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Simultaneous state and process fault estimation in LPV systems using robust QPV observers

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Abstract

This paper undertakes the problem of simultaneous estimation of state and process faults in linear parameter varying systems. For this purpose, a novel strategy that exploits recent results on the design of observers for quadratic parameter varying systems is developed, and a complete design procedure is described. First, it is shown that by treating the process faults as additional states to be estimated, the arising augmented state-space model is indeed expressed as a quadratic parameter varying system. Hence, the estimates provided by a quadratic parameter varying observer based on the so-called linear output error injection principle would comprise both the actual state and the process faults estimates. Robust design conditions that minimize the effect of disturbances and measurement noise on some linear, and possibly parameter-varying, combination of error variables are obtained using a Lyapunov-based approach. Then, it is shown that the design problem can be reduced to a finite set of linear matrix inequalities that can be solved using available computational tools. The final part of the paper exhibits an illustrative example, which clearly exposes the potential applicability and performance of the developed approach.

KEYWORDS:

Quadratic parameter varying (QPV) systems, linear parameter varying (LPV) systems, fault estimation, process faults, state observers, robustness.

1 | INTRODUCTION

Owing to an inevitably increasing demand for reliability and safety in industrial systems, fault detection, isolation and estimation (FDIE)[1,2] (also known as *fault diagnosis*[3,4]) have raised to be topics of a large interest for both the academy and the industry. This interest is also proliferating due to the increasing research attention concerning fault tolerant control (FTC)[5,6,7]. Indeed, FDIE, and in particular, fault estimation constitutes the main ingredient of the so-called integrated FTC[8,9]. The appealing property of the integrated FTC is the fact that it avoids fault detection and isolation in favour of using fault estimation only.

Classical model-based fault diagnosis methods aim at generating a signal, called *residual*, which is different from zero when a fault occurs[10,11,12]. By exploiting well-known geometric properties, it is possible to associate various residual behaviors with a given set of faults, thus enabling fault isolation[13,14,15]. Although obtaining information about the presence (detection) and location (isolation) of the occurring faults is sufficient for performing some kind of fault-tolerant control reconfiguration, on-line fault estimation is a crucial component in active FTC schemes. Indeed, active FTC aims at attaining good compensation of the faults, with minimal loss of performance. In the light of the above discussion, many works have focused on estimating the faults by treating them as uncertain parameters or signals.

The system fault estimation problem can be divided into three sub-problems pertaining an estimation of actuator, sensor and process faults. The estimation of actuator faults has been addressed using different methods. Indeed, Zhang et al.[16] proposed a fast adaptive mechanism for estimating loss of actuator effectiveness. Lee et al.[17] proposed a sliding mode scheme based on a robust descriptor observer. Witczak et al.[18] presented a linear matrix inequality (LMI)-based observer design strategy capable of achieving simultaneous estimation of faults and states while decoupling the effect of an unknown input and reducing an unappealing impact of an exogenous disturbance. Subsequently, Rotondo et al.[19] formulated fault estimation problem as a parameter estimation one. Their work enabled the use of the so-called set-membership approach that, apart from fault estimates, provides the so-called uncertainty intervals shaping a feasible behavior of the fault, which is consistent with the measurements. The notion of quadratic boundedness (QB) was used by Buciakowski et al.[20] to derive a feasible set for the state and fault error, respectively. As a result, the uncertainty intervals for both the state and the actuator fault can be estimated efficiently. The above approach was extended towards a simultaneous estimation of actuator, sensor and states of the system by Pazera and Witczak [21], who showed that $\mathcal{H}_\infty$-based fault estimation could be perceived as a particular case of the QB-based one.

Even though most of the available results are for additive fault signals, some works have considered multiplicative faults (see, e.g.,[22,23] and the references therein). Within a large set of solutions for sensor fault estimation, a particular attention is focused on sliding mode[24,25] and unknown input observer approaches[26,27]. Notably, there are also several works investigating a simultaneous estimation of sensor and actuator faults (see, e.g.,[28,29,30,21]. and the references therein).

Logically, the number of solutions for process fault estimation should proliferate equally, but unfortunately this does not happen. In this case, the fault is affecting some internal component of the system, which for linear systems can be simply interpreted as unpermitted changes of the system matrices. This means that, even for linear systems, the problem of simultaneous estimation of states and process faults boils down to a nonlinear estimation problem. This is due to the multiplication of the process faults by the state variables. To tackle this issue, Shi and Patton[31] considered the case of LTI systems affected by process faults, and showed how an adaptive observer structure where the observer gain and the adaptive law were computed using LMIs could be employed to achieve simultaneous state and process fault estimation. Later, Pazera and Witczak[32] showed that the nonlinear estimation problem could be reduced to a linear problem by introducing a suitable system reparameterization, so that instead of estimating directly the fault signal, its product with the state variables was estimated. In spite of the appeal brought by the relative simplicity, this method suffered from the fact that, instead of getting a single estimate for the elements of the fault vector, a different estimate was obtained for each state variable. This drawback was mitigated by computing the final fault estimate as the mean value of the computed estimates, although without any theoretical guarantee of getting an unbiased estimate of the process faults. This approach was extended later by Pazera et al.[33] to cope with the case of simultaneous sensor and process faults, whereas Witczak et al.[34] analyzed the structural properties of dynamical systems under process faults. Finally, the work [35] exploited the QB approach to obtain an uncertainty interval for the fault estimate which could be fed to the FTC module.

From the analysis of the literature, it seems that there are two main shortcomings in the available approaches. First, so far only the case of time-invariant systems affected by process faults has been considered, and there are no results concerning possibly parameter- or time-varying plants affected by this type of faults; and second, most of the available approaches are based on a change of variables that transforms the product of state and process fault signals into a single variable, from whose estimate a possibly biased fault estimate is recovered a posteriori. It is clear that the development of an approach which is capable of dealing with systems that vary in time while estimating directly the process fault signal without introducing any repameterization would constitute an advancement of the state-of-the-art in the field of process fault estimation.

As it was already mentioned, the main difficulty with estimating directly the process fault signal $f(t)$ arises from the fact that, even when the attention is restricted to linear systems, such a fault causes the appearance of a nonlinear term in the state equation, which involves some kind of multiplication between the fault itself and the state variables contained in the vector $x(t)$. However, it should be noted that such nonlinearity can be treated as a special case of quadratic one. Recently, a set of useful developments concerning quadratic parameter varying (QPV) systems were published[36,37,38,39,40]. Indeed, QPV systems can be perceived as valuable extensions of the celebrated LPV systems[41,42]. Such an extension is attributed with nonlinear quadratic terms formed by products between state variables. In particular, the work[40] tackles the state estimation problem for QPV systems. The appealing property of this approach is the fact that a desired convergence rate within a predefined polytopic error space is achieved.

The motivation of the present work arises from noticing the fact that the QPV observer proposed by Rotondo and Johansen[40] can be adapted to tackle the simultaneous estimation of states and process faults. Indeed, by considering the process faults as additional states to be estimated, the faulty system's state equation can be converted into an augmented state equation with products between augmented state variables. Unlike additive faults, this augmentation does not preserve the linearity, since it introduces quadratic terms in the augmented state equation. If the non-faulty system is assumed to be described by an LPV representation (many nonlinear systems can be reshaped into an LPV structure by means of automated techniques[43,44]), then the arising augmented state-space model is a QPV system. As a consequences, a QPV observer would return an augmented state estimate which comprises both the state and the process faults estimates. The main goal of this paper is to assess the applicability of QPV observers to the problem of simultaneous state and process fault estimation in LPV systems. It is shown that the direct application of the QPV observer proposed by Rotondo and Johansen[40] is impeded by the special structure of the matrices appearing in the dynamics of the estimation error, so that some assumptions alike to persistency of excitation conditions must be introduced so that the observer design becomes feasible. Thus, the overall design procedure is modified adequately by obtaining LMI-based sufficient conditions for the design of a QPV observer that achieves the above-stated goal. It should be noted that the design procedure presented in the work[40] considered the ideal case in which the observed system is not affected by unknown disturbances and noise. With the goal of improving the robustness of the observer against these undesired effects, we propose a design procedure that attempts at minimizing their effect on some linear, possibly parameter-varying, combination of error variables. The conceived procedure can be thought of as an extension of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal design[45] to the QPV case.

Thus, the contributions of this paper can be summarized as follows.

- It is shown that the problem of simultaneous state and process fault observer-based estimation in LPV systems can be reformulated as a state estimation problem in an augmented QPV system whose state vector comprises the state and fault estimation errors.
- The observer design problem is adapted to work about a non-zero operating point, in order to ensure that the convergence of the estimation error to zero is not impeded by some elements of the state vector approaching zero values.
- The presence of unknown disturbances and measurement noise are taken into account by including the minimization of their effect on some combination of error variables in the design procedure.

The structure of the paper is as follows. Section 2 states the overall estimation problem and shows that the application of a QPV observer to an LPV system affected by process fault leads to estimation error dynamics that are described by a QPV system affected by the unknown augmented state which acts as if it were an exogenous input. Section 3 provides the robust observer design conditions. In Section 4, the design conditions are shaped as a finite set of LMIs, which are tractable with the widely available computational tools. The developed approach is exemplified using an illustrative example in Section 5. Section 6 provides concluding remarks about the approach proposed in this paper and suggests prospective research directions.

Notation: The notation is fairly standard. Given a vector v or a matrix A, v^T and A^T denote their transposes. If A is invertible, A^{-1} denotes its inverse. He $\{A\}$ stands as a shorthand for $A + A^T$. The symbols $>$ and $<$ are used for scalars, whereas $\succ$ and $\prec$ are used for matrices, in which case they must be interpreted in the sense of definiteness. Co $\left\{v_1, v_2, \cdots, v_n\right\}$ denotes the convex hull with vertices v_1, v_2, ..., v_n.

2 | PROBLEM FORMULATION

Let us consider an LPV system affected by possible process faults:

$$\dot{x}(t) = A\left(\theta(t)\right) x(t) + B\left(\theta(t)\right) u(t) + G\left(\theta(t)\right) w(t) + \sum_{k=1}^{n_f} A_{f,k}\left(\theta(t)\right) f_k(t) x(t) \tag{1}$$

$$y(t) = C\left(\theta(t)\right) x(t) + H\left(\theta(t)\right) w(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$ stands for the state vector, $u(t) \in \mathbb{R}^r$ is the (known) control input vector, $w(t) \in \mathbb{R}^p$ is an (unknown) exogenous input, which can represent disturbances and/or measurement noise, and $y(t) \in \mathbb{R}^m$ denotes the measured output vector, respectively. Furthermore, $f_i(t) \in \mathbb{R}$, $i = 1, \dots, n_f$, denotes (unknown) process faults, which act onto the system through the fault distribution matrix functions $A_{f,k}\left(\theta(t)\right)$, $k = 1, \dots, n_f$. Notice that all the matrix functions appearing in the system (1)-(2) are scheduled by a time-varying parameter vector $\theta \in \Theta$.

The problem addressed in this paper is defined as the simultaneous estimation of the system states and the process faults with (1) along with the measured (2). To settle this problem, let us define a fault vector $f(t) \in \mathbb{R}^{n_f}$ as follows:

$$f(t) = \left[\, f_1(t)\ f_2(t)\ \cdots\ f_{n_f}(t) \,\right]^T \tag{3}$$

Subsequently, $f(t)$ can be used to extend the state vector to form $\bar{x}(t) \in \mathbb{R}^{n+n_f}$:

$$\bar{x}(t) = \begin{bmatrix} x(t)^T & f(t)^T \end{bmatrix}^T \tag{4}$$

Then, by imposing a standard assumption concerning the fault rate of change, i.e., $\dot{f}(t) \approxeq 0$, the dynamics of $\bar{x}(t)$ can be described by:

$$\dot{\bar{x}}(t) = \bar{A}(\theta(t))\,\bar{x}(t) + \bar{N}(\theta(t), \bar{x}(t))\,\bar{x}(t) + \bar{B}(\theta(t))\,u(t) + \bar{G}(\theta(t))\,w(t) \tag{5}$$

$$y(t) = \bar{C}(\theta(t))\,\bar{x}(t) + \bar{H}(\theta(t))\,w(t) \tag{6}$$

where:

$$\bar{A}(\theta(t)) = \begin{bmatrix} A(\theta(t)) & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{B}(\theta(t)) = \begin{bmatrix} B(\theta(t)) \\ 0 \end{bmatrix} \quad \bar{C}(\theta(t)) = \begin{bmatrix} C(\theta(t)) & 0 \end{bmatrix} \quad \bar{G}(\theta(t)) = \begin{bmatrix} G(\theta(t)) \\ 0 \end{bmatrix} \quad \bar{H}(\theta(t)) = \begin{bmatrix} H(\theta(t)) & 0 \end{bmatrix}$$

$$\bar{N}(\theta(t), \bar{x}(t)) = \begin{bmatrix} \bar{x}(t)^T N_1(\theta(t)) \\ \bar{x}(t)^T N_2(\theta(t)) \\ \vdots \\ \bar{x}(t)^T N_n(\theta(t)) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{matrix} N_1(\theta(t)) = \begin{bmatrix} 0 & N_1^{12}(\theta(t)) \\ 0 & 0 \end{bmatrix} \\ N_2(\theta(t)) = \begin{bmatrix} 0 & N_2^{12}(\theta(t)) \\ 0 & 0 \end{bmatrix} \\ \vdots \\ N_n(\theta(t)) = \begin{bmatrix} 0 & N_n^{12}(\theta(t)) \\ 0 & 0 \end{bmatrix} \end{matrix}$$

with:

$$N_i^{12}(\theta(t)) = \begin{bmatrix} a_{f,1}^{(i,1)}(\theta(t)) & a_{f,2}^{(i,1)}(\theta(t)) & \cdots & a_{f,n_f}^{(i,1)}(\theta(t)) \\ a_{f,1}^{(i,2)}(\theta(t)) & a_{f,2}^{(i,2)}(\theta(t)) & \cdots & a_{f,n_f}^{(i,2)}(\theta(t)) \\ \vdots & \vdots & \ddots & \vdots \\ a_{f,1}^{(i,n)}(\theta(t)) & a_{f,2}^{(i,n)}(\theta(t)) & \cdots & a_{f,n_f}^{(i,n)}(\theta(t)) \end{bmatrix}$$

where $a_{f,k}^{(i,j)}$ signifies i, j-th element of $A_{f,k}(\theta(t))$.

The augmented system (5)-(6) is a QPV system, so that the observer presented by Rotondo and Johansen[40] can be used. This observer is based on the classical *linear output error injection principle*, which causes that:

$$\dot{\hat{\bar{x}}}(t) = \bar{A}(\theta(t))\,\hat{\bar{x}}(t) + \bar{N}\left(\theta(t), \hat{\bar{x}}(t)\right)\hat{\bar{x}}(t) + \bar{B}(\theta(t))\,u(t) + L(\theta(t))\left[y(t) - \bar{C}(\theta(t))\,\hat{\bar{x}}(t)\right] \tag{7}$$

where $\hat{\bar{x}}(t) \in \mathbb{R}^{n+n_f}$ denotes the estimate of $\bar{x}(t)$. Then, by forming $e(t) = \bar{x}(t) - \hat{\bar{x}}(t)$, it follows that the dynamical system that describes the evolution of $e(t)$ is also a QPV system, affected by the unknown state $\bar{x}(t)$, which acts as if it were an exogenous input together with the exogenous signal $w(t)$:

$$\dot{e}(t) = \left[\bar{A}(\theta(t)) - L(\theta(t))\,\bar{C}(\theta(t)) - \bar{N}(\theta(t), e(t))\right] e(t) + \tilde{N}(\theta(t), e(t))\,\bar{x}(t) + \left[\bar{G}(\theta(t)) - L(\theta(t))\,\bar{H}(\theta(t))\right] w(t) \tag{8}$$

with:

$$\tilde{N}(\theta(t), e(t)) = \begin{bmatrix} e(t)^T \mathrm{He}\{N_1(\theta(t))\} \\ e(t)^T \mathrm{He}\{N_2(\theta(t))\} \\ \vdots \\ e(t)^T \mathrm{He}\{N_n(\theta(t))\} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

where the following identity has been exploited:

$$\boldsymbol{\bar{N}(\theta(t), \bar{x}(t))\,\bar{x}(t) - \bar{N}\left(\theta(t), \hat{\bar{x}}(t)\right)\hat{\bar{x}}(t) = \tilde{N}(\theta(t), e(t))\,\bar{x}(t) - \bar{N}(\theta(t), e(t))\,e(t)} \tag{10}$$

Since the dynamics of the estimation error is available, the objective of the prospective section is to provide the observer design procedure.

The problem addressed in this paper is the one of choosing the observer gain $L(\theta(t))$ that provides convergence of the estimation error to zero with some guaranteed decay rate (i.e. the convergence of $\hat{\bar{x}}(t)$ to $\bar{x}(t)$) when $w(t) = 0$, while at the same time minimizing the effect of $w(t)$ on a performance output signal $z(t)$ defined as follows:

$$z(t) = C_z(\theta(t))\,e(t) \tag{11}$$

This latter problem can be expressed as the minimization of an upper bound $\gamma > 0$ on the $\mathcal{L}_2$ gain from w to z:

$$\frac{\|z\|_2^2}{\|w\|_2^2} = \frac{\int_0^\infty z(t)^T z(t)dt}{\int_0^\infty w(t)^T w(t)dt} < \gamma \tag{12}$$

which is related to $\mathcal{H}_\infty$ optimization. Alternatively, we can minimize the value of $\mu > 0$ such that:

$$\text{trace}\left(C_z(\theta) X C_z(\theta)^T\right) < \mu \qquad \forall \theta \in \Theta \tag{13}$$

where X denotes a generalized controllability gramian[46] for (8) which means that, by defining:

$$V(e(t)) = e(t)^T X^{-1} e(t) \tag{14}$$

it satisfies the inequality:

$$\dot{V}(\theta(t), e(t), \bar{x}(t)) < w(t)^T w(t) \tag{15}$$

The term $C_z(\theta) X C_z(\theta)^T$ can be thought of as a generalized output controllability gramian, so that through the minimization of μ, the amount of energy needed by the signal $w(t)$ to steer the estimation error away from the origin of the error space would be increased[47]. This latter problem is related to $\mathcal{H}_2$ optimization.

Remark 1. Notice that, in order to keep the mathematical formulation as general as possible, we allow the matrix $C_z(\theta(t))$ to be parameter-dependent. In most cases of practical interest, this matrix is actually chosen as a constant matrix. For instance, if the effect of w on the state estimation error is to be minimized, then we would have $C_z = \begin{bmatrix} I & 0 \end{bmatrix}$, whereas if its effect on the fault estimation error is to be minimized, then $C_z = \begin{bmatrix} 0 & I \end{bmatrix}$, where I and 0 denote identity/zero matrices of appropriate dimensions.

3 | OBSERVER DESIGN

Due to the quadratic terms in (8), it is not possible to assess convergence of $e(t)$ to zero in the whole error and state spaces. At the same time, due to the special structures of the matrices $\bar{A}(\theta(t))$, $\bar{C}(\theta(t))$, $N_i(\theta(t))$, $i = 1, \dots, n$, observability of faults can be obtained only thanks to the action of the quadratic term, which vanishes to zero when some elements of $x(t)$ approach zero values. For this reason, convergence of the estimation error $e(t)$ to zero can be ensured only if the system is working about a non-zero operating point, which leads to introducing the following polytopes and assumptions, that can be perceived as conditions about the persistency of excitation:

$$\mathcal{P}_{\bar{x}} = \text{Co}\{\bar{x}_{(1)}, \bar{x}_{(2)}, \dots, \bar{x}_{(p)}\} = \{\bar{x} \in \mathbb{R}^{n+n_f} : a_k^T(\bar{x} - c) \le 1, k = 1, \dots, q\} \tag{16}$$

$$\mathcal{P}_e = \text{Co}\{\bar{x}_{(1)} - c, \bar{x}_{(2)} - c, \dots, \bar{x}_{(p)} - c\} = \{\bar{x} \in \mathbb{R}^{n+n_f} : a_k^T \bar{x} \le 1, k = 1, \dots, q\} \tag{17}$$

where $\bar{x}_{(i)}$, $i = 1, \dots, p$, denote points of $\mathbb{R}^{n+n_f}$ (*vertex representation*), $\text{Co}\{\cdot\}$ denotes the convex hull operation, a_k, $k = 1, \dots, q$, are appropriate vectors (*half-space representation*) and c denotes a non-zero point such that $c \in \mathcal{P}_{\bar{x}}$, so that $0 \in \mathcal{P}_e$ (for example, c can be chosen as the geometric center of the polytope $\mathcal{P}_{\bar{x}}$).

Assumption 1: The time-varying $\theta(t)$ is perceived as a known parameter vector, which can vary within Θ.

Assumption 2: The state $\bar{x}(t)$ belongs to:

$$\mathcal{X} = \{\bar{x} \in \mathbb{R}^{n+n_f} : (\bar{x} - c)^T Q^{-1} (\bar{x} - c) \le 1\} \tag{18}$$

with $Q \succ 0$ and $\mathcal{P}_{\bar{x}} \subset \mathcal{X}$.

Remark 2. Notice that Assumption 2 constrains the state to lie inside an ellipsoidal region centered in the operating point of the system. Under the assumption of bounded disturbances, the satisfaction of this assumption can be ensured by designing the controller using a quadratic boundedness specification, as proposed first by Brockman and Corless[48].

3.1 | $\mathcal{H}_\infty$ optimal design

Under the above assumptions, the following theorem is proposed for designing an observer gain $L(\theta(t))$ which provides convergence of $e(t)$ to zero with some guaranteed decay rate α for any initial augmented state $\bar{x}(0) \in \mathcal{P}_{\bar{x}}$ with initial estimation error $e(0) \in \mathcal{P}_e$, while at the same time ensuring the fulfillment of (12).

Theorem 1. Given $\gamma > 0$, let $P \succ 0$, $0 < \eta < 1$, and the matrix function $\Gamma(\theta) \in \mathbb{R}^{n\times m}$ satisfy $\forall i, j \in \{1, \dots, p\}$, $\forall k \in \{1, \dots, q\}$ and $\forall \theta \in \Theta$:

$$\begin{bmatrix} P & \bar{x}_{(i)} - c \\ \left(\bar{x}_{(i)} - c\right)^T & 1 \end{bmatrix} \succ 0 \tag{19}$$

$$\begin{bmatrix} P & \eta P a_k \\ \eta a_k^T P & 1 \end{bmatrix} \geq 0 \tag{20}$$

$$\begin{bmatrix} Q & \eta Q a_k \\ \eta a_k^T Q & 1 \end{bmatrix} \geq 0 \tag{21}$$

$$\eta\alpha P + \text{He}\{\eta \left[P\bar{A}(\theta) - \Gamma(\theta)\bar{C}(\theta)\right] + P\Psi(\cdot)\} \prec 0 \tag{22}$$

$$\begin{bmatrix} \text{He}\{\eta \left[P\bar{A}(\theta) - \Gamma(\theta)\bar{C}(\theta)\right] + P\Psi(\cdot)\} + \eta C_z(\theta)^T C_z(\theta) & \sqrt{\eta}\left[P\bar{G}(\theta) - \Gamma(\theta)\bar{H}(\theta)\right] \\ \sqrt{\eta}\left[\bar{G}(\theta)^T P - \bar{H}(\theta)^T \Gamma(\theta)^T\right] & -\gamma I \end{bmatrix} \prec 0 \tag{23}$$

with:

$$\Psi(\cdot) = \begin{bmatrix} \left((\eta-1)c + \bar{x}_{(j)}\right)^T \text{He}\{N_1(\theta)\} - \left(\bar{x}_{(i)} - c\right)^T N_1(\theta) \\ \left((\eta-1)c + \bar{x}_{(j)}\right)^T \text{He}\{N_2(\theta)\} - \left(\bar{x}_{(i)} - c\right)^T N_2(\theta) \\ \vdots \\ \left((\eta-1)c + \bar{x}_{(j)}\right)^T \text{He}\{N_n(\theta)\} - \left(\bar{x}_{(i)} - c\right)^T N_n(\theta) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{24}$$

Then, the observer structure (7) with $L(\theta(t)) = P^{-1}\Gamma(\theta(t))$ causes the estimation error $e(t)$ to converge towards zero with a guaranteed decay rate α for any $\bar{x}(0) \in \mathcal{P}_{\bar{x}}$ and $e(0) \in \mathcal{P}_e$ while ensuring the constraint (12) on the $\mathcal{L}_2$ gain from w to z.

Proof: Let us start with formulating the following candidate Lyapunov function:

$$V(e(t)) = e(t)^T P e(t) \tag{25}$$

According to Boyd et al.[49], Eq. (19) ensures that:

$$\mathcal{P}_e \subset \mathcal{E} = \{e \in \mathbb{R}^{n+n_f} : V(e) \leq 1\} \tag{26}$$

Let us notice that due to Assumption 2 and (26), if the candidate Lyapunov function in (25) satisfies:

$$\dot{V}(\theta(t), e(t), \bar{x}(t)) + \alpha V(e(t)) < 0 \tag{27}$$

for all $\theta \in \Theta$, $e \in \mathcal{E}$ and $\bar{x} \in \mathcal{X}$, then $e(t)$ would converge to zero with guaranteed decay rate α for any $\bar{x}(0) \in \mathcal{P}_{\bar{x}}$, $e(0) \in \mathcal{P}_e$. On the other hand, if the candidate Lyapunov function in (25) satisfies:

$$\dot{V}(\theta(t), e(t), \bar{x}(t)) + z(t)^T z(t) - \gamma w(t)^T w(t) < 0 \tag{28}$$

for all $\theta \in \Theta$, $e \in \mathcal{E}$ and $\bar{x} \in \mathcal{X}$, then the condition (12) would hold.

By taking into account the dynamics of $e(t)$ in (8), the following calculations can be performed:

$$\begin{aligned} \dot{V}(\theta(t), e(t), \bar{x}(t)) + \alpha V(e(t)) = &\ \alpha e(t)^T P e(t) + \text{He}\{e(t)^T P \tilde{N}(\theta(t), e(t))\, \bar{x}(t)\} \\ &+ \text{He}\{e(t)^T P \left[\bar{A}(\theta(t)) - L(\theta(t))\, \bar{C}(\theta(t)) - \bar{N}(\theta(t), e(t))\right] e(t)\} \end{aligned} \tag{29}$$

Since:

$$\tilde{N}(\theta(t), e(t))\, \bar{x}(t) = \tilde{N}(\theta(t), \bar{x}(t))\, e(t) \tag{30}$$

one can rewrite (29) as a quadratic form in $e(t)$, such that (27) becomes equivalent to the following condition:

$$\alpha P + \text{He}\{P \left[\tilde{N}(\theta, \bar{x}) + \bar{A}(\theta) - L(\theta)\, \bar{C}(\theta) - \bar{N}(\theta, e)\right]\} \prec 0 \tag{31}$$

where the product of unknown variables P and $\bar{L}(\theta)$ can be get rid of by using the change of variable $\Gamma(\theta) = PL(\theta)$, thus obtaining:

$$\alpha P + \text{He}\{P \left[\bar{A}(\theta) - \bar{N}(\theta, e) + \tilde{N}(\theta, \bar{x})\right] - \Gamma(\theta)\bar{C}(\theta)\} \prec 0 \tag{32}$$

In order to ensure that (32) holds for any $e \in \mathcal{E}$, $\bar{x} \in \mathcal{X}$, let us introduce the following enlarged versions of (16)-(17):

$$\tilde{\mathcal{P}}_{\bar{x}} = \mathrm{Co}\{c + \rho\left(\bar{x}_{(1)} - c\right), \dots, c + \rho\left(\bar{x}_{(p)} - c\right)\} \\ = \{\bar{x} \in \mathbb{R}^{n+n_f} : \eta a_k^T(\bar{x} - c) = \frac{a_k^T}{\rho}(\bar{x} - c) \leq 1, k = 1, \dots, q\} \tag{33}$$

$$\tilde{\mathcal{P}}_e = \mathrm{Co}\{\rho(\bar{x}_{(1)} - c), \dots, \rho(\bar{x}_{(p)} - c)\} = \{\bar{x} \in \mathbb{R}^{n+n_f} : \eta a_k^T \bar{x} = \frac{a_k^T}{\rho}\bar{x} \leq 1, k = 1, \dots, q\} \tag{34}$$

By exploiting the Schur complements, (20)-(21) are equivalent to:

$$\eta^2 a_k^T P a_k \leq 1 \tag{35}$$

$$\eta^2 a_k^T Q a_k \leq 1 \tag{36}$$

and guarantee that $\mathcal{E} \subseteq \tilde{\mathcal{P}}_e$ and $\mathcal{X} \subseteq \tilde{\mathcal{P}}_{\bar{x}}$. Hence, one can rewrite (32) at the vertices of the polytopes $\tilde{\mathcal{P}}_e$ and $\tilde{\mathcal{P}}_{\bar{x}}$, obtaining (22).

Similarly, by taking into account the dynamics of $e(t)$, the following can be obtained from (28):

$$\mathrm{He}\{e(t)^T P\left[\bar{A}(\theta(t)) - L(\theta(t))\bar{C}(\theta(t)) - \bar{N}(\theta(t), e(t))\right] e(t)\} + \mathrm{He}\{e(t)^T P \tilde{N}(\theta(t), e(t))\bar{x}(t)\} \\ + \mathrm{He}\left\{e(t)^T P\left[\bar{G}(\theta(t)) - L(\theta(t))\bar{H}(\theta(t))\right] w(t)\right\} + e(t)^T C_z(\theta(t))^T C_z(\theta(t)) e(t) - \gamma w(t)^T w(t) < 0 \tag{37}$$

which, following steps similar to the ones described above, can be shown to be equivalent to:

$$\begin{bmatrix} \mathrm{He}\{P\left[\bar{A}(\theta) - \bar{N}(\theta, e) + \tilde{N}(\theta, \bar{x})\right] - \Gamma(\theta)\bar{C}(\theta)\} + C_z(\theta)^T C_z(\theta) & P\bar{G}(\theta) - \Gamma(\theta)\bar{H}(\theta) \\ \bar{G}(\theta)^T P - \bar{H}(\theta)^T\Gamma(\theta)^T & -\gamma I \end{bmatrix} \prec 0 \tag{38}$$

Then, (38) can be ensured to hold for any $e \in \mathcal{E}$, $\bar{x} \in \mathcal{X}$ by rewriting it at the vertices of the polytopes $\tilde{\mathcal{P}}_e$ and $\tilde{\mathcal{P}}_{\bar{x}}$, thus obtaining (23), which completes the proof. □

The $\mathcal{H}_\infty$ optimal design problem can be recast as the problem of finding the minimum value of the scalar $\gamma > 0$ which preserves feasibility of the matrix inequalities in Theorem 1.

3.2 | $\mathcal{H}_2$ optimal design

The following theorem is proposed for designing an observer gain $L(\theta(t))$ which provides convergence of $e(t)$ to zero with some guaranteed decay rate α for any initial augmented state $\bar{x}(0) \in \mathcal{P}_{\bar{x}}$ with initial estimation error $e(0) \in \mathcal{P}_e$, while at the same time ensuring the fulfillment of (13)-(15).

Theorem 2. Let $P \succ 0$, $Z \succ 0$, $0 < \eta < 1$, and the matrix function $\Gamma(\theta) \in \mathbb{R}^{n\times m}$ satisfy $\forall i, j \in \{1, \dots, p\}$, $\forall k \in \{1, \dots, q\}$ and $\forall \theta \in \Theta$ the matrix inequalities (19)-(22) and:

$$\mathrm{trace}(Z) < \mu \tag{39}$$

$$\begin{bmatrix} -Z & C_z(\theta) \\ C_z(\theta)^T & -P \end{bmatrix} \prec 0 \tag{40}$$

$$\begin{bmatrix} \mathrm{He}\{\eta\left[P\bar{A}(\theta) - \Gamma(\theta)\bar{C}(\theta)\right] + P\Psi(\cdot)\} & \sqrt{\eta}\left[P\bar{G}(\theta) - \Gamma(\theta)\bar{H}(\theta)\right] \\ \sqrt{\eta}\left[\bar{G}(\theta)^T P - \bar{H}(\theta)^T\Gamma(\theta)^T\right] & -I \end{bmatrix} \prec 0 \tag{41}$$

with $\Psi(\cdot)$ defined as in (24). Then, the observer structure (7) with $L(\theta(t)) = P^{-1}\Gamma(\theta(t))$ causes the estimation error $e(t)$ to converge towards zero with a guaranteed decay rate α for any $\bar{x}(0) \in \mathcal{P}_{\bar{x}}$ and $e(0) \in \mathcal{P}_e$ while ensuring that (13)-(15) hold.

Proof: The first part of the proof follows the proof of Theorem 1 by explaining the inclusion of the matrix inequalities (19)-(22), and it is thus omitted. According to[50], (13) is equivalent to the existence of $Z \succ 0$ such that:

$$C_z(\theta) X C_z(\theta)^T \prec Z \tag{42}$$

where Z satisfies (39). From direct comparison between (14) and (25), it is clear that $X = P^{-1}$, so that (42) becomes:

$$C_z(\theta) P^{-1} C_z(\theta)^T \prec Z \tag{43}$$

and, through Schur complements, (40) is obtained. On the other hand, by means of Schur complements and the change of variable $\Gamma(\theta) = PL(\theta)$, the inequality (15) can be shown to be equivalent to:

$$\begin{bmatrix} \mathrm{He}\{P\left[\bar{A}(\theta) - \bar{N}(\theta, e) + \tilde{N}(\theta, \bar{x})\right] - \Gamma(\theta)\bar{C}(\theta)\} & P\bar{G}(\theta) - \Gamma(\theta)\bar{H}(\theta) \\ \bar{G}(\theta)^T P - \bar{H}(\theta)^T\Gamma(\theta)^T & -I \end{bmatrix} \prec 0 \tag{44}$$

Then, (44) can be ensured to hold for any $e \in \mathcal{E}$, $\bar{x} \in \mathcal{X}$ by rewriting it at the vertices of the polytopes $\tilde{P}_e$ and $\tilde{P}_{\bar{x}}$, thus obtaining (41), which completes the proof. □

The $\mathcal{H}_2$ optimal design problem can be recast as the problem of finding the minimum value of the scalar $\mu > 0$ which preserves feasibility of the matrix inequalities in Theorem 2.

4 | DESIGN CONSIDERATIONS

The main difficulty with the direct application of Theorems 1-2 is that they require the assessment of an infinite number of conditions due to the dependency of (22)-(23) and (40)-(41) on θ. In order to reduce the number to finite, let us consider a polytopic assumption on the dependency of matrix functions $\bar{A}(\theta)$, $\bar{C}(\theta)$, $\bar{G}(\theta)$, $\bar{H}(\theta)$, $N_1(\theta), \dots, N_n(\theta)$, $\Gamma(\theta)$, $C_z(\theta)$ on θ, i.e.:

$$\begin{pmatrix} \bar{A}(\theta(t)) \\ \bar{C}(\theta(t)) \\ \bar{G}(\theta(t)) \\ \bar{H}(\theta(t)) \\ N_1(\theta(t)) \\ \vdots \\ N_n(\theta(t)) \\ \Gamma(\theta(t)) \\ C_z(\theta(t)) \end{pmatrix} = \sum_{l=1}^{N} \mu_l(\theta(t)) \begin{pmatrix} \bar{A}_l \\ \bar{C}_l \\ \bar{G}_l \\ \bar{H}_l \\ N_{1,l} \\ \vdots \\ N_{n,l} \\ \Gamma_l \\ C_{z,l} \end{pmatrix} \tag{45}$$

with some finite N and:

$$\sum_{l=1}^{N} \mu_l(\theta(t)) = 1, \quad \mu_l(\theta(t)) \geq 0, \quad \forall l = 1, \dots, N, \quad \forall \theta \in \Theta \tag{46}$$

Since the term $\Gamma(\theta)\bar{C}(\theta)$ in (22), under the assumption (45), corresponds to a double polytopic sum, Polya's theorem on definite quadratic forms is applied in the following. As discussed comprehensively by Sala and Ariño[51], the application of Polya's theorem provides a set of sufficient conditions to assess the definiteness of double sums, which are progressively less conservative while a complexity parameter $s \in \mathbb{N}$ increases. Moreover, such conditions are asymptotically exact, i.e., there exists a finite s ensuring that they become necessary and sufficient.

Given $s \in \mathbb{N}$, let us define the following sets:

$$\mathbb{P}_s = \left\{ \vec{p} = \left[p_1, p_2, \dots, p_s\right] \in \mathbb{N}^s | 1 \leq p_k \leq s \quad \forall k = 1, \dots, s \right\} \tag{47}$$

$$\mathbb{P}_s^+ = \left\{ \vec{p} \in \mathbb{P}_s | p_k \leq p_{k+1}, k = 1, \dots, s-1 \right\} \tag{48}$$

and let us denote as $\mathcal{P}(\vec{p}) \subset \mathbb{P}_s$ the set of permutations, with possible repeated elements, of the multi-index $\vec{p}$. Then, the following corollaries hold.

Corollary 1. Given $\gamma > 0$ and $s \in \mathbb{N}$, let $P \succ 0$, $0 < \eta < 1$, and the matrices $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}^{n \times m}$ satisfy $\forall i, j \in \{1, \dots, p\}$, $\forall k \in \{1, \dots, q\}$ and $\forall \vec{p} \in \mathbb{P}_s^+$:

$$\eta \alpha P + \sum_{\vec{r} \in \mathcal{P}(\vec{p})} \text{He}\{\eta \left[P\bar{A}_{r_1} - \Gamma_{r_1}\bar{C}_{r_2}\right] + P\Psi_{ij,r_1}\} \prec 0 \tag{49}$$

$$\sum_{\vec{r} \in \mathcal{P}(\vec{p})} \begin{bmatrix} \text{He}\{\eta \left[P\bar{A}_{r_1} - \Gamma_{r_1}\bar{C}_{r_2}\right] + P\Psi_{ij,r_1}\} & \sqrt{\eta}\left[P\bar{G}_{r_1} - \Gamma_{r_1}\bar{H}_{r_2}\right] & \sqrt{\eta}C_{z,r_1}^T \\ \sqrt{\eta}\left[\bar{G}_{r_1}^T P - \bar{H}_{r_2}^T \Gamma_{r_1}^T\right] & -\gamma I & 0 \\ \sqrt{\eta}C_{z,r_1} & 0 & -I \end{bmatrix} \prec 0 \tag{50}$$

and the matrix inequalities (19)-(21), with:

$$\Psi_{ij,r_1} = \begin{bmatrix} \left((\eta-1)c+\bar{x}_{(j)}\right)^T \mathrm{He}\{N_{1,r_1}\} - \left(\bar{x}_{(i)}-c\right)^T N_{1,r_1} \\ \left((\eta-1)c+\bar{x}_{(j)}\right)^T \mathrm{He}\{N_{2,r_1}\} - \left(\bar{x}_{(i)}-c\right)^T N_{2,r_1} \\ \vdots \\ \left((\eta-1)c+\bar{x}_{(j)}\right)^T \mathrm{He}\{N_{n,r_1}\} - \left(\bar{x}_{(i)}-c\right)^T N_{n,r_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{51}$$

Then, the observer shaped by (7) along with:

$$L(\theta(t)) = \sum_{l=1}^{N} \mu_l(\theta(t))\, L_l, \quad L_l = P^{-1}\Gamma_l \tag{52}$$

causes that the estimation error $e(t)$ converges to zero with guaranteed decay rate α for any $\bar{x}(0) \in \mathcal{P}_{\bar{x}}$ and $e(0) \in \mathcal{P}_e$, while ensuring the constraint (12).

Proof: This corollary is a consequence of the fact that (49)-(50) are obtained from (22)-(23) through Schur complements and by applying Polya's theorem on definite quadratic forms, as proposed, e.g., by Sala and Ariño[51]. □

Corollary 2. Given $\gamma > 0$ and $s \in \mathbb{N}$, let $P \succ 0$, $Z \succ 0$, $0 < \eta < 1$, and the matrices $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}^{n\times m}$ satisfy $\forall i, j \in \{1, \dots, p\}$, $\forall k \in \{1, \dots, q\}$ and $\forall \vec{p} \in \mathbb{P}_s^+$ the matrix inequalities (19)-(21), (39), (49) and:

$$\sum_{\vec{r}\in\mathcal{P}(\vec{p})} \begin{bmatrix} -Z & C_{z,r_1} \\ C_{z,r_1}^T & -P \end{bmatrix} \prec 0 \tag{53}$$

$$\sum_{\vec{r}\in\mathcal{P}(\vec{p})} \begin{bmatrix} \mathrm{He}\{\eta\left[P\bar{A}_{r_1} - \Gamma_{r_1}\bar{C}_{r_2}\right] + P\Psi_{ij,r_1}\} & \sqrt{\eta}\left[P\bar{G}_{r_1} - \Gamma_{r_1}\bar{H}_{r_2}\right] \\ \sqrt{\eta}\left[\bar{G}_{r_1}^T P - \bar{H}_{r_2}^T\Gamma_{r_1}^T\right] & -I \end{bmatrix} \prec 0 \tag{54}$$

with Ψ_{ij,r_1} defined as in (51). Then, the observer shaped by (7) with $L(\theta(t))$ computed using (52) causes the estimation error $e(t)$ to converge towards zero with a guaranteed decay rate α for any $\bar{x}(0) \in \mathcal{P}_{\bar{x}}$ and $e(0) \in \mathcal{P}_e$ while ensuring that (13)-(15) hold.

Proof: This corollary is a consequence of the fact that (53)-(54) are obtained from (40)-(41) by applying Polya's theorem on definite quadratic forms, as proposed, e.g., by Sala and Ariño[51]. □

Notice that, owing to the product ηP and the presence of $\sqrt{\eta}$, (20), (49)-(50) and (54) represent nonlinear matrix inequalities. However, it is possible to grid η in the interval $[0, 1]$, and apply iteratively one among Corollary 1 and Corollary 2 for each fixed η. In this way, (20), (49)-(50) and (54) simply become LMIs. Thus, together with (19), (21) and (53) their solutions can be obtained using, e.g., YALMIP[52]/SeDuMi[53].

5 | ILLUSTRATIVE EXAMPLE

This section concerns a numerical verification of the proposed approach. Let us start with considering a QPV system shaped by (1)–(2), with:

$$A(\theta(t)) = \begin{bmatrix} -4-\theta_1(t) & 10 & 2+2\theta_2(t) \\ -1 & -1-\theta_2(t) & 1.5+2\theta_1(t) \\ 1 & 1 & -4-3\theta_1(t) \end{bmatrix}, \quad B(\theta(t)) = \begin{bmatrix} -1.2-\theta_1(t) & 0 & 0.7 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_{f,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{f,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{f,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.15 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}, \quad H = \begin{bmatrix} 0.025 & 0 & 0 \\ 0 & 0.015 & 0 \\ 0 & 0 & 0.025 \end{bmatrix}$$

with $\theta_1, \theta_2 \in [0, 1]$ and $N_1^{12} = A_{f,1}$, $N_2^{12} = A_{f,2}$, $N_3^{12} = A_{f,3}$. Following Assumption 2, the augmented state variable $\bar{x}(t)$ belongs to the ellipsoid defined by (18), with:

$$Q = \mathrm{diag}\,(0.2, 0.2, 0.2, 0.1, 0.1, 0.1), \quad c = [10,\ 10,\ 10,\ 0,\ 0,\ 0]^T$$

which puts constraints on the trajectories for the state variable $x(t)$ and the possible fault magnitudes for which the approach proposed in this paper provides theoretical guarantees of convergence of the estimated variable to the real ones.

Let us start by assuming that $w(t) = 0$, which gives a simplified form (1)–(2) as follows:

$$\dot{x}(t) = A\left(\theta(t)\right) x(t) + B\left(\theta(t)\right) u(t) + \sum_{k=1}^{n_f} A_{f,k}\left(\theta(t)\right) f_k(t) x(t) \tag{55}$$

$$y(t) = C\left(\theta(t)\right) x(t) \tag{56}$$

Using the design procedure described in Section 3, the following observer matrices are obtained with $\eta = 0.2$, $\alpha = 0.1$, $\gamma = 0.2$:

$$L_1 = \left[\begin{array}{ccc} 71.0525 & -30.1291 & -11.5840 \\ 103.3840 & -24.6506 & -19.2940 \\ -120.2296 & 22.5926 & 42.0515 \\ \hline 110.9573 & -96.8190 & -33.2168 \\ -27.9764 & 74.9819 & 21.2993 \\ -97.3858 & 53.6423 & 44.2481 \end{array}\right], \quad L_2 = \left[\begin{array}{ccc} 64.3704 & -25.0137 & -9.8594 \\ 86.1731 & -10.3352 & -14.5522 \\ -85.2083 & -8.2030 & 33.3839 \\ \hline 106.8744 & -93.9764 & -34.1212 \\ -34.3997 & 81.1466 & 24.2239 \\ -78.4953 & 37.2649 & 40.1757 \end{array}\right],$$

$$L_3 = \left[\begin{array}{ccc} 75.4617 & -35.8183 & -13.0476 \\ 115.4117 & -37.3239 & -22.2768 \\ -141.1897 & 44.7093 & 46.2986 \\ \hline 113.7898 & -99.8003 & -33.4420 \\ -23.1945 & 69.9437 & 18.6991 \\ -108.5528 & 65.4271 & 47.7827 \end{array}\right], \quad L_4 = \left[\begin{array}{ccc} 75.4617 & -35.8183 & -13.0476 \\ 115.4117 & -37.3239 & -22.2768 \\ -141.1897 & 44.7093 & 46.2986 \\ \hline 113.7898 & -99.8003 & -33.4420 \\ -23.1945 & 69.9437 & 18.6991 \\ -108.5528 & 65.4271 & 47.7827 \end{array}\right],$$

In order to exemplify the behavior of the proposed state/fault QPV estimator, let us consider simulations for which the system and observer initial conditions are given by:

$$x(0) = [10,\ 10,\ 10]^T, \quad \bar{x}(0) = [11,\ 10.5,\ 12,\ 0,\ 0,\ 0]^T.$$

while the trajectories for the varying parameters are given by:

$$\theta_1(t) = 0.5 + 0.5\sin(2\pi t/5), \quad \theta_2(t) = 0.5 + 0.5\cos(2\pi t/3).$$

The input signal $u(t)$ is calculated as:

$$\begin{aligned} U_p &= 0.065, \\ u_1(t) &= u_1^{eq}(\theta(t)) + U_p(5\cos(t) + 5\cos(4t - 12\pi/9) + 5\cos(7t - 42\pi/9)), \\ u_2(t) &= u_2^{eq}(\theta(t)) + U_p(\cos(2t - 2\pi/9) + \cos(5t - 20\pi/9) + \cos(8t - 56pi/9)), \\ u_3(t) &= u_3^{eq}(\theta(t)) + U_p(\cos(3t - 6\pi/9) + \cos(6t - 30\pi/9) + \cos(9t - 72pi/9)). \end{aligned}$$

where $u_1^{eq}(\theta(t))$, $u_2^{eq}(\theta(t))$ and $u_3^{eq}(\theta(t))$ are the parameter-dependent input values which drive the state trajectory to an equilibrium point corresponding to c, while the remaining terms (depending on U_p) denote additive perturbations.

For the sake of comparative study, the following fault scenarios (FS) are introduced:

$$\textbf{FS1} \rightarrow \begin{cases} f_1(t) = \begin{cases} -0.1 & 100 \le t \le 140, \\ 0 & \text{otherwise.} \end{cases} \\ f_2(t) = \begin{cases} 0.2 & 120 \le t \le 160, \\ 0 & \text{otherwise.} \end{cases} \\ f_3(t) = \begin{cases} -0.05 & 130 \le t \le 175, \\ 0 & \text{otherwise.} \end{cases} \end{cases} \qquad \textbf{FS2} \rightarrow \begin{cases} f_1(t) = \begin{cases} -0.005(t-50) & 50 \le t \le 70, \\ 0 & \text{otherwise.} \end{cases} \\ f_2(t) = \begin{cases} 0.0025(t-60) & 60 \le t \le 80, \\ 0 & \text{otherwise.} \end{cases} \\ f_3(t) = \begin{cases} -0.001(t-70) & 70 \le t \le 100, \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Let us begin the simulation study with the fault–free case, i.e., $f_k(t) = 0$, which will be denoted hereafter as FS0. Figures 1 and 2 portray the state and fault estimate, respectively. By observing the original values and their estimates, it is evident that both the state and the process faults are estimated correctly. Indeed, the process fault estimate converges to zero after some short settling time caused by the discrepancy between real and assumed initial conditions. Now let us proceed to the faulty case, i.e., $f_k \neq 0$. Figures 3–5 show the faults and their estimates for FS1 and FS2, along with the state variables and their estimates for FS2. The achieved estimation results confirm clearly the theoretical results provided for the developed approach. This appealing property makes the proposed approach a solid candidate for the prospective application in an integrated process FTC scheme.

Subsequently, Figure 6 plots the value of the function $(\bar{x}(t) - c)^T Q^{-1} (\bar{x}(t) - c)$ which describes whether $\bar{x}(t) \in \mathcal{X}$ or not, depending on whether the value of this function is less than or bigger than 1, respectively. It is shown that the trajectories of $\bar{x}(t)$ remain inside the ellipsoid $\mathcal{X}$ in all the three scenarios FS0, FS1 and FS2, thus theoretical convergence guarantees are provided. It is worth remarking that the proposed approach would work in practice for a much wider set of trajectories, not necessarily contained inside $\mathcal{X}$. For instance, Figure 7 shows state and fault estimation results for a simulation of FS2 for which the value of U_p was increased up to 1 and the fault magnitude was increased of an order of magnitude when compared with the previous results. These results show that, although the augmented state trajectory is not contained in $\mathcal{X}$ (see Figure 7(c) for the values of the function $(\bar{x}(t) - c)^T Q^{-1} (\bar{x}(t) - c)$, which now become much bigger than 1), the proposed approach is still able to provide good state and fault estimates, albeit with the loss of theoretical guarantees of convergence.

For the sake of completeness, let us consider the full representation of (1)–(2) (i.e. $w(t) \neq 0$). Using the design procedure described in Section 3 along with $\mathcal{H}_2$ constraints (39)–(41), the following observer matrices are obtained:

$$L_1 = \left[\begin{array}{ccc} 427.2646 & -259.7506 & -16.3848 \\ -129.0876 & 549.1894 & -14.4332 \\ 2.4575 & 2.3203 & 40.20427 \\ \hline 862.9942 & -526.4240 & -35.0245 \\ -247.4478 & 1091.7887 & -30.6016 \\ 0.1198 & 0.1033 & 31.6956 \end{array}\right], \quad L_2 = \left[\begin{array}{ccc} 427.2820 & -259.5539 & -46.5346 \\ -129.2089 & 547.9813 & -0.7498 \\ 4.1287 & 2.3621 & 40.2119 \\ \hline 863.0264 & -526.0396 & -96.9897 \\ -247.6797 & 1091.3691 & -4.1523 \\ 0.2363 & 0.1062 & 31.6956 \end{array}\right],$$

$$L_3 = \left[\begin{array}{ccc} 425.7364 & -260.0593 & -1.1649 \\ -128.5972 & 549.3101 & -51.8865 \\ 2.4981 & 4.1526 & 37.2134 \\ \hline 861.9118 & -527.0171 & -5.1480 \\ -246.5044 & 1092.02101 & -106.6068 \\ 0.1226 & 0.2310 & 31.6960 \end{array}\right], \quad L_4 = \left[\begin{array}{ccc} 425.7576 & -259.8632 & -31.3115 \\ -128.7288 & 548.1022 & -38.2032 \\ 4.1696 & 4.1944 & 37.2211 \\ \hline 861.9515 & -526.6341 & -67.1056 \\ -246.7571 & 1091.6018 & -80.1577 \\ 0.2391 & 0.2340 & 31.6959 \end{array}\right],$$

with $\eta = 0.2$, $\alpha = 0.1$, $\gamma = 0.2$ and $\mu = 2.0024$ where the following value has been used for the matrix C_z:

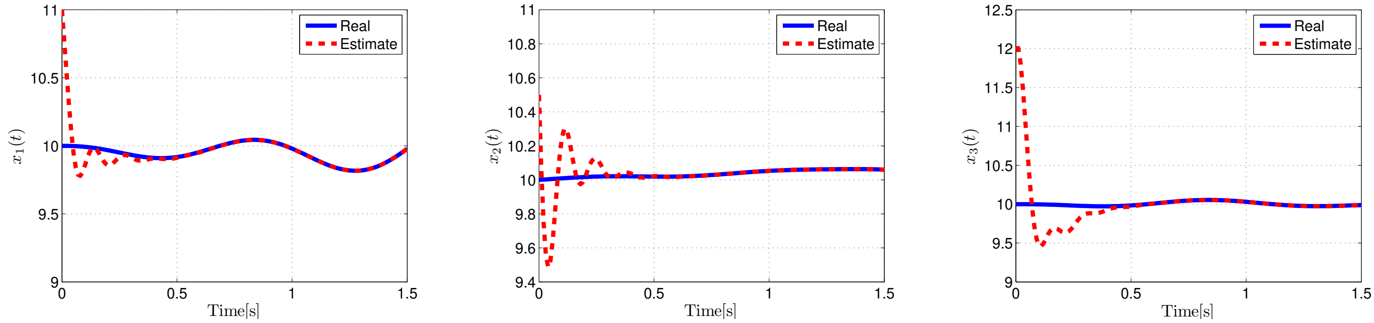
$$C_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that the effect of the disturbance on the estimation error for the fault $f_3(t)$ is minimized. For the purpose of further comparison, two simulation cases are used:

SC1 - In this case, the fault estimation was obtain for FS2 and for observer gain matrices without $\mathcal{H}_2$ norm minimization;

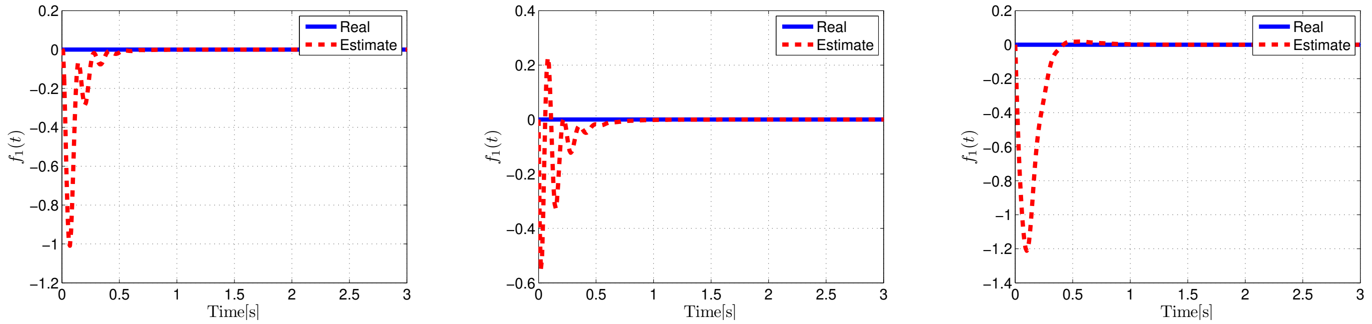
SC2 - In this case, the fault estimation was obtain for FS2 and for observer gain matrices with $\mathcal{H}_2$ norm minimization.

By selecting $w(t)$ as a uniformly distributed random vector, where each element takes values in the interval $[-2.5, 2.5]$, Figure 8 shows the fault estimation results as well as the values of $||e_{f,3}||$ for FS2 and SC1, SC2. The results show that the $\mathcal{H}_2$ optimal observer is able to reject the disturbance better than the non-optimal observer.



(a) (b) (c)

FIGURE 1 (a) $x_1(t)$ and its estimate $\hat{x}_1(t)$, (b) $x_2(t)$ and its estimate $\hat{x}_2(t)$, (c) $x_3(t)$ and its estimate $\hat{x}_3(t)$ (FS0, $t \in [0, 1.5]$)



(a) (b) (c)

FIGURE 2 (a) $f_1(t)$ and its estimate $\hat{f}_1(t)$, (b) $f_2(t)$ and its estimate $\hat{f}_2(t)$, (c) $f_3(t)$ and its estimate $\hat{f}_3(t)$ (FS0, $t \in [0, 3]$)

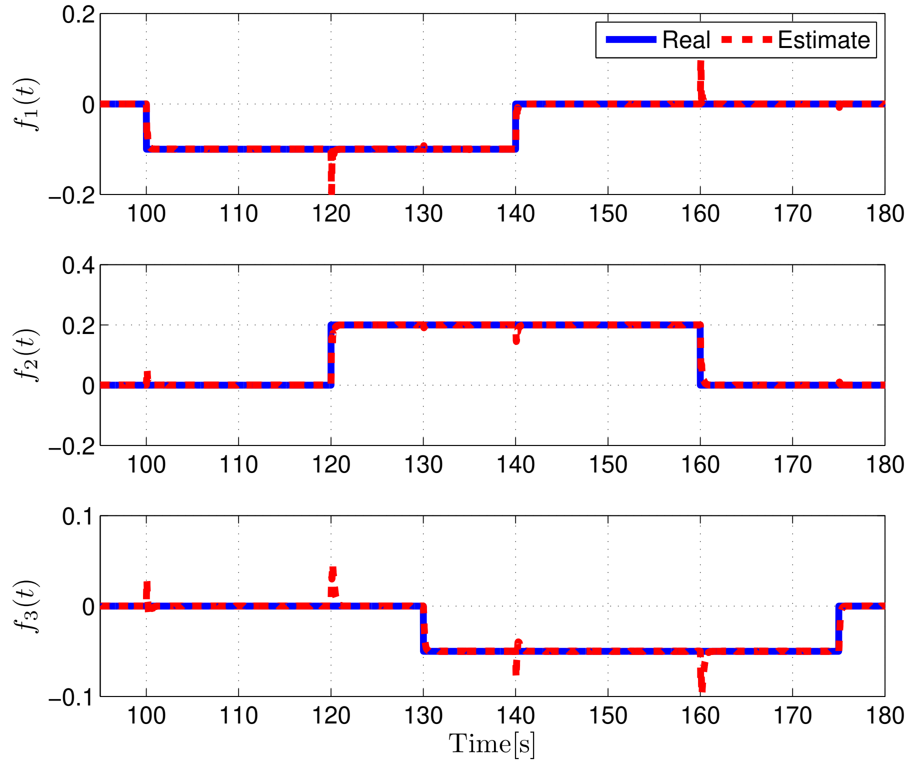


FIGURE 3 (Top) $f_1(t)$ and its estimate $\hat{f}_1(t)$, (Middle) $f_2(t)$ and its estimate $\hat{f}_2(t)$, (Bottom) $f_3(t)$ and its estimate $\hat{f}_3(t)$ (FS1, $t \in [105, 180]$)

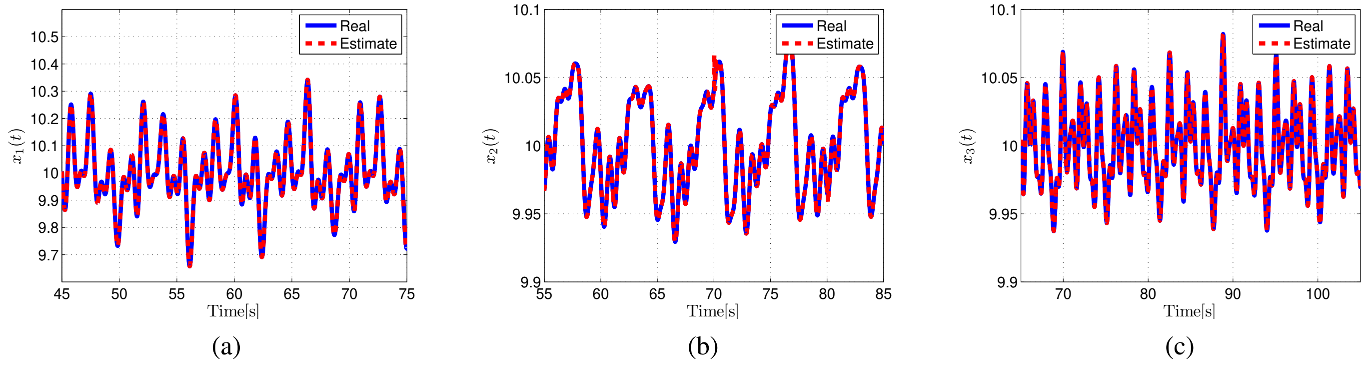


FIGURE 4 (a) $x_1(t)$ and its estimate $\hat{x}_1(t)$, (b) $x_2(t)$ and its estimate $\hat{x}_2(t)$, (c) $x_3(t)$ and its estimate $\hat{x}_3(t)$ (FS2, $t \in [45, 75]$)

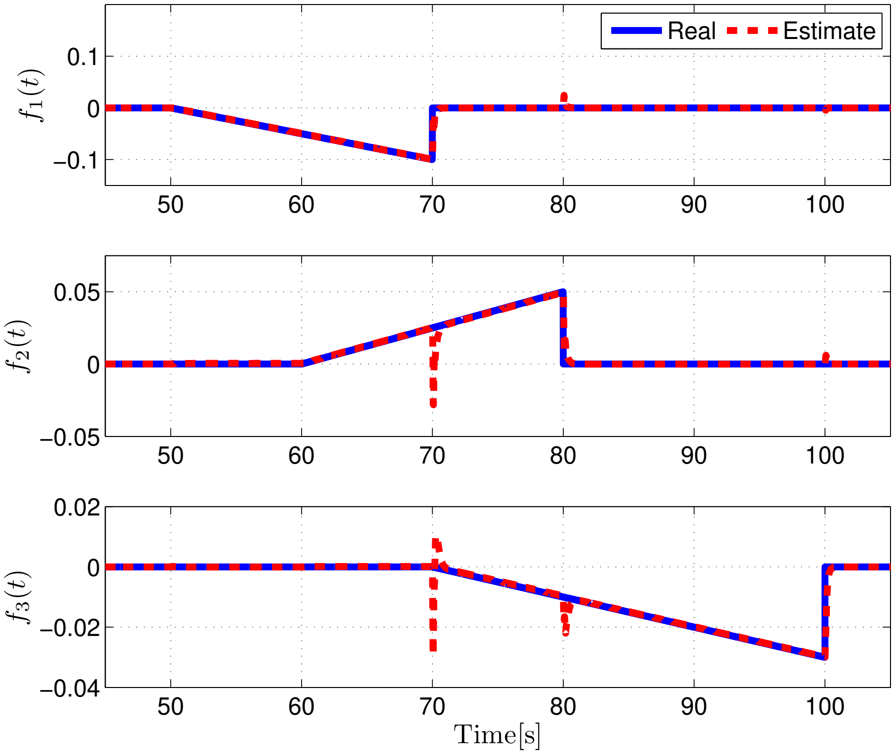


FIGURE 5 (Top) $f_1(t)$ and its estimate $\hat{f}_1(t)$, (Middle) $f_2(t)$ and its estimate $\hat{f}_2(t)$, (Bottom) $f_3(t)$ and its estimate $\hat{f}_3(t)$ (FS2, $t \in [45, 105]$)

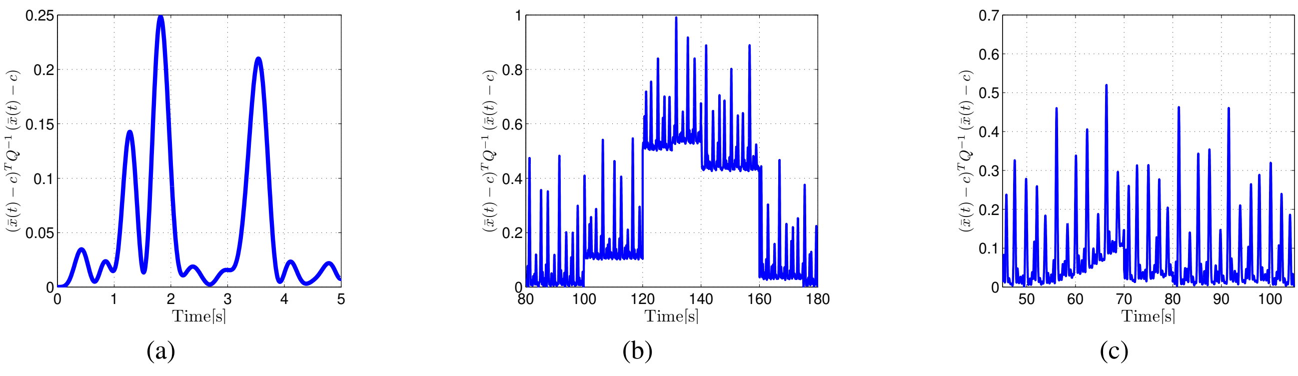


FIGURE 6 Value of the function describing the ellipsoid (18): (a) FS0, $t \in [0, 5]$, (b) FS1, $t \in [80, 180]$, (c) FS2, $t \in [45, 105]$

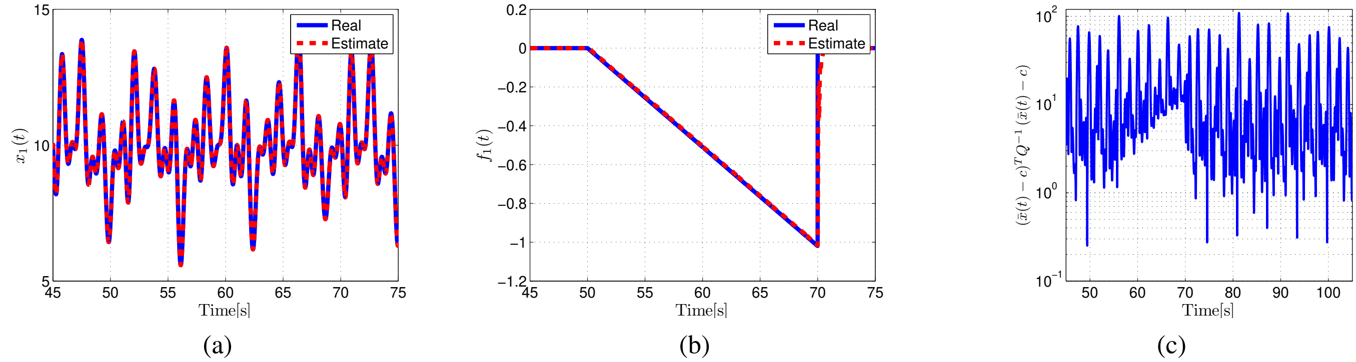


FIGURE 7 (a) $x_1(t)$ and its estimate $\hat{x}_1(t)$ (FS2, $t \in [45, 75]$), (b) $f_1(t)$ and its estimate $\hat{f}_1(t)$ (FS2, $t \in [45, 75]$), (c) Value of the function describing the ellipsoid (18) - logarithmic scale (FS2, $t \in [45, 105]$)

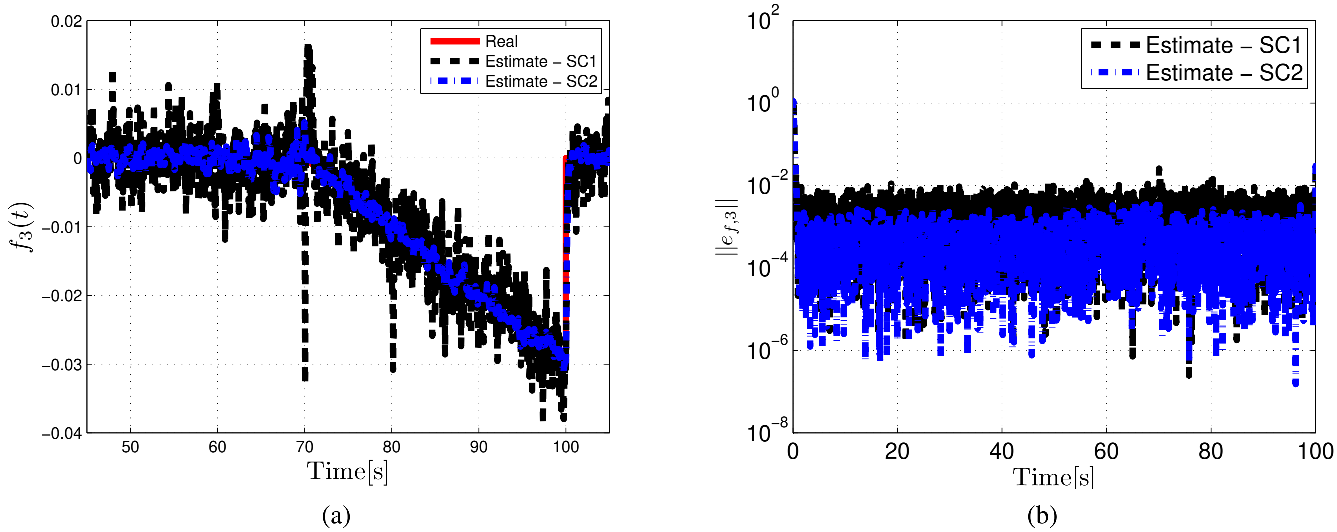


FIGURE 8 Comparison of SC1 and SC2: (a) $\mathbf{f_3(t)}$ and its estimates $\mathbf{\hat{f}_3(t)}$, (FS2, $\mathbf{t} \in [\mathbf{45}, \mathbf{75}]$); (b) $||e_{f,3}||$, (FS2, $\mathbf{t} \in [\mathbf{0}, \mathbf{100}]$)

6 | CONCLUSIONS

The main objective of this paper was to undertake the problem of simultaneous estimation of process faults and states in linear parameter varying systems. Contrarily to the approaches available in the literature, the estimation problem was tackled by reformulating the original problem of the simultaneous process fault and state estimation for linear parameter varying systems into an equivalent state estimation problem for quadratic parameter varying systems. This allows avoiding the chain of restrictive assumptions imposed by the methods presented in the literature, which lead to a suboptimal solution of the problem. In the proposed approach, after transforming the original problem into a quadratic parameter varying equivalent, it is assumed that the state trajectory is bounded by a known ellipsoid along with supplementary linear constraints, which make the estimation problem tractable. Under these assumptions, it is proven that the estimation error of the proposed observer converges asymptotically to zero. This appealing behaviour is confirmed by the simulation study, which have shown that the proposed approach can handle simultaneous process faults.

Future research will aim at increasing the robustness of the proposed approach against structured uncertainties by seeking an interval formulation that can exploit adequately known bounds on the uncertainty affecting the system. Moreover, the integration of the proposed process fault estimation within an active FTC system that can tolerate possible delays and some degree of inaccuracy in the fault estimation will be investigated.

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