FROM SYMMETRIES TO SCATTERING AMPLITUDES

A Lie-algebraic categorisation of symmetry-breaking patterns that create enhanced soft limits for NG bosons.

by

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ABSTRACT

The standard calculation of scattering amplitudes in quantum field theory is carried out using a perturbative expansion, that at successive orders contains an escalating number of terms to calculate.

The amplitudes depend on an action, that specifies the properties and interactions of the particles involved in the scattering events. It is worthwhile to establish more direct relationships between the qualities of an action and its scattering amplitudes, not just to refine the formalism, but to simplify certain calculations. To that end, the long wavelength behaviour of the scattering amplitudes of Nambu-Goldstone bosons are investigated. NG bosons are massless scalar particles that exist due to, and whose interactions are principally determined by, the spontaneous breaking of symmetries; they make for good models to study the relations between symmetries and amplitudes. The subject here is specifically NG models with enhanced soft limits, meaning that, as the momentum of one of the particles entering a scattering event goes to zero, the amplitude must vanish as a higher power of that momentum.

This thesis lays out the context and procedures that lead from symmetry breaking to effective actions and finally to the calculation of scattering amplitudes. Using this framework, the relation between Lie algebras and the enhanced soft limits was researched. In the case of a single physical NG boson, a full Lie algebraic categorisation of the models with enhanced scaling of the amplitudes in the soft limit was found: the only non-trivial models were the familiar DBI and galileon actions. For multiple physical NG bosons with enhanced soft limits a Lie algebra was discovered, which is purely determined by its internal, non-redundant symmetries, its affine representation and an invariant symmetric 2-tensor of said representation. From this algebra two infinite classes of models can be derived that are generalisations of the known DBI and galileon multi-flavour theories. The constraints on the amplitudes thus suffice to create categorisations of the Lie algebras. Due to the special properties of their soft limits, the resulting models may be interesting to high-energy physics and cosmology.

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CONTENTS

INTRODUCTION

Science aims to bring understanding and experience into correspondence with each other, introducing neither concepts nor predictions that are irrelevant to that correspondence. In physics, this mirroring is established by working from the two sides towards a convergence. Firstly, by a quantitative characterisation of phenomena in nature, through experiment. Secondly, by using quantifiable concepts to calculate predictions for the values found in experiment, employing theory.

For the phenomenon of particles it is a given that they are very light, or even without mass, and so very small relative to the human scale. Because massless particles travel at light speed in a vacuum, the theoretical description must involve special relativity. Furthermore, the small size of the particles means that a good description also requires quantum mechanics. The theoretical domain that unites both these frameworks is quantum field theory (QFT).
QFT is capable of calculating the probability of a scattering event, given the particles that go in and come out of it. However, to complete the calculation, the perturbative approach to QFT demands that every possible way that these particles could have interacted must be accounted for. In consequence, for an event involving many particles, the combinatorics of the possible interactions lead to a very quick growth of the number of calculations. The search for new, underlying structures in QFT that lead to novel and simpler methods of calculation may yield both better understanding and usefulness of the theory.

The subject of this thesis is the behaviour of the scattering amplitudes of Nambu-Goldstone (NG) bosons in the long wavelength limit. The NG bosons are massless scalar particles, which are special because their actions can be constructed purely on the basis of spontaneous symmetry breaking. A physics model, given either in the form of an action or equations of motion, yields a space of solutions. If there are solutions that have fewer symmetries than the model, the absent symmetries are said to be broken. The difference between the full symmetry before and the remaining symmetries after breaking, suffices to model the space of possible NG bosons. In the special case that broken symmetries don't commute with space-time translation, they don't give rise to new NG bosons; these symmetries are therefore referred to as redundant.

In any case, there exists a clear sequence from the mathematical concept of symmetry breaking, the construction of models, and then through QFT to scattering amplitudes and probabilities. This makes the NG bosons a suitable subject to investigate the relations between symmetries and scattering amplitudes.

PART I of the thesis introduces the requisite mathematics, and shows how it can construct actions for NG bosons. PART II sets forth the manner in which observables, or rather the scattering amplitudes, may be derived from the actions. Then follows a short discussion of cosmology and how redundant symmetries may find a natural context there, which means that the related NG bosons could occur as a phenomenon. The first two parts provide the necessary background for PART III, which contains the actual research [1,2]. An NG boson model can have an Adler's zero, which means that its scattering amplitudes vanish if the momentum of one of the external particles goes to zero (the single soft limit). The amplitudes can also have enhanced soft limits, meaning that they go to zero quicker than the momentum does. The original impetus for the research was the finding that, starting from the enhanced soft limits of scattering amplitudes, a set of theories was be found that have redundant symmetries [3]. The question that will be answered here, approaches matters from the opposite direction: what symmetry breaking patterns, involving redundant symmetries, lead to scattering amplitudes that have enhanced soft limits?

A more detailed version of the thesis, when broken down into chapters, takes the following form:

CHAPTER 1 CALCULUS The notation and concepts from differential geometry are introduced, that are useful for doing calculus on manifolds; the exposition is informal, so although statements are grounded, they are not proved. The scope of this chapter is limited to vectors, differential forms and what's required to formulate Stokes' theorem in these terms.

CHAPTER 2 SYMMETRY Using the context of differential equations and functions, the ideas of invariance and symmetry are introduced. Then it is explained how different symmetries combine into Lie group and how a Lie algebra can be found for each group. After this, spontaneous symmetry breaking and NG bosons are introduced. It's shown how, given a Lie algebra and its breaking pattern, differential forms can be found using the Maurer Cartan form. These forms are a basis for the construction of possible actions for the NG bosons, thus making a path back from Lie algebras to differential equations. The last section defines redundant symmetries and the inverse Higgs constraints that accompany them.

CHAPTER 3 TOPOLOGY This chapter deals with topological invariants, global properties that don't change under the smooth deformation of the manifold. Then homology, which classifies manifolds by their topological invariants, is introduced. Using Stokes' theorem, the duality between manifolds and differential forms is established and with it the notion of cohomology for forms. Then homotopy, which deals with mapping specific cycles onto manifolds, is brought in. Finally, this is combined with the cohomology for forms to introduce Wess-Zumino terms, which are additional building blocks for the NG actions.

CHAPTER 4 EFFECTIVE FIELD THEORY After briefly reviewing some basic QFT, the effective field theory perspective is assumed. Starting from an effective action and its equations of motion, the concepts of Jacobi and Green's functions are derived. It's shown how boundary conditions can be introduced as a source term, which naturally leads to the Legendre transform of the action. The appropriate series expansions then reveal that this transformed action contains the model's scattering amplitudes. The chapter concludes with a discussion on how symmetries, particularly the spacetime kind, affect the scattering amplitudes.

CHAPTER 5 COSMOLOGY To begin, a basic background to the theory of cosmic inflation is sketched out. The early universe seems to have gone through a period of rapid expansion, which explains the features of its current large scale structure; this growth continues, in a much slower manner, to this day. The mathematical means to model this expansion take the form of an inflaton field. An hypothesis that the Poincaré symmetry group of Minkowski space might be the remainder of a larger, broken group of symmetries is formulated. Under this conjecture the inflaton could be the instantion of not just any NG boson, but specifically one whose construction involves redundant symmetries.

CHAPTER 6 THEORIES WITH ENHANCED SOFT LIMITS In this final chapter specific NG models with enhanced soft limits are constructed. Each subsequent section, systematically extends the Poincaré algebra with additional generators. The extended algebras are then broken, back to the Poincaré algebra, which ultimately yields a set of Lorentz invariant models for NG bosons.
The requirement that the scattering amplitudes of these models have an enhanced soft limit, is a strong enough constraint to give a classification of the underlying Lie algebras. In the case of a single, physical NG boson, the only resulting algebras lead to the familiar DBI and Galileon models. However, in the multi NG boson case, with multiple redundant generators, the possible solutions take the form of two classes, of infinite size; one class of generalised multi- flavour DBI theories and one of generalised multi-galileon theories.

PART I
MATHEMATICS AND MODELS

CHAPTER 1

CALCULUS

Although matter without spacetime extension is inconceivable, space and time exist even without substance; spatiotemporality is an essential quality of existence. Any mathematical description of the universe will therefore include geometry, which concerns itself with the formalisation of space.

In modern physics spacetime is identified with an active gravitational field, where the dynamics of objects are defined by relative position and movement. The particles, matter broken down to its smallest components, are themselves also excitations of other fields. Part of fundamental physics thus revolves around finding the shape and configuration of the field space, to set a stage for the modelling of phenomena.

These spaces, which may have arbitrary dimension and a complex shape, fall in the domain of differential geometry. This is an extensive branch of mathematics, that generalises the study of analysis beyond the flat spaces. The following chapter is a short, informal overview of the concepts required to understand the calculus.

1.1 Manifolds

Manifolds are smooth shapes, in a manner of speaking. Locally a manifold is homeomorphic, deformable, to a flat space. A small patch U of an n-dimensional manifold M should be able to be laid out, by a mapping φ, on a flat space $\mathbb{R}^n$. This mapping of a piece of the manifold, (U_i, φ_i), is called a chart. The collection of charts $\{(U_i, \varphi_i)\}$ that covers the manifold is called an atlas.

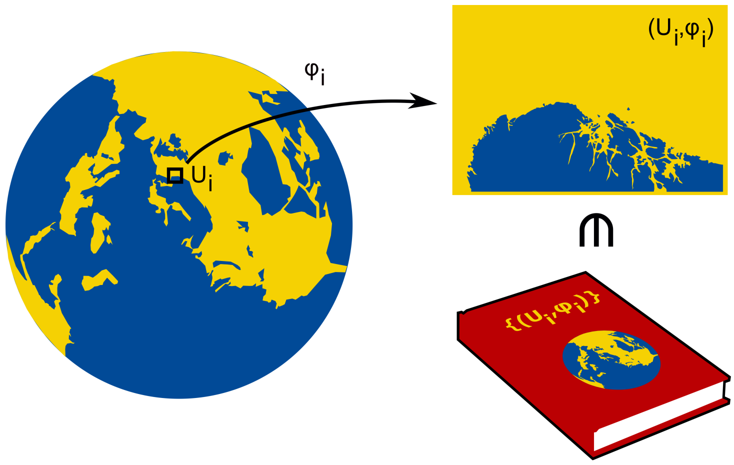

Figure 1.1: The nomenclature draws from cartography, by design. If the globe is a manifold M, its chart is (U_i, φ_i) and the collection of charts forms the atlas.

The parts of different charts, that overlap on a piece of the manifold, should be deformable into one another. This is how the smoothness of the manifold manifests. In other words, the transition mapping $t_{ij} = \varphi_i \circ \varphi_j^{-1}$ from one chart to the other should be infinitely differentiable.

1.2 Vectors

The everyday example of velocity forms the paradigm for vectors. Choose a path $\vec{x}(t)$ in flat space, and see how the position changes in an infinitesimal moment; at any point along the path this defines the velocity vector $\vec{v} = \frac{d\vec{x}}{dt}$. The change over time of a function $f(x)$, along the path, is then given by a directional derivative $\frac{df}{dt} = \vec{v} \cdot \nabla f$.

This generalises to manifolds by choosing a curve $c(\lambda) : \mathbb{R} \to M$ through a point p. Define both chart coordinates on the manifold, $\varphi(c(\lambda)) = x^\mu(\lambda)$, and a function $f : M \to \mathbb{R}$. The change of this function along the path c is again a directional derivative.

$$\left.\frac{df(c)}{d\lambda}\right|_p = \left.\frac{df(\varphi^{-1}(\varphi(c)))}{d\lambda}\right|_p = \left.\frac{dx^\mu(\lambda)}{d\lambda}\frac{\partial f(\varphi^{-1}(x))}{\partial x^\mu}\right|_p = X(f) \tag{1.1}$$

The operator $X = X^\mu \partial_\mu = \left.\frac{dx^\mu}{d\lambda}\right|_p \partial_\mu$ yields, by its action at p, the tangent vector at that point along the curve $c(\lambda)$. The set of all curves through p gives a set of vectors that locally forms a vector space: the tangent space at the point p in M, denoted as T_pM. One set of basis vectors consists of the derivatives $\hat{e}_i = \partial_i$, called the coordinate basis. The operator $X = X^\mu \hat{e}_\mu$ is a vector in this basis, with components X^μ.

1.3 Exterior forms and differentials

For any vector space T_pM, with a basis $\{\hat{e}_\mu\}$, a dual space T_p^*M with basis $\{\hat{e}^{*\mu}\}$ can be defined. Their inner product is defined as $\langle \hat{e}^{*\nu}, \hat{e}_\mu \rangle = \delta^\nu_\mu$. The dual space is the space of all linear maps from the vector space to the real numbers. Take a vector $X = X^\mu \hat{e}_\mu$ and dual vector $Y = Y_\mu \hat{e}^{*\mu}$, then

$$\langle X, Y \rangle = X^\mu Y_\nu \langle \hat{e}_\mu, \hat{e}^{*\nu} \rangle = X^\mu Y_\mu \in \mathbb{R}. \tag{1.2}$$

Note that $X^\mu Y_\mu = C$, for some constant C, is the equation of a hyperplane. In this equation X is a vector, that takes values in $\mathbb{R}^n$. The inner product with Y is a map from $\mathbb{R}^n$ to $\mathbb{R}$, with a null space isomorphic to $\mathbb{R}^{n-1}$. The dual vectors are characterized by this space, up to a factor, so they can be represented by the $(n-1)$ dimensional hyperplane.

Figure 1.2: In any dimension the vector can be represented by an arrow. In 3 dimensions its dual vectors take the form of planes.

The definition of the vector, (1.1), states that the action of a vector on a function gives a scalar $X(f) = X^\mu \partial_\mu f$. The dual basis components here are $\partial_\mu f$, those of a gradient, which follow naturally from $df = (\partial_\mu f) dx^\mu$. The dual to the coordinate basis is then $\hat{e}^{*\mu} = dx^\mu$, so that the inner product with the gradient yields the same:

$$\langle X, df \rangle = (X^\mu \partial_\nu f) \langle \partial_\mu, dx^\nu \rangle = X^\mu \partial_\mu f = X(f). \tag{1.3}$$

The dual vectors in the differential basis are called 1-forms, and take the general form $\omega = \omega_\mu dx^\mu$.

The 1-forms can, like the vectors, be thought of as having both magnitude and "direction". To define a product between forms, some notion of the product of directions is required. The idea that the square of a direction does not exist, or rather is zero, already suffices for this purpose.
This multiplication of forms, called the exterior or wedge product $\wedge$, should therefore satisfy $\omega \wedge \omega = 0$. Writing the form as the sum of two others, $\omega = \alpha + \beta$, shows that the wedge product is anti-symmetric.

$$\omega \wedge \omega = \alpha \wedge \beta + \beta \wedge \alpha = 0 \qquad \rightarrow \qquad \alpha \wedge \beta = -\beta \wedge \alpha \tag{1.4}$$

By repeated multiplication one can construct forms of arbitrary order. These are all completely anti-symmetric multilinear maps.

$$\omega = \frac{1}{k!} \omega_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \tag{1.5}$$

This is an element of the vector space of k-forms, at the point p, denoted by Ω_p^k. The exterior product between the forms $\zeta \in \Omega_p^l$ and $\eta \in \Omega_p^m$ yields a form $\theta \in \Omega_p^{l+m}$. The general commutation rule for the exterior product is

$$\zeta \wedge \eta = (-1)^{lm} \eta \wedge \zeta. \tag{1.6}$$

At each order the forms can be depicted by a space with an orientation, similar to how the plane represents a 1-form in three dimensions. Each increase in the order of the form corresponds to a decrease in the dimension of space used in its representation. A general k-form, in an n-dimensional space, can be depicted by $(n-k)$-dimensional submanifolds. The wedge product of forms is then represented by the submanifold that is the intersection of the submanifolds that portray the constituent forms in the product. In three dimensions the 1-forms are planes, so the 2-form is the intersection line of those planes. The 3-form is the intersection of three planes, or two planes and one line, which is a point.

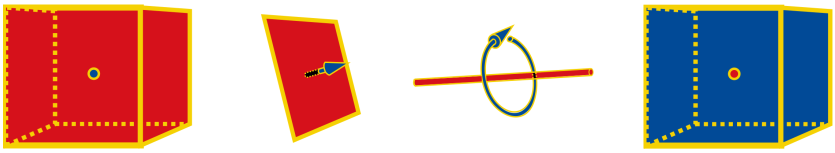

Figure 1.3: In D dimensions, an n-form can be represented locally by an fragment of a $(D-n)$-dimensional submanifold. If $D=3$, then the 0-form is a volume, 1-form is a plane, the 2-form is a line and the 3-form is a point; here in red. Each has an associated dual picture, in blue.

Since a 1-form, $\omega = \omega^\mu dx_\mu$, contains a basis of differentials, these are the basis elements of the exterior algebra. It is the gradient d which changes the scalar 0-form x^μ to a one form dx^μ, so the derivative d carries the anti-commutativity. This exterior derivative more generally maps forms to higher orders, $\Omega_p^k \to \Omega_p^{k+1}$, and acts on them as

$$d\omega = \frac{1}{k!}\partial_\lambda \omega_{\mu_1 \ldots \mu_k} dx^\lambda \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}. \tag{1.7}$$

The commutation of the exterior derivative is defined by

$$d(\zeta \wedge \eta) = d\zeta \wedge \eta + (-1)^{lm} d\eta \wedge \zeta = d\zeta \wedge \eta + (-1)^l \zeta \wedge d\eta. \tag{1.8}$$

This implies that the exterior derivative is a nilpotent operator.

$$dd = -dd \qquad \rightarrow \qquad d^2 = 0 \tag{1.9}$$

1.4 Pullback

Let a mapping between the spaces M and N be denoted by $\phi : M \to N$; M is the base manifold and N is the target manifold. These two spaces are parametrised by the coordinates x^μ and y^α respectively, and may differ in dimension.
Such a map also induces a map of the dual tangent space, going from N to M.

$$\phi^* : \Omega^k_{\phi(p)}(N) \to \Omega^k_p(M) \tag{1.10}$$

This is called a pullback, it maps between the forms on the two manifolds. In the simplest case it acts on a 0-form, which is a function $f : N \to \mathbb{R}$.

$$\phi^*(f(y)) = f(y(x)) \tag{1.11}$$

The mapping ϕ need not be one-to-one; when it isn't invertible, it's not just a change of coordinates. The pullback does, however, share the following properties with a change of coordinates:

$$\phi^*(\zeta \wedge \eta) = \phi^*(\zeta) \wedge \phi^*(\eta), \qquad \& \qquad d\phi^*(\omega) = \phi^*(d\omega). \tag{1.12}$$

Now the pullback of any k-form can be calculated. For example, the pullback of the 1-form is

$$\phi^*(\omega_\alpha(y)dy^\alpha) = \omega_\alpha(y(x)) \left(\frac{\partial y^\alpha}{\partial x^\mu} dx^\mu \right). \tag{1.13}$$

1.5 Integration and Stokes' theorem

The familiar notation for a multiple integral of a function $f(x_1, ..., x_m)$, over a m-dimensional domain c in the D-dimensional manifold N, is:

$$\int ... \int_c \omega(x_1, ..., x_m) d^m x. \tag{1.14}$$

The integrand and differentials in this integral can be replaced by an m-form ω.

$$\int_c \omega = \int ... \int_c \omega(x_1, ..., x_m) dx_1 \wedge ... \wedge dx_m \tag{1.15}$$

The orientation in the multiple integral (1.14) was totally implicit, now it is also determined by the ordering of the differentials in the form; one exchange of adjacent differentials in the wedge product generates a minus sign.

Let the m-form $\omega = \frac{\omega_{\alpha_1 ... \alpha_m}}{m!} dy^{\alpha_1} ... dy^{\alpha_m}$ now be defined in terms of the coordinates $y^\alpha = (y_1, ..., y_D)$ on N, which can be parametrised in terms of $x^\mu = (x_1, ..., x_m)$ in c by a mapping ϕ. The integral of this form can be written as

$$\int_c \phi^*(\omega) = \int_c \frac{\omega_{\alpha_1 ... \alpha_m}}{m!} \frac{dy^{\alpha_1}}{dx^{\mu_1}} ... \frac{dy^{\alpha_m}}{dx^{\mu_m}} dx^{\mu_1} ... dx^{\mu_m}. \tag{1.16}$$

The anti-symmetry of the wedge product means that the coordinate transformation of the form naturally introduces the appropriate jacobian determinant into the integral.

A standard result for the integration of forms is the generalized Stokes' theorem. Let ∂c be the boundary of the m-dimensional domain c, and $\omega \in \Omega^{m-1}$. Then

$$\int_c d\omega = \int_{\partial c} \omega. \tag{1.17}$$

This theorem is an m-dimensional generalisation of the familiar integral theorems of vector calculus: in one dimension it is the gradient theorem, in two dimensions it's the curl theorem and in three it's the divergence theorem. The generalised Stokes' theorem holds for the same reason as in the lower dimensional instances; it will only be sketched loosely out here:

Heuristically, the integral reduces to a form on an infinitesimal domain and the infinitesimal version of Stokes' theorem is [1]

$$d\omega = \omega\left(x_1 + \frac{dx_1}{2}, ..., x_m\right) - \omega\left(x_1 - \frac{dx_1}{2}, ..., x_m\right) +$$
$$... + \omega\left(x_1, ..., x_m + \frac{dx_m}{2}\right) - \omega\left(x_1, ..., x_m - \frac{dx_m}{2}\right).$$

This sum contains $2m$ terms, one for each $m-1$ dimensional face of the infinitesimal m-dimensional cube around x_μ. On this infinitesimal domain, Stokes' theorem is analogous to the definition of an m-dimensional derivative via a symmetric difference quotient.
The integral over the volume c can be divided into such infinitesimal m-cubes. Inside c the boundaries of adjacent infinitesimal m-cubes lie against each other, but they have an opposite orientation, and so their contributions to the sum cancel each other out. The remainder, which has no neighbours, lies along the boundary ∂c. The sum of $d\omega$ over c therefore equals the sum of ω over ∂c [2].

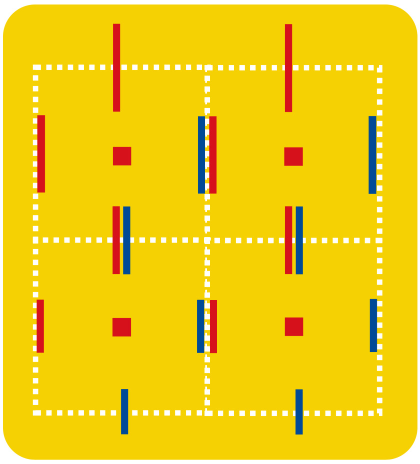

Figure 1.4: In D=2 the 1-form ω is rendered as a line element, red and blue representing opposite orientations. They are drawn along the boundaries, in white, of four 2-cubes. The 2-form $d\omega$ is locally represented by a point, in red, it is the sum of the (oriented) 1-forms along the boundary of a single square. The total domain c is the sum of four 2-cubes. The sum of those forms along the internal boundaries cancels, leaving only the part due to the outer contour ∂c. The total of dots inside c equals the sum of the 1-forms along ∂c.

[1] the differential dx_μ denotes an infinitesimal distance here, not a 1-form.
[2] For an actual proof of the generalized theorem see [4], for instance.

CHAPTER 2

Symmetry

The equivalence of an object, before and after a definite transformation, is called symmetry. Everyday objects are never exactly symmetric, but their useful description is never exact either. When the deviations in objects can be overlooked, the symmetries simplify and order the description of objects.

Figure 2.1: Symmetry implements composition in an otherwise random silhouette.

In theoretical physics the object of study is nature's law. These laws are often written in the form of differential equations, formulated in terms of variables, and spacetime coordinates and derivatives. By the principle of least action, some such equations can equivalently be derived from a functional called the action.

The description of nature should mimic its symmetry. A differential equation is symmetric if, after an alteration to the variables or coordinates, the set of solutions is mapped into itself; the individual solutions however are mapped into other ones. Invariance under a transform of the variables or the coordinates is called, respectively, internal or spacetime symmetry.

A sufficient change to a symmetric system can disturb its balance, and mar the symmetry of its description. The addition of these deviations is called symmetry breaking. The addition of a new term, that is not invariant, to a differential equation is called explicit symmetry breaking. When a solution lacks the symmetry of the equation it derives from, the symmetry was spontaneously broken.

The symmetries of differential equations are described by Lie groups. The set of all symmetries of an object forms a group. A Lie group is a group that is also a smooth manifold. The symmetries of a Lie group are continuous, loosely speaking. The rotations of a circle form a Lie group, for instance, and they are a continuous function of the angle.
A continuous transformation defines a curve, like the circle, and with it a tangent vector at each point, which points along it. If there are multiple symmetry transformations for an object, their consecutive application should leave it invariant too; a combined transformation is a symmetry in its own right. This symmetry structure manifests itself locally, between the tangent vectors of the different curves. The relations among these vectors form the Lie algebra of the group.

Starting from a characterisation of nature by its invariances, leaves the differential equations to be determined. For spontaneously broken symmetries, forms that are invariant under symmetry transformations can be constructed from Lie algebras using a so-called coset construction.
These forms can be interpreted as the set of invariant actions, that yield a span of possible differential equations.

2.1 INVARIANTS AND CLASSES

The essential description of something is, in a sense, it's definition, but it may take different forms. Intensional definitions state the necessary and sufficient conditions for an object to be part of a class. The abstraction of such a definition may lead to nonsense [1]. Extensional definitions, by contrast, consist of the explicit list of objects that make up the class.

The empirical description of a system starts with the accumulation of observations. Similarly, its model can be defined, extensionally, by the ensemble Ω of its presumed states ω_i [5].

$$\Omega = (\omega_1, ..., \omega_N) \tag{2.1}$$

By assigning quantity to the configurations, measurement is modelled by a mapping

$$\phi : \Omega \to \mathbb{R}. \tag{2.2}$$

The ϕs are the observables of a system. The qualities of the system are contained in the relations between them; these are expressed by equations

$$\mathcal{E}^i(\phi_1, ..., \phi_n) = 0. \tag{2.3}$$

If some observable stays constant, whilst all others change, it is a parameter α that characterises the total system.

$$\mathcal{E}^i_{\alpha_1 ... \alpha_m}(\phi_{n+1}, ..., \phi_n) = 0. \tag{2.4}$$

The remaining observables are interdependent. Some set ϕ_i, the dependent variables, can be expressed as a function of the others, the independent variables.

$$\phi_i = Y_i(\phi_{m+1}, ..., \phi_{m+r}) \qquad \text{where } (m + r < i < n) \tag{2.5}$$

The independent variables can be split into those that must vary over the various ω_i, x, and those that can remain constant β. Different βs give different solutions, so the relations between them characterise the space of solutions. The equations of state for the ϕ are

$$\mathcal{E}^i_{\alpha_1 ... \alpha_m \beta_1 ... \beta_r}(x_1, ..., x_s, \phi_1, ..., \phi_t) = 0. \tag{2.6}$$

[1] When Plato defined man as a featherless biped, Diogenes the Cynic presented him with a plucked rooster, stating "Behold, a man!"

Given the states of some system, an extensional definition, a set of parameters that characterise the system can be identified by invariance. Such a system can then be placed inside a larger class, that is defined by those invariances.
If from the properties that characterise the class, the intensional definition, the space of all elements in the class can be generated, any particular model can be identified by using information from measurements. In this way one can build a general theory from abstract definitions, and be certain that the description of some particular observed object is in there. This motivates the following study of symmetries.

2.2 The symmetries of differential equations

The general form of a differential equation is

$$\mathcal{E}^i_{\boldsymbol{\alpha}}(x_\mu)[\phi^a_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x_\mu)] = 0. \tag{2.7}$$

The ϕ^a are functions of the coordinates x_μ, dependent on boundary values fixed by $\boldsymbol{\beta} = \{\beta_1...\beta_n\}$ and choice of $\boldsymbol{\alpha} = \{\alpha_1...\alpha_m\}$. The $\mathcal{E}^i$ are differential operators which may depend on parameters $\boldsymbol{\alpha}$ and act via x^μ.

Nature, or its equations, should not depend on the choice of a coordinate system, for either spacetime or the fields; this is called the principle of covariance. Those changes of variables, that map each solution into an associated one, are the symmetries.

A redefinition changes the coordinates in an equation. The new variables ϕ' and x' become a function of the old ϕ and x. In the simplest case this transformation, away from the original choice of coordinates, is parametrised by a single parameter s.

$$\begin{aligned} x' &= X[\phi, ..., \partial^n\phi, x, s], & x &= X[\phi, ..., \partial^n\phi, x, 0] \\ \phi'(x') &= \Phi[\phi, ..., \partial^n\phi, x, s], & \phi(x) &= \Phi[\phi, ..., \partial^n\phi, x, 0] \\ &\vdots & &\vdots \\ \partial^m\phi'(x') &= \Phi_m[\phi, ..., \partial^n\phi, x, s], & \partial^m\phi(x) &= \Phi_m[\phi, ..., \partial^n\phi, x, 0] \end{aligned} \tag{2.8}$$

The functions Φ and X are smooth, invertible functions of s, that return the old variables at the origin. If the symmetry can act on one point, independently of the others, the parameter $s(x,\phi)$ is an arbitrary function of the variables; this is called a gauge symmetry. In the following the s will be an independent parameter, creating a global symmetry that acts simultaneously at each point.

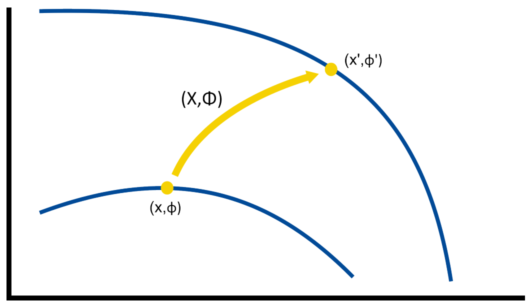

Figure 2.2: A redefinition as a mapping of variables

The Φ_m are the most general form of a global redefinition, the Lie-Bäcklund transformations. The new coordinates, fields and their derivatives are redefined in terms of the older ones, and $n > m$. If the derivatives are redefined in terms of derivatives up to their own order, so that $n \leq m$, these are called contact transformations. For this account, it will suffice to regard the most basic instance, when $n = 0$. This means the redefinition of the variables doesn't depend on derivatives of the field, and such symmetries are called point transformations

The invariance under a transformation is best studied in the linear regime, when s is infinitesimal. This minimal redefinition is constructed by a Maclaurin expansion, where terms $\mathcal{O}(s^2)$ and higher are disregarded.

$$\begin{aligned} x' &= x + s\frac{\partial X}{\partial s}[\phi, x, 0] + \mathcal{O}(s^2) &&= x + s\chi(x) \\ \phi'(x') &= \phi(x) + s\frac{\partial \Phi}{\partial s}[\phi, ..., \partial^n\phi, x, 0] + \mathcal{O}(s^2) &&= \phi(x) + s\varphi(\phi, x) \end{aligned} \tag{2.9}$$

A symmetry is called linear, if the infinitesimal variation φ depends on terms up to first order in the fields.

$$\varphi = c_0 + c_1\phi \tag{2.10}$$

The symmetry is non-linear when the higher orders of the field are involved.

$$\varphi = c_0 + c_1\phi + c_2\phi^2 + c_3\phi^3 + ... \tag{2.11}$$

If the symmetry is only a redefinition of the fields it is called an internal transformation. When the coordinates are part of the transform it is a spacetime transformation. [6]

Active and passive transforms

The position of an object, field or otherwise, is described using a coordinate system. A change in the relation between the axes and the object can be expressed either actively or a passively; either the world changes, or its description. In the active view the object moves whilst the coordinates stay fixed, in the passive view the object is immobile but the coordinate system changes.

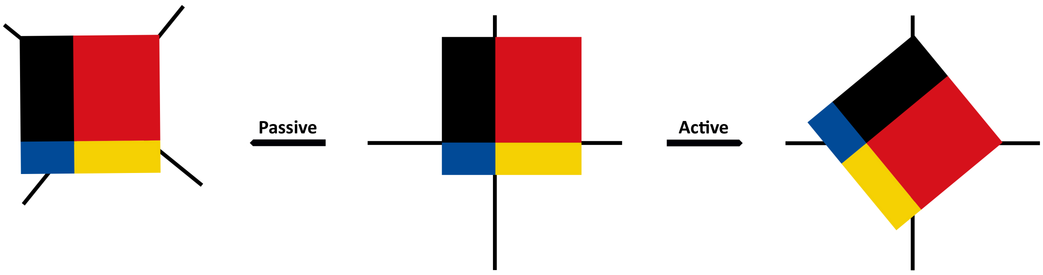

Figure 2.3: Left: passive, the axes moved. Right: active, the cube moved.

The independent variables x constitute the spacetime coordinates for the variables ϕ. The transformations of spacetime coordinates are passive, those of the fields are active.

By writing a transformation in terms of the same coordinates, before and after, a purely active description can be found for it:

$$\begin{aligned} \phi'(x') &= \phi(x) + s\varphi(x) \\ &= \phi(x' - s\chi(x)) + s\varphi(x' - s\chi(x)) \\ &= \phi(x') + s(\varphi(x') - \chi(x')\partial\phi(x')) + \mathcal{O}(s^2). \end{aligned} \tag{2.12}$$

Although this redefinition now depends on the derivatives of the fields, the benefit is that it commutes with derivation

$$(\partial_x \phi(x))' = \partial_{x'}(\phi'(x')). \tag{2.13}$$

Spacetime symmetries were originally defined as transformations of the coordinates. In the active perspective only the fields change and this definition loses meaning. In its place, any variation of the fields, that depends on the coordinates, is called a spacetime symmetry.

Orbits

Symmetry connects corresponding points on different solutions. Points on solutions, that are connected by a symmetry transform, form a curve through the solution space called an orbit.

In the family of solutions to a differential equation, symmetry manifests either in a single solution, or in a set of them. Invariant solutions are mapped into themselves by symmetry, so they are orbits. The trivial transformation of the solution is $\Phi[\phi, x, 0] = \phi(x)$, the expansion (2.9) to a different point of the same solution, $\phi(x'')$, yields the condition on invariant solutions.

$$\phi'(x') \quad = \Phi[\phi, x, s] \quad = \phi(x''(s)) \tag{2.14}$$

$$\varphi(\phi, x) = \frac{\partial \Phi}{\partial s}[\phi, x, 0] = \left.\frac{\partial x''}{\partial s}\right|_{s=0} \frac{\partial \phi(x)}{\partial x} \tag{2.15}$$

Therefore these transformations correspond to active coordinate transforms (2.12), where $\chi = -\partial_s x''$. These can't derive from point transformations of the fields. The other symmetries map one solution of a differential equation, into one another. Their inclusion makes it that the invariance condition must be expressed in terms of the left-hand side (LHS) of (2.7). When neglecting the indices, the transformation of this term can be written as:

$$\begin{aligned} \mathcal{E}(x')[\phi'(x')] &= \mathcal{E}(X[x, s])[\Phi[\phi, x, s]] \\ &= \mathcal{E}(x)[\phi] + \left.\frac{d\mathcal{E}(x')[\phi']}{ds}\right|_{s=0} s + \mathcal{O}(s^2). \end{aligned} \tag{2.16}$$

Because at $s = 0$ the functions Φ and X simply give back the old variables, the first term of the series expansion itself also reduces to the original expression. The differential equation is invariant if, when $\mathcal{E}(x)[\phi] = 0$, then $\mathcal{E}(x')[\phi'(x')] = 0$. This statement can be simplified for infinitesimal transformations, using the series expansion. The infinitesimal invariance criterion for differential equations is:

$$\mathcal{E}(x)[\phi] = 0, \qquad \left.\frac{d\mathcal{E}(x')[\phi']}{ds}\right|_{s=0} = 0 \tag{2.17}$$

This criterion defines the infinitesimal symmetry transformations in (2.9), in terms of the fields and coordinates.

Example: Define a differential equation

$$\mathcal{E}(x)[\phi] = \frac{d\phi}{dx} + \frac{x}{\phi} = 0. \tag{2.18}$$

A straightforward integration yields the quadratic equation, which describes a circle in the ϕx-plane of radius R:

$$\phi^2 + x^2 = R^2 \tag{2.19}$$

Using (2.9) the second part of the infinitesimal invariance criterion can be written as a differential equation for the symmetry variations:

$$\left.\frac{d\mathcal{E}(x')[\phi']}{ds}\right|_{s=0} = \frac{d\varphi}{dx} - \frac{d\chi}{dx}\frac{d\phi}{dx} + \frac{\chi}{\phi} - \frac{x}{\phi^2}\varphi = 0. \tag{2.20}$$

Using the original differential equation, this can be integrated to

$$\varphi\phi + x\chi = r^2. \tag{2.21}$$

The r is an integration constant. There are infinitely many solutions; they consist of φ solved as a function of an independent χ, or vice versa.

$$\varphi(\chi) = \frac{r^2 - x\chi}{\phi} \tag{2.22}$$

Two possible solutions are

$$\varphi = x, \chi = -\phi, r = 0 \qquad \& \qquad \varphi = \phi, \chi = x, r = R. \tag{2.23}$$

The first infinitesimal transformation maps points to adjacent points on the circle. The orbit coincides with the circle, and the symmetry is trivial. The second symmetry connects adjacent circles and the orbits are lines through the origin.

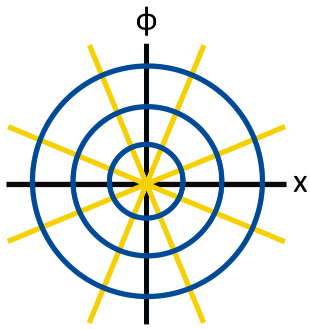

Figure 2.4: The trivial orbit, in blue, coincides with a solution. The yellow orbits connects different solutions.

2.3 Symmetry groups

An object can have multiple, qualitatively different symmetries. Performing two, different transformations consecutively must leave the object invariant too. The combination of two transformations into a single one, is therefore a symmetry again. A group is the structure built by the composition of symmetries.

A set of elements $(g_1, ..., g_N)$ with such a rule of composition γ forms a group $\mathbb{G}$, if it satisfies the following conditions:

I *Closure:*
If g_i and g_j are in $\mathbb{G}$, their combination $\gamma(g_i, g_j) = g_k$ is also in $\mathbb{G}$.

II *Associativity:*
For any g_i, g_j and g_k from $\mathbb{G}$ the order of composition is not important, if the sequence is the same: $\gamma(g_i, \gamma(g_j, g_k)) = \gamma(\gamma(g_i, g_j), g_k)$.

III *Identity:*
There is an identity element e in $\mathbb{G}$, that leaves any element of $\mathbb{G}$ it acts on unchanged: $\gamma(e, g) = g = \gamma(g, e)$

IV *Inverse:*
Each element g_i of $\mathbb{G}$, has a unique inverse g_i^{-1} that is also part of $\mathbb{G}$: $\gamma(g_i, g_i^{-1}) = e = \gamma(g_i^{-1}, g_i)$

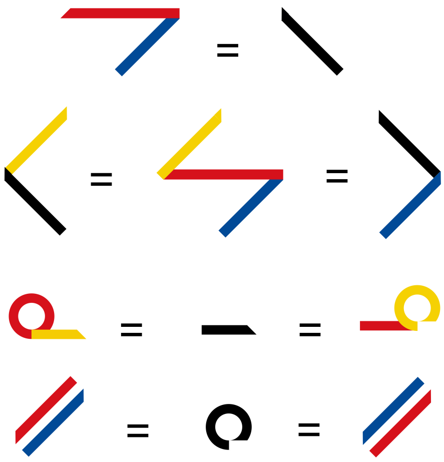

Figure 2.5: Illustration of the axioms, for vectors in the plane composed by addition. The identity, the zero vector, is indicated by a circle.

A subgroup is a subset $\mathbb{H}$ of the group $\mathbb{G}$, with the same composition, that satisfies the group requirements by itself. The trivial subgroups of $\mathbb{G}$ are e and $\mathbb{G}$ itself, however there are typically also other, more interesting subgroups.

For clarity, the composition was written explicitly as a function. For convenience it is usually written implicitly as

$$\gamma(g_1, g_2) = g_1 g_2.$$

Continuous groups

Smooth invertible mappings, the diffeomorphisms, can form groups; these are called Lie groups. Start with the variables and coordinates, that define a point $p = (x^\mu, \phi^i(x))$ in the manifold M. The elements of the group consist of the mappings $p' = U(p, \mathbf{s})$. The point p' can similarly be mapped into another point, and thus it follows that the elements are composed by the composition of functions. The parameters $\mathbf{s} = \{s_1, ..., s_N\}$ in the mapping, parametrize the path leading from the original point. A mapping joins these parameters by a composition, generating a new parameter set $\gamma(\mathbf{s}^1, \mathbf{s}^2) = (\gamma_1(\mathbf{s}^1, \mathbf{s}^2), ..., \gamma_N(\mathbf{s}^1, \mathbf{s}^2)) = \mathbf{s}^3$. Both the mappings U and the parameters $\mathbf{s}$ satisfy the group conditions

I	$U(U(p,\mathbf{s}^i),\mathbf{s}^j) = U(p,\gamma(\mathbf{s}^i,\mathbf{s}^j))$	$\gamma(\mathbf{s}^i,\mathbf{s}^j) = \mathbf{s}^k$
II	$U(U(p,\mathbf{s}^i),\gamma(\mathbf{s}^j,\mathbf{s}^k)) = U(U(p,\gamma(\mathbf{s}^i,\mathbf{s}^j)),\mathbf{s}^k)$	$\gamma(\mathbf{s}^i,\gamma(\mathbf{s}^j,\mathbf{s}^k)) = \gamma(\gamma(\mathbf{s}^i,\mathbf{s}^j),\mathbf{s}^k)$
III	$U(p,0) = p$	$\gamma(\mathbf{s},\mathbf{0}) = \gamma(\mathbf{0},\mathbf{s}) = \mathbf{s}$
IV	$U(U(p,\mathbf{s}),-\mathbf{s}) = U(U(p,-\mathbf{s}),\mathbf{s}) = p$	$\gamma(\mathbf{s},-\mathbf{s}) = \gamma(-\mathbf{s},\mathbf{s}) = \mathbf{0}$

The mappings don't rely on the initial point p, it is just an origin for the coordinates provided by the parameters $\mathbf{s}$. Geometrically, a Lie group is a set of smooth transformations that forms a differentiable manifold [7, 8].

Example: The elements of the Lie group SO(2) are the rotations around an axis. The parameter s parametrizes the angle of rotation. The Lie group is thus a circle, and each point on the circle is a rotation by an angle s.

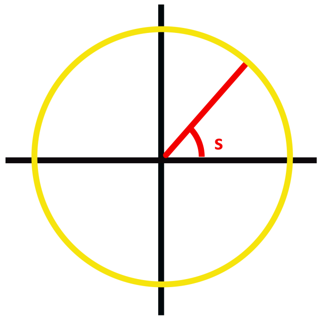

Figure 2.6: SO(2)

A Lie group can be analytically expanded around the origin, in the parameters **s**

$$p'_i = U_i(p_j, \mathbf{s}) \approx p_i + s^\alpha \frac{\partial p'_i}{\partial s^\alpha}\bigg|_{s=0} + \ldots \tag{2.24}$$

The tangent space at the origin, that consists of the possible infinitesimal displacements around the point p', is represented by the matrix

$$h^i_\alpha(p') = \frac{\partial p'_i}{\partial s^\alpha}. \tag{2.25}$$

These define a set of vectorfields, called generators, corresponding to the independent, infinitesimal shifts

$$X_\alpha = h^i_\alpha(p')\frac{\partial}{\partial p'^i}. \tag{2.26}$$

For a single generator, the Taylor expansion can be written in terms of the generators using a pullback.

$$p'' = p'(s+t) = \sum_{m=0}^{\infty} \frac{(t\frac{\partial}{\partial s})^m}{m!} p'(s) \tag{2.27}$$

$$= \sum_{m=0}^{\infty} \frac{(t\, h^i \partial_i)^m}{m!} p' = e^{tX} p' = \hat{U}(t) p' \tag{2.28}$$

This transformation creates a curve, parametrized by t. This curve is an orbit.

Figure 2.7: Orbits of a rotation, on sphere S^2

For multiple generators, such an expansion can be written in different ways:

$$p' = \hat{U}(\mathbf{s})p = \left(e^{\sum_{n=1}^{N} \alpha_n X_n}\right) p \tag{2.29}$$
$$= \left(\prod_{n=1}^{N} e^{\beta_n X_n}\right) p.$$

While both representations will connect points p and p', all different factorizations represent different paths between the two points. In consequence $\alpha_1 \neq \beta_1$. Moreover, the order of the exponentials matters. In general

$$e^{\beta_1 X_1} e^{\beta_2 X_2} \neq e^{\beta_2 X_2} e^{\beta_1 X_1}. \tag{2.30}$$

Lie algebra

The set of infinitesimal generators forms an algebra, not a group. An algebra is a vector space, with a product rule for the vectors. The Lie algebra product is called the commutator.

$$[X_a, X_b] = X_a X_b - X_b X_a = \left(h_a^i \frac{\partial h_b^j}{\partial p^i} - h_b^i \frac{\partial h_a^j}{\partial p^i}\right) \partial_j \tag{2.31}$$

The closure under group multiplication [B.1], implies that the Lie algebra is closed under the Lie bracket:

$$[X_a, X_b] = f_{ab}^c X_c. \tag{2.32}$$

A Lie algebra $\mathfrak{g}$ is constructed from a set of generators $X_1, ..., X_N$, that fulfil these demands:

I If X_a and X_b are in $\mathfrak{g}$, then $\alpha X_a + \beta X_b$ is also in $\mathfrak{g}$.

II The summation commutes, so $X_a + X_b = X_b + X_a$.

III The summation is associative, $X_a + (X_b + X_c) = (X_a + X_b) + X_c$.

IV The commutator gives elements of the Lie algebra, $[X_a, X_b] \in \mathfrak{g}$.

V The commutator is skew-symmetric, $[X_a, X_b] = -[X_b, X_a]$.

VI The commutator is bilinear, $[\alpha X_a + \beta X_b, X_c] = \alpha[X_a, X_c] + \beta[X_b, X_c]$.

VII The commutator satisfies the Jacobi identity,
$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$.

A basis of generators $Y_b = M_b^a X_a$, where M is by an invertible matrix, forms an equivalent Lie algebra with different structure constants [B.2].
The skew-symmetry implies for the structure constants that $f_{ab}^c = -f_{ba}^c$. The Jacobi identity can be written in terms of the structure constants.

$$f_{ad}^e f_{bc}^d + f_{bd}^e f_{ca}^d + f_{cd}^e f_{ab}^d = 0 \tag{2.33}$$

The structure constants form a matrix representation of the symmetry generators. Writing $X_a = (f_a)_d^e$, the Lie brackets correspond to the Jacobi identity.

Example: The algebra $\mathfrak{so}(3)$ of the group of three dimensional rotations SO(3) is spanned by the generators $\{L_1, L_2, L_3\}$. Their commutation relations are

$$[L_i, L_j] = i\epsilon_{ijk} L_k. \tag{2.34}$$

The ϵ_{ijk} is the 3-dimensional Levi-Civita symbol.

Subalgebras

A subset $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra if the commutator of any generators from $\mathfrak{h}$ gives another element in $\mathfrak{h}$; this is denoted $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. An algebra consisting of commuting elements, $[\mathfrak{h}, \mathfrak{h}] = 0$, is called abelian.
A subalgebra is ideal, under the stricter condition that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. The trivial ideal subalgebras are 0 and the whole algebra. The subgroup of generators that commute with all others in the group, $[\mathfrak{g}, \mathfrak{h}] = 0$, is called the centre.
If a Lie algebra contains only trivial ideal subalgebras, it is called simple. A semisimple Lie algebra is a direct product of simple ones.

A natural ideal of an algebra consists of all generators created by the Lie brackets, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^{(1)}$. The algebra $\mathfrak{g}^{(1)}$ is called a derived algebra. Iterating this procedure yields the derived series of $\mathfrak{g}$:

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \qquad \ldots \qquad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]. \tag{2.35}$$

If $\mathfrak{g}^{(1)} = 0$ the algebra is abelian. If this series terminates with some $\mathfrak{g}^{(n)} = 0$, then the algebra $\mathfrak{g}$ is called solvable. Each subsequent derived algebra is a subalgebra of the previous one:

$$\mathfrak{g}^{(n)} \subset \mathfrak{g}^{(n-1)} \subset \ldots \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}. \tag{2.36}$$

İnönü-Wigner contraction

Any algebra can be written in many equivalent forms, through the invertible changes of basis M. A non-invertible transformation that actually yields a different Lie algebra is the İnönü-Wigner group contraction.
A Lie algebra $\mathfrak{g}$ with a subalgebra $\mathfrak{h}$ and a complementary subspace $\mathfrak{i}$ can be decomposed as $\mathfrak{g} = \mathfrak{h} + \mathfrak{i}$. The generic form of the commutation relations is

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subseteq \mathfrak{h}, \\ [\mathfrak{h}, \mathfrak{i}] &\subseteq \mathfrak{h} + \mathfrak{i}, \\ [\mathfrak{i}, \mathfrak{i}] &\subseteq \mathfrak{h} + \mathfrak{i}. \end{aligned} \tag{2.37}$$

A redefinition of the complementary generators $\mathfrak{i}^{(0)} = \epsilon\mathfrak{i}$, followed by a limit $\epsilon \to 0$, creates the contracted algebra

$$\begin{aligned} &[\mathfrak{h},\mathfrak{h}] &&\subseteq \mathfrak{h} & & &&\subseteq \mathfrak{h},\\ &[\mathfrak{h},\mathfrak{i}^{(0)}] &&\subseteq \epsilon\mathfrak{h}+\mathfrak{i}^{(0)} &&\xrightarrow{\epsilon\to 0} &&\subseteq \mathfrak{i}^{(0)},\\ &[\mathfrak{i}^{(0)},\mathfrak{i}^{(0)}] &&\subseteq \epsilon^2\mathfrak{h}+\epsilon\mathfrak{i}^{(0)} & & &&= 0. \end{aligned} \tag{2.38}$$

Example: One subalgebra of $\mathfrak{so}(3)$ is $\mathfrak{h} = L_3$. Rescaling the other generators and taking the limit yields

$$\begin{aligned} &[L_2^{(0)},L_3] &&= iL_1^{(0)} & & &&= iL_1^{(0)},\\ &[L_3,L_1^{(0)}] &&= iL_2^{(0)} &&\xrightarrow{\epsilon\to 0} &&= iL_2^{(0)},\\ &[L_1^{(0)},L_2^{(0)}] &&= i\epsilon^2 L_3 & & &&= 0. \end{aligned} \tag{2.39}$$

This is the algebra of the special euclidean group, $\mathfrak{se}(2)$, that consists of the rotation L_3 and two translations, $L_1^{(0)}$ and $L_2^{(0)}$, in the plane.

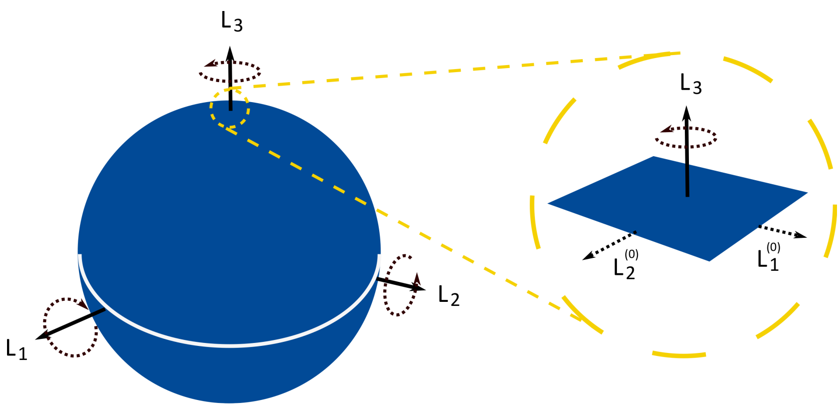

Figure 2.8: Locally, the sphere resembles the plane.

2.4 Symmetry breaking and non-linear realisations

Spontaneous symmetry breaking in field theories is often illustrated by a particle moving in the "Mexican hat potential". This potential has two distinct extrema: the top and the brim of the sombrero. In a stable state, the value of the field will lie either in the point at the top or in one of the point on the ring in the brim.
The point solution is mapped into itself by a rotation, which is therefore a trivial symmetry. If the solution is a point in the ring, the rotation will map that point to some other point on the ring; an equivalent, but different solution.
So, when the solution decays from the top to the brim, the symmetry of the solution reduces to a symmetry of the equations. This is called spontaneous symmetry breaking, and the generator of rotation is described as broken.

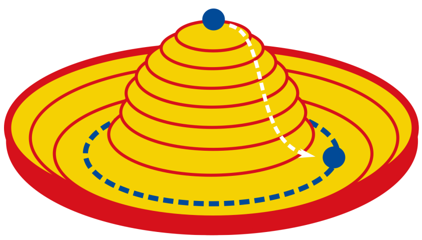

Figure 2.9: A yellow Mexican hat potential, with red level curves. The solution at the top, blue dot, can decay to one in the brim. A rotation will map that dot to any other point along the blue dashed line.

Given that the state is in a minimum in the brim, the rotation connects different equivalent solutions. The angle, which was a parameter of rotation, becomes a field coordinate of the ground state.

These observations can be generalized. Assume the existence of two solutions, ground state 1 and 2, and the groups $\mathbb{G}$ and $\mathbb{H}$, where $\mathbb{H} \subset \mathbb{G}$. Take it that state 1 and 2 are trivially invariant under transformations of the group $\mathbb{G}$ and $\mathbb{H}$, respectively. Given that both states are solutions to the same equation, the symmetries that are in $\mathbb{G}$ but not in $\mathbb{H}$ must be broken. Acting on ground state 2 with a broken symmetry will therefore generate a different, but equivalent solution. The parameters of the broken symmetry will become field coordinates for this space of ground state solutions. The fields moving around in this lowest energy state are called Nambu-Goldstone (NG) bosons.

The non-linear realisation of internal symmetries

Take a decomposition of a Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{i}$, as before. The generators of $\mathfrak{h}$ and $\mathfrak{i}$ are H_α and I_a, respectively. Any element of the group $\mathbb{G}$ can be now written as an exponent

$$g = e^{iv^a I_a} e^{iu^\alpha H_\alpha}. \tag{2.40}$$

The total Lie group $\mathbb{G}$ breaks spontaneously to a subgroup $\mathbb{H}$. The solutions, representing the state of a system, should remain invariant under the transformations from $\mathbb{H}$. For the group, this means that two transformations from $\mathbb{G}$ should be equivalent if they only differ by a factor h from $\mathbb{H}$. This relation, $g_1 \sim g_2$ if $g_1 h = g_2$, defines a set of equivalence classes that are called (left) cosets:

$$\chi_g = \{gh | h \in \mathbb{H}\}. \tag{2.41}$$

The manifold that remains after factoring out the equivalence is called the coset space, or quotient space, denoted by $\mathbb{G}/\mathbb{H}$. Each point in this space is a coset and the action of the group maps one coset to another.

$$g_2 \chi_{g_1} = \chi_{g_2 g_1} = \chi_{g_3} \tag{2.42}$$

There is a redundancy in the representation of one point in the quotient space by the unbroken group $\mathbb{G}$. This excess is removed by choosing a single element of the coset as its representative, at the cost of more complex symmetry transformations; this is a type of gauge fixing. Let the elements from $\mathbb{G}$ obey the transformation rule $g_2 g_1 = g_3$. These g are related to the representative elements χ of their cosets by $\chi_{g_1} = g_1 h_1$ and $\chi_{g_3} = g_3 h_3$. The rule for transformations between the chosen χs, under the action of the group, then follows:

$$g_2 \chi_{g_1} = g_3 h_1 = \chi_3 h_3^{-1} h_1 \qquad \rightarrow \qquad \chi' = g \chi h^{-1}. \tag{2.43}$$

In the exponential representation, the points χ in the coset space are parametrized by the broken generators I^a and the coordinates π_a:

$$U(\pi) = e^{i\pi_a I^a}. \tag{2.44}$$

In this notation, the transformation of the fields under the group action is

$$gU(\pi) = e^{i\pi'_a(\pi,g) I^a} e^{i\sigma_\alpha(\pi,g) H^\alpha} \quad \rightarrow \quad U(\pi') = gU(\pi) e^{-i\sigma_\alpha(\pi,g) H^\alpha}. \tag{2.45}$$

Note that this gives a general, non-linear transformation of the coordinates on the coset space $\pi \rightarrow \pi'(\pi, g)$. These are point transformations, since they only depend on the fields.

The vectors in the tangent space, at the origin of the group, formed the generators in the Lie algebra. The cotangent space at some other point in the manifold is given by $dg(p)$. Transporting it back to the origin for reference, gives $g(p)^{-1}dg(p) = \omega$. The differential 1-form ω is called the Maurer-Cartan (MC) form, it too characterizes the local structure of the group $\mathbb{G}$.
The MC form in terms of (2.44) encodes the action of the symmetry on the quotient manifold.

$$\omega \equiv -iU^{-1}dU = \omega_\alpha^H H^\alpha + \omega_a^I I^a \tag{2.46}$$

The transformation of the parametrization, $U \to U'$, dictates the change of the MC forms. The transform g is independent of the fields π, and falls away.

$$\omega' = -iU^{-1}(\pi')dU(\pi') = e^{i\sigma_\alpha H^\alpha}(\omega - id)e^{-i\sigma_\alpha H^\alpha} \tag{2.47}$$

The various components transform as follows

$$g : \omega_\alpha^H H^\alpha \to \quad e^{i\sigma_\alpha H^\alpha}(\omega_\alpha^H H^\alpha - id)e^{-i\sigma_\alpha H^\alpha}, \tag{2.48}$$

$$g : \omega_a^I I^a \quad \to \quad e^{i\sigma_\alpha H^\alpha}(\omega_a^I I^a)e^{-i\sigma_\alpha H^\alpha}. \tag{2.49}$$

The unbroken part of the form transforms like a gauge connection.
The only symmetry left at the level of the forms should be that of the subgroup $\mathbb{H}$. Under the group action, the broken forms must therefore linearly transform by a matrix D, as if they were vectors in the representation space of the unbroken group:

$$\omega_b'^I I^b = e^{i\sigma_\alpha H^\alpha}(\omega_a^I I^a)e^{-i\sigma_\alpha H^\alpha} = D_b^a \omega_a^I I^b. \tag{2.50}$$

The expansion of the middle term, by means of the Baker-Campbell-Hausdorff (BCH) formula [B.3], implies that this is true when $[H_\alpha, I_a] = f_{\alpha a}^b I_b$. This suggests that the algebra should take the form:

$$[H_\alpha, H_\beta] = f_{\alpha\beta}^\gamma H_\gamma, \qquad [H_\alpha, I_a] = f_{\alpha a}^b I_b, \qquad [I_a, I_b] = f_{ab}^i G_i. \tag{2.51}$$

This assumption is justified in the case of semi-simple Lie algebras, that have fully anti-symmetric structure constants. The transformation rules of the broken forms also imply that they provide a set of covariant derivatives for the NG-fields

$$\nabla \pi^a = \omega_I^a. \tag{2.52}$$

An arbitrary "matter" field, coupled to the NG-fields in the low energy limit, should transform in a similar way under the symmetry. Choose a field η for this, which is part of a linear representation D of the subgroup $\mathbb{H}$ and transforms under the group action as

$$g : \eta \to \eta' = D(e^{i\sigma_\alpha H^\alpha})\eta. \tag{2.53}$$

The derivative of this fields transforms as

$$g : d\eta \to D(e^{i\sigma_\alpha H^\alpha})\Big(d + D(e^{-i\sigma_\alpha H^\alpha})dD(e^{i\sigma_\alpha H^\alpha})\Big)\eta. \tag{2.54}$$

Because of the second term in the parentheses, this derivative doesn't transform covariantly. However, the transformation of $\omega^H_\alpha H^\alpha$ can cancel this part out, if added. So, using the unbroken components of the MC form as a connection, the covariant derivative becomes

$$\nabla\eta = (d + i\omega^H_\alpha D(H^\alpha))\eta. \tag{2.55}$$

To sum up, the MC form provides a set of building blocks, defined by their transformation properties. The broken fields change non-linearly under the group, but their covariant derivatives transform linearly.

$$g : \pi \quad \to \quad \pi'(\pi, g) \tag{2.56}$$

$$h : \pi \quad \to \quad D(e^{i\sigma_\alpha H^\alpha})\pi \tag{2.57}$$

$$g : \nabla\pi \to \quad D(e^{i\sigma_\alpha(\pi,g)H^\alpha})\nabla\pi \tag{2.58}$$

The unbroken fields, and their covariant derivatives, transform under the group action as linear representations of the unbroken subgroup:

$$g : \eta \quad \to \quad D(e^{i\sigma_\alpha H^\alpha})\eta, \tag{2.59}$$

$$g : \nabla\eta \to \quad D(e^{i\sigma_\alpha H^\alpha})\nabla\eta. \tag{2.60}$$

Action

To find the equations of motion for broken fields, the Nambu-Goldstone bosons, actions can be constructed from the MC forms [9], [10]. The action should be invariant under the full symmetry group $\mathbb{G}$. Terms that are invariant under group action from $\mathbb{G}$ can be constructed by contracting the η, $D\eta$ and $D\pi$ with the metric or tensors that are invariant under transformations (2.58), (2.59) and (2.60) from the subgroup $\mathbb{H}$.

The content of the MC forms is determined by the broken generators, but the way that the forms are put together follows from the unbroken subgroup. The simplest action is the contraction with a positive definite matrix c_{ab}, that is invariant under the group action [11].

$$\mathcal{L} = c_{ab}\frac{\omega^a_I \omega^b_I}{2} \tag{2.61}$$

The final lagrangean, using covariant derivatives contracted on the spacetime with metric $g_{\mu\nu}$, should also be invariant under spacetime symmetries.

$$\mathcal{L} = c_{ab} g^{\mu\nu} \frac{\nabla_\mu \pi^a \nabla_\nu \pi^b}{2} \tag{2.62}$$

For compact, semisimple, internal groups all parametrizations are equivalent. For such groups the most general invariant lagrangean can be assembled from the coset contruction. For non-compact groups, like the group of continuous plane isometries from the example, this isn't necessarily be the case.

Example: The continuous symmetries of a plane are an unusual choice for the demonstration of internal symmetries, which are normally compact groups, but they allow for neat illustrations. The isometries of a plane, without reflections, form the algebra $\mathfrak{se}(2)$. Let L, T_1 and T_2 represent the generators for rotation and horizontal and vertical translation, respectively. The non-zero commutation relations are

$$[L, T_2] = iT_1, \qquad [L, T_1] = -iT_2. \tag{2.63}$$

The 3 generators are vector fields, defined at each point in the plane; to span this manifold a basis of 2 vectors already suffices.

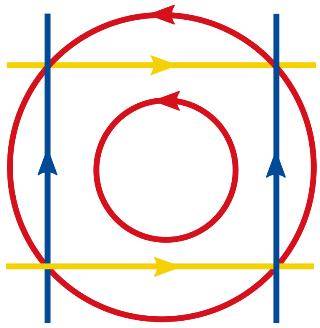

Figure 2.10: The flow lines for the vector fields of rotation (red), vertical and horizontal translation (blue and yellow)

There are two distinct, one generator, subalgebras in $\mathfrak{se}(2)$. In the first one, the unbroken generator is $H = L$. In the second one, the unbroken generator is either $H = T_1$ or $H = T_2$, they are equivalent.
The parameters of the broken generators provide a set of coordinates, which clothe some 2 dimensional manifold. For a manifold in cartesian coordinates, with standard basis vectors $\hat{e}_i$ and curvilinear coordinates $\mathbf{r} = r^i(q^j)\hat{e}_i$, the line element

is $d\mathbf{r} = dr^j \frac{\partial x^i}{\partial q^j}\hat{e}_i$. The distance interval is then defined as $ds^2 = \delta_{ij} dr^j dr^i$, and the lagrangean for a free particle on the manifold is $L = \frac{ds^2}{dt^2}$. Action (2.61) is constructed similarly, and by analogy the broken part of the MC form can be thought of as a line element.

In the case that $H = L$, the manifold is simply a plane. If $H = T_2$, it is not so straightforward because T_1 and L don't commute. Two choices for the parametrisation are

$$U_c(x_c, \theta_c) = e^{i\theta_c L} e^{i x_c T_1} \quad \& \quad U_s(x_s, \theta_s) = e^{i x_s T_1} e^{i\theta_s L}. \tag{2.64}$$

The calculation of the MC forms, for both possibilities, yields:

$$\omega_c = d\theta_c L + dx_c T_1 + x_c d\theta_c T_2 \quad \& \quad \omega_s = d\theta_s L + \cos(\theta_s) dx_s T_1 + \sin(\theta_s) dx_s T_2.$$

The covariant derivatives of the broken fields are therefore:

$$\begin{aligned} \nabla\theta_c &= d\theta_c & \nabla\theta_s &= d\theta_s, \\ \nabla x_c &= dx_c, & \nabla x_s &= \cos(\theta_s) dx_s. \end{aligned} \tag{2.65}$$

The simplest lagrangeans, pulled back to the time coordinate, then are

$$\mathcal{L}_c = \dot{x}_c^2 + \dot{\theta}_c^2, \qquad \& \qquad \mathcal{L}_s = \dot{\theta}_s^2 + \cos^2(\theta)\dot{x}_s^2. \tag{2.66}$$

The first lagrangean describes a free particle moving on a cylinder of radius 1, the second one describes a free particle moving on a sphere of radius 1. They are, however, not topologically different because the range of the fields is not restricted by a definition. These are different realisations of the same symmetry breaking pattern and, presumably, by adding all terms to the action the same particle behaviours can be described by both parametrisations.

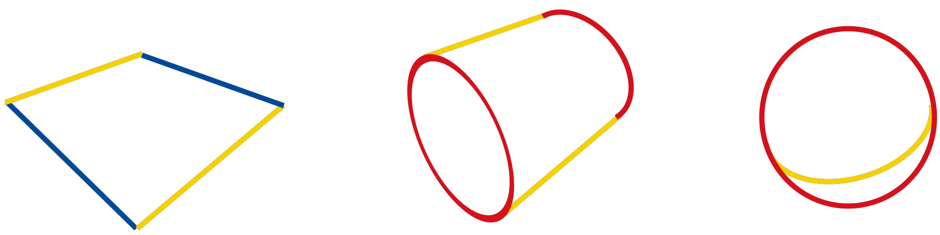

Figure 2.11: The manifolds the free particles move in. Left: the manifold for $H = L$. Center: the manifold for $H = T$ and parametrisation U_c. Right: the manifold for $H = T$ and parametrisation U_s.

Spacetime symmetry breaking

When the effect of gravity is neglible, spacetime can be approximated by Minkowski space. The group of symmetries of Minkowski spacetime is the Poincaré group; a semidirect product of the group of translations, that has the spacetime coordinates as parameters, and the Lorentz group $O(1,3)$. The symmetry generators of the Poincaré algebra consist of the four-vector P_μ, for translations, and the antisymmetric tensor $J^{\mu\nu}$, for Lorentz transformations.

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [J_{\mu\nu}, P_\lambda] &= i(g_{\nu\lambda}P_\mu - g_{\mu\lambda}P_\nu) \\ [J_{\mu\nu}, J_{\kappa\lambda}] &= i(g_{\mu\lambda}J_{\nu\kappa} + g_{\nu\kappa}J_{\mu\lambda} - g_{\mu\kappa}J_{\nu\lambda} - g_{\nu\lambda}J_{\mu\kappa}) \end{aligned} \tag{2.67}$$

When a symmetry transformation mixes coordinates and fields, the coset construction must be altered [12] [13]. The set of broken, internal generators Z is supplemented by the generator of translation, P^μ, and the coordinates x^μ. The coset space is now parametrized by

$$U(\pi, x) = e^{ix_\mu P^\mu} e^{i\pi_a I^a}. \tag{2.68}$$

The construction of the invariant terms, and their transformation rules, follows as before. Note that the generator of Lorentz transformations $J^{\mu\nu}$ is now appended to the set of unbroken generators H.

$$\begin{aligned} gU(\pi, x) &= e^{ix'_\mu P^\mu} e^{i\pi'_a I^a} e^{i\sigma_\alpha H^\alpha} e^{i\sigma_{\mu\nu} J^{\mu\nu}} \\ &\rightarrow \\ U(\pi', x') &= gU(\pi, x) e^{-i\sigma_{\mu\nu} J^{\mu\nu}} e^{-i\sigma_\alpha H^\alpha} \end{aligned} \tag{2.69}$$

This relation yields the transformations of x and π,

$$\pi(x) \rightarrow \pi'(x') = \pi'(\pi(x), x, g). \tag{2.70}$$

The spacetime MC form[2] is

$$\omega \equiv -iU^{-1}dU = \omega^H_\alpha H^\alpha + \omega^I_a I^a + \omega^P_\mu P^\mu + \frac{\omega^J_{\mu\nu}}{2} J^{\mu\nu}. \tag{2.71}$$

The forms transform under the action of the group as

$$\begin{aligned} g : \omega^H_\alpha H^\alpha + \omega^J_{\mu\nu} J^{\mu\nu} &\rightarrow \quad e^{i\sigma_\alpha H^\alpha} e^{i\sigma_{\mu\nu} J^{\mu\nu}} (\omega^H_\alpha H^\alpha + \omega^J_{\mu\nu} J^{\mu\nu} - id) e^{-i\sigma_{\mu\nu} J^{\mu\nu}} e^{-i\sigma_\alpha H^\alpha}, \\ g : \omega^I_a I^a &\rightarrow \quad e^{i\sigma_\alpha H^\alpha} e^{i\sigma_{\mu\nu} J^{\mu\nu}} (\omega^I_a I^a) e^{-i\sigma_{\mu\nu} J^{\mu\nu}} e^{-i\sigma_\alpha H^\alpha}, \\ g : \omega^P_\lambda P^\lambda &\rightarrow \quad e^{i\sigma_\alpha H^\alpha} e^{i\sigma_{\mu\nu} J^{\mu\nu}} (\omega^P_\lambda P^\lambda) e^{-i\sigma_{\mu\nu} J^{\mu\nu}} e^{-i\sigma_\alpha H^\alpha}. \end{aligned} \tag{2.72}$$

[2]There are 16 possible combinations of the indices of $J_{\mu\nu}$, but it is anti-symmetric. This means it contains only 6 unique generators, that each appear twice; the contraction $\omega^{\mu\nu}_J J_{\mu\nu}$ sums these copies together. The inclusion of the factor $\frac{1}{2}$ in the MC form, makes the entries of $\omega^{\mu\nu}_J$ equal to the components along the unique generators in $J_{\mu\nu}$.

The generator of translations should remain invariant under any internal symmetry transformations, and transform as a Lorentz vector:

$$\omega_\lambda^P \to D(e^{\sigma_{\mu\nu}J^{\mu\nu}})\omega_\lambda^P. \tag{2.73}$$

Although ω^P must transform covariantly, the conventional 1-form dx_μ needn't any longer. This makes only the former an appropriate basis form and it can be interpreted as a dual vielbein basis:

$$\omega_\mu^P = e_\mu^\tau dx_\tau. \tag{2.74}$$

It expresses the relation between the metric $g_{\sigma\tau}$ in the Minkowski space and the induced metric $G_{\mu\nu}$ on the coset space.

$$G_{\mu\nu} = g_{\sigma\tau}e_\mu^\sigma e_\nu^\tau. \tag{2.75}$$

The inverse vielbein is a set of vectors $e_\tau^\mu \partial_\mu$ that forms a basis for the tangent space. These have the property

$$e_\mu^\sigma e_\sigma^\nu = \delta_\mu^\nu, \qquad\qquad e_\mu^\sigma e_\tau^\mu = \delta_\sigma^\tau. \tag{2.76}$$

The covariant derivative for the broken forms $\omega_Z^a = (\omega_Z)_\mu^a dx^\mu + (\omega_Z)_b^a d\pi^b$ is now

$$\omega_Z^a = \omega_P^\mu \nabla_\mu \pi^a = dx^\tau e_\tau^\mu \nabla_\mu \pi^a \qquad \to \qquad \nabla_\mu \pi^a = e_\mu^\tau \left((\omega^Z)_b^a \partial_\tau \pi^b + (\omega^Z)_\tau^a\right). \tag{2.77}$$

For a matter field η that transforms in the representation of the extended invariant group, made up of $\mathbb{H}$ and $O(1,3)$, the covariant derivative is defined via

$$\omega^\mu \nabla_\mu \eta = (d - i\omega_\alpha^H H^\alpha - i\omega_{\mu\nu}^J J^{\mu\nu})\eta. \tag{2.78}$$

The building blocks for the construction of invariant terms are now ω_P, ω_Z and $\nabla_\mu \eta$.

Action

Invariant terms are constructed via the contraction with tensors τ that are invariant under the action of the subgroup. The dx^μ is no longer a good basis form, and the volume element $dV = dx_1 dx_2 dx_3 dx_4$ isn't $\mathbb{G}$ invariant either. The correct volume form, using the vielbein interpretation, is

$$dV = \epsilon_{\mu\nu\rho\sigma}\omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma = |e|dx^4. \tag{2.79}$$

Invariant NG-boson actions can be constructed by any invariant contraction of broken forms into a 4-form λ. The λ is a Lie algebra valued form on the target manifold T; in the coset construction this is the quotient manifold $\mathbb{G}/\mathbb{H}$. These fields are located in some base manifold M, in this case Minkowski space. The mapping from the base space to the target space is

$$\phi : \; M \quad \rightarrow \quad T. \tag{2.80}$$

This map can be used to map the differential forms on T back to base M, using the pullback ϕ^*. The action is constructed by pulling the 4-form λ back to the base manifold M, and then performing an integration.

$$S_{NG} = \int_M \phi^* \lambda = \int_M \phi^* \left(\tau^{ABCD} \omega_A \wedge \omega_B \wedge \omega_C \wedge \omega_D \right) = \int_M \mathcal{L} \; d^4 x \tag{2.81}$$

Inverse Higgs constraint

In combining spacetime coordinates and variables, different symmetries may become related to one another; a redundancy enters the description. In the case of internal symmetries, the degrees of freedom of different broken generators are independent. For spacetime symmetries this is no longer the case, and the set of variables can be reduced. The obvious way to do this would through the equations of motion; there is an alternative, however. The inverse Higgs constraint (IHC) provides the algebraic means to remove the unphysical degrees of freedom [14].
Take a Lie algebra that contains the commutation relation

$$[P_\mu, I_1] \supset I_2. \tag{2.82}$$

For such models the description is overcomplete, there is typically a relation such that $\pi_1^\mu \sim \partial^\mu \pi_2 + ...$ The field π_1 can be removed in favour of a function of derivatives of the field π_2. This constraint is implemented by setting the MC form associated to the generator I_2 to zero:

$$\omega_{I_2} = 0. \tag{2.83}$$

The form is covariant and so the solution of the IHC applies, no matter the action of the group. Symmetries that are part of the group but whose parameter doesn't become a NG field, like I_1, are called redundant. Note that this redundancy implies that the fields will no longer transform as point transformations, since after the substitution of the IHC they may depend on the derivatives of the field.

A caveat also applies to the non-linear realisation of spacetime symmetries. The parametrisation of the coset space may lead to differing IHCs, and a different choice for what degrees of freedom are redundant. It isn't proven that the coset construction is universal, and any non-linear realisation can be deduced from a particular parametrization and ordering of the coset [15].

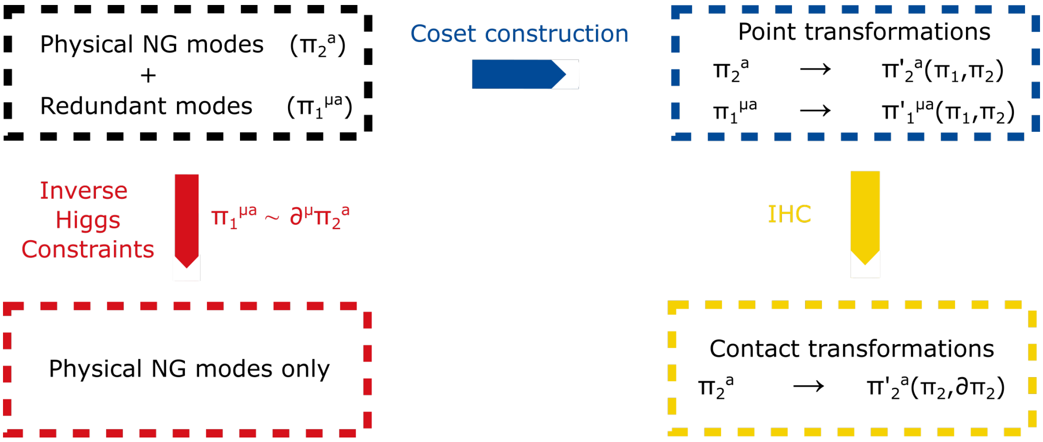

Figure 2.12: The Lie algebra and the parametrisation of the coset space determine the symmetry transformation of the fields. These are point transformations; they are algebraic functions of the coordinates of the coset space. The implementation of the inverse Higgs constraints removes the redundant modes, in favour of derivatives of the physical fields. In doing so the symmetry transformations become contact transformations, dependent on derivatives of the field.

CHAPTER 3

TOPOLOGY

Topology is the study of the global properties of geometrical objects. These objects, in their most general form, are called topological spaces. The differentiable manifolds, which perform as matter fields and spacetime in physics, are a subset of these abstract spaces.

The global properties of some object are not affected by a small local change. By extension, any smooth distortion of the topological space will leave these characteristics invariant. The continuous deformation of an object, which is reversible, corresponds to a one-to-one mapping called a homeomorphism.
So, if two figures can be deformed into one another, they have the same topological properties. In this way, all spaces that can be warped into each other are topologically equivalent; they form a class. The properties that are conserved under deformation are topological invariants; these are shared among all members of a class. Moreover, if two spaces have different topological invariants, they aren't homeomorphic.

The topological invariants can be easily illustrated in three dimensions. Imagine a malleable ball; it can freely be deformed but its surface can't tear, nor bond to itself elsewhere. The ball can then be moulded to any shape, but it is impossible to create a hole through the ball. Likewise, the hole in a malleable torus can't be removed by a smooth deformation.

The invariant quality is the number of holes in the manifold. To measure this one can imagine placing loops on the surface of the ball and the torus, that can only move in those surfaces. The fact that all loops on the circle can be contracted to a point, indicates that there is no hole. After all, on the torus the loop around the holes can't be contracted. The study comparing the structures of manifolds and boundaries is called homology.

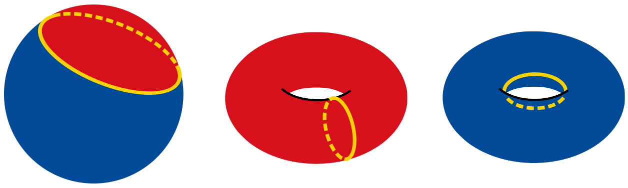

Figure 3.1: A sphere has no 2D holes, but a torus has two of them.

The actions for NG-bosons are built from differential forms. These forms depend on spacetime and quotient spaces, and those can also be made to bend. The actions which are invariant under such continuous deformations are called topological. These terms are important because they are purely geometrical, and if they are present they affect the behaviour of a theory in the same low energy regime as the NG-bosons.

3.1 HOMOLOGY

A defining quality of a boundary is that it does not have a boundary itself. The sets of manifolds that don't have a boundary thus qualify to make up the boundary of another manifold. For example, a circle has no boundary and can form the boundary of a disc. Homology, in it's simplest form, is the study of comparing such familiar "boundary manifolds" to the structures in a manifold, to find its topological properties.

For the study of general manifolds, it is convenient to dissect them into basic building blocks. A manifold M can be constructed from k-chains. A k-chain, C_k, is a smooth k-dimensional surface, formed from the smooth deformation of a union of k-dimensional blocks.

The operator that picks the boundary $(k-1)$-chain from a k-chain is defined as

$$\partial_k : \quad C_k \quad \rightarrow \quad C_{k-1}. \tag{3.1}$$

For the boundary operator to function on chains it must account for their orientation. The boundaries of chains that lie inside a manifold overlap one another. Overlapping boundaries with opposite orientations cancel each other out, so only the external boundary of the overall manifold remains. Those chains that have no boundary, $\partial_k C_k = 0$, are called cycles.

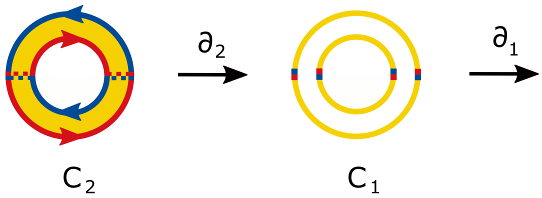

Figure 3.2: Red and blue indicate the orientation of the boundary. Overlaying boundaries of opposite orientations cancel each other. The annulus constructed from two bent rectangles is a 2-chain. Its boundary are two 1-chains, two rings constructed from two lines each. The boundaries of the lines cancel each other, so the rings are cycles.

Placing the cycles in the manifold gives two options. If every point the cycle surrounds is in the manifold, it is the boundary of that volume. If it frames a hole, it is not the boundary of a volume in the manifold; it is part of the boundary of the manifold. Define the set Z_k as all k-chains that have no boundary, and the set B_k as those k-chains that are the boundary of some volume in M:

$$Z_k = \ker \partial_k \qquad \& \qquad B_k = \operatorname{im} \partial_{k+1}.$$

The $\ker$ and im denote the kernel and image of the mappings, respectively. The homology group H_k consists of equivalence classes of cycles which are defined up to boundaries inside M.

$$H_k = \frac{Z_k}{B_k} = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}} \tag{3.2}$$

This mathematical statement abstractly expresses an intuitive picture. Imagine two cycles in the manifold M, one from Z_k and one from B_k, placed so that they overlap partially. The overlap cancels out, due to the orientation of the cycles, and a new cycle from the set Z_k is left; this cycle is a deformation of the previous one. The equivalence up to any cycle from B_k means that all deformations of the Z_k-cycle are part of the same equivalence class. All cycles that are the boundary of a volume in M can deformed away in this manner, like the circle on the sphere. The cycles around the holes reduce to a class for each boundary of a hole in the manifold; the class contains all possible deformations of that boundary. The elements of the homology group $H_k(M)$ are thus the classes of cycles around the k-dimensional holes in M. The term homology refers to the fact that different cycles in a class are homologous: they have the same relation to a hole.

The elements in Z_0 are points. Those points that aren't the boundary of a line count for the homology class H_0. If a manifold M is split into n components, the group $H_0(M) = \mathbb{Z}^n$ contains the possible numbers of loose points in each component. The elements of Z_1 are loops, and H_1 is the space of the number of loops around the holes in a manifold. There are many possible elements in Z_2, for example the sphere or tori with n holes; this homology class represents the cavities inside a manifold. It depends on the manifold which cycles are applicable and the adaptation of these simple ideas to complex chains becomes subtle, quickly.

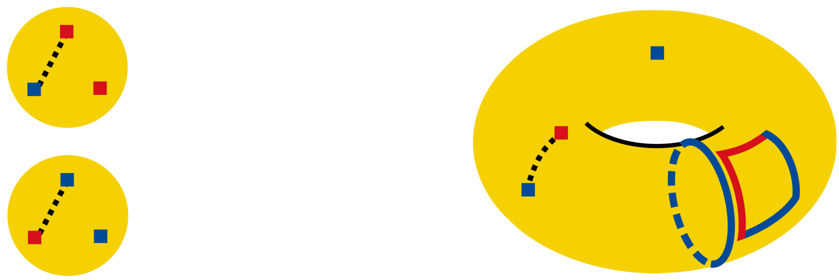

Figure 3.3: Left: A manifold consisting of two 2D discs, its homology group is $H_0(M) = \mathbb{Z} \times \mathbb{Z}$. Although there are three cycles, points, in each disc, two are the boundary of a line and vanish under the equivalence relation. Accounting for the orientations, this configuration is part of the class (-1,1) in H_0. Right: The manifold is a torus and its homology groups are $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z} \times \mathbb{Z}$, $H_2 = \mathbb{Z}$. The cycles are part of the classes (1), $(0, 1)$ and (0) in H_0, H_1 and H_2 respectively, because two points and one circle are boundaries inside the manifold.

3.2 Duality

The set of chains C and the set of differential forms Ω are both graded vector spaces, and the exterior differential and the boundary operator subspaces of different orders:

$$\begin{aligned} &\Omega = \bigoplus_{r\in\mathbb{N}} \Omega^r && C = \bigoplus_{k\in\mathbb{N}} C_k \\ &d\Omega^r \subset \Omega^{r+1} && \partial C_k \subset C_{k-1} \\ &d^2 = 0 && \partial^2 = 0 \end{aligned} \tag{3.3}$$

Using the conventional definition of integration (1.15), the integral over a manifold of an r-form α over a domain D, which is an r-chain gives a constant

$$\int_D \alpha \in \mathbb{R}.$$

This can be interpreted as an inner product between a chain and a form: $\langle D, \alpha \rangle$. If ω is an $(r-1)$-form and c is a r-chain, then Stokes' theorem states that

$$\int_c d\omega = \int_{\partial c} \omega \qquad \to \qquad \langle c, d\omega \rangle = \langle \partial c, \omega \rangle. \tag{3.4}$$

This relation implies that the exterior derivative d is the adjoint of the boundary operator. The nilpotency of the boundary operator, i.e. boundaries having no boundary, can now be derived from the algebra of the exterior derivative $\langle \partial^2 c, \omega \rangle = \langle c, d^2\omega \rangle = 0$.

3.3 Cohomology

The duality indicates the existence of forms that have properties that are analogous to the different cycles:

Homology:		*Cohomology*:	
Chain	c	Cochain (form)	ω
Cycle	$\partial c = 0$	Cocycle (closed)	$d\omega = 0$
Boundary	$c = \partial b$	Coboundary (exact)	$\omega = d\alpha$

The cochains correspond to the vector space of r-forms Ω^r. The r-cocycles, which are called closed forms, vanish under the exterior derivative; they are part of the r^{th} cocycle group Z^r.

The coboundaries or exact forms, that make up the coboundary group B^r, can be written as $\omega = d\alpha$. The form α is called a potential. The action of the exterior derivative on the cochains in Ω^r creates a structure in the sequence

$$0 \xrightarrow{i} \Omega^0 \xrightarrow{d_1} \Omega^1 \xrightarrow{d_2} \dots \xrightarrow{d_n} \Omega^n \xrightarrow{d_{n+1}} 0.$$

This sequence is called the de Rham complex. Inside the complex the groups are nested as follows: $B^r \subset Z^r \subset \Omega^r$.

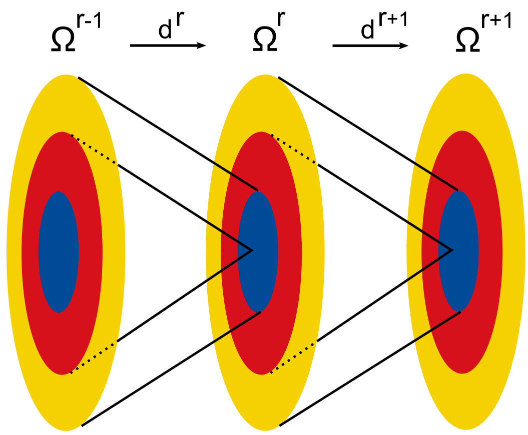

Figure 3.4: One disc, containing all 3 colours, represents the space of n-forms Ω^n. The exterior derivative d^{n+1} maps these into the space of $(n+1)$-forms, indicated by black lines, forming the de Rham complex. The action of the exterior derivative defines subspaces in Ω^r: $Z^r :=$ ker d^{r+1}, the union of red and blue, and $B^r :=$ im d^r in blue. Their quotient, in red, is the cohomology group H^r.

The (de Rham) cohomology group, which is dual to the homology group, is defined by

$$H^r = \frac{Z^r}{B^r} = \frac{\ker d^{r+1}}{\mathrm{im}\ d^r}. \tag{3.5}$$

It follows from Stokes' theorem that the integration of a closed form ω along a boundary ∂b inside a manifold M is zero. Similarly, the integration of an exact form $d\alpha$ over a cycle c must be zero.

$$\int_{\partial b} \omega = 0, \qquad\qquad \int_c d\alpha = 0. \tag{3.6}$$

The existence of an integral along a cycle c, of a closed form ω, that isn't zero,

$$\int_c \omega \neq 0, \tag{3.7}$$

therefore implies simultaneously that there are cycles in the manifold that aren't a boundary, around the holes, and that there are cocycles that aren't coboundaries. The cohomology group H^r consists of such cocycles, and they are in fact the duals to the cycles around the holes. For each hole in M such a non-zero integral of a closed form exists, so in fact the homology group H_r and the cohomology group H^r are dual too. This is a loosely statement of the content of the de Rham's theorem and its corollary [4].

3.4 Homotopy

Let a cycle c be a submanifold of M. This submanifold can be viewed as the embedding of a lower dimensional manifold K into M, by a map

$$\phi : K \to M. \tag{3.8}$$

There are many different mappings of K into the manifold, and all resulting submanifolds c are homeomorphisms of K. However, these cycles can't all be deformed into one another if there are holes in M. The equivalence between cycles up to a boundary, the homology, can now become an equivalence based on the mapping. If it is possible to continuously deform two mappings into each other, they are homotopic; the deformation itself is called a homotopy.
In the context of physics, the configuration of matter is described by the value of a field as a function of space and time. The base space of the mapping therefore corresponds to some combination of space and time, and takes the form of some simply connected manifold $K \in \mathbb{R}^D$ that extends out to infinity. In the case of low-energy physics, the fields can't be excited everywhere, and must approach a constant value along the boundary of K. The fusion of the entire boundary of K into a single point thus doesn't impose undue constraints on the mapping. The mapping from K is therefore equivalent to mapping from a D-dimensional sphere S^D; this sphere is called the compactification of K.

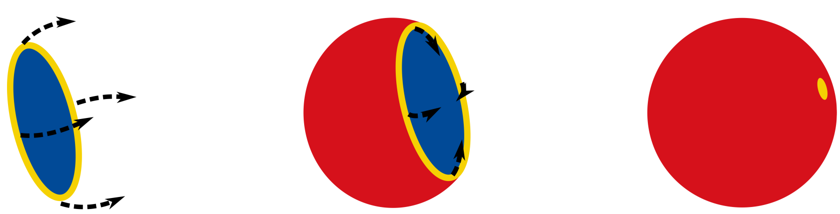

Figure 3.5: The base manifold K, on the left, is a two dimensional plane. K compactifies into the sphere S^2 by its boundary pulling together into one point.

The n-sphere is a cycle with coordinates α, and the map $\phi : S^D \to M$ places it in M at the coordinates $\phi(\alpha)$. A homotopy between two such mappings, ϕ' and ϕ'', can be tracked by a parameter s with a domain $[0, 1]$. Introduce a new map:

$$\tilde{\phi} : S^D \times [0,1] \to M \qquad \& \qquad (\alpha, s) \to \tilde{\phi}(\alpha, s) = \phi_s(\alpha). \tag{3.9}$$

This describes the deformation, under the conditions that $\phi(\alpha, 0) = \phi'$ and $\phi(\alpha, 1) = \phi''$. The mappings of S^D that are homotopic to each other form an equivalence class, and the set of all mapping equivalence classes forms the D-th homotopy group $\pi_D(M)$. If a spacetime can be compactified into some S^D, then the homotopy group only contains the mappings into a field space M with physical significance; the homology group, in contrast, registers all holes in M.

3.5 WESS-ZUMINO TERMS

The integral of a D-form over an D-dimensional domain may form an action, as was the case for the NG bosons (2.81). If the compactified space S^D wraps around a hole in M and the corresponding D-form λ is closed but not exact, the simplest topological action takes the shape of (3.7):

$$S = \int_{S^D} \phi^*(\lambda) = \int \mathcal{L} d^D x. \tag{3.10}$$

Different mappings don't affect the value of the integral, so this action is topological. These terms, however, are already constructed as part of the standard NG-actions; to introduce new topological terms, homotopies are needed.

The embedding of cycles inside a manifold M gives an account of its topology. Each of these cycles has its own topology, which in turn can be uncovered by placing cycles inside of it. A manifold thus contains a hierarchy of submanifolds. One branch of this hierarchy may contain two manifolds M_A and M_B, both cycles, embedded in M as $\phi(M_B) \subset M_A$ and $\varphi(M_A) \subset M$.

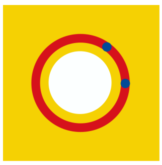

Figure 3.6: The embedding of 0-dimensional points in the 1-dimensional circle, which is itself embedded in a 2-dimensional plane (with a hole).

Let the submanifold M_B be D-dimensional, and any mapping $\phi(M_B)$ into the $(D+1)$-dimensional manifold M_A be contractible to a point. The contraction of the submanifold is described by a homotopy map ϕ_s, where $\phi_1 = \phi$ and ϕ_0 maps all of M_B into some point on M_A. If $M_B = S^D$, the contractibility can be expressed in terms of the homotopy group as $\pi_D(M_A) = 0$.
In deforming the mapping of M_B to a point, it sweeps out a surface N on M_A. This deformation may proceed by infinitesimal steps, because it is continuous, and in such a way that each step makes up an infinitesimally thin segment of N. The integration of a closed $(D+1)$-form ω over the domain N can now be split up into two integrations; one over the coordinates of the cycle $\phi_s(M_B)$ and one over the homotopy parameter s.
The Poincaré lemma states that, since the coordinate patch N in M_A is contractible to a point, any closed form ω is locally exact on N [4]. It follows that $\omega = d\lambda$ on N, so

$$S_{WZ} = q\int_N \omega = q\int_0^1 \int_{\phi_s(M_B)} \omega = q\int_{M_B}\int_0^1 \phi_s^*(d\lambda) \tag{3.11}$$

$$= q\int_{M_B} [\phi_s^*(\lambda)]_0^1 \;= q\int_{M_B} \phi_1^*(\lambda). \tag{3.12}$$

The pullback ϕ_0^* pulls the forms into a point, a 0-dimensional space; it doesn't contribute to the overall integral. The q is a coupling constant.

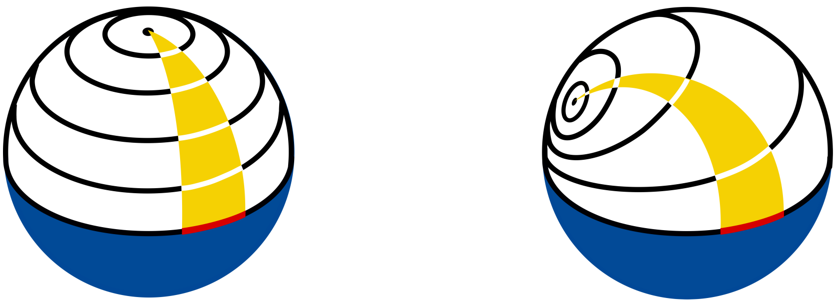

Figure 3.7: The circle M_B is mapped onto the equator of the sphere M_A. As the circle contracts to a point, it divides the white area N into segments. The red line element sweeps out a different yellow area for two different contractions. The integral over s, $\int_0^1 \phi_s^*(\omega)$, would thus yield different results for non-closed forms in both cases. For closed forms, instead, only the coordinates may differ.

The final expression doesn't yield any new actions for exact forms, because these can always be written as $\omega = d\lambda$; the D-form λ is already generated by the standard construction for the NG actions.
When ω is a cohomological term, however, no global coordinate representation exists such that $\omega = d\lambda$; it is only valid locally in some specific chart. These terms form the Wess-Zumino (WZ) actions, S_{WZ}. They are topological actions because they don't depend on the specific homotopy, only on the map ϕ_1 at the boundary. The WZ-action has a final property, only mentioned here for completeness: If the integral is normalized, so $\langle M_A, \omega \rangle = 1$, its coupling constant q is quantised [16] [17].

$$S_{WZ} = (2\pi n) \int_N \omega \qquad \text{where } n \in \mathbb{Z} \tag{3.13}$$

This holds for most common cases, when the target manifold is compact or can be compactified, but exceptions exist [18].

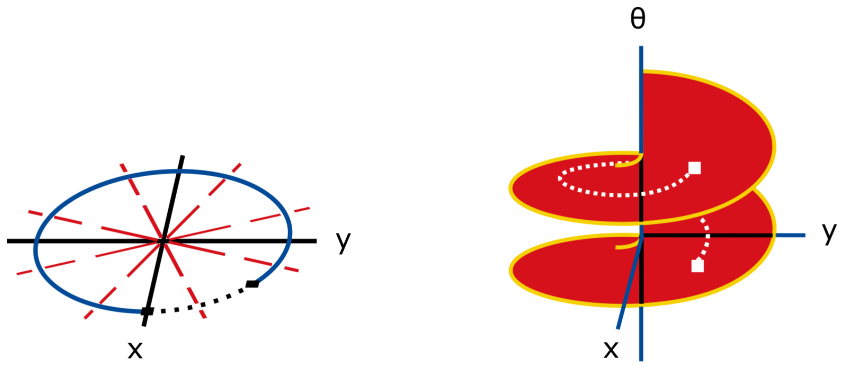

Figure 3.8: Left: The manifold M_A is a circle and M_B is a 0-sphere, a pair of points. The manifold N is an arc on this circle, in dotted points, between the angles 0 and Θ. The red dashes represent the cohomology form ω. Right: The $\omega = \frac{-ydx+xdy}{x^2+y^2}$ can locally be written in the exact form $\omega = d\arctan(\frac{y}{x}) = d\theta$; θ is the counterclockwise angle from the x-axis. The ω isn't globally exact, because θ actually isn't a proper function. The angle is multivalued: when going around the origin in a loop, and returning to the same coordinate, the value of θ changes by 2π. The WZ term is $S_{WZ} \propto \int_N \tilde{\phi}^* \omega = \int_0^1 d\left(\frac{1+s}{2}\Theta\right) + \int_0^1 d\left(\frac{1-s}{2}\Theta\right) = \Theta$.

PART II
OBSERVABLES AND PHENOMENON

CHAPTER 4

EFFECTIVE FIELD THEORY

As yet, the mathematics has served to create models that have a special invariances. To transition from mathematics to physics, these models must translate into predicted observations. In the context of physics, an observation typically corresponds to a set of numbers that result from a measurement of some signal, induced or not, from some specific configuration of matter. If all difficulties on the side of the observer could be neglected, such as the setup of an experiment and the interpretation of its results, they would have direct access to the quantities that characterise the phenomenon. Even in that ideal case, there can be a gap between the predictions from the model and directly observable quantities.

To wit, in classical physics a model of interacting particles can be constructed purely in terms of their coordinates and velocities, and these are also directly the observables of the system. However, in quantum mechanics such a simple correspondence does not exist. Although the position and velocity could still be both the variables of a model and the observables, the predictions made for these observables are probabilistic, even for an ideal observer. Therefore an additional protocol is required to calculate the predictions, given the variables; this is where quantum probability amplitudes come in. When the matter is modeled by quantum fields, instead of particles, the same concepts apply.

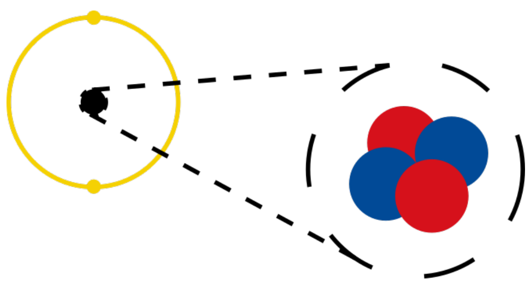

Figure 4.1: A schematic representation of a helium-4 atom. Its nucleus consists of two protons and two neutrons.

The use of a quantum mechanical description is especially required on smaller scales. However, over distant scales the world looks very different due to the clustering of matter. For instance, atoms are the building blocks of chemistry; the calculation of the electronic structure of the hydrogen atom was one of the earliest successes of quantum mechanics. However, famously, the atom can be split which opens up the field of nuclear physics. This is to say that the degrees of freedom at different scales can differ as well in the quantum regime: atomic nuclei may consist of protons and neutrons but in normal chemical reactions they don't play an active role, and only the atomic nucleus as a whole needs to be accounted for.

At a particular scale, the behaviour of a system can be modeled to arbitrary precision using the observables at that scale as variables; the result is called an effective theory. Given an underlying system that is stable, and can be characterised by constants, the oscillations around this equilibrium make up a new layer of physics that can be described using an effective theory. Here the Nambu-Goldstone bosons will be investigated and, in that same vein, their models are low energy effective field theories.

The terms of the NG boson models are determined by the broken symmetries and, also when these models involve quantum fields, those symmetries control the properties of measurements. More specifically it is possible that processes involving an observable NG boson no longer contribute to the possible events that make up the predictions of quantum field theory, the probability amplitudes, if the momentum of said boson goes to zero; this is called the Adler's zero.

These idea form an outline that will be expanded upon in this chapter. What follows will be a short sketch of quantum field theory, the philosophy behind effective field theories and the context to Adler's zero. Then starting from effective field theory, the correspondence between boundary values and measurements will be explicated. Then it will first be shown how the structure of the effective field can be decomposed into modes and then how calculations involving many modes quickly balloon. Using the mode decomposition, the path from the effective action to the probability amplitudes can then be completed. With this in hand, the effect of the symmetry on the field theory and the amplitudes can then be reviewed. This leads to the relation between the shift symmetry of a model and the Adler's zero, and the further effects of additional spacetime symmetries on the amplitudes.

4.1 Quantum field theory

Quantum mechanics is the mathematical framework that expresses the peculiar nature of matter and measurement: The state of a system is characterised by all its observable properties, and encoded as a state vector $|n\rangle$ in Hilbert space $\mathcal{H}$. A system exists in some superposition of possible states:

$$|\psi\rangle = \sum_n c_n |n\rangle. \tag{4.1}$$

An ideal measurement of a property returns one value, and reduces the superposition to the fraction of states consistent with that value. The overlap between states allows for a measurement consistent with a subset of states $|\psi_b\rangle$, given a system in the subset of states $|\psi_a\rangle$. A scalar product between states assigns a probability $\mathcal{P}$ and an amplitude $\mathscr{A}$ to such an event:

$$\mathcal{P}_{ab} = |\mathscr{A}_{ab}|^2 = |\langle\psi_a|\psi_b\rangle|^2. \tag{4.2}$$

A measurement on whatever system $|\psi\rangle$ may yield any of the possibilities $|n\rangle$ with certain probabilities $\mathcal{P}_n$ but, with probability 1, some state must be found; this is called unitarity.

$$\sum_n \mathcal{P}_n = \sum_n |\langle\psi|n\rangle|^2 = 1 \tag{4.3}$$

Matter exists in spacetime, which can be approximated by a Minkowski space if gravity is negligible. In Minkowski space, relativistic invariance is imposed on matter. The state vector as a function of spacetime is called the wave function $\psi(x) = \langle x|\psi\rangle$. A Lorentz invariant quantum system, consisting of a single particle, should be described by a wave function that satisfies a relativistic wave equation.

A free wave has a fixed momentum but indeterminate position, so the wave function expands beyond the future light cone. Some states of the particle then travel faster than light, they have spacelike trajectories. The order of events, that have spacelike separation, depends on the velocity of timelike observers. Some observers will therefore see a particle travelling back in time.

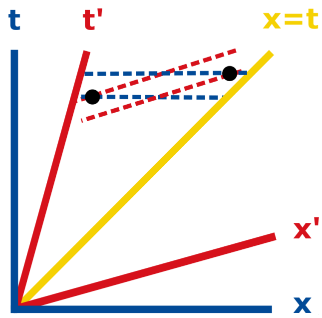

Figure 4.2: A Minkowski diagram containing two events, indicated by black dots, with a spacelike separation. The red coordinate system moves, relative to the blue one. The dotted lines are equal time lines in the two systems, indicating the time ordering of the events differs between the systems.

Rather than letting it break causality, the particle can be viewed as a negative energy state travelling forwards in time, an anti-particle. Positive and negative energy states can annihilate each other in pairs, or be created from nothing. A superposition of non-spacelike positive and negative energy states can cancel outside the lightcone, restoring relativity to the observables and the wave function. The possibility of particle creation and annihilation means that the wave function now can't just be a single particle, but must accommodate variable numbers of particles and anti-particles. In aggregate, the superposition of numerous free one-particle waves makes up a field. The geometry of spacetime thus requires relativistic quantum mechanics to be a quantum field theory (QFT) [19].

The properties of the many free, relativistic particles are now the objects of measurements, and the many-particle wave function is a distribution over their possible number and configuration. Given a system $|k_1...k_M\rangle$ of M particles characterised by their 4-momenta k, the amplitude for them scattering into N particles with 4-momenta p at a later time is

$$\mathscr{A}(\mathbf{p},\mathbf{k}) = {}_{out}\langle p_1...p_N|k_1...k_M\rangle_{in}. \tag{4.4}$$

If the time between the in and out states is very large, infinite, the scattering matrix (S-matrix) contains the evolution of particle states that are defined at a common time

$$\mathscr{A}(\mathbf{p},\mathbf{k}) = {}_{in}\langle p_1...p_N|\mathcal{S}|k_1...k_M\rangle_{in}. \tag{4.5}$$

The S-matrix contains only scattering events where all in- and outgoing particles take part in the scattering. If the particles split into clusters that have no mutual interactions, their results should be uncorrelated.

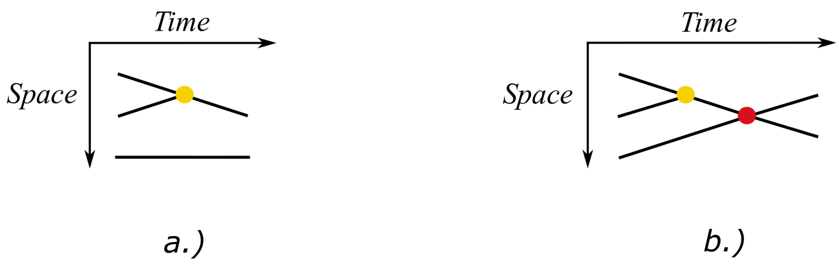

Figure 4.3: Two processes, a and b, each have 3 particles going in and 2 coming out. An intersection of trajectories, in a dot, is an interaction between the particles. In b all 5 external particles are part of one scattering event. In a the 5 particle scattering breaks into one freely moving particle, and two particles merging into one. These are loose clusters, which are completely separate and uncorrelated.

The S-matrix directly relates in- and out-states, without any account of the intermediate. However, this can be provided by the representation of the amplitude in the form of a path integral; it describes the in-between as a superposition of all possible histories between the in and outgoing particles. The path integral also yields all possible histories that evade measurement. These histories are completely independent, disconnected from any external states, and together make up the quantum vacuum. The introduction of an external source field J, that couples to the matter field ϕ, creates a new vacuum. This coupling to J supplants the coupling to an external state, and the in- and out-state amplitudes can be derived from this vacuum. The action of each history adds a phase factor to the path integral so the vacuum amplitude, driven by an external source J, is

$$\langle vac|vac\rangle_J = \int \mathcal{D}\phi \; e^{iS[\phi]+\int J\phi \; d^4x} = e^{i\Gamma[\varphi]+\int J\varphi \; d^4x}. \tag{4.6}$$

The integration over all histories, leaves one with a quantum effective action Γ. This action is a function of expectation value φ of the field ϕ in the vacuum driven by the source.

Typical observables of QFT are the differential cross section $d\sigma$ and decay rate $d\lambda$. The cross section is the fraction of scattering events occurring in colliding particle beams, per time period per area. Similarly, the differential decay rate is the probability one particle decays to N different particles, in a time interval. The S-matrix determines the probability of scattering events, and thus the observables.

$$d\sigma \propto |\mathscr{A}(\mathbf{p},\mathbf{k})|^2 \qquad \& \qquad d\lambda \propto |\mathscr{A}(\mathbf{p},k_1)|^2 \tag{4.7}$$

4.2 Effective field theory

There is no fundamental model of physics, for now at least, but if there were it wouldn't suit all calculations. Nature separates into several, sequestered energy scales. Each scale has its own relevant degrees of freedom and interactions; it is not always possible or worthwhile to derive those from some higher energy theory. Accordingly, it may be better to write a model directly in terms of the relevant degrees of freedom. Such a description will be incomplete, because the fundamental constituents are missing, but may still be exact, if the relevant observables can be modelled to an arbitrary degree of precision.

Models of this type can also be used in quantum field theory, based on Weinberg's folk theorem [20]:
"If one writes down the most general possible Lagrangian, including *all* terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with perturbative unitarity, analyticity, cluster decomposition, and the assumed symmetry properties. "
The theorem is an assumption, it has no proof but no good counter examples either.

The description of physics at a distinct scale commonly starts by identifying the relevant invariances and coordinates φ_a, at that scale. An effective, quantum action can be written in terms of these fields and symmetries.
In natural units, the physical dimensions can be expressed as powers of the mass dimension. The action Γ is dimensionless and, in d spacetime dimensions, the mass dimension of the volume measure is $[d^dx] = -d$, so it follows for the lagrangean that $[\mathcal{L}] = d$. This constrains the possible lagrangeans that can be constructed from just spacetime derivatives, $[\partial_\mu] = 1$, and scalar fields, $[\varphi_a] = (d-2)/2$.

The introduction of new physics, new degrees of freedom, at some higher energy also introduces a dimensionful energy scale Λ at which this occurs. If the energy gap to Λ were infinitely large, all interaction between the scales would fall away. The behaviour of the system would then be described by only the first terms of the lagrangean, denoted by $\mathcal{L}_{D\leq 4}$. However, if there is a slight influence this can be introduced by a perturbation. Given that there will be interaction between the scales, the low energy physics is then described, to arbitrary accuracy, by the addition of correction terms to the action. These terms factorise into a part $\mathcal{L}_D$ with mass dimension D that depends on the fields, derivatives and dimensionless constants, and a factor that depends on the scale Λ, where $[\Lambda] = 1$. The perturbation is then ordered by increasing powers of Λ^{-1} [21]:

$$\Gamma = \int \left(\sum_{D\geq 0} \frac{\mathcal{L}_D}{\Lambda^{D-d}} \right) d^4x \tag{4.8}$$

$$= \int \left(\mathcal{L}_{D\leq 4} + \frac{\mathcal{L}_5}{\Lambda} + \frac{\mathcal{L}_6}{\Lambda^2} + ... \right) d^4x. \tag{4.9}$$

The physics at smaller scales does not become involved in the events occurring in the low energy sector. That is to say, its effects appear local when the low energy degrees of freedom are observed so it only appears via the couplings in the effective lagrangean. Constructing a QFT this way, at a particular scale and only with the relevant operators and symmetries, doesn't overreach and simplifies calculations; it is called an effective field theory (EFT).

The folk theorem, an intensional definition of the class of quantum field theories, has thus led to a construction of the possible actions Γ, an extensional definition. The choice of coupling constants in Γ determines which member of the class is realized.

4.3 Spontaneous symmetry breaking

Spontaneous symmetry breaking guarantees the existence of NG bosons and a general effective action, consistent with the symmetries, can be constructed using the non-linear realisation of the symmetries. The coset construction only requires the symmetry breaking pattern, not any model of the physics that appears at the higher energy scale Λ. Without a model of the small scale physics, the coefficients of the interaction terms of Γ can't be calculated. Instead of doing any calculations involving high energies to determine these coefficients, they are therefore rather to be taken as as free parameters, to be determined by experiment.

NG modes propagate in the broken vacuum, along the equivalent minima of some potential, so they must be massless. The relativistic dispersion relation, $E^2 = p^2$, implies that these particles vanish as their momentum goes to zero $p \to 0$. Massless particles are the only relevant degrees of freedom for an effective field theory at the lowest energies. The higher energy degrees of freedom are the massive particles, so the smallest mass gap determines the characteristic scale Λ.

The energy of massless interactions is determined by the 4-momenta of the in- and outgoing modes. The low energy approximation of the dynamics of the NG-bosons is therefore an expansion in the powers of the momenta around zero. The vanishing of the scattering amplitude containing a NG boson and any number of particles in the limit that the energy of the boson goes to zero, the soft limit, is called Adler's zero. However, taking the soft limit of a proces involving a NG boson does not guarantee Adler's zero, there are exceptions [22–24]. Let the first momentum in a scattering process, involving N particles with momenta $\mathbf{p} = (z\tilde{p}_1, \tilde{p}_2(z)..., \tilde{p}_N(z))$, scale linearly with the factor z. The momenta $\tilde{p}_2(z)$ to $\tilde{p}_N(z)$ deform with z so all momenta remain on mass-shell and the overall energy and momentum in the process are conserved. The final constraint is that only $p_1 = z\tilde{p}_1$ vanishes together with z. The Maclaurin expansion of the scattering amplitudes in powers of z yields a symbolic expression.

$$\mathscr{A}(\mathbf{p}) \propto z^\sigma + \mathcal{O}(z^{\sigma+1}) \tag{4.10}$$

If the particles in the scattering are NG-bosons, and thus satisfy the Adler zero condition, then $\sigma \geq 1$. In case $\sigma \geq 2$, the amplitudes are said to have enhanced soft limits.

4.4 The variation of the action

The quantum effective action is a functional of classical fields. The quantum mechanical quantity closest to a classical variable, is the probabilistically expected value of a measurement on a system. The fields correspond to these expectation values; because they are classical the particle picture from QFT doesn't carry over, but the fields can still be conceived as a sum of modes. The action, that doesn't explicitly depend on spacetime coordinates, for a single scalar field in normal condensed notation [C.1] is

$$\Gamma[\varphi] = \int_\Sigma \mathcal{L}(\varphi, \varphi_{,1}, ..., \varphi_{,N})\, d^4x. \tag{4.11, S}$$

The quantum effective action usually can't be expressed as a local density, this is only possible when the heavier degrees of freedom have been integrated out; for models involving only NG bosons, heavy particles simply never even appeared. The vanishing of the variation of the action yields the equations for the field φ:

$$\begin{aligned}\delta\Gamma[\varphi] = &\int \delta\varphi \sum_{m=0}^{\infty}\left((-1)^m \frac{\partial\mathcal{L}}{\partial\varphi_{,m}}{}_{,m}\right) d^4x \\ &+ \int \sum_{n=1}^{\infty} \delta\varphi_{,n-1}\left(\sum_{m=0}^{\infty}(-1)^m \frac{\partial\mathcal{L}}{\partial\varphi_{,m+n}}{}_{,m}\right) d\Sigma_1 \\ =&0.\end{aligned} \tag{4.12, S}$$

The term $d\Sigma_1$ is a three-dimensional surface element on the spacetime boundary. The action is stationary in the bulk, for any variation $\delta\varphi$, if the fields obey the dynamical equations of motion (EoM).

$$\sum_{m=0}^{\infty}(-1)^m \frac{\partial\mathcal{L}}{\partial\varphi_{,m}}{}_{,m} = \sum_{l,m=0}^{\infty}(-1)^m \frac{\partial^2\mathcal{L}}{\partial\varphi_{,l}\partial\varphi_{,m}}\varphi_{,l+m} = 0. \tag{4.13, S}$$

The vanishing of the boundary terms suffices to uniquely solve the EoM [25]. Boundary conditions that are externally forced on the system don't allow for variation at the boundary. If the m^{th} derivative of the field at the boundary is imposed, its variation should vanish.

$$\text{Imposed conditions:} \qquad \delta\varphi_{,m} = 0 \tag{4.14, S}$$

Without forcing, boundary conditions also follow naturally from the action itself. The vanishing of the coefficients of $\delta\varphi_{,m}$ assures a stationary action, like for the EoM.

$$\text{Natural conditions, } (n \in \mathbb{Z}^+): \qquad \sum_{m=0}^{\infty}(-1)^m \frac{\partial\mathcal{L}}{\partial\varphi_{,m+n}}{}_{,m} = 0 \tag{4.15, S}$$

Lagrangeans that have a polynomial expansion in the fields φ, starting only at quadratic order, yield the EoM.

$$\mathcal{L} = \sum_{\substack{a=2\\ b_1,\dots,b_a=0}}^{\infty} C_{b_1\dots b_a}\partial^{b_1}\varphi\dots\partial^{b_a}\varphi \tag{4.16, S}$$

$$\sum_{m=0}^{\infty}(-1)^m \frac{\partial\mathcal{L}}{\partial\varphi_{,m}}{}_{,m} = \sum_{\substack{m=0\\ a=2\\ b_i=0}}^{\infty}\sum_{j=1}^{a}(-1)^m C_{b_i}\left(\partial^{b_1}\varphi\dots\delta_m^{b_j}\dots\partial^{b_a}\varphi\right)_{,m} = 0 \tag{4.17, S}$$

Each term in the EoM depends on the field φ. The vacuum solution $\varphi = 0$, is therefore a special solution to these equations.
Using the view of the field as an aggregate state of modes, it is clear that an alteration to the composition of that state may change the expectation value. Conversely, an infinitesimal change to the solution of the EoM represents an effective representation of the smallest changes to the composition of the state.

4.5 Jacobi fields and boundary conditions

Different solutions to the equations of motion correspond to different underlying states. A small alteration in the state will only slightly alter the expectation value, so similar states correspond to adjacent solutions. Different solutions of the EoM, φ_0 and φ_1, are trivially related through

$$\varphi_1 = \varphi_0 + \epsilon \left(\frac{\varphi_1 - \varphi_0}{\epsilon} \right) = \varphi_0 + \epsilon\phi. \tag{4.18, S}$$

The parameter ϵ controls the difference ϕ between the solutions. The equation that relates these different solutions of the EoM, to the first order in ϵ, follows from the variation

$$\begin{aligned} \delta\Gamma[\varphi_1] &= \delta\Gamma[\varphi_0 + \epsilon\phi] \\ &= \delta\Gamma[\varphi_0] + \epsilon \sum_{n}^{\infty} \int \delta \left(\frac{\delta\mathcal{L}}{\delta\varphi_{0,n}} \phi_{,n} \right) d^4x + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.19, S}$$

If the variation for both solutions is to vanish, in the bulk, then to the first order their difference must obey the Jacobi equation:

$$\sum_{m,n=0} (-1)^m \left(\frac{\partial^2\mathcal{L}}{\partial\varphi_{,m}\partial\varphi_{,n}} \phi_{,n} \right)_{,m} \Bigg|_{\varphi_0} = 0. \tag{4.20, S}$$

This on-shell variation ϕ is called the Jacobi field. The field φ is on-shell when it reduces to a solution φ_0 of the EoM.
For a particular solution, the Jacobi field is the bridge to its neighbours. The pointwise catenation of these fields therefore spans the breadth between different solutions, given some base solution to build it on.

The Jacobi equation defined around the vacuum solution $\varphi = 0$, for a lagrangean defined by the couplings C_{b_i}, is a homogeneous linear differential equation with constant coefficients.

$$L^0[\phi] = \sum_{m,n=0}^{\infty} (-1)^m C_{mn} \phi_{\epsilon,n+m} = 0 \tag{4.21, S}$$

This Jacobi field is the smallest excitation of the expectation value of the field above the vacuum, the first mode; it is the smallest, because it is only true in the regime where ϵ is infinitesimal and the higher order terms in (4.19, S) can be neglected.

For a weakly coupled theory, when this perturbative expansion makes sense, the smallest perturbations above the vacuum correspond to free particles. Free particles experience no interactions, and are described by the freely propagating part of the lagrangean:

$$\mathcal{L} = \sum_{m,n=0}^{\infty} C_{mn}\partial^m\varphi\partial^n\varphi \qquad \to \qquad \sum_{m,n=0}^{\infty}(-1)^m C_{mn}\varphi_{,m+n} = 0. \quad \text{(4.22, S)}$$

The Jacobi equation, around the vacuum, can be therefore interpreted as the free, single particle wave equation.

Sources and Green's functions

The Jacobi field represents a small disturbance of the system. An external influence that creates the disturbance is modelled by a source, or driving function $J(x)$. The general driven Jacobi equation is an inhomogeneous linear partial differential equation of the form

$$L[\phi] = J(x). \quad (4.23)$$

L is a general linear differential operator of order M, dependent on spacetime functions $\varphi_0(x)$, defined as

$$L[\phi] = \sum_{m=0}^{M} L_m[\varphi_0]\partial^m\phi, \quad \text{(4.24, S)}$$
$$L_m[\varphi_0] = \sum_{n=0}^{m}(-1)^n\frac{\partial^2\mathcal{L}}{\partial\varphi_{,n}\,\partial\varphi_{,m-n}} + \sum_{n=0}^{\infty}(-1)^n\frac{\partial^2\mathcal{L}}{\partial\varphi_{,n}\,\partial\varphi_{,m}}{}_{,n}\Bigg|_{\varphi_0}.$$

Between real functions an inner product can be defined.

$$\langle\psi,\phi\rangle = \int_\Omega(\psi\phi)d^4x$$

The partial integration of a linear differential operator, nested in this scalar product, yields its Lagrange identity.

$$\int \left(\psi L[\phi] - \phi \bar{L}[\psi]\right) d^4x = \int_\Omega \partial^\mu (\psi \overleftrightarrow{W}_\mu \phi) d^4x \tag{4.25}$$

$$\psi \overleftrightarrow{W}_1 \phi = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} (-1)^{m-n} \partial^{m-n} (L_{m+1}[\varphi_0]\psi) \partial_n \phi \tag{4.26, S}$$

$$\bar{L}[\psi] = \sum_{m=0}^{M} (-1)^m \partial_m (L^m[\varphi_0]\psi) \tag{4.27, S}$$

The difference at the boundary, W_μ, is called the bilinear concomitant or Wronskian [26,27]. The adjoint $\bar{L}$ to a differential operator L, given an inner product, is defined as

$$\langle \psi, L\phi \rangle = \langle \bar{L}\psi, \phi \rangle. \tag{4.28}$$

The operator $\bar{L}$ is only a formal adjoint, because this definition is only satisfied for chosen functions ϕ and ψ that vanish at the boundary of the spacetime region Ω. The operator L is self-adjoint if $L = \bar{L}$.

The solutions to inhomogeneous linear differential equations are constructed on the basis of the superposition of Dirac delta functions, $\delta(x - x')$, and the inverse to the differential operator L_x [1]. This inverse is a class of integral kernels $G(x - x')$, because it isn't uniquely defined, called the Green's function.

$$L_x[G(x - x')] = \delta(x - x') \tag{4.29}$$

Together with the Lagrange identity this solves the inhomogeneous linear differential equation:

$$\begin{aligned}\phi(x) = \int_\Omega \phi(x')\delta(x' - x) d^4x' \qquad &= \int_\Omega \phi(x') L_{x'}[G(x' - x)] d^4x' \\ = \int_\Omega G(x' - x) \bar{L}_{x'}[\phi(x')] d^4x' &+ \int_{\partial\Omega} \phi(x') \overleftrightarrow{W}^{\mu'} G(x' - x) d\Sigma'_\mu.\end{aligned} \tag{4.30}$$

For a self-adjoint operator, the problem is solved

$$\phi(x) = \int_\Omega G(x' - x) J(x') d^n x' + \int_{\partial\Omega} \phi(x') \overleftrightarrow{W}^{\mu'} G(x' - x) d\Sigma'_\mu. \tag{4.31}$$

A solution exists even without a source, when $J = 0$. It depends on the values the field $\phi(x')$ takes on the boundary $\partial\Omega$. Cauchy boundary conditions, that

[1] The subscript on L indicates it acts on the variable x

impose the boundary values on the field and its derivatives on only one spacelike hypersurface Σ, then solve the Jacobi equation as

$$\phi(x) = \int_{\Sigma} \phi_{\Sigma}(x') \overleftrightarrow{W}^{\mu'} G(x - x') d\Sigma_{\mu'}. \tag{4.32}$$

The function ϕ_{Σ} prescribes the values ϕ takes on the hypersurface Σ [27].

Self-adjoint operators are attended by symmetric Green's functions.

$$\begin{aligned} G(x - x') - G(x' - x) &= \int \left(G(y - x')\delta(y - x) - G(y - x)\delta(y - x') \right) d^4y \qquad (4.33) \\ &= \int \left(G(y - x') L_y[G(y - x)] - G(y - x) L_y[G(y - x')] \right) d^4y \\ &= \int_{\partial\Omega} G(y - x') \overleftrightarrow{W}^{\mu} G(y - x) d\Sigma_{\mu} \end{aligned}$$

For suitable (self-adjoint) boundary conditions the RHS vanishes. The symmetry of the Green's function expresses a reciprocity, the effect at x from an influence at x' equals the effect at x from an influence on x'. This reversibility of cause and effect occurs in conservative systems. In terms of the Jacobi fields, it indicates a microscopic reversibility.

Sources and boundary conditions

The operation of the differential operator L on solution (4.31) yields the source.

$$\begin{aligned} L_x[\phi(x)] &= J(x) + \int_{\partial\Omega} (\phi_{\Sigma}(x') \overleftrightarrow{W}^{\mu'} \delta(x - x')) d\Sigma_{\mu'} \qquad (4.34) \\ &= J(x) + J^{\infty}(x) \end{aligned}$$

The new source J^{∞} induces the boundary conditions. The integration variable x' lies on the boundary $\partial\Omega$; the source only switches on, on said surface. The boundary source carries the ∞ label, because in its application to particle physics the boundary is placed at infinity.

Adjusted natural boundary conditions

The driving force $J(x)$ for the EoM derives from an added linear term in the lagrangean.

$$W[\varphi, J] = \Gamma[\varphi] + \int \left(J(x)\varphi(x) \right) d^4x$$

$$\sum_{m=0}^{\infty} (-1)^m \frac{\partial \mathcal{L}}{\partial \varphi_{,m}}{}_{,m} = J(x) \qquad \text{(4.35, S)}$$

The source term, for boundaries, takes a suggestive form

$$\begin{aligned}\int \left(\varphi(x) J^{\infty}(x)\right) d^4x &= \int \left(\varphi(x) \int_{\partial\Omega} (\varphi_\Sigma(x') \overleftrightarrow{W}^{\mu'} \delta(x-x')) d\Sigma_{\mu'}\right) d^4x \\ &= \int_{\Omega} (\varphi_\Sigma(x) \overleftrightarrow{W}^{\mu} \varphi(x)) d\Sigma_\mu \\ &= \int_{\Omega} \varphi_\Sigma(x) L[\varphi(x)] - \varphi(x) L[\varphi_\Sigma(x)] d^4x. \end{aligned} \tag{4.36}$$

This formulation expresses the structure hidden in J^∞ outright. The boundary source is expressible as a total derivative of W^μ. In this representation not the EoM but the variation at the boundary changes:

$$\int \sum_{n=1}^{\infty} \left(\left(\sum_{m=0}^{\infty} (-1)^m \frac{\partial \mathcal{L}}{\partial \varphi_{,m+n}}{}_{,m} \right) \delta\varphi^{,n-1} - (\varphi_\Sigma \overleftrightarrow{W}^1 \delta\varphi) \right) d\Sigma_1 = 0. \tag{4.37, S}$$

In consequence, the natural boundary conditions are modified. They need no longer vanish, but must equal the conditions of a free particle at the boundary.

Free particles at the boundaries

A measurement requires a coupling between a phenomenon and a measuring apparatus. In measurements on scattering processes, the outbound particles couple to a macroscopic detector. The interaction region and the detectors in the experiment correspond, respectively, to the bulk and the boundary of the action. The measurement of quanta in the detector implies the quenching of interactions before the boundary, so the fields cross it as free waves.
The free waves are solutions to the Jacobi equation. The source J^∞ creates the condition for the field to match to a free field at the boundary (4.21, S).

$$L^0[\varphi_\Sigma] = 0. \tag{4.38}$$

This simplifies the form of the source term in the lagrangean.

$$\int d^4x [J^\infty(x) \varphi(x)] = \int_\Omega \varphi_\Sigma(x) L^0[\varphi(x)] d^4x \tag{4.39}$$

Example The basic lagrangean and EoM for a free scalar field are written as

$$\mathcal{L} = \frac{\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2}{2} \qquad \rightarrow \qquad (\Box + m^2)\varphi = 0. \tag{4.40}$$

This is the familiar Klein-Gordon equation from quantum field theory [28], expressed using the d'Alembertian $\Box = \partial_\mu \partial^\mu$, except that here the field φ is classical. This means that instead of using creation and annihilation operators, $a_p^\dagger$ and a_p, the solution is formulated in terms of two commuting functions of the momentum, $a_p = a(\vec{p})$ and its complex conjugate $a_p^* = a^*(\vec{p})$.

$$\varphi(x) = \int \left(a_p e^{-ip^\mu x_\mu} + a_p^* e^{ip^\mu x_\mu}\right) \frac{d^3p}{(2\pi)^3} \tag{4.41}$$

The relativistic momenta p^μ satisfy $p_\mu p^\mu = m^2$. This is a free field, meaning it equals the Jacobi fields. The source term for a Klein-Gordon field is

$$\begin{aligned} \int_\Omega [J^\infty(x)\varphi(x)] d^4x &= \int_\Omega \phi(x)(\Box + m^2)\varphi(x) d^4x \\ &= \int_{\partial\Omega} (\phi\partial^\mu\varphi - \varphi\partial^\mu\phi)\, d\Sigma_\mu. \end{aligned} \tag{4.42}$$

The source alters the boundary conditions to

$$\int_{\partial\Omega} (\phi\delta\varphi,_\mu + (\varphi,_\mu - \phi,_\mu)\delta\varphi) d\Sigma_\mu. \tag{4.43}$$

Both conditions vanish by fixing the derivatives on the boundary, and imposing that $\varphi,_\mu = \phi,_\mu$.

4.6 The Legendre transform of the action

The Legendre transform takes a functional and its variables and introduces new variables to yield a new functional, given certain conditions.

The variational, or functional, derivative is more useful for studying the structure of functionals than the spacetime derivatives used in their integrands. In DeWitt notation (D) these derivatives are written very compactly C.2.

The functional $F[\varphi]$ is a functional of N fields $\varphi_i = (\varphi_1(x), ..., \varphi_N(x))$. This means that $F[\varphi]$ may also contain operations on the φ, such as spacetime derivations or integrations, on the condition that ultimately the fields are mapped to a number [C.2]. The functional derivative of F generates a new set of N quantities

$$Y_i(x) = F_{,i}. \tag{4.44, D}$$

These quantities are a functional of the fields $Y_i[\varphi_j]$. They can function as a new variable for F if this mapping is invertible, so the fields can be written as a functional $\varphi_i[Y_j]$. By the inverse function theorem [29] an inverse map exists, at least in some patch, if the jacobian determinant of the mapping is not zero

$$\det\left(\frac{\delta Y_i}{\delta\varphi_j}\right) = \det\left(F_{,ij}\right) \neq 0. \tag{4.45, D}$$

The Legendre transform defines the new functional V of Y_i as

$$V[Y] = F[\varphi] + Y^i \varphi_i. \tag{4.46, D}$$

The effective action $\Gamma[\varphi]$ is a functional of N fields $\varphi_i = (\varphi_1, ..., \varphi_N)$. If $\Gamma_{,ij} \neq 0$, then its Legendre transform W is

$$W[J] = \Gamma[\varphi] + J^i \varphi_i. \tag{4.47, D}$$

The new variables for this functional are the sources $J_i(x)$.

$$\Gamma_{,i} = -J_i \tag{4.48, D}$$

This is of course the EoM for the Legendre transformed action $W[J]$ from (4.35, S). The variables, found by taking the partial derivatives of $W[J]$, are the fields again.

$$\frac{\delta W}{\delta J^i} = \left(\frac{\delta \Gamma}{\delta \varphi_j} + J^j \right) \frac{\delta \varphi_j}{\delta J^i} + \varphi_i = \varphi_i \tag{4.49}$$

These defining relations also imply that the hessian of the action is the negative inverse to the hessian of its dual.

$$\begin{aligned} \frac{\delta}{\delta \varphi_j} \left(\frac{\delta W}{\delta J^i} \right) &= \frac{\delta^2 W}{\delta J^i \delta J^k} \frac{\delta J^k}{\delta \varphi^j} = -\frac{\delta^2 W}{\delta J^i \delta J^j} \frac{\delta^2 \Gamma}{\delta \varphi_k \delta \varphi_j} \\ &= \frac{\delta \varphi_i}{\delta \varphi_j} \qquad\qquad = \delta_i^j \end{aligned} \tag{4.50}$$

The two descriptions are completely equivalent. The EoM directly yields the source, for any chosen function for $\varphi(x)$. Conversely, the choice of the source determines the solution to the EoM, $\varphi[J]$.

Solving the EoM

In DeWitt notation the effective action can be written as a series expansion

$$\Gamma[\varphi] = \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma_{,i_1 \dots i_n}[0] \varphi^{i_1} \dots \varphi^{i_n}. \tag{4.51, D}$$

Legendre's coordinate transform (4.48, D) and the EoM have the same form

$$\Gamma_{,jk}[0] \varphi^k + \sum_{n=2}^{\infty} \frac{\Gamma_{,j i_1 \dots i_n}[0]}{n!} \varphi^{i_1} \dots \varphi^{i_n} = -J_j. \tag{4.52, D}$$

The Jacobi differential operator around the solution φ is denoted $\Gamma_{,ij}[\varphi]$. The solutions are constructed using its Green's functions.

$$G^{ii'}(x^{\mu}-x^{\mu'}) = G^{ii'} \qquad \rightarrow \qquad \Gamma_{,ij}G^{jk} = -\delta_i^k \tag{4.53, D}$$

The inversion of the Jacobi operator is the first step towards the solution.

$$\varphi^i = G[0]^{ij}J_j + \sum_{n=2}^{\infty} G[0]^{ij}\frac{\Gamma_{,ji_1...i_n}[0]}{n!}\varphi^{i_1}...\varphi^{i_n} \tag{4.54, D}$$

Iteratively filling in the new solution on the RHS will express the solution in terms of the sources. The structure of the field in terms of the sources follows naturally from the Legendre transformed action $W[J]$. The series expansion of this functional is

$$W[J] = \sum_{n=0}^{\infty}\frac{W_{,j_1...j_n}[0]}{n!}J^{j_1}...J^{j_n}. \tag{4.55, D}$$

The field follows from the Legendre transform.

$$\varphi_i[J] = \sum_{n=0}^{\infty}\frac{W_{,ij_1...j_n}[0]}{n!}J^{j_1}...J^{j_n} \tag{4.56, D}$$

The action of derivatives on $W[J]$ determines the coefficients of this expansion. The Legendre transform requires particular relations for the first and second coefficients of this expansion. The comparison of the first terms of the Maclaurin expansion in (4.54, D) and (4.56, D) yields the following relationships, if the source is zero:

$$\begin{aligned} W^{,i}[0] &= 0 = \varphi[0], \\ W^{,ij}[0] &= G^{ij}[0]. \end{aligned} \tag{4.57, D}$$

The latter relation holds for any source: the Jacobi operator, $\Gamma_{,ij}[\varphi]$, is inverse to its Green's functions (4.29) and $W_{,ij}$ simultaneously (4.50). Therefore the $W_{ij}[J]$ corresponds to some Green's function.

$$\frac{\partial^2 W}{\partial J_i \partial J_j} = G^{ij} \tag{4.58, D}$$

This Green's function, called the full propagator, is defined with respect to the background φ. Its derivatives are proportional to those of the Jacobi operator, because of the inverse matrix relation.

$$dG^{ij}\Gamma_{,jk} + G^{ij}d\Gamma_{,jk} = 0 \qquad \rightarrow \qquad dG^{ij} = G^{ia}d\Gamma_{,ab}G^{bj} \tag{4.59, D}$$

The higher coefficients $W^{ij_1 \ldots j_n}$ depend on the derivatives of the Green's functions and the effective action with respect to J.

$$\frac{\partial \Gamma_{,i_1 \ldots i_n}}{\partial J_j} = \frac{\partial \varphi^k}{\partial J_j} \frac{\partial \Gamma_{,i_1 \ldots i_n}}{\partial \varphi^k} \qquad (4.60,\ \mathrm{D})$$
$$= G^{jk} \Gamma_{,i_1 \ldots i_n k}$$
$$\frac{\partial G^{il}}{\partial J^m} = G^{ij} \frac{\partial \Gamma_{,jk}}{\partial J^m} G^{kl} \qquad (4.61,\ \mathrm{D})$$
$$= G^{ia} G^{jb} G^{kc} \Gamma_{,abc}$$

These relations suffice to carry out the derivations to any order. The first four terms of the expansion are:

$$\begin{aligned} W^{,i} &= \varphi^i, \qquad (4.62,\ \mathrm{D}) \\ W^{,ij} &= G^{ij}, \\ W^{,ijk} &= G^{ia} G^{jb} G^{kc} \Gamma_{,abc}, \\ W^{,ijkl} &= G^{ia} G^{jb} G^{kc} G^{ld} \left(\Gamma_{,abcd} + G^{rs} \left(\Gamma_{,abr} \Gamma_{,scd} + \Gamma_{,car} \Gamma_{,sbd} + \Gamma_{,bcr} \Gamma_{,sad} \right) \right). \end{aligned}$$

This edifice of indices grows ever more labyrinthine, as the order of the derivative increases. The structure of the $W^{,i_1 \ldots i_n}$ can be represented by collections of tree graphs; these are graphs in which all lines are connected, but that are split in two by any single cut.
The lines in the graph are the Green's functions. The $\Gamma_{,i_1 \ldots i_n}$ are points where n-lines meet, called the n-vertices.

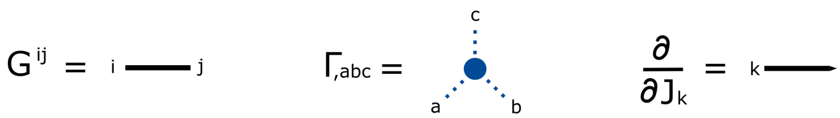

Figure 4.4: The Green's function is a line between i and j. The Γ_{abc} is a 3-vertex with 3 slots (a, b and c) for the insertion of lines. The source is an external line, a leg, with index k.

The action of the derivative on the Green's function or the effective action (4.60, D, 4.61, D) corresponds to the insertion of a new leg in the line or vertex, respectively. Any order of the expansion W can be represented by a summation of graphs. The derivative $\frac{\partial}{\partial J_l}$ acts on each term in the summation, or each part of the graph.

Figure 4.5: The derivative inserts a new external line into an element.

The coefficients $W_{,ij_0...j_n}$ are connected diagrams, so the Legendre transformed action $W[J]$ is called the generating functional of connected Green's functions. The sources J are introduced as an artifice; the original dynamics are only recovered if $J = 0$ in the bulk . The coefficients $W_{,ij_0...j_n}[0]$ are therefore physical. The boundary sources J^∞ only serve to impose conditions at the boundary, and vanish elsewhere.

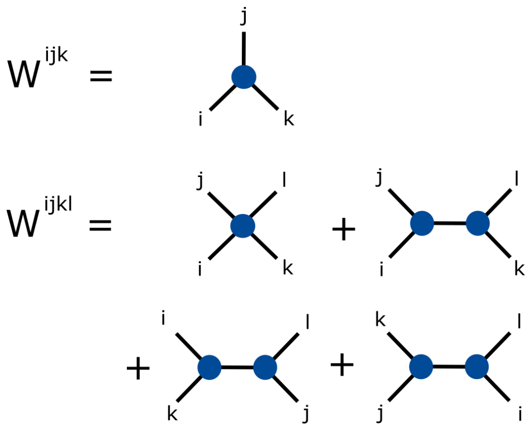

Figure 4.6: The diagrams at order 4 are generated by inserting a line l in the vertex first, and subsequently in the legs k, j and i of the diagram at order 3.

If the external legs are amputated, meaning that the external Green's functions are removed, the graphs for $W_{,ij_0...j_n}$ correspond to tree-level Feynman diagrams. The sources J^∞ contain the in-and-out particles in the diagrams for the collisions processes. Their interaction in these collisions is determined by the vertex functions $\Gamma_{,ii_1...i_n}[0]$. The overall field φ^i is a sum of particles coming in via the legs with index i in the diagrams.

$$\varphi^i = \sum_{n=0}^{\infty} \frac{W^{,ij_0...j_n}[0]}{n!} J^\infty_{j_0}...J^\infty_{j_n} \qquad (4.63, D)$$

The number of possible tree graphs

The expansion of $W[J]$ involves an increasing number of possible diagrams per order, with an increasing number of parts. For instance, an illustration of W^{ijklm}, in the manner of figure 4.6, already requires 26 figures each consisting of between 6 to 10 parts. Although the actual calculation of $W^{ijklm}[0]$ can be simplified, it remains taxing when done by hand. A computer, however, can still calculate it quickly at this scale. To get a sense of the increasing difficulty of the calculation, the number of diagrams at each order can be estimated.
The set of all diagrams at a given order of $W^{i_1 \dots i_E}$, meaning with E external lines, has cardinality N_E. This set contains a total of I_E and $N_E E$ internal and external lines, respectively, and an aggregate V_E vertices over all diagrams.
The action of the derivative $\frac{\partial}{\partial J_l}$ generates the progressive orders of the expansion. Therefore the total number of elements, lines and vertices, contained in the set at order E equals the number of diagrams at order $E+1$.

$$N_{E+1} = N_E E + I_E + V_E \tag{4.64}$$

In a single graph there are V vertices and I internal lines, and its total number of elements is $N = V + I + E$. The number of vertices in a tree graph is one greater than the number of internal lines, $V - I = 1$. This implies that the vertices make up a fraction less than half of any graph, $\frac{V}{N} = \frac{1}{2}(1 - \frac{E-1}{N})$. The same reasoning, extended to the set of graphs at order E, implies that

$$V_E - I_E = N_E \tag{4.65}$$

$$\rightarrow \qquad \frac{V_E}{N_{E+1}} = \frac{1}{2}\left(1 - \frac{E-1}{N_{E+1}/N_E}\right) \tag{4.66}$$

$$\rightarrow \qquad 0 < \frac{E-1}{N_{E+1}/N_E} < 1. \tag{4.67}$$

The average diagram at order E contains V_E/N_E vertices and I_E/N_E internal lines, and a total of N_{E+1}/N_E elements. Equations (4.64) and (4.65) indicate that the average graph obeys the equations for a single graph.
The increase of the number of possible tree diagrams, as a function of E, can be estimated using these averages. One N-element diagram at order E, that contains V vertices, generates V diagrams at order $E+1$ that have $N+1$ elements and $N_{E+1} - V_E$ diagrams that have $N+3$ elements. This means that the increase in the average number of terms per order can be estimated:

$$\frac{N_{E+2}}{N_{E+1}} - \frac{N_{E+1}}{N_E} \approx \frac{\left(1(V_E) + 3(N_{E+1} - V_E)\right)/N_E}{N_{E+1}/N_E} \tag{4.68}$$

$$= 2 + \frac{E-1}{N_{E+1}/N_E}. \tag{4.69}$$

So, the average diagram gains between 2 and 3 elements every order. This non-linear recurrence relation, in combination with two initial values, yields the following numbers, rounded to integers.

E	3	4	5	6	7	8	9
N_E	1	4	26	233	2658	36804	599146

A computer code, developed in Mathematica to generate the set of diagrams, was able to generate diagrams up to the ninth order on a laptop:

E	3	4	5	6	7	8	9
# diagrams	1	4	26	236	2752	39208	660032

The recurrence relation for N_E thus underestimates the actual number of diagrams, as it iterates on itself. Using the computer's table and N_E, yields an approximate number of $13 \cdot 10^6$ diagrams at tenth order. A scattering amplitude calculation of that size takes days on a laptop, and any further calculation is only feasible on more powerful machines.
A qualitative sense for the number of diagrams can be found using the limit on their growth. The substitution of a constant χ for $2 + \frac{E-1}{N_{E+1}/N_E}$ in (4.68) makes the equation soluble in terms of the gamma function and initial values N_B and N_{B+1}.

$$\frac{N_{E+2}}{N_{E+1}} = \frac{N_{E+1}}{N_E} + \chi \tag{4.70}$$

$$N_{E+B} = N_B\, \chi^E \frac{\Gamma\left(\frac{N_{B+1}}{N_B\, \chi} + E\right)}{\Gamma\left(\frac{N_{B+1}}{N_B\, \chi}\right)} \tag{4.71}$$

The constant χ is constrained by (4.67), so $2 < \chi < 3$. The upper limit to the growth, when E becomes large, can be simplified using Stirling's approximation, $n! \sim \sqrt{n}\left(\frac{n}{e}\right)^n$, and $\chi = 3$.

$$N_{E+3} \sim \left(\frac{3}{e}\right)^E E^{E+\frac{5}{6}} \tag{4.72}$$

The number of possible tree diagrams grows faster than a factorial. This does, however, represent an absolute maximum to the number of calculations for a single field, because in practice a model won't generate all possible diagrams.

4.7 AMPLITUDES AND THE S-MATRIX

All possible particle processes feed into the expected value of the field. These scattering processes, represented by the diagrams, were generated by the functional $W[J]$. The derivatives of W define the connected correlation functions of a quantum field theory.

$$\begin{aligned} W^{,i_1...i_n}[0] &= \langle \Omega | T\{\varphi^{i_1}...\varphi^{i_n}\}|\Omega\rangle_c \\ &= G_C^{i_1...i_n} \end{aligned} \tag{4.73, D}$$

The $T\{...\}$ is the time ordering operation, Ω is the vacuum of the interacting theory and the subscript c indicates these are connected correlations. The last identity indicates these are sometimes called n-point Greens functions, as an extension of $W^{ij} = G^{ij}$

Amplitude calculations involving path integrals entail many more types of correlation functions. The effective action incorporates the quantum corrections as part of the classical action, so there are no loop diagrams, and its Legendre transform $W[J]$ contains no disconnected diagrams.

Disconnected diagrams consist of several, loose processes. These may be different processes that are measured simultaneously, or vacuum processes that don't connect to measurement at all. Each process separately is part of an amplitude, but their co-occurrence does not add up to a new higher order interaction. This is the cluster decomposition principle, it implies that the connected tree diagrams alone suffice to calculate the correlations functions.

The integration of the boundary sources (4.34) in the generating functional of connected Green's functions $W[J^\infty]$ again reveals a different structure.

$$\begin{aligned} W[J] &= \sum_{n=0}^{\infty} \frac{W^{,j_1...j_n}[0]}{n!} J^{j_1}...J^{j_n} \\ &= \sum_{n=0}^{\infty} \left(\int_{\partial\Omega} d\Sigma_{\mu_1}...d\Sigma_{\mu_n} \varphi_\Sigma^{j_1}(x_1) \overleftrightarrow{W}^{\mu_1}...\varphi_\Sigma^{j_n}(x_n)\overleftrightarrow{W}^{\mu_n} \right) \frac{W_{,j_1...j_n}[0]}{n!} \end{aligned} \tag{4.74, D}$$

Let there only be boundary sources J^∞. These introduce free particles, Jacobi fields, at the boundary (4.38).

$$W[J^\infty] = \sum_{n=0}^{\infty}\left(\int_\Omega d^4x_1...d^4x_n\phi^{j_1}(x_1)...\phi^{j_n}(x_n)L_{x_1}...L_{x_n}\frac{W_{,j_1...j_n}[0]}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\frac{\mathscr{A}_{j_1...j_n}}{n!}\phi^{j_1}...\phi^{j_n} \tag{4.75, D}$$

For a single scalar field, the Jacobi operator is $L = (\Box + m^2)$ and the free field is

$$\phi_i = \int \left(a_{ip}e^{-ip^\mu x_\mu} + a^*_{ip}e^{ip^\mu x_\mu}\right)\frac{d^3p}{(2\pi)^3}. \tag{4.76}$$

The substitution of this explicit form of ϕ_i in equation (4.75, D) makes it possible to write $W[J^\infty]$ as a functional of a_{ip} and a^*_{ip}, denoted by $W[a, a^*]$. The Taylor expansion of this functional, in powers of a and a^*, is expressed using the functional derivatives defined by the following relations:

$$\frac{\delta a_{ip}}{\delta a_{jk}} = \delta_i^j\delta^3(p-k) \qquad \frac{\delta a_{ip}}{\delta a^*_{jk}} = 0 \tag{4.77}$$
$$\frac{\delta a^*_{ip}}{\delta a_{jk}} = 0 \qquad \frac{\delta a^*_{ip}}{\delta a^*_{jk}} = \delta_i^j\delta^3(p-k) \tag{4.78}$$

The coefficients of the N^{th}-order of the expansion of $W[a, a^*]$, in the case of a theory of a single field φ, are proportional to the Lehmann-Symanzik-Zimmerman reduction formulas.

$$\left.\frac{\delta}{\delta a^*_{p_1}}...\frac{\delta}{\delta a^*_{p_n}}\frac{\delta}{\delta a_{k_{n+1}}}...\frac{\delta}{\delta a_{k_N}}W[a, a*]\right|_{a=a^*=0} \tag{4.79}$$
$$= \left[\int \tfrac{d^4x_1}{(2\pi)^3}e^{ip_1x_1}(\Box_1+m^2)\right]...\left[\int \tfrac{d^4x_N}{(2\pi)^3}e^{-ik_Nx_N}(\Box_N+m^2)\right]\langle\Omega|T\{\varphi(x_1)...\varphi(x_N)\}|\Omega\rangle$$
$$= \left(\frac{i}{2\pi}\right)^{3N}\langle p_1, ..., p_{n-1}|\mathcal{S}|k_n...k_N\rangle$$

These $\langle p_1, ...|\mathcal{S}|...k_N\rangle$ are the S-matrix elements for N asymptotic momentum states. The definition of $W[a, a^*]$, provided by (4.75, D), shows that they are proportional to the Fourier transform of $\mathscr{A}(\mathbf{x})$.

$$\langle p_1, ..., p_{n-1}|\mathcal{S}|k_n...k_N\rangle = i^N\int d^4x_1e^{ip_1x_1}...\int d^4x_Ne^{-ik_Nx_N}\mathscr{A}(\mathbf{x})$$
$$= i^N\mathscr{A}(\mathbf{p}, \mathbf{k}) \tag{4.80}$$

The coefficients $\mathscr{A}(\mathbf{x})$ are therefore the position-space scattering amplitudes, modulo a complex phase.

4.8 Noether's first theorem

With a known procedure to go from a field theory to its scattering amplitudes, the influence of the symmetries on the amplitudes can be clarified. To begin, however, the effect of a symmetry transformation on the action will be reviewed in brief through Noether's theorem.

A variation h_i, controlled by an infinitesimal parameter ϵ, changes the field φ_i as

$$\varphi_i \to \varphi_i + \epsilon h_i(\varphi, x). \tag{4.81}$$

The smallest variation of the effective action $\Gamma[\varphi]$ is the first coefficient of the expansion in ϵ. Using the supercondensed notation for spacetime coordinates, this can be compactly expressed as:

$$\begin{aligned} \left.\frac{d\Gamma[\varphi+\epsilon h]}{d\epsilon}\right|_{\epsilon=0} &= \int \sum_{m=0}^{\infty} \frac{\partial \mathcal{L}}{\partial \varphi_{i,m}} h_{i,m} d^4x \\ &= \int h_i \sum_{m=0}^{\infty} \left((-1)^m \frac{\partial \mathcal{L}}{\partial \varphi_{i,m}}{}_{,m} \right) d^4x \\ &\quad + \int \sum_{n=1}^{\infty} h_{i,n-1} \left(\sum_{m=0}^{\infty} (-1)^m \frac{\partial \mathcal{L}}{\partial \varphi_{i,m+n}}{}_{,m} \right) d\Sigma_1. \end{aligned} \tag{4.82, S}$$

This expression simplifies if the variation is due to a symmetry, or if the fields are put on-shell. The symmetry transformations are off-shell variations, linking equivalent points everywhere. If h is an infinitesimal symmetry transformation, its effect on the action is equal to a change in the natural boundary conditions, by the addition of a total derivative to the lagrangean [2].

$$\sum_{m=0}^{\infty} \frac{\partial \mathcal{L}}{\partial \varphi_{i,m}} h_{i,m} = K^1_{,1} \tag{4.83, S}$$

If the fields φ are put on-shell in (4.82, S), indicated by $\approx$ below, the variation due to h vanishes in the bulk. The remainder is a total derivative of the Noether current, $\mathscr{J}^{\mu}$.

[2] This does not account for the possibility of a scaling symmetry, when $\frac{d\Gamma}{d\epsilon} \propto \Gamma$.

$$\sum_{m=0}^{\infty} \frac{\partial \mathcal{L}}{\partial \varphi_{i,m}} h_{i,m} \approx \left(\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{\partial \mathcal{L}}{\partial \varphi_{i,m+n}}{}_{,m}\, h_{i,n-1} \right)_{,1} = \mathscr{J}^1_{,1} \qquad \text{(4.84, S)}$$

These two total derivatives are equal on-shell, so for a symmetry variation h:

$$\left(\mathscr{J}^1 - K^1\right)_{,1} = -\int h_i \sum_{m=0}^{\infty} \left((-1)^m \frac{\partial \mathcal{L}}{\partial \varphi_{i,m}}{}_{,m} \right) d^4x \approx 0. \qquad \text{(4.85, S)}$$

The vanishing of the LHS implies that the derivative acts on a conserved current, denoted by J^μ.

$$J^1 = \mathscr{J}^1 - K^1 = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{\partial \mathcal{L}}{\partial \varphi_{i,m+n}}{}_{,m}\, h_{i,n-1} - K^1 \qquad \text{(4.86, S)}$$

Because $\partial_\mu J^\mu = \partial_0 J^0 - \nabla \cdot \mathbf{J} = 0$ the space integral $Q = \int_V J^0 d\Sigma_0$ over a volume V is conserved over time, if there is no source inside V.

$$\partial_t Q = \int_V (\nabla \cdot \mathbf{J}) dV = \int_{\partial V} \mathbf{J} \cdot d\mathbf{A} = 0 \qquad \text{(4.87)}$$

This is Noether's theorem: For each continuous symmetry of the system a connate conserved quantity exists.

The relation between the infinitesimal symmetry transformations and the Jacobi fields can be clarified, using an equation derived from (4.83, S). The symmetry transformation adds a total derivative to the action. This derivative can't influence the EoM, so its variation in the bulk should vanish identically off-shell.

$$\begin{aligned} \frac{\delta}{\delta \varphi^i} \frac{d\Gamma[\varphi, \epsilon]}{d\epsilon}\bigg|_0 &= \sum_{m,n=0}^{\infty} (-1)^m \left[\frac{\partial^2 \mathcal{L}}{\partial \varphi^i_{,m} \partial \varphi^j_{,n}} h^j_{,n} + (-1)^n \frac{\partial \mathcal{L}}{\partial \varphi^j_{,n}}{}_{,n} \frac{\partial h^j}{\partial \varphi^i_{,m}} \right]_{,m} \\ &= 0 \end{aligned} \qquad \text{(4.88, S)}$$

This equation places constraints on h^i, not on φ^i. However, this becomes the Jacobi equation when the EoM are imposed. The global infinitesimal symmetry transformation reduces to a Jacobi field when placed on-shell, reaching from a particular solution of the EoM to its neighbour. The last term in the first line determines the difference between Jacobi fields and symmetries. This term vanishes if the h^j are field independent transformations: pure spacetime symmetries are naturally also Jacobi fields.

4.9 ASYMPTOTIC SYMMETRIES

The action $\Gamma[\varphi]$ is a functional of the fields, and the generating functional of connected Green's functions $W[\phi]$ can be written as a functional of the free modes at infinity. Given the infinitesimal symmetry transformation of φ^i, the change of $\Gamma[\varphi]$ at the boundary is directly calculable. To find the change of $W[\phi]$, it is necessary to find the change induced by the symmetry on the free fields at the boundary. These symmetries of the fields in the limits of the infinite past and future, when the the interactions are suppressed and the fields are free, are called the asymptotic symmetries.
Equation (4.52, D) was the starting point to expressing φ^i in terms of ϕ^i, but from it the inverse relation may also be found. Because the Jacobi field should issue from the particular solution to the EoM, there should be no source; the inclusion of J_i^∞ would already impose the values that ϕ^i should take.

$$\Gamma_{,ji}[0]\varphi^i = -\sum_{n=2}^{\infty} \frac{\Gamma_{,ji_1...i_n}[0]}{n!}\varphi^{i_1}...\varphi^{i_n} \qquad \text{(4.89, D)}$$
$$\varphi^i = \phi^i + G[0]^{ij}\sum_{n=2}^{\infty} \frac{\Gamma_{,ji_1...i_n}[0]}{n!}\varphi^{i_1}...\varphi^{i_n} = \phi^i - G[0]^{ij}\Gamma_{,jk}[0]\varphi^k$$

A rearrangement of terms then yields the Jacobi field as the projection of some particular solution.

$$\phi^i = \left(\delta^i_k + G[0]^{ij}\Gamma_{,jk}[0]\right)\varphi^k \qquad \text{(4.89, D)}$$

The order of operations is important here; $\Gamma_{,jk}[0]$ acts to the right. An infinitesimal symmetry variation of the field, as in (4.81), therefore transforms the Jacobi field to

$$\phi^i \quad \to \quad \phi^i + \epsilon\varsigma^i, \qquad \text{where} \quad \varsigma^i = \left(\delta^i_k + G[0]^{ij}\Gamma_{,jk}[0]\right)h^k. \qquad \text{(4.89, D)}$$

The Jacobi field is defined on-shell, so the h^i in the equation above must also be on-shell; this reduces it to some Jacobi field around the solution φ.
The suppression of the interactions, so that the field corresponds to the Jacobi field at distant times, means that in this region the lagrangean tails off to just a kinetic term.

$$\Gamma[\varphi] \qquad \to \qquad \frac{\Gamma_{,ij}[0]}{2}\varphi^i\varphi^j \qquad \text{(4.89, D)}$$

For this action the EoM and the equations for the symmetry transformations and the Jacobi fields are:

$$\Box\varphi^i = 0, \tag{4.90}$$

$$\Box h^i = \partial_\mu \left(\Box\varphi^j \frac{\partial h^i}{\partial \varphi^j_{;\mu}} \right), \tag{4.91}$$

$$\Box\phi^i = 0. \tag{4.92}$$

The symmetry equation is difficult to solve but one solution, $h^i = a_\mu \partial^\mu \varphi^i$, is apparent just by inspection; it is the active form of a symmetry that shifts the coordinates by a constant. Although equation (4.88, S) identifies such complex symmetries, it ignores the simple scaling symmetries; these can be found by a symmetry analysis of the EoM, as described in section [2.2].
The field independent symmetries, when h^i is only a function of spacetime coordinates, take the same form as the solutions to the equation of motion. The contact transformations of the field, made up of the scaling and spacetime symmetries, can be written as

$$\varphi \quad \rightarrow \quad (1+\lambda)\varphi + \alpha + \beta_\mu x^\mu + \sum_{n=2}^{\infty} s_{\mu_1 \ldots \mu_n} x^{\mu_1} \ldots x^{\mu_n}. \tag{4.93}$$

The λ parameterises an infinitesimal scaling of the field. The other part of the transformation is simply a solution of $\Box h^i = 0$ but, for future purposes, written as an expansion in the spacetime coordinates; the $s_{\mu_1 \ldots \mu_n}$ are completely symmetric, traceless matrices.

This simplification of the symmetry followed from explicitly casting off the higher order terms of the action, following the assumption that this is what happens in the asymptotic regime. The Jacobi fields are the inexplicit form in which the asymptotic fields enter the description of the interacting fields, the general form of their transformations now follows from equation (4.93).

$$\varsigma = \lambda\phi + \alpha + \beta_\mu x^\mu + \sum_{n=2}^{\infty} s_{\mu_1 \ldots \mu_n} x^{\mu_1} \ldots x^{\mu_n} \tag{4.94}$$

4.10 ADLER'S ZERO AND ENHANCED SOFT LIMITS

Let ς, the asymptotic change, for now just be an arbitrary function:

$$\varsigma_i[\phi, x] = \sum_{n=0}^{\infty} \frac{\varsigma_i^{,j_1 \dots j_n}[0, x]}{n!} \phi_{j_1} \dots \phi_{j_n}. \tag{4.95, D}$$

The series expansion of the generating functional of Green's functions (4.75, D) therefore changes under this variation by

$$\begin{aligned} \left. \frac{dW[\phi + \epsilon\varsigma]}{d\epsilon} \right|_0 &= \sum_{n=2}^{\infty} \frac{\mathscr{A}^{i_1 \dots i_n}}{(n-1)!} \phi_{i_1} \dots \phi_{i_{n-1}} \varsigma_{i_n} \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \frac{\varsigma_j^{,i_1 \dots i_m} \mathscr{A}^{i_{m+1} \dots i_n j}}{(n-m)!(m)!} \right) \phi_{i_1} \dots \phi_{i_n}. \end{aligned} \tag{4.96, D}$$

If the asymptotic variation is a symmetry of the functional W, the amplitudes are not affected by this transformation and the expression should vanish. The symmetries of W map from Jacobi functions into other Jacobi functions. This means that the asymptotic variation takes the form of (4.94), so ς is at most linear in the Jacobi fields. The conclusion of this argument, from Kallosh [30], is that the vanishing of each order of the power series (4.96, D) leads to the following constraint on the scattering amplitudes:

$$\left(\frac{\varsigma_j \mathscr{A}^{i_1 \dots i_n j}}{n!} + \frac{\varsigma_j^{,i_1} \mathscr{A}^{i_2 \dots i_n j}}{(n-1)!} \right) \phi_{i_1} \dots \phi_{i_n} = 0 \tag{4.97, D}$$

The above result simplifies if there is no scaling symmetry, so $\varsigma_j^{,i}[0, x] = 0$. The remaining $\varsigma_j[0, x]$ are spacetime symmetries. The simplest member of this class of symmetries is a constant shift of the field, which is a property of the basic single NG boson corresponding to a broken $U(1)$ group. In a similar way, this applies to scalar field models that are invariant under a shift of the field by a higher order polynomial in the spacetime coordinates.

To write the constraint on the scattering amplitudes in momentum space, it is convenient to use $\varsigma[0, x]$ in an alternate form:

$$\begin{aligned} \varsigma_j[0, x] = \varsigma_j(x) &= \sum_{m=0}^{\infty} \left. \frac{\varsigma_{j,m}}{m!} \right|_{x=0} x^m \\ &= \lim_{k \to 0} \sum_{m=0}^{\infty} \left. \frac{\varsigma_{j,m}}{i^m m!} \right|_0 \frac{\partial^m}{\partial k^m} e^{-ikx}. \end{aligned} \tag{4.98, S}$$

Starting from (4.97, D) with $\varsigma_j^{;i} = 0$ and either taking the derivative n times with respect to a_{p_i} as in (4.79), or Fourier transforming the term inside the brackets to fix the outside momenta, straightforwardly gives the momentum space relation:

$$\int_{-\infty}^{\infty} e^{ip_1x_1}...e^{-ik_nx_n}\varsigma_j(x)\mathscr{A}^{i_1...i_nj}(x_1,...,x_n,x)d^4x_1...d^4x_nd^4x$$
$$= \lim_{k\to 0}\sum_{m=0}^{\infty}\frac{\varsigma_{j;m}}{m!}\bigg|_0 \frac{\partial^m}{i^m\partial k^m}\mathscr{A}(p_1,...,k_n,k) = 0. \qquad (4.99,\ \mathrm{S})$$

The external leg with momentum k is unexceptional, the same applies for any leg. As a consequence of the shift symmetry, $\varsigma = \alpha$, the S-matrix elements must therefore vanish as the momentum approaches zero.

$$\lim_{k\to 0}\alpha\mathscr{A}(p_1,...,k_n,k) = 0 \qquad \to \qquad \mathscr{A}(\mathbf{p},\mathbf{k}) \sim k + \mathcal{O}(k^2) \qquad (4.100)$$

If the action for the NG-bosons is invariant under a constant shift of the field, then the amplitudes for the scattering events satisfy the Adler's zero condition, generically.

The existence of extended shift symmetries, when ς is a polynomial function of the coordinates, implies that additional terms in the expansion of the amplitude vanish. The extension of the symmetry transformation to $\varsigma(x) = \alpha + \beta_\mu x^\mu$ implies an additional constraint on the amplitudes.

$$\lim_{k\to 0}\beta^\mu\frac{\partial}{\partial k^\mu}\mathscr{A}(p_1,...,k_n,k) = 0 \qquad \to \qquad \mathscr{A}(\mathbf{p},\mathbf{k}) \sim k^2 + \mathcal{O}(k^3) \quad (4.101)$$

The independent parameters α and β_μ in ς correspond to separate symmetries of the action. If the next order of ς, quadratic in the coordinates, also manifests as a separate symmetry this increases the scaling of the amplitude by another order; each extension of this chain of symmetries implies that the power series of the scattering amplitudes start at a higher order of the momentum. For the NG bosons, this means that the existence of extended shift symmetries leads to enhanced soft limits, so that $\sigma \geq 2$ in the definitions from (4.10). Note, however, that this argument doesn't obtain all possible ways to achieve an enhanced soft limit, nor does it even imply the existence of symmetry breaking patterns with the desired $\varsigma(x)$.

CHAPTER 5

COSMOLOGY

Mankind and its instruments are bound to earth, with only few exceptions. For now, the knowledge of what lies far beyond the solar system is purely passive, gained only through the observation of signals that were not generated as part of a controlled experiment. Assuming that all of nature's variety manifests on earth, the framework developed here is then used to interpret all observations. Any difference between observation and prediction can indicate that a change in our knowledge or understanding of nature is required.
The Copernican revolution removed the earth as the absolute centre of cosmology. Ultimately any notion of a centre was abandoned, however, research over the last century has revealed an apparent origin to the universe. This 'Big Bang' cosmological model fundamentally relies on the discovery of the recession of the stars, and of the cosmic microwave background (CMB) radiation.

Cosmic microwave background radiation means electromagnetic radiation that has a blackbody spectrum, peaking around the millimeter wavelength, that comes from each direction of the sky. The CMB is commonly characterized by its blackbody temperature [1]. The mean temperature of the CMB is 2.72548 K and, on average, this value doesn't fluctuate at any point by more than 1 part in 100000 [31].

[1]The wavelength λ of the spectral peak and the temperature T of the blackbody are related by Wien's law: $\lambda \approx \frac{2.9\ mm\cdot K}{T}$

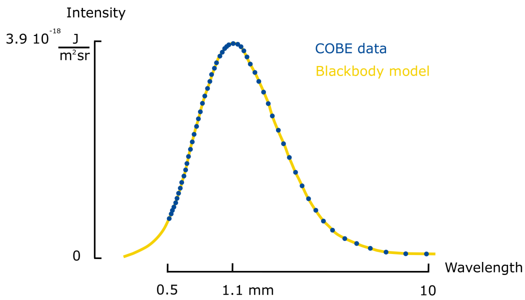

Figure 5.1: The CMB is the closest observed match to the ideal blackbody. The figure is adapted from [32], which is based on the data from [33].

Earlier observations, had already revealed that stars in far-off galaxies recede with a velocity that is proportional to their distance from earth. This is due to the expansion of the universe, which is an increase in the scale of space itself. In reverse, this means that in the past all matter was packed very closely together.

If matter is dense and hot enough it exists in a plasma state: a gas of unbound electrons, baryons and photons. The light continually scatters off the charged particles. The plasma cools down, through expansion, until the charged particles bind together into electrically neutral hydrogen atoms; this happens at temperatures approaching 3000 K. Once the charged particles are bound together, matter becomes transparent to most wavelengths of light. These earliest free photons then permeate the whole universe, creating the background radiation. The ratio of the blackbody wavelengths, from then and now, indicates that the universe scaled up by around a factor 1000 since the photon decoupling.

The CMB reveals two very remarkable qualities of the universe. The first one is the isotropy of its temperature profile. Causal interactions between matter can't travel faster than the speed of light. The particle horizon at the time of decoupling, the greatest possible distance that light could have travelled since the beginning of the universe, subtends an angle less than 2° on the current sky. Since areas further apart couldn't have interacted, this raises the 'horizon problem´: How can causally disconnected parts of the universe have reached a thermal equilibrium?

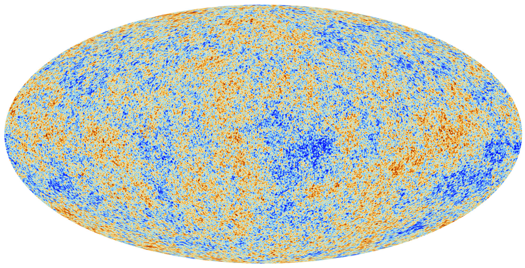

Figure 5.2: This is a Mollweide projection of the CMB temperature on the sky. The average temperature is 2.73 K. The maximum size of the deviations is 300 μK; a variation by 1 part out of 10000. (Image: ESA/Planck collaboration [34]).

The inhomogeneity of the CMB temperature across the sky, the anisotropy, originates in small variations of the cosmic mass density. Before the photon decoupling this mass consisted of the plasma, but in the most part of dark matter. Matter is called dark when it interacts through gravitation but not, or only negligibly, through other forces. Inside the plasma, however, there is a significant radiation pressure force, due to the scattering of photons between the charges.
The initial inhomogeneity of matter has no particular physical scale, until gravity attracts matter to the denser regions, emphasizing the differences. The momentum of the contraction causes it to pass by the equilibrium, where the gravitational force is in balance with the internal pressure of the plasma. This induces an acoustic oscillation in the plasma, which has a predictable characteristic size. The temperature of the photons, when they decouple, indicates which phase of the oscillation that part of space was in.

Since the physical size of the inhomogeneity can be calculated it makes for a good standard ruler. The size can be used to determine the geometry of spacetime between us and the decoupling. The light from the decoupling follows the geodesics. In case of the euclidean geometry these are straight lines, but if spacetime is curved the path of light bends too. It is therefore possible to determine the the curvature of the intervening spacetime by comparing the observed and theoretical size of the inhomogeneity.

It seems that to a close approximation the universe is flat [2]. The curvature of the universe is determined by its matter and energy content. Matter contracts under the force of gravity; it is opposed by an unexplained influence that drives the expansion of space, called dark energy. If a universe is flat it stays flat, in a balance between dark energy and matter, but any deviation would increase as the universe evolved. In those cases the universe would either collapse, if overdense, or expand so quickly that no galaxies would form, if underdense. If the universe isn't flat by an accidental initial condition, then there is a 'flatness problem´: Why does the universe have the exact matter density required for it to be flat?

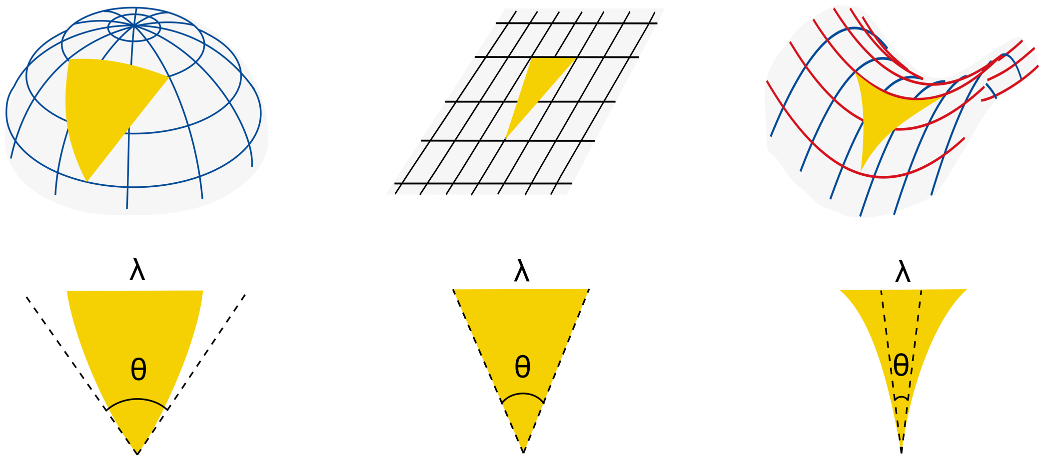

Figure 5.3: The comparison of the observed angular size θ of the inhomogeneity, due to the oscillation in the plasma, and its estimated size λ indicates the type and size of the curvature of the space.

The standard solution to these problems is the theory of cosmic inflation. It is a prologue to the hot big bang, during which the universe expanded by a factor e^{70} in a fraction of a second. This enormous expansion of space means that the entirety of the visible universe corresponds to an infinitesimal patch of space in the early universe. A patch of a smooth, curved manifold will approximate a flat space as it becomes very small; this solves the flatness problem. Since the horizon decreases with the expansion of space, its problem is also resolved: early on all areas of the CMB would have been close enough to equilibrate.

[2]The precise facts seem to be still under development [35]

In order to model the expansion mathematically, a new 'inflaton' field ϕ is introduced. The generic action for such a field, coupled to gravity, is written as

$$S = \int (R + \mathcal{L}_I(\phi, \partial\phi, ...) \sqrt{g}\, d^4x. \tag{5.1}$$

Here g is the determinant of the spacetime metric $g_{\mu\nu}$, and R is the curvature scalar. The inflationary model is a way to obtain the homogeneous qualities of the universe, at the time of decoupling, from earlier, random initial conditions. That is a rather abstract motivation, the practical reason for inflation is that it provides an origin for the inhomogeneity in the early universe.

Like all other matter in the universe, the inflaton field should fundamentally be governed by quantum mechanics. The quantum fluctuations of the inflaton field around the vacuum can generate the primordial density perturbations, which impel the start of the oscillations in the plasma. Ultimately, these density perturbations are the seeds that grow into the current large-scale structure of the universe, consisting of a distribution of galaxies.

Although it is not apparent within the solar system, the current universe is still expanding; it's even accelerating, in fact. The inflaton, or rather the dark energy that it models, drives this expansion to this day.

5.1 Spacetime and symmetry

So, the observation of the universe, at cosmological scales, has shown that spacetime seems to be expanding. This is not apparent at a local scale; locally, empty spacetime looks like Minkowski space. Yet, Minkowski space itself is not so apparent either; on the human scale, at non-relativistic speeds, instead only the galilean geometry of classical physics is recognizable.
This is a nested sequence of scales and geometries and the last two are already familiar. The isometries of the Minkowski spacetime form a group, whose symmetry generators form the Poincaré algebra. The generators consist of the four-vector P_μ, for translations, and the antisymmetric tensor $J^{\mu\nu}$, for Lorentz transformations.

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[J_{\mu\nu}, P_\lambda] &= \frac{i}{2}(\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\alpha\beta})g_{\lambda\rho}P_\sigma, \\
[J_{\mu\nu}, J_{\kappa\lambda}] &= -\frac{i}{4}(\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\alpha\beta})(\epsilon_{\kappa\lambda\gamma\delta}\epsilon^{\zeta\eta\gamma\delta})g_{\rho\zeta}J_{\sigma\eta}.
\end{aligned} \tag{5.2}$$

The generator of Lorentz transformations, $J_{\mu\nu}$ can be decomposed into the parts that contain the rotations, J_{mn}, and the Lorentz boosts J_{m0}. The Latin alphabet indices only span the space components. The rotations form a subalgebra, so redefining $J^{(0)}_{\mu 0} = \epsilon J_{\mu 0}$ and taking the limit $\epsilon \to 0$, yields the contracted algebra:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[J^{(0)}_{m0}, P_\lambda] &= 0, \\
[J_{mn}, P_\lambda] &= \frac{i}{2}(\epsilon_{mn\alpha\beta}\epsilon^{rs\alpha\beta})g_{\lambda r}P_s, \\
[J_{mn}, J_{kl}] &= -\frac{i}{4}(\epsilon_{mn\alpha\beta}\epsilon^{rs\alpha\beta})(\epsilon_{kl\gamma\delta}\epsilon^{vw\gamma\delta})g_{rv}J_{sw}, \\
[J^{(0)}_{m0}, J_{kl}] &= -\frac{i}{4}(\epsilon_{m0ab}\epsilon^{r0ab})(\epsilon_{kl\gamma\delta}\epsilon^{vw\gamma\delta})g_{rv}J^{(0)}_{0w}, \\
[J^{(0)}_{m0}, J^{(0)}_{k0}] &= 0.
\end{aligned} \tag{5.3}$$

In this limit the Lorentz boost becomes a Galilei boost, and the Lie algebra reduces to that of the galilean group. This group relates different reference frames in classical physics; frames that move at velocities much smaller than the speed of light. Any observer measures time passing at a constant rate, $t' = t$, and positions are related by $x' = x + vt$. These rules are the common sense relations between different observers known from everyday life.

In the example of the İnönü-Wigner contraction of the algebra of the sphere (2.39), the algebra of the plane was recovered. Living on a sphere looks like living on a plane, for someone much smaller than the sphere. Similarly, the relativistic Minkowski space looks like absolute space and time, for an observer severely slower than the speed of light. In these cases the larger symmetry group is unnoticeable because of a limit but there is another possibility, namely that the symmetry is broken. This suggests an new perspective on established foundations: given some observed symmetry group, is it possible that it is only a subgroup of some larger Lie group? [3]

On cosmological scales, the expanding spacetime is assumed to be modeled by a Friedmann-Lemaître-Robertson-Walker (FLRW) space. The metric of these spaces is determined by their curvature k and spatial expansion $a(t)$:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \tag{5.4}$$

This spacetime is coupled to the inflaton via the Einstein-Hilbert action (5.1). As stated, the known symmetry of the local, flat space time is represented by the Poincaré group[4]. If this is an unbroken subgroup of a larger, broken Lie group, there will also be NG bosons; due to spacetime symmetry breaking some of these fields will be redundant. The physical bosons that are created in this way may function as inflaton fields, driving the expansion of the universe. In conclusion then, the phenomenon of inflation could be a possible physical manifestation of NG bosons involving redundant broken symmetries.

[3] The existence of Minkowski space was already hypothesized by Helmholtz in 1876, based on symmetry arguments [36, 37]. A more recent argument that traces of the Lorentz transformation are already present in the Galilei invariance of the action for a classical particle, based on the relation between active and passive symmetry transformations, can be found in [38].

[4] The Minkowski spacetime is flat, it has zero curvature. It is also possible to use the (anti) de Sitter space as a jumping-off point [39–43]. Like the Minkowski space, such a space is a maximally symmetric lorentzian spacetime with a constant curvature.

PART III
RESEARCH

CHAPTER 6

THEORIES WITH ENHANCED SOFT LIMITS

The calculation of scattering amplitudes is difficult, when it is not trivial. Both the construction of the possible terms in the effective action, and their expansion into S-matrix elements are labour intensive processes. The structure of these calculations is an active subject of investigation, with the aim to reduce the effort expended on laborious number crunching. As a result many approaches have been proposed, that centre around finding underlying simplifications to existing methods (for some reviews see [44–46]) or developing altogether new frameworks for performing calculations [47–53]. For effective field theories of NG bosons the underlying, governing principle is clear: spontaneous symmetry breaking rules all [1].

The symmetry breaking generates not only the NG bosons, but also the form of their interactions. These interactions become weak at low energies. If the scattering amplitudes vanish altogether when the momentum of one external boson goes to zero, the Adler's zero condition is satisfied; this was the case $\sigma \leq 1$ in (4.10).
If a single field theory has an asymptotic shift symmetry then its scattering amplitudes will satisfy the Adler zero condition, as was shown in (4.100). Of course, if a lagrangean only contains fields with derivatives acting on it, this realisation of the soft limit is trivial.
Theories with enhanced soft limits, $\sigma \leq 2$, can be trivial in the same manner. The general form of a single field lagrangean (4.16, S) is

$$\mathcal{L} = \sum_{\substack{a=2 \\ b_1,\dots,b_a=0}}^{\infty} C_{b_1 \dots b_a} \partial^{b_1}\varphi \dots \partial^{b_a}\varphi. \tag{6.1, S}$$

For a particular term in the summation, the total number of derivatives is $t = \sum_{n=1}^{a} b_n$, so the average number of derivatives per term is

$$\rho = \frac{t}{a}. \tag{6.2}$$

[1] Except the coupling constants.

The softness of the scattering amplitudes, σ from (4.10), is proportional to the highest order of the spacetime coordinates, $\sigma - 1$, appearing in the symmetries of the field (4.100). Because $\partial^\rho x^{\sigma-1} = 0$ if $\sigma \leq \rho$, an action must be non-trivially invariant under this symmetry if $\sigma > \rho$.

Cheung et al. [3, 54] started from amplitudes with non-trivial enhanced soft limits to find the set of scalar theories that generate them. This set contained the galileon and Dirac-Born-Infeld (DBI) theories. Both theories were already in use in cosmology where they model the dark energy scalar field, the inflaton, that drives the accelerating expansion of the universe [55, 56]. On a practical note, it seems that the both theories in their standard form are currently all but ruled out by CMB observations [57–59], although specialised scenarios are still being investigated [60].

The DBI and galileon theories have an extended shift symmetry, too. For the simplest of the two, the galileons, it takes following form:

$$\varphi \quad \rightarrow \quad \varphi + a + b^\mu x_\mu. \tag{6.3}$$

Spacetime-dependent symmetries don't commute with the generator of spacetime translations P_μ. Since the breaking of those symmetries does not result in additional NG bosons, through the IHC, they are redundant.

The goal is to determine the relation between redundant broken symmetries and the enhanced soft limits. The classification by Cheung et al. [3] had already achieved much of this, starting from the amplitude viewpoint, however, starting from the symmetries displays the underlying governing principle. The conclusion will be that the restriction to enhanced soft limits is strict enough, that the set of symmetries that generates them can be fully classified.

6.1 Method

In order to research the relations between Lie algebras and the scattering amplitudes, a complete line of reasoning must be established between them. This line roughly follows the order of concepts introduced in the first chapters. The specific procedure is as follows:

- Starting from the Poincaré algebra, introduce additional infinitesimal generators and construct all possible commutators. The imposition of the Jacobi relations on the commutators makes it a new Lie algebra $\mathfrak{g}$. There are natural subalgebras $\mathfrak{h}$ in $\mathfrak{g}$, in the form of the Poincaré algebra or any of its subalgebras.

- The breaking of the new symmetries generates the quotient space $\mathbb{G}/\mathbb{H}$. Using the coset construction for spacetime symmetry breaking the MC forms, or covariant derivatives, in this space are constructed. The symmetry transformations of the fields are also established.
- The appropriate contraction of these forms produces the possible invariant terms. These invariants make up actions, that may contain the conventional NG lagrangeans and the topological WZ terms.

Using a computer code, written for this particular purpose, the following checks are performed:

- Construct the particular tree diagrams that correspond to the action under investigation.
- Introduce random external momenta that contain a factor z, following the procedure previously outlined in section [4.3]. Then calculate the scattering amplitudes, order by order, in the perturbative expansion.
- Find the dependency of the amplitude on the scaling factor z, taking a Taylor expansion around $z = 0$. Determine the scaling behaviour of the first term in the expansion, this determines the softness of the scattering.

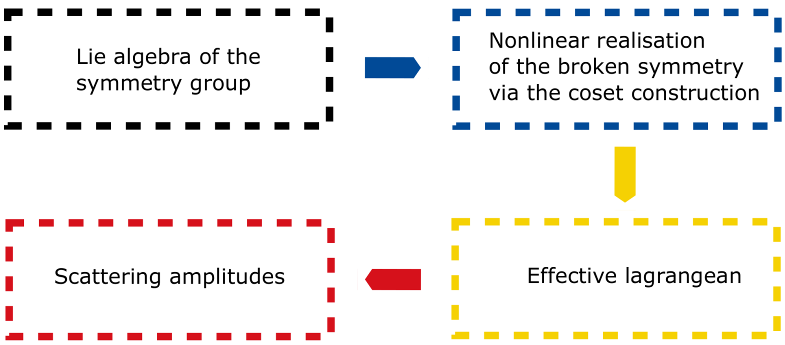

Figure 6.1: The procedure, beginning from a choice of symmetries, fixes the form of the scattering amplitudes, apart from the coupling constants.

The spacetime symmetries, that form the Poincaré group, are the basis on which the extended groups $\mathbb{G}$ will be built. Extending the Poincaré algebra, made up of generators $\{P_\lambda, J_{\mu\nu}\}$, with additional generators should create the Lie algebra $\mathfrak{g}$ of the new group. The Poincare group has a subgroup $\mathbb{H}$, consisting of the Lorentz transformations $H_\alpha = J_{\mu\nu}$. The Lie algebra can thus be decomposed as

$\mathfrak{g} = \mathfrak{h} + \mathfrak{i}$; the generator set I_a contains the spacetime translations P_μ and any newly added generators. The inclusion of the spacetime translations in $\mathbb{I}$ does not mean that those symmetries are broken, rather that they are non-linearly realised. The Lorentz invariance places strong constraints on the possible forms that $\mathfrak{g}$ can take. In the mould of (2.51), the general set of commutation relations will take the form:

$$\begin{aligned} [H_\alpha, H_\beta] &= c^\gamma_{\alpha\beta} H_\gamma, \\ [H_\alpha, I_b] &= c^c_{\alpha b} I_c, \\ [I_a, I_b] &= c^\alpha_{ab} H_\alpha + c^c_{ab} I_c. \end{aligned} \tag{6.4}$$

The complexity of the new generators I_a may be extended in two ways:

- By increasing the rank of each new redundant generator: Add a scalar, then add a vector, then a 2-tensor, and so on. The first part of the text, (6.2), extends the algebra this way.
- By increasing the number of generators: Add many scalars and many vectors. The second section (6.3) expands both the broken and unbroken parts of the algebra.

6.2 Single physical field

6.2.1 No redundant generators $(\sigma = 1)$

The first possible extension is a Lorentz invariant theory for a single NG boson, produced from the spontaneous breaking of one symmetry. The generator of that broken symmetry is the scalar $I = Q$. Later constructions, that contain a single physical scalar field and multiple redundant ones, must also contain this generator; this is just an intermediate step to those circumstances. The commutators express the common spacetime symmetries (2.67), and the fact that the scalar is invariant under Lorentz transformations

$$[J_{\mu\nu}, Q] = 0. \tag{6.5}$$

The other new commutator, $[P_\mu, Q]$ is not uniquely fixed by the Lorentz invariance alone, and will depend on the specific set of redundant generators. If only Q is added, the most general set of commutators is

$$[P_\mu, Q] = idP_\mu. \tag{6.6}$$

These commutators satisfy the Jacobi identity for any value of d. The non-linearly realised generators are therefore Q and P_μ, in this instance. One choice for the parametrisation of the coset space is

$$U(x_\mu, \theta) = e^{ix^\mu P_\mu} e^{i\theta Q}. \tag{6.7}$$

The transformation of the fields are defined by the multiplication of U by an element of the symmetry group, so $e^{i\alpha Q}U$ in this case.

$$\begin{aligned} x^\mu &\rightarrow x^\mu e^{d\alpha} && (6.8)\\ \theta &\rightarrow \theta + \alpha && (6.9) \end{aligned}$$

The MC form is

$$\begin{aligned} \omega &= -iU^{-1}dU && (6.10)\\ &= (d\theta)Q + (e^{-d\theta}dx^\mu)P_\mu. \end{aligned}$$

The covariant derivative, constructed from ω_Q, is $\nabla_\mu\theta = e^{d\theta}\partial_\mu\theta$. The action, formed using the invariant integration measure formulated in terms of the Levi-Civita symbol (208), is

$$\int det(e^\alpha_\mu) d^4x = \int \frac{e^{-4d\theta}}{-4!} \epsilon^{\mu\nu\rho\sigma} dx_\mu dx_\nu dx_\rho dx_\sigma. \tag{6.11}$$

The kinetic term of the action, for instance, can then be written as

$$S = \int \frac{(e^{d\theta}\partial_\mu\theta)^2}{2} e^{-4d\theta} d^4x. \tag{6.12}$$

The symmetry transform indicates that the generator Q not only induces a shift in the field but also generates dilations; it rescales spacetime. If the broken symmetry generates a shift of the field it represents a soft NG boson; this is the case when $d = 0$. The actions constructed under those terms only depend on the derivatives of the field.

6.2.2 One redundant generator $(\sigma = 2)$

The first extension to the single NG boson introduces a redundant broken generator. This generator must be a Lorentz vector, denoted by K_μ, and its commutator with P_μ must be proportional to Q to make it redundant; the IHC from section [2.4] then eliminates the redundant degrees of freedom. The set of possible commutation relations between the broken generators, that maintains Lorentz invariance, is:

$$\begin{aligned}
[J_{\mu\nu}, K_\lambda] &= i(g_{\nu\lambda}K_\mu - g_{\mu\lambda}K_\nu), \\
[P_\mu, K_\nu] &= i(a g_{\mu\nu}Q + bJ_{\mu\nu} + c\epsilon_{\mu\nu\rho\sigma}J_{\rho\sigma}), \\
[P_\mu, Q] &= i\left(dP_\mu + eK_\mu\right), \\
[K_\mu, K_\nu] &= i\left(fJ_{\mu\nu} + g\epsilon_{\mu\nu\kappa\lambda}J^{\kappa\lambda}\right), \\
[K_\mu, Q] &= i(hP_\mu + iK_\mu).
\end{aligned} \tag{6.13}$$

These brackets form a Lie algebra when they obey the Jacobi identity. This is the case when the following relations between the red coefficients hold:

$$\begin{aligned}
c &= 0, & b(d+i) + ef &= 0, \\
g &= 0, & f + ah &= 0, \\
ae &= 0, & b - ai &= 0, \\
be &= 0, & b + ad &= 0.
\end{aligned} \tag{6.14}$$

There are multiple solutions to these equations, but only some are relevant to the investigation of the soft limits. There is only a redundant symmetry if the coefficient $a \neq 0$; the parameter of the additional broken symmetry K_μ may then be removed via the IHC. This can't occur if the coefficient $a = 0$, so this case is only included as an appendix [D.2].

Relevant algebras

If the coefficient $a \neq 0$, then the solutions to the equation set (6.14) are

$$\begin{aligned}
c &= 0, & b &= ai, \\
e &= 0, & f &= -ah, \\
g &= 0, & d &= -i.
\end{aligned} \tag{6.15}$$

This set of parameters can be reduced. Rescaling, $K_\mu \to aK_\mu$, and then defining the two new parameters, $u = i$ and $v = \frac{h}{a}$, yields the following non-trivial commutation relations between the broken generators:

$$\begin{aligned}
[P_\mu, K_\nu] &= i(g_{\mu\nu}Q + uJ_{\mu\nu}), \\
[P_\mu, Q] &= -iuP_\mu, \\
[K_\mu, K_\nu] &= -ivJ_{\mu\nu}, \\
[K_\mu, Q] &= i(vP_\mu + uK_\mu).
\end{aligned} \tag{6.16}$$

This constitutes a complete classification of the Lie algebraic extensions of the Poincaré group that have one physical NG-boson and one redundant degree of freedom. It can be subdivided into several distinct cases by setting either u or v, or both, to zero. In each case, the remaining coefficients can then be absorbed into the generators.

$(u \neq 0)$

The redefinition of the vector generator $K_\mu \to K_\mu - \frac{v}{2u}P_\mu$ makes sense if $u \neq 0$. Scaling the generators, $Q \to uQ$ and $K_\mu \to uK_\mu$, reduces the algebra to

$$\begin{aligned} [P_\mu, K_\nu] &= i(g_{\mu\nu}Q + J_{\mu\nu}), \\ [P_\mu, Q] &= -iP_\mu, \\ [K_\mu, K_\nu] &= 0, \\ [K_\mu, Q] &= iK_\mu. \end{aligned} \tag{6.17}$$

These commutation relations form the conformal Lie algebra, which represents the local structure of the conformal Lie group SO(4,2). The Q generates dilatations and K_μ generates special conformal transformations.
After breaking the conformal group to the Poincaré group the dilatation field becomes the NG mode, whilst the fields of the special conformal transformation become redundant.

$(u = 0, v \neq 0)$

Setting u to zero and rescaling the generators, $Q \to \sqrt{|v|}Q$ and $K_\mu \to \sqrt{|v|}K_\mu$, reduces the algebra to:

$$\begin{aligned} [P_\mu, K_\nu] &= ig_{\mu\nu}Q, \\ [P_\mu, Q] &= 0, \\ [K_\mu, K_\nu] &= \mp iJ_{\mu\nu}, \\ [K_\mu, Q] &= \pm iP_\mu. \end{aligned} \tag{6.18}$$

These are two versions of the five dimensional Poincaré algebra, differing only by their sign; this sign still stems from v. This algebra is an extension of the four dimensional Poincaré algebra. The momentum operator P_μ expands from four to five dimensions by the addition of $P_4 = Q$, and similarly the angular momentum operator $J_{\mu\nu}$ gains a term $J_{\mu 4} = K_\mu$. The sign in the algebra corresponds to the choice of the sign of the metric in the fifth dimension $g_{44} = \pm 1$. The breaking of these symmetries creates a low-energy EFT that passes for the fluctuations of a four-dimensional brane in a five-dimensional spacetime, where SO(4,1) or SO(3,2) takes up the role that SO(3,1) had in four dimensions. The EFT matches a familiar model, called the DBI scalar, in either case. This algebra can also be derived via an İnönü-Wigner contraction (2.3) of the original algebra (6.16), using the complementary generators $P_\mu^{(0)} = \epsilon P_\mu$ and $Q^{(0)} = \epsilon Q$.

$(u = 0, v = 0)$

In the final case, the only non-zero commutator is

$$[P_\mu, K_\nu] = ig_{\mu\nu}Q. \tag{6.19}$$

This is the galileon algebra, it is the simplest non-trivial extension of the Poincaré group. By defining the set of complementary generators $K_\mu^{(0)} = \epsilon K_\mu$ and $Q^{(0)} = \epsilon Q$, it too can be derived through an İnönü-Wigner contraction.

In summary, the unique subsets of the algebra are:

$(u \neq 0)$	$\rightarrow$	The conformal algebra	$\rightarrow$	The dilaton EFT
$(u = 0, v \neq 0)$	$\rightarrow$	The 5D Poincaré algebra	$\rightarrow$	The DBI EFT
$(u = 0, v = 0)$	$\rightarrow$	The galileon algebra	$\rightarrow$	The galileon theories

The conformal algebra does not have a soft limit, which is related to the non-commutation of P_μ and Q. The dilatation generator doesn't commute with translations, $[P_\mu, Q] = -iP_\mu$, like in the $\sigma = 1$ case at the beginning of section 6.2.1. This means that the infinitesimal dilatation transformation depends on the coordinate; the explicit infinitesimal transformations, calculated later in (6.29), are $x_\mu \rightarrow x_\mu - \alpha x_\mu$ and $\theta \rightarrow \theta + \alpha$. This means that, according to 2.12, the active infinitesimal transformation of the field is $\theta \rightarrow \theta + \alpha + \alpha x^\mu \partial_\mu \theta$; this is also a symmetry of the free field, so it holds in the asymptotic limit. The constraint on the amplitudes, from (4.97, D), for this symmetry takes the specific form:

$$\lim_{k \to 0} \mathscr{A}(p_1, ..., k_n, k) = -p_1^\mu \frac{\partial}{\partial p_1^\mu} \mathscr{A}(p_2, ..., k_n, p_1). \tag{6.20}$$

Since the field doesn't shift purely by a constant, the existence of a soft limit of the field is no longer guaranteed. More broadly, for any non-commuting P_μ and Q there may not be a soft limit [22, 23]. Therefore, in further exploration of the possible algebras, the assumption will be that $[P_\mu, Q] = 0$. The last two theories are familiar models, from [3]. The calculation of their scattering amplitudes confirms that they have an enhanced soft limit of $\sigma = 2$.

The choice of the free variables u and v of the original algebra (6.16) determines what class of theory it is, so the space of theories can be represented in a chart in the uv-plane.

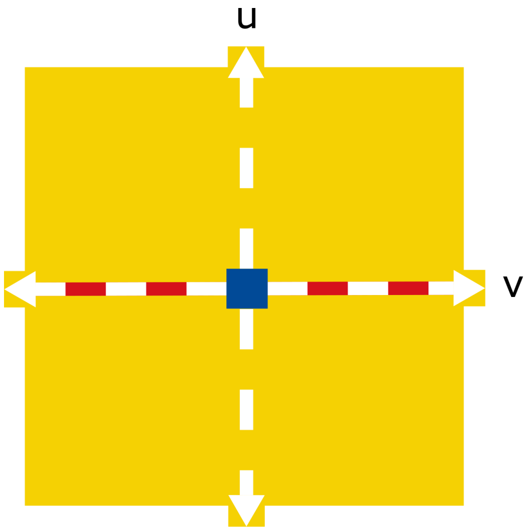

Figure 6.2: The "phase diagram" of algebras, with axes along the white arrows: The part outside the v-axis corresponds to the conformal algebra. Along the u-axis lies the 5D Poincaré algebra, except at the origin, where it is the galileon algebra.

Coset construction

If first the general Maurer-Cartan form is calculated using the algebra (6.16), the MC form of all subcases can be derived by choosing the appropriate values for u and v. The coset is parametrised by the element

$$U(x_\mu, \theta, \xi_\mu) = e^{ix^\mu P_\mu} e^{i\theta Q} e^{i\xi^\mu K_\mu}. \tag{6.21}$$

The corresponding MC form can be decomposed along the generators.

$$\begin{aligned} \omega &= -iU^{-1}dU \\ &= \frac{\omega_J^{\mu\nu}}{2} J_{\mu\nu} + \omega_P^\mu P_\mu + \omega_K^\mu K_\mu + +\omega_Q^\mu Q \end{aligned} \tag{6.22}$$

The specific contents of the one-forms are

$$\begin{aligned}
\omega_P^\mu &= e^{u\theta}dx^\mu - e^{u\theta}\frac{\xi^\mu\xi\cdot dx}{\xi^2}\left(1-\cos\sqrt{v\xi^2}\right) + v\frac{\sin\sqrt{v\xi^2}}{\sqrt{v\xi^2}}\xi^\mu d\theta, \qquad (6.23)\\
\omega_K^\mu &= d\xi^\mu - \frac{u}{v}e^{u\theta}\left(2\frac{\xi^\mu\xi\cdot dx}{\xi^2} - dx^\mu\right)\left(1-\cos\sqrt{v\xi^2}\right),\\
&\quad + \left(\frac{\xi^\mu\xi\cdot d\xi}{\xi^2} - d\xi^\mu\right)\left(1-\frac{\sin\sqrt{v\xi^2}}{\sqrt{v\xi^2}}\right) + u\frac{\sin\sqrt{v\xi^2}}{\sqrt{v\xi^2}}\xi^\mu d\theta,\\
\omega_Q &= \cos\sqrt{v\xi^2}d\theta - e^{u\theta}\frac{\sin\sqrt{v\xi^2}}{\sqrt{v\xi^2}}\xi\cdot dx,\\
\omega_J^{\mu\nu} &= ue^{u\theta}\frac{\sin\sqrt{v\xi^2}}{\sqrt{v\xi^2}}\left(\xi^\mu dx^\nu - \xi^\nu dx^\mu\right) - \frac{1-\cos\sqrt{v\xi^2}}{\xi^2}\left(\xi^\mu d\xi^\nu - \xi^\nu d\xi^\mu\right).
\end{aligned}$$

The IHC constraint, $\omega_Q = 0$, can be written as

$$\frac{\tan\sqrt{v\xi^2}}{\sqrt{v\xi^2}}\xi_\mu = e^{-u\theta}\partial_\mu\theta. \qquad (6.24)$$

This should remove the degrees of freedom ξ_μ but, when something may be written more simply using ξ^μ, it will sometimes still be expressed in those terms.

Geometry

The form ω_P contains the vielbein (2.74) expressible as

$$\begin{aligned}
e_\mu^\alpha &= e^{u\theta}\left(\left(\delta_\mu^\alpha - \frac{\xi_\mu\xi^\alpha}{\xi^2}\right) + \frac{1}{\cos\sqrt{v\xi^2}}\frac{\xi_\mu\xi^\alpha}{\xi^2}\right) \qquad (6.25)\\
&= e^{u\theta}\left(\mathscr{P}_{\perp\mu}^\alpha + \frac{\mathscr{P}_{\parallel\mu}^\alpha}{\cos\sqrt{v\xi^2}}\right).
\end{aligned}$$

The terms $\mathscr{P}_\perp$ and $\mathscr{P}_\parallel$ have the properties:

$$\mathscr{P}_{\perp\mu}^\alpha\mathscr{P}_{\perp\alpha}^\beta = \mathscr{P}_{\perp\mu}^\beta, \qquad \mathscr{P}_{\parallel\mu}^\alpha\mathscr{P}_{\parallel\alpha}^\beta = \mathscr{P}_{\parallel\mu}^\beta, \qquad \mathscr{P}_{\parallel\mu}^\alpha\mathscr{P}_{\perp\alpha}^\beta = 0,$$
$$\mathscr{P}_{\perp\mu}^\alpha\xi^\mu = 0, \qquad \mathscr{P}_{\parallel\mu}^\alpha\xi^\mu = \xi^\alpha.$$

They are projectors, that project onto directions perpendicular and parallel to the vector ξ^μ. The induced metric $G_{\mu\nu}$, as defined in (2.75), on the coset space is

$$\begin{aligned}
G_{\mu\nu} &= e^{2u\theta}\left(\mathscr{P}_{\perp\mu\nu} + \frac{\mathscr{P}_{\parallel\mu\nu}}{\cos^2\sqrt{v\xi^2}}\right) \qquad (6.26)\\
&= e^{2u\theta}g_{\mu\nu} + v\partial_\mu\theta\partial_\nu\theta.
\end{aligned}$$

The three classes identified in the algebra yield different metrics. In case of the galileons, $(u = 0, v = 0)$, the metric reduces to the Minkowski metric. If $(u = 0, v \neq 0)$, the DBI case, the gradient of θ may be interpreted as the distension of the distances in Minkowski space. Finally, the metric confirms the geometrical interpretation of the field θ as the dilaton, when $(u \neq 0, v = 0)$. With the vielbein in hand, the covariant derivative of ξ^μ, defined by $\omega_K^\alpha \equiv e_\mu^\alpha dx^\nu \nabla_\nu \xi^\mu$ can be determined.

$$\nabla_\mu \xi^\nu = \frac{u}{v}\left(1 - \cos\sqrt{v\xi^2}\right)\delta_\mu^\nu + \left(\mathscr{P}^\nu_{\perp\alpha}\frac{\sin\sqrt{v\xi^2}}{\sqrt{v\xi^2}} + \mathscr{P}^\nu_{\parallel\alpha}\cos\sqrt{v\xi^2}\right)e^{-u\theta}\partial_\mu\xi^\alpha \tag{6.27}$$

The first NG actions

The ϕ^* in the following is always a pullback to the Minkowski space. The lowest order of the action, in terms of the covariant derivatives, simply consist of the volume element:

$$\begin{aligned} S_0 &= \int \phi^*(\epsilon_{\mu\nu\rho\sigma}\omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma) = \int d^4x\sqrt{-G} \\ &= \int d^4x e^{4u\theta}\sqrt{1 + ve^{-2u\theta}\partial_\mu\theta\partial^\mu\theta}. \end{aligned} \tag{6.28}$$

The integrand is a lagrangean in flat spacetime. The lagrangean takes a different form, depending which class it is.

$$\begin{array}{llll} (u \neq 0) & \rightarrow & \mathcal{L}_0 = e^{4u\theta} & \rightarrow \text{ Conformal} \\ (u = 0, v \neq 0) & \rightarrow & \mathcal{L}_0 = \sqrt{1 + v\partial_\mu\theta\partial^\mu\theta} & \rightarrow \text{ DBI} \\ (u = 0, v = 0) & \rightarrow & \mathcal{L}_0 = 1 & \rightarrow \text{ Galileon} \end{array}$$

The next action, first order in the covariant derivatives, is

$$\begin{aligned} S_1 &= \int \phi^*(\epsilon_{\mu\nu\rho\sigma}\omega_K^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma) = \int d^4x\sqrt{-G}\nabla_\mu\xi^\mu \\ &= \int d^4x e^{4u\theta}\left\{\frac{4u}{v}\left(\sqrt{1 + ve^{-2u\theta}\partial_\mu\theta\partial^\mu\theta} - 1\right)\right. \\ &\left. + \sqrt{v}e^{-2u\theta}\left(\partial_\mu\partial^\mu\theta - u(\partial^\mu\theta)^2 - ve^{-2u\theta}\frac{\partial^\mu\theta\partial^\nu\theta(\partial_\mu\partial_\nu\theta - u\partial_\mu\theta\partial_\nu\theta)}{1 + ve^{-2u\theta}\partial_\mu\theta\partial^\mu\theta}\right)\right\}. \end{aligned}$$

The lagrangeans for the separate classes take more comprehensible forms.

$$\begin{array}{llll} (u \neq 0) & \rightarrow & \mathcal{L}_1 = 2ue^{2u\theta}\partial_\mu\theta\partial^\mu\theta & \rightarrow \text{ Conformal} \\ (u = 0, v \neq 0) & \rightarrow & \mathcal{L}_1 = \sqrt{v}\left(\partial_\mu\partial^\mu\theta - v\dfrac{\partial^\mu\theta\partial_\mu\partial_\nu\theta\partial^\nu\theta}{1 + v\partial_\mu\theta\partial^\mu\theta}\right) & \rightarrow \text{ DBI} \\ (u = 0, v = 0) & \rightarrow & \mathcal{L}_1 = 0 & \rightarrow \text{ Galileon} \end{array}$$

The conformal and DBI actions match those in [61], but the actions for the galileons are trivial. More interesting actions for the galileon algebras, however, will be constructed later in the form of WZ terms. Another way those may be constructed, is by the finding the first order Taylor expansion in v of the DBI action [62].

The symmetry transformations

The action of Q, parametrised by α, induces the transformations of the fields, like in (2.45).

$$\begin{aligned} x_\mu &\rightarrow e^{-u\alpha}x_\mu \\ \xi_\mu &\rightarrow \xi^\mu \\ \theta &\rightarrow \theta+\alpha \end{aligned} \tag{6.29}$$

These transformations are the same as found in (6.8); if $u \neq 0$ the generator Q generates spacetime dilatations. The transformations generated by K^μ, parametrised by β_μ, are to the first order in β:

$$\begin{aligned} x_\mu &\rightarrow x_\mu + u\beta\cdot x x_\mu - \frac{u}{2}x^2\beta_\mu - \frac{v}{u}e^{-u\theta}\beta_\mu \sinh(u\theta) + \mathcal{O}(\beta^2), \\ \xi_\mu &\rightarrow \xi_\mu + e^{-u\theta}\beta_\mu + u\xi\cdot x\beta_\mu - u\xi\cdot\beta x_\mu + \mathcal{O}(\beta^2,\xi^2), \\ \theta &\rightarrow \theta + \beta\cdot x + \mathcal{O}(\beta^2). \end{aligned} \tag{6.30}$$

In this general case, when $u \neq 0$, the transformation of the field ξ^μ is only given to the second order in ξ^2. In case $u = 0$, which corresponds to the enhanced soft limit, the infinitesimal transformation of all the fields to all orders in ξ^μ are:

$$\begin{aligned} x_\mu &\rightarrow x_\mu - \frac{\sin\sqrt{v\xi^2}}{\sqrt{v\xi^2}}v\theta\beta_\mu + (\cos\sqrt{v\xi^2}-1)\frac{\beta_\mu\beta_\nu}{\beta^2}x^\nu, \\ \xi_\mu &\rightarrow \xi_\mu + \left(\frac{\sqrt{v\xi^2}}{\tan\sqrt{v\xi^2}}\mathscr{P}^\nu_{\perp\mu} + \mathscr{P}^\nu_{\parallel\mu}\right)\beta_\nu + \mathcal{O}(\beta^2), \\ \theta &\rightarrow \theta\cos\sqrt{v\beta^2} + \beta\cdot x\frac{\sin\sqrt{v\beta^2}}{\sqrt{v\beta^2}}. \end{aligned} \tag{6.31}$$

The MC structure equations

The classification of the actions is not complete until it is checked whether 1-forms, generated as part of the MC form, can generate WZ terms. In four dimensional spacetime, this requires the categorization of all invariant 5-forms that are part of the cohomology group for a given Lie algebra. The cohomology is defined using exterior derivatives, so fortunately there is an equation for the exterior derivative of the Maurer-Cartan forms. Write the commutation relations of the generators as:

$$[G_i, G_j] = ic_{ij}^k G_k. \tag{6.32}$$

The Maurer-Cartan form for the broken symmetry group is:

$$\omega = -iU^{-1}dU = \omega^i G_i. \tag{6.33}$$

The exterior derivative of this form is

$$\begin{aligned} d\omega &= -idU^{-1}dU = i(U^{-1}dU)U^{-1}dU = -i\,\omega \wedge \omega \qquad (6.34) \\ &= -\frac{i}{2}\,\omega^i \wedge \omega^j\,(G_iG_j - G_jG_i) \\ &= \frac{c_{ij}^k}{2}\omega^i \wedge \omega^j G_k. \end{aligned}$$

A decomposition of this equation, along the components G_i of the Lie algebra, allows it to be written purely in terms of the 1-forms ω^i.

$$d\omega^k = \frac{c_{ij}^k}{2}\omega^i \wedge \omega^j \tag{6.35}$$

These are called the Maurer Cartan structure equations.

The Wess-Zumino terms

The algebra (6.16) contains three subcategories. The use of this algebra allows the investigation of the cohomology group in all cases, at once.
To find the kernel and image of mapping of 4- and 5-forms under the exterior differential, respectively, the relevant space that these forms span needs to be established. The ω_P^μ and ω_K^μ make up the linearly independent, Lorentz invariant basis of 4-forms e^i.

$$\begin{aligned} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \end{pmatrix} &\equiv \epsilon_{\mu\nu\rho\sigma} \begin{pmatrix} \omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma \\ \omega_K^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma \\ \omega_K^\mu \wedge \omega_K^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma \\ \omega_K^\mu \wedge \omega_K^\nu \wedge \omega_K^\rho \wedge \omega_P^\sigma \\ \omega_K^\mu \wedge \omega_K^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma \end{pmatrix} \\ &\equiv \epsilon \cdot \left(\omega_P^4, \omega_K^1 \wedge \omega_P^3, \omega_K^2 \wedge \omega_P^2, \omega_K^3 \wedge \omega_P^1, \omega_K^4\right)^T \end{aligned} \tag{6.36}$$

The span of Lorentz invariant 5-forms is made up from linear combinations of the basis elements $\mathfrak{g}^i = \omega_Q \wedge e^i$

The exterior derivatives of the 1-forms are found from the structure equation:

$$\begin{aligned} d\omega_P^\mu &= \omega_Q \wedge (u\, \omega_P^\mu - v\, \omega_K^\mu) + \omega_J^{\mu\lambda} \wedge \omega_P^\lambda, \\ d\omega_K^\mu &= \omega_Q \wedge (-u\, \omega_K^\mu) + \omega_J^{\mu\lambda} \wedge \omega_K^\lambda, \\ d\omega_Q &= \omega_P^\mu \wedge \omega_K^\mu. \end{aligned} \tag{6.37}$$

The exterior derivative of the basis e_i then follows, after a sizable calculation:

$$\begin{aligned} de^i &= \omega_Q \wedge \begin{pmatrix} 4u & -4v & 0 & 0 & 0 \\ 0 & 2u & -3v & 0 & 0 \\ 0 & 0 & 0 & -2v & 0 \\ 0 & 0 & 0 & -2u & -v \\ 0 & 0 & 0 & 0 & -4u \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \end{pmatrix} \\ &= \omega_Q \wedge \left(M_j^{\ i} e^j \right) \end{aligned} \tag{6.38}$$

It follows, straightforwardly, that the entirety of the invariant 5-form basis consists of closed forms.

$$\begin{aligned} d\mathfrak{g}^i &= (\omega_P^\mu \wedge \omega_K^\mu) \wedge e^i - \omega \wedge \omega_Q \wedge \left(M_j^{\ i} e^j \right) \\ &= 0 \end{aligned} \tag{6.39}$$

This is true, irrespective of the choice of constants u and v, so all these terms are part of the cocycle group Z^5.

The next step in constructing the topological WZ terms is finding the forms that are part of the coboundary group B^5. In this case those are the $\mathfrak{g}^i$ that can't be obtained from e^i by an exterior derivation. The forms in the cocycle group Z^5, with those in the coboundary group B^5 factored out yields the Lorentz invariant elements of the cohomology group, $H^5 = \frac{Z^5}{B^5}$, that constitute the set of WZ terms:

$$\begin{array}{lclcl} (u \neq 0, v = 0) & \rightarrow & \{\mathfrak{g}^1, \mathfrak{g}^2, \mathfrak{g}^4, \mathfrak{g}^5\} \in B^5 & \rightarrow & \mathfrak{g}^3 \in H^5 \\ (u = 0, v \neq 0) & \rightarrow & \{\mathfrak{g}^2, \mathfrak{g}^3, \mathfrak{g}^4, \mathfrak{g}^5\} \in B^5 & \rightarrow & \mathfrak{g}^1 \in H^5 \\ (u = 0, v = 0) & \rightarrow & \{\mathfrak{g}^i\} \notin B^5 & \rightarrow & \{\mathfrak{g}^i\} \in H^5 \end{array} \tag{6.40}$$

After integrating and implementing the IHC, in the conformal case when $(u \neq 0, v = 0)$, the action finally becomes

$$\begin{aligned} S &= \int \phi^*(\mathfrak{g}^3) \\ &= \frac{1}{24} \int \left(\frac{u(\partial_\mu \theta)^4}{2} - \Box\theta(\partial_\mu\theta)^2 \right) \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma. \end{aligned} \tag{6.41}$$

The same calculation in the case of the DBI theory, when $(u = 0, v \neq 0)$, yields

$$\begin{aligned} S &= \int \phi^*(\mathfrak{g}^1) \\ &= \int (\theta)\epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma. \end{aligned} \tag{6.42}$$

Finally, in the case of the galileons, each $\mathfrak{g}_i$ creates an action $S_i = \int \phi^*(\mathfrak{g}_i)$. The full form of these actions is:

$$\begin{aligned} S_1 &= \int \theta\epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \\ S_2 &= -\frac{1}{8}\int (\partial_\mu\theta)^2 \, \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \\ S_3 &= \frac{1}{24}\int (\partial_\mu\theta)^2\Box\theta \, \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \\ S_4 &= -\frac{1}{16}\int (\partial_\mu\theta)^2 \left((\Box\theta)^2 - (\partial_\mu\partial_\nu\theta)^2\right) \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \\ S_5 &= -\frac{1}{48}\int (\partial_\mu\theta)^2 \left((\Box\theta)^3 - 3\Box\theta(\partial_\mu\partial_\nu\theta)^2 + 2\partial_\mu\partial^\nu\theta\partial_\nu\partial^\rho\theta\partial_\rho\partial^\mu\theta\right) \\ &\qquad \times \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma. \end{aligned} \tag{6.43}$$

These results were originally found in [61]. The terms were worked out for the sake of completeness; these basic elements of the WZ galileons will return in later constructions.

6.2.3 Two redundant generators $(\sigma = 3)$

To obtain theories that have even softer scattering amplitudes, an additional redundant generator is added. The distinct possible rank two tensor generators can either be scalar, anti-symmetric or traceless symmetric in it components; these alternatives are called spin zero, spin one or spin two generators, respectively. Section [4.10] shows that the addition of a new generator increases the softness by one order, if it generates an asymptotic shift symmetry that is one order higher in the spacetime coordinates than that of the previously added generator. In the particular case of $\sigma = 3$, the new generator should generate a shift of the field that is quadratic in spacetime coordinates. The asymptotic free field ϕ is invariant under the shift $\phi \to \phi + s_{\mu\nu}x^\mu x^\nu$; $s_{\mu\nu}$ is a traceless symmetric matrix as in (4.93). In the coset construction, the transformation of fields under the action of a group can be calculated from (2.45). In this context the $s_{\mu\nu}$ is thus a parameter that controls the global symmetry transformation, like for instance α in (6.29). The parameter $s_{\mu\nu}$ contracts with the rank two generator, so it must

be traceless and symmetric as well; the contraction of a traceless symmetric matrix and either a scalar or anti-symmetric matrix reduces to zero.
Although it was shown that the stacking of ever higher spacetime dependent shift symmetries generates increasing degrees of softness, it was not shown that this is the only way of achieving this. To achieve generality, all cases were fully worked through to check for enhanced soft limits. The scalar and anti-symmetric extension are included in the appendices [D.3] and [D.4]; they didn't generate theories with enhanced soft limits.

Spin-two multiplet of redundant generators

The Poincaré algebra, consisting of the generators $J_{\mu\nu}$ and P_μ, is now extended by Q, K_μ and $S_{\mu\nu}$. The redundant symmetric tensor $S_{\mu\nu} = S_{\nu\mu}$ is also traceless, so $g_{\mu\nu}S^{\mu\nu} = 0$. The form of the commutator between $J_{\mu\nu}$ and $S_{\mu\nu}$ is determined by Lorentz invariance.

$$[J_{\mu\nu}, S_{\kappa\lambda}] = i\left(-g_{\mu\lambda}S_{\nu\kappa} + g_{\nu\kappa}S_{\mu\lambda} - g_{\mu\kappa}S_{\nu\lambda} + g_{\nu\lambda}S_{\mu\kappa}\right) \tag{6.44}$$

The Lorentz invariance, and the tracelessness of $S_{\mu\nu}$, also fixes the remainder of the commutation relations. This possibility space is parametrised by a set of red coefficients, as before.

$$\begin{aligned}
[P_\mu, K_\nu] &= i(a g_{\mu\nu}Q + bJ_{\mu\nu} + c\epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma} + jS_{\mu\nu}) \\
[P_\mu, Q] &= i\left(dP_\mu + eK_\mu\right) \\
[K_\mu, K_\nu] &= i\left(fJ_{\mu\nu} + g\epsilon_{\mu\nu\kappa\lambda}J^{\kappa\lambda}\right) \\
[K_\mu, Q] &= i(hP_\mu + iK_\mu) \\
[S_{\mu\nu}, S_{\kappa\lambda}] &= i\,[\,k\left(g_{\mu\lambda}J_{\nu\kappa} + g_{\nu\kappa}J_{\mu\lambda} + g_{\mu\kappa}J_{\nu\lambda} + g_{\nu\lambda}J_{\mu\kappa}\right) \\
&\qquad + l\left(g_{\mu\lambda}\epsilon_{\nu\kappa\alpha\beta} + g_{\nu\kappa}\epsilon_{\mu\lambda\alpha\beta} + g_{\mu\kappa}\epsilon_{\nu\lambda\alpha\beta} + g_{\nu\lambda}\epsilon_{\mu\kappa\alpha\beta}\right)J^{\alpha\beta}\,] \\
[S_{\mu\nu}, P_\lambda] &= i[m(g_{\mu\lambda}P_\nu + g_{\nu\lambda}P_\mu - \frac{1}{2}g_{\mu\nu}P_\lambda) + n(g_{\mu\lambda}K_\nu + g_{\nu\lambda}K_\mu - \frac{1}{2}g_{\mu\nu}K_\lambda)] \\
[S_{\mu\nu}, K_\lambda] &= i[o(g_{\mu\lambda}P_\nu + g_{\nu\lambda}P_\mu - \frac{1}{2}g_{\mu\nu}P_\lambda) + p(g_{\mu\lambda}K_\nu + g_{\nu\lambda}K_\mu) - \frac{1}{2}g_{\mu\nu}K_\lambda] \\
[S_{\mu\nu}, Q] &= iqS_{\mu\nu}
\end{aligned} \tag{6.45}$$

The field associated with K_μ becomes redundant, via the IHC, if the condition $a \neq 0$ is met. The commutation relations satisfy the Jacobi identity, under this condition, when the red parameters solve the following equations:

$$\begin{aligned} b &= jm, & c &= 0, & d &= -\frac{5jm}{2a}, \\ e &= 0, & f &= jo, & g &= 0, \\ h &= -\frac{5jo}{2a}, & i &= \frac{5jm}{2a}, & k &= m^2 + no, \\ l &= 0, & nj &= 0, & p &= -m, \\ & & q &= 0. & & \end{aligned} \tag{6.46}$$

The implementation of this solution make the commutators into a Lie algebra.

$$\begin{aligned} [P_\mu, K_\nu] &= i(ag_{\mu\nu}Q + jmJ_{\mu\nu} + jS_{\mu\nu}) \\ [P_\mu, Q] &= -i\frac{5jm}{2a}P_\mu \\ [K_\mu, K_\nu] &= ijoJ_{\mu\nu} \\ [K_\mu, Q] &= i\frac{5j}{2a}(-oP_\mu + mK_\mu) \\ [S_{\mu\nu}, S_{\kappa\lambda}] &= ik\left(g_{\mu\lambda}J_{\nu\kappa} + g_{\nu\kappa}J_{\mu\lambda} + g_{\mu\kappa}J_{\nu\lambda} + g_{\nu\lambda}J_{\mu\kappa}\right) \\ [S_{\mu\nu}, P_\lambda] &= i[m(g_{\mu\lambda}P_\nu + g_{\nu\lambda}P_\mu - \frac{1}{2}g_{\mu\nu}P_\lambda) + n(g_{\mu\lambda}K_\nu + g_{\nu\lambda}K_\mu - \frac{1}{2}g_{\mu\nu}K_\lambda)] \\ [S_{\mu\nu}, K_\lambda] &= i[o(g_{\mu\lambda}P_\nu + g_{\nu\lambda}P_\mu - \frac{1}{2}g_{\mu\nu}P_\lambda) - m(g_{\mu\lambda}K_\nu + g_{\nu\lambda}K_\mu - \frac{1}{2}g_{\mu\nu})K_\lambda] \\ [S_{\mu\nu}, Q] &= 0 \end{aligned} \tag{6.47}$$

Again, the condition $[P_\mu, Q] = 0$ corresponds to the existence of an Adler's zero [22, 23], so either $j = 0$ or $m = 0$. If only $m = 0$, then the condition $jn = 0$ implies that $n = 0$ too. In consequence the commutator $[S_{\mu\nu}, P_\lambda] = 0$, so the generator $S_{\mu\nu}$ is not redundant and the theory will contain multiple species of NG bosons; this is beyond the scope of this investigation[2].

The physically interesting algebra must then satisfy $j = 0$, and the remaining non-zero commutators out of the set are:

$$\begin{aligned} [P_\mu, K_\nu] &= iag_{\mu\nu}Q, \\ [S_{\mu\nu}, S_{\kappa\lambda}] &= ik\left(g_{\mu\lambda}J_{\nu\kappa} + g_{\nu\kappa}J_{\mu\lambda} + g_{\mu\kappa}J_{\nu\lambda} + g_{\nu\lambda}J_{\mu\kappa}\right), \\ [S_{\mu\nu}, P_\lambda] &= i[m(g_{\mu\lambda}P_\nu + g_{\nu\lambda}P_\mu - \frac{1}{2}g_{\mu\nu}P_\lambda) + n(g_{\mu\lambda}K_\nu + g_{\nu\lambda}K_\mu - \frac{1}{2}g_{\mu\nu}K_\lambda)], \\ [S_{\mu\nu}, K_\lambda] &= i[o(g_{\mu\lambda}P_\nu + g_{\nu\lambda}P_\mu - \frac{1}{2}g_{\mu\nu}P_\lambda) - m(g_{\mu\lambda}K_\nu + g_{\nu\lambda}K_\mu - \frac{1}{2}g_{\mu\nu})K_\lambda]. \end{aligned} \tag{6.48}$$

[2] The algebra is still not uniquely defined and, if $o \neq 0$, it seems possible to make a change of basis in the (P_μ, K_μ)-space so that $S_{\mu\nu}$ can be made redundant again. This creates the problem that the new commutator $[\tilde{P}_\mu, \tilde{P}_\nu] \neq 0$, so the generator $\tilde{P}_\mu$ no longer represents translations in spacetime.

The red coefficients may take any value, as long as $a \neq 0$ and $n \neq 0$. Since $[K_\mu, K_\nu] = 0$ a change of basis in the (P_μ, K_μ)-space won't affect the underlying Poincaré algebra. Using the transformation of the basis derived in the appendix, equation (211), the Lie algebra may be simplified to:

$$\begin{aligned}
[P_\mu, K_\nu] &= i g_{\mu\nu} Q, \\
[S_{\mu\nu}, S_{\kappa\lambda}] &= is\left(g_{\mu\lambda} J_{\nu\kappa} + g_{\nu\kappa} J_{\mu\lambda} + g_{\mu\kappa} J_{\nu\lambda} + g_{\nu\lambda} J_{\mu\kappa}\right), \\
[S_{\mu\nu}, P_\lambda] &= i(g_{\mu\lambda} K_\nu + g_{\nu\lambda} K_\mu - \frac{1}{2} g_{\mu\nu} K_\lambda), \\
[S_{\mu\nu}, K_\lambda] &= is(g_{\mu\lambda} P_\nu + g_{\nu\lambda} P_\mu - \frac{1}{2} g_{\mu\nu} P_\lambda).
\end{aligned} \tag{6.49}$$

The constant s takes the values from the set $\{-1, 0, 1\}$. The coefficient a and the constant λ (introduced by the similarity transformation (211)) were removed by scaling Q and $S_{\mu\nu}$ by them, respectively.

Coset construction

The coset space is parametrised by

$$U(x, \theta, \xi, \beta_{\mu\nu}) = e^{ix_\mu P^\mu} e^{i\theta Q} e^{i\xi_\mu K^\mu} e^{i\beta_{\mu\nu} S^{\mu\nu}}. \tag{6.50}$$

The coordinates $\beta_{\mu\nu}$ associated with $S_{\mu\nu}$ also form a symmetric, traceless tensor. The left multiplication of U by $e^{i\alpha Q}$ only generates the shift $\theta \to \theta + \alpha$. The equivalent transformation of the fields in U induced by the generator K_μ, parametrised by β_μ, generates the following two transformations.

$$\begin{aligned}
\xi_\mu &\to \xi_\mu + \beta_\mu \\
\theta &\to \theta + \beta_\mu x^\mu
\end{aligned} \tag{6.51}$$

Finally, the field transformation induced by left multiplication of U by $e^{i\omega_{\mu\nu} S^{\mu\nu}}$ is

$$\begin{aligned}
x_\mu &\to \cosh(\sqrt{s}\omega)^{\mu\nu} x_\nu - \sqrt{s}\sinh(\sqrt{s}\omega)^{\mu\nu}\xi_\nu, \\
\xi_\mu &\to -\sinh(\sqrt{s}\omega)^{\mu\nu}\frac{x_\nu}{\sqrt{s}} + \cosh(\sqrt{s}\omega)^{\mu\nu}\xi_\nu, \\
\theta &\to \theta - \frac{[\sinh(\sqrt{s}\omega)\cosh(\sqrt{s}\omega)]^{\mu\nu}}{2}\left(\frac{x_\mu x_\nu}{\sqrt{s}} + \sqrt{s}\xi_\mu\xi_\nu\right) \\
&\quad + \sinh^2(\sqrt{s}\omega)^{\mu\nu}\xi_\mu x_\nu, \\
\beta_{\mu\nu} &\to \beta_{\mu\nu} + \omega_{\mu\nu} + \mathcal{O}(\omega^2, \beta^2).
\end{aligned} \tag{6.52}$$

The form of these transformations closely mirrors those from the spin-zero extension (D.3). The MC form for this algebra is decomposed along the generators in the following way

$$\omega = \omega_P^\mu P_\mu + \omega_K^\mu K_\mu + \omega_Q Q + \frac{1}{2}\left(\omega_S^{\mu\nu} S_{\mu\nu} + \omega_J^{\mu\nu} J_{\mu\nu}\right). \tag{6.53}$$

The explicit expressions for these 1-forms are:

$$\begin{aligned}
\omega_P^\mu &= \cosh(\sqrt{s}\beta)^{\mu\nu} dx_\nu + \sqrt{s}\sinh(\sqrt{s}\beta)^{\mu\nu} d\xi_\nu, \\
\omega_K^\mu &= \frac{\sinh(\sqrt{s}\beta)^{\mu\nu}}{\sqrt{s}} dx_\nu + \cosh(\sqrt{s}\beta) d\xi^\mu, \\
\omega_Q &= d\theta - \xi \cdot dx, \\
\omega_S^{\mu\nu} &= \frac{[B^{-1}\sinh(\sqrt{s}B)]^{\mu\nu}_{\alpha\beta}}{\sqrt{s}} d\beta^{\alpha\beta}, \\
\omega_J^{\mu\nu} &= [B^{-1}[\cosh(\sqrt{s}B) - I]]^{\mu\nu}_{\alpha\beta} d\beta^{\alpha\beta}.
\end{aligned} \tag{6.54}$$

The functions are to be interpreted as power series, with tensors $B^{\alpha\beta}_{\mu\nu}$ as their argument; the I is the identity matrix. The tensor B was only introduced for the sake of notational convenience, its full form is

$$B^{\alpha\beta}_{\mu\nu} = \beta^\alpha_\mu \delta^\beta_\nu - \beta^\beta_\nu \delta^\alpha_\mu. \tag{6.55}$$

The redundant degrees of freedom ξ^μ are eliminated by the IHC

$$\omega_Q = 0 \qquad \rightarrow \qquad \xi^\mu = \partial_\mu \theta. \tag{6.56}$$

The physical field θ then transforms under the action of $S_{\mu\nu}$ as

$$\theta \qquad \rightarrow \qquad \theta - \frac{\omega^{\mu\nu}}{2}\left(x_\mu x_\nu + s\partial_\mu\theta\partial_\nu\theta\right) + \mathcal{O}(\omega^2). \tag{6.57}$$

This is the expression for the hidden symmetry of the special galileon, which was first found in [63].

Geometry

The vielbein, derivable from ω_P^α, is

$$e_\mu^\alpha = \cosh(\sqrt{s}\beta)^{\alpha\nu} g_{\mu\nu} + \sqrt{s}\sinh(\sqrt{s}\beta)^{\alpha\nu}\partial_\mu\xi_\nu. \tag{6.58}$$

The metric can be derived from this, $G_{\mu\nu} = g_{\alpha\beta} e^\alpha_\mu e^\beta_\nu$, but not much is gained from an explicit representation. The fields ξ_μ and $\beta_{\mu\nu}$ are the redundant degrees of freedom. The covariant derivatives of these fields, $\nabla_\mu \xi^\nu$ and $\nabla_\lambda \beta^{\mu\nu}$, are best expressed in the implicit form; they aren't used in calculation here.

$$e^\alpha_\nu \nabla_\mu \xi^\nu \quad = \cosh(\sqrt{s}\beta)^{\alpha\nu}\partial_\mu \xi_\nu + \sqrt{s}\sinh(\sqrt{s}\beta)^{\alpha\nu} \tag{6.59}$$

$$e^\alpha_\mu e^\beta_\nu \nabla_\lambda \beta^{\mu\nu} = \frac{[B^{-1}\sinh(\sqrt{s}B)]^{\alpha\beta}_{\mu\nu}}{\sqrt{s}}\partial_\lambda \beta^{\mu\nu} \tag{6.60}$$

The redundant field $\beta_{\mu\nu}$ can be removed by setting the symmetric, traceless part of $\nabla_\mu \xi^\nu$ to zero. The action can then be constructed from the singlet and anti-symmetric part of the covariant derivative $\nabla_\mu \xi^\nu$, and $\nabla_\lambda \beta^{\mu\nu}$. Finally, the ω_J can be used for the spin connection in the covariant derivatives.

$$\omega^{\mu\nu}_\lambda = [B^{-1}[\cosh(\sqrt{s}B) - I]]^{\mu\nu}_{\alpha\beta}\partial_\lambda \beta^{\alpha\beta} \tag{6.61}$$

The solution of the second IHC implies that $\beta_{\mu\nu}$ is a function of $\partial_\mu \xi_\nu$. After implementing the first IHC, $\xi_\mu = \partial_\mu \theta$, $\beta_{\mu\nu}$ becomes a function of $\partial_\mu \partial_\nu \theta$. This means that all covariant derivatives used to construct actions will depend on the second derivative of the physical field θ. In consequence, no kinetic terms can be constructed and there are no special soft NG actions. For example, in the case $s = 0$ the MC forms reduce to

$$\begin{gathered} \omega^\mu_P = dx^\mu, \quad \omega^\mu_K = d\xi^\mu + \beta^{\mu\nu}dx_\nu, \\ \omega_Q = d\theta - \xi \cdot dx, \quad \omega^{\mu\nu}_S = d\beta^{\mu\nu}, \quad \omega^{\mu\nu}_J = 0. \end{gathered} \tag{6.62}$$

The covariant derivate is $\nabla^\nu \xi^\mu = \partial^\nu \xi^\mu + \beta^{\mu\nu}$, so the IHCs are solved by

$$\begin{aligned} \xi^\mu = \partial^\mu \theta, \qquad \beta^{\mu\nu} &= \frac{1}{2}\left(\frac{g^{\mu\nu}\partial^\alpha \xi_\alpha}{2} - \partial^\nu \xi^\mu - \partial^\mu \xi^\nu\right) \\ &= \frac{g^{\mu\nu}\Box\theta}{4} - \partial^\nu \partial^\mu \theta. \end{aligned} \tag{6.63}$$

The first IHC has removed the antisymmetric part of $\nabla_\mu \xi^\nu$, the second IHC its symmetric traceless part. The remaining covariant derivatives are therefore

$$\nabla_\mu \xi^\mu = \Box\theta \qquad \& \qquad \nabla_\lambda \beta^{\mu\nu} = \partial_\lambda \left(\frac{g^{\mu\nu}\Box\theta}{4} - \partial^\nu \partial^\mu \theta\right). \tag{6.64}$$

The terms $\nabla_\lambda \beta^{\mu\nu}$ have three derivatives per field, so they have a trivial enhanced soft limit. The action of the d'Alembertian on a free field annihilates it. Any action of the form

$$\mathcal{L} = \frac{\partial_\mu \theta \partial^\mu \theta}{2} + f(\Box\theta), \tag{6.65}$$

can be rewritten to a free theory by an appropriate redefinition of the field. Although these terms don't establish an interesting theory with enhanced soft limits on their own, since they reduce to the free theory, but they may function as interaction terms for an action built using WZ terms.

The WZ terms

The spin-two WZ terms will now be constructed using the familiar scheme. The building blocks for the 5-forms are ω_Q, ω_P^μ, ω_K^μ and $\omega_S^{\mu\nu}$. The last two forms, if expressed in terms of the physical field θ, lead to lagrangean densities containing two derivatives per field; this makes them uninteresting as candidates for enhanced soft limits. The remainder combines into the same basis again, $\omega_5^i = \omega_Q \wedge e^i$, although the content of the individual forms differs. The exterior derivatives of the 1-forms are:

$$d\begin{pmatrix}\omega_P^\kappa \\ \omega_K^\kappa\end{pmatrix} = \begin{pmatrix}\omega_{J\,\lambda}^\kappa & s\omega_{S\,\lambda}^\kappa \\ \omega_{S\,\lambda}^\kappa & \omega_{J\,\lambda}^\kappa\end{pmatrix} \wedge \begin{pmatrix}\omega_P^\lambda \\ \omega_K^\lambda\end{pmatrix}, \tag{6.66}$$

$$d\omega_Q = g_{\kappa\lambda}\omega_P^\kappa \wedge \omega_K^\lambda. \tag{6.67}$$

Using some identities of the Levi-Civita symbol [D.1], the transformation of the 4-basis under derivation can be calculated.

$$de^i = \epsilon_{\mu\nu\rho\sigma}\omega_{S\,\lambda}^\mu \wedge \omega_K^\lambda \wedge \begin{pmatrix}4s & 0 & 0 \\ 0 & 3s & 0 \\ -\frac{2}{3} & 0 & 2s \\ 0 & -3 & 0 \\ 0 & 0 & -12\end{pmatrix}\begin{pmatrix}\omega_P^3 \\ \omega_K \wedge \omega_P^2 \\ \omega_K^2 \wedge \omega_P\end{pmatrix} \tag{6.68}$$

The matrix in the above expression has two left eigenvectors with eigenvalue 0,

$$v_i^+ = (1, 4s^{\frac{1}{2}}, 6s, 4s^{\frac{3}{2}}, s^2) \qquad \& \qquad v_i^- = (1, -4s^{\frac{1}{2}}, 6s, -4s^{\frac{3}{2}}, s^2). \tag{6.69}$$

The exterior derivative of the 5-forms is

$$d\omega_5^i = -\omega_Q \wedge de^i. \tag{6.70}$$

This means that the null vectors $v_i^\pm\omega_5^i$ make up the kernel of the mapping by the exterior derivative, and they are part of Z^5.

Q is part of the centre of the algebra, so ω_Q can't be generated as part of a derivative of any other form. The ω_5^i can therefore only be generated from 4-forms that already contain ω_Q; those forms then contain three ω_P^μs or ω_K^μs at most. The exterior derivative of the 4-forms could only generate the ω_5^i if the exterior derivatives of ω_P^μ or ω_K^μ raised the number of those factors in the form; in fact only the exterior derivative of ω_Q does this. The null-vectors are therefore not in the coboundary group, and so they are part of the cohomology group. These Wess-Zumino terms can be compactly written as:

$$v_i^+\omega_5^i = \omega_Q \wedge \epsilon \cdot \left(\omega_P + \sqrt{s}\omega_K\right)^4, \tag{6.71}$$

$$v_i^-\omega_5^i = \omega_Q \wedge \epsilon \cdot \left(\omega_P - \sqrt{s}\omega_K\right)^4. \tag{6.72}$$

A further simplification can be made by introducing the following notation:

$$(E^{\pm})^{\mu}_{\nu} = \left(e^{\pm\sqrt{s}\beta}\right)^{\mu}_{\nu}, \tag{6.73}$$
$$z^{\mu}_{\pm} \quad = x^{\mu} \pm \sqrt{s}\xi^{\mu}. \tag{6.74}$$

In these terms the combination of 1-forms in $v_i^{\pm}$ can be written as

$$\omega_P^{\mu} + \sqrt{s}\omega_K^{\mu} = (e^{\sqrt{s}\beta})^{\mu}_{\nu}(dx^{\nu} + \sqrt{s}d\xi^{\nu}) = (E^{+})^{\mu}_{\nu}dz^{\nu}_{+}, \tag{6.75}$$
$$\omega_P^{\mu} - \sqrt{s}\omega_K^{\mu} = (e^{-\sqrt{s}\beta})^{\mu}_{\nu}(dx^{\nu} - \sqrt{s}d\xi^{\nu}) = (E^{-})^{\mu}_{\nu}dz^{\nu}_{-}. \tag{6.76}$$

Both terms have the same structure in this notation, differing only by a sign. This simplifies the WZ terms as well; the sign is temporarily omitted, since it carries no significance in the calculation.

$$\begin{aligned} v_i\omega_5^i &= \omega_Q \wedge \left(\epsilon_{\mu\nu\rho\sigma}E^{\mu}_{\alpha}E^{\nu}_{\beta}E^{\rho}_{\gamma}E^{\sigma}_{\delta}dz^{\alpha} \wedge dz^{\beta} \wedge dz^{\gamma} \wedge dz^{\delta}\right) \\ &= \omega_Q \wedge \det(E)\left(\epsilon_{\mu\nu\rho\sigma}dz^{\mu} \wedge dz^{\nu} \wedge dz^{\rho} \wedge dz^{\sigma}\right) \end{aligned} \tag{6.77}$$

Using the identity $\det(e^A) = e^{\mathrm{tr}(A)}$, and because β^{μ}_{ν} is traceless, it follows that

$$\det(E^{\pm}) = \det(e^{\pm\sqrt{s}\beta^{\mu}_{\nu}}) = e^{\mathrm{tr}(\pm\sqrt{s}\beta^{\mu}_{\nu})} = 1. \tag{6.78}$$

Thus the WZ terms, (6.71) and (6.72), simplify to

$$v_i^{\pm}\omega_5^i = (d\theta - \xi \cdot dx) \wedge \epsilon \cdot \left(dx \pm \sqrt{s}d\xi\right)^4. \tag{6.79}$$

This means that the $v_i^{\pm}\omega_5^i$ can be written as a sum of the galileon 5-forms $\mathfrak{g}^i$, defined earlier in section [6.2.2]. It is convenient to express the new forms in those terms, because the integrals of the $\mathfrak{g}^i$ were already found in (6.43).

$$v_i^{+}\omega_5^i - v_i^{-}\omega_5^i = 8(\mathfrak{g}_2 + s\mathfrak{g}_4) \tag{6.80}$$
$$v_i^{+}\omega_5^i + v_i^{-}\omega_5^i = 2(\mathfrak{g}_1 + 6s\mathfrak{g}_2 + s^2\mathfrak{g}_4) \tag{6.81}$$

The lagrangean of $\mathfrak{g}_1$ is $\mathcal{L}_1 = \theta$, this is called a tadpole term; it functions as a constant source. These terms are not fit for purpose, because the source would couple to scattering processes that should only involve NG bosons. The 5-form $v_i^{+}\omega_5^i - v_i^{-}\omega_5^i$ is therefore the unique WZ term for the galileon algebra extended by a redundant, traceless, symmetric tensor.
The extended algebra still depends on $s \in \{-1, 0, 1\}$. In the case that $s = 0$ the topological term is reduced to the kinetic term of the action for the field θ. If $s = \pm 1$ the action corresponds to that of the special galileon [63].

$$\mathcal{L}_{WZ} = \frac{(\partial_{\mu}\theta)^2}{2} + s\frac{(\partial_{\mu}\theta)^2}{4}\left((\Box\theta) - (\partial_{\mu}\partial_{\nu}\theta)^2\right) \tag{6.82}$$

6.3 Multiple physical fields

The previous section focused on increasing the softness of the scattering amplitudes for a single-field theory by increasing the rank and number of the redundant generators. The exploration of the possibilities turned up no new non-trivial theories with enhanced soft limits beyond the familiar galileon and DBI models. Although the Lie algebra can be extended with three or more redundant generators, so that $\sigma > 3$, this only leads to models that realize an enhanced soft scattering limit in a trivial way [54]; their only symmetries would be pure extended shift symmetries [64].

Besides increasing the number of redundant generators, increasing the number of NG bosons also generates a space for new theories. A number of the new physical fields, or flavours, may have enhanced soft limits. A trivial extension would simply contain a set of independent copies of the single field models. However, a more interesting multi-flavour theory would be one where the broken NG generators don't commute with one another, so their internal algebra is non-abelian. The goal of this section is to create a systematic catalogue for the multiple physical NG bosons, some of which have enhanced soft limits.

Lie algebra extension for multiple NG bosons

The physical NG bosons arise as a consequence of the breaking of the internal symmetry generators Q_i. The multiple redundant broken generators are denoted by $K_{\mu A}$. The most general form of the algebra that consists of these generators, and those from the Poincaré group, is:

$$
\begin{aligned}
[J_{\mu\nu}, J_{\kappa\lambda}] &= i\left(g_{\mu\lambda}J_{\nu\kappa} + g_{\nu\kappa}J_{\mu\lambda} - g_{\mu\kappa}J_{\nu\lambda} - g_{\nu\lambda}J_{\mu\kappa}\right), && (6.83)\\
[J_{\mu\nu}, P_{\lambda}] &= i\left(g_{\nu\lambda}P_{\mu} - g_{\mu\lambda}P_{\nu}\right), && (6.84)\\
[J_{\mu\nu}, K_{\lambda A}] &= i\left(g_{\nu\lambda}K_{\mu A} - g_{\mu\lambda}K_{\nu A}\right), && (6.85)\\
[J_{\mu\nu}, Q_i] &= 0, && (6.86)\\
[P_{\mu}, P_{\nu}] &= 0, && (6.87)\\
[P_{\mu}, K_{\nu A}] &= i(a^i_A g_{\mu\nu}Q_i + b_A J_{\mu\nu} + c_A \epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma}, && (6.88)\\
[P_{\mu}, Q_i] &= i\left(d_i P_{\mu} + e_i^A K_{\mu A}\right), && (6.89)\\
[K_{\mu A}, K_{\nu B}] &= i\left(f_{AB}J_{\mu\nu} + g_{AB}\epsilon_{\mu\nu\kappa\lambda}J^{\kappa\lambda} + \Xi^i_{AB}g_{\mu\nu}Q_i\right), && (6.90)\\
[K_{\mu A}, Q_i] &= i(h_{Ai}P_{\mu} + i^B_{Ai}K_{\mu B}), && (6.91)\\
[Q_i, Q_j] &= i\Lambda^k_{ij}Q_k. && (6.92)
\end{aligned}
$$

To reveal the similarity, the choice of letters for the red coefficients mimics that in the single field case. The f_{AB} and g_{AB}, however, have now become symmetric

2-tensors. The qualitatively new structure constants that appear for the multi-field case are Λ_{ij}^{k} and Ξ_{AB}^{i}. The Ξ_{AB}^{i} is antisymmetric under the exchange of indices $A \leftrightarrow B$, and Λ_{ij}^{k} is antisymmetric under the the switch $i \leftrightarrow j$. Like in the single boson case, the assumption is that the Q_i generate a uniform symmetry so that the scattering amplitudes satisfy the Adler's zero condition. This is imposed by the following constraints on the structure constants:

$$d_i = 0, \qquad e_i^A = 0. \tag{6.93}$$

The demand that the commutators satisfy the Jacobi identities, and so form a Lie algebra, imposes an additional set of conditions.

$$\begin{aligned}
b_A = 0, \qquad c_A = 0, \qquad g_{AB} &= 0, && (6.94)\\
\Lambda_{ij}^{m}\Lambda_{mk}^{n} + \Lambda_{jk}^{m}\Lambda_{mi}^{n} + \Lambda_{ki}^{m}\Lambda_{mj}^{n} &= 0, && (6.95)\\
i_{Ai}^{B} i_{Bj}^{C} - i_{Aj}^{B} i_{Bi}^{C} &= \Lambda_{ij}^{k} i_{Ak}^{C}, && (6.96)\\
i_{Ai}^{B} h_{Bj} - i_{Aj}^{B} h_{Bi} &= \Lambda_{ij}^{k} h_{Ak}, && (6.97)\\
a_A^i i_{Bi}^{C} &= 0, && (6.98)\\
a_B^k i_{Ai}^{B} + a_A^j \Lambda_{ij}^{k} &= 0, && (6.99)\\
a_A^k h_{Bi} - a_B^k h_{Ai} + i_{Ai}^{C}\Xi_{BC}^{k} - i_{Bi}^{C}\Xi_{AC}^{k} &= \Lambda_{ij}^{k}\Xi_{AB}^{j}, && (6.100)\\
f_{AB} &= -a_A^i h_{Bi}, && (6.101)\\
h_{Ai}\Xi_{BC}^{i} &= 0, && (6.102)\\
f_{AC}\delta_B^D - f_{BC}\delta_A^D &= \Xi_{AB}^{i} i_{Ci}^{D}. && (6.103)
\end{aligned}$$

Equality (6.95) states that the internal generators Q_i form a Lie algebra, because they satisfy the Jacobi identity. Equation (6.96) indicates that $(t_i)_A^B = -i i_{Ai}^B$ forms a representation of the Lie algebra of the Q_i. These observations imply a geometrical interpretation for a class of Lie algebras with multi-flavour enhanced soft limits. First, however, the classification of theories with multiple NG bosons, of which only one has an enhanced limit, is carried through.

6.3.1 Single redundant generator ($\sigma = 2$)

If only one field has an enhanced soft limit, in a theory containing multiple NG fields, this means that there are multiple Q_i and one K_μ. This means that the index A becomes superfluous, and in turn the constraints from the Jacobi identity simplify considerably.

$$\begin{aligned}
&a^k i_i + a^j \Lambda^k_{ij} = 0, && a^i i_i = 0, \\
&b = 0, && c = 0, \\
&f = -a^i h_i, && g = 0, \\
&i_i h_j - i_j h_i = \Lambda^k_{ij} h_k, && \Lambda^k_{ij} i_k = 0, \\
&\Lambda^m_{ij}\Lambda^n_{mk} + \Lambda^m_{jk}\Lambda^n_{mi} + \Lambda^m_{ki}\Lambda^n_{mj} = 0, && \Xi^i = 0.
\end{aligned} \tag{6.104}$$

The constraint $\Xi^i = 0$ doesn't follow from the Jacobi identities, Ξ^i_{AB} is anti-symmetric so it vanishes when A is one-dimensional. The vector a_i determines which direction, in the space spanned by the Q_i, couples to the redundant degree of freedom introduced by K^μ; it is convenient to introduce $\tilde{Q} = a_i Q^i$ for this generator. The existence of an enhanced soft limit requires the existence of a $\tilde{Q}$, so a_i cannot vanish completely. This means that the above constraints are not all independent. The contraction of the first constraint in (6.104) with i_k replicates the other constraint in the first line.

$$(a^k i_i + a^j \Lambda^k_{ij}) i_k = a^k (a^i i_i) = 0 \tag{6.105}$$

The definition $-f = a^i h_i = v$ makes the parallel to the single boson case more obvious. The Lie algebra now takes the form:

$$\begin{aligned}
[P_\mu, K_\nu] &= i g_{\mu\nu} \tilde{Q}, \\
[K_\mu, K_\nu] &= -iv J_{\mu\nu}, \\
[K_\mu, \tilde{Q}] &= iv P_\mu, \\
[K_\mu, Q_i] &= i\left(h_i P_\mu + i_i K_\mu\right), \\
[Q_i, Q_j] &= i \Lambda^k_{ij} Q_k.
\end{aligned} \tag{6.106}$$

By design the first three commutators now reproduce the structure from equation (6.16), in the case that $u = 0$. The last two indicate that this is the algebra of a multi-flavour system, that may have a non-abelian internal symmetry. Depending on whether v is zero or not, this system is an extension of the galileon or DBI model respectively.

DBI-like systems

If $v \neq 0$ the system has a DBI-like structure. The first constraint in (6.104) now implies another condition:

$$h_k \left(a^k i_i + a^j \Lambda^k_{ij}\right) = 2v i_i = 0 \qquad \rightarrow \qquad i_i = 0. \tag{6.107}$$

The only unsolved constraint equations, that contained i_i, remaining are

$$a^j \Lambda_{ij}^k = 0, \qquad h_k \Lambda_{ij}^k = 0. \tag{6.108}$$

After implementing the constraints, and making the redefinition $\tilde{Q}_i = Q_i - \frac{h_i}{v}\tilde{Q}$, the Lie algebra (6.106) can be expressed as:

$$\begin{aligned} [P_\mu, K_\mu] &= ig_{\mu\nu}\tilde{Q}, & [K_\mu, \tilde{Q}_i] &= 0, \\ [K_\mu, K_\nu] &= -ivJ_{\mu\nu}, & [\tilde{Q}, \tilde{Q}_j] &= 0, \\ [K_\mu, \tilde{Q}] &= ivP_\mu, & [\tilde{Q}_i, \tilde{Q}_j] &= i\Lambda_{ij}^k \tilde{Q}_k. \end{aligned} \tag{6.109}$$

This is a trivial extension of the DBI algebra from section [6.2.2]. It consists of a DBI part made up from the generators $\tilde{Q}$ and K_μ and the internal symmetry part generated by $\tilde{Q}_i$. The algebra is a direct sum of these parts; there is no coupling because the generators from different sectors commute. Using the definition $\tilde{U}(\tilde{\theta}) \equiv e^{i\tilde{\theta}^a \tilde{Q}_a}$, the coset can be parametrised as

$$U(x^\mu, \theta, \xi^\mu, \tilde{\theta}^a) = e^{ix^\mu P_\mu} e^{i\theta\tilde{Q}} e^{i\xi^\mu K_\mu} e^{i\tilde{\theta}^a \tilde{Q}_a} = U_{DBI}(x^\mu, \theta, \xi^\mu)\tilde{U}(\tilde{\theta}^a). \tag{6.110}$$

The MC form then decomposes into the sum

$$\begin{aligned} \omega_{MC} &= -iU^{-1}dU = -iU_{DBI}^{-1}dU_{DBI} - i\tilde{U}^{-1}d\tilde{U} \\ &= \omega_{DBI} + \Omega. \end{aligned} \tag{6.111}$$

The overall MC form splits into ω_{DBI} and Ω. The forms in ω_{DBI} were already calculated in (6.23). The Ω is simply the MC form for the broken generators of the internal symmetry. In this notation the $\tilde{Q}_i$ are generic generators of the internal symmetry, which can be divided into the unbroken $\tilde{Q}_\alpha$ and the broken $\tilde{Q}_a$.

$$\Omega \equiv -ie^{-i\tilde{\theta}\cdot\tilde{Q}} de^{i\tilde{\theta}\cdot\tilde{Q}} \tag{6.112}$$

These forms can be used to construct actions, along the lines sketched out in section [2.4] for broken internal symmetries.

The galileon-like system

The galileon models are characterized by the vanishing of v. The constraints from the Jacobi identity can't be solved in this case, but the commutation relations simplify considerably. Again, the field gains an enhanced soft limit for $\tilde{\theta}$, with the broken generator $\tilde{Q}$. Under the assumption that the other internal symmetry generators are independent and form a Lie algebra, so $[\tilde{Q}_i, \tilde{Q}_j] = i\Lambda_{ij}^k \tilde{Q}_k$, the

MC form for this case can be worked out too. The coset space is parametrised by

$$U(x,\theta,\tilde{\theta},\xi) \equiv e^{ix^\mu P_\mu} e^{i\xi^\mu K_\mu} e^{i\theta\tilde{Q}} e^{i\tilde{\theta}^a \tilde{Q}_a} = \tilde{U}_{gal}\tilde{U}. \tag{6.113}$$

As before, the $\tilde{Q}_a$ are the broken generators of the internal symmetry group and their MC form takes the form of Ω. Using this parametrisation, the MC form of Galileon like systems is

$$\begin{aligned} \omega &= -iU^{-1}dU \\ &= \tilde{U}^{-1}\tilde{\omega}_{gal}\tilde{U} + \Omega. \end{aligned} \tag{6.114}$$

The forms Ω take the same form they would in the case of the DBI-like systems. Although the commutation relations of P_μ, K_μ and $\tilde{Q}$ copy the structure of the single field galileon, the MC form $\tilde{\omega}_{gal}$ is ordered differently for the convenience of calculation. The expressions for the 1-forms of these generators are

$$\omega_P^\mu = dx^\mu + h_a\tilde{\theta}^a \frac{e^{-i_b\tilde{\theta}^b} - 1}{i_c\tilde{\theta}^c} d\xi^\mu, \tag{6.115}$$

$$\omega_K^\mu = e^{-i_a\tilde{\theta}^a} d\xi^\mu, \tag{6.116}$$

$$\omega_{\tilde{Q}} = e^{-i_a\tilde{\theta}^a}(d\theta - \xi\cdot dx). \tag{6.117}$$

These components of the MC form contain exponential factors $e^{i_b\tilde{\theta}^b}$ because $\tilde{\omega}_{gal}$ is wedged between $\tilde{U}$ and its inverse, and the generators of $\tilde{U}$ don't commute with $\tilde{Q}$:

$$[\tilde{Q}_i, \tilde{Q}] = -ii_i\tilde{Q}. \tag{6.118}$$

The broken components of the MC form transform covariantly under the action of the unbroken subgroup (2.48). The commutation relation $[K_\mu, Q_\alpha] = i(h_\alpha P_\mu + i_\alpha K_\mu)$ induces a non-covariant transformation of ω_P^μ, so the vielbein can only be covariant under the action of the unbroken internal symmetry when $h_\alpha = 0$.

The redundant modes ξ^μ are removed by the IHC $\omega_{\tilde{Q}} = 0$. After implementing this constraint all other parts of the MC form only depend on the second derivatives of θ, like in the single field galileon case. Similarly, if a kinetic term exists, it can only be found via the WZ construction.

WZ terms for the galileon-like system

The Maurer-Cartan structure equations for the Lie algebra (6.106) are:

$$d\begin{pmatrix}\omega_P^\mu\\ \omega_K^\mu\end{pmatrix} = -\Omega^i\wedge\begin{pmatrix}0 & h_i\\ 0 & i_i\end{pmatrix}\begin{pmatrix}\omega_P^\mu\\ \omega_K^\mu\end{pmatrix}, \tag{6.119}$$

$$d\omega_{\tilde{Q}} = g_{\mu\nu}\omega_P^\mu\wedge\omega_K^\nu - i_i\Omega^i\wedge\omega_{\tilde{Q}}, \tag{6.120}$$

$$d\Omega^k = \frac{\Lambda_{ij}^k}{2}\Omega^i\wedge\Omega^j. \tag{6.121}$$

Lorentz invariant, topological actions must be constructed from the basis consisting of the 4-forms (6.36) wedged with $\omega_{\tilde{Q}}$ or one of the forms in Ω. The 5-forms must not only be invariant under the Lorentz transformations, but also under the unbroken internal symmetries. Because of the commutation relations

$$[K_\mu, Q_\alpha] = i\left(h_\alpha P_\mu + i_\alpha K_\mu\right) \qquad \& \qquad [\tilde{Q}_\alpha, \tilde{Q}] = -i i_i\tilde{Q} \tag{6.122}$$

the forms $\omega_{\tilde{Q}}$ and ω_K^μ change by a factor $e^{i\epsilon^\alpha i_\alpha}$ under the internal symmetry transformation by $e^{i\epsilon^\alpha\tilde{Q}_\alpha}$, like in (2.50). A model is therefore only invariant if $i_\alpha = 0$.

The exterior derivative of the 4-basis can be written compactly as

$$de^k = -\Omega^i\wedge\left((k-1)i_i e^k + (5-k)h_i e^{k+1}\right). \tag{6.123}$$

The exterior derivative of the 5-forms $\omega_5^i = \omega_{\tilde{Q}}\wedge e^i$ then is

$$d\omega_5^k = \omega_{\tilde{Q}}\wedge\Omega^i\wedge\left(k i_i e^k + (5-k)h_i e^{k+1}\right). \tag{6.124}$$

Using a Gram-Schmidt decomposition on the vector h_i and i_i yields

$$i_i = i_i, \qquad j_i = h_i - \frac{i\cdot h}{i^2}i_i. \tag{6.125}$$

This pair is orthogonal, $i_i j^i = 0$. Using the parameter $s = i_i h^i/i^2$, the vector h_i takes the form

$$h_i = j_i + s i_i. \tag{6.126}$$

In these terms, the derivative becomes

$$\begin{aligned} d\omega_5^k =& (\omega_{\tilde{Q}}\wedge i_i\Omega^i)\wedge\left(k e^k + s(5-k)e^{k+1}\right)\\ &+ (5-k)(\omega_{\tilde{Q}}\wedge j_i\Omega^i\wedge e^{k+1})\\ =& I_l^k(\omega_{\tilde{Q}}\wedge i_i\Omega^i\wedge e^l) + J_l^k(\omega_{\tilde{Q}}\wedge j_i\Omega^i\wedge e^l). \end{aligned} \tag{6.127}$$

The matrices I and J, implicitly defined in the previous equation, take the following forms:

$$I = \begin{pmatrix} 1 & 4s & 0 & 0 & 0 \\ 0 & 2 & 3s & 0 & 0 \\ 0 & 0 & 3 & 2s & 0 \\ 0 & 0 & 0 & 4 & s \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \qquad \& \qquad J = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.128}$$

The kernel of I is empty, and the null vector of J is

$$v^J_k = (0, 0, 0, 0, 1). \tag{6.129}$$

This means that WZ terms only exist if $i_i = 0$, which in turn implies that $s = 0$ too. Furthermore, when $j_i \neq 0$ the integration of the 5-form $v^J_k \omega^k_5$ only yields another interaction term for the action, not a kinetic term. Setting j_i to zero too, means that the algebra reduces to the direct sum of the galileon algebra for a single field and the internal symmetry. The galileon and internal NG sectors then interact only through strictly invariant terms in the lagrangean, and the enhanced soft limits can only be realized trivially.

6.3.2 Multiple redundant generators $(\sigma = 2)$

It is possible to construct a general solution to the constraints due to the Jacobi identity on the multi-flavour Lie algebra (6.83). It turns out that these constraints actually imply a particular formulation of the commutation relations, in terms of a set of redefined generators, that forms a solution.

The matrices $(t_i)^B_A = -i i^A_{Bi}$ form a representation of the generators Q_i of the internal algebra. In other words, the constraint (6.96) can be rewritten to

$$[t_i, t_j] = -i\Lambda^k_{ij} t_k. \tag{6.130}$$

In this particular representation the generators $Q_A = a^i_A Q_i$ vanish, because $a^i_A (t_i)^C_B = 0$ (6.98). The equality $(t_i)^C_B f_{AC} + (t_i)^C_A f_{BC} = 0$ is a synthesis of (6.101), (6.96), (6.98) and (6.99); it expresses that f_{AB} is an invariant tensor of this representation of the internal symmetry. Finally, the relations (6.96) and (6.97) can be combined into the block matrices $(T_i)^A_B$. These form a representation of the internal Lie algebra, too.

$$(T_i)^A_B = \left(\begin{array}{c|c} t^A_{iB} & 0 \\ \hline h_{iB} & 0 \end{array}\right) \qquad \rightarrow \qquad [T_i, T_j] = i\Lambda^k_{ij} T_k \tag{6.131}$$

Starting from the remaining constraints implied by the Jacobi identities, a new set of commutation relations may now be formulated for the generators. To begin, the equations (6.99), (6.98) and (6.92) together imply that the new generator Q_A commutes with other generators of the internal space as follows:

$$a^k_B i^B_{Ai} Q_k = -a^j_A \Lambda^k_{ij} Q_k = i a^j_A [Q_i, Q_j] \qquad (6.132)$$
$$\rightarrow$$
$$[Q_i, Q_A] = (t_i)^B_A Q_B \qquad \& \qquad [Q_A, Q_B] = 0.$$

The constraint (6.100) together with the commutator (6.92) creates a commutation relation for the new generators $Q_{AB} = \Xi^k_{AB} Q_k$:

$$(a^k_A h_{Bi} - a^k_B h_{Ai} + i^C_{Ai} \Xi^k_{BC} - i^C_{Bi} \Xi^k_{AC}) Q_k = \Lambda^k_{ij} \Xi^j_{AB} Q_k \qquad (6.133)$$
$$\rightarrow$$
$$i(h_{Bi} Q_A - h_{Ai} Q_B) - t^C_{Ai} Q_{BC} + t^C_{Bi} Q_{AC} = [Q_i, Q_{AB}].$$

The equations (6.101), (6.98) and (6.89) establish another commutation rule.

$$-i f_{AB} P_\mu = i a^i_A h_{Bi} P_\mu = [K_{\mu B}, Q_A] \qquad (6.134)$$

Finally, so do (6.102), (6.103) and (6.91):

$$i \left(f_{AC} \delta^D_B - f_{BC} \delta^D_A \right) K_{\mu D} = \Xi^i_{AB} i^D_{Ci} K_{\mu D} = [K_{\mu C}, Q_{AB}]. \qquad (6.135)$$

The geometry of the multi-flavour symmetries

The commutation relations of the vectors Q_A and tensors Q_{AB} express the restrictions imposed by the Jacobi identity. In consequence, it will be advantageous to write out their role in the non-linearly realised part of the Lie algebra.

$$\begin{aligned}
[P_\mu, K_{\nu A}] &= i g_{\mu\nu} Q_A && (6.136)\\
[K_{\mu A}, K_{\nu B}] &= i \left(f_{AB} J_{\mu\nu} + g_{\mu\nu} Q_{AB} \right) && (6.137)\\
[K_{\mu A}, Q_B] &= -i f_{AB} P_\mu && (6.138)\\
[K_{\mu A}, Q_{BC}] &= i \left(f_{BA} K_{\mu C} - f_{CA} K_{\mu B} \right) && (6.139)\\
[Q_A, Q_B] &= 0 && (6.140)\\
[Q_A, Q_{BC}] &= i \left(f_{AB} Q_C - f_{AC} Q_B \right) && (6.141)\\
[Q_{AB}, Q_{CD}] &= i \left(f_{AD} Q_{BC} + f_{BC} Q_{AD} - f_{AC} Q_{BD} - f_{BD} Q_{AC} \right) && (6.142)\\
[Q_i, L_{AB}] &= (T_i^T L + L T_i)_{AB} && (6.143)
\end{aligned}$$

The matrices T_i were defined in (6.131), the operators L_{AB} are defined as

$$L_{AB} = \left(\begin{array}{c|c} Q_{AB} & i Q_A \\ \hline -i Q_B & 0 \end{array} \right). \qquad (6.144)$$

The T_i form a representation of the Lie algebra of the internal generators Q_i. The Q_A are the shift symmetries, in a "galileon space" parametrised by the galileon-like fields θ_A. The matrices T_i are affine mappings, linear transformations with a translation added, on a vector space that is isomorphic to the galileon space. They define the adjoint of the action of Q_i on Q_A and Q_{AB} via (6.143); L_{AB} transforms as an anti-symmetric 2-tensor under the representation T_i.

It follows that the Lie algebra for the Poincaré algebra, enhanced by a set of broken generators Q_i and broken redundant generators $K_{\mu A}$, is completely determined by

- The internal algebra, generated by the Q_i
- The affine representation of the internal algebra T_i
- The symmetric 2-tensor f_{AD}, which is invariant under the representation T_i.

The Lie algebra reveals one more feature of the structure of the geometry: the commutators (6.140 - 6.142) mimic the form of the Poincaré algebra. In this analogy spacetime corresponds to the galileon space:

- The shifts in galileon space Q_A mimic the spacetime translations P_μ.
- The Q_{AB} imitate the Lorentz transformations $J_{\mu\nu}$.
- The tensor f_{AB} takes over the role of the metric $g_{\mu\nu}$.

The P_μ, $J_{\mu\nu}$, Q_A and Q_{AB} together generate the isometry group of the direct sum of Minkowski and galileon space, with a metric $g_{\mu\nu} \oplus f_{AB}$ [3]. The generator $K_{\mu A}$ acts as a rotation between the Minkowski and galileon space.
Given this general structure of multiflavour theories, with redundant symmetries, the next step is to investigate some special classes.

Generalised DBI theory

Under the assumption that f_{AB} is non-singular, it has an inverse f^{AB}; they function as a metric on the galileon space and can raise and lower indices. The internal generators Q_i can be redefined, like in section [6.3.1], to

$$\tilde{Q}_i \equiv Q_i + h_{Ai} f^{AB} Q_B. \tag{6.145}$$

[3] The probe brane construction [62] of DBI/Galileons starts from a 5D metric, which includes the field as a coordinate. The tensor f_{AB} is not necessarily non-singular however and the generators Q_{AB} may be zero or linearly dependent, so the algebraic construction carries over to a higher dimensional spacetime in an uneven way [D.5].

The old Q_i now splits into two sets, Q_A and $\tilde{Q}_i$, because $a^i_A\tilde{Q}_i = 0$. Since $Q_{AB} = \Xi^i_{AB}Q_i = \Xi^i_{AB}\tilde{Q}_i$, the rotations in galileon space are part of the second group. The new commutation relations are

$$\begin{aligned} [\tilde{Q}_i, K_{\mu A}] &= (t_i)^B_A K_{\mu B}, & [\tilde{Q}_i, Q_A] &= (t_i)^B_A Q_B, \\ [\tilde{Q}_i, Q_{AB}] &= (t_i)^C_A Q_{CB} + (t_i)^C_B Q_{AC}, & [\tilde{Q}_i, \tilde{Q}_j] &= \Lambda^k_{ij}\tilde{Q}_k. \end{aligned} \tag{6.146}$$

The Lie algebra of the internal symmetry has the structure of a semi-direct sum of the abelian subalgebra, generated by the Q_A, and the subalgebra generated by the $\tilde{Q}_i$. The $\tilde{Q}_i$ act on other generators through the representation t_i. This algebra is called the generalised DBI theory, because it extends the original algebra (6.18) to multiple flavours of the shift generator Q_A. The choice of the internal algebra $\tilde{Q}_i$, its representation t_i and the metric f_{AB} obviates the need to solve the Jacobi constraints.

The coset space of this algebra is parametrised by

$$U(x^\mu, \theta^A, \xi^{\mu A}, \theta^a) \equiv e^{ix^\mu P_\mu} e^{i\theta^A Q_A} e^{i\xi^{\mu A}K_{\mu A}} e^{i\theta^a \tilde{Q}_a}. \tag{6.147}$$

The $\tilde{Q}_a$ are those generators out of $\tilde{Q}_i$ that are spontaneously broken. The MC form, $\omega = -iU^{-1}dU$, can once again be decomposed on the basis of the generators of the Lie algebra. The significant components for the construction of the simplest action are ω^μ_P and ω^A_Q; the latter is required for the IHC, which removes the redundant degrees of freedom $\xi^{\mu A}$. The full MC form can be found in appendix [D.6]. Before writing out the forms, it is practical to introduce a shorthand notation:

$$\hat{ch}(x) \equiv \cosh(\sqrt{x}), \qquad \hat{sh}(x) \equiv \frac{\sinh(\sqrt{x})}{\sqrt{x}}, \tag{6.148}$$

$$\Pi^\nu_\mu \equiv f_{AB}\xi^A_\mu \xi^{\nu B}, \qquad \amalg^B_A \equiv f_{AC}\xi^{\mu C}\xi^B_\mu. \tag{6.149}$$

These functions are only defined via their series expansion, as before. The components of the MC form are

$$\omega^\mu_P = (\hat{ch}\Pi)^\mu_\nu dx^\nu - (\hat{sh}\Pi)^\mu_\nu f_{AB}\xi^{\nu B}d\theta^A, \tag{6.150}$$

$$\omega^A_Q = (e^{-i\theta^a t_a})^A_B \left[(\hat{ch}\amalg)^B_C d\theta^C - (\hat{sh}\amalg)^B_C \xi^C_\mu dx^\mu\right]. \tag{6.151}$$

Once again, the field transformations under the action of a generator T follow from working out $e^{i\alpha\cdot T}U$. In particular, when $T = P_\mu$ this generates a spacetime translation $x^\mu \to x^\mu + \alpha^\mu$, and if $T = Q_A$ it generates a shift in the field

$\theta_A \to \theta_A + \alpha_A$. The action of the other internal generators $\tilde{Q}_i$, with parameter ϵ_i, transforms the fields as

$$\begin{aligned} \theta_A &\to (e^{\epsilon^i t_i})_A^B \theta_B, \\ \xi_A^\mu &\to (e^{\epsilon^i t_i})_A^B \xi_B^\mu. \end{aligned} \tag{6.152}$$

The spacetime coordinates are invariant under this transformation. The change of the fields θ^a to θ'^a under the action of $\tilde{Q}_i$ is determined by the equation

$$e^{\epsilon^i \tilde{Q}_i} e^{i\theta^a \tilde{Q}_a} = e^{i\theta'^a \tilde{Q}_a}. \tag{6.153}$$

This is solved by the BCH formula, in principle, although the actual calculation is very involved; choosing a specific internal algebra may simplify this situation. The transformations induced by $K_{\mu A}$, with parameter $\beta_{\mu A}$, take the form:

$$\begin{aligned} x_\mu &\to (\hat{ch}\Pi_\beta)^\mu_\nu x^\nu + (\hat{sh}\Pi_\beta)^\mu_\nu f_{AB} \beta^{\nu B} \theta^A, \\ \xi_\mu^A &\to \xi_\mu^A + \beta_\mu^A + \mathcal{O}(\beta^2, \xi^2), \\ \theta^A &\to (\hat{ch}\text{II}_\beta)^A_B \theta^B + (\hat{sh}\text{II}_\beta)^A_B \beta_\mu^B x^\mu. \end{aligned} \tag{6.154}$$

The Π_β and II_β above are shorthand for

$$(\Pi_\beta)^\mu_\nu \equiv f_{AB} \beta_\mu^A \beta^{\nu B} \quad \& \quad (\text{II}_\beta)_A^B \equiv f_{AC} \beta^{\mu C} \beta_\mu^B. \tag{6.155}$$

Action

The redundant degrees of freedom, ξ_A^μ, are removed by the IHC $\omega_Q^A = 0$:

$$\partial_\mu \theta^A = \left(\frac{\hat{sh}\text{II}}{\hat{ch}\text{II}} \right)^A_B \xi^B. \tag{6.156}$$

The generalisation of the single-flavour DBI action can then be constructed via the volume measure

$$S_{DBI} = \int \epsilon_{\mu\nu\rho\sigma} \phi^* (\omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma) = \int \sqrt{-|G|} d^4x. \tag{6.157}$$

As before, the vielbein derives from the ω_P^μ.

$$e_\mu^\alpha = (\hat{ch}\Pi)_\mu^\alpha - (\hat{sh}\Pi)_\nu^\alpha f_{AB} \xi^{\nu B} \partial_\mu \theta^A \tag{6.158}$$

The induced metric $G_{\mu\nu}$ can then be written as a function of the physical fields θ^A, after the implementation of the IHC.

$$G_{\mu\nu} = g_{\alpha\beta} e_\mu^\alpha e_\nu^\beta = g_{\mu\nu} - f_{AB} \partial_\mu \theta^A \partial_\nu \theta^B \tag{6.159}$$

This is a direct generalisation of the single field metric (6.26). The action becomes

$$S_{DBI} = \int \sqrt{-|g_{\mu\nu} - f_{AB}\partial_\mu\theta^A\partial_\nu\theta^B|}d^4x. \tag{6.160}$$

This action was also found in [54]. The algebraic method allows one to extend beyond this one action, to a more general setting. Rather than assuming that only the Q_A are broken generators, the $\tilde{Q}_i$ (and so Q_{AB}) may be broken too. The additional NG modes θ^a don't appear in the metric, but through a minimal coupling

$$S_{DBI-like} = \int \mathcal{L}_{inv}(\theta_A, \theta_a)\sqrt{-|G|}d^4x. \tag{6.161}$$

The leading-order action for the extra-dimensional fields θ^a, which also couples them to the θ_A, is

$$S_{DBI-like} = \int c_{ab}\Omega^a_\mu\Omega^{\mu b}\sqrt{-|G|}d^4x. \tag{6.162}$$

The c_{ab} is a rank two invariant tensor that is invariant under the action of the unbroken subalgebra of $\tilde{Q}_i$, and Ω^i is the part of the MC form along the generators $\tilde{Q}_i$. By including more factors of Ω^i, and figuring out terms like $\omega_K^{\mu A}$ and $\omega_J^{\mu\nu}$, higher orders of the action may be constructed.

Generalized galileon theory

A different choice for the coefficients of the Lie algebra, yields a different class of theories.

$$h_{Ai} = 0, \qquad f_{AB} = 0, \qquad \Xi^i_{AB} = 0. \tag{6.163}$$

Given the first equation, the second one follows from Jacobi constraint (6.101). It ensures that the Q_i and $K_{\mu A}$ form a closed algebra, and therefore generate a purely internal symmetry. The last assumption simplifies matters, because it divides the algebras; the Q_i and $K_{\mu A}$ now separately form closed algebras[4]. The remaining, nontrivial, commutators then contain the last of the constraints.

$$\begin{aligned}
[P_\mu, K_{\nu A}] &= ig_{\mu\nu}Q_A \\
[\tilde{Q}_i, K_{\mu A}] &= (t_i)_A^B K_{\mu B} \\
[Q_i, Q_A] &= (t_i)_A^B Q_B
\end{aligned} \tag{6.164}$$

It follows that the generators Q_A form an abelian ideal of the internal symmetry group generated by Q_i. If the internal symmetry group is compact, the Q_A must

[4]This algebra also ensues after an İnönü-Wigner contraction using the substitutions $Q_A^{(0)} = \epsilon Q_A$, $Q_{AB}^{(0)} = \epsilon Q_{AB}$ and $K_{\mu A}^{(0)} = \epsilon K_{\mu A}$

necessarily belong to the centre of the algebra. The NG bosons with enhanced soft limits must therefore derive from the U(1) phase factors from the symmetry group.
However, for non-compact groups the enhanced NG bosons are not due to such factors. For instance, the internal Lie group may be ISO(n) [61], a non-semisimple group that is isomorphic to SO(n)$\ltimes\mathbb{R}^n$. The SO(n) rotations remain unbroken, and the breaking of the n internal translations produces the NG modes. The translations transform under the rotations like a vector.
More generally, if the internal symmetry algebra is the semidirect sum of $\tilde{Q}_i$ and Q_A matters simplify considerably [5]. Let the algebra $\mathfrak{q}$ of group $\mathbb{Q}$, consisting of the generators $\tilde{Q}_i$, have a real representation R in n dimensions. The abelian algebra of translations on the n-dimensional space $\mathbb{R}^n$ is denoted by $\mathfrak{t}^n$, and generated by the Q_A. The internal group is then $\mathbb{Q}\ltimes\mathbb{R}^n$, and the algebra of Q_i is the semidirect sum of $\mathfrak{q}$ and $\mathfrak{t}^n$.
The multigalileon models in the literature are of this type and are typically constructed by breaking only the Q_A and leaving the $\tilde{Q}_i$ unbroken. For example the $\tilde{Q}_i$ generate SO(n) or SU(n), and the Q_A defines their adjoint or fundamental representation [65–67]. Starting from the Lie algebra of the internal group $\mathbb{Q}\ltimes\mathbb{R}^n$ a multi-flavour model of NG bosons may be constructed, whatever the broken generators in $\mathfrak{q}$ are.

The coset space for the generalized galileon theory is parametrised by

$$U(x^\mu,\theta^A,\xi^{\mu A},\theta^a)\equiv e^{ix^\mu P_\mu}e^{i\theta^A Q_A}e^{i\xi^{\mu A}K_{\mu A}}e^{i\theta^a\tilde{Q}_a}. \tag{6.165}$$

The non-trivial components of the MC form are:

$$\begin{aligned}
\omega_P^\mu &= dx^\mu, \\
\omega_{K\mu}^A &= (e^{-i\theta^a t_a})^A_B d\xi^B_\mu, \\
\omega_Q^A &= (e^{-i\theta^a t_a})^A_B (d\theta^B-\xi^B_\mu dx^\mu).
\end{aligned} \tag{6.166}$$

The MC form for the components $\tilde{Q}_i$ is Ω, as in (6.112). The transformations of the fields under the action of Q_A and $K_{\mu A}$, parametrised by α^A, and β^A_μ respectively, are

$$\theta^A\to\theta^A+\alpha^A+\beta^A_\mu x_\mu \quad \text{and} \quad \xi^A_\mu\to\xi^A_\mu+\beta^A_\mu. \tag{6.167}$$

These symmetries don't affect the fields θ_a, because of the parametrisation. The transformation induced by $\tilde{Q}_i$ transforms the θ_a fields as (2.45). The symmetry

[5] This internal algebra needn't necessarily be a semidirect sum of two algebras, a simple counterexample is the Heisenberg algebra. The central charges of the central extension of a Lie algebra are naturally part of the abelian Q_A, but don't satisfy the condition either. Thanks to Qiaochu Yuan & Torsten Schoeneberg for pointing out these examples.

transformation $e^{i\epsilon^i\tilde{Q}_i}$ does transform the fields ξ^μ_A and θ_A linearly, according to the representation R.

$$\theta^A \to \left(e^{i\epsilon^i t_i}\right)^A_B \theta^B, \qquad \xi^A_\mu \to \left(e^{i\epsilon^i t_i}\right)^A_B \xi^A_\mu \tag{6.168}$$

None of these symmetry transformations affect the spacetime coordinate x^μ. In summary, the Ω only transforms non-trivially under the action of the $\tilde{Q}_i$ and the fields θ^A and ξ^A_μ transform linearly under the entire internal algebra $\mathfrak{q}$, but like n independent galileon copies under the Q_A and $K_{\mu A}$ transformations. These transformation rules together imply that the galileon fields decouple from the other NG modes, and that the galileon actions will take a familiar form from earlier (6.41).

The IHC, $\omega^A_Q = 0$, eliminates the redundant degrees of freedom.

$$\xi^{\mu A} = \partial^\mu \theta^A \tag{6.169}$$

The remaining forms, ω^μ_p, $\omega^A_{K\mu}$ and Ω, are the building blocks for the strictly invariant actions. The first, $\omega^\mu_P = dx^\mu$, is trivial and, after implementation of the IHC and a pullback, the second is $\omega^A_{K\mu} = (e^{-i\theta_a t^a})^A_B \partial_\mu\partial_\nu\theta^B dx^\nu = \omega^A_{K\mu\nu}dx^\nu$. This means that, like in the single-flavour case, it isn't possible to directly construct a kinetic term for the galileons from these forms. The third set of building blocks, contained in Ω, can be used to construct invariant actions in a standard manner, for details see [68]. Interaction terms may also be constructed from the $\omega^A_{K\mu\nu}$, by multiplying several of them together and contracting their indices with invariant tensors of the unbroken subgroup. A simple example is $\delta_{AB}\omega^A_{K\mu\nu}\omega^{B\mu\nu}_K$; in this contraction the phase factors $(e^{i\theta_a t^a})^A_B$ fall away, so these terms don't induce any interactions between the galileons and the other NG fields. However, such interactions between the two sectors are possible if the (t_i) can be reduced with respect to the unbroken subgroup of $\mathfrak{q}$. The example invariant could take the form $c_{AB}\omega^A_{K\mu\nu}\omega^{B\mu\nu}_K$, where c_{AB} is a symmetric invariant tensor under the unbroken subalgebra of $\mathfrak{q}$. Other interaction terms between the galileon and the ordinary NG sector can easily be constructed by multiplying invariant terms from different sectors together.

WZ-terms

The method of construction for the WZ terms is the same as before, and naturally extends the single field case. The need to contract over the internal indices, that also occur on the Q_A, makes the 5-forms a natural starting point.

$$\begin{pmatrix} \omega_5^1 \\ \omega_5^2 \\ \omega_5^3 \\ \omega_5^4 \\ \omega_5^5 \end{pmatrix} \equiv \epsilon_{\mu\nu\rho\sigma}\omega_Q^A \wedge \begin{pmatrix} c_A\omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma \\ c_{AB}\omega_K^{B\mu} \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma \\ c_{ABC}\omega_K^{B\mu} \wedge \omega_K^{C\nu} \wedge \omega_P^\rho \wedge \omega_P^\sigma \\ c_{ABCD}\omega_K^{B\mu} \wedge \omega_K^{C\nu} \wedge \omega_K^{D\rho} \wedge \omega_P^\sigma \\ c_{ABCDE}\omega_K^{B\mu} \wedge \omega_K^{C\nu} \wedge \omega_K^{D\rho} \wedge \omega_K^{E\sigma} \end{pmatrix} \tag{6.170}$$

The Maurer-Cartan structure equations for the broken galileon forms are:

$$\begin{aligned} d\omega_P^\mu &= \omega_J^{\mu\nu} \wedge \omega_P^\lambda = 0, \\ d\omega_{K\mu}^A &= -i(t_i)^A_B \Omega^i \wedge \omega_{K\mu}^B, \\ d\omega_Q^A &= \omega_P^\mu \wedge \omega_{K\mu}^A - i(t_i)^A_B \Omega^i \wedge \omega_Q^B. \end{aligned} \tag{6.171}$$

The 5-forms ω_5 must be invariant under the transformations of the unbroken subgroup of the algebra. The Ω that arises in the derivative depends on the specific choice of the internal algebra, but the first term of its Taylor expansion, $\Omega = d\theta^a \tilde{Q}_a + \mathcal{O}(\theta^2)$, is linear in the gradient of the broken fields. If the ω_5 is closed, then it must also be invariant under the transformations under the broken part of $\mathfrak{q}$, defined by the t_i in the representation R. The invariance and closedness together mean that the constants $c_{A_1 \ldots A_n}$ are invariant under the whole algebra $\mathfrak{q}$ in the representation R.
An explicit calculation shows that these 5-forms are part of the cohomology group. It will be convenient to define a new set of variables:

$$\tilde{\theta}^A \equiv \left(e^{i\epsilon^i t_i}\right)^A_B \theta^B, \qquad \tilde{\xi}^A_\mu \equiv \left(e^{i\epsilon^i t_i}\right)^A_B \xi^B_\mu. \tag{6.172}$$

The integration of the 5-forms can then be written as:

$$\begin{aligned} \mathfrak{g}_1 &= \epsilon_{\mu\nu\rho\sigma} c_A \tilde{\theta}^A dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \\ \mathfrak{g}_2 &= \epsilon_{\mu\nu\rho\sigma} c_{AB} \left(\tilde{\theta}^A \omega_K^{B\mu} \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma + \frac{1}{8}\tilde{\xi}^A \cdot \tilde{\xi}^B dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\right), \\ \mathfrak{g}_3 &= \epsilon_{\mu\nu\rho\sigma} c_{ABC} \left(\tilde{\theta}^A \omega_K^{B\mu} \wedge \omega_K^{C\nu} \wedge dx^\rho \wedge dx^\sigma + \frac{1}{3}\tilde{\xi}^A \cdot \tilde{\xi}^B \omega_K^{C\mu} \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\right), \\ \mathfrak{g}_4 &= \epsilon_{\mu\nu\rho\sigma} c_{ABCD} \left(\tilde{\theta}^A \omega_K^{B\mu} \wedge \omega_K^{C\nu} \wedge \omega_K^{D\rho} \wedge dx^\sigma + \frac{3}{4}\tilde{\xi}^A \cdot \tilde{\xi}^B \omega_K^{C\mu} \wedge \omega_K^{D\nu} \wedge dx^\rho \wedge dx^\sigma\right), \\ \mathfrak{g}_5 &= \epsilon_{\mu\nu\rho\sigma} c_{ABCDE} \left(\tilde{\theta}^A \omega_K^{B\mu} \wedge \omega_K^{C\nu} \wedge \omega_K^{D\rho} \wedge \omega_K^{E\sigma} + 2\tilde{\xi}^A \cdot \tilde{\xi}^B \omega_K^{C\mu} \wedge \omega_K^{D\nu} \wedge \omega_K^{E\rho} \wedge dx^\sigma\right). \end{aligned} \tag{6.173}$$

Note that invariance requires that the coefficients $c_{AB...}$ are fully symmetric under the exchange of indices [6]. Because the $c_{AB...}$ are invariant under the entire representation R of the algebra $\mathfrak{q}$ the factors $e^{-i\theta_a t^a}$ drop out of these expressions; they no longer depend on θ_a at all. After the implementation of the IHC, and a pullback to the Minkowski spacetime manifold, the multi-galileon langrangeans can be extracted from the forms $\mathfrak{g}_k$. After some partial integrations, these lagrangeans may be written in a shorthand notation:

$$\mathcal{L}_k^{WZ} = c_{A_1...A_k}\theta^{A_1}G_{k-1}^{A_2...A_k}. \tag{6.174}$$

The $G^{A_1...A_n}$ are the anti-symmetric products of the second derivatives of θ^A.

$$\begin{aligned} G_k^{A_1...A_k} \equiv& \frac{1}{(4-k)!}\epsilon_{\alpha_1...\alpha_k\mu_{k+1}...\mu_4}\epsilon^{\beta_1...\beta_k\mu_{k+1}...\mu_4} & (6.175)\\ &\times \left(\partial_{\beta_1}\partial^{\alpha_1}\theta^{A_1}\right)...\left(\partial_{\beta_k}\partial^{\alpha_k}\theta^{A_k}\right) & \\ G_0 \equiv& 1 & (6.176)\end{aligned}$$

This is the general form of the multi-galileon lagrangeans, for any internal algebra $\mathfrak{q}$ and its real finite dimensional representation R.

These multi-galileons were all constructed in analogy to the single field galileon model, however the distinction between fields introduced by the internal indices allows for a broader set of 5-form candidates.
Rather than using the Levi-Civita symbol to contract over the spacetime indices the other invariant indices, $g_{\mu\nu}$ and $g_{\mu\nu}g_{\rho\sigma}$ may be used. Now the Lorentz invariant tensor need no longer be antisymmetric, the other invariant tensors $c_{A_1...A_n}$ aren't necessarily symmetric either.

$$\begin{aligned} &c_{ABCD}\omega_Q^A\wedge\omega_Q^B\wedge\omega_Q^C\wedge\omega_{K\mu}^D\wedge\omega_P^\mu, \qquad && c_{ABCD}\omega_Q^A\wedge\omega_{K\mu}^B\wedge\omega_{K\nu}^C\wedge\omega_K^{D\mu}\wedge\omega_P^\nu, \qquad (6.177)\\ &c_{ABC}\omega_Q^A\wedge\omega_{K\mu}^B\wedge\omega_{K\nu}^C\wedge\omega_P^\mu\wedge\omega_P^\nu, && c_{ABCDE}\omega_Q^A\wedge\omega_{K\mu}^B\wedge\omega_{K\nu}^C\wedge\omega_K^{D\mu}\wedge\omega_K^{E\nu}.\end{aligned}$$

There are many more possibilities; it isn't practical to work them all out. However, the examples on the top line in (6.177) don't lead to new galileon lagrangeans. The integration of those 5-forms shows that these actions are total derivatives and don't affect the scattering amplitudes.

Multi-galileon WZ terms will exist for most common cases, because δ_{AB} is an invariant tensor of any representation of a compact Lie algebra. The general case is more complex, and will depend on the specific structure of the Lie algebra.
The non-galileon sector contains infinitely many invariant actions, only dependent on the variables θ_a, constructed from the components of the MC form Ω.

[6] The constants in front of the $\tilde{\xi}^A\cdot\tilde{\xi}^B$ in $\mathfrak{g}_k$ are consistent with the expression for these in D dimensions: $\frac{k-1}{2(D-k+2)}$. See for example [69]

The actions in this sector may of course also be topological in nature [70, 71]. The interactions between the galileon and non-galileon sector of the theory can be constructed in the higher orders of the derivative expansion; these interaction terms may consist of invariants made from $\omega^A_{K\mu}$, and their products with invariants built from the components of Ω. Finally, it is also possible to construct WZ terms that mix the θ_A and θ_a, using both the forms ω^A_Q and Ω, as shown in [72] for non-relativistic theories; doing so places an additional constraint on the algebra. The established WZ terms, however, work without such constraints and yield multi-galileon actions for an infinite class of Lie algebras.

Twisted galileon models

In the general galileon model Ξ^i_{AB}, h_{Ai} and f_{AB} vanished. Removing the first restriction allows the Q_{AB} to exist, and a new model emerges.

$$\begin{aligned}
[P_\mu, K_{\nu A}] &= i g_{\mu\nu} Q_A \\
[K_{\mu A}, K_{\nu B}] &= i g_{\mu\nu} Q_{AB} \\
[Q_i, K_{\mu A}] &= (t_i)^B_A K_{\mu B} \\
[Q_i, Q_A] &= (t_i)^B_A Q_B \\
[Q_i, Q_{AB}] &= (t_i)^C_A Q_{CB} + (t_i)^C_B Q_{AC}
\end{aligned} \tag{6.178}$$

The algebra now contains the additional, 'twisted' commutator $[K_{\mu A}, K_{\nu B}]$. The other commutators remain zero: the Q_A and Q_{AB} still commute with $K_{\mu A}$ and each other. The coset element is parametrised by

$$U(x^\mu, \theta^A, \theta^{AB}, \xi^{\mu A}, \theta^a) \equiv e^{ix^\mu P_\mu} e^{i\theta^A Q_A} e^{\frac{i}{2}\theta^{AB} Q_{AB}} e^{i\xi^{\mu A} K_{\mu A}} e^{i\theta^a \tilde{Q}_a}. \tag{6.179}$$

The MC form is similar to that of the generalised galileons, but gains an additional component along the new generator Q_{AB}.

$$\begin{aligned}
\omega^\mu_P &= dx^\mu \\
\omega^A_{K\mu} &= (e^{-i\theta^a t_a})^A_B d\xi^B_\mu \\
\omega^A_Q &= (e^{-i\theta^a t_a})^A_B (d\theta^B - \xi^B_\mu dx^\mu) \\
\omega^{AB}_Q &= (e^{-i\theta^a t_a})^A_C (e^{-i\theta^a t_a})^B_D \left[d\theta^{CD} + \frac{1}{2}(\xi^C_\mu d\xi^{\mu D} - \xi^D_\mu d\xi^{\mu C}) \right]
\end{aligned} \tag{6.180}$$

The last 1-form introduces a coupling between the galileon fields θ^A and the normal NG modes θ^{AB}; the investigation of the symmetry transformation will show that the two also mix under the action of $K_{\mu A}$, through $\xi^{\mu A}$. The action of the shift generators on the coset element, $e^{i\alpha^A Q_A} U$ or $e^{i\alpha_\mu P_\mu} U$, only induces one transformation of the fields each: $\theta^A \rightarrow \theta^A + \alpha^A$ and $x^\mu \rightarrow x^\mu + \alpha^\mu$ respectively. In the same vein, the internal rotation generators Q_{AB} only generate the transformation $\theta_{AB} \rightarrow \theta_{AB} + \alpha_{AB}$. The remainder of the internal generators $\tilde{Q}_i$ generates the following transformations, parametrised by ϵ^i on the internal space:

$$\begin{aligned} \theta_A &\rightarrow (e^{\epsilon^i t_i})_A^B \theta_B, \\ \xi_A^\mu &\rightarrow (e^{\epsilon^i t_i})_A^B \xi_B^\mu, \\ \theta_{AB} &\rightarrow \left(e^{i\epsilon^i E_i}\right)_{AB}^{CD} \theta_{CD}. \end{aligned} \tag{6.181}$$

The new tensor in the last transformation is defined by $(E_i)_{AB}^{CD} = (t_i)_A^C \delta_B^D - (t_i)_B^C \delta_A^D$. The transformation of the fields θ_a depends on the structure of the internal algebra. The coordinate x^μ remains unchanged. Finally, the transformations induced by $K_{\mu A}$, parametrised by β_μ^A, are:

$$\begin{aligned} \theta_A &\rightarrow \theta^A + \beta_\mu^A x^\mu, \\ \xi_\mu^A &\rightarrow \xi_\mu^A + \beta_\mu^A, \\ \theta_{AB} &\rightarrow \theta_{AB} + \frac{\xi^{\mu A}\beta_\mu^B - \xi^{\mu B}\beta_\mu^A}{2}. \end{aligned} \tag{6.182}$$

The implementation of the inverse Higgs constraint, $\xi_\mu^B = \partial_\mu \theta^B$, makes $\omega_{K\mu}$ depend on two derivatives per field θ_A, as before. The new form ω_Q^{AB}, however, will have fewer than two derivatives per θ on average; a lagrangean constructed using this form inherits that property.

The simplest example of an algebra with a twisted commutator contains an internal symmetry group, isomorphic to SO(2)$\ltimes\mathbb{R}^2$. The Q_1 and Q_2 generate the translations, and the generator of rotations is Q_{12}. The kinetic and interaction terms for the galileons are simply the WZ terms. The kinetic term for the fields θ_{AB} is

$$\mathcal{L}_{twist} = \frac{1}{2}\left[\partial_\mu \theta^{12} + \frac{1}{2}(\partial^\nu \theta^1 \partial_\mu \partial_\nu \theta^2 - \partial^\nu \theta^2 \partial_\mu \partial_\nu \theta^1)\right]^2. \tag{6.183}$$

This terms also introduces interactions between the galileon fields and θ_{12}. Although θ_1 and θ_2 have a galilean shift symmetry, the computation of the amplitude using the computer code, described in the method section [6.1], shows that the kinetic term due to the twisted field spoils the enhanced soft limit of the other fields. This is because the action (6.183) brings in cubic interactions, which lead to collinear singularities in the soft limit [3, 73]. These interactions derive from the contents of form ω_Q^{AB} from (6.180), and can therefore be expected for any internal Lie algebra in the twisted galileon model.

6.4 Summary and conclusion

The overall result is a classification of effective field theories of scalars with enhanced soft limits in Minkowski spacetime, based on the Lie algebra. This classification is based on the extension of the broken symmetry, that creates the NG bosons, with additional broken and redundant generators.

In the instance of a single, physical, NG field a full categorisation of the models with enhanced soft limits was produced. The categorisation contained no new models, but offers a unified perspective on the DBI and galileon theories. The inclusion of additional redundant generators showed that the special galileon theory was unique in realising a doubly enhanced soft limit.
The first extension couples multiple NG bosons to a single mode with an enhanced soft limit, it leads to a trivial coupling between the enhanced and normal sectors of the theory.

In the most general case, multi-flavour models involving multiple redundant generators, it was shown that a useful structure resides inside the general Lie algebra. This structure is a set of commutators, and its existence in the algebra is equivalent to solving the Jacobi constraints. These commutators are fully defined by the internal algebra, its affine representation and an invariant 2-tensor of that representation. Using this structure, two classes were found that form a generalisation of the DBI and galileon multi-flavour theories. It seems that these classes are unique in consisting solely of NG bosons with enhanced soft limits, however, there are still possibilities that fell outside this cataloguing effort.
For all that, an infinite catalogue of theories, that have an enhanced scaling of the scattering amplitude in the soft limit, may be constructed using only some mild constraints on the Lie algebra.

PART IV
APPENDIX

A Conventions

The equations are written in natural units. In the context of quantum field theory this means that the system of units is specifically defined, such that the reduced Plank constant $\hbar$ and the speed of light c take value one.

$$\hbar = c = 1 \tag{184}$$

The Minkowski metric $g_{\mu\nu}$ is defined differently in particle physics and general relativity; multiplying the chosen metric by minus one, amounts to switching between these conventions. Here the particle physics convention will be assumed, which has a positive proper time interval, so

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{185}$$

In the case of spacetime indices, the Einstein summation is used. This means that a repeated index in a term implies that this index is summed over all its values. So, for instance

$$A_\mu B^\mu = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3. \tag{186}$$

The final convention fixes the sign in the exponent of the Fourier transform and the scaling of its momentum space measure by $1/(2\pi)^4$, so

$$f(x) = \int_{-\infty}^{\infty} f(p) e^{-ipx} \frac{d^4p}{(2\pi)^4}, \qquad f(p) = \int_{-\infty}^{\infty} f(x) e^{ipx} d^4x. \tag{187}$$

B Symmetry

B.1 Lie algebra closure

The transformation $\hat{U}(t) = e^{tX}$ transforms point p' to p'' (2.28). The transformations $\hat{U}(x)$, for values $0 < x < t$, also map p' to some point; the $\hat{U}(x)p'$ can be thought of as a path departing from point p'. Closure under group multiplication means that the combination of different paths, should equal some direct path:

$$e^{\alpha^a X_a} e^{\beta^b X_b} p' = e^{\gamma^c X_c} p' \qquad \rightarrow \qquad \gamma^c X_c = \ln(e^{\alpha^a X_a} e^{\beta^b X_b}). \tag{188}$$

Write $A = \alpha^a X_a,\ B = \beta^b X_b$ and $C = \gamma^c X_c$. Take the series expansion , careful of the operator ordering, to the second order:

$$\begin{aligned} C &= \ln(1 + A + B + AB + \frac{A^2 + B^2}{2} + ...) \\ &= (A + B + AB + \frac{A^2 + B^2}{2} + ...) - \frac{1}{2}(A + B + ...)^2 + ... \\ &= A + B + \frac{1}{2}(AB - BA) + \mathcal{O}(X^3). \end{aligned} \tag{189}$$

Since C is a sum of generators, the commutator $AB - BA$ must reduce to a linear combination of generators. This implies that

$$[X_a, X_b] = f^c_{ab} X_c. \tag{190}$$

In fact, this condition suffices to reduce all higher orders of X as well [74].

B.2 Basis transformation

The transformation of the basis, $Y_b = M^a_b X_a$, means that

$$[Y_u, Y_v] = a^w_{uv} Y_w \qquad \rightarrow \qquad [X_a, X_b] = (M^{-1})^u_a (M^{-1})^v_b a^w_{uv} M^c_w X_c. \tag{191}$$

The new structure constant, $f^c_{ab} = (M^{-1})^u_a (M^{-1})^v_b a^w_{uv} M^c_w$ inherits its anti-symmetry from a^w_{uv}. The change of basis changes the form of the structure constants, but the Jacobi identity is not affected; that identity holds for any triple of elements in the Lie algebra.

$$\begin{aligned} f^e_{ad} f^d_{bc} + f^e_{bd} f^d_{ca} + f^e_{cd} f^d_{ab} &= (M^{-1})^u_a (M^{-1})^x_b (M^{-1})^y_c M^e_w \left(a^w_{uv} a^v_{xy} + a^w_{xv} a^v_{yu} + a^w_{yv} a^v_{ux} \right) \\ &= 0 \end{aligned} \tag{192}$$

B.3 The matrix exponent formulas

Hadamard's lemma to the Baker-Campbell-Hausdorff formula states that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + ... \tag{193}$$

Using the notation $Ad_A B = [A, B]$, this can be written as a series

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{(Ad_A)^n}{n!} B = e^{Ad_A} B. \tag{194}$$

For calculating the MC forms, when the generators in A don't commute, the following formula is useful:

$$e^A d e^{-A} = -\int_0^1 dt \left[e^{At} dA e^{-At} \right]. \tag{195}$$

C Effective field theory

C.1 Supercondensed notation

Commuting derivatives, with indices, are only distinguished by their contraction with different tensors. Supercondensed notation reduces indistinguishable derivatives to their number.

$$\phi_{,\mu_1 \dots \mu_n} \quad \xrightarrow{S} \quad \phi_{,n} = \partial_n \phi \tag{196}$$

The contraction between indices, assuming the Einstein convention, is

$$A^{\mu_1 \dots \mu_m} B^{\nu_1 \dots \nu_n} \partial_{\mu_1 \dots \mu_m \nu_1 \dots \nu_n} \phi \quad \xrightarrow{S} \quad A^m B^n \phi_{,m+n}. \tag{197}$$

The use of this notation is indicated by the tag (S)

C.2 DeWitt condensed notation

The action S is a functional. This is a function of functions; it associates a number to a set of functions.

$$S[\varphi_1, \dots \varphi_n] = \int_\Omega \mathcal{L}(\varphi_1, \dots \varphi_n) d^4x \in \mathbb{R} \tag{198}$$

The lagrangean $\mathcal{L}$ is a function of the fields φ_i and their spacetime derivatives. The functional derivative is defined as

$$\frac{\delta F[\varphi(x)]}{\delta \varphi(x')} = \lim_{\epsilon \to 0} \frac{F[\varphi(x) + \epsilon \delta(x - x')] - F[\varphi(x)]}{\epsilon}. \tag{199}$$

This is equivalent to taking the variation, and generates boundary terms too. Take the example of a simple kinetic term.

$$S[\varphi] \quad = \int_\Omega \frac{\partial_\mu \varphi \partial^\mu \varphi}{2} d^4x \tag{200}$$

$$\frac{\delta S[\varphi(x)]}{\delta \varphi(x')} = \int_{\partial\Omega} \delta(x - x') \partial_\mu \varphi \, d\Sigma^\mu - \Box' \varphi = 0 \tag{201}$$

The boundary term exists only on $\partial\Omega$, and vanishes naturally if the integrand is exactly zero. Otherwise, it is assumed that the function's value is already imposed at the boundary and not subject to a variation.
The indices in DeWitt condensed notation concurrently apply to spacetime and fieldspace. It is an effective script for functional analysis, that strongly pares back the notation for derivatives.

$$\frac{\delta}{\delta \varphi^j(x')} \varphi^i(x) = \delta^i_j \delta(x - x') \quad \xrightarrow{D} \quad \frac{\delta}{\delta \varphi^j} \varphi^i = \varphi^i{}_{,j} = \delta^i_j \tag{202}$$

The functional and spacetime derivative are distinguished by the use of latin or greek indices, respectively. Similarly, a repetition of indices sums over both the internal indices and spacetime.

$$A^i_j B^j = \sum_j \int_\Omega A^i_j(x,y) B^j(y) d^4y \tag{203}$$

The use of this notation is indicated by the tag (D). The supercondensed DeWitt notation (SD) reduces a list of indices/derivatives to their number.

$$\frac{\delta^2 F}{\delta\varphi^i(x)\delta\varphi^j(y)} \quad \xrightarrow{D} \quad F_{,ij} \quad \xrightarrow{SD} \quad F_{,2} \tag{204}$$

The Legendre transform introduces dual functionals, $F[\varphi]$ and $V[Y]$, and functions φ_i and Y_i that have the same indices. For convenience, in condensed notation the derivatives are taken with respect to the argument of the functional.

$$F_{,i} = \frac{\delta F}{\delta \varphi^i} \qquad\qquad V_{,i} = \frac{\delta V}{\delta Y^i} \tag{205}$$

D Theories with enhanced soft limits

D.1 Levi-Civita identities

The contraction of two Levi-Civita symbols, down to four free indices, can be written compactly as:

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} &= -4!, \\ \epsilon^{\alpha\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} &= -3!\ \delta^\alpha_\mu, \\ \epsilon^{\alpha\beta\rho\sigma}\epsilon_{\mu\nu\rho\sigma} &= -2!\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right). \end{aligned} \tag{206}$$

Let Y^μ be a 1-form and T^α_β a traceless 1-form. The identities for the rearrangement of forms are:

$$\begin{aligned} Y^\mu \wedge Y^\nu \wedge Y^\rho \wedge Y^\sigma &= \epsilon^{\mu\nu\rho\sigma} Y^1 \wedge Y^2 \wedge Y^3 \wedge Y^4, \\ Y^\mu \wedge Y^\nu \wedge Y^\rho &= -\frac{\epsilon^{\mu\nu\rho\lambda}\epsilon_{\alpha\beta\gamma\lambda}}{6} Y^\alpha \wedge Y^\beta \wedge Y^\gamma, \\ Y^\mu \wedge Y^\nu &= -\frac{\epsilon^{\mu\nu\kappa\lambda}\epsilon_{\alpha\beta\kappa\lambda}}{4} Y^\alpha \wedge Y^\beta, \\ T^\beta_\alpha \wedge Y^\alpha &= \frac{\epsilon^{\alpha\beta\kappa\lambda}\epsilon_{\mu\nu\kappa\lambda}}{2} T^\mu_\alpha \wedge Y^\nu. \end{aligned} \tag{207}$$

The determinant of a 4×4 matrix M can be found using the Levi-Civita symbol:

$$\det(M) = \frac{\epsilon^{\mu\nu\rho\sigma} M^\alpha_\mu M^\beta_\nu M^\gamma_\rho M^\delta_\sigma \epsilon_{\alpha\beta\gamma\delta}}{-4!}. \tag{208}$$

D.2 Irrelevant solutions in the case of one redundant generator

The generator K_μ is not redundant and broken, when $a = 0$. This means its NG fields can't fall away in the final theory, because there is no IHC to implement. The introduction of such fields is therefore beside the point, when studying the single field case, and the following is only included for completeness.
The Jacobi identity imposes the following constraints:

$$a = b = c = f = g = 0. \tag{209}$$

The remaining coefficients (d, e, h and i) can take any arbitrary value. The commutators that contain them can be written in matrix form:

$$\left[\begin{pmatrix} P_\mu \\ K_\mu \end{pmatrix}, Q\right] = i\begin{pmatrix} d & e \\ h & i \end{pmatrix}\begin{pmatrix} P_\mu \\ K_\mu \end{pmatrix}. \tag{210}$$

The transformation of the basis of P_μ and K_μ by some matrix A, results in a similarity transformation of the matrix of coefficients. The invariance of the determinant of a matrix under such a transformation means its characteristic equation and eigenvalues are invariant too. A 2×2 matrix can therefore not generally be reduced beyond dependence on 2 variables.

Theorem: A similarity transform can bring any real 2×2 matrix M into the form

$$M \to A^{-1}MA = \begin{pmatrix} \kappa & \lambda \\ s\lambda & \kappa \end{pmatrix} \tag{211}$$

Where $\kappa = \frac{\mathrm{tr}M}{2}$, λ is a positive real number and $s = sgn\left[(\mathrm{tr}M)^2 - 4\det(M)\right]$. The parameter s takes the values 1, 0 and -1.

- $s = 1$: The matrix M has the eigenvalues $\kappa \pm \lambda$.
- $s = 0$: The matrix M has one eigenvalue κ: λ can be transformed away.
- $s = -1$: The matrix M has the eigenvalues $\kappa \pm i\lambda$.

A straightforward calculation proves these statements and is omitted for that reason.

The new form of the Lie algebra is

$$\left[\begin{pmatrix} P_\mu \\ K_\mu \end{pmatrix}, Q\right] = i\begin{pmatrix} \kappa & \lambda \\ s\lambda & \kappa \end{pmatrix}\begin{pmatrix} P_\mu \\ K_\mu \end{pmatrix}. \tag{212}$$

One similarity transform of the basis $B = (P_\mu, K_\mu)$ is $e^{-i\epsilon Q}Be^{-i\epsilon Q} \approx B + i\epsilon[B,Q] + \mathcal{O}(\epsilon^2)$. In this context the commutator of Q and B can be interpreted as an infinitesimal change of the basis, consisting of a scaling by a factor $1+\kappa$ and a rotation through an angle λ. The angle λ may be hyperbolic if $s=-1$ rather than $s=1$, and degenerate into a shearing if $s=0$. These transformation properties are better expressed in terms of the fields. The coset space is parametrised by

$$U(x,\xi,\theta) = e^{ix_\mu P^\mu} e^{i\xi_\mu K^\mu} e^{i\theta Q}. \tag{213}$$

The transformation of the fields under the action of Q (2.45) takes an exact form, for any size of the parameter α:

$$\begin{aligned} x_\mu &\to e^{\alpha\kappa}\left[\cosh(\alpha\sqrt{s}\lambda)x_\mu + \sqrt{s}\sinh(\alpha\sqrt{s}\lambda)\xi_\mu\right], \\ \xi_\mu &\to e^{\alpha\kappa}\left[\frac{1}{\sqrt{s}}\sinh(\alpha\sqrt{s}\lambda)x_\mu + \cosh(\alpha\sqrt{s}\lambda)\xi_\mu\right], \\ \theta &\to \theta + \alpha. \end{aligned} \tag{214}$$

The first two transformations confirm the transformation picture that was indicated by the commutation relations. The θ simply shifts, as would be expected from the NG field.
The K_μ commutes with all other broken generators, except Q. The algebra indicates that the action of K_μ shears the basis of (Q, P_μ, K_μ) in the direction of the latter two. Calculating the transformation of the coordinates and fields under K_μ, parametrised by β^μ yields:

$$\begin{aligned} x_\mu &\to x_\mu, \\ \xi_\mu &\to \xi_\mu + \beta_\mu, \\ \theta &\to \theta. \end{aligned} \tag{215}$$

D.3 Spin-zero multiplet of redundant generators

Add additional generators Q, K_μ and X to the Poincaré algebra. The new commutator with the generator of Lorentz transformations is

$$[J_{\mu\nu}, X] = 0. \tag{216}$$

The most general form of the remaining bracket structure, constrained by the Lorentz invariance, is:

$$\begin{aligned}
[P_\mu, K_\nu] &= i(a g_{\mu\nu} Q + b J_{\mu\nu} + c\epsilon_{\mu\nu\rho\sigma} J_{\rho\sigma} + j g_{\mu\nu} X), \\
[P_\mu, Q] &= i\left(d P_\mu + e K_\mu\right), \\
[K_\mu, K_\nu] &= i\left(f J_{\mu\nu} + g\epsilon_{\mu\nu\kappa\lambda} J^{\kappa\lambda}\right), \\
[K_\mu, Q] &= i(h P_\mu + i K_\mu), \\
[P_\mu, X] &= i(k P_\mu + l K_\mu), \\
[K_\mu, X] &= i(m P_\mu + n K_\mu), \\
[Q, X] &= i(o Q + p X).
\end{aligned} \tag{217}$$

In order that the fields associated to X and Q are redundant, they must be eliminated via the IHC. The IHC exists when $a \neq 0$ and $l \neq 0$. The structure becomes a Lie algebra when the brackets satisfy the Jacobi identity. This requirement places the following constraints on the parameters:

$$\begin{aligned}
b &= 0, & c &= 0, & d &= \frac{ek}{l}, \\
f &= 0, & g &= 0, & h &= \frac{em}{l}, \\
i &= \frac{en}{l}, & j &= -\frac{ae}{l}, & o &= k+n, \\
& & p &= -\frac{e}{l}\left(k+n\right).
\end{aligned}$$

The redefinition $Q \to Q - \frac{e}{l}X$ always exists because $l \neq 0$. Under this redefinition, the non-zero commutators are:

$$\begin{aligned}
[P_\mu, K_\nu] &= i a g_{\mu\nu} Q, \\
[P_\mu, X] &= i(k P_\mu + l K_\mu), \\
[K_\mu, X] &= i(m P_\mu + n K_\mu), \\
[Q, X] &= i(k+n) Q.
\end{aligned} \tag{218}$$

This is an extension of the galileon algebra (6.19). Using the similarity transform (211) and the accompanying theorem, this expression can be further simplified. Absorbing a into Q and λ into X, it becomes

$$\begin{aligned}
[P_\mu, K_\nu] &= i g_{\mu\nu} Q, \\
[P_\mu, X] &= i(\kappa P_\mu + K_\mu), \\
[K_\mu, X] &= i(s P_\mu + \kappa K_\mu), \\
[Q, X] &= 2i\kappa Q.
\end{aligned} \tag{219}$$

The parameter κ can be any real number and $s \in \{-1, 0, 1\}$. The coset element for this algebra is

$$U(x, \theta, \xi, \phi) = e^{ix_\mu P^\mu} e^{i\theta Q} e^{i\xi_\mu K^\mu} e^{i\phi X}. \tag{220}$$

The decomposition of the MC form for this parametrisation is

$$\begin{aligned} \omega &= -iU^{-1}dU \\ &= \omega_P^\mu P_\mu + \omega_K^\mu K_\mu + \omega_Q Q + \omega_X X. \end{aligned} \tag{221}$$

The components of this form are:

$$\begin{aligned} \omega_P^\mu &= e^{-\kappa\phi} \left(\cosh(\sqrt{s}\phi) dx^\mu - \sinh(\sqrt{s}\phi)\sqrt{s} d\xi^\mu\right), \\ \omega_K^\mu &= e^{-\kappa\phi} \left(\cosh(\sqrt{s}\phi) d\xi^\mu - \sinh(\sqrt{s}\phi)\frac{dx^\mu}{\sqrt{s}}\right), \\ \omega_Q &= e^{-2\kappa\phi}(d\theta - \xi \cdot dx), \\ \omega_X &= d\phi. \end{aligned} \tag{222}$$

The redundant fields, ξ_μ and ϕ, are substituted using the IHC. Using the full four constraints in $\omega_K^\mu = 0$ would be too strict, since it must only eliminate a single field ϕ. The MC form contains the covariant derivative, $\omega_K^\alpha \equiv e_\mu^\alpha dx^\nu \nabla_\nu \xi^\mu$, so because the vielbein won't vanish the constraint is levied over to it. The only Lorentz scalar that forms the constraint is $\nabla_\mu \xi^\mu = 0$. In the simplest case, when $s = 0$, the covariant derivative is $\nabla_\mu \xi^\alpha = \partial_\mu \xi^\alpha - \phi \delta_\mu^\alpha$.

$$\begin{aligned} \omega_Q = 0 \qquad &\rightarrow \qquad \xi^\mu = \partial_\mu \theta \\ \nabla_\mu \xi^\mu = 0 \qquad &\rightarrow \qquad \phi = \frac{\partial_\mu \xi^\mu}{4} = \frac{\Box\theta}{4} \end{aligned} \tag{223}$$

When $s \neq 0$, solving the second IHC is difficult. Regardless, ϕ will be a function of $\partial_\mu \xi^\nu$ because these are the only variables in the equation.

The action for the NG-bosons is constructed from the forms ω_X, ω_P^μ and ω_K^μ. After the implementation of the IHC only the traceless part of ω_K^μ, remains. Where previously the forms depended on ϕ and the gradient of ξ^ν and ϕ, after the constraint all depend on $\partial_\mu \xi^\nu = \partial_\mu \partial^\nu \theta$ or its derivative. A proper kinetic term can't be constructed from fields with second derivatives acting on it. In conclusion then, there are no models for a Lorentz invariant single NG boson extended by a redundant vector K^μ and scalar X.

The action of Q only shifts its associated field, $\theta \to \theta + \alpha$. The action of K_μ however affects both:

$$\begin{aligned} x_\mu &\to x_\mu, \\ \xi_\mu &\to \xi_\mu + \beta_\mu, \\ \theta &\to \theta + \beta \cdot x, \\ \phi &\to \phi. \end{aligned} \tag{224}$$

The transformation of the fields under X, parametrised by ω is

$$\begin{aligned} x_\mu &\to e^{\kappa\omega}\left(\cosh(\sqrt{s}\omega)x_\mu + \sinh(\sqrt{s}\omega)\sqrt{s}\xi_\mu\right), \\ \xi_\mu &\to e^{\kappa\omega}\left(\sinh(\sqrt{s}\omega)\frac{x_\mu}{\sqrt{s}} + \cosh(\sqrt{s}\omega)\xi_\mu\right), \\ \theta &\to e^{2\kappa\omega}\left(\theta + \frac{1}{2}\left(\frac{x^2}{\sqrt{s}} + \sqrt{s}\xi^2\right)\sinh(\sqrt{s}\omega)\cosh(\sqrt{s}\omega) + \xi \cdot x \sinh^2(\sqrt{s}\omega)\right), \\ \phi &\to \phi + \omega. \end{aligned} \tag{225}$$

When $s = 0$, these rules reduce to

$$\begin{aligned} &x_\mu \to e^{\kappa\omega}x_\mu, && \xi_\mu \to e^{\kappa\omega}\left(\xi_\mu + \omega x_\mu\right), \\ &\theta \to e^{2\kappa\omega}\left(\theta + \frac{1}{2}\omega x^2\right), && \phi \to \phi + \omega. \end{aligned} \tag{226}$$

WZ-terms of the spin-zero multiplet

The assemblage of the WZ terms follows the same strategy that was used in the case of the redundant spin-two generator $S_{\mu\nu}$. The exterior derivative of the 1-forms is:

$$\begin{aligned} d\begin{pmatrix} \omega_P^\mu \\ \omega_K^\mu \end{pmatrix} &= \begin{pmatrix} \kappa & s \\ 1 & \kappa \end{pmatrix}\begin{pmatrix} \omega_P^\mu \wedge \omega_X \\ \omega_K^\mu \wedge \omega_X \end{pmatrix}, \\ d\omega_Q &= g_{\mu\nu}\omega_P^\mu \wedge \omega_K^\nu + 2\kappa\omega_Q \wedge \omega_X, \\ d\omega_X &= 0. \end{aligned} \tag{227}$$

The exterior derivative of the basis of 4-forms e_i yields

$$de^i = -\omega_X \wedge \begin{pmatrix} 4\kappa & 4s & 0 & 0 & 0 \\ 1 & 4\kappa & 3s & 0 & 0 \\ 0 & 2 & 4\kappa & 2s & 0 \\ 0 & 0 & 3 & 4\kappa & s \\ 0 & 0 & 0 & 4 & 4\kappa \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix} = -\omega_X \wedge M^i_j e^j. \tag{228}$$

This implies that the 5-forms $\omega_X \wedge e^i$ are closed. Yet, these won't yield interesting models, because writing these in terms of θ after integration leads to lagrangean densities that average two derivatives per field. The 5-forms $\omega_5^i = \omega_Q \wedge e^i$ contain fewer than 2 derivatives per field, and are suitable. Furthermore, calculation (228) shows that they aren't the result of the derivative of an invariant 4-form. The operation of the exterior derivative on the MC forms gives

$$v_i d\omega_5^i = v_i \left(2\kappa\delta_j^i + M_j^i\right) \omega_Q \wedge \omega_X \wedge e^j. \tag{229}$$

The closed forms then correspond to the left eigenvectors v_i of the matrix $2\kappa I + M$, with eigenvalue 0. The 5 eigenvalues of the matrix can be written compactly:

$$\lambda_\pm^m = 2(3\kappa \pm m\sqrt{s}) \qquad \text{where } m \in \{0, 1, 2\}. \tag{230}$$

Because κ is a coefficient in the Lie algebra it should be a real number. Therefore only the $s = 0$ and $s = 1$ create acceptable solutions to $\lambda_\pm^m = 0$.
First, when $s = 1$ then $\kappa = \mp\frac{m}{3}$ yields five possible values. For these values the closed 5-forms $\omega_5(s, \kappa)$ may neatly be written as

$$\omega_5(1, \kappa) = \epsilon \cdot \omega_Q \wedge (\omega_K + \omega_P)^{2-3\kappa} \wedge (\omega_K - \omega_P)^{2+3\kappa}. \tag{231}$$

It turns out that the dependence on the factor ϕ falls out of these 5 forms.
Secondly, when $s = 0$ there is the only value $\kappa = 0$. The only closed 5-form is

$$\omega_5(0, 0) = \omega_Q \wedge e^1. \tag{232}$$

This form doesn't depend on ϕ either. In fact, all solutions to the eigenvalue equation can be written as a sum of the original galileon forms:

$$\begin{aligned} \omega_5(1, \pm\frac{2}{3}) &= \mathfrak{g}_1 \mp 4\mathfrak{g}_2 + 6\mathfrak{g}_3 \mp 4\mathfrak{g}_4 + \mathfrak{g}_5, \\ \omega_5(1, \pm\frac{1}{3}) &= -\mathfrak{g}_1 \pm 2\mathfrak{g}_2 \mp 2\mathfrak{g}_4 + \mathfrak{g}_5, \\ \omega_5(\pm 1, 0) &= \mathfrak{g}_1 \mp 2\mathfrak{g}_3 + \mathfrak{g}_5, \\ \omega_5(0, 0) &= \mathfrak{g}_1. \end{aligned} \tag{233}$$

Each WZ term contains $\mathfrak{g}_1$, whatever the value of κ and s is. Integrating this form of the extra dimension leads to the tadpole term $\mathcal{L} = \theta$. The extension of the galileon algebra by the scalar generator X therefore doesn't constitute a non-trivial model for interacting NG bosons.

D.4 Spin-one multiplet of redundant generators

Another possible extension of the Poincaré algebra includes the generators Q, K_μ together with an anti-symmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$. The Lorentz invariance dictates the commutation relations of $A_{\mu\nu}$ and $J_{\mu\nu}$:

$$[J_{\mu\nu}, A_{\kappa\lambda}] = i\left(g_{\mu\lambda}A_{\nu\kappa} + g_{\nu\kappa}A_{\mu\lambda} - g_{\mu\kappa}A_{\nu\lambda} - g_{\nu\lambda}A_{\mu\kappa}\right). \tag{234}$$

The constraints, imposed by the Lorentz invariance on the remainder of the bracket relations, allow the following possible commutators:

$$\begin{aligned}
[P_\mu, K_\nu] &= i(a g_{\mu\nu}Q + bJ_{\mu\nu} + c\epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma} + jA_{\mu\nu} + c\epsilon_{\mu\nu\rho\sigma}A^{\rho\sigma}), \\
[P_\mu, Q] &= i\left(dP_\mu + eK_\mu\right), \\
[K_\mu, K_\nu] &= i\left(fJ_{\mu\nu} + g\epsilon_{\mu\nu\kappa\lambda}J^{\kappa\lambda} + sA_{\mu\nu} + t\epsilon_{\mu\nu\rho\sigma}A^{\rho\sigma}\right), \\
[K_\mu, Q] &= i(hP_\mu + iK_\mu), \\
[A_{\mu\nu}, A_{\kappa\lambda}] &= i\left[\, k\left(g_{\mu\lambda}J_{\nu\kappa} + g_{\nu\kappa}J_{\mu\lambda} - g_{\mu\kappa}J_{\nu\lambda} - g_{\nu\lambda}J_{\mu\kappa}\right)\right. \\
&\quad + l\left(g_{\mu\lambda}\epsilon_{\nu\kappa\alpha\beta} + g_{\nu\kappa}\epsilon_{\mu\lambda\alpha\beta} - g_{\mu\kappa}\epsilon_{\nu\lambda\alpha\beta} - g_{\nu\lambda}\epsilon_{\mu\kappa\alpha\beta}\right)J^{\alpha\beta} \\
&\quad + u\left(g_{\mu\lambda}A_{\nu\kappa} + g_{\nu\kappa}A_{\mu\lambda} - g_{\mu\kappa}A_{\nu\lambda} - g_{\nu\lambda}A_{\mu\kappa}\right) \\
&\quad \left. + v\left(g_{\mu\lambda}\epsilon_{\nu\kappa\alpha\beta} + g_{\nu\kappa}\epsilon_{\mu\lambda\alpha\beta} - g_{\mu\kappa}\epsilon_{\nu\lambda\alpha\beta} - g_{\nu\lambda}\epsilon_{\mu\kappa\alpha\beta}\right)A^{\alpha\beta}\,\right], \\
[A_{\mu\nu}, P_\lambda] &= i[m(g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu) + n(g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu)], \\
[A_{\mu\nu}, K_\lambda] &= i[o(g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu) + p(g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu)], \\
[A_{\mu\nu}, Q] &= i\left(qA_{\mu\nu} + w\epsilon_{\mu\nu\rho\sigma}A^{\rho\sigma} + xJ_{\mu\nu} + y\epsilon_{\mu\nu\kappa\lambda}J^{\kappa\lambda}\right).
\end{aligned} \tag{235}$$

It must be that $a \neq 0$ and $o \neq 0$, so the redundant degrees of freedom can be removed via the IHC. The constraints on the algebra, that make it into a Lie algebra, reduce to the following identities:

$$\text{If } a \neq 0 \qquad \rightarrow \qquad \begin{cases} m = p \\ n = o = 0 \end{cases} \tag{236}$$

The combination of the two constraints means that the additional degree of freedom can't be redundant, in any basis in the span of the generators K_μ and P_μ. Therefore the extension of the Poincaré algebra that includes $A_{\mu\nu}$ won't yield any single field NG boson models, and it is unsuitable.

D.5 The (4+N) dimensional 'isometry' algebra

Given the 4 spacetime translations P_μ and the N shift generators Q_A, the direct sum of the Minkowski and the galileon space has $4+N$ dimensions. The "metric" of this space may be written as

$$g_{\mathfrak{ab}} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & -f_{AB} \end{pmatrix}. \tag{237}$$

This metric inherits the properties of f_{AB}, so it may also be degenerate. On this $(4+N)$-dimensional space new translation and rotation generators, $P_{\mathfrak{a}}$ and $J_{\mathfrak{ab}}$ respectively, are defined as:

$$P_{\mathfrak{a}} = \begin{pmatrix} P_\mu \\ -Q_A \end{pmatrix}, \qquad J_{\mathfrak{ab}} = \begin{pmatrix} J_{\mu\nu} & -K_{\mu B} \\ -K_{A\nu} & -Q_{AB} \end{pmatrix}. \tag{238}$$

The anti-symmetry of $J_{\mathfrak{ab}} = -J_{\mathfrak{ba}}$, implies that in this arrangement $K_{\mu A} = -K_{A\mu}$. The Poincaré algebra for Minkowski spacetime, $[P_\mu, Q_i] = 0$ and (6.136-6.142), are all contained in the 'isometry algebra' of the generators of the sum space:

$$\begin{aligned} &[P_{\mathfrak{a}}, P_{\mathfrak{b}}] = 0, \\ &[J_{\mathfrak{ab}}, P_{\mathfrak{c}}] = i(g_{\mathfrak{bc}} P_{\mathfrak{a}} - g_{\mathfrak{ac}} P_{\mathfrak{b}}), \\ &[J_{\mathfrak{ab}}, J_{\mathfrak{cd}}] = i(g_{\mathfrak{ad}} J_{\mathfrak{bc}} + g_{\mathfrak{bc}} J_{\mathfrak{ad}} - g_{\mathfrak{ac}} J_{\mathfrak{bd}} - g_{\mathfrak{bd}} J_{\mathfrak{ac}}). \end{aligned} \tag{239}$$

This is only an isometry algebra in the sense that it mimics that structure, because the metric can be singular. The awkward minus signs in the redefinition (238) helped rewrite the algebra, a rescaling of the original generators could of course remove them. The only remaining commutators are

$$\begin{aligned} &[Q_i, Q_j] = i\Lambda_{ij}^k Q_k, \\ &[Q_i, P_{\mathfrak{a}}] = iI_{i\mathfrak{a}}^{\mathfrak{b}} P_{\mathfrak{b}}, \\ &[Q_i, J_{\mathfrak{ab}}] = i(PH_i^T - H_i P^T)_{\mathfrak{ab}} - i(I_{\mathfrak{a}}^{\mathfrak{c}} J_{\mathfrak{cd}} \Sigma_{\mathfrak{b}}^{\mathfrak{d}} + \Sigma_{\mathfrak{a}}^{\mathfrak{c}} J_{\mathfrak{cd}} I_{\mathfrak{d}}^{\mathfrak{b}}). \end{aligned} \tag{240}$$

The matrices H, I and Σ are defined as:

$$H_{i\mathfrak{a}} = \begin{pmatrix} 0 \\ h_{iA} \end{pmatrix}, \qquad I_{i\mathfrak{a}}^{\mathfrak{b}} = \begin{pmatrix} 0 & 0 \\ 0 & i_{iA}^B \end{pmatrix}, \qquad \Sigma_{\mathfrak{a}}^{\mathfrak{b}} = \begin{pmatrix} \delta_\mu^\nu & 0 \\ 0 & -\delta_A^B \end{pmatrix}. \tag{241}$$

Accordingly, the conclusion may be formulated differently. Assume that P_μ and Q_i commute. Only those Lie algebras containing a subalgebra, which expresses the isometries of a (possibly degenerate) extension of Minkowski space, generate Lorentz invariant redundant modes.

D.6 Generalised DBI MC form

Let the coset space of the generalised DBI theory be parametrised by

$$U(x^\mu, \theta^A, \xi^{\mu A}, \theta^a) \equiv e^{ix^\mu P_\mu} e^{i\theta^A Q_A} e^{i\xi^{\mu A} K_{\mu A}} e^{i\theta^a \tilde{Q}_a}. \tag{242}$$

The Maurer Cartan form then becomes

$$\begin{aligned} \omega_{MC} &= -iU^{-1} dU \\ &= \omega_P^\mu P_\mu + \omega_Q^A Q_A + \omega_K^{\mu A} K_{\mu A} + \omega_J^{\mu\nu} J_{\mu\nu} + \omega_Q^{AB} Q_{AB} + \omega_{\tilde{Q}}^a \tilde{Q}_a. \end{aligned} \tag{243}$$

Explicitly, the different 1-forms in the MC form are:

$$\begin{aligned}
\omega_P^\mu &= (\hat{ch}\Pi)^\mu_\nu dx^\nu - (\hat{sh}\Pi)^\mu_\nu f_{AB}\xi^{\nu B} d\theta^A, \\
\omega_Q^A &= (e^{-i\theta^a t_a})^A_B \left[(\hat{ch}\Pi)^B_C d\theta^C - (\hat{sh}\Pi)^B_C \xi^C_\mu dx^\mu\right], \\
\omega_K^{\mu A} &= (e^{-i\theta^a t_a})^A_B \left(\frac{\sin\sqrt{L}}{\sqrt{L}}\right)^{B\mu}_{C\nu} d\xi^{C\nu}, \\
\omega_J^{\mu\nu} &= \left(\frac{\cos\sqrt{L}-1}{L}\right)^{B\mu}_{A\rho} \xi^\nu_B d\xi^{\rho A}, \\
\omega_Q^{AB} &= (e^{-i\theta^a E_a})^{AB}_{CD}\xi^D_\nu \left(\frac{\cos\sqrt{L}-1}{L}\right)^{C\nu}_{E\rho} d\xi^{\rho E}, \\
\omega_{\tilde{Q}}^a &= \left(\frac{e^{-i\theta^i\Lambda_i}-1}{-i\theta^i\Lambda_i}\right)^a_b d\theta^b.
\end{aligned} \tag{244}$$

Here

$$L^{Z\omega}_{A\mu} = \xi^{\nu B}\xi^{\rho C}\left(f_{AB}\delta^Z_C(g_{\mu\rho}\delta^\omega_\nu - g_{\nu\rho}\delta^\omega\mu) + g_{\mu\nu}\delta^\omega_\rho(f_{AC}\delta^Z_B - f_{BC}\delta^Z_A)\right), \tag{245}$$

and

$$(E_a)^{CD}_{AB} = (t_a)^C_A\delta^D_B - (t_a)^C_B\delta^D_A. \tag{246}$$

D.7 Galileons from DBI

At the level of the Lie algebra, an İnönü-Wigner contraction in the case of the DBI (6.18) generates the galileon model (6.19). In the single field case, this corresponds to the limit $v \to 0$, but for the multi-field models one must let $\Xi_{AB} \to 0$, or equivalently $Q_{AB} = 0$, and $f_{AB} \to 0$. On the level of the actions, this manifests as the limit of the DBI yielding the galileon action [62]. The series expansion of the multi-DBI action (6.160), in the first orders of f_{AB}, is

$$\begin{aligned}
S &= S\Big|_{f=0} + \frac{\partial S}{\partial f_{AB}}\Big|_{f=0} f_{AB} + \mathcal{O}(f^2) \\
&= \int \left(1 - f_{AB}\frac{\partial_\mu\theta^A\partial^\mu\theta^B}{2}\right) d^4x + \mathcal{O}(f^2).
\end{aligned} \tag{247}$$

The galileon action derived via this expansion is the term linear in f_{AB}. This relation between the DBI and galileon seemingly contains a contradiction: the symmetry transform leaves the DBI action exactly invariant, whilst it generates a surface term for the galileon action. The infinitesimal form of the DBI symmetry transformations (6.154) is:

$$\begin{aligned} x^\mu \quad &\rightarrow \quad x^\mu + \beta^{\mu B} f_{AB} \theta^A, \\ \theta^A \quad &\rightarrow \quad \theta^A + \beta^A_\mu x^\mu. \end{aligned} \tag{248}$$

In the limit $f_{AB} \rightarrow 0$ these reduce to the transformation rules of the the galileon (6.167). The orders of the action transform under the infinitesimal symmetry as:

$$\begin{aligned} S\Big|_{f=0} \quad &\rightarrow \quad \int d^4x(1 + f_{AB}\beta^{\mu B}\partial_\mu\theta^A) + \mathcal{O}(\beta^2), \\ \frac{\partial S}{\partial f_{AB}}\Big|_{f=0} f_{AB} \quad &\rightarrow \quad -\int d^4x \left(f_{AB}\frac{\partial_\mu\theta^A\partial^\mu\theta^B}{2} + f_{AB}\beta^{\mu A}\partial_\mu\theta^B \right) + \mathcal{O}(f^2), \end{aligned} \tag{249}$$

The action of the symmetry on the zeroth order term generates a surface term. This term cancels against the term generated by symmetry transform of the first order of the action. The limit of the DBI action removes the higher order terms, and leaves a term that transforms like the WZ term. All galileon models can be derived from higher order DBI actions, and their symmetries are related in a similar way.
For a multi-DBI theory the zero limit may be taken for only some of the components of f_{AB}, in hopes of finding a mixed model of DBI and galileon fields.
The simplest example is a two field multi-DBI. By an appropriate change of base of the generators Q_A, the metric may generally be written in a diagonal form. The metric for two fields, containing a parameter v, is:

$$f_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}. \tag{250}$$

Then the induced metric is

$$(G_v)_{\mu\nu} = g_{\mu\nu} - \partial_\mu\theta^1\partial_\nu\theta^1 - v\partial_\mu\theta^2\partial_\nu\theta^2. \tag{251}$$

The series expansion action in v is

$$\begin{aligned} S &= \int \sqrt{1 - (\partial_\mu\theta^1)^2}\left(1 - \frac{v}{2}G_0^{\mu\nu}\partial_\mu\theta^2\partial_\nu\theta^2\right) d^4x + \mathcal{O}(v^2) \\ &= S_0 + vS_1 + ... \end{aligned} \tag{252}$$

The infinitesimal symmetry transformations in this scenario become

$$\begin{aligned} x^\mu \quad &\rightarrow \quad x^\mu + \beta^{\mu 1}\theta^1 + \beta^{\mu 2} v\theta^2, \\ \theta^A \quad &\rightarrow \quad \theta^A + \beta^A_\mu x^\mu. \end{aligned} \tag{253}$$

The zeroth order action now transforms as

$$S_0 \rightarrow \int d^4x \left(\sqrt{1-(\partial_\mu\theta^1)^2} + v\beta^{\mu 2}\partial_\mu\theta^2\sqrt{1-(\partial_\nu\theta^1)^2} + v\beta^{\mu 2}\frac{\partial_\mu\theta^1(\partial_\nu\theta^1\partial^\nu\theta^2)}{\sqrt{1-(\partial_\nu\theta^1)^2}} \right). \quad (254)$$

The term that is to the first order in v is no longer a total derivative, so its counterpart in the transformation S_1 won't be either. This also means that S_1 won't be an invariant term by itself either. The explicit action, found by taking the limit and inverting $G_{0\mu\nu}$, is

$$S_1 = -\frac{1}{2}\int d^4x \left[(\partial_\mu\theta^2)^2\sqrt{1-(\partial_\nu\theta^1)^2} + \frac{(\partial_\mu\theta^1\partial_\nu\theta^2)^2}{\sqrt{1-(\partial_\nu\theta^1)^2}} \right]. \quad (255)$$

The field θ^2 is not a galileon. The action isn't invariant under a shift of the field that is linear in the spacetime coordinate, and its amplitudes scale like a normal NG boson. This is a simple example, but the reasoning can be extended to more fields. The construction of a mixed model, consisting of galileons and DBIs, by simply taking limits is therefore not generically possible.

D.8 Multi-galileons with a central extension

In the main text it was assumed that the internal generators Q_i separate into two sets, Q_A and $\tilde{Q}_i$, that each form a Lie algebra independently. This is not a given and infinitely many algebras don't satisfy these conditions. The simplest probe into that domain takes the following form: Split the Q_i into two independent sets, Q_A and $\tilde{Q}_i$. The Q_A must form the centre of the group, the representation t_i becomes trivial, and the $\tilde{Q}_i$ no longer form a subalgebra:

$$[\tilde{Q}_i, Q_A] = 0 = [Q_A, Q_B], \qquad [\tilde{Q}_i, \tilde{Q}_j] = \Lambda^k_{ij}\tilde{Q}_k + \Lambda^A_{ij}Q_A. \quad (256)$$

The brackets take such a form, for instance, when the Q_A are the generators that form a central extension of the Lie algebra $[\tilde{Q}_i, \tilde{Q}_j] = \Lambda^k_{ij}\tilde{Q}_k$. The possible extensions of the $\tilde{Q}_i$ algebra are determined by the second cohomology group H^2 (defined in (3.5)), see for instance [75]. The MC form, using the parametrisation (6.165), yields

$$\omega = -iU^{-1}dU = dx^\mu P_\mu + d\xi^{\mu A}K_{\mu A} + (d\theta^A - \xi^{\mu A}dx_\mu)Q_A + \Omega. \quad (257)$$

The implementation of the IHC, $\omega^A_Q = 0$, means that the $\omega^{\mu A}_K = d\xi^{\mu A}$ will depend on second derivatives of θ_A. This means that for such theories, the construction won't provide kinetic terms from NG actions; it must come from WZ terms. The 5-forms, in the case that $f^A_{ij} = 0$, take the form from (6.170); the combination ω^2_5 furnishes the kinetic term.

Now, in the case of a centrally extended internal symmetry group, it is no longer guaranteed that the forms are part of the cohomology group. The MC structure equations are, in this parametrisation:

$$\begin{aligned} d\omega_Q^A &= \omega_P^\mu \wedge \omega_{K\mu}^A + \frac{1}{2}\Lambda_{jk}^A \omega_{\tilde{Q}}^j \wedge \omega_{\tilde{Q}}^k, \\ d\omega_{\tilde{Q}}^i &= \frac{1}{2}\Lambda_{jk}^i \omega_{\tilde{Q}}^j \wedge \omega_{\tilde{Q}}^k, \\ d\omega_P^\mu &= 0, \\ d\omega_K^\mu &= 0. \end{aligned} \tag{258}$$

A possible extension of ω_5^2 is now

$$\tilde{\omega}_5^2 = \epsilon_{\mu\nu\rho\sigma}(c_{AB}\omega_Q^A + c_{iB}\omega_{\tilde{Q}}^i) \wedge \omega_K^{B\mu} \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma. \tag{259}$$

This form is closed if

$$c_{AB}\Lambda_{jk}^A + c_{iB}\Lambda_{jk}^i = 0. \tag{260}$$

The c_{AB} must non-singular, invertible, to provide kinetic terms for all fields θ^A. The structure constants of the central extension, can then be solved in terms of the 'old' algebra.

$$\Lambda_{ij}^A = -c_k^A \Lambda_{ij}^k \tag{261}$$

The commutation relation reduces to

$$[\tilde{Q}_i, \tilde{Q}_j] = \Lambda_{ij}^k(\tilde{Q}_k - c_k^A Q_A). \tag{262}$$

After the redefinition $\tilde{Q}'_i = \tilde{Q}_i - c_i^A Q_A$, the $\tilde{Q}'_i$ form a subalgebra again. The term therefore only yields a kinetic term, in case that the central extension is trivial. In this simple example, the non-trivial extensions are therefore ruled out.

D.9 Dependent twisted galileon models

In the case of the twisted galileons it was assumed that the generators Q_A and Q_{AB} were independent. This is not a necessity, it may be that

$$Q_{AB} = \lambda_{AB}^C Q_C. \tag{263}$$

This would be a new theory, that only contains physical fields with enhanced soft limits. This means that the Jacobi constraints now place constraints on the tensor λ_{AB}^C. Since $h_{Ai} = 0$, the constraint (6.133) yields

$$(t_i)_A^D \lambda_{DB}^C + (t_i)_B^D \lambda_{AD}^C - (t_i)_D^C \lambda_{AB}^D = 0. \tag{264}$$

This means that λ^C_{AB} should be an invariant under the representation t_i of the internal symmetry. The MC form now picks up a term:

$$\omega^A_Q = (e^{-\theta^a t_a})^A_B \left(d\theta^B - \xi^B_\mu dx^\mu + \frac{1}{2}\lambda^B_{CD}\xi^B_\nu \partial_\mu \xi^{\nu C} \right). \tag{265}$$

The resulting IHC, $\omega^A_Q = 0$, is rather difficult to solve.

$$\xi^A_\mu - \frac{1}{2}\lambda^A_{BC}\xi^B_\nu \partial_\mu \xi^{\nu C} = \partial_\mu \theta^A \tag{266}$$

To find ξ^A_μ in terms of $\partial_\mu \theta^A$ a set of non-linear differential equations must be solved. The MC structure equations become

$$\begin{aligned}
d\omega^A_{K\mu} &= -i\tilde{\Omega}^A_B \wedge \omega^B_{K\mu}, \\
d\omega^A_Q &= \omega^\mu_P \wedge \omega^A_{K\mu} - i\tilde{\Omega}^A_B \wedge \omega^B_Q + +\frac{1}{2}\lambda^A_{BC}\omega^B_{K\mu} \wedge \omega^{\mu C}_K, \\
d\omega^i_{\tilde{Q}} &= \frac{1}{2}\Lambda^i_{jk}\omega^j_{\tilde{Q}} \wedge \omega^k_{\tilde{Q}}.
\end{aligned} \tag{267}$$

Here $\tilde{\Omega}^A_B \equiv \omega^i_{\tilde{Q}}(t_i)^A_B$. The 5-forms (6.170) are not closed for any arbitrary choice of λ^C_{AB}, and the addition of terms containing $\omega^i_{\tilde{Q}}$ can't remedy this. This means that the closedness of the 5-form imposes a constraint on λ^C_{AB}. This equation doesn't seem to have a solution in Minkowski spacetime other than $\lambda^C_{AB} = 0$, so it is unlikely that this algebra leads to any interesting theories.

PART V
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