}$. The orbit Gx is now identified with the orbit $\gamma(Gx) = \gamma(G)\gamma(x)$, while the orbit

$\tilde{G}\tilde{x}$ remains unchanged. So $\tilde{x}$ is now an element of $\mathfrak{g}$ (the new copy) and is minimal in $\tilde{G}\tilde{x}$ w.r.t the new inner product $\kappa_{\tilde{\theta}'}(-,-)$. This proves the corollary. □

It is easy to check that the previous two lemmas also hold for the action of $G, \tilde{G}$ on $\mathcal{V}$ and $\tilde{\mathcal{V}}$ respectively, equipped with the inner products $\langle\langle -,-\rangle\rangle_\theta$ and $\langle\langle -,-\rangle\rangle_{\tilde{\theta}}$ respectively. To see why, conjugate our Cartan involution of $\mathfrak{g}$ by say $\gamma \in Ad(G)$. Suppose we have two conjugate real orbits $G\mathcal{R} \subseteq \mathcal{V}$ and $\tilde{G}\tilde{\mathcal{R}} \subseteq \tilde{\mathcal{V}}$. Assume that our setup is conjugated by γ (i.e we conjugate the groups and the Lie algebras) then the real orbit $G\mathcal{R}$ is now conjugate to the real orbit $\gamma(\tilde{G}) \cdot \gamma \circ \tilde{\mathcal{R}}$.

Moreover if say $\mathcal{R}$ is minimal w.r.t $\langle\langle -,-\rangle\rangle_\theta$ and if θ' denotes the conjugated Cartan involution of $\mathfrak{g}$ then for any $X \in P_0$:

$$
\begin{aligned}
0 = \langle\langle X \cdot \mathcal{R}, \mathcal{R}\rangle\rangle_\theta &= \sum_j \langle X \cdot \mathcal{R}(e_j), \mathcal{R}(e_j)\rangle_\theta \\
&= -\sum_j \kappa(\gamma(X \cdot \mathcal{R}(e_j)), \gamma(\theta(\mathcal{R}(e_j))) \\
&= -\sum_j \kappa(\gamma([X, \mathcal{R}(e_j)]), \gamma(\theta(\mathcal{R}(e_j)))) \\
&= -\sum_j \kappa(\gamma(X) \cdot \gamma(\mathcal{R}(e_j))), \gamma(\theta(\gamma^{-1}\gamma(\mathcal{R}(e_j))))) \\
&= \sum_j \kappa_{\theta'}(\gamma(X) \cdot \gamma(\mathcal{R}(e_j))), \gamma(\mathcal{R}(e_j))) \\
&= \langle\langle \gamma(X) \cdot \gamma \circ \mathcal{R}, \gamma \circ \mathcal{R}\rangle\rangle_{\theta'}.
\end{aligned}
$$

This shows that $\gamma \circ \mathcal{R}$ is minimal w.r.t $\langle\langle -,-\rangle\rangle_{\theta'}$.

Let now $\mathcal{R} \in \mathcal{V}$ and $\tilde{\mathcal{R}} \in \tilde{\mathcal{V}}$ then we can decompose into symmetric/antisymmetric parts:

$$
\begin{gathered}
\mathcal{R} = R_+ + R_- \ \textit{w.r.t} \ \langle -,-\rangle_\theta \\
\textit{and} \\
\tilde{\mathcal{R}} = \tilde{R}_+ + \tilde{R}_- \ \textit{w.r.t} \ \langle -,-\rangle_{\tilde{\theta}}.
\end{gathered}
$$

Lemma 5.11. *Suppose $\mathcal{R} \in \mathcal{V}$. Let $\gamma \in Aut(\mathfrak{g})$ and $\theta' = \gamma\theta\gamma^{-1}$ be another Cartan involution. Then $\gamma \circ \mathcal{R} \circ \gamma^{-1}$ is symmetric/antisymmetric w.r.t $\kappa_{\theta'}(-,-)$ if and only if $\mathcal{R}$ is symmetric/antisymmetric w.r.t $\kappa_\theta(-,-)$.*

PROOF. Let $x, y \in \mathfrak{g}$ then

$$\kappa_{\theta'}(\gamma\mathcal{R}\gamma^{-1}(x), y) = -\kappa(\gamma\mathcal{R}\gamma^{-1}(x), \gamma\theta\gamma^{-1}(y)) =$$

$$= -\kappa(\mathcal{R}(\gamma^{-1}(x)), \theta(\gamma^{-1}(y))) = \kappa_\theta(\mathcal{R}(\gamma^{-1}(x)), \gamma^{-1}(y)).$$

Now since γ is bijective the result follows. □

The lemma shows that by conjugating a Cartan involution we loose no information about the symmetric/antisymmetric parts of an endomorphism of the Lie algebra w.r.t the Killing form. We will use the previous lemmas in the next theorem.

Recall if we set $G = \tilde{U}$ (the compact real form of $G^{\mathbb{C}}$) associated to $\tilde{G}$, then Lemma 5.9 holds naturally as we have seen. In particular our inner products $\langle -, - \rangle_\theta = \kappa_\theta(-, -)$ and $\langle -, - \rangle_{\tilde{\theta}} = \kappa_{\tilde{\theta}}(-, -)$ both restrict from the inner product $\langle -, - \rangle_{\theta^{\mathbb{R}}} = \frac{1}{2}\kappa_{\theta^{\mathbb{R}}}(-, -)$. We will use this property now.

The following theorem is an application of Proposition 5.5.

Theorem 5.5. *Let $\tilde{G}$ be an arbitrary real form of $G^{\mathbb{C}}$ and $G = \tilde{U}$ be the compact real form of $G^{\mathbb{C}}$. Let $\tilde{G}\tilde{\mathcal{R}} \subseteq \tilde{\mathcal{V}}$ and $G\mathcal{R} \subseteq \mathcal{V}$ be two conjugate real orbits. Then one can assume $\tilde{\mathcal{R}} = \mathcal{R}$ and*

$$\tilde{R}_+ = R_+ \quad \textit{and} \quad \tilde{R}_- = R_-.$$

Proof. Since $G = \tilde{U}$ is a compact real form of $G^{\mathbb{C}}$ then it follows that we can choose a minimal vector $\mathcal{X}$ in the intersection of the real orbits. In particular $\mathcal{X} \in \mathcal{V} \cap \tilde{\mathcal{V}}$. Now since $G\mathcal{R} \subseteq \mathcal{M}(G, \mathcal{V})$ then we can assume that $\mathcal{X} = \mathcal{R}$ i.e $\mathcal{R} \in \tilde{\mathcal{V}} \cap \tilde{G}\tilde{\mathcal{R}}$ is minimal. So $\mathcal{R} = \tilde{g} \cdot \tilde{\mathcal{R}}$ for a suitable $\tilde{g} \in \tilde{G}$. We may assume w.l.o.g that $\mathcal{R} = \tilde{\mathcal{R}}$, since we can always conjugate the Cartan involution of $\tilde{\mathfrak{g}}$ so that $\tilde{\mathcal{R}}$ becomes minimal. Now as an element of $\mathcal{V}^{\mathbb{C}}$ then $\mathcal{R}$ has the form

$$\mathcal{R} = R_+ + iR_+ + R_- + iR_-.$$

The maps $T_\pm = R_\pm + iR_\pm$ are clearly the symmetric(+)/antisymmetric(-) parts of $\mathcal{R}$ w.r.t the inner product $\langle -, - \rangle_{\theta^{\mathbb{R}}}$ on the realification of $V^{\mathbb{C}}$. We can similarly decompose:

$$\tilde{\mathcal{R}} = \tilde{R}_+ + i\tilde{R}_+ + \tilde{R}_- + i\tilde{R}_-$$

as an element in $\mathcal{V}^{\mathbb{C}}$. Define similarly $\tilde{T}_\pm = \tilde{R}_\pm + i\tilde{R}_\pm$, then these are also the symmetric/antisymmetric parts of $\tilde{\mathcal{R}}$ w.r.t $\langle -, - \rangle_{\theta^{\mathbb{R}}}$. But since $\mathcal{R} = \tilde{\mathcal{R}}$ then $\tilde{T}_\pm$ must coincide with $T_\pm$ by uniqueness of symmetric/antisymmetric parts. In particular by restricting $\tilde{T}_\pm$ and $T_\pm$ to $\tilde{V}$ and V respectively on $\langle -, - \rangle_{\theta^{\mathbb{R}}}$, we obtain that $T_{\pm_{|_{\tilde{V}}}} = \tilde{R}_\pm$ and similarly $\tilde{T}_{\pm_{|_V}} = R_\pm$. So we get as required

$$R_\pm = \tilde{R}_\pm.$$

□

We note if $\mathfrak{g}$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$ that $\langle -, - \rangle_\theta$ is just the Killing form: $-\kappa(-,-)$. So if $\tilde{\mathcal{R}} : \tilde{V} \to \tilde{V}$ is an endomorphism satisfying the criteria in the previous theorem, then the symmetric/antisymmetric parts w.r.t the Killing form κ coincide with the symmetric/antisymmetric parts w.r.t $\kappa_{\tilde{\theta}}(-,-) = -\kappa(-, \tilde{\theta}(-))$.

Here is an example for which the theorem immediately applies.

Example 5.6. Let $\tilde{\mathfrak{g}} = \mathfrak{o}(p,q)$ for $p+q=n$ and $\mathfrak{g} = \mathfrak{o}(n)$. Suppose $\tilde{\mathcal{R}}$ is an endomorphism of $\mathfrak{o}(p,q)$ which satisfies the criteria of the theorem. Then the symmetric/antisymmetric parts of $\tilde{\mathcal{R}}$ w.r.t the Killing form $\kappa(-,-)$ coincide with the symmetric/antisymmetric parts w.r.t $\kappa_{\tilde{\theta}}(-,-)$, where $\tilde{\theta}$ can be chosen to be the Cartan involution: $\tilde{\theta} : X \to \bar{X}$.

APPENDIX A

A root system

For more details about the theory of root systems we refer to [**3**], chapter 11.

Let V be an inner product space with inner product (-,-) and let $0 \neq \alpha \in V$.

Definition A.1. A linear map $s_\alpha : V \to V$ is said to be a reflection in the hyperplane $\langle\alpha\rangle^\perp$, if it fixes pointwise $\langle\alpha\rangle^\perp \subset V$ and is an involution such that $s(\alpha) = -\alpha$.

Proposition A.1. *$s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$ for all $x \in V$.*

PROOF. It is easy to check that s_α is a reflection in the hyperplane $\langle\alpha\rangle^\perp$. Now since $V = \langle\alpha\rangle \oplus \langle\alpha\rangle^\perp$ then obviously such a reflection must be unique. □

Definition A.2. A subset $\mathcal{R} \subset V$ is said to be a **root system** if the following axioms are satisfied.

(1) $0 \notin \mathcal{R}$ and $|\mathcal{R}| < \infty$ together with $\langle\mathcal{R}\rangle = V$.

(2) If $\alpha \in \mathcal{R}$ and $C \in \mathbb{R}$ then $C\alpha \in \mathcal{R}$ if and only if $C = \pm 1$.

(3) There are reflections for each $\alpha \in \mathcal{R}$, s_α in the hyperplane $\langle\alpha\rangle^\perp$ such that the restriction to $\mathcal{R}$ is a bijection, i.e there is a natural homomorphism

$$\langle s_\alpha | \alpha \in \mathcal{R}\rangle \to S_{|\mathcal{R}|}.$$

(4) For any $\alpha, \beta \in \mathcal{R}$ we have $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.

The vectors in $\mathcal{R}$ are said to be the **roots** of the root system.

APPENDIX B

Elementary representation theory of Lie algebras

1. A representation of a Lie algebra

In this section we assume a Lie algebra L can be over any field $\mathbb{K}$ unless otherwise stated. Most of what is written here is based on [**3**], chapter 7.

Definition B.1. A **representation** of a Lie algebra L over $\mathbb{K}$ is a Lie homomorphism

$$\psi : L \to \mathfrak{gl}(V)$$

for some vector space V over $\mathbb{K}$. We say a representation is **faithful** if ψ is a monomorphism. We also say that L has a n-dimensional representation if $Dim(V) = n$.

Example B.1. Any matrix Lie algebra $L \subset \mathfrak{gl}(n, \mathbb{K})$ has a natural faithful representation $L \hookrightarrow \mathfrak{gl}(V)$ for V a vector space over $\mathbb{K}$ of dimension n. Indeed we can send a matrix x to the linear map $V \to V$ represented by the matrix x.

Definition B.2. Let L be a Lie algebra and V be a vector space. We say that V is an L-module if there is a map $L \times V \to_{\psi} V$ which is bilinear such that for every $x, y \in L$ and $v \in V$ we have $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$, where $[x, y] \cdot v$ is a shortcut for writing $\psi([x, y], v)$.

Proposition B.1. *Let L be a Lie algebra then a vector space V is an L-module if and only if L has a representation $L \to \mathfrak{gl}(V)$.*

Proof. Assume V is an L-module then there is a bilinear map $L \times V \to_{\psi} V$ satisfying $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for all $x \in L$ and $v \in V$. Now consider the map $\psi(x, -) : V \to V$ for $x \in L$. This is a linear map because ψ is bilinear and so for $\lambda \in \mathbb{K}$ and $v \in V$ we have

$$\psi(x, \lambda v) = x \cdot (\lambda v) = \lambda(x \cdot v) = \lambda\psi(x, v).$$

So define the map $L \to \mathfrak{gl}(V)$ by sending $x \to \psi(x, -)$. This map is clearly linear again because ψ is bilinear, and for every $x, y \in L$ and $v \in V$ we have,

$$\psi([x, y], v) = [x, y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v) = \psi(x, \psi(y, v)) - \psi(y, \psi(x, v)) = [\psi(x, -), \psi(y, -)](v).$$

So we deduce that the map is a representation of L as required. Now conversely if $L \to_\psi \mathfrak{gl}(V)$ is a representation of L, then we can define a map $L \times V \to V$ by sending $(x, v) \to \psi(x)(v)$. This map is clearly bilinear as each map $\psi(x)$ is linear and ψ is linear. Also since $\psi([x, y]) = \psi(x) \circ \psi(y) - \psi(y) \circ \psi(x)$ we see that $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for every $x, y \in L$ and $v \in V$. So V is an L-module as required. □

Definition B.3. Let L be Lie algebra with representation $L \to_\psi \mathfrak{gl}(V)$ and suppose there is a vector subspace $W \leq V$ such that $\psi(L)(W) \subset W$, then we say that the representation ψ restricted to W: $L \to \mathfrak{gl}(W)$, is a subrepresentation of ψ. Similarly if V is an L-module, $L \times V \to_\beta V$ then a submodule W is a vector subspace $W \leq V$ such that $\beta(L, W) \subset W$.

If an L-module V does not have any non-trivial proper submodules then V is said to be **irreducible** or **simple**, the same definition goes for irreducible representations.

Consider the adjoint map $Ad : L \to \mathfrak{gl}(L)$ of L, then this is a representation of L. In particular if L is semisimple then $ker(Ad) = Z(L) = 0$ so that Ad is a faithful representation of L. We also note that a submodule $\tilde{L} \subset L$ w.r.t Ad is just an ideal of L. Indeed we have $Ad(L)(\tilde{L}) = [L, \tilde{L}] \subset \tilde{L}$. This shows that the adjoint representation of a Lie algebra L is irreducible if and only if L is simple.

Proposition B.2. *Every finite dimensional representation of a solvable Lie algebra over $\mathbb{C}$ has a 1-dimensional subrepresentation. In particular every irreducible finite dimensional representation of a complex solvable Lie algebra is 1-dimensional.*

PROOF. Let $L \to_\psi \mathfrak{gl}(V)$ be a representation then it is straightforward to show that $\psi(L^{(k)}) = \psi(L)^{(k)}$ for each $k \geq 1$ so that $\psi(L)$ is solvable. Now use Lie's Theorem to choose a basis $x_1, x_2, \ldots x_n$ of V such that simultaneously every element in $\psi(L)$ is represented by an upper triangular matrix. Now we may consider the subspace $W = \langle x_1 \rangle \leq V$ so $\psi(x)(x_1) = \lambda x_1 \in W$ for some $\lambda \in \mathbb{C}$. Hence W is a 1-dimensional L-submodule of V as required. □

Definition B.4. Let L be a Lie algebra and V an L-module then we say that V is **completely reducible** if $V = \oplus_j^n V_j$ for irreducible submodules V_j of V. Otherwise V is said to be **indecomposable**. The same definition goes for a representation of L, viewing it as an L-module.

It turns out that every finite dimensional representation of a complex semisimple Lie algebra is completely reducible into one dimensional submodules. This is known as Weyl's theorem.

Definition B.5. Let L be a Lie algebra together with L-modules V and W. Then a linear map $V \to_\psi W$ such that for all $x \in L$ and $v \in V$:

$$\psi(x \cdot v) = x \cdot \psi(v)$$

is said to be a **homomorphism** of L-modules. Moreover if ψ is bijective then V and W are said to be **isomorphic** as L-modules, we write $V \cong W$. Similarly two representations are defined to be isomorphic if they are isomorphic as L-modules.

Note that if $\psi, \tilde{\psi}$ are two isomorphic representations of L, then viewing them as L-modules there is a linear isomorphism $V \to_\gamma \tilde{V}$ such that

$$\gamma(\psi(x)(v)) = \tilde{\psi}(x)(\gamma(v))$$

for $x \in L$ and $v \in V$. This tells us that $\gamma \circ \psi = \tilde{\psi} \circ \gamma$ so ψ and $\tilde{\psi}$ are conjugate via γ. This means that if two representations are isomorphic then for every $x \in L$ the matrix of the linear map $\psi(x)$ must be similar to the matrix representing $\tilde{\psi}(x)$.

The following lemma is immediate.

Lemma B.1. *If $V \to_\psi W$ is a homomorphism of L-modules then $ker(\psi)$ and $\psi(V)$ are L-submodules of V and W respectively.* □

Lemma B.2. ***[Schur's Lemma]****. If V is a finite irreducible L-module over $\mathbb{C}$ with $Dim(V) = n$, then a linear map $V \to_\psi V$ is a homomorphism of L-modules if and only if $\psi = \lambda 1_V$ for some $\lambda \in \mathbb{C}$.*

Proof. The direction ($\Leftarrow$) is clear as V is an L-module. So consider the direction ($\Rightarrow$). First since we are working over $\mathbb{C}$ we can choose an eigenvalue of ψ say $\lambda \in \mathbb{C}$ with eigenspace $V_\lambda \neq 0$. Now because ψ is a homomorphism then for any $x \in L$ and $v \in V_\lambda$ we have

$$x \cdot \psi(v) = \lambda(x \cdot v) = \psi(x \cdot v).$$

Hence $x \cdot V_\lambda \subset V_\lambda$ for all $x \in L$. This shows that V_λ is a non-trivial L-submodule of V, but as V is irreducible then $V_\lambda = V$. This shows that $\psi = \lambda 1_V$ as required. □

Observe that if $x \in Z(L)$ and V is an L-module then $V \to V$ given by $v \to x \cdot v$ is naturally a homomorphism of L-modules. Indeed $[x, y] \cdot v = 0$ so $x \cdot (y \cdot v) = y \cdot (x \cdot v)$ for all $y \in L$ and $v \in V$. In particular the following corollary is immediate from Schur's lemma.

Corollary B.1. *Let L be a complex Lie algebra with irreducible representation $L \to_\psi \mathfrak{gl}(V)$. Then any element in $\psi(Z(L))$ is diagonalisable.* □

Proposition B.3. *Every 1-dimensional representation of a Lie algebra L with $L = L'$ is trivial. Moreover if $L' \neq L$ then L has an infinite number of non-isomorphic 1-dimensional representations, in fact we have a bijection,*

$$\{L \to \mathbb{K} \textit{ of } 1 \textit{ dimensional reps}\} \leftrightarrow (L/L')^*.$$

PROOF. Let $L \to_\psi \mathbb{K}$ be a 1-dimensional representation of L. Now since $\mathbb{K}$ is abelian and ψ is a Lie homomorphism then we must have $\psi(L') = 0$. Now if L is a Lie algebra with $L' = L$ then ψ is trivial, so the first part is proved. For the second part assume $L' \neq L$. We first note that if p is the quotient map $L \to L/L'$ and $\alpha \in (L/L')^*$ then this gives rise to a 1-dimensional representation of L namely the composition,

$$L \to_p L/L' \to_\alpha \mathbb{K}.$$

Conversely if ψ is a 1-dimensional representation of L then $\psi(x) = \alpha(p(x))$ for $\alpha \in (L/L')^*$ defined by $\alpha(x+L') = \psi(x)$ for $x \in L$. This is a well-defined linear functional as $\psi(L') = 0$. Now if we have two 1-dimensional reps of L say $\psi \neq \hat{\psi}$ which are isomorphic then we must have:

$$\lambda\psi(x) = \lambda\hat{\psi}(x)$$

for some $0 \neq \lambda \in \mathbb{K}$. So this shows that two 1-dimensional representations are isomorphic if and only if they are equal. It now follows that there is a bijection between 1-dimensional representations of L and linear functionals $(L/L')^*$ as claimed, in particular $(L/L')^*$ is infinite as $Dim(L') < Dim(L)$. The proof is complete. $\square$

In particular if L is a complex semisimple Lie algebra then every 1-dimensional representation is trivial, since $L = L'$. We end our discussion of representation theory by stating an interesting theorem, which states that every Lie algebra over a field of characteristic 0 is in fact a matrix Lie algebra.

Theorem B.1. ***[Ado's Theorem]***. *Given any finite dimensional Lie algebra L over a field $\mathbb{K}$ with $Char(K) = 0$, there is a faithful representation*

$$L \hookrightarrow \mathfrak{gl}(n, \mathbb{K}).$$

Hence every Lie algebra is linear, in the sense that it has a representation in terms of matrices equipped with the commutator bracket $[-,-]$.

Bibliography

[1] Sigurdur Helgason,*Differential geometry, Lie groups and symmetric spaces*, Graduate studies in mathematics, AMS, 1978.

[2] R.W.Richardson and P.J.Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math Soc. 42 (1990) 409-429.

[3] Karin Erdmann and Mark J. Wildon, *Introduction to Lie algebras*, Springer undergraduate mathematics series, 2006.

[4] Andrew Baker, *Matrix groups: An introduction to Lie Group Theory*, Springer undergraduate mathematics series, 2002.

[5] Anthony W. Knapp, *Lie groups: beyond and introduction*, 2nd, Birkhauser Verlag, 2002.

[6] A.L. Onishchik and E.B. Vinberg, *Lie groups and Lie algebras III*, Springer Verlag, Volume 41, 1991.

[7] Michael R. Jablonski, *Real Geometric Invariant Theory and Ricci Soliton Metrics on Two-step Nilmanifolds*, Chapel Hill, 2008.

[8] S. Hervik, M. Ortaggio and L. Wylleman, *Minimal tensors and purely electric or magnetic spacetimes of arbitrary dimension*, (2013), 165014, [arXiv:1203.3563 [gr-qc]].

[9] Joachim Hilgert and Karl-Hermann Nebb, *Structure and Geometry of Lie groups*, Springer monographs in mathematics, 2012.

[10] P.Eberlein and M.Jablonski, *Closed orbits of semisimple group actions and the real Hilbert-Mumford function*, AMS Subject Classification: 14L24, 14L35, 57S20.

[11] R.Goodman and N.R. Wallach, *Symmetry, Representations, and Invariants*, Graduate texts in mathematics, Springer, Vol. 255, 2009.

[12] Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, The Annals of mathematics (1962), no. 3, 485-535, 2nd ser.