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Complex Analysis the Hard Way

A Measure Theoretical Approach

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Abstract. In this thesis we present a sophisticated, non-standard treatment of complex analysis using modern tools from measure theory and advanced analysis. This approach opens the path to deep and powerful results, including the regularity theorem and the global solution to the inhomogeneous Cauchy-Riemann equation on a disc and on the complex plane. Moreover, the resulting theory is suitable for generalisation to the further study of Riemann surfaces, and of complex geometry more generally.

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0. Introduction

The aim of this thesis is a development of complex analysis in the plane using techniques from measure theory and advanced analysis. Although this approach to complex analysis is more sophisticated than the standard one and requires substantial additional background, it has the benefit of allowing us to prove much more powerful and general results. This greater generality also makes the definitions and results readily extendable to the study of Riemann surfaces and higher-dimensional complex analysis and geometry.

Chapter 1 covers some necessary background about the linear algebra of complex vector spaces. This is applied in Chapter 2 to the introduction of complex tangent vectors, covectors and differential forms on smooth manifolds. Further notions about smooth manifolds are also discussed in Chapter 2, including orientability, measurable sets and measurable differential forms. The available tools from measure theory allow us to develop integration on smooth manifolds that are not necessarily second countable, as discussed in this chapter, culminating with a proof of Stokes' theorem in this more general setting. Chapter 3 contains a development of advanced analysis in Euclidean space, focusing on locally integrable functions and their smooth regularisation using mollifiers, and on linear differential operators.

In Chapter 4, we use many of the tools and results developed in the previous chapters to prove numerous impressive and compelling results involving integration on subsets of $\mathbb{C}$, mind-blowing properties of holomorphic functions, and solutions to the inhomogeneous Cauchy-Riemann equation. We start with some basic facts about holomorphic functions, and use polar coordinates to establish the local integrability of $1/z$ on $\mathbb{C}$. Using Stokes' theorem, we prove the Cauchy integral formula and Cauchy's theorem for the general case of C^1 functions. We then prove the existence of a local solution of the inhomogeneous Cauchy-Riemann equation for C^k functions, and also Montel's theorem, among other results. Using the full power of the material developed in Chapter 3, we prove the outstandingly deep and cool fact that a weak solution to the homogeneous Cauchy-Riemann equation is equal almost everywhere to a holomorphic function, which in turn implies the regularity theorem for solutions of the inhomogeneous equation. Following this, we establish the mean value property about holomorphic functions and a generalised version of Riemann's extension theorem. We conclude with a treatment of complex power series and their notorious and renowned consequences: among others, the global solution to the inhomogeneous Cauchy-Riemann equation on a disc and on $\mathbb{C}$, the identity theorem, the open mapping theorem and the maximum principle.

It is assumed that the reader has a solid background in real analysis, topology and basic measure theory, in addition to being familiar with the notion of a smooth manifold.

The general presentation of the material is based on [2]. However, the proofs of all results in this thesis have been constructed completely independently by the author without reference to any source, except for a very small number of isolated cases in which a trick was required that the author could not have been expected to work out in isolation within a reasonable period of time. Even in these very few cases, only a small push in the right direction sufficed for the author to complete the proof, since in searching for the right argument she had already acquired a deep and comprehensive understanding of the problem, and she would tell her supervisor off quite hard if he tried to reveal slightly more information than necessary.

The author is nevertheless very thankful for her supervisor's guidance and patience.

1.1. **Real and Complex Vector Spaces, Realifications and Complexifications.** Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces over $\mathbb{F}$. Recall that then the set of $\mathbb{F}$-linear maps from $\mathcal{V}$ to $\mathcal{W}$ is denoted by $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$, and it is itself a vector space over $\mathbb{F}$. The *(algebraic) dual space* of $\mathcal{V}$ is defined to be $\mathcal{V}^* := \mathrm{Hom}(\mathcal{V}, \mathbb{F})$. If $\mathcal{V}$ has finite dimension $n \in \mathbb{N}$ and $e_1, \dots, e_n \in \mathcal{V}$ is a basis for $\mathcal{V}$, then we can define the linear functionals $\lambda_1, \dots, \lambda_n$ in $\mathcal{V}^*$ characterised by

$$\lambda_j(e_k) := \delta_k^j = \begin{cases} 1 & \text{if } j = k\,, \\ 0 & \text{if } j \neq k\,, \end{cases} \qquad j, k \in \{1, \dots, n\}.$$

These n linear functionals then form a basis for the vector space $\mathcal{V}^*$, called the basis *dual* to $e_1, \dots, e_n$. It follows that the dual space of $\mathcal{V}$ has the same dimension as $\mathcal{V}$.

For each vector $v \in \mathcal{V}$, we can define the map

$$g_v : \mathcal{V}^* \to \mathbb{F}\,, \quad f \mapsto f(v)\,.$$

This map is linear, since for $f, h \in \mathcal{V}^*$ and $c \in \mathbb{F}$,

$$g_v(cf + h) = (cf + h)(v) = cf(v) + h(v) = cg_v(f) + g_v(h)\,.$$

Thus, $g_v \in (\mathcal{V}^*)^*$. We then obtain a map

$$\phi : \mathcal{V} \to (\mathcal{V}^*)^*\,, \quad v \mapsto g_v\,.$$

Proposition 1.1. *If $\mathcal{V}$ is a vector space over $\mathbb{F}$ of finite dimension $n \in \mathbb{N}$, then the map ϕ as defined above is an isomorphism of vector spaces.*

Proof. We first show linearity of ϕ. Let $u, v \in \mathcal{V}$, and $c \in \mathbb{F}$. Then, for all $f \in \mathcal{V}^*$

$$\begin{aligned} (\phi(cu + v))(f) &= g_{cu+v}(f) \\ &= f(cu + v) \\ &= cf(u) + f(v) \\ &= cg_u(f) + g_v(f) \\ &= (cg_u + g_v)(f) \\ &= (c\phi(u) + \phi(v))(f)\,, \end{aligned}$$

so $\phi(cu + v) = c\phi(u) + \phi(v)$. To show injectivity of ϕ, assume $v \in \mathcal{V}$ and $\phi(v) = 0$. Then, $g_v = 0$, so

$$\begin{aligned} v &= \lambda_1(v)e_1 + \cdots + \lambda_n(v)e_n \\ &= g_v(\lambda_1)e_1 + \cdots + g_v(\lambda_n)e_n \\ &= 0\,. \end{aligned}$$

For surjectivity, let $g \in (\mathcal{V}^*)^*$, and $v := g(\lambda_1)e_1 + \cdots + g(\lambda_n)e_n \in \mathcal{V}$. Then, for all $f \in \mathcal{V}^*$,

$$\begin{aligned} g(f) &= g(f(e_1)\lambda_1 + \cdots + f(e_n)\lambda_n) \\ &= f(e_1)g(\lambda_1) + \cdots + f(e_n)g(\lambda_n) \\ &= f(v) \\ &= g_v(f)\,, \end{aligned}$$

so $g = g_v = \phi(v)$. $\square$

If $\mathcal{V}$ has dimension 1 and $v \in \mathcal{V} \setminus \{0\}$, then for all $u \in \mathcal{V}$ there is a unique $c \in \mathbb{F}$ such that $u = cv$, since v is a basis for $\mathcal{V}$. We then define $\frac{u}{v} := c$. This defines a map

$$\psi_v : \mathcal{V} \to \mathbb{F}\,, \quad u \mapsto \frac{u}{v}\,,$$

which is linear, since it is just the basis for $\mathcal{V}^*$ dual to v. We may denote the map ψ_v by v^{-1}.

Remark 1.2. Suppose now that $\mathcal{V}$ is infinite-dimensional with basis $\{e_\alpha\}_{\alpha\in A}$, where A is an (infinite) indexing set. Then, every vector $v \in \mathcal{V}$ can be written uniquely as a finite linear combination

$$v = v^{j_1} e_{j_1} + \cdots + v^{j_m} e_{j_m}\,,$$

where $m \in \mathbb{N}$, $\{j_1, \ldots, j_m\} \subset A$, and $v^{j_1}, \ldots, v^{j_m} \in \mathbb{F}$. We may then define $v^\alpha := 0$ for $\alpha \notin \{j_1, \cdots, j_m\}$. Then, we can again define linear functionals $\{\lambda_\alpha\}_{\alpha\in A} \subset \mathcal{V}^*$ by

$$\lambda_\alpha(v) := v^\alpha$$

for each $\alpha \in A$ and $v \in \mathcal{V}$, characterised by

$$\lambda_\alpha(e_\beta) = \delta^\alpha_\beta\,, \qquad \alpha, \beta \in A\,.$$

However, these linear functionals are not a basis for $\mathcal{V}^*$, since they fail to span it: any finite linear combination

$$a^{j_1}\lambda_{j_1} + \cdots + a^{j_m}\lambda_{j_m}$$

(where $m \in \mathbb{N}$, $\{j_1, \ldots, j_m\} \subset A$, and $a^{j_1}, \ldots, a^{j_m} \in \mathbb{F}$) will send every vector e_α with $\alpha \notin \{j_1, \cdots, j_m\}$ to 0; and thus the linear functional

$$f : \mathcal{V} \to \mathbb{F}\,, \quad \sum_{k=1}^m v^{j_k} e_{j_k} \mapsto \sum_{k=1}^m v^{j_k}\,,$$

which has $f(e_\alpha) = 1$ for all $\alpha \in A$, cannot be spanned by $\{\lambda_\alpha\}_{\alpha\in A}$. For each vector $v \in \mathcal{V}$, we may again define the linear map $g_v \in (\mathcal{V}^*)^*$ by

$$g_v : \mathcal{V}^* \to \mathbb{F}\,, \quad f \mapsto f(v)\,,$$

obtaining again a linear map

$$\phi : \mathcal{V} \to (\mathcal{V}^*)^*\,, \quad v \mapsto g_v\,.$$

However, in this case ϕ is not an isomorphism, since, as we now show, it is not surjective. Let $\{\gamma_\beta\}_{\beta\in B} \subset \mathcal{V}^*$ be a basis for $\mathcal{V}^*$ containing $\{\lambda_\alpha\}_{\alpha\in A}$ (such a basis exists because the set $\{\lambda_\alpha\}_{\alpha\in A}$ is linearly independent), and define $h \in (\mathcal{V}^*)^*$ to be the linear map on $\mathcal{V}^*$ characterised by

$$h(f) := \begin{cases} 0 & \text{for all } f \in \{\lambda_\alpha\}_{\alpha\in A}\,, \\ 1 & \text{for all } f \in \{\gamma_\beta\}_{\beta\in B} \setminus \{\lambda_\alpha\}_{\alpha\in A}\,. \end{cases}$$

If h is in the image of ϕ, then there is a vector $v \in \mathcal{V}$ with $g_v = \phi(v) = h$, and then for all $\alpha \in A$

$$v^\alpha = \lambda_\alpha(v) = g_v(\lambda_\alpha) = h(\lambda_\alpha) = 0\,,$$

so $v = 0$. However, since $h \neq 0$ and ϕ is linear, we cannot have $\phi(0) = h$. Thus, $h \notin \phi(\mathcal{V})$.

Proposition 1.3. *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$, with addition map $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and scalar multiplication map $\cdot : \mathbb{C} \times \mathcal{V} \to \mathcal{V}$. Then, the set $\mathcal{V}$ together with $+$ and the restriction of $\cdot$ to $\mathbb{R} \times \mathcal{V}$ is a real vector space $\mathcal{V}_{\mathbb{R}}$. If $\{e_\alpha\}_{\alpha \in A}$ is a basis for $\mathcal{V}$, where A is a suitable indexing set, then the set $\{e_\alpha\}_{\alpha \in A} \cup \{ie_\alpha\}_{\alpha \in A}$ is a basis for $\mathcal{V}_{\mathbb{R}}$. As a consequence, $\mathcal{V}_{\mathbb{R}}$ has dimension $\dim \mathcal{V}_{\mathbb{R}} = 2 \dim \mathcal{V}$.*

Proof. Since $\mathcal{V}_{\mathbb{R}}$ has the same addition map as $\mathcal{V}$, to show it is a vector space we only need to check the axioms that involve scalar multiplication, namely that if $u, v \in \mathcal{V}_{\mathbb{R}}$ and $a, b \in \mathbb{R}$,

(i) $a(bv) = (ab)v$,
(ii) $1v = v$,
(iii) $a(u + v) = au + av$,
(iv) $(a + b)v = av + bv$.

All these axioms follow from the fact that $\mathcal{V}$ is a complex vector space and $\mathbb{R}$ is a subfield of $\mathbb{C}$. Thus, $\mathcal{V}_{\mathbb{R}}$ is a real vector space. Consider the set $\{e_\alpha\}_{\alpha \in A} \cup \{ie_\alpha\}_{\alpha \in A} \subset \mathcal{V}_{\mathbb{R}}$. Observe that this is a disjoint union, since if $\alpha, \alpha' \in A$ and $e_\alpha = ie_{\alpha'}$, then we either have $1 = -i = 0$ if $\alpha \neq \alpha'$, or $1 - i = 0$ if $\alpha = \alpha'$, which are both false. Moreover, if $\alpha, \alpha' \in A$ and $ie_\alpha = ie_{\alpha'}$, then we must have $\alpha = \alpha'$, which shows that $\{ie_\alpha\}_{\alpha \in A}$ has cardinality $|A|$. Thus, the union $\{e_\alpha\}_{\alpha \in A} \cup \{ie_\alpha\}_{\alpha \in A}$ has cardinality $2|A| = 2 \dim \mathcal{V}$. To show that $\{e_\alpha\}_{\alpha \in A} \cup \{ie_\alpha\}_{\alpha \in A}$ spans $\mathcal{V}_{\mathbb{R}}$, consider an arbitrary vector $v \in \mathcal{V}_{\mathbb{R}}$. Since $\mathcal{V}_{\mathbb{R}} = \mathcal{V}$ as sets, we have $v \in \mathcal{V}$. Thus, v has a representation

$$v = v^{j_1} e_{j_1} + \cdots + v^{j_m} e_{j_m} ,$$

for some $m \in \mathbb{N}$, $\{j_1, \ldots, j_m\} \subset A$, and $v^{j_1}, \ldots, v^{j_m} \in \mathbb{C}$. For $k \in \{1, \ldots, m\}$, let $a^{j_k} := \mathrm{Re}(v^{j_k})$ and $b^{j_k} := \mathrm{Im}(v^{j_k})$. Then,

$$\begin{aligned} v &= (a^{j_1} + ib^{j_1}) e_{j_1} + \cdots + (a^{j_m} + ib^{j_m}) e_{j_m} \\ &= a^{j_1} e_{j_1} + b^{j_1} i e_{j_1} + \cdots + a^{j_m} e_{j_m} + b^{j_m} i e_{j_m} , \end{aligned}$$

where $a^{j_k}, b^{j_k} \in \mathbb{R}$ for all $k \in \{1, \ldots, m\}$. It remains to show that the set $\{e_\alpha\}_{\alpha \in A} \cup \{ie_\alpha\}_{\alpha \in A} \subset \mathcal{V}_{\mathbb{R}}$ is linearly independent. To see this, observe that every finite linear combination C of vectors in $\{e_\alpha\}_{\alpha \in A} \cup \{ie_\alpha\}_{\alpha \in A}$ with real coefficients can be rewritten as a finite linear combination of vectors in $\{e_\alpha\}_{\alpha \in A}$ with complex coefficients, by simply writing each sum $ae_\alpha + bie_\alpha$ as $(a + ib)e_\alpha$, for $a, b \in \mathbb{R}$ and $\alpha \in A$. Thus, if $C = 0$, then by linear independence of $\{e_\alpha\}_{\alpha \in A}$ all these complex coefficients $a + ib$ are zero, hence so are all their real and imaginary parts a and b, which were the real coefficients in C. This concludes the proof that $\{e_\alpha\}_{\alpha \in A} \cup \{ie_\alpha\}_{\alpha \in A}$ is a basis for $\mathcal{V}_{\mathbb{R}}$, and since this basis has cardinality $2 \dim \mathcal{V}$, we have $\dim \mathcal{V}_{\mathbb{R}} = 2 \dim \mathcal{V}$. □

Definition 1.4. For a complex vector space $\mathcal{V}$, the associated real vector space $\mathcal{V}_{\mathbb{R}}$ given by Proposition 1.3 is called the *realification* of $\mathcal{V}$, or the *underlying real vector space* of $\mathcal{V}$.

Proposition 1.5. *Let $\mathcal{V}$ be a real vector space with addition map $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and scalar multiplication map $\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$.*

(i) *The abelian group $\mathcal{V} \oplus \mathcal{V}$, together with the scalar multiplication map*

$$\cdot : \mathbb{C} \times (\mathcal{V} \oplus \mathcal{V}) \to \mathcal{V} \oplus \mathcal{V} , \quad (a + ib)(u, v) := (au - bv, av + bu) ,$$

for $a, b \in \mathbb{R}$ and $(u, v) \in \mathcal{V} \oplus \mathcal{V}$, is a complex vector space $\mathcal{V}_{\mathbb{C}}$.

(ii) *The map*

$$\iota : \mathcal{V} \to \mathcal{V}_{\mathbb{C}}\,, \quad v \mapsto (v, 0)\,,$$

is injective and linear with respect to vector addition and to multiplication by scalars in $\mathbb{R}$*, that is, for all* $u, v \in \mathcal{V}$ *and* $c \in \mathbb{R}$*,*

$$\iota(cu + v) = c\iota(u) + \iota(v)\,.$$

Proof. (i) Since $\mathcal{V} \oplus \mathcal{V}$ is an abelian group with respect to $+$, it only remains to check the vector space axioms that involve multiplication by scalars, that is, that for all $u, v, u', v' \in \mathcal{V}$ and $z, w \in \mathbb{C}$,
(1) $z(w(u,v)) = (zw)(u,v)$,
(2) $1(u,v) = (u,v)$,
(3) $z((u,v) + (u',v')) = z(u,v) + z(u',v')$,
(4) $(z+w)(u,v) = z(u,v) + w(u,v)$.
Let $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$. For (1), we have

$$\begin{aligned} z(w(u,v)) &= (a+ib)(cu - dv, cv + du) \\ &= (acu - adv - bcv - bdu, acv + adu + bcu - bdv) \\ &= ((ac - bd)u - (ad + bc)v, (ac - bd)v + (ad + bc)u) \\ &= (ac - bd + i(ad + bc))(u,v) \\ &= (zw)(u,v)\,. \end{aligned}$$

For (2),

$$1(u,v) = (1u - 0v, 1v + 0u) = (u,v)\,.$$

For (3),

$$\begin{aligned} z((u,v) + (u',v')) &= (a+ib)(u+u', v+v') \\ &= (au + au' - bv - bv', av + av' + bu + bu') \\ &= (au - bv, av + bu) + (au' - bv', av' + bu') \\ &= (a+ib)(u,v) + (a+ib)(u',v') \\ &= z(u,v) + z(u',v')\,. \end{aligned}$$

Finally, for (4),

$$\begin{aligned} (z+w)(u,v) &= (a + c + i(b+d))(u,v) \\ &= (au + cu - bv - dv, av + cv + bu + du) \\ &= (au - bv, av + bu) + (cu - dv, cv + du) \\ &= (a+ib)(u,v) + (c+id)(u,v) \\ &= z(u,v) + w(u,v)\,. \end{aligned}$$

(ii) Let $u, v \in \mathcal{V}$. Then, if $\iota(u) = \iota(v)$, we have $(u, 0) = (v, 0)$, which implies $u = v$. Thus, ι is injective. Moreover, if $u, v \in \mathcal{V}$ and $c \in \mathbb{R}$, we have

$$\begin{aligned}
\iota(cu + v) &= (cu + v, 0) \\
&= (cu, 0) + (v, 0) \\
&= (cu - 00, c0 + 0u) + (v, 0) \\
&= c(u, 0) + (v, 0) \\
&= c\iota(u) + \iota(v) .
\end{aligned}$$

□

Remark 1.6. Since the map

$$\iota : \mathcal{V} \to \mathcal{V}_{\mathbb{C}} , \quad v \mapsto (v, 0)$$

in Proposition 1.5 (ii) is injective and its image is the set $S := \{(v, 0) \,|\, v \in \mathcal{V}\} \subset \mathcal{V}_{\mathbb{C}}$, for each $v \in \mathcal{V}$ we may denote the element $(v, 0) \in S$ by v. Then, if $u, v \in \mathcal{V}$ and $c \in \mathbb{R}$, the notations $u + v \in \mathcal{V}_{\mathbb{C}}$ and $cv \in \mathcal{V}_{\mathbb{C}}$ have two possible interpretations: if addition and scalar multiplication take place in $\mathcal{V}$, then $u + v = (u + v, 0)$ and $cv = (cv, 0)$; and if they take place in $\mathcal{V}_{\mathbb{C}}$, then $u + v = (u, 0) + (v, 0)$ and $cv = c(v, 0)$. However, by linearity of ι, these two interpretations are actually the same, hence there is no ambiguity in the notation. Then, for each $u, v \in \mathcal{V}$ and $z, w \in \mathbb{C}$, we may write $zu + wv$ to denote the element $z(u, 0) + w(v, 0) \in \mathcal{V}_{\mathbb{C}}$. We can then write an arbitrary element $(u, v) \in \mathcal{V}_{\mathbb{C}}$ as

$$(u, v) = (u, 0) + (0, v) = (u, 0) + i(v, 0) = u + iv .$$

This representation is unique, since if $(u, v) = u' + iv'$ for some $u', v' \in \mathcal{V}$, then $(u, v) = (u', v')$, hence $u = u'$ and $v = v'$.

Definition 1.7. Let $\mathcal{V}$ be a real vector space. Then, the associated complex vector space $\mathcal{V}_{\mathbb{C}}$ given by Proposition 1.5 (i) is called the *complexification* of $\mathcal{V}$. If $w = u + iv \in \mathcal{V}_{\mathbb{C}}$, for $u, v \in \mathcal{V}$, we call $\mathrm{Re}(w) := u$ and $\mathrm{Im}(w) := v$ the *real part* and the *imaginary part* of w respectively. We also call $\bar{w} := u - iv$ the *conjugate* of w.

Proposition 1.8. *Let $\mathcal{V}$ be a real vector space, and let $\{e_\alpha\}_{\alpha \in A}$ be a basis for $\mathcal{V}$, where A is a suitable indexing set. Then, the set $\{e_\alpha\}_{\alpha \in A}$, regarded to be a subset of the complexification $\mathcal{V}_{\mathbb{C}}$ (that is, the set $\{(e_\alpha, 0)\}_{\alpha \in A} \subset \mathcal{V}_{\mathbb{C}}$), is a basis for $\mathcal{V}_{\mathbb{C}}$. As a consequence,* $\dim \mathcal{V}_{\mathbb{C}} = \dim \mathcal{V}$.

Proof. We first show that $\{e_\alpha\}_{\alpha \in A}$ spans $\mathcal{V}_{\mathbb{C}}$. Let $w = u + iv \in \mathcal{V}_{\mathbb{C}}$, for some $u, v \in \mathcal{V}$. Then,

$$u = u^{j_1} e_{j_1} + \cdots + u^{j_n} e_{j_n} \quad \text{and} \quad v = v^{k_1} e_{k_1} + \cdots + v^{k_m} e_{k_m} ,$$

for some $n, m \in \mathbb{N}$, $\{j_1, \ldots, j_n, k_1, \ldots, k_m\} \subset A$, and $u^{j_1}, \ldots, u^{j_n}, v^{k_1}, \ldots, v^{k_m} \in \mathbb{R}$; and then,

$$\begin{aligned}
w &= u^{j_1} e_{j_1} + \cdots + u^{j_n} e_{j_n} + i(v^{k_1} e_{k_1} + \cdots + v^{k_m} e_{k_m}) \\
&= u^{j_1} e_{j_1} + \cdots + u^{j_n} e_{j_n} + iv^{k_1} e_{k_1} + \cdots + iv^{k_m} e_{k_m} ,
\end{aligned}$$

so w can be written as a linear combination (with complex coefficients) of vectors in $\{e_\alpha\}_{\alpha \in A}$. Moreover, if $C := z^{j_1} e_{j_1} + \cdots + z^{j_n} e_{j_n} \in \mathcal{V}_{\mathbb{C}}$ is an arbitrary linear combination of vectors in $\{e_\alpha\}_{\alpha\in A}$, for $z^{j_1},\dots,z^{j_n}\in\mathbb{C}$, and $C=0$, we have

$$\begin{aligned}0&=z^{j_1}e_{j_1}+\cdots+z^{j_n}e_{j_n}\\&=(a^{j_1}+ib^{j_1})e_{j_1}+\cdots+(a^{j_n}+ib^{j_n})e_{j_n}\\&=a^{j_1}e_{j_1}+\ldots+a^{j_n}e_{j_n}+i(b^{j_1}e_{j_1}+\ldots+b^{j_n}e_{j_n})\,,\end{aligned}$$

where $a^{j_k}:=\mathrm{Re}(z^{j_k})$ and $b^{j_k}:=\mathrm{Im}(z^{j_k})$, for $k\in\{1,\dots,n\}$. Thus, as vectors in $\mathcal{V}$,

$$a^{j_1}e_{j_1}+\ldots+a^{j_n}e_{j_n}=0\quad\text{and}\quad b^{j_1}e_{j_1}+\ldots+b^{j_n}e_{j_n}=0\,,$$

which implies $a^{j_k}=b^{j_k}=0$ for all $k\in\{1,\dots,n\}$, by linear independence of $\{e_\alpha\}_{\alpha\in A}$ in $\mathcal{V}$. Thus, $z^{j_k}=0$ for all $k\in\{1,\dots,n\}$, which proves linear independence of $\{e_\alpha\}_{\alpha\in A}$ in $\mathcal{V}_\mathbb{C}$. Thus, $\{e_\alpha\}_{\alpha\in A}$ is a basis for $\mathcal{V}_\mathbb{C}$, which implies $\dim\mathcal{V}_\mathbb{C}=A=\dim\mathcal{V}$. □

Definition 1.9. Let $\mathcal{V}$ and $\mathcal{W}$ be complex vector spaces. Then, a map $\phi:\mathcal{V}\to\mathcal{W}$ is said to be a *conjugate linear isomorphism* if it is bijective and for all $u,v\in\mathcal{V}$ and $z\in\mathbb{C}$,

$$\phi(zu+v)=\bar{z}\phi(u)+\phi(v)\,.$$

Proposition 1.10. *Let $\mathcal{V}$ be a real vector space. Then, the map*

$$\phi:\mathcal{V}_\mathbb{C}\to\mathcal{V}_\mathbb{C}\,,\quad w\mapsto\bar{w}$$

is a conjugate linear isomorphism.

Proof. Let $w,w'\in\mathcal{V}_\mathbb{C}$, and suppose $w=u+iv$ and $w'=u'+iv'$, for $u,v,u',v'\in\mathcal{V}$. If $\phi(w)=\phi(w')$, then

$$\begin{aligned}\bar{w}=\bar{w}'\implies&\ \overline{u+iv}=\overline{u'+iv}\\\implies&\ u-iv=u'-iv'\\\implies&\ u=u'\quad\text{and}\quad v=v'\,,\end{aligned}$$

so $w=w'$. Thus, ϕ is injective. Moreover, $\bar{w}\in\mathcal{V}_\mathbb{C}$ and

$$\phi(\bar{w})=\overline{u-iv}=u+iv=w\,,$$

which shows surjectivity of ϕ. Furthermore, if $z=a+ib\in\mathbb{C}$, for $a,b\in\mathbb{R}$, then

$$\begin{aligned}\phi(zw+w')&=\overline{zw+w'}\\&=\overline{(a+ib)(u+iv)+u'+iv'}\\&=\overline{au-bv+u'+i(av+bu+v')}\\&=au-bv+u'-i(av+bu+v')\\&=(a-ib)(u-iv)+u'-iv'\\&=\bar{z}\bar{w}+\bar{w}'\\&=\bar{z}\phi(w)+\phi(w').\end{aligned}$$

□

For the remainder of this text, if S is any set, we will denote by $\mathbb{1}_S$ the identity function $S\to S$.

Proposition 1.11. *Let $\mathcal{V}$ and $\mathcal{W}$ be two real vector spaces. If $\alpha, \beta \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$, then the map*

$$\lambda(\alpha, \beta) : \mathcal{V}_{\mathbb{C}} \to \mathcal{W}_{\mathbb{C}}, \quad u + iv \mapsto \alpha(u) - \beta(v) + i(\beta(u) + \alpha(v))$$

is (complex) linear. Moreover, the map $\lambda : [Hom(\mathcal{V}, \mathcal{W})]_{\mathbb{C}} \to \mathrm{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}})$ sending an element $\alpha + i\beta \in [Hom(\mathcal{V}, \mathcal{W})]_{\mathbb{C}}$ to the map $\lambda(\alpha, \beta) \in \mathrm{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}})$ defined above is an isomorphism of vector spaces.

Proof. We omit the proof that $\lambda(\alpha, \beta)$ is linear, since it can be done by direct computation.

To show that $\lambda : [\mathrm{Hom}(\mathcal{V}, \mathcal{W})]_{\mathbb{C}} \to \mathrm{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}})$ is bijective, it suffices to find an inverse, that is, a map $\lambda^{-1} : \mathrm{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}) \to [\mathrm{Hom}(\mathcal{V}, \mathcal{W})]_{\mathbb{C}}$ such that $\lambda^{-1} \circ \lambda = \mathbb{1}_{[\mathrm{Hom}(\mathcal{V}, \mathcal{W})]_{\mathbb{C}}}$ and $\lambda \circ \lambda^{-1} = \mathbb{1}_{\mathrm{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}})}$. An element $f \in \mathrm{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}})$ is a complex linear function mapping each pair $u+iv \in \mathcal{V}_{\mathbb{C}}$ to some $f(u+iv) = f_1(u+iv)+if_2(u+iv) \in \mathcal{W}_{\mathbb{C}}$, where $f_1(u+iv) := \mathrm{Re}(f(u+iv))$ and $f_2(u+iv) := \mathrm{Im}(f(u+iv))$ are both vectors in $\mathcal{W}$. We thus obtain from f two functions f_1 and f_2 mapping $\mathcal{V}_{\mathbb{C}}$ to $\mathcal{W}$. Since f is complex linear, for each $u + iv \in \mathcal{V}_{\mathbb{C}}$ we have

$$\begin{aligned}
-f_2(u + iv) + if_1(u + iv) &= i(f_1(u + iv) + if_2(u + iv)) \\
&= if(u + iv) \\
&= f(i(u + iv)) \\
&= f(-v + iu) \\
&= f_1(-v + iu) + if_2(-v + iu)\,,
\end{aligned}$$

so

$$-f_2(u + iv) = f_1(-v + iu) \quad \text{and} \quad f_1(u + iv) = f_2(-v + iu) \tag{1}$$

for all $u + iv \in \mathcal{V}_{\mathbb{C}}$ (note that these two equalities are actually equivalent). We define the functions

$$\begin{aligned}
\hat{f}_1 &: \mathcal{V} \to \mathcal{W}\,, \quad v \mapsto f_1(v + i0)\,, \\
\hat{f}_2 &: \mathcal{V} \to \mathcal{W}\,, \quad v \mapsto f_2(v + i0)\,.
\end{aligned}$$

Then, for all $u, v \in \mathcal{V}$ and $c \in \mathbb{R}$,

$$\begin{aligned}
\hat{f}_1(cu + v) + i\hat{f}_2(cu + v) &= f_1(cu + v + i0) + if_2(cu + v + i0) \\
&= f(cu + v + i0) \\
&= f(c(u + i0) + v + i0) \\
&= cf(u + i0) + f(v + i0) \\
&= c(f_1(u + i0) + if_2(u + i0)) + f_1(v + i0) + if_2(v + i0) \\
&= cf_1(u + i0) + f_1(v + i0) + i(cf_2(u + i0) + f_2(v + i0)) \\
&= c\hat{f}_1(u) + \hat{f}_1(v) + i(c\hat{f}_2(u) + \hat{f}_2(v))\,,
\end{aligned}$$

which shows that $\hat{f}_1$ and $\hat{f}_2$ are (real) linear, that is $\hat{f}_1, \hat{f}_2 \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$. We can now define the map

$$\lambda^{-1} : \mathrm{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}) \to [\mathrm{Hom}(\mathcal{V}, \mathcal{W})]_{\mathbb{C}}\,, \quad f \mapsto \hat{f}_1 + i\hat{f}_2\,.$$

We check that λ and λ^{-1} are indeed inverses. If $\alpha + i\beta \in [\mathrm{Hom}(\mathcal{V},\mathcal{W})]_{\mathbb{C}}$, then we have

$$(\lambda^{-1}\circ\lambda)(\alpha+i\beta) = \lambda^{-1}(\lambda(\alpha,\beta)) = \widehat{\lambda(\alpha,\beta)}_1 + i\widehat{\lambda(\alpha,\beta)}_2\,.$$

Observe that for all $v \in \mathcal{V}$,

$$\begin{aligned}\widehat{\lambda(\alpha,\beta)}_1(v) &= \lambda(\alpha,\beta)_1(v+i0) = \alpha(v) - \beta(0) = \alpha(v)\,,\\ \widehat{\lambda(\alpha,\beta)}_2(v) &= \lambda(\alpha,\beta)_2(v+i0) = \beta(v) + \alpha(0) = \beta(v)\,,\end{aligned}$$

so $\widehat{\lambda(\alpha,\beta)}_1 = \alpha$ and $\widehat{\lambda(\alpha,\beta)}_2 = \beta$ as elements in $\mathrm{Hom}(\mathcal{V},\mathcal{W})$. This shows that $\lambda^{-1}\circ\lambda = \mathbb{1}_{[\mathrm{Hom}(\mathcal{V},\mathcal{W})]_{\mathbb{C}}}$. Furthermore, if $f \in \mathrm{Hom}(\mathcal{V}_{\mathbb{C}},\mathcal{W}_{\mathbb{C}})$, then

$$(\lambda\circ\lambda^{-1})(f) = \lambda(\hat{f}_1 + i\hat{f}_2) = \lambda(\hat{f}_1,\hat{f}_2).$$

For $u + iv \in \mathcal{V}_{\mathbb{C}}$, we have

$$\begin{aligned}\lambda(\hat{f}_1,\hat{f}_2)(u+iv) &= \hat{f}_1(u) - \hat{f}_2(v) + i(\hat{f}_2(u) + \hat{f}_1(v))\\ &= f_1(u+i0) - f_2(v+i0) + i(f_2(u+i0) + f_1(v+i0))\\ &= f_1(u+i0) + f_1(0+iv) + i(f_2(u+i0) + f_2(0+iv)) \quad \text{(by (1))}\\ &= f(u+i0) + f(0+iv)\\ &= f(u+iv)\,,\end{aligned}$$

which shows that indeed $\lambda(\hat{f}_1,\hat{f}_2) = f$, hence $\lambda\circ\lambda^{-1} = \mathbb{1}_{\mathrm{Hom}(\mathcal{V}_{\mathbb{C}},\mathcal{W}_{\mathbb{C}})}$. Thus, λ is bijective, and linearity can be shown by direct computation. □

Remark 1.12. (i) Using notation from Proposition 1.11, for each $\alpha+i\beta \in [\mathrm{Hom}(\mathcal{V},\mathcal{W})]_{\mathbb{C}}$ we obtain a map $\lambda(\alpha,\beta) \in \mathrm{Hom}(\mathcal{V}_{\mathbb{C}},\mathcal{W}_{\mathbb{C}})$. Defining $\overline{\lambda}(\alpha,\beta) := \lambda(\alpha,-\beta)$, we have

$$\overline{\lambda(\alpha,\beta)(w)} = \overline{\lambda}(\alpha,\beta)(\bar{w}) \quad \text{for all } w \in \mathcal{V}_{\mathbb{C}}.$$

(ii) As one can check, for any $\alpha \in \mathrm{Hom}(\mathcal{V},\mathcal{W})$, the map α is injective if and only if the map $\lambda(\alpha) := \lambda(\alpha,0) : \mathcal{V}_{\mathbb{C}} \to \mathcal{W}_{\mathbb{C}}$ is injective, and α is surjective if and only if $\lambda(\alpha)$ is surjective.

(iii) As a complex vector space, $\mathbb{C}$ can be regarded to be the complexification of $\mathbb{R}$ (not as a field, since a priori the product of two elements in $\mathbb{R}_{\mathbb{C}}$ is not defined). Then, we have

$$(\mathcal{V}^*)_{\mathbb{C}} = [\mathrm{Hom}(\mathcal{V},\mathbb{R})]_{\mathbb{C}} \cong \mathrm{Hom}(\mathcal{V}_{\mathbb{C}},\mathbb{R}_{\mathbb{C}}) = \mathrm{Hom}(\mathcal{V}_{\mathbb{C}},\mathbb{C}) = (\mathcal{V}_{\mathbb{C}})^*\,.$$

1.2. **Exterior Products.**

Throughout Subsection 1.2, we let $\mathbb{F} := \mathbb{R}$ or $\mathbb{C}$ and we fix a vector space $\mathcal{V}$ over $\mathbb{F}$.

Definition 1.13. A function $\theta : \mathcal{V}\times\mathcal{V} \to \mathbb{F}$ is said to be *bilinear* if it is linear in each entry, that is, if for all $u, v, w \in \mathcal{V}$ and $c \in \mathbb{F}$,

$$\begin{aligned}\theta(u+v,w) &= \theta(u,w) + \theta(v,w)\,,\\ \theta(u,v+w) &= \theta(u,v) + \theta(u,w)\,,\\ \theta(cu,v) &= c\,\theta(u,v) = \theta(u,cv)\,.\end{aligned}$$

The set of bilinear functions on $\mathcal{V}\times\mathcal{V}$, which we denote by $\mathcal{V}^*\otimes\mathcal{V}^*$, is a subspace of the vector space (over $\mathbb{F}$) of $\mathbb{F}$-valued functions on $\mathcal{V}\times\mathcal{V}$. We call $\mathcal{V}^*\otimes\mathcal{V}^*$ the *tensor product* of $\mathcal{V}^*$ with itself, and we also call an element in $\mathcal{V}^*\otimes\mathcal{V}^*$ a 2-*tensor on* $\mathcal{V}$.

Definition 1.14. Let $\theta \in \mathcal{V}^* \otimes \mathcal{V}^*$. The 2-tensor θ is said to be *symmetric* if for all $u, v \in \mathcal{V}$, $\theta(v,u) = \theta(u,v)$. Moreover, θ is said to be *alternating* (or *skew-symmetric*) if for all $u, v \in \mathcal{V}$, $\theta(v,u) = -\theta(u,v)$. We call this last equality the *alternating property.* An alternating 2-tensor on $\mathcal{V}$ is also called a 2-*covector on* $\mathcal{V}$.

We denote the set of 2-covectors on $\mathcal{V}$ by $\Lambda^2\mathcal{V}^*$. As one can check, $\Lambda^2\mathcal{V}^*$ is a subspace of $\mathcal{V}^* \otimes \mathcal{V}^*$.

Proposition 1.15. *Let $\alpha, \beta \in \mathcal{V}^*$. Then, the function*

$$(\alpha \wedge \beta) : \mathcal{V} \times \mathcal{V} \to \mathbb{F}\,, \quad (u,v) \mapsto \alpha(u)\beta(v) - \alpha(v)\beta(u)\,,$$

is a 2-covector on $\mathcal{V}$.

Proof. Bilinearity follows directly from linearity of α and β. Moreover, for each $(u,v) \in \mathcal{V} \times \mathcal{V}$, we have

$$(\alpha \wedge \beta)(v,u) = \alpha(v)\beta(u) - \alpha(u)\beta(v) = -(\alpha \wedge \beta)(u,v)\,,$$

so the alternating property is fulfilled. □

Proposition 1.16. *For all $\alpha, \beta, \gamma \in \mathcal{V}^*$ and $c \in \mathbb{F}$,*

$$\begin{aligned}(\alpha + \beta) \wedge \gamma &= \alpha \wedge \gamma + \beta \wedge \gamma\,,\\ (c\alpha) \wedge \beta &= c(\alpha \wedge \beta)\,,\\ \alpha \wedge \beta &= -\beta \wedge \alpha\,.\end{aligned}$$

Proof. For each $(u,v) \in \mathcal{V} \times \mathcal{V}$, we have

$$\begin{aligned}((\alpha + \beta) \wedge \gamma)(u,v) &= (\alpha + \beta)(u)\gamma(v) - (\alpha + \beta)(v)\gamma(u)\\ &= (\alpha(u) + \beta(u))\gamma(v) - (\alpha(v) + \beta(v))\gamma(u)\\ &= \alpha(u)\gamma(v) + \beta(u)\gamma(v) - \alpha(v)\gamma(u) - \beta(v)\gamma(u)\\ &= (\alpha \wedge \gamma)(u,v) + (\beta \wedge \gamma)(u,v)\\ &= (\alpha \wedge \gamma + \beta \wedge \gamma)(u,v)\,.\end{aligned}$$

The two remaining equalities are proved similarly by direct computation. □

Remark 1.17. It follows from Proposition 1.16 that for $\alpha, \beta, \gamma \in \mathcal{V}^*$ and $c \in \mathbb{F}$, we also have

$$\begin{aligned}\alpha \wedge (\beta + \gamma) &= \alpha \wedge \beta + \alpha \wedge \gamma\,,\\ \alpha \wedge (c\beta) &= c(\alpha \wedge \beta)\,.\end{aligned}$$

Proposition 1.18. (i) *If* $\dim \mathcal{V} = 1$*, then* $\Lambda^2\mathcal{V}^* = \{0\}$*.*

(ii) *If* $\dim \mathcal{V} = 2$ *and $\{e_1, e_2\} \subset \mathcal{V}$ is a basis for $\mathcal{V}$ with dual basis $\{\alpha^1, \alpha^2\}$ for $\mathcal{V}^*$, then $\{\alpha^1 \wedge \alpha^2\}$ is a basis for $\Lambda^2\mathcal{V}^*$, with $\theta = \theta(e_1, e_2)\, \alpha^1 \wedge \alpha^2$ for each $\theta \in \Lambda^2\mathcal{V}^*$. As a consequence,* $\dim \Lambda^2\mathcal{V}^* = 1$.

Proof. (i) Suppose that $\dim \mathcal{V} = 1$ and $\{e\} \subset \mathcal{V}$ is a basis for $\mathcal{V}$. Since $\Lambda^2\mathcal{V}^*$ is a vector space, it is nonempty; and for each $\theta \in \Lambda^2\mathcal{V}^*$ and $(u,v) = (ae, be) \in \mathcal{V} \times \mathcal{V}$, for $a, b \in \mathbb{F}$, we have

$$\theta(u,v) = \theta(ae, be) = ab\, \theta(e,e) = 0\,,$$

since $\theta(e,e) = 0$ by the alternating property. Thus, $\theta = 0$.

(ii) Suppose now that $\mathcal{V}$ has dimension 2 and $\{e_1, e_2\}$ is a basis for $\mathcal{V}$ with dual basis $\{\alpha^1, \alpha^2\}$ for $\mathcal{V}^*$. Then,

$$(\alpha^1 \wedge \alpha^2)(e_1, e_2) = \alpha^1(e_1)\alpha^2(e_2) - \alpha^1(e_2)\alpha^2(e_1) = 1\,,$$

so $\alpha^1 \wedge \alpha^2 \neq 0$. If $\theta \in \Lambda^2\mathcal{V}^*$ and $(u, v) \in \mathcal{V} \times \mathcal{V}$, we have

$$\begin{aligned}\theta(u, v) &= \theta(\alpha^1(u)e_1 + \alpha^2(u)e_2, \alpha^1(v)e_1 + \alpha^2(v)e_2)\\ &= \alpha^1(u)\alpha^1(v)\theta(e_1, e_1) + \alpha^1(u)\alpha^2(v)\theta(e_1, e_2)\\ &\quad + \alpha^2(u)\alpha^1(v)\theta(e_2, e_1) + \alpha^2(u)\alpha^2(v)\theta(e_2, e_2)\\ &= \alpha^1(u)\alpha^2(v)\theta(e_1, e_2) - \alpha^2(u)\alpha^1(v)\theta(e_1, e_2)\\ &= \theta(e_1, e_2)(\alpha^1 \wedge \alpha^2)(u, v)\,,\end{aligned}$$

so $\theta = \theta(e_1, e_2)(\alpha^1 \wedge \alpha^2)$. Thus, since the singleton $\{\alpha^1 \wedge \alpha^2\} \subset \Lambda^2\mathcal{V}^*$ is linearly independent and spans $\Lambda^2\mathcal{V}^*$, it is a basis for $\Lambda^2\mathcal{V}^*$, and hence $\dim \Lambda^2\mathcal{V}^* = 1$.

□

Proposition 1.19. *Suppose $\mathcal{V}$ is a vector space over $\mathbb{R}$. Then,*

$$(\mathcal{V}^* \otimes \mathcal{V}^*)_{\mathbb{C}} \cong \mathcal{V}^*_{\mathbb{C}} \otimes \mathcal{V}^*_{\mathbb{C}}$$

and

$$(\Lambda^2\mathcal{V}^*)_{\mathbb{C}} \cong \Lambda^2\mathcal{V}^*_{\mathbb{C}}\,.$$

Proof. Consider the map

$$\phi : (\mathcal{V}^* \otimes \mathcal{V}^*)_{\mathbb{C}} \to \mathcal{V}^*_{\mathbb{C}} \otimes \mathcal{V}^*_{\mathbb{C}}\,,$$

$$\begin{aligned}\theta + i\mu \mapsto \Big((u + iv, \tilde{u} + i\tilde{v}) \mapsto{}& \theta(u, \tilde{u}) - \theta(v, \tilde{v}) - \mu(u, \tilde{v}) - \mu(v, \tilde{u})\\ &+ i(\theta(u, \tilde{v}) + \theta(v, \tilde{u}) + \mu(u, \tilde{u}) - \mu(v, \tilde{v}))\Big)\,,\end{aligned}$$

for $\theta, \mu \in \mathcal{V}^* \otimes \mathcal{V}^*$ and $u, v, \tilde{u}, \tilde{v} \in \mathcal{V}$. It is left to the reader to check that for each $\theta, \mu \in \mathcal{V}^* \otimes \mathcal{V}^*$, the function $\phi(\theta + i\mu) : \mathcal{V}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}} \to \mathbb{C}$ is indeed bilinear, and that ϕ is a linear map of vector spaces. The map

$$\phi^{-1} : \mathcal{V}^*_{\mathbb{C}} \otimes \mathcal{V}^*_{\mathbb{C}} \to (\mathcal{V}^* \otimes \mathcal{V}^*)_{\mathbb{C}}\,,$$

$$\tau \mapsto \Big((u, v) \mapsto \mathrm{Re}(\tau(u + i0, v + i0))\Big) + i\Big((u, v) \mapsto \mathrm{Im}(\tau(u + i0, v + i0))\Big)\,,$$

for $\tau \in \mathcal{V}^*_{\mathbb{C}} \otimes \mathcal{V}^*_{\mathbb{C}}$ and $u, v \in \mathcal{V}$, gives the inverse of ϕ. It is also left to the reader to check that the restrictions $\phi|_{(\Lambda^2\mathcal{V}^*)_{\mathbb{C}}}$ and $\phi^{-1}|_{\Lambda^2\mathcal{V}^*_{\mathbb{C}}}$ give an isomorphism $(\Lambda^2\mathcal{V}^*)_{\mathbb{C}} \cong \Lambda^2\mathcal{V}^*_{\mathbb{C}}$. □

We now assume that $\dim \mathcal{V} \leq 2$. We set $\Lambda^0\mathcal{V}^* := \mathbb{F}$, $\Lambda^1\mathcal{V}^* := \mathcal{V}^*$, and $\Lambda^p\mathcal{V}^* := \{0\}$ (the trivial vector space over $\mathbb{F}$) for $p \in \mathbb{Z}_{\geq 3}$. For $p \in \mathbb{Z}_{\geq 0}$, we call $\Lambda^p\mathcal{V}^*$ the *pth exterior power of* $\mathcal{V}^*$, and we call an element in $\Lambda^p\mathcal{V}^*$ a *p-covector on* $\mathcal{V}$. For $c \in \Lambda^0\mathcal{V}^* = \mathbb{F}$ and $\alpha \in \Lambda^p\mathcal{V}^*$ for $p \in \mathbb{Z}_{\geq 0}$, we define $c \wedge \alpha := c\alpha =: \alpha \wedge c \in \Lambda^p\mathcal{V}^*$. For $\alpha \in \Lambda^p\mathcal{V}^*$ and $\beta \in \Lambda^q\mathcal{V}^*$, for $p, q \in \mathbb{Z}_{\geq 0}$ with $p + q \geq 3$, we define $\alpha \wedge \beta := 0 \in \Lambda^{p+q}\mathcal{V}^*$. Then, for every $p, q \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \Lambda^p\mathcal{V}^*$ and $\beta \in \Lambda^q\mathcal{V}^*$, the wedge product $\alpha \wedge \beta$ is well defined and an element in $\Lambda^{p+q}\mathcal{V}^*$, and if $\tilde{\alpha} \in \Lambda^p\mathcal{V}^*$, $\tilde{\beta} \in \Lambda^q\mathcal{V}^*$ and $\gamma \in \Lambda^r\mathcal{V}^*$ for $r \in \mathbb{Z}_{\geq 0}$, we have (as one can check)

$$
\begin{aligned}
(\alpha + \tilde{\alpha}) \wedge \beta &= \alpha \wedge \beta + \tilde{\alpha} \wedge \beta\,, \\
\alpha \wedge (\beta + \tilde{\beta}) &= \alpha \wedge \beta + \alpha \wedge \tilde{\beta}\,, \\
(c\alpha) \wedge \beta &= c(\alpha \wedge \beta) = \alpha \wedge (c\beta)\,, \quad \text{for } c \in \mathbb{F}\,, \\
\alpha \wedge \beta &= (-1)^{pq}\beta \wedge \alpha \qquad (\textit{anticommutativity}), \\
(\alpha \wedge \beta) \wedge \gamma &= \alpha \wedge (\beta \wedge \gamma) \qquad (\textit{associativity}).
\end{aligned}
$$

Moreover, if $\mathbb{F} = \mathbb{R}$ (and still assuming $\dim \mathcal{V} \leq 2$), we have $(\Lambda^p\mathcal{V}^*)_{\mathbb{C}} \cong \Lambda^p\mathcal{V}^*_{\mathbb{C}}$ for all $p \in \mathbb{Z}_{\geq 0}$ (the case of $p = 2$ is given by Proposition 1.19, and the remaining cases follow directly from previously established isomorphisms).

Proposition 1.20. *Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces over $\mathbb{F}$. A linear map $L : \mathcal{V} \to \mathcal{W}$ induces for each $p \in \{0, 1, 2\}$ a linear map $L^* : \Lambda^p\mathcal{W}^* \to \Lambda^p\mathcal{V}^*$ given by*

(i) *$L^*c := c$ for $c \in \Lambda^0\mathcal{W}^* = \mathbb{F}$, if $p = 0$;*
(ii) *$(L^*\alpha)(v) := \alpha(L(v))$ for $\alpha \in \Lambda^1\mathcal{W}^* = \mathcal{W}^*$ and $v \in \mathcal{V}$, if $p = 1$;*
(iii) *$(L^*\beta)(u, v) := \beta(L(u), L(v))$ for $\beta \in \Lambda^2\mathcal{W}^*$ and $u, v \in \mathcal{V}$, if $p = 2$.*

Proof. For (i), we have $L^* = \mathbb{1}_{\mathbb{F}}$, the identity map on $\mathbb{F} = \Lambda^0\mathcal{V}^* = \Lambda^0\mathcal{W}^*$, which is linear. For (ii) and (iii), it is left to the reader to check that $L^*\alpha \in \Lambda^1\mathcal{V}^*$ and $L^*\beta \in \Lambda^2\mathcal{V}^*$, and that the resulting maps $L^* : \Lambda^1\mathcal{W}^* \to \Lambda^1\mathcal{V}^*$ and $L^* : \Lambda^2\mathcal{W}^* \to \Lambda^2\mathcal{V}^*$ are linear. □

Definition 1.21. We call the map $L^* : \Lambda^p\mathcal{W}^* \to \Lambda^p\mathcal{V}^*$ in Proposition 1.20 the *pullback map of L*, and for $\alpha \in \Lambda^p\mathcal{W}^*$, we call $L^*\alpha \in \Lambda^p\mathcal{V}^*$ the *pullback of α*.

Proposition 1.22. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are two vector spaces over $\mathbb{F}$, and let $L : \mathcal{V} \to \mathcal{W}$ be a linear map. If $p, q \in \{0, 1, 2\}$ such that $p + q \leq 2$, and if $\alpha \in \Lambda^p\mathcal{W}^*$ and $\beta \in \Lambda^q\mathcal{W}^*$, then*

$$L^*(\alpha \wedge \beta) = (L^*\alpha) \wedge (L^*\beta)$$

as $(p + q)$-covectors on $\mathcal{V}$.

Proof. The case when $p = 0$ or $q = 0$ follows from linearity of the pullback map L^*. The only remaining case is when $p = q = 1$. If $\alpha \in \Lambda^1\mathcal{W}^*$ and $\beta \in \Lambda^1\mathcal{W}^*$, then for each $u, v \in \mathcal{V}$,

$$
\begin{aligned}
(L^*(\alpha \wedge \beta))(u, v) &= (\alpha \wedge \beta)(L(u), L(v)) \\
&= \alpha(L(u))\beta(L(v)) - \alpha(L(v))\beta(L(u)) \\
&= (L^*\alpha)(u)(L^*\beta)(v) - (L^*\alpha)(v)(L^*\beta)(u) \\
&= ((L^*\alpha) \wedge (L^*\beta))(u, v)\,.
\end{aligned}
$$

□

In this section, we generalise some definitions and results from the theory of smooth manifolds, and also introduce new ones. It should be remarked that unless otherwise specified, we will not assume smooth manifolds to be second countable, only Hausdorff, locally Euclidean (of one unique dimension) topological spaces with a smooth differentiable structure.

It should also be remarked that throughout this text, *a neighbourhood of a point p in S*, where S is a topological space containing the point p, is intended to mean *an open subset of S containing p.*

2.1. **Tangent and Cotangent Vectors.**

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open, and let $x^1, \ldots, x^n$ be the standard coordinates on $\mathbb{R}^n$. For $j \in \{1, \ldots, n\}$, we say that a function $f : \Omega \to \mathbb{C}$ has a *partial derivative* with respect to x^j at a point $p \in \Omega$ if the (real-valued) functions $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$ both have a partial derivative with respect to x^j at p. We then define

$$\frac{\partial f}{\partial x^j}(p) := \frac{\partial u}{\partial x^j}(p) + i\frac{\partial v}{\partial x^j}(p) \in \mathbb{C}$$

(we would write $\frac{d}{dx}$ instead of $\frac{\partial}{\partial x^j}$ if $\Omega \subset \mathbb{R}$). Note that if f has a partial derivative with respect to x^j at all points in Ω, then we obtain a new complex-valued function $\frac{\partial f}{\partial x^j}$ on Ω. If $\Omega \subset \mathbb{C}$ is open, we may regard Ω as an open subset of $\mathbb{R}^2$, and we obtain a similar definition of the partial derivatives of a function $f : \Omega \to \mathbb{C}$ with respect to the standard coordinates x and y on $\mathbb{C}$.

Definition 2.2. Let Ω be an open subset of $\mathbb{R}^n$ or $\mathbb{C}$, and consider a function $f : \Omega \to \mathbb{R}$ or $\mathbb{C}$. Then,

(i) f is said to be C^0 if f is continuous on Ω;
(ii) for $k \in \mathbb{N}$, f is said to be C^k if f is continuous and has continuous partial derivatives of all orders up to k on Ω;
(iii) f is said to be C^∞ if f is continuous and has continuous partial derivatives of all orders on Ω, that is, if f is C^k for all $k \in \mathbb{N}_0$.

For $k \in \mathbb{N}_0 \cup \{\infty\}$, the set of C^k real- or complex-valued functions on Ω is denoted by $C^k(\Omega, \mathbb{R})$ and $C^k(\Omega, \mathbb{C})$ respectively, or just $C^k(\Omega)$ if there is no possibility of confusion.

Note that for all $k, \ell \in \mathbb{N}_0 \cup \{\infty\}$, if $k \geq \ell$ (defining $\infty > m$ for all integers m), then $C^k(\Omega, \mathbb{F}) \subset C^\ell(\Omega, \mathbb{F})$, where Ω is an open subset of $\mathbb{R}^n$ or $\mathbb{C}$ and $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

We have the following two propositions, which we state without proof.

Proposition 2.3. *Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then, for an open subset Ω of $\mathbb{R}^n$ or $\mathbb{C}$, the set $C^k(\Omega, \mathbb{F})$ is a vector space over $\mathbb{F}$ under the usual operations of function addition and of scalar multiplication of functions by numbers in $\mathbb{F}$.*

Proposition 2.4. *Let Ω be an open subset of $\mathbb{R}^n$ or $\mathbb{C}$, and let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Then, a function $f \in C^1(\Omega, \mathbb{F})$ is real-differentiable, regarding $\mathbb{C}$ to be $\mathbb{R}^2$ if $\mathbb{C}$ is the domain or target space.*

Let M be a smooth manifold of dimension $n \in \mathbb{Z}_{\geq 0}$, and let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. For an open subset $U \subset M$, denote by $C^\infty_{\mathbb{F}}(U)$ the algebra of smooth $\mathbb{F}$-valued functions on U. For each $p \in M$, we define an equivalence relation $\sim_p$ on the set

$$\{f \in C^\infty_{\mathbb{F}}(U) \,|\, U \subset M \text{ is a neighbourhood of } p\}$$

given by

$$f \sim_p g \quad \Longleftrightarrow \quad f = g \text{ on some neighbourhood of } p,$$

for each two functions f and g in this set. The equivalence class $[f]$ of a C^∞ function f on a neighbourhood of p is called the *germ* of f at p. We call the set of germs of C^∞ functions at p the *stalk of* C^∞ at p, and denote it by $C^\infty_p(M)$, or $C^\infty_{\mathbb{F},p}(M)$ if we wish to specify the field $\mathbb{F}$. If $[f], [g] \in C^\infty_p(M)$ for some C^∞ functions $f : U \to \mathbb{F}$ and $g : V \to \mathbb{F}$ on neighbourhoods U and V of p, we define

(i) $\quad [f] + [g] := [\, f|_{U \cap V} + g|_{U \cap V} \,]$,

(ii) $\quad [f] \cdot [g] := [\, f|_{U \cap V} \cdot g|_{U \cap V} \,]$,

(iii) $\quad c[f] := [cf] \quad$ for each $c \in \mathbb{F}$.

One can then check that these operations are well defined and give $C^\infty_p(M)$ the structure of an algebra over $\mathbb{F}$, which is in particular a vector space over $\mathbb{F}$. A *tangent vector over* $\mathbb{F}$ at p, or a *real* (if $\mathbb{F} = \mathbb{R}$) or *complex* (if $\mathbb{F} = \mathbb{C}$) *tangent vector* at p, is defined to be a linear functional $v : C^\infty_p(M) \to \mathbb{F}$ that fulfils the Leibniz rule, that is, for all $[f], [g] \in C^\infty_p(M)$

$$v([f] \cdot [g]) = v([f])g(p) + f(p)v([g])\,.$$

Thus, the set of tangent vectors over $\mathbb{F}$ at p, which for now we denote by $T_{\mathbb{F},p}M$, is a subset of the dual space $(C^\infty_p(M))^*$. In fact, direct computation shows that if $u, v \in T_{\mathbb{F},p}M \subset (C^\infty_p(M))^*$ and $a, b \in \mathbb{F}$, then $au + bv \in T_{\mathbb{F},p}M$ (that is, $au + bv$ also fulfils the Leibniz rule), which, together with the fact that $0 \in T_{\mathbb{F},p}M$, means that $T_{\mathbb{F},p}M$ is actually a subspace of $(C^\infty_p(M))^*$. Moreover, we have the following proposition:

Proposition 2.5. *Let M be a smooth manifold and let $p \in M$. Then,*

$$(T_{\mathbb{R},p}M)_{\mathbb{C}} \cong T_{\mathbb{C},p}M\,.$$

Proof. Consider the map

$$\phi : (T_{\mathbb{R},p}M)_{\mathbb{C}} \to T_{\mathbb{C},p}M\,,$$
$$u + iv \mapsto \Big([f] \mapsto u([\mathrm{Re}(f)]) - v([\mathrm{Im}(f)]) + i(u([\mathrm{Im}(f)]) + v([\mathrm{Re}(f)])) \Big)\,,$$

where $u, v \in T_{\mathbb{R},p}M$ and $[f] \in C^\infty_{\mathbb{C},p}(M)$. If $f : U \to \mathbb{C}$ is smooth on a neighbourhood U of p, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are smooth real-valued functions on U; and if $f \sim_p g$ for some other smooth complex-valued function g on a neighbourhood of p, then $\mathrm{Re}(f) \sim_p \mathrm{Re}(g)$ and $\mathrm{Im}(f) \sim_p \mathrm{Im}(g)$ as real-valued functions. This shows that $\phi(u + iv)$ is well defined as a map from $C^\infty_{\mathbb{C},p}(M)$ to $\mathbb{C}$. One can also check that $\phi(u + iv)$ is linear and fulfils the Leibniz rule, so that indeed $\phi(u + iv) \in T_{\mathbb{C},p}M$. Linearity of ϕ can also be checked explicitly. The map

$$\phi^{-1} : T_{\mathbb{C},p}M \to (T_{\mathbb{R},p}M)_{\mathbb{C}}\,,$$
$$w \mapsto \Big([h] \mapsto \mathrm{Re}(w([h + i0])) \Big) + i\Big([h] \mapsto \mathrm{Im}(w([h + i0])) \Big)\,, \quad [h] \in C^\infty_{\mathbb{R},p}(M)\,,$$

gives the inverse of ϕ. $\square$

We now fix a smooth manifold M of dimension n and a point $p \in M$.

Using the correspondence given by Proposition 2.5, we may denote by T_pM the vector space of real tangent vectors at p, and by $(T_pM)_{\mathbb{C}}$ the vector space of complex tangent vectors at p. We call T_pM the *tangent space (to M) at p*, and $(T_pM)_{\mathbb{C}}$ the *complexified tangent space (to M) at p*. We also call $T_p^*M := (T_pM)^*$ the *cotangent space (to M) at p*, and $((T_pM)_{\mathbb{C}})^* \cong (T_p^*M)_{\mathbb{C}}$ the *complexified cotangent space (to M) at p*. We may call elements in T_p^*M and $(T_p^*M)_{\mathbb{C}}$ *real* and *complex cotangent vectors at p* respectively.

Again letting $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, if $v \in T_{\mathbb{F},p}M$ and f is a C^∞ $\mathbb{F}$-valued function on a neighbourhood of p, we may write $v(f)$ to denote $v([f])$. We define the *differential of f at p* to be the (real or complex) cotangent vector $(df)_p \in (T_{\mathbb{F},p}M)^*$ given by

$$(df)_p(u) := u(f)\,, \quad u \in T_{\mathbb{F},p}M\,.$$

If $(U, \phi) = (U, x^1, \dots, x^n)$ is a chart about p in M and f is an $\mathbb{F}$-valued function on a neighbourhood V of p, for each $j \in \{1, \dots, n\}$ we define

$$\left.\frac{\partial f}{\partial x^j}\right|_p := \left.\frac{\partial (f \circ \phi^{-1})}{\partial r^j}\right|_{\phi(p)}$$

if the partial derivative on the right-hand side exists, where the function $f \circ \phi^{-1}$ is defined on the open subset $\phi(U \cap V)$ of $\mathbb{R}^n$, and r^j is the jth standard coordinate on $\mathbb{R}^n$. One can then check that for each $j \in \{1, \dots, n\}$ the map

$$\left.\frac{\partial}{\partial x^j}\right|_p : C^\infty_{\mathbb{F},p}(M) \to \mathbb{F}\,, \quad [f] \mapsto \left.\frac{\partial f}{\partial x^j}\right|_p$$

is well defined, linear and fulfils the Leibniz rule, so that $\frac{\partial}{\partial x^j}\big|_p \in T_{\mathbb{F},p}M$. Note that the notation $\frac{\partial}{\partial x^j}\big|_p$ as an element in $T_{\mathbb{C},p}M \cong (T_pM)_{\mathbb{C}}$ may also mean the element $\frac{\partial}{\partial x^j}\big|_p + i0$, for $\frac{\partial}{\partial x^j}\big|_p \in T_pM$. However, for each $[f] \in C^\infty_{\mathbb{C},p}(M)$, regarding $\frac{\partial}{\partial x^j}\big|_p$ to be a real tangent vector, we have

$$\left(\left.\frac{\partial}{\partial x^j}\right|_p + i0\right)([f]) = \left.\frac{\partial(\mathrm{Re}(f))}{\partial x^j}\right|_p + i\left.\frac{\partial(\mathrm{Im}(f))}{\partial x^j}\right|_p = \left.\frac{\partial f}{\partial x^j}\right|_p,$$

so there is no ambiguity in the notation. Since we know from the theory of smooth manifolds that $\{\frac{\partial}{\partial x^j}\big|_p\}_{j=1}^n \subset T_pM$ is a basis for T_pM and $\{(dx^j)_p\}_{j=1}^n \subset T_p^*M$ is a basis for T_p^*M, it follows from Proposition 1.8 that these are also bases for the respective complexifications $(T_pM)_{\mathbb{C}}$ and $(T_p^*M)_{\mathbb{C}} \cong ((T_pM)_{\mathbb{C}})^*$. Moreover, the basis $\{(dx^j)_p\}_{j=1}^n \subset ((T_pM)_{\mathbb{C}})^*$ is dual to the basis $\{\frac{\partial}{\partial x^j}\big|_p\}_{j=1}^n \subset (T_pM)_{\mathbb{C}}$. Note that if $u + iv$ is a vector in $(T_pM)_{\mathbb{C}}$, for $u, v \in T_pM$, then for each $j \in \{1, \dots, n\}$

$$\begin{aligned}
((dx^j)_p + i0)(u + iv) &= (dx^j)_p(u) + i(dx^j)_p(v) \\
&= u(x^j) + iv(x^j) \\
&= (u + iv)(x^j + i0) \\
&= (d(x^j + i0))_p(u + iv)\,,
\end{aligned}$$

so we may also denote by $(dx^j)_p \in ((T_pM)_{\mathbb{C}})^*$ the differential at p of x^j as a C^∞ complex-valued function at p.

Let again $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart about p in M, and f be a smooth $\mathbb{F}$-valued function on a neighbourhood V of p. Since $\{\frac{\partial}{\partial x^j}\big|_p\}_{j=1}^n \subset T_{\mathbb{F},p}M$ is a basis for

$T_{\mathbb{F},p}M$, for each $v \in T_{\mathbb{F},p}M$ we have $v = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\Big|_p$ for some unique $v^1, \dots, v^n \in \mathbb{F}$. Thus,

$$v(f) = \left(\sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\bigg|_p\right)(f) = \sum_{j=1}^n v^j \frac{\partial f}{\partial x^j}\bigg|_p .$$

Then, if g is a C^1 $\mathbb{F}$-valued function on some neighbourhood W of p, we may define

$$v(g) := \sum_{j=1}^n v^j \frac{\partial g}{\partial x^j}\bigg|_p .$$

As one can check, the number $v(g) \in \mathbb{F}$ is independent of the choice of chart about p, and if g is smooth it agrees with our previous definition of $v(g)$. We can then define the $\mathbb{F}$-linear map

$$(dg)_p : T_{\mathbb{F},p}M \to \mathbb{F}\,, \quad (dg)_p(u) := u(g)\,, \quad u \in T_{\mathbb{F},p}M\,.$$

We call $(dg)_p \in (T_{\mathbb{F},p}M)^*$ the *differential of g at p*, thus extending the definition of the differential of a smooth function on a neighbourhood of p to the case when the function is only C^1.

Definition 2.6. Let M and N be smooth manifolds, and let $F : M \to N$ be a C^1 map. For each $p \in M$, we define the *differential of F at p* to be the $\mathbb{F}$-linear map of vector spaces

$$F_{*,p} : T_{\mathbb{F},p}M \to T_{\mathbb{F},F(p)}N\,, \quad (F_{*,p}v)([f]) := v(f \circ F)\,,$$

for each $v \in T_{\mathbb{F},p}M$ and $[f] \in C^\infty_{\mathbb{F},F(p)}N$.

Remark 2.7. (i) In Definition 2.6, if $\Omega \subset N$ is the domain of f, then the function $f \circ F$ is C^1 on the open subset $F^{-1}(\Omega) \subset M$, so $v(f \circ F)$ is defined. Moreover, one can check that $(F_{*,p}v)([f])$ is independent of the choice of representative for $[f]$ and hence well defined, and that $F_{*,p}v$ is indeed a tangent vector (over $\mathbb{F}$) at $F(p)$ in N. We may write F_* instead of $F_{*,p}$ if there is no possibility of confusion.

(ii) The differential $F_{*,p}$ is either a map $T_pM \to T_{F(p)}N$ or $(T_pM)_{\mathbb{C}} \to (T_{F(p)}N)_{\mathbb{C}}$, depending on our choice of $\mathbb{F}$. Denoting the former by $F^{\mathbb{R}}_{*,p}$ and the latter by $F^{\mathbb{C}}_{*,p}$, one can check that for all $v \in T_pM$

$$F^{\mathbb{C}}_{*,p}(v + i0) = F^{\mathbb{R}}_{*,p}(v) + i0\,,$$

or in other words,

$$F^{\mathbb{C}}_{*,p} \circ \iota_p = \iota_{F(p)} \circ F^{\mathbb{R}}_{*,p}$$

as maps $T_pM \to (T_{F(p)}N)_{\mathbb{C}}$, where $\iota_p : T_pM \to (T_pM)_{\mathbb{C}}$ and $\iota_{F(p)} : T_{F(p)}N \to (T_{F(p)}N)_{\mathbb{C}}$ are the inclusion maps.

Proposition 2.8. *Let $F : M \to N$ be a C^1 map of manifolds, with $m := \dim M$ and $n := \dim N$, and let $p \in M$. If $(U, x^1, \dots, x^m)$ and $(V, y^1, \dots, y^n)$ are charts about p in M and about $F(p)$ in N respectively, then the $n \times m$ matrix representing the differential $F_* : T_{\mathbb{F},p}M \to T_{\mathbb{F},F(p)}N$ with respect to the bases $\{\frac{\partial}{\partial x^j}|_p\}_{j=1}^m$ for $T_{\mathbb{F},p}M$ and $\{\frac{\partial}{\partial y^k}|_{F(p)}\}_{k=1}^n$ for $T_{\mathbb{F},F(p)}N$ is given by*

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}\Big|_p & \cdots & \frac{\partial F^1}{\partial x^m}\Big|_p \\ \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial x^1}\Big|_p & \cdots & \frac{\partial F^n}{\partial x^m}\Big|_p \end{pmatrix},$$

where for each $k \in \{1, \dots, n\}$ we let $F^k := y^k \circ F : F^{-1}(V) \to \mathbb{R} \subset \mathbb{F}$.

Proof. By facts from linear algebra, we know that for $j \in \{1, \dots, m\}$ and $k \in \{1, \dots, n\}$, the kth element in the jth column of the matrix A representing F_* with respect to the chosen bases is the coefficient multiplying $\frac{\partial}{\partial y^k}|_{F(p)}$ in the representation of $F_*(\frac{\partial}{\partial x^j}|_p)$ with respect to the basis $\{\frac{\partial}{\partial y^\ell}|_{F(p)}\}_{\ell=1}^n$ for $T_{\mathbb{F},F(p)}N$, that is,

$$A_j^k = \left(F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right)\right)(y^k) = \frac{\partial}{\partial x^j}\bigg|_p (y^k \circ F) = \frac{\partial F^k}{\partial x^j}\bigg|_p .$$

□

Remark 2.9. One may use Proposition 2.8 to show that, in notation from Definition 2.6, if g is a C^1 $\mathbb{F}$-valued function on a neighbourhood of $F(p)$ in N, then we also have

$$(F_{*,p}v)(g) = v(g \circ F) .$$

Definition 2.10. Let M be a smooth manifold. We define the *tangent bundle* of M by

$$TM := \bigcup_{p \in M} T_pM ,$$

and the *complexified tangent bundle* of M by

$$(TM)_{\mathbb{C}} := \bigcup_{p \in M} (T_pM)_{\mathbb{C}} .$$

We also define the projection maps

$$\Pi_{TM} : TM \to M , \quad \Pi_{TM}(u) := p \quad \text{if } u \in T_pM \text{ for } p \in M;$$

$$\Pi_{(TM)_{\mathbb{C}}} : (TM)_{\mathbb{C}} \to M , \quad \Pi_{(TM)_{\mathbb{C}}}(v) := p \quad \text{if } v \in (T_pM)_{\mathbb{C}} \text{ for } p \in M.$$

We now recall the definition of a C^∞ vector bundle of rank $r \in \mathbb{Z}_{\geq 0}$.

Definition 2.11. Let $r \in \mathbb{Z}_{\geq 0}$. A C^∞ *vector bundle of rank* r is a triple (E, M, Π) consisting of C^∞ manifolds E and M and a smooth surjective map $\Pi : E \to M$ such that

(i) for each $p \in M$, the preimage $\Pi^{-1}(\{p\}) \subset E$, called the *fiber at* p and denoted merely by $\Pi^{-1}(p)$, is a real vector space of dimension r;
(ii) for every point $p \in M$ there exist a neighbourhood $U \subset M$ of p and a diffeomorphism $\varphi : \Pi^{-1}(U) \to U \times \mathbb{R}^r$, where $U \times \mathbb{R}^r$ has the product manifold structure and $\pi : U \times \mathbb{R}^r \to U$ is the projection map onto U, such that $\Pi = \pi \circ \varphi$ on $\Pi^{-1}(U)$ and for every $q \in U$ the restriction

$$\varphi\big|_{\Pi^{-1}(q)} : \Pi^{-1}(q) \to \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism. The open set U is then called a *trivialising open set for* E, and the map φ is called a *trivialisation of* E *over* U.

It is assumed that the reader is familiar with the usual construction of a topology and C^∞ manifold structure on the tangent bundle TM of a smooth n-manifold M, and with the proof that (TM, M, Π_{TM}) is then a smooth vector bundle of rank n. We proceed analogously for the case of $(TM)_{\mathbb{C}}$. For each chart $(U, \phi) = (U, x^1, \dots, x^n)$ on M, a vector $v \in \Pi^{-1}_{(TM)_{\mathbb{C}}}(U)$ is in $(T_pM)_{\mathbb{C}}$ for exactly one $p \in U$, with $p = \Pi_{(TM)_{\mathbb{C}}}(v)$. Since $(T_pM)_{\mathbb{C}}$ is an n-dimensional complex vector space with basis $\{\frac{\partial}{\partial x^j}\big|_p\}_{j=1}^n$, by Proposition

1.3 it is also a real vector space of dimension $2n$ with basis $\{\frac{\partial}{\partial x^j}\big|_p\}_{j=1}^n \cup \{i\frac{\partial}{\partial x^j}\big|_p\}_{j=1}^n$. Thus, we have

$$v = \sum_{j=1}^{n} \left(a^j(v) \frac{\partial}{\partial x^j}\bigg|_p + b^j(v) i \frac{\partial}{\partial x^j}\bigg|_p \right)$$

for some unique $a^j(v), b^j(v) \in \mathbb{R}$, $j \in \{1, \ldots, n\}$. In fact, for each $j \in \{1, \ldots, n\}$ we have $a^j(v) = \mathrm{Re}(v(x^j)) = \mathrm{Re}((dx^j)_p(v))$ and $b^j(v) = \mathrm{Im}(v(x^j)) = \mathrm{Im}((dx^j)_p(v))$. We may then consider the map

$$\tilde{\phi} : \Pi^{-1}_{(TM)_\mathbb{C}}(U) \to \phi(U) \times \mathbb{R}^{2n} \subset \mathbb{R}^{3n} ,$$

$$v \mapsto ((x^1 \circ \Pi_{(TM)_\mathbb{C}})(v), \ldots, (x^n \circ \Pi_{(TM)_\mathbb{C}})(v), a^1(v), b^1(v), \ldots, a^n(v), b^n(v)) .$$

Since $\tilde{\phi}$ is a bijection, we may use it to transfer the topology of $\phi(U) \times \mathbb{R}^{2n}$ to $\Pi^{-1}_{(TM)_\mathbb{C}}(U)$, that is, we can define the open sets in $\Pi^{-1}_{(TM)_\mathbb{C}}(U)$ to be the preimages under $\tilde{\phi}$ of the open sets in $\phi(U) \times \mathbb{R}^{2n}$. We then define the collection

$$B := \{A \subset (TM)_\mathbb{C} \,|\, A \text{ is open in } \Pi^{-1}_{(TM)_\mathbb{C}}(U) \text{ for some chart } (U, \phi) \text{ on } M\} .$$

It can be shown that B fulfils the necessary conditions to be the basis for a topology on $(TM)_\mathbb{C}$, and that $(TM)_\mathbb{C}$ with this topology is Hausdorff. Moreover, it follows from the construction of this topology that for each chart (U, ϕ) on M the subspace topology on the open subset $\Pi^{-1}_{(TM)_\mathbb{C}}(U) \subset (TM)_\mathbb{C}$ is the same as the one we transferred using the bijection $\tilde{\phi} : \Pi^{-1}_{(TM)_\mathbb{C}}(U) \to \phi(U) \times \mathbb{R}^{2n}$ (the proof relies on the fact that if (V, ψ) is another chart on M, then the map

$$\tilde{\phi} \circ \tilde{\psi}^{-1} : \tilde{\psi}(\Pi^{-1}_{(TM)_\mathbb{C}}(U \cap V)) \to \tilde{\phi}(\Pi^{-1}_{(TM)_\mathbb{C}}(U \cap V))$$

is a homeomorphism). Thus, $\tilde{\phi}$ is a homeomorphism from an open subset of $(TM)_\mathbb{C}$ to an open subset of $\mathbb{R}^{3n}$, so $(\Pi^{-1}_{(TM)_\mathbb{C}}(U), \tilde{\phi})$ is a chart on $(TM)_\mathbb{C}$. One can also check that if $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is an atlas in the differentiable structure of M, for a suitable indexing set A, then $\{(\Pi^{-1}_{(TM)_\mathbb{C}}(U_\alpha), \tilde{\phi}_\alpha)\}_{\alpha \in A}$ is an atlas in $(TM)_\mathbb{C}$ and hence makes the complexified tangent bundle of M a C^∞ manifold of dimension $3n$. Moreover, $\Pi_{(TM)_\mathbb{C}} : (TM)_\mathbb{C} \to M$ becomes a C^∞ surjective map of manifolds, and the triple $((TM)_\mathbb{C}, M, \Pi_{(TM)_\mathbb{C}})$ becomes a C^∞ vector bundle of rank $2n$: for each chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on M, the open subset $\Pi^{-1}_{(TM)_\mathbb{C}}(U) \subset (TM)_\mathbb{C}$ is a trivialising open set for $(TM)_\mathbb{C}$, with trivialisation

$$((\phi^{-1} \times \mathbb{1}_{\mathbb{R}^{2n}}) \circ \tilde{\phi}) : \Pi^{-1}_{(TM)_\mathbb{C}}(U) \to U \times \mathbb{R}^{2n} ,$$

$$v \mapsto (\Pi_{(TM)_\mathbb{C}}(v), \mathrm{Re}(v(x^1)), \mathrm{Im}(v(x^1)), \ldots, \mathrm{Re}(v(x^n)), \mathrm{Im}(v(x^n))) .$$

Definition 2.12. Let M be a smooth manifold. We define the *cotangent bundle* of M by

$$T^*M := \bigcup_{p \in M} T_p^*M ,$$

and the *complexified cotangent bundle* of M by

$$(T^*M)_\mathbb{C} := \bigcup_{p \in M} (T_p^*M)_\mathbb{C} .$$

We also define the projection maps

$$\Pi_{T^*M} : T^*M \to M , \quad \Pi_{T^*M}(u) := p \quad \text{if } u \in T_p^*M \text{ for } p \in M;$$

$$\Pi_{(T^*M)_\mathbb{C}} : (T^*M)_\mathbb{C} \to M , \quad \Pi_{(T^*M)_\mathbb{C}}(v) := p \quad \text{if } v \in (T_p^*M)_\mathbb{C} \text{ for } p \in M.$$

For a smooth n-manifold M, we construct a topology and a differentiable structure on T^*M and $(T^*M)_{\mathbb{C}}$ in an analogous way to the cases of TM and $(TM)_{\mathbb{C}}$ respectively. Then, T^*M and $(T^*M)_{\mathbb{C}}$ become smooth manifolds of dimensions $2n$ and $3n$ respectively, and the triples (T^*M, M, Π_{T^*M}) and $((T^*M)_{\mathbb{C}}, M, \Pi_{(T^*M)_{\mathbb{C}}})$ become smooth vector bundles of ranks n and $2n$ respectively. For each chart (U, ϕ) on M, we obtain charts $(\Pi_{T^*M}^{-1}(U), \hat{\phi})$ on T^*M and $(\Pi_{(T^*M)_{\mathbb{C}}}^{-1}(U), \hat{\phi}_{\mathbb{C}})$ on $(T^*M)_{\mathbb{C}}$ defined by

$$\hat{\phi} : \Pi_{T^*M}^{-1}(U) \to \phi(U) \times \mathbb{R}^n ,$$

$$u \mapsto \left((\phi \circ \Pi_{T^*M})(u), u\left(\frac{\partial}{\partial x^1}\bigg|_{\Pi_{T^*M}(u)} \right), \ldots, u\left(\frac{\partial}{\partial x^n}\bigg|_{\Pi_{T^*M}(u)} \right) \right)$$

for $u \in \Pi_{T^*M}^{-1}(U)$, and

$$\hat{\phi}_{\mathbb{C}} : \Pi_{(T^*M)_{\mathbb{C}}}^{-1}(U) \to \phi(U) \times \mathbb{R}^{2n} ,$$

$$v \mapsto \Bigg((\phi \circ \Pi_{(T^*M)_{\mathbb{C}}})(v), \mathrm{Re}\left(v\left(\frac{\partial}{\partial x^1}\bigg|_{\Pi_{(T^*M)_{\mathbb{C}}}(v)} \right)\right), \mathrm{Im}\left(v\left(\frac{\partial}{\partial x^1}\bigg|_{\Pi_{(T^*M)_{\mathbb{C}}}(v)} \right)\right),$$
$$\ldots, \mathrm{Re}\left(v\left(\frac{\partial}{\partial x^n}\bigg|_{\Pi_{(T^*M)_{\mathbb{C}}}(v)} \right)\right), \mathrm{Im}\left(v\left(\frac{\partial}{\partial x^n}\bigg|_{\Pi_{(T^*M)_{\mathbb{C}}}(v)} \right)\right)\Bigg)$$

for $v \in \Pi_{(T^*M)_{\mathbb{C}}}^{-1}(U)$. The subsets $\Pi_{T^*M}^{-1}(U) \subset T^*M$ and $\Pi_{(T^*M)_{\mathbb{C}}}^{-1}(U) \subset (T^*M)_{\mathbb{C}}$ are also trivialising open sets with respective trivialisations

$$((\phi^{-1} \times \mathbb{1}_{\mathbb{R}^n}) \circ \hat{\phi}) : \Pi_{(T^*M)}^{-1}(U) \to U \times \mathbb{R}^n$$

and

$$((\phi^{-1} \times \mathbb{1}_{\mathbb{R}^{2n}}) \circ \hat{\phi}_{\mathbb{C}}) : \Pi_{(T^*M)_{\mathbb{C}}}^{-1}(U) \to U \times \mathbb{R}^{2n} .$$

Recall that if (E, M, Π) is a smooth vector bundle of rank $r \in \mathbb{Z}_{\geq 0}$ and $U \subset M$ is open, a *section of E (or of (E, M, Π)) over U* is a map $s : U \to E$ such that $\Pi \circ s$ is the inclusion map $\iota : U \to M$, that is, such that for all $p \in U$, $s(p) \in \Pi^{-1}(p)$. If $U = M$, we may only say that s is a *section* of E. Since for each $p \in M$ the fiber $\Pi^{-1}(p)$ is a real vector space, if s and t are two sections of E over U and $c \in \mathbb{R}$, we may define the sections $s + t$ and cs of E over U by $(s + t)(p) := s(p) + t(p)$ and $(cs)(p) := cs(p)$ respectively for each $p \in U$. This gives the set of sections of E over U the structure of a real vector space. If s is a section of E over U and $f : U \to \mathbb{R}$ is a function, we may also define the section fs of E over U by $(fs)(p) := f(p)s(p)$ for each $p \in U$. For $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a section $s : U \to E$ is said to be C^k if it is C^k as a map of manifolds. The set $\Gamma^k(U, E)$ of C^k sections of E over U is a subspace of the vector space of sections of E over U. If $s \in \Gamma^k(U, E)$ and $f : U \to \mathbb{R}$ is a C^k function, one can show that $fs \in \Gamma^k(U, E)$, and as a result $\Gamma^k(U, E)$ is also a module over the ring $C^k_{\mathbb{R}}(U)$ of C^k real-valued functions on U. A *frame* for E over U is a collection $\{s_1, \ldots, s_r\}$ of sections of E over U such that for each $p \in U$ the collection $\{s_1(p), \ldots, s_r(p)\} \subset \Pi^{-1}(p)$ is a basis for the (real) vector space $\Pi^{-1}(p)$. If $\{s_1, \ldots, s_r\}$ is a frame for E over U, then each section s for E over U can be written as $\sum_{j=1}^r f^j s_j$ for some unique real-valued functions $f_1, \ldots, f_r$ on U. A frame for E over U is said to be C^k if all the sections in the frame are C^k. If U is a trivialising open set for E with trivialisation $\varphi : \Pi^{-1}(U) \to U \times \mathbb{R}^r$ and $\{e_1, \ldots, e_r\}$ denotes the standard basis for $\mathbb{R}^r$, then for each $p \in U$ we may use the vector space isomorphism $\varphi|_{\Pi^{-1}(p)} : \Pi^{-1}(p) \to \{p\} \times \mathbb{R}^r$ to map the basis $\{(p, e_1), \ldots, (p, e_r)\}$ for $\{p\} \times \mathbb{R}^r$ to a basis for the fiber $\Pi^{-1}(p)$. Then, for each $j \in \{1, \ldots, r\}$ we can define

$t_j(p) := \varphi^{-1}((p, e_j))$, and we obtain a frame $\{t_1, \ldots, t_r\}$ for E over U. As one can check, the frame $\{t_1, \ldots, t_r\}$ is C^∞, and we call it the C^∞ *frame over* U *of the trivialisation* φ. We recall the following two propositions:

Proposition 2.13. *Let (E, M, Π) be a smooth vector bundle of rank $r \in \mathbb{Z}_{\geq 0}$, and let $U \subset M$ be open. If $\{s_1, \ldots, s_r\}$ is a C^∞ frame for E over U and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, then a section s of E over U is C^k if and only if $s = \sum_{j=1}^r f^j s_j$ for some C^k real-valued functions $f^1, \ldots, f^r$ on U.*

Proposition 2.14. *Let (E, M, Π) be a smooth vector bundle of rank $r \in \mathbb{Z}_{\geq 0}$, let $\Omega \subset M$ be open, and let $s : \Omega \to E$ be a section of E over Ω. For $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the following statements are equivalent:*

(i) *s is C^k on Ω;*
(ii) *for each trivialising open set $U \subset M$ with trivialisation $\varphi : \Pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $\Omega \cap U \neq \emptyset$, the section $s\big|_{\Omega \cap U}$ of E over $\Omega \cap U$ can be written as $s\big|_{\Omega \cap U} = \sum_{j=1}^r f^j \cdot t_j\big|_{\Omega \cap U}$ for some C^k real-valued functions $f^1, \ldots, f^r$ on $\Omega \cap U$, where $\{t_1, \ldots, t_r\}$ is the C^∞ frame over U of the trivialisation φ;*
(iii) *for each $p \in \Omega$ there exists a trivialising open set $U \subset M$ with trivialisation $\varphi : \Pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $p \in U$ and the section $s\big|_{\Omega \cap U}$ of E over $\Omega \cap U$ can be written as $s\big|_{\Omega \cap U} = \sum_{j=1}^r f^j \cdot t_j\big|_{\Omega \cap U}$ for some C^k real-valued functions $f^1, \ldots, f^r$ on $\Omega \cap U$, where $\{t_1, \ldots, t_r\}$ is the C^∞ frame over U of the trivialisation φ.*

For a smooth vector bundle (E, M, Π) of rank $r \in \mathbb{Z}_{\geq 0}$ and an arbitrary subset $A \subset M$, we may also define a *section of* E *over* A to be a map $s : A \to E$ such that for all $p \in A$, $s(p) \in \Pi^{-1}(p)$. The set of sections of E over A is also a real vector space and a module over the ring of real-valued functions on A. Even though A is not a manifold in general, it is a topological space with the subspace topology inherited from M, so we may define a section s of E over A to be continuous if it is continuous as a map $s : A \to E$. The set $\Gamma^0(A, E)$ of continuous sections of E over A is a subspace of the vector space of sections of E over A, and a module over the ring of continuous real-valued functions on A. The notions of *frame for* E *over* A and *continuity* of such a frame are defined exactly as for the case when $A \subset M$ is open. Moreover, we have the following proposition:

Proposition 2.15. *Let (E, M, Π) be a smooth vector bundle of rank $r \in \mathbb{Z}_{\geq 0}$, let $A \subset M$, and let $s : A \to E$ be a section of E over A. Then, the following statements are equivalent:*

(i) *s is continuous on A;*
(ii) *for each trivialising open set $U \subset M$ with trivialisation $\varphi : \Pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $A \cap U \neq \emptyset$, the section $s\big|_{A \cap U}$ of E over $A \cap U$ can be written as $s\big|_{A \cap U} = \sum_{j=1}^r f^j \cdot t_j\big|_{A \cap U}$ for some continuous real-valued functions $f^1, \ldots, f^r$ on $A \cap U$, where $\{t_1, \ldots, t_r\}$ is the C^∞ frame over U of the trivialisation φ;*
(iii) *for each $p \in A$ there exists a trivialising open set $U \subset M$ with trivialisation $\varphi : \Pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $p \in U$ and the section $s\big|_{A \cap U}$ of E over $A \cap U$ can be written as $s\big|_{A \cap U} = \sum_{j=1}^r f^j \cdot t_j\big|_{A \cap U}$ for some continuous real-valued functions $f^1, \ldots, f^r$ on $A \cap U$, where $\{t_1, \ldots, t_r\}$ is the C^∞ frame over U of the trivialisation φ.*

Definition 2.16. Let M be a C^∞ manifold, and let $A \subset M$.

(i) A *(real) vector field on* A, or a *vector field over* $\mathbb{R}$ *on* A, is defined to be a section $u : A \to TM$ of TM over A. A *complex vector field on* A, or a *vector field over* $\mathbb{C}$ *on* A, is a section $v : A \to (TM)_\mathbb{C}$ of $(TM)_\mathbb{C}$ over A.

(ii) A *(real) differential form of degree* 1 *on* A, or a *(real)* 1*-form on* A, or a 1*-form over* $\mathbb{R}$ *on* A, is a section $\omega : A \to T^*M$ of T^*M over A. A *complex differential form of degree* 1 *on* A, or a *complex* 1*-form on* A, or a 1*-form over* $\mathbb{C}$ *on* A, is a section $\tau : A \to (T^*M)_\mathbb{C}$ of $(T^*M)_\mathbb{C}$ over A.

A real or complex vector field v on A, and a real or complex 1-form ω on A, are defined to be *continuous* if they are continuous as sections of the corresponding smooth vector bundles. If $A \subset M$ is open and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, then v and ω are defined to be C^k if they are C^k as sections of the corresponding smooth vector bundles.

Definition 2.17. Let M be a smooth manifold and f a C^1 $\mathbb{F}$-valued function on an open set $U \subset M$. Then, the *differential of* f is defined to be the real (if $\mathbb{F} = \mathbb{R}$) or complex (if $\mathbb{F} = \mathbb{C}$) 1-form df on U mapping each $p \in U$ to the differential $(df)_p \in (T_{\mathbb{F},p}M)^*$ of f at p.

2.2. **Differential Forms on Smooth Curves and Surfaces.**

Throughout Subsection 2.2, we fix a smooth manifold M of dimension $n \in \{1, 2\}$, and we let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$.

For each point $p \in M$, since $\dim T_pM = n \leq 2$, we may consider the exterior powers $\Lambda^r T_p^*M$ and $\Lambda^r (T_pM)^*_\mathbb{C} \cong (\Lambda^r T_p^*M)_\mathbb{C}$ for $r \in \mathbb{Z}_{\geq 0}$. We identify $\Lambda^0 T_p^*M = \mathbb{R}$ and $\Lambda^0(T_pM)^*_\mathbb{C} = \mathbb{C}$ with $\{p\} \times \mathbb{R}$ and $\{p\} \times \mathbb{C}$ respectively, to distinguish them from $\Lambda^0 T_q^*M$ and $\Lambda^0(T_qM)^*_\mathbb{C}$ for some different point $q \in M$. If $n = 1$, then by Proposition 1.18(i) the vector spaces $\Lambda^2 T_p^*M$ and $\Lambda^2(T_pM)^*_\mathbb{C}$ are trivial, and we write $\Lambda^2 T_p^*M = \{p\} \times \{0\}$ (as a real vector space) and $\Lambda^2(T_pM)^*_\mathbb{C} = \{p\} \times \{0\}$ (as a complex vector space). If $n = 2$ and (U, x^1, x^2) is a chart about p in M, then by Proposition 1.18(ii) the spaces $\Lambda^2 T_p^*M$ and $\Lambda^2(T_pM)^*_\mathbb{C}$ are 1-dimensional with respective bases $\{(dx^1)_p \wedge (dx^2)_p\} \subset \Lambda^2 T_p^*M$ (for $(dx^1)_p, (dx^2)_p \in T_p^*M$) and $\{(dx^1)_p \wedge (dx^2)_p\} \subset \Lambda^2(T_pM)^*_\mathbb{C}$ (for $(dx^1)_p, (dx^2)_p \in (T_p^*M)_\mathbb{C}$).

Remark 2.18. For a point $p \in M$ and $r \in \mathbb{Z}_{\geq 0}$, we write $\Lambda^r(T_pM)^*_\mathbb{C}$ to denote the *rth* exterior power of $(T_pM)_\mathbb{C}$, instead of writing $\Lambda^r(T_p^*M)_\mathbb{C}$, since we have only defined the notation $\Lambda^r\mathcal{V}$ when $\mathcal{V} = \mathcal{W}^*$ for some vector space $\mathcal{W}$ over $\mathbb{F}$.

We defined in Subsection 2.1 the cotangent bundle T^*M and the complexified cotangent bundle $(T^*M)_\mathbb{C}$ of M, which we may also write as

$$\Lambda^1 T^*M := \bigcup_{p \in M} \Lambda^1 T_p^*M = \bigcup_{p \in M} T_p^*M = T^*M$$

and

$$\Lambda^1 (T^*M)_\mathbb{C} := \bigcup_{p \in M} \Lambda^1 (T_pM)^*_\mathbb{C} = \bigcup_{p \in M} (T_p^*M)_\mathbb{C} = (T^*M)_\mathbb{C}$$

respectively. As we have seen, $\Lambda^1 T^*M$ and $\Lambda^1(T^*M)_\mathbb{C}$ are smooth manifolds, and also smooth vector bundles together with their respective projection maps $\Pi_{(T^*M)} : T^*M \to$

M and $\Pi_{(T^*M)_{\mathbb{C}}} : (T^*M)_{\mathbb{C}} \to M$. We now define the sets

$$\Lambda^0 T^*M := \bigcup_{p\in M} \Lambda^0 T_p^*M = \bigcup_{p\in M} (\{p\} \times \mathbb{R}) = M \times \mathbb{R}\,,$$
$$\Lambda^0 (T^*M)_{\mathbb{C}} := \bigcup_{p\in M} \Lambda^0 (T_pM)^*_{\mathbb{C}} = \bigcup_{p\in M} (\{p\} \times \mathbb{C}) = M \times \mathbb{C}\,,$$
$$\Lambda^2 T^*M := \bigcup_{p\in M} \Lambda^2 T_p^*M\,,$$
$$\Lambda^2 (T^*M)_{\mathbb{C}} := \bigcup_{p\in M} \Lambda^2 (T_pM)^*_{\mathbb{C}}\,.$$

The respective maps $\Pi_{\Lambda^0 T^*M}, \Pi_{\Lambda^0 (T^*M)_{\mathbb{C}}}, \Pi_{\Lambda^2 T^*M}$ and $\Pi_{\Lambda^2 (T^*M)_{\mathbb{C}}}$ projecting each of these sets onto M are defined analogously to the cases of T^*M and $(T^*M)_{\mathbb{C}}$. For $r \in \{0, 1, 2\}$, we call $\Lambda^r T^*M$ the *rth exterior power of* T^*M, and $\Lambda^r (T^*M)_{\mathbb{C}}$ the *rth exterior power of* $(T^*M)_{\mathbb{C}}$. We give the sets $\Lambda^0 T^*M = M \times \mathbb{R}$ and $\Lambda^0 (T^*M)_{\mathbb{C}} = M \times \mathbb{C}$ the product topologies and product manifold structures of $M \times \mathbb{R}$ and $M \times \mathbb{R}^2$ respectively, and the triples $(\Lambda^0 T^*M, M, \Pi_{\Lambda^0 T^*M})$ and $(\Lambda^0 (T^*M)_{\mathbb{C}}, M, \Pi_{\Lambda^0 (T^*M)_{\mathbb{C}}})$ become smooth vector bundles of ranks 1 and 2 respectively, with global trivialisations given by the identity maps on $\Pi^{-1}_{\Lambda^0 T^*M}(M) = M \times \mathbb{R}$ and $\Pi^{-1}_{\Lambda^0 (T^*M)_{\mathbb{C}}}(M) = M \times \mathbb{R}^2$ respectively. If $n = 1$, then as sets $\Lambda^2 T^*M = \bigcup_{p\in M} (\{p\} \times \{0\}) = M \times \{0\}$ and $\Lambda^2 (T^*M)_{\mathbb{C}} = \bigcup_{p\in M} (\{p\} \times \{0\}) = M \times \{0\}$, so we may transfer to both $\Lambda^2 T^*M$ and $\Lambda^2 (T^*M)_{\mathbb{C}}$ the topology and differentiable structure from M, and we may also regard the triplets $(\Lambda^2 T^*M, M, \Pi_{\Lambda^2 T^*M})$ and $(\Lambda^2 (T^*M)_{\mathbb{C}}, M, \Pi_{\Lambda^2 (T^*M)_{\mathbb{C}}})$ as smooth vector bundles of rank 0 with global trivialisations. If $n = 2$, then the procedure to give $\Lambda^2 T^*M$ and $\Lambda^2 (T^*M)_{\mathbb{C}}$ topologies and smooth manifold structures is analogous to the cases of $TM, (TM)_{\mathbb{C}}, T^*M$ and $(T^*M)_{\mathbb{C}}$, using the fact that for each chart $(U, \phi) = (U, x^1, x^2)$ on M we obtain bijections

$$\tilde{\phi} : \Pi^{-1}_{\Lambda^2 T^*M}(U) \to \phi(U) \times \mathbb{R}\,,$$
$$u \mapsto \left((\phi \circ \Pi_{\Lambda^2 T^*M})(u), u\left(\frac{\partial}{\partial x^1}\bigg|_{\Pi_{\Lambda^2 T^*M}(u)}, \frac{\partial}{\partial x^2}\bigg|_{\Pi_{\Lambda^2 T^*M}(u)} \right) \right),$$

and

$$\tilde{\phi}_{\mathbb{C}} : \Pi^{-1}_{\Lambda^2 (T^*M)_{\mathbb{C}}}(U) \to \phi(U) \times \mathbb{R}^2\,,$$
$$v \mapsto \Bigg((\phi \circ \Pi_{\Lambda^2 (T^*M)_{\mathbb{C}}})(v), \operatorname{Re}\left(v\left(\frac{\partial}{\partial x^1}\bigg|_{\Pi_{\Lambda^2 (T^*M)_{\mathbb{C}}}(v)}, \frac{\partial}{\partial x^2}\bigg|_{\Pi_{\Lambda^2 (T^*M)_{\mathbb{C}}}(v)} \right) \right),$$
$$\operatorname{Im}\left(v\left(\frac{\partial}{\partial x^1}\bigg|_{\Pi_{\Lambda^2 (T^*M)_{\mathbb{C}}}(v)}, \frac{\partial}{\partial x^2}\bigg|_{\Pi_{\Lambda^2 (T^*M)_{\mathbb{C}}}(v)} \right) \right) \Bigg)\,.$$

Then, $\Lambda^2 T^*M$ and $\Lambda^2 (T^*M)_{\mathbb{C}}$ become smooth manifolds of dimensions $n+1 = 3$ and $n+2 = 4$ respectively, and the triples $(\Lambda^2 T^*M, M, \Pi_{\Lambda^2 T^*M})$ and $(\Lambda^2 (T^*M)_{\mathbb{C}}, M, \Pi_{\Lambda^2 (T^*M)_{\mathbb{C}}})$ become smooth vector bundles of ranks 1 and 2 respectively.

In conclusion, for each $r \in \{0, 1, 2\}$ we obtain smooth manifolds $\Lambda^r T^*M$ and $\Lambda^r (T^*M)_{\mathbb{C}}$ of dimensions $n + \binom{n}{r}$ and $n + 2\binom{n}{r}$ respectively (where we let $\binom{1}{2} := 0$), and the triples $(\Lambda^r T^*M, M, \Pi_{\Lambda^r T^*M})$ and $(\Lambda^r (T^*M)_{\mathbb{C}}, M, \Pi_{\Lambda^r (T^*M)_{\mathbb{C}}})$ are smooth vector bundles of ranks $\binom{n}{r}$ and $2\binom{n}{r}$ respectively.

Definition 2.19. For $A \subset M$ and $r \in \{0,1,2\}$, we define a *(real) differential form of degree r on A,* or a *(real) r-form on A*, or an *r-form over $\mathbb{R}$ on A*, to be a section $\omega : A \to \Lambda^r T^*M$ of the smooth vector bundle $(\Lambda^r T^*M, M, \Pi_{\Lambda^r T^*M})$ over A. We also define a *(complex) differential form of degree r on A,* or a *(complex) r-form on A*, or an *r-form over $\mathbb{C}$ on A*, to be a section $\tau : A \to \Lambda^r(T^*M)_{\mathbb{C}}$ of the smooth vector bundle $(\Lambda^r(T^*M)_{\mathbb{C}}, M, \Pi_{\Lambda^r(T^*M)_{\mathbb{C}}})$ over A. If σ is a real or complex r-form on A and $p \in A$, we denote the value of σ at p by σ_p. The r-form σ is said to be *continuous (on A)* if it is continuous as a section, and if $A \subset M$ is open and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, then σ is said to be *C^k (on A)* if it is C^k as a section. We denote by $\mathcal{E}^0_r(A, \mathbb{F})$ the space of continuous r-forms over $\mathbb{F}$ on A, and if $A \subset M$ is open, we denote by $\mathcal{E}^k_r(A, \mathbb{F})$ the space of C^k r-forms over $\mathbb{F}$ on A.

Remark 2.20. Note that Definition 2.19 generalises Definition 2.16(ii).

Remark 2.21. Since $\Lambda^0 T^*_pM = \mathbb{R} \cong \{p\} \times \mathbb{R}$ and $\Lambda^0(T_pM)^*_{\mathbb{C}} = \mathbb{C} \cong \{p\} \times \mathbb{C}$ for each $p \in M$, if $A \subset M$ then we may identify real or complex 0-forms on A with real- or complex-valued functions on A respectively, via

$$\omega \leftrightarrow f \quad \text{if } \omega_p = (p, f(p)) \text{ for all } p \in A,$$

for each real or complex 0-form ω and each real- or complex-valued function f on A, respectively. Moreover, it follows from Proposition 2.15 that a real or complex 0-form ω on A is continuous as a section if and only if its corresponding function f is continuous on A, and if $A \subset M$ is open and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, it follows from Proposition 2.14 that ω is C^k as a section if and only if f is a C^k function on A.

Remark 2.22. As we saw in Subsection 2.1, if (E, M, Π) is a smooth vector bundle and $A \subset M$, then the real vector space of sections of E over A is also a module over the ring of real-valued functions on A, the space $\Gamma^0(A, E)$ is a module over $C^0_{\mathbb{R}}(A)$, and if $A \subset M$ is open and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, then $\Gamma^k(A, E)$ is a module over $C^k_{\mathbb{R}}(A)$. If $r \in \{0,1,2\}$ and ω is a complex r-form on $A \subset M$, then for each complex-valued function $f : A \to \mathbb{C}$ we may also define the complex r-form $f\omega$ on A by $(f\omega)_p := f(p)\omega_p$ for each $p \in A$, since the fiber $\Lambda^r(T_pM)^*_{\mathbb{C}}$ is a complex vector space. Then, $\mathcal{E}^0_r(A, \mathbb{C})$ becomes also a module over $C^0_{\mathbb{C}}(A)$, and if $A \subset M$ is open and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, then $\mathcal{E}^k_r(A, \mathbb{C})$ is a module over $C^k_{\mathbb{C}}(A)$.

Definition 2.23. Let $A \subset M$ and $r \in \{0,1,2\}$. If ω is a complex differential form of degree r on A, we define the *real part of* ω, denoted $\mathrm{Re}(\omega)$, to be the real r-form on A given by

$$(\mathrm{Re}(\omega))_p := \mathrm{Re}(\omega_p) \in \Lambda^r(T^*_pM)\,, \quad p \in A\,,$$

where at each point $p \in A$ we regard $\omega_p \in \Lambda^r(T_pM)^*_{\mathbb{C}}$ to be its corresponding element in $(\Lambda^r(T^*_pM))_{\mathbb{C}} \cong \Lambda^r(T_pM)^*_{\mathbb{C}}$, so that $\mathrm{Re}(\omega_p) \in \Lambda^r(T^*_pM)$ is defined. Similarly, we define the *imaginary part of* ω, denoted $\mathrm{Im}(\omega)$, to be the real r-form on A given by

$$(\mathrm{Im}(\omega))_p := \mathrm{Im}(\omega_p) \in \Lambda^r(T^*_pM)\,, \quad p \in A\,.$$

Definition 2.24. Let $A \subset M$, and suppose ω and τ are differential forms over $\mathbb{F}$ on A, of degrees r and s respectively, for $r, s \in \{0,1,2\}$ and $r + s \leq 2$. We then define the *wedge product* $\omega \wedge \tau$ of ω with τ to be the $(r+s)$-form over $\mathbb{F}$ on A given by

$$(\omega \wedge \tau)_p := \omega_p \wedge \tau_p$$

for each $p \in A$.

Remark 2.25. Using notation from Definition 2.24, it follows from the properties of the wedge product of an r-covector with an s-covector on a real or complex vector space that if $\tilde{\omega}$ is another r-form on A and $\tilde{\tau}$ another s-form on A, then

$$\begin{aligned} (\omega + \tilde{\omega}) \wedge \tau &= \omega \wedge \tau + \tilde{\omega} \wedge \tau \,, \\ \omega \wedge (\tau + \tilde{\tau}) &= \omega \wedge \tau + \omega \wedge \tilde{\tau} \,, \\ (f\omega) \wedge \tau = f(\omega \wedge \tau) &= \omega \wedge (f\tau) \,, \quad \text{for any function } f : A \to \mathbb{F} \,, \\ \omega \wedge g = g \wedge \omega &= g\,\omega \,, \qquad \text{for any 0-form } g : A \to \mathbb{F} \,, \\ \omega \wedge \tau &= (-1)^{rs} \tau \wedge \omega \qquad (\textit{anticommutativity}), \end{aligned}$$

and if $t \in \{0, 1, 2\}$ with $r + s + t \leq 2$ and σ is a t-form on A, then

$$(\omega \wedge \tau) \wedge \sigma = \omega \wedge (\tau \wedge \sigma) \qquad (\textit{associativity}).$$

Proposition 2.26. *Let $A \subset M$ and $r, s \in \{0, 1, 2\}$ with $r + s \leq 2$. If $\omega \in \mathcal{E}^0_r(A, \mathbb{F})$ and $\tau \in \mathcal{E}^0_s(A, \mathbb{F})$, then $\omega \wedge \tau \in \mathcal{E}^0_{r+s}(A, \mathbb{F})$. Moreover, if $A \subset M$ is open and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, then for each $\omega \in \mathcal{E}^k_r(A, \mathbb{F})$ and $\tau \in \mathcal{E}^k_s(A, \mathbb{F})$ we have $\omega \wedge \tau \in \mathcal{E}^k_{r+s}(A, \mathbb{F})$.*

Proof. The case when $r = 0$ or $s = 0$ follows from Remarks 2.21 and 2.22. The only remaining case is when $r = s = 1$. If $n = 1$ then $\omega \wedge \tau$ is the zero 2-form over $\mathbb{F}$ on A, and for $n = 2$ the result can be proved using Propositions 2.15 and 2.14, by direct computation of $\omega \wedge \tau$ on the intersection of A with each coordinate open set in M. The details are left to the reader. □

Definition 2.27. Suppose $F : M \to N$ is a C^1 map of smooth manifolds of dimensions $\dim M, \dim N \in \{1, 2\}$. If $r \in \{0, 1, 2\}$ and ω is an r-form over $\mathbb{F}$ on a subset $A \subset N$, we define the *pullback of ω (under F)* to be the r-form $F^*\omega$ over $\mathbb{F}$ on $F^{-1}(A) \subset M$ defined by

$$(F^*\omega)_p := (F_{*,p})^* \omega_{F(p)} \,, \quad p \in F^{-1}(A) \,,$$

that is, for each point $p \in F^{-1}(A)$ we use the differential $F_{*,p} : T_{\mathbb{F},p}M \to T_{\mathbb{F},F(p)}N$, which is a linear map of vector spaces, to pull back the r-covector $\omega_{F(p)}$ on $T_{\mathbb{F},F(p)}N$ to an r-covector $(F_{*,p})^* \omega_{F(p)} =: (F^*\omega)_p$ on $T_{\mathbb{F},p}M$.

Remark 2.28. Using notation from Definition 2.27, if $S \subset A$ then for each $p \in F^{-1}(S)$ we have

$$\begin{aligned} \Big(F^*\Big(\omega\big|_S\Big)\Big)_p &= (F_{*,p})^* \Big(\omega\big|_S\Big)_{F(p)} \\ &= (F_{*,p})^* \omega_{F(p)} \\ &= (F^*\omega)_p \\ &= \Big((F^*\omega)\big|_{F^{-1}(S)}\Big)_p \,, \end{aligned}$$

so $F^*\Big(\omega\big|_S\Big) = (F^*\omega)\big|_{F^{-1}(S)}$ as r-forms on $F^{-1}(S)$.

Remark 2.29. Note that in Definition 2.27, if $r = 0$ then ω is an $\mathbb{F}$-valued function f on A, and for each $p \in F^{-1}(A)$ we have

$$(F^*f)_p = (F_{*,p})^*(f(F(p))) = f(F(p)) \,,$$

so the 0-form F^*f on $F^{-1}(A)$ is the composition function $f \circ F : F^{-1}(A) \to \mathbb{F}$. This shows in particular that if f is a continuous 0-form on A, then its pullback $F^*f = f \circ F$

is a continuous 0-form on $F^{-1}(A) \subset M$; and if $A \subset N$ is open, $k, \ell \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $\ell \geq 1$, F is C^ℓ and f is a C^k 0-form on A, then $F^*f = f \circ F$ is a $C^{\min\{k,\ell\}}$ 0-form on $F^{-1}(A) \subset M$.

Proposition 2.30. *Let $F : M \to N$ be a C^1 map of smooth manifolds of dimensions $\dim M, \dim N \in \{1, 2\}$, let $\Omega \subset N$ be open, and let $f : \Omega \to \mathbb{F}$ be a C^1 function. Then, on $F^{-1}(\Omega)$,*

$$F^*(df) = d(F^*f)\,.$$

Proof. Let $p \in F^{-1}(\Omega)$ and $v \in T_{\mathbb{F},p}M$. Then,

$$\begin{aligned}(F^*(df))_p(v) &= ((F_{*,p})^*(df)_{F(p)})(v)\\ &= (df)_{F(p)}(F_{*,p}v)\\ &= (F_{*,p}v)(f)\\ &= v(f \circ F)\\ &= d(f \circ F)_p(v)\\ &= d(F^*f)_p(v)\,,\end{aligned}$$

so $(F^*(df))_p = d(F^*f)_p$. ☐

Proposition 2.31. *Let $F : M \to N$ be a C^1 map of smooth manifolds of dimensions $\dim M, \dim N \in \{1, 2\}$, and let $A \subset N$. If $r, s \in \{0, 1, 2\}$ with $r + s \leq 2$, and ω and τ are differential forms over $\mathbb{F}$ on A of respective degrees r and s, then*

$$F^*(\omega \wedge \tau) = (F^*\omega) \wedge (F^*\tau)$$

as $(r + s)$-forms on $F^{-1}(A)$.

Proof. For each $p \in F^{-1}(A)$,

$$\begin{aligned}(F^*(\omega \wedge \tau))_p &= (F_{*,p})^*(\omega \wedge \tau)_{F(p)}\\ &= (F_{*,p})^*(\omega_{F(p)} \wedge \tau_{F(p)})\\ &= ((F_{*,p})^*\omega_{F(p)}) \wedge ((F_{*,p})^*\tau_{F(p)}) \quad \text{(by Proposition 1.22)}\\ &= (F^*\omega)_p \wedge (F^*\tau)_p\\ &= ((F^*\omega) \wedge (F^*\tau))_p\,.\end{aligned}$$

☐

Proposition 2.32. *Let M and N be smooth manifolds of dimensions m and n respectively, with $m, n \in \{1, 2\}$. Suppose $F : M \to N$ is a C^1 map, let $A \subset N$, and let ω be an r-form over $\mathbb{F}$ on A, for $r \in \{0, 1, 2\}$. Then,*

(i) *if ω is continuous, the r-form $F^*\omega$ on $F^{-1}(A)$ is also continuous;*
(ii) *if $A \subset N$ is open, $k, \ell \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $\ell \geq 1$, F is C^ℓ and ω is C^k, then $F^*\omega$ is $C^{\min\{k,\ell-1\}}$ on $F^{-1}(A)$.*

Proof. The case when $r = 0$ is given by Remark 2.29. Suppose $r \in \{1, 2\}$. We prove only (i), since the proof of (ii) is analogous. If $r = 2$ and $n = 1$ or $m = 1$, then $F^*\omega$ is the zero 2-form over $\mathbb{F}$ on $F^{-1}(A)$, which is continuous. We therefore consider only the remaining cases. Let $p \in F^{-1}(A)$, and choose charts $(U, x^1, \ldots, x^m)$ about p in M and

$(V, y^1, \ldots, y^n)$ about $F(p)$ in N such that $F(U) \subset V$. Since ω is continuous on A, by Proposition 2.15 we have

$$\begin{aligned} \omega|_{A\cap V} &= f^1 \cdot dy^1|_{A\cap V} + \cdots + f^n \cdot dy^n|_{A\cap V} && \text{if } r = 1, \\ \omega|_{A\cap V} &= g \cdot dy^1|_{A\cap V} \wedge dy^2|_{A\cap V} && \text{if } r = 2 \text{ and } n = 2, \end{aligned}$$

for some continuous $\mathbb{F}$-valued functions $f^1, \ldots, f^n$ and g on $A \cap V$. If $r = 1$, then

$$\begin{aligned} (F^*\omega)|_{F^{-1}(A\cap V)} &= F^*\Big(\omega|_{A\cap V}\Big) \\ &= F^*(f^1 \cdot dy^1|_{A\cap V} + \cdots + f^n \cdot dy^n|_{A\cap V}) \\ &= (F^*f^1)\Big(F^*\Big(dy^1|_{A\cap V}\Big)\Big) + \cdots + (F^*f^n)\Big(F^*\Big(dy^n|_{A\cap V}\Big)\Big) \\ &= (f^1 \circ F) \cdot (F^*(dy^1))|_{F^{-1}(A\cap V)} + \cdots + (f^n \circ F) \cdot (F^*(dy^n))|_{F^{-1}(A\cap V)} \\ &= (f^1 \circ F) \cdot (dF^1)|_{F^{-1}(A\cap V)} + \cdots + (f^n \circ F) \cdot (dF^n)|_{F^{-1}(A\cap V)}, \end{aligned}$$

where we set $F^j := y^j \circ F : F^{-1}(V) \to \mathbb{R} \subset \mathbb{F}$ for each $j \in \{1, \ldots, n\}$. Then, for each $q \in F^{-1}(A) \cap U \subset F^{-1}(A \cap V)$,

$$\begin{aligned} (F^*\omega)_q &= \Big((F^*\omega)|_{F^{-1}(A\cap V)}\Big)_q \\ &= (f^1 \circ F)(q) \cdot (dF^1)_q + \cdots + (f^n \circ F)(q) \cdot (dF^n)_q \\ &= (f^1 \circ F)(q) \cdot \left(\sum_{k=1}^m \frac{\partial F^1}{\partial x^k}\bigg|_q (dx^k)_q\right) + \cdots + (f^n \circ F)(q) \cdot \left(\sum_{k=1}^m \frac{\partial F^n}{\partial x^k}\bigg|_q (dx^k)_q\right) \\ &= \left(\sum_{j=1}^n (f^j \circ F)(q) \cdot \frac{\partial F^j}{\partial x^1}\bigg|_q\right)(dx^1)_q + \cdots + \left(\sum_{j=1}^n (f^j \circ F)(q) \cdot \frac{\partial F^j}{\partial x^n}\bigg|_q\right)(dx^n)_q\,, \end{aligned}$$

so on $F^{-1}(A) \cap U$,

$$F^*\omega = \left(\sum_{j=1}^n (f^j \circ F) \cdot \frac{\partial F^j}{\partial x^1}\right)dx^1 + \cdots + \left(\sum_{j=1}^n (f^j \circ F) \cdot \frac{\partial F^j}{\partial x^n}\right)dx^n\,.$$

Since for all $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ the functions $f^j \circ F : F^{-1}(A \cap V) \to \mathbb{F}$ and $\frac{\partial F^j}{\partial x^k} : U \to \mathbb{F}$ are continuous, they are also continuous when restricted to $F^{-1}(A) \cap U \subset F^{-1}(A \cap V)$. It then follows from Proposition 2.15 that the 1-form $F^*\omega$ on $F^{-1}(A)$ is continuous. For $r = n = m = 2$, we use a similar argument:

$$\begin{aligned} (F^*\omega)|_{F^{-1}(A\cap V)} &= F^*\Big(\omega|_{A\cap V}\Big) \\ &= F^*\Big(g \cdot dy^1|_{A\cap V} \wedge dy^2|_{A\cap V}\Big) \\ &= (F^*g)\,\Big(F^*\Big(dy^1|_{A\cap V}\Big)\Big) \wedge \Big(F^*\Big(dy^2|_{A\cap V}\Big)\Big) \\ &= (g \circ F) \cdot (F^*(dy^1))|_{F^{-1}(A\cap V)} \wedge (F^*(dy^2))|_{F^{-1}(A\cap V)} \\ &= (g \circ F) \cdot (dF^1)|_{F^{-1}(A\cap V)} \wedge (dF^2)|_{F^{-1}(A\cap V)}\,, \end{aligned}$$

so for each $q \in F^{-1}(A) \cap U \subset F^{-1}(A \cap V)$,

$$\begin{aligned}
(F^*\omega)_q &= \Big((F^*\omega)\big|_{F^{-1}(A\cap V)} \Big)_q \\
&= (g \circ F)(q) \cdot (dF^1)_q \wedge (dF^2)_q \\
&= (g \circ F)(q) \cdot \left(\frac{\partial F^1}{\partial x^1}\bigg|_q (dx^1)_q + \frac{\partial F^1}{\partial x^2}\bigg|_q (dx^2)_q \right) \wedge \left(\frac{\partial F^2}{\partial x^1}\bigg|_q (dx^1)_q + \frac{\partial F^2}{\partial x^2}\bigg|_q (dx^2)_q \right) \\
&= (g \circ F)(q) \cdot \left(\frac{\partial F^1}{\partial x^1}\bigg|_q \frac{\partial F^2}{\partial x^2}\bigg|_q - \frac{\partial F^1}{\partial x^2}\bigg|_q \frac{\partial F^2}{\partial x^1}\bigg|_q \right) (dx^1)_q \wedge (dx^2)_q \,.
\end{aligned}$$

Thus, on $F^{-1}(A) \cap U$

$$F^*\omega = (g \circ F) \left(\frac{\partial F^1}{\partial x^1}\frac{\partial F^2}{\partial x^2} - \frac{\partial F^1}{\partial x^2}\frac{\partial F^2}{\partial x^1} \right) dx^1 \wedge dx^2 \,,$$

where the functions $g \circ F : F^{-1}(A \cap V) \to \mathbb{F}$ and $\left(\frac{\partial F^1}{\partial x^1}\frac{\partial F^2}{\partial x^2} - \frac{\partial F^1}{\partial x^2}\frac{\partial F^2}{\partial x^1} \right) : U \to \mathbb{F}$ are continuous on their respective domains and hence so is the product of their restrictions to $F^{-1}(A) \cap U$. Again, it follows from Proposition 2.15 that $F^*\omega$ is continuous on $F^{-1}(A)$. $\square$

Proposition 2.33. *Let $\Omega \subset M$ be open, and let $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. If $f \in \mathcal{E}_0^k(\Omega, \mathbb{F})$ is a C^k $\mathbb{F}$-valued function on Ω, then $df \in \mathcal{E}_1^{k-1}(\Omega, \mathbb{F})$, where we set $\infty - 1 := \infty$.*

Proof. We prove only the case when $n = 2$ and $\mathbb{F} = \mathbb{C}$, since the other cases are analogous. Let (U, x^1, x^2) be a chart in M. If $\Omega \cap U \neq \emptyset$ and $p \in \Omega \cap U$, we have

$$\begin{aligned}
(df)_p &= \left((df)_p \left(\frac{\partial}{\partial x^1}\bigg|_p \right) \right) (dx^1)_p + \left((df)_p \left(\frac{\partial}{\partial x^2}\bigg|_p \right) \right) (dx^2)_p \\
&= \frac{\partial f}{\partial x^1}\bigg|_p (dx^1)_p + \frac{\partial f}{\partial x^2}\bigg|_p (dx^2)_p \\
&= \frac{\partial \operatorname{Re}(f)}{\partial x^1}\bigg|_p (dx^1)_p + \frac{\partial \operatorname{Im}(f)}{\partial x^1}\bigg|_p i(dx^1)_p + \frac{\partial \operatorname{Re}(f)}{\partial x^2}\bigg|_p (dx^2)_p + \frac{\partial \operatorname{Im}(f)}{\partial x^2}\bigg|_p i(dx^2)_p \,,
\end{aligned}$$

so on $\Omega \cap U$,

$$df = \frac{\partial \operatorname{Re}(f)}{\partial x^1} dx^1 + \frac{\partial \operatorname{Im}(f)}{\partial x^1} idx^1 + \frac{\partial \operatorname{Re}(f)}{\partial x^2} dx^2 + \frac{\partial \operatorname{Im}(f)}{\partial x^2} idx^2 \,.$$

Since f is C^k on $\Omega \cap U$, its partial derivatives are C^{k-1} on $\Omega \cap U$. The result then follows from Proposition 2.14. $\square$

Remark 2.34. Using Proposition 2.33, if $\Omega \subset M$ is open we may define an $\mathbb{F}$-linear map of vector spaces

$$d : \mathcal{E}_0^1(\Omega, \mathbb{F}) \to \mathcal{E}_1^0(\Omega, \mathbb{F})$$

sending each C^1 $\mathbb{F}$-valued function f on Ω to the continuous 1-form df on Ω.

Proposition 2.35. *Suppose that M has dimension $n = 2$, and let $\Omega \subset M$ be open. There exists a unique linear map $d : \mathcal{E}_1^1(\Omega, \mathbb{F}) \to \mathcal{E}_2^0(\Omega, \mathbb{F})$ fulfilling*

(i) *for all $f \in \mathcal{E}_0^2(\Omega, \mathbb{F})$, $d^2(f) := d(df) = 0$;*
(ii) *for each $f \in \mathcal{E}_0^1(\Omega, \mathbb{F})$ and $\omega \in \mathcal{E}_1^1(\Omega, \mathbb{F})$, $d(f\omega) = (df) \wedge \omega + fd\omega$;*
(iii) *if $U \subset \Omega$ is open and $\omega \in \mathcal{E}_1^1(\Omega, \mathbb{F})$, then as 2-forms on U, $d(\omega|_U) = (d\omega)|_U$.*

The proof of Proposition 2.35 is analogous to the one for the case when $\mathbb{F} = \mathbb{R}$, which it is assumed the reader is familiar with, so we omit it. The resulting linear map $d : \mathcal{E}^1_1(\Omega, \mathbb{F}) \to \mathcal{E}^0_2(\Omega, \mathbb{F})$ is given by

$$(d\omega)_p := \left(\frac{\partial f^2}{\partial x^1}\bigg|_p - \frac{\partial f^1}{\partial x^2}\bigg|_p \right) (dx^1)_p \wedge (dx^2)_p \,,$$

for $\omega \in \mathcal{E}^1_1(\Omega, \mathbb{F})$, $p \in \Omega$ and any chart (U, x^1, x^2) about p in M, where f^1 and f^2 are the unique (C^1) $\mathbb{F}$-valued functions on $\Omega \cap U$ such that $\omega|_{\Omega \cap U} = f^1\, dx^1 + f^2\, dx^2$, that is,

$$f^1(q) := \omega_q\left(\frac{\partial}{\partial x^1}\bigg|_q \right) \quad \text{and} \quad f^2(q) := \omega_q\left(\frac{\partial}{\partial x^2}\bigg|_q \right)$$

for each $q \in \Omega \cap U$.

Definition 2.36. Suppose $\Omega \subset M$ is open. For $r \in \{0, 1\}$ and a differential form $\omega \in \mathcal{E}^1_r(\Omega, \mathbb{F})$, we define the *exterior derivative of* ω, denoted by $d\omega$, to be

(i) the differential $d\omega \in \mathcal{E}^0_1(\Omega, \mathbb{F})$ of ω as an $\mathbb{F}$-valued function on Ω, if $r = 0$;
(ii) the image $d\omega$ of ω under the map $d : \mathcal{E}^1_1(\Omega, \mathbb{F}) \to \mathcal{E}^0_2(\Omega, \mathbb{F})$ given in Proposition 2.35, if $r = 1$ and $n = 2$;
(iii) the zero 2-form on Ω (the only 2-form in $\mathcal{E}^0_2(\Omega, \mathbb{F}) = \{0\}$) if $r = n = 1$.

Proposition 2.37. *Suppose M and N are smooth manifolds of respective dimensions $m, n \in \{1, 2\}$, and let $F : M \to N$ be a C^2 map. If $\Omega \subset N$ is open and $\omega \in \mathcal{E}^1_1(\Omega, \mathbb{F})$, then*

$$F^*(d\omega) = d(F^*\omega) \,.$$

Proof. Note that since F is C^2 and ω is C^1, by Proposition 2.32 the 1-form $F^*\omega$ on $F^{-1}(\Omega)$ is C^1, so the 2-form $d(F^*\omega)$ is defined. If $m = 1$, then $F^*(d\omega)$ and $d(F^*\omega)$ are both the zero 2-form on $F^{-1}(\Omega)$, so we consider the remaining cases. Assume $m = 2$, and let $p \in F^{-1}(\Omega)$. We may choose charts (U, x^1, x^2) about p in M and (V, y), if $n = 1$, or (V, y^1, y^2), if $n = 2$, about $F(p)$ in N such that $F(U) \subset V \subset \Omega$. If $n = 1$, then $d\omega$ is the zero 2-form on Ω, so $F^*(d\omega) = 0$ on $F^{-1}(\Omega)$. On the other hand, on V we have

$$\omega|_V = f\, dy$$

for some C^1 function $f : V \to \mathbb{F}$, so for all $q \in U \subset F^{-1}(V)$

$$\begin{aligned} (F^*\omega)_q &= (F_{*,q})^*\omega_{F(q)} \\ &= (F_{*,q})^*(f(F(q))(dy)_{F(q)}) \\ &= f(F(q))(F^*(dy))_q \\ &= f(F(q))(d\tilde{F})_q \,, \end{aligned}$$

where $\tilde{F} := y \circ F$. This gives

$$(F^*\omega)|_U = (f \circ F)\frac{\partial \tilde{F}}{\partial x^1}\, dx^1 + (f \circ F)\frac{\partial \tilde{F}}{\partial x^2}\, dx^2 \,.$$

Thus,

$$(d(F^*\omega))_p = \left(\frac{\partial}{\partial x^1}\left((f \circ F)\frac{\partial \tilde{F}}{\partial x^2} \right)\bigg|_p - \frac{\partial}{\partial x^2}\left((f \circ F)\frac{\partial \tilde{F}}{\partial x^1} \right)\bigg|_p \right) (dx^1)_p \wedge (dx^2)_p \,,$$

and direct computation gives

$$\frac{\partial}{\partial x^1}\left((f\circ F)\frac{\partial \tilde{F}}{\partial x^2}\right)\bigg|_p - \frac{\partial}{\partial x^2}\left((f\circ F)\frac{\partial \tilde{F}}{\partial x^1}\right)\bigg|_p = 0\,,$$

so on $F^{-1}(\Omega)$

$$d(F^*\omega) = 0 = F^*(d\omega)\,.$$

If $n = 2$, then

$$\omega\big|_V = g^1\,dy^1 + g^2\,dy^2$$

for some C^1 $\mathbb{F}$-valued functions g^1 and g^2 on V. Computing as before the pullback $(F^*\omega)_q = (F_{*,q})^*(\omega_{F(q)})$ at each $q \in U$, we obtain

$$(F^*\omega)\big|_U = h^1\,dx^1 + h^2\,dx^2\,,$$

where h^1 and h^2 are the C^1 $\mathbb{F}$-valued functions on U

$$\begin{aligned} h^1 &:= (g^1\circ F)\frac{\partial F^1}{\partial x^1} + (g^2\circ F)\frac{\partial F^2}{\partial x^1}\,,\\ h^2 &:= (g^1\circ F)\frac{\partial F^1}{\partial x^2} + (g^2\circ F)\frac{\partial F^2}{\partial x^2}\,. \end{aligned}$$

Thus,

$$(d(F^*\omega))_p = \left(\frac{\partial h^2}{\partial x^1}\bigg|_p - \frac{\partial h^1}{\partial x^2}\bigg|_p\right)(dx^1)_p \wedge (dx^2)_p\,.$$

We also have

$$(d\omega)_{F(p)} = \left(\frac{\partial g^2}{\partial y^1}\bigg|_{F(p)} - \frac{\partial g^1}{\partial y^2}\bigg|_{F(p)}\right)(dy^1)_{F(p)} \wedge (dy^2)_{F(p)}\,,$$

so

$$\begin{aligned} (F^*(d\omega))_p &= (F_{*,p})^*(d\omega)_{F(p)}\\ &= \left(\frac{\partial g^2}{\partial y^1}\bigg|_{F(p)} - \frac{\partial g^1}{\partial y^2}\bigg|_{F(p)}\right)(dF^1)_p \wedge (dF^2)_p\\ &= \left(\frac{\partial g^2}{\partial y^1}\bigg|_{F(p)} - \frac{\partial g^1}{\partial y^2}\bigg|_{F(p)}\right)\left(\frac{\partial F^1}{\partial x^1}\bigg|_p\frac{\partial F^2}{\partial x^2}\bigg|_p - \frac{\partial F^1}{\partial x^2}\bigg|_p\frac{\partial F^2}{\partial x^1}\bigg|_p\right)(dx^1)_p \wedge (dx^2)_p\,, \end{aligned}$$

and if you are very bored and do not have anything better to do with your life right now, you can check that this last expression is indeed equal to the one we obtained for $(d(F^*\omega))_p$ above, so that

$$F^*(d\omega) = d(F^*\omega)$$

on $F^{-1}(\Omega)$. □

2.3. **Integral of a 1-form Along a Curve.**

Definition 2.38. (i) Let X be a topological space. A *(parametrised) path* or *(parametrised) curve* in X is a continuous map $\gamma : [a,b] \to X$, for some $a, b \in \mathbb{R}$, $a < b$. If $\gamma(a) = p$ and $\gamma(b) = q$, for $p, q \in X$, we say that γ is a path from p to q, and we call p and q the *initial point* and the *terminal point* of γ respectively. If $p = q$, we call γ a *loop* or *closed curve based at p*.

(ii) If M is a smooth manifold and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a path $\gamma : [a,b] \to M$ is said to be C^k if there exist an open set $U \subset \mathbb{R}$ with $[a,b] \subset U$ and a C^k map $\tilde{\gamma} : U \to M$ such that $\tilde{\gamma}|_{[a,b]} = \gamma$. If there exists a partition $a = t_0 < \cdots < t_m = b$ of $[a,b]$ for some $m \in \mathbb{N}$ such that $\gamma|_{[t_{j-1},t_j]}$ is a C^k path for all $j \in \{1,\ldots,m\}$, then we say that γ is a *piecewise C^k path.*

Definition 2.39. Let $I \subset \mathbb{R}$ be an interval and M a C^∞ manifold of dimension n. Suppose $\gamma : I \to M$ is a map admitting a C^1 extension, that is, there exist an open set $U \subset \mathbb{R}$ with $I \subset U$ and a C^1 map $\tilde{\gamma} : U \to M$ such that $\tilde{\gamma}|_I = \gamma$. For each $t_0 \in I$, we define the *tangent vector to γ (over $\mathbb{F}$) at t_0* by

$$\dot{\gamma}(t_0) := \tilde{\gamma}_{*,t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right) \in T_{\mathbb{F},\gamma(t_0)}M\,,$$

where t denotes the standard coordinate on $U \subset \mathbb{R}$ and we regard $\frac{d}{dt}\big|_{t_0}$ to be a tangent vector in $T_{t_0}U$ or in $(T_{t_0}U)_{\mathbb{C}}$ depending on our choice of $\mathbb{F}$.

Remark 2.40. (i) Referring to Definition 2.39, note that on the interior of I the differentials $\tilde{\gamma}_*$ and γ_* are the same. Moreover, if $t_0 \in I$ and $(V, x^1, \ldots, x^n)$ is a chart about $\gamma(t_0)$ in M, we have

$$\dot{\gamma}(t_0) = \sum_{j=1}^{n} \frac{d(x^j \circ \tilde{\gamma})}{dt}\bigg|_{t_0} \frac{\partial}{\partial x^j}\bigg|_{\tilde{\gamma}(t_0)}\,,$$

where we have restricted $\tilde{\gamma}$ to $\tilde{\gamma}^{-1}(V)$. We also restrict γ to $\gamma^{-1}(V)$, and if the point t_0 is an endpoint of I, we define the one-sided derivative (for $s \in \gamma^{-1}(V) \setminus \{t_0\}$)

$$\frac{d(x^j \circ \gamma)}{dt}\bigg|_{t_0} := \lim_{s \to t_0} \frac{(x^j \circ \gamma)(s) - (x^j \circ \gamma)(t_0)}{s - t_0}\,, \quad j \in \{1, \ldots, n\}\,,$$

whose existence is guaranteed, since

$$\lim_{s \to t_0} \frac{(x^j \circ \gamma)(s) - (x^j \circ \gamma)(t_0)}{s - t_0} = \lim_{s \to t_0} \frac{(x^j \circ \tilde{\gamma})(s) - (x^j \circ \tilde{\gamma})(t_0)}{s - t_0} = \frac{d(x^j \circ \tilde{\gamma})}{dt}\bigg|_{t_0}\,.$$

Then, for any $t_0 \in I$ and chart $(V, x^1, \ldots, x^n)$ about $\gamma(t_0)$, we have

$$\sum_{j=1}^{n} \frac{d(x^j \circ \tilde{\gamma})}{dt}\bigg|_{t_0} \frac{\partial}{\partial x^j}\bigg|_{\tilde{\gamma}(t_0)} = \sum_{j=1}^{n} \frac{d(x^j \circ \gamma)}{dt}\bigg|_{t_0} \frac{\partial}{\partial x^j}\bigg|_{\gamma(t_0)}\,,$$

which in particular shows that the definition of $\dot{\gamma}(t_0)$ is independent of the choice of C^1 extension $\tilde{\gamma}$ even when t_0 is an endpoint of I.

(ii) By Remark 2.7 (ii), for any $t_0 \in I$ we have

$$\dot{\gamma}(t_0)_{\mathbb{C}} = \dot{\gamma}(t_0)_{\mathbb{R}} + i0\,,$$

where $\dot{\gamma}(t_0)_{\mathbb{F}}$ denotes the tangent vector to γ over $\mathbb{F}$ at t_0.

Definition 2.41. Let M be a C^∞ manifold of dimension $n \in \{1, 2\}$, and suppose α is a continuous 1-form over $\mathbb{F}$ on M. Let $\gamma : [a,b] \to M$ be a piecewise C^1 path in M, and let $a = s_0 < \cdots < s_m = b$ be a partition of $[a,b]$, for $m \in \mathbb{N}$, such that $\gamma_k := \gamma|_{[s_{k-1},s_k]}$ is a C^1 path for all $k \in \{1, \ldots, m\}$. We define the *(line) integral of α along γ* by

$$\int_\gamma \alpha := \sum_{k=1}^{m} \int_{s_{k-1}}^{s_k} \alpha_{\gamma(t)}(\dot{\gamma}_k(t))\, dt\,,$$

where for each $k \in \{1, \ldots, m\}$ and $t \in [s_{k-1}, s_k]$, $\dot{\gamma_k}(t)$ is the tangent vector to γ_k over $\mathbb{F}$ at t.

Remark 2.42. Referring to Definition 2.41, let $k \in \{1, \ldots, m\}$ and $t_0 \in [s_{k-1}, s_k]$. Denote by γ_k the restriction $\gamma|_{[s_{k-1}, s_k]}$, and let $(V, x^1, \ldots, x^n)$ be a chart about $\gamma_k(t_0) = \gamma(t_0)$ in M. By Remark 2.40 (i), we have

$$\dot{\gamma_k}(s) = \sum_{j=1}^{n} \frac{d(x^j \circ \gamma_k)}{dt}\bigg|_s \frac{\partial}{\partial x^j}\bigg|_{\gamma_k(s)} \quad \text{for all } s \in \gamma_k^{-1}(V)\,.$$

Since the 1-form α is continuous, on V we have

$$\alpha|_V = f^1\, dx^1 + \cdots + f^n\, dx^n$$

for some continuous $\mathbb{F}$-valued functions $f^1, \ldots, f^n$ on V. Thus, for each $s \in \gamma_k^{-1}(V)$,

$$\begin{aligned}\alpha_{\gamma(s)}(\dot{\gamma_k}(s)) &= \left(\sum_{\ell=1}^{n} f^\ell(\gamma_k(s))(dx^\ell)_{\gamma_k(s)}\right)\left(\sum_{j=1}^{n} \frac{d(x^j \circ \gamma_k)}{dt}\bigg|_s \frac{\partial}{\partial x^j}\bigg|_{\gamma_k(s)}\right) \\ &= \sum_{j=1}^{n} f^j(\gamma_k(s)) \frac{d(x^j \circ \gamma_k)}{dt}\bigg|_s .\end{aligned}$$

Then, since for each $j \in \{1, \ldots, n\}$ the functions $f^j \circ \gamma_k : \gamma_k^{-1}(V) \to \mathbb{F}$ and $\frac{d(x^j \circ \gamma_k)}{dt} : \gamma_k^{-1}(V) \to \mathbb{R} \subset \mathbb{F}$ are continuous, the function $s \mapsto \alpha_{\gamma(s)}(\dot{\gamma_k}(s)) \in \mathbb{F}$ is continuous on $\gamma_k^{-1}(V)$, which is an open subset of $[s_{k-1}, s_k]$ containing t_0. Since $t_0 \in [s_{k-1}, s_k]$ was arbitrary, $s \mapsto \alpha_{\gamma_k(s)}(\dot{\gamma_k}(s))$ is a continuous $\mathbb{F}$-valued function on $[s_{k-1}, s_k]$, which shows that the integral $\int_\gamma \alpha$ indeed exists.

2.4. **Measurability in a Smooth Manifold.**

Definition 2.43. Let M be a smooth manifold with $n := \dim M$.

(i) A subset $S \subset M$ is said to be *measurable* if for every chart (U, ϕ) in M the set $\phi(U \cap S) \subset \mathbb{R}^n$ is Lebesgue measurable.
(ii) A measurable set $S \subset M$ is said to have (or to be of) *measure* 0 if for every chart (U, ϕ) in M the set $\phi(U \cap S)$ has Lebesgue measure 0 on $\mathbb{R}^n$.
(iii) If $S \subset M$ is measurable, then a property P that may or may not hold for points or subsets of S is said to hold *almost everywhere (in S)* if it holds on $S \setminus A$ for some subset $A \subset S$ of measure 0.

We state the following theorem from measure theory, without proof.

Theorem 2.44. *Let Ω and Ω' be open subsets of $\mathbb{R}^n$ and $F : \Omega \to \Omega'$ a diffeomorphism. If X is $\mathbb{R}$, $\overline{\mathbb{R}}$ or $\mathbb{C}$, and if $f : \Omega' \to X$ is a measurable function, then the composition $f \circ F : \Omega \to X$ is also a measurable function. If $X = \mathbb{R}$ or $\overline{\mathbb{R}}$ and f is measurable and nonnegative, then*

$$\int_{\Omega'} f\, d\lambda = \int_{\Omega} (f \circ F)|\mathcal{J}_F|\; d\lambda\,, \tag{2}$$

where $\mathcal{J}_F : \Omega \to \mathbb{R} \subset X$ is the function mapping each point $p \in \Omega$ to the Jacobian determinant of F at p. Moreover, if $X = \mathbb{R}$ or $\mathbb{C}$ and f is integrable on Ω', then the function $(f \circ F)|\mathcal{J}_F|$ on Ω is integrable and Equation (2) *also holds. As a consequence,* *if $E \subset \Omega$ is (Lebesgue) measurable then so is $F(E) \subset \Omega'$, and if E has measure 0 then so does $F(E)$.*

Proposition 2.45. *Let M be a smooth manifold of dimension n.*

(i) *For a subset $S \subset M$, the following statements are equivalent:*
   (a) *S is measurable,*
   (b) *for all $p \in S$ there exists a chart (U, ϕ) about p in M such that $\phi(U \cap S)$ is (Lebesgue) measurable in $\mathbb{R}^n$.*

(ii) *For a measurable subset $S \subset M$, the following statements are equivalent:*
   (a) *S has measure 0,*
   (b) *for all $p \in S$ there exists a chart (U, ϕ) about p in M such that $\phi(U \cap S)$ has (Lebesgue) measure 0 in $\mathbb{R}^n$.*

(iii) *Suppose $S \subset M$ is measurable and X and Y are topological spaces. If $F : S \to X$ is a measurable map and $G : X \to Y$ a continuous map, then the composition $G \circ F : S \to Y$ is measurable.*

(iv) *If N is another smooth manifold and $H : M \to N$ is a diffeomorphism, then the image of a measurable subset of M under H is measurable in N.*

Proof. (i) Statement (b) follows directly from (a) and Definition 2.43 (i). We prove (b)$\Rightarrow$(a). Let $S \subset M$, and suppose that for each $p \in S$ there exists a chart (U_p, ϕ_p) about p in M such that $\phi_p(U_p \cap S) \subset \mathbb{R}^n$ is measurable. Consider an arbitrary chart (V, ψ) in M. For each $p \in S$,

$$\phi_p(V \cap S \cap U_p) = \phi_p(U_p \cap V \cap S \cap U_p) = \phi_p(U_p \cap V) \cap \phi_p(S \cap U_p)\,.$$

Since $\phi_p(U_p \cap V)$ is open in $\mathbb{R}^n$, it is measurable, so $\phi_p(V \cap S \cap U_p)$ is an intersection of two measurable subsets and hence measurable. We also have

$$V \cap S = V \cap S \cap \left(\bigcup_{p \in S} U_p \right) = \bigcup_{p \in S} (V \cap S \cap U_p)\,,$$

so

$$\begin{aligned} \psi(V \cap S) &= \psi\Bigg(\bigcup_{p \in S} (V \cap S \cap U_p) \Bigg) \\ &= \bigcup_{p \in S} \psi(V \cap S \cap U_p) \\ &= \bigcup_{p \in S} (\psi(V \cap S) \cap \psi(U_p \cap V))\,. \end{aligned}$$

Thus, the collection $\mathcal{C} := \{\psi(V \cap S) \cap \psi(U_p \cap V)\}_{p \in S}$ is an open cover for $\psi(V \cap S)$; and since $\psi(V \cap S) \subset \mathbb{R}^n$ is second countable, $\mathcal{C}$ has a countable subcover $\{\psi(V \cap S) \cap \psi(U_p \cap V)\}_{p \in A}$, for some countable subset $A \subset S$. Then,

$$\begin{aligned} \psi(V \cap S) &= \bigcup_{p \in A} (\psi(V \cap S) \cap \psi(U_p \cap V)) \\ &= \bigcup_{p \in A} \psi(V \cap S \cap U_p) \\ &= \bigcup_{p \in A} (\psi \circ \phi_p^{-1})(\phi_p(V \cap S \cap U_p))\,. \end{aligned}$$

Since for each $p \in A$ the set $\phi_p(V \cap S \cap U_p)$ is measurable and the map $\psi \circ \phi_p^{-1} : \phi_p(U_p \cap V) \to \psi(U_p \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$, by Theorem 2.44 the set $(\psi \circ \phi_p^{-1})(\phi_p(V \cap S \cap U_p))$ is measurable. Thus, $\psi(V \cap S)$ is a countable union of measurable sets and hence measurable.

(ii) Again, the proof of (a)⇒(b) is direct, so we give a proof for (b)⇒(a). If $S \subset M$ is measurable and for each $p \in S$ there exists a chart (U_p, ϕ_p) about p in M such that $\phi_p(U_p \cap S)$ has measure 0, then for an arbitrary chart (V, ψ) in M we may apply the same argument as in the proof of (i) to conclude that

$$\psi(V \cap S) = \bigcup_{p \in A} \psi(V \cap S \cap U_p),$$

for some countable subset $A \subset S$. For each $p \in A$, we have $\phi_p(V \cap S \cap U_p) \subset \phi_p(S \cap U_p)$, so $\lambda(\phi_p(V \cap S \cap U_p)) = 0$. Then, since $\psi \circ \phi_p^{-1} : \phi_p(U_p \cap V) \to \psi(U_p \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$, by Theorem 2.44 we also have $\lambda(\psi(V \cap S \cap U_p)) = \lambda\big((\psi \circ \phi_p^{-1})(\phi_p(V \cap S \cap U_p))\big) = 0$. Then,

$$\begin{aligned}\lambda(\psi(V \cap S)) &= \lambda\Bigg(\bigcup_{p \in A} \psi(V \cap S \cap U_p)\Bigg) \\ &\leq \sum_{p \in A} \lambda(\psi(V \cap S \cap U_p)) \\ &= 0\,.\end{aligned}$$

(iii) If $U \subset Y$ is an open subset, then by continuity of G the preimage $G^{-1}(U) \subset X$ is open, so by measurability of F, $(G \circ F)^{-1}(U) = F^{-1}(G^{-1}(U))$ is measurable.

(iv) Suppose $S \subset M$ is measurable, and let $q \in H(S)$. Let $p := H^{-1}(q)$, and choose a chart (U, ϕ) about p in M. Then, $\phi(U \cap S)$ is measurable. Since $H^{-1} : N \to M$ is a diffeomorphism, so is the restriction $H^{-1} : H(U) \to U$, and then the composition $\phi \circ H^{-1} : H(U) \to \phi(U) \subset \mathbb{R}^n$ is also a diffeomorphism. Thus, the pair $(H(U), \phi \circ H^{-1})$ is a chart in N, and it contains $H(p) = q$. Moreover,

$$(\phi \circ H^{-1})(H(U) \cap H(S)) = (\phi \circ H^{-1})(H(U \cap S)) = \phi(U \cap S),$$

which is measurable. In conclusion, $(H(U), \phi \circ H^{-1})$ is a chart about q in N such that $(\phi \circ H^{-1})(H(U) \cap H(S)) \subset \mathbb{R}^n$ is measurable. Since q was an arbitrary point in $H(S)$, $H(S)$ is measurable in N by (i).

□

Proposition 2.46. *Let M be a smooth manifold, and let $\mathcal{A}$ be the collection of measurable subsets of M. Then,*

(i) *$\mathcal{A}$ is a σ-algebra;*
(ii) *$\mathcal{A}$ contains all the Borel subsets of M;*
(iii) *if $S \in \mathcal{A}$ and S has measure 0, then any subset $R \subset S$ is also in $\mathcal{A}$ and has measure 0;*
(iv) *if $\{S_j\}_{j \in J}$ is a countable collection of sets in $\mathcal{A}$ of measure 0, then their union $\bigcup_{j \in J} S_j \in \mathcal{A}$ also has measure 0.*

Proof. (i) Let (U, ϕ) be any chart in M. We have $\phi(U \cap M) = \phi(U)$, which shows that $M \in \mathcal{A}$. If $S \in \mathcal{A}$, then $\phi(U \cap S)$ is measurable and

$$\phi(U \cap S^c) = \phi(U) \setminus \phi(U \cap S) = \phi(U) \cap (\phi(U \cap S))^c,$$

so $\phi(U \cap S^c)$ is also measurable, and thus S^c is measurable. Moreover, if $\{S_j\}_{j \in J} \subset \mathcal{A}$ is a countable collection of measurable subsets of M, then

$$\begin{aligned}\phi\Big(U \cap \Big(\bigcup_{j \in J} S_j\Big)\Big) &= \phi\Big(\bigcup_{j \in J}(U \cap S_j)\Big) \\ &= \bigcup_{j \in J} \phi(U \cap S_j)\,,\end{aligned}$$

which is measurable. Thus, the union $\bigcup_{j \in J} S_j \subset M$ is also measurable. In conclusion, $\mathcal{A}$ satisfies the axioms of a σ-algebra.

(ii) Suppose $\Omega \subset M$ is open. Then, if (U, ϕ) is any chart in M, the set $U \cap \Omega$ is open in U, so $\phi(U \cap \Omega)$ is open in $\mathbb{R}^n$ and hence measurable. Thus, $\mathcal{A}$ contains all the open subsets of M, and since $\mathcal{A}$ is a σ-algebra, it must then contain the σ-algebra generated by the collection of open subsets of M. Thus, $\mathcal{A}$ contains all the Borel subsets of M.

(iii) Suppose $S \subset M$ is measurable and has measure 0, and let $R \subset S$. If (U, ϕ) is a chart in M, then $U \cap R \subset U \cap S$, so $\phi(U \cap R) \subset \phi(U \cap S)$. Then, since $\phi(U \cap S)$ is measurable and has measure 0, $\phi(U \cap R)$ is also measurable and of measure 0 by completeness of the Lebesgue measure on $\mathbb{R}^n$. Thus, R is measurable and has measure 0.

(iv) Suppose $\{S_j\}_{j \in J}$ is a countable collection of measurable subsets of M of measure 0. If (U, ϕ) is a chart in M, then

$$\phi\Big(U \cap \Big(\bigcup_{j \in J} S_j\Big)\Big) = \bigcup_{j \in J} \phi(U \cap S_j)$$

has Lebesgue measure 0 in $\mathbb{R}^n$ as a countable union of sets of Lebesgue measure 0. Thus, $\bigcup_{j \in J} S_j$ has measure 0.

□

From now on, we regard a smooth manifold M to also be a measurable space whose σ-algebra consists of all the measurable subsets of M.

Proposition 2.47. *Let M be a smooth manifold, $S \subset M$ a measurable subset, and X a topological space. A map $F : S \to X$ is measurable if and only if for every chart (U, ϕ) in M, the restriction $F|_{U \cap S} : U \cap S \to X$ is measurable.*

Proof. For each open subset $\Omega \subset X$ and chart (U, ϕ) in M, we have

$$\Big(F|_{U \cap S}\Big)^{-1}(\Omega) = U \cap S \cap F^{-1}(\Omega) = U \cap F^{-1}(\Omega)\,.$$

If F is measurable, then the set $U \cap F^{-1}(\Omega)$ is measurable, which shows that $F|_{U \cap S}$ is a measurable map. If $F|_{U \cap S}$ is measurable for every chart (U, ϕ) in M, then for every chart (U, ϕ) in M the set $U \cap F^{-1}(\Omega)$ is measurable, which implies that $\phi(U \cap F^{-1}(\Omega)) \subset \mathbb{R}^n$ is measurable. Thus, $F^{-1}(\Omega)$ is a measurable set, which shows that F is a measurable map. □

Remark 2.48. As one can check, the statements in Proposition 2.47 are also equivalent to the condition that for every point $p \in S$ there exists a chart (U_p, ϕ_p) about p in M such that the restriction $F|_{U_p \cap S} : U_p \cap S \to X$ is measurable.

Definition 2.49. Let M be a smooth manifold of dimension $n \in \{1, 2\}$, and let $S \subset M$ be a measurable set.

(i) For $r \in \{1, 2\}$, a differential r-form ω over $\mathbb{F}$ on S is said to be *measurable* if for every chart $(U, \phi) = (U, x^1, \dots, x^n)$ in M, on $U \cap S$ we have

$$\begin{aligned} \omega &= \sum_{j=1}^{n} f^j \, dx^j \quad && \text{if } r = 1 \,, \\ \omega &= g \, dx^1 \wedge dx^2 \quad && \text{if } n = r = 2 \,, \end{aligned}$$

for some $\mathbb{F}$-valued measurable functions $f^1, \dots, f^n, g$ on $U \cap S$. If $n = 1$, we define the zero 2-form on S to be *measurable*.

(ii) A 0-form ω over $\mathbb{F}$ on S is said to be *measurable* if for every chart (U, ϕ) in M the restriction $\omega|_{U \cap S}$ is measurable as an $\mathbb{F}$-valued function on $U \cap S$.

Remark 2.50. By Proposition 2.47 and Remark 2.48, if M is a smooth manifold of dimension $n \in \{1, 2\}$, then a 0-form ω on a measurable subset $S \subset M$ is measurable if and only if ω is measurable as an $\mathbb{F}$-valued function on S, and if and only for every point $p \in S$ there exists a chart (U_p, ϕ_p) about p in M such that the restriction $\omega|_{U_p \cap S}$ is measurable as an $\mathbb{F}$-valued function.

Proposition 2.51. *Let M be a smooth manifold of dimension $n \in \{1, 2\}$ and $S \subset M$ a measurable set.*

(i) *For $r \in \{1, 2\}$ and $r \leq n$, a differential r-form ω over $\mathbb{F}$ on S is measurable if and only if for every point $p \in S$ there exists a chart $(U_p, \phi_p) = (U, x_p^1, \dots, x_p^2)$ about p in M such that on $U_p \cap S$*

$$\begin{aligned} \omega &= \sum_{j=1}^{n} f^j \, dx_p^j \quad && \textit{if } r = 1 \,, \\ \omega &= g \, dx_p^1 \wedge dx_p^2 \quad && \textit{if } n = r = 2 \,, \end{aligned}$$

for some $\mathbb{F}$-valued measurable functions $f^1, \dots, f^n, g$ on $U_p \cap S$.

(ii) *Suppose N is another smooth manifold with $\dim N = \dim M$ and $F : N \to M$ is a diffeomorphism. Then, for $r \in \{0, 1, 2\}$, an r-form ω over $\mathbb{F}$ on S is measurable if and only if its pullback $F^*\omega$ is measurable on $F^{-1}(S) \subset N$.*

(iii) *If $r \in \{0, 1, 2\}$ and ω and τ are two measurable r-forms over $\mathbb{F}$ on S, then the r-form $\omega + \tau$ on S is also measurable.*

(iv) *Let $r, s \in \{0, 1, 2\}$ and $r + s \leq 2$. If ω and τ are respectively an r-form and an s-form over $\mathbb{F}$ on S, and if ω and τ are both measurable, then the $(r + s)$-form $\omega \wedge \tau$ on S is also measurable.*

The proof of Proposition 2.51 is left to the reader.

2.5. **Lebesgue Integration on Curves and Surfaces.**

Throughout Subsection 2.5, we let M denote an arbitrary smooth manifold of dimension $n \in \{1, 2\}$.

Suppose (U,ϕ) and (V,ψ) are two charts in M. Then, for each $p \in U \cap V$ the transition matrix between the bases $\{\frac{\partial}{\partial x^j}|_p\}_{j=1}^n$ and $\{\frac{\partial}{\partial y^j}|_p\}_{j=1}^n$ for T_pM is given by

$$\begin{pmatrix} \frac{\partial y^1}{\partial x^1}\Big|_p & \cdots & \frac{\partial y^1}{\partial x^n}\Big|_p \\ \vdots & \ddots & \vdots \\ \frac{\partial y^n}{\partial x^1}\Big|_p & \cdots & \frac{\partial y^n}{\partial x^n}\Big|_p \end{pmatrix},$$

which is also the Jacobian matrix of the smooth map $(\psi \circ \phi^{-1}) : \phi(U \cap V) \to \psi(U \cap V)$ at $\phi(p)$. Denoting by $\mathcal{J}_{\psi\circ\phi^{-1}} : \phi(U \cap V) \to \mathbb{R}$ the function mapping each $r \in \phi(U \cap V)$ to the Jacobian determinant of $\psi \circ \phi^{-1}$ at r, we have $\mathcal{J}_{\psi\circ\phi^{-1}} \neq 0$ on $\phi(U \cap V)$. Since $\mathcal{J}_{\psi\circ\phi^{-1}}$ is continuous, if $\phi(U \cap V)$ is connected we must have $\mathcal{J}_{\psi\circ\phi^{-1}} > 0$ or $\mathcal{J}_{\psi\circ\phi^{-1}} < 0$. Moreover, on $\psi(U\cap V)$ we have $\mathcal{J}_{\phi\circ\psi^{-1}} = (1/\mathcal{J}_{\psi\circ\phi^{-1}})\circ\phi\circ\psi^{-1}$, which implies that if $\mathcal{J}_{\psi\circ\phi^{-1}}$ is everywhere positive or everywhere negative on $\phi(U \cap V)$, then $\mathcal{J}_{\phi\circ\psi^{-1}}$ is respectively everywhere positive or everywhere negative on $\psi(U \cap V)$.

Definition 2.52. (i) Two charts (U,ϕ) and (V,ψ) in M are said to have *compatible orientations* if $\mathcal{J}_{\psi\circ\phi^{-1}} > 0$ on $\phi(U\cap V)$, or equivalently, if $\mathcal{J}_{\phi\circ\psi^{-1}} > 0$ on $\psi(U\cap V)$.
(ii) An atlas $\mathfrak{U}$ in M is said to be *oriented* if every two charts in $\mathfrak{U}$ have compatible orientations.

We may define an equivalence relation on the set of oriented atlases in M, where two oriented atlases $\mathfrak{U}_1$ and $\mathfrak{U}_2$ in M are *equivalent*, denoted $\mathfrak{U}_1 \sim \mathfrak{U}_2$, if the atlas $\mathfrak{U}_1 \cup \mathfrak{U}_2$ is also oriented.

Definition 2.53. If there exists an oriented atlas in M, then M is said to be *orientable*. If no oriented atlas exists, M is said to be *non-orientable*. An equivalence class of oriented atlases in M is called an *orientation (in M)*. If M is orientable, then M together with a choice of orientation is said to be an *oriented* manifold. If M is oriented, a chart in an atlas in the orientation of M said to be *positively oriented*.

Definition 2.54. If N and M are oriented smooth manifolds of dimension $n \in \{1,2\}$ and $F : N \to M$ is a diffeomorphism, we say that F is *orientation-preserving* if for every positively oriented chart (U,ϕ) in M, the induced chart $(F^{-1}(U), \phi \circ F)$ in N is positively oriented.

Remark 2.55. Note that if M has a global chart (M,ϕ) in its differentiable structure, then $\{(M,\phi)\}$ is an oriented atlas in M, and hence M is orientable.

Definition 2.56. For $n \in \{1,2\}$, we call the orientation on $\mathbb{R}^n$ given by the oriented atlas $\{(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})\}$ the *standard orientation* on $\mathbb{R}^n$.

Unless otherwise mentioned, for $n \in \{1,2\}$ we will assume that $\mathbb{R}^n$ is equipped with the standard orientation.

Remark 2.57. Using notation from Definition 2.54, suppose $F : N \to M$ is orientation-preserving, and let (V,ψ) be a positively oriented chart in N. Then, for each positively oriented chart (U,ϕ) in M, the chart $(F^{-1}(U), \phi \circ F)$ in N is also positively oriented, so the transition map

$$\psi \circ (\phi \circ F)^{-1} : (\phi \circ F)(F^{-1}(U) \cap V) \to \psi(F^{-1}(U) \cap V)$$

has positive Jacobian determinant everywhere. We may rewrite this map as

$$\psi \circ F^{-1} \circ \phi^{-1} : \phi(U \cap F(V)) \to (\psi \circ F^{-1})(U \cap F(V)) .$$

This is precisely the transition map between the charts (U, ϕ) and $(F(V), \psi \circ F^{-1})$ in M, so these two charts have compatible orientations. Since this holds for any positively oriented chart (U, ϕ) in M, the chart $(F(V), \psi \circ F^{-1})$ in M is positively oriented. This shows that a diffeomorphism $F : N \to M$ is orientation-preserving if and only if its inverse $F^{-1} : M \to N$ is orientation-preserving.

Lemma 2.58. *Suppose N and M are smooth manifolds of dimension $n \in \{1, 2\}$ and $F : N \to M$ is a diffeomorphism. If M is orientable and $\mathfrak{U} := \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is an oriented atlas in M, then the induced atlas $\mathfrak{V} := \{(F^{-1}(U_\alpha), \phi_\alpha \circ F)\}_{\alpha \in A}$ in N is also oriented. As a consequence, N is also orientable.*

Proof. Let $(U, \phi), (V, \psi) \in \mathfrak{U}$. Then, the transition map between the charts $(F^{-1}(U), \phi \circ F)$ and $(F^{-1}(V), \psi \circ F)$ in N is given by

$$\psi \circ F \circ (\phi \circ F)^{-1} : (\phi \circ F)(F^{-1}(U) \cap F^{-1}(V)) \to (\psi \circ F)(F^{-1}(U) \cap F^{-1}(V)) ,$$

which is precisely the transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ between the charts (U, ϕ) and (V, ψ), and hence its Jacobian determinant is positive everywhere. Thus, $\mathfrak{V}$ is oriented. □

Remark 2.59. Referring to Lemma 2.58, suppose we have made a choice of orientation $\mathscr{O}$ in M. If $\mathfrak{U}$ and $\mathfrak{U}'$ are two oriented atlases in $\mathscr{O}$, then their union $\mathfrak{U} \cup \mathfrak{U}'$ is also oriented. Letting $\mathfrak{V}$ and $\mathfrak{V}'$ denote the oriented atlases in N induced respectively from $\mathfrak{U}$ and from $\mathfrak{U}'$, the union $\mathfrak{V} \cup \mathfrak{V}'$ is the atlas induced from $\mathfrak{U} \cup \mathfrak{U}'$ and hence it is oriented by Lemma 2.58. Thus, the atlases $\mathfrak{V}$ and $\mathfrak{V}'$ belong to the same orientation $\mathscr{O}'$ in N. It follows that all the oriented atlases in N induced from oriented atlases in $\mathscr{O}$ belong to $\mathscr{O}'$. We then call $\mathscr{O}'$ the *induced orientation* in N from the diffeomorphism $F : N \to M$.

Remark 2.60. Suppose M is oriented and $\Omega \subset M$ is open. Let $\{\mathfrak{U}_\alpha\}_{\alpha \in A}$ be the collection of all oriented atlases $\mathfrak{U}_\alpha = \{(U_\beta, \phi_\beta)\}_{\beta \in B_\alpha}$ in the orientation of M. Then, as one can check, for each $\alpha \in A$ the induced atlas $\mathfrak{U}_\alpha^\Omega := \{(U_\beta \cap \Omega, \phi_\beta|_{U_\beta \cap \Omega})\}_{\beta \in B_\alpha}$ in Ω is oriented, and for all $\alpha_1, \alpha_2 \in A$, the oriented atlases $\mathfrak{U}_{\alpha_1}^\Omega$ and $\mathfrak{U}_{\alpha_2}^\Omega$ are equivalent. As a result, Ω is also orientable, and all the oriented atlases in Ω in the collection $\{\mathfrak{U}_\alpha^\Omega\}_{\alpha \in A}$ belong to a unique orientation in Ω, which we call the *induced orientation (from M)*. Unless otherwise specified, we will give open sets of oriented smooth manifolds the induced orientation.

Proposition 2.61. *If M is orientable and connected, then M has exactly two orientations.*

Proof. Since M is orientable, M has an oriented atlas

$$\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A} = \{(U_\alpha, x_\alpha^1, \dots, x_\alpha^n)\}_{\alpha \in A} .$$

For each $\alpha \in A$, we let $\tilde{\phi}_\alpha := -\phi_\alpha = -x_\alpha^1$ if $n = 1$, and $\tilde{\phi}_\alpha := (-x_\alpha^1, x_\alpha^2)$ if $n = 2$. Then, the atlas

$$-\mathfrak{U} := \{(U_\alpha, \tilde{\phi}_\alpha)\}_{\alpha \in A}$$

is also oriented and $\mathfrak{U} \not\sim -\mathfrak{U}$, so $\mathfrak{U}$ and $-\mathfrak{U}$ represent two different orientations in M. It remains to show that for any oriented atlas $\mathfrak{V}$ in M, we have either $\mathfrak{V} \sim \mathfrak{U}$ or $\mathfrak{V} \sim -\mathfrak{U}$.

Suppose $\mathfrak{V}$ is an oriented atlas in M. For each $p \in M$, choose charts $(V_p, \psi_p) \in \mathfrak{V}$ and $(U_{\alpha_p}, \phi_{\alpha_p}) \in \mathfrak{U}$ about p. Since $\phi_{\alpha_p}(U_{\alpha_p} \cap V_p)$ is open and contains $\phi_{\alpha_p}(p)$, we may choose an open ball $B_p \subset \phi_{\alpha_p}(U_{\alpha_p} \cap V_p)$ (which is connected) containing $\phi_{\alpha_p}(p)$. Then, $D_p := \phi_{\alpha_p}^{-1}(B_p) \subset U_{\alpha_p} \cap V_p$ is an open subset of both U_{α_p} and V_p, and $p \in D_p$. Letting $\tilde{\psi}_p := \psi_p|_{D_p}$, the pair $(D_p, \tilde{\psi}_p)$ is then another chart about p in M, and the transition map

$$\tilde{\psi}_p \circ \phi_{\alpha_p}^{-1} : \phi_{\alpha_p}(U_{\alpha_p} \cap D_p) = \phi_{\alpha_p}(D_p) = B_p \to \tilde{\psi}_p(U_{\alpha_p} \cap D_p) = \psi_p(D_p)$$

has a connected domain and hence it must fulfil either $\mathcal{J}_{\tilde{\psi}_p \circ \phi_{\alpha_p}^{-1}} > 0$ or $\mathcal{J}_{\tilde{\psi}_p \circ \phi_{\alpha_p}^{-1}} < 0$. If (U_β, ϕ_β) is an arbitrary chart in $\mathfrak{U}$, then we may write the transition map

$$\tilde{\psi}_p \circ \phi_\beta^{-1} : \phi_\beta(U_\beta \cap D_p) \to \tilde{\psi}_p(U_\beta \cap D_p)$$

as the composition $F_2 \circ F_1$, where F_1 is the restriction of the transition map $\phi_{\alpha_p} \circ \phi_\beta^{-1}$ to $\phi_\beta(U_\beta \cap D_p) \subset \phi_\beta(U_\beta \cap U_{\alpha_p})$, and F_2 is the restriction of the transition map $\tilde{\psi}_p \circ \phi_{\alpha_p}^{-1}$ to $\phi_{\alpha_p}(U_\beta \cap D_p) \subset \phi_{\alpha_p}(D_p)$. Since F_1 has positive Jacobian determinant, we have

$$\begin{aligned}
&\mathcal{J}_{\tilde{\psi}_p \circ \phi_{\alpha_p}^{-1}} > 0 \quad \Rightarrow \quad \mathcal{J}_{F_2} > 0 \quad \Rightarrow \quad \mathcal{J}_{F_2 \circ F_1} > 0 \quad \Rightarrow \quad \mathcal{J}_{\tilde{\psi}_p \circ \phi_\beta^{-1}} > 0\,, \\
&\mathcal{J}_{\tilde{\psi}_p \circ \phi_{\alpha_p}^{-1}} < 0 \quad \Rightarrow \quad \mathcal{J}_{F_2} < 0 \quad \Rightarrow \quad \mathcal{J}_{F_2 \circ F_1} < 0 \quad \Rightarrow \quad \mathcal{J}_{\tilde{\psi}_p \circ \phi_\beta^{-1}} < 0 \quad \Leftrightarrow \quad \mathcal{J}_{\tilde{\psi}_p \circ \tilde{\phi}_\beta^{-1}} > 0\,.
\end{aligned}$$

Thus, precisely one of the following holds:

(a) $\mathcal{J}_{\tilde{\psi}_p \circ \phi_\beta^{-1}} > 0$ for every chart $(U_\beta, \phi_\beta) \in \mathfrak{U}$;

(b) $\mathcal{J}_{\tilde{\psi}_p \circ \tilde{\phi}_\beta^{-1}} > 0$ for every chart $(U_\beta, \tilde{\phi}_\beta) \in -\mathfrak{U}$.

The collection $\tilde{\mathfrak{V}} := \{(D_p, \tilde{\psi}_p)\}_{p \in M}$ is then an atlas in M that is oriented and equivalent to $\mathfrak{V}$, and for each $p \in M$ the chart $(D_p, \tilde{\psi}_p)$ fulfils either (a) or (b). We define the function

$$f : M \to \{0, 1\}\,, \quad f(p) := \begin{cases} 0 & \text{if } (D_p, \tilde{\psi}_p) \text{ fulfils (a)}\,, \\ 1 & \text{if } (D_p, \tilde{\psi}_p) \text{ fulfils (b)}\,. \end{cases}$$

We claim that f is a locally constant. Indeed, assume $p \in M$ and $f(p) = 0$. If there exists $q \in D_p$ such that $f(q) = 1$, then $(D_p, \tilde{\psi}_p)$ fulfils (a) and $(D_q, \tilde{\psi}_q)$ fulfils (b). In particular, the transition map $G_1 := \tilde{\psi}_p \circ \phi_{\alpha_p}^{-1}$ has positive Jacobian determinant, and $G_2 := \phi_{\alpha_p} \circ \tilde{\psi}_q^{-1}$ has negative Jacobian determinant. We may restrict the map G_1 to $\phi_{\alpha_p}(D_p \cap D_q) \subset \phi_{\alpha_p}(D_p) = \phi_{\alpha_p}(U_{\alpha_p} \cap D_p)$, and G_2 to $\tilde{\psi}_q(D_p \cap D_q) \subset \tilde{\psi}_q(U_{\alpha_p} \cap D_q)$. Then, the composition

$$G_1|_{\phi_{\alpha_p}(D_p \cap D_q)} \circ G_2|_{\tilde{\psi}_q(D_p \cap D_q)} : \tilde{\psi}_q(D_p \cap D_q) \to \tilde{\psi}_p(D_p \cap D_q)$$

has negative Jacobian determinant. This is a contradiction, since the above map is precisely the transition map $\tilde{\psi}_p \circ \tilde{\psi}_q^{-1}$, which has positive Jacobian determinant. Thus, we must have $f(q) = 0$ for all $q \in D_p$. A similar argument shows that if $f(p) = 1$, then $f(q) = 1$ for all $q \in D_p$. Thus, f is locally constant on M, and since M is connected, f must then be constant. If $f = 0$ on M, then $\tilde{\mathfrak{V}} \sim \mathfrak{U}$, and if $f = 1$ on M, then $\tilde{\mathfrak{V}} \sim -\mathfrak{U}$. Since $\mathfrak{V} \sim \tilde{\mathfrak{V}}$, we have $\mathfrak{V} \sim \mathfrak{U}$ or $\mathfrak{V} \sim -\mathfrak{U}$. $\square$

If $S \subset M$ and ω is a nowhere-vanishing n-form over $\mathbb{F}$ on S, then for each $p \in S$ the n-covector ω_p is a basis for the 1-dimensional vector space $\Lambda^n(T_{\mathbb{F},p}M)^*$. As such, if τ is any n-form over $\mathbb{F}$ on S, we may define the function

$$\frac{\tau}{\omega} : S \to \mathbb{F}\,, \quad p \mapsto \frac{\tau_p}{\omega_p}\,,$$

and then on S we have

$$\tau = \frac{\tau}{\omega}\,\omega\,.$$

For each chart $(U, \phi) = (U, x^1, \ldots, x^n)$ in M, we will denote by ω_ϕ the nowhere-vanishing n-form over $\mathbb{F}$ on U given by

$$\begin{aligned} \omega_\phi &:= dx^1 && \text{if } n = 1, \\ \omega_\phi &:= dx^1 \wedge dx^2 && \text{if } n = 2. \end{aligned}$$

Then, if (V, ψ) is another chart in M, on $U \cap V$ we have

$$\omega_\phi = \frac{\omega_\phi}{\omega_\psi}\,\omega_\psi$$

with

$$\frac{\omega_\phi}{\omega_\psi} = \mathcal{J}_{\phi\circ\psi^{-1}} \circ \psi : U \cap V \to \mathbb{R}\,.$$

It follows that the charts (U, ϕ) and (V, ψ) have compatible orientations if and only if the function ω_ϕ/ω_ψ is everywhere positive on $U \cap V$.

Proposition 2.62. (i) *If there exists a continuous nowhere-vanishing n-form over $\mathbb{R}$ on M, then M is orientable.*

(ii) *If M is second countable and oriented, then there exists a C^∞ nowhere-vanishing n-form ω over $\mathbb{R}$ on M such that for every positively oriented chart (U, ϕ) in M, we have $\omega/\omega_\phi > 0$ on U.*

Proof. (i) Suppose ω is a continuous nowhere-vanishing n-form over $\mathbb{R}$ on M. For each $p \in M$, choose a chart $(U_p, \phi_p) = (U_p, x_p^1, \ldots, x_p^n)$ about p in M such that U_p is connected. Since $\omega/\omega_{\phi_p} : U_p \to \mathbb{R}$ is continuous and nowhere-vanishing, we must have either $\omega/\omega_{\phi_p} > 0$ everywhere on U_p or $\omega/\omega_{\phi_p} < 0$ everywhere on U_p. If $\omega/\omega_{\phi_p} < 0$, redefine (U_p, ϕ_p) by replacing x_p^1 by $-x_p^1$. Then, $\omega/\omega_{\phi_p} > 0$ on U_p, and since ω is nowhere-vanishing, we may also define the function $\omega_{\phi_p}/\omega = 1/(\omega/\omega_{\phi_p})$, which is also everywhere positive on U_p. We show that the resulting atlas $\mathfrak{U} := \{(U_p, \phi_p)\}_{p\in M}$ is oriented. If $(U_p, \phi_p), (U_q, \phi_q) \in \mathfrak{U}$, then on $U_p \cap U_q$

$$\frac{\omega_{\phi_p}}{\omega_{\phi_q}} = \frac{\omega_{\phi_p}}{\omega}\,\frac{\omega}{\omega_{\phi_q}}\,,$$

which is a product of positive functions and hence positive on $U_p \cap U_q$. Thus, the charts (U_p, ϕ_p) and (U_q, ϕ_q) have compatible orientations.

(ii) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be the collection of all positively oriented charts in M. Since M is second countable, there exists a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$ for M. The C^∞ nowhere-vanishing real n-form ω on M defined by

$$\omega_p := \sum_{\substack{\alpha \in A \text{ with} \\ p \in \mathrm{supp}\rho_\alpha}} \rho_\alpha(p)\,\omega_{\phi_\alpha}(p)\,, \quad p \in M\,,$$

fulfils $\omega/\omega_{\phi_\alpha} > 0$ on U_α for all $\alpha \in A$. We leave the details for the reader to check. $\square$

Definition 2.63. A C^∞ nowhere-vanishing real n-form on M is called a *volume form (on M)*.

Definition 2.64. Suppose M is oriented and $S \subset M$.

(i) A real n-form ω on S is said to be *positive*, denoted $\omega > 0$, if for every positively oriented chart (U, ϕ) in M intersecting S, we have $\omega/\omega_\phi > 0$ on $U \cap S$. The real n-form ω is said to be *nonnegative*, denoted $\omega \geq 0$, if $\omega/\omega_\phi \geq 0$ on $U \cap S$ for every positively oriented chart (U, ϕ) in M intersecting S. Moreover, ω is said to be *negative*, denoted $\omega < 0$, if $-\omega > 0$, and ω is said to be *nonpositive*, denoted $\omega \leq 0$, if $-\omega \geq 0$.

(ii) For $p \in M$, a real n-covector $\alpha \in \Lambda^n T_p^* N$ is said to be respectively *positive, nonnegative, negative*, or *nonpositive*, if it is so as a real n-form on $\{p\} \subset M$, that is, if for every positively oriented chart $(U, \phi) = (U, x^1, \dots, x^n)$ in M about p, the real number $\alpha/(\omega_\phi)_p = \alpha\big(\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\big)$ is respectively positive, nonnegative, negative, or nonpositive.

(iii) For $p \in M$, an ordered basis $\{v^1, \dots, v^n\}$ for T_pM is said to be *positively oriented* if for every positive real n-covector $\alpha \in \Lambda^n T_p^* M$, we have $\alpha(v^1, \dots, v^n) > 0$.

(iv) If ω and τ are two real n-forms on S, we say that $\omega > \tau$, $\omega \geq \tau$, $\omega < \tau$, or $\omega \leq \tau$, respectively if $\omega - \tau > 0$, $\omega - \tau \geq 0$, $\omega - \tau < 0$, or $\omega - \tau \leq 0$.

Proposition 2.65. *Suppose M is oriented and $S \subset M$.*

(i) *A real n-form ω on S is positive if and only if for every $p \in S$ there exists a positively oriented chart (U_p, ϕ_p) about p in M such that $\omega/\omega_{\phi_p} > 0$ on $U_p \cap S$. Similarly, ω is nonnegative if and only if for every $p \in S$ there exists a positively oriented chart (U_p, ϕ_p) about p in M such that $\omega/\omega_{\phi_p} \geq 0$ on $U_p \cap S$.*

(ii) *For a point $p \in M$, an ordered basis $\{v^1, \dots, v^n\}$ for T_pM is positively oriented if and only if there exists a positively oriented chart (U, ϕ) about p in M such that $(\omega_\phi)_p(v^1, \dots, v^n) > 0$.*

Proof. (i) Suppose ω is a real n-form on S such that for every $p \in S$ there exists a positively oriented chart (U_p, ϕ_p) about p in M such that $\omega/\omega_{\phi_p} > 0$ on $U_p \cap S$. If (V, ψ) is another positively oriented chart in M intersecting S, then for each $p \in V \cap S$ we have $(\omega_{\phi_p})_p/(\omega_\psi)_p > 0$, so

$$\frac{\omega}{\omega_\psi}(p) = \frac{\omega_p}{(\omega_\psi)_p} = \frac{\omega_p}{(\omega_{\phi_p})_p} \frac{(\omega_{\phi_p})_p}{(\omega_\psi)_p} > 0\,.$$

Thus, $\omega/\omega_\psi > 0$ on $V \cap S$, so $\omega > 0$. The case when for every $p \in S$ there exists a positively oriented chart (U_p, ϕ_p) about p in M such that $\omega/\omega_{\phi_p} \geq 0$ on $U_p \cap S$ is similar. The rest of the proof is left to the reader.

(ii) ($\Rightarrow$) Suppose $\{v^1, \dots, v^n\}$ is a positively oriented ordered basis for T_pM. If (U, ϕ) is any positively oriented chart about p in M, then $(\omega_\phi)_p \in \Lambda^n T_p^* M$ is positive, so $(\omega_\phi)_p(v^1, \dots, v^n) > 0$.

($\Leftarrow$) Let $\{v^1, \dots, v^n\}$ be an ordered basis for T_pM, and suppose there exists a positively oriented chart (U, ϕ) about p in M such that $(\omega_\phi)_p(v^1, \dots, v^n) > 0$. If $\alpha \in \Lambda^n T_p^* M$ is a positive n-covector, then $\alpha/(\omega_\phi)_p > 0$, so

$$\alpha(v^1, \dots, v^n) = \frac{\alpha}{(\omega_\phi)_p}\,(\omega_\phi)_p(v^1, \dots, v^n) > 0\,.$$

Thus, the ordered basis $\{v^1, \ldots, v^n\}$ is positively oriented.

$\square$

Definition 2.66. Suppose M is oriented. Let $S \subset M$ and suppose ω is a real n-form on S. We define the *positive part of* ω, denoted ω^+, to be the nonnegative real n-form on S given by

$$(\omega^+)_p := \begin{cases} \omega_p & \text{if } \omega_p \geq 0 \\ 0 & \text{if } \omega_p < 0 \end{cases} , \quad p \in S\,.$$

We also define the *negative part of* ω, denoted ω^-, to be the nonnegative real n-form on S given by

$$(\omega^-)_p := \begin{cases} -\omega_p & \text{if } \omega_p \leq 0 \\ 0 & \text{if } \omega_p > 0 \end{cases} , \quad p \in S\,.$$

Remark 2.67. Using notation from Definition 2.66, if (U, ϕ) is a positively oriented chart in M then on $U \cap S$ we have $\omega = f_\phi\, \omega_\phi$, for $f_\phi := (\omega/\omega_\phi) : U \cap S \to \mathbb{R}$. Then, on $U \cap S$

$$\omega^+ = f_\phi^+\, \omega_\phi\,, \quad \omega^- = f_\phi^-\, \omega_\phi\,.$$

If ω is continuous on S, then f_ϕ is continuous on $U \cap S$, which implies that f_ϕ^+ and f_ϕ^- are also continuous. Since we can cover S by positively oriented charts, it follows that ω^+ and ω^- are also continuous on S. If $S \subset M$ is measurable and ω is measurable, then f_ϕ is measurable on $U \cap S$, so f_ϕ^+ and f_ϕ^- are also measurable. Thus, if ω is measurable on S, then ω^+ and ω^- are also measurable on S.

Proposition 2.68. *Suppose M is oriented, $S \subset M$ is measurable and ω is a measurable, nonnegative real n-form on S. Letting t and (t^1, t^2) denote respectively the standard coordinates on $\mathbb{R}$ and $\mathbb{R}^2$, define $\omega_{\mathbb{R}} := dt$ and $\omega_{\mathbb{R}^2} := dt^1 \wedge dt^2$. If (U, ϕ) and (V, ψ) are two positively oriented charts in M, then*

$$\int_{\phi(U\cap V\cap S)} \frac{(\phi^{-1})^*\big(\omega|_{U\cap V\cap S}\big)}{\omega_{\mathbb{R}^n}}\, d\lambda = \int_{\psi(U\cap V\cap S)} \frac{(\psi^{-1})^*\big(\omega|_{U\cap V\cap S}\big)}{\omega_{\mathbb{R}^n}}\, d\lambda$$

Proof. On $U \cap V \cap S$, we have

$$\omega = f\, \omega_\phi = g\, \omega_\psi\,,$$

where $f := \omega/\omega_\phi : U \cap V \cap S \to \mathbb{R}$ and $g := \omega/\omega_\psi : U \cap V \cap S \to \mathbb{R}$ are nonnegative and measurable, with

$$g = f\, \frac{\omega_\phi}{\omega_\psi} = f \cdot (\mathcal{J}_{\phi\circ\psi^{-1}} \circ \psi)|_{U\cap V\cap S}\,.$$

Computing the pullbacks $(\phi^{-1})^*\big(\omega|_{U\cap V\cap S}\big)$ and $(\psi^{-1})^*\big(\omega|_{U\cap V\cap S}\big)$ explicitly and applying change of variables (Theorem 2.44) via the diffeomorphism $\phi\circ\psi^{-1} : \psi(U\cap V) \to \phi(U\cap V)$, the result follows. $\square$

Definition 2.69. Suppose M is oriented and $S \subset M$ is measurable. Let ω be a measurable, real nonnegative n-form on S. Let t and (t^1, t^2) denote respectively the standard coordinates on $\mathbb{R}$ and $\mathbb{R}^2$, and let $\omega_{\mathbb{R}} := dt$ and $\omega_{\mathbb{R}^2} := dt^1 \wedge dt^2$.

(i) Suppose $S \subset U$ for some positively oriented chart (U, ϕ) in M. We then define

$$\int_S \omega := \int_{\phi(S)} \frac{(\phi^{-1})^*\omega}{\omega_{\mathbb{R}^n}}\, d\lambda\,,$$

which by Proposition 2.68 is independent of the choice of positively oriented chart (U, ϕ) with $S \subset U$.

(ii) Let $\Upsilon(S)$ be the collection of tuples $(S_1, \ldots, S_m)$ of finitely many mutually disjoint measurable subsets $S_1, \ldots, S_m \subset S$ such that for all $j \in \{1, \ldots, m\}$, $S_j \subset U$ for some positively oriented chart (U, ϕ) in M. We then define the *integral of ω over S* by

$$\int_S \omega := \sup_{(S_1,\ldots,S_m)\in\Upsilon(S)} \sum_{j=1}^{m} \int_{S_j} \omega \in [0, +\infty]$$

Remark 2.70. (1) Note that in Definition 2.69, if $S \subset U$ for some positively oriented chart (U, ϕ) in M, then the definitions of $\int_S \omega$ given in (i) and (ii) agree.

(2) In Definition 2.69 (ii), the set

$$P := \left\{ \sum_{j=1}^{m} \int_{s_j} \omega \,\middle|\, (s_1, \ldots, s_m) \in \Upsilon(S) \right\}$$

is nonempty, since we always have $\emptyset \in \Upsilon$ (as a tuple of one element), so that $\int_\emptyset \omega \in P$. Thus, the supremum in Definition 2.69 (ii) is that of a nonempty set.

Definition 2.71. Suppose M is oriented and $S \subset M$ is measurable. Let ω be a measurable real or complex n-form on S.

(i) Suppose ω is real. We say that ω is *integrable* if

$$\int_S \omega^+ < +\infty \quad \text{and} \quad \int_S \omega^- < +\infty\,.$$

If ω is integrable, we define the *integral of ω over S* by

$$\int_S \omega := \int_S \omega^+ - \int_S \omega^- \in \mathbb{R}\,.$$

(ii) If ω is complex, we say that ω is *integrable* if the measurable real n-forms $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ on S are integrable, and if this is the case we define the *integral of ω over S* to be

$$\int_S \omega := \int_S \mathrm{Re}(\omega) + i \int_S \mathrm{Im}(\omega) \in \mathbb{C}\,.$$

Proposition 2.72. *Suppose M is oriented and $S \subset M$ is measurable. Let $\mathcal{A}_S$ denote the σ-algebra on S of measurable subsets of S. If ω is a measurable, real nonnegative n-form on S, then*

(i) *the function*

$$\lambda_\omega : \mathcal{A}_S \to [0, +\infty]\,, \quad T \mapsto \int_T \omega\,,$$

is a measure on $\mathcal{A}_S$;

(ii) *if $f : S \to \mathbb{R}$ is a nonnegative measurable function on S, then*

$$\int_S f \, d\lambda_\omega = \int_S f\,\omega\,.$$

Proof. (i) Choosing any positively oriented chart (U, ϕ) in M, we have

$$\lambda_\omega(\emptyset) = \int_\emptyset \omega = \int_{\phi(\emptyset)} \frac{(\phi^{-1})^*\big(\omega|_\emptyset\big)}{\omega_{\mathbb{R}^n}} \, d\lambda = 0\,.$$

It remains to show that if $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of mutually disjoint measurable subsets of S, then

$$\lambda_\omega\Big(\bigcup_{k\in\mathbb{N}} T_k\Big) = \sum_{k=1}^{\infty} \lambda_\omega(T_k)\,.$$

We have

$$\lambda_\omega\Big(\bigcup_{k\in\mathbb{N}} T_k\Big) = \int_{\bigcup_{k\in\mathbb{N}} T_k} \omega = \sup P$$

and

$$\lambda_\omega(T_k) = \int_{T_k} \omega = \sup P_k$$

for each $k \in \mathbb{N}$, where

$$P := \left\{ \sum_{j=1}^{m} \int_{s_j} \omega \;\middle|\; (s_1, \ldots, s_m) \in \Upsilon\Big(\bigcup_{k\in K} T_k\Big) \right\} \subset [0, +\infty]$$

and

$$P_k := \left\{ \sum_{j=1}^{m} \int_{s_j} \omega \;\middle|\; (s_1, \ldots, s_m) \in \Upsilon(T_k) \right\} \subset [0, +\infty]$$

for each $k \in \mathbb{N}$. Thus, we need to prove that

$$\sup P = \sum_{k=1}^{\infty} \sup P_k\,.$$

For this, we will show that

(a) every element $x \in P$ can be written as $x = \sum_{k=1}^{\infty} a_k$ for some sequence $\{a_k\}_{k\in\mathbb{N}} \subset [0, +\infty]$ such that $a_k \in P_k$ for each $k \in \mathbb{N}$, and

(b) if $K \subset \mathbb{N}$ is a finite set and $b_k \in P_k$ for each $k \in K$, then $\sum_{k\in K} b_k \in P$.

To show (a), choose any $x = \sum_{j=1}^{m} \int_{s_j} \omega$ in P, for $(s_1, \ldots, s_m) \in \Upsilon\big(\bigcup_{k\in K} T_k\big)$. For each $j \in \{1, \ldots, m\}$, we have $s_j \subset U_j$ for some positively oriented chart (U_j, ϕ_j) in M. We may write s_j as the disjoint union $s_j = \bigcup_{k\in\mathbb{N}}(s_j \cap T_k)$, so that $\phi_j(s_j) = \bigcup_{k\in\mathbb{N}} \phi_j(s_j \cap T_k)$ is also a disjoint union. We then have

$$\begin{aligned}
\int_{s_j} \omega &= \int_{\phi_j(s_j)} f_j \, d\lambda\,, && \text{where } f_j := \frac{(\phi^{-1})^*\big(\omega|_{s_j}\big)}{\omega_{\mathbb{R}^n}}\,,\\
&= \int_{\phi_j(s_j)} \sum_{k=1}^{\infty} f_j \chi_{\phi_j(s_j\cap T_k)} \, d\lambda \\
&= \sum_{k=1}^{\infty} \int_{\phi_j(s_j)} f_j \chi_{\phi_j(s_j\cap T_k)} \, d\lambda && \text{(by the Monotone Convergence Theorem)}\\
&= \sum_{k=1}^{\infty} \int_{\phi_j(s_j\cap T_k)} f_j|_{\phi_j(s_j\cap T_k)} \, d\lambda \\
&= \sum_{k=1}^{\infty} \int_{\phi_j(s_j\cap T_k)} \frac{(\phi^{-1})^*\big(\omega|_{s_j\cap T_k}\big)}{\omega_{\mathbb{R}^n}} \, d\lambda \\
&= \sum_{k=1}^{\infty} \int_{s_j\cap T_k} \omega\,.
\end{aligned}$$

For each $k \in \mathbb{N}$, let

$$a_k := \sum_{j=1}^{m} \int_{s_j \cap T_k} \omega \,,$$

which is in P_k, since $(s_1 \cap T_k, \ldots, s_m \cap T_k) \in \Upsilon(T_k)$. Then,

$$\begin{aligned} x &= \sum_{j=1}^{m} \int_{s_j} \omega \\ &= \sum_{j=1}^{m} \left(\lim_{N\to\infty} \sum_{k=1}^{N} \int_{s_j \cap T_k} \omega \right) \\ &= \lim_{N\to\infty} \sum_{j=1}^{m} \sum_{k=1}^{N} \int_{s_j \cap T_k} \omega \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} \sum_{j=1}^{m} \int_{s_j \cap T_k} \omega \\ &= \sum_{k=1}^{\infty} a_k \,. \end{aligned}$$

Thus, (a) is proved. For (b), suppose $K \subset \mathbb{N}$ is finite and $b_k \in P_k$ for each $k \in K$. For each $k \in K$, we have

$$b_k = \sum_{j=1}^{m_k} \int_{s_{kj}} \omega$$

for some $(s_{k1}, \ldots, s_{km_k}) \in \Upsilon(T_k)$. Then, we have

$$\{s_{kj}\}_{\substack{k\in K \\ j\in\{1,\ldots,m_k\}}} \in \Upsilon\left(\bigcup_{k\in K} T_k \right),$$

so

$$\sum_{k\in K} b_k = \sum_{k\in K} \sum_{j=1}^{m_k} \int_{s_{kj}} \omega \in P \,,$$

which concludes the proof of (b). Statement (a) guarantees that $\sum_{k=1}^{\infty} \sup P_k$ is an upper bound for P, while from (b) it follows that no real number $r < \sum_{k=1}^{\infty} \sup P_k$ can be an upper bound for P. The details are left to the reader. In conclusion,

$$\sup P = \sum_{k=1}^{\infty} \sup P_k \,.$$

(ii) We have

$$\int_S f\omega = \sup Q$$

and

$$\int_S f \, d\lambda_\omega = \sup R \,,$$

for

$$Q := \left\{ \sum_{j=1}^{m} \int_{s_j} f\omega \;\middle|\; (s_1, \ldots, s_m) \in \Upsilon(S) \right\} \subset [0, +\infty]$$

and

$$R := \left\{ \int_S h\, d\lambda_\omega \,\middle|\, h \in H^+(S) \text{ and } h \le f \right\} \subset [0, +\infty]\,,$$

where $H^+(S)$ denotes the set of nonnegative simple functions on S, that is, the set of functions $h : S \to [0, +\infty)$ that are measurable and only take finitely many values. Suppose x is a nonzero element in R. Then, $x = \int_S h\, d\lambda_\omega$ for some nonzero $h \in H^+(S)$ such that $h \le f$. We may then write $h = \sum_{i=1}^r \alpha_i\, \chi_{T_i}$, where $r \in \mathbb{N}$, $\{\alpha_1, \ldots, \alpha_r\}$ is the set of positive values taken by h, and $T_i := h^{-1}(\{\alpha_i\})$ for each $i \in \{1, \ldots, r\}$. Then,

$$x = \int_S h\, d\lambda_\omega = \sum_{i=1}^r \alpha_i\, \lambda_\omega(T_i) = \sum_{i=1}^r \alpha_i \int_{T_i} \omega$$

If $i \in \{1, \ldots, r\}$ and $t \subset T_i$ is a measurable set such that $t \subset U$ for some positively oriented chart (U, ϕ) in M, then

$$\begin{aligned} \alpha_i \int_t \omega &= \alpha_i \int_{\phi(t)} \frac{(\phi^{-1})^*\big(\omega|_t\big)}{\omega_{\mathbb{R}^n}}\, d\lambda \\ &= \int_{\phi(t)} \alpha_i\, \frac{(\phi^{-1})^*\big(\omega|_t\big)}{\omega_{\mathbb{R}^n}}\, d\lambda \\ &= \int_{\phi(t)} \frac{(\phi^{-1})^*\big(\alpha_i\, \omega|_t\big)}{\omega_{\mathbb{R}^n}}\, d\lambda \\ &= \int_t \alpha_i\, \omega\,. \end{aligned}$$

Moreover, for all $q \in \phi(t)$ we have $\phi^{-1}(q) \in t \subset T_i$, so $\alpha_i = h(\phi^{-1}(q)) \le f(\phi^{-1}(q))$. Thus,

$$\begin{aligned} \frac{(\phi^{-1})^*\big(\alpha_i\, \omega|_t\big)}{\omega_{\mathbb{R}^n}}(q) &= \alpha_i \frac{(\phi^{-1})^*\big(\omega|_t\big)}{\omega_{\mathbb{R}^n}}(q) \\ &\le f(\phi^{-1}(q)) \frac{(\phi^{-1})^*\big(\omega|_t\big)}{\omega_{\mathbb{R}^n}}(q) \\ &= \frac{(\phi^{-1})^*\big((f\omega)|_t\big)}{\omega_{\mathbb{R}^n}}(q) \end{aligned}$$

for all $q \in \phi(t)$, so

$$\begin{aligned} \int_t \alpha_i\, \omega &= \int_{\phi(t)} \frac{(\phi^{-1})^*\big(\alpha_i\, \omega|_t\big)}{\omega_{\mathbb{R}^n}}\, d\lambda \\ &\le \int_{\phi(t)} \frac{(\phi^{-1})^*\big((f\omega)|_t\big)}{\omega_{\mathbb{R}^n}} d\lambda \\ &= \int_t f\, \omega\,. \end{aligned}$$

We will consider two cases:
(1) $+\infty \in R$,
(2) every element in R is finite.

We first prove the statement for the case when (1) holds. From (1), we have $\sup R = +\infty$. We show that then $\sup Q = +\infty$. Choose $M \in (0, +\infty)$. Since $+\infty \in R$ and $+\infty \neq 0$, as we saw before we have

$$+\infty = \int_S h \, d\lambda_\omega = \sum_{i=1}^{r} \alpha_i \int_{T_i} \omega \,,$$

where $h \in H^+(S)$ is nonzero, $h \leq f$, and the numbers r and $\{\alpha_i\}_{i=1}^r$ and subsets $\{T_i\}_{i=1}^r$ of S are defined as above. Then, there must be some $i \in \{1, \ldots, r\}$ such that $\int_{T_i} \omega = +\infty$, which allows us to choose a tuple $(t_1, \ldots, t_m) \in \Upsilon(T_i)$ such that

$$\sum_{j=1}^{m} \int_{t_j} \omega > \frac{M}{\alpha_i} \,.$$

Since $(t_1, \ldots, t_m) \in \Upsilon(S)$, we have

$$\begin{aligned} Q \ni \sum_{j=1}^{m} \int_{t_j} f \, \omega &\geq \sum_{j=1}^{m} \int_{t_j} \alpha_i \, \omega \\ &= \alpha_i \sum_{j=1}^{m} \int_{t_j} \omega \\ &> M \,. \end{aligned}$$

Thus, $\sup Q = +\infty = \sup R$. We now assume that (2) holds. Choose $\varepsilon \in (0, +\infty)$. We claim that for all $x \in R$, there exists $y \in Q$ such that $y > x - \varepsilon$. If $x = 0$, we may choose any $y \in Q \neq \emptyset$. Suppose then that $x \in R$ and $x > 0$. We may write

$$x = \int_S h \, d\lambda_\omega = \sum_{i=1}^{r} \alpha_i \int_{T_i} \omega$$

as before. For each $i \in \{1, \ldots, r\}$, we have $\int_{T_i} \omega < +\infty$, so we may choose $(t_{i1}, \ldots, t_{im_i}) \in \Upsilon(T_i)$ such that

$$\sum_{j=1}^{m_i} \int_{t_{ij}} \omega > \left(\int_{T_i} \omega \right) - \frac{\varepsilon}{r \, \alpha_i} \,,$$

so that

$$\alpha_i \sum_{j=1}^{m_i} \int_{t_{ij}} \omega > \left(\alpha_i \int_{T_i} \omega \right) - \frac{\varepsilon}{r} \,.$$

Then, since the collection $\{t_{ij} \mid \ i \in \{1, \ldots, r\} \text{ and } j \in \{1, \ldots, m_i\}\}$ is in $\Upsilon(S)$, we have

$$\begin{aligned} Q \ni \sum_{i=1}^{r} \sum_{j=1}^{m_i} \int_{t_{ij}} f \, \omega &\geq \sum_{i=1}^{r} \sum_{j=1}^{m_i} \int_{t_{ij}} \alpha_i \, \omega \\ &= \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{m_i} \int_{t_{ij}} \omega \\ &> \left(\sum_{i=1}^{r} \alpha_i \int_{T_i} \omega \right) - \varepsilon \\ &= x - \varepsilon \,. \end{aligned}$$

Thus, the claim is proved. It follows that no element $x \in R$ can fulfil $x > \sup Q$, since then we would be able to find $y \in Q$ with $y > \sup Q$. Thus, $\sup Q$ is an upper bound for R. To prove that $\sup R = \sup Q$, we will show that for all $y \in Q$ there exists a sequence in R converging to y. Choose $y \in Q$. Then,

$$y = \sum_{j=1}^{m} \int_{s_j} f\,\omega$$

for some $(s_1, \ldots, s_m) \in \Upsilon(S)$. For each $j \in \{1, \ldots, m\}$, we have $s_j \subset U_j$ for some positively oriented chart (U_j, ϕ_j) in M, so

$$\int_{s_j} f\,\omega = \int_{\phi_j(s_j)} \frac{(\phi_j^{-1})^*\big((f\omega)\big|_{s_j}\big)}{\omega_{\mathbb{R}^n}}\,d\lambda = \int_{\phi_j(s_j)} (f \circ \phi_j^{-1}) \frac{(\phi_j^{-1})^*\big(\omega\big|_{s_j}\big)}{\omega_{\mathbb{R}^n}}\,d\lambda\,.$$

Define

$$g_j := \frac{(\phi_j^{-1})^*\big(\omega\big|_{s_j}\big)}{\omega_{\mathbb{R}^n}} : \phi_j(s_j) \to [0, +\infty)\,.$$

Since $f \circ \phi_j^{-1}$ is a nonnegative measurable function on $\phi_j(s_j)$, there exists an increasing sequence $\{h_k^j\}_{k\in\mathbb{N}}$ of nonnegative simple functions on $\phi_j(s_j)$ converging to $f \circ \phi_j^{-1}$. Then, the sequence $\{h_k^j\, g_j\}_{k\in\mathbb{N}}$ is also increasing and it converges to $(f \circ \phi_j^{-1})\, g_j$, so by the Monotone Convergence Theorem,

$$\int_{\phi_j(s_j)} h_k^j\, g_j\, d\lambda \to \int_{\phi_j(s_j)} (f \circ \phi_j^{-1})\, g_j\, d\lambda = \int_{s_j} f\,\omega \quad \text{as } k \to \infty\,.$$

It follows that

$$c_k := \sum_{j=1}^{m} \int_{\phi_j(s_j)} h_k^j\, g_j\, d\lambda \to \sum_{j=1}^{m} \int_{s_j} f\,\omega = y \quad \text{as } k \to \infty\,.$$

We now fix $k \in \mathbb{N}$, and choose $j \in \{1, \ldots, m\}$. Write

$$h_k^j = \sum_{i=1}^{m_{jk}} \alpha_{jki}\, \chi_{T_{jki}}\,,$$

where $\alpha_{jk1}, \ldots, \alpha_{jkm_{jk}} \in [0, +\infty)$ are the finitely many values taken by h_k^j, and $T_{jki} := (h_k^j)^{-1}(\{\alpha_{jki}\})$ for each $i \in \{1, \ldots, m_{jk}\}$. Then,

$$\begin{aligned}
\int_{\phi_j(s_j)} h_k^j \, g_j \, d\lambda &= \int_{\phi_j(s_j)} \sum_{i=1}^{m_{jk}} \alpha_{jki} \, \chi_{T_{jki}} \, g_j \, d\lambda \\
&= \sum_{i=1}^{m_{jk}} \int_{\phi_j(s_j)} \alpha_{jki} \, \chi_{T_{jki}} \, g_j \, d\lambda \\
&= \sum_{i=1}^{m_{jk}} \int_{T_{jki}} \alpha_{jki} \, g_j\big|_{T_{jki}} \, d\lambda \\
&= \sum_{i=1}^{m_{jk}} \alpha_{jki} \int_{T_{jki}} \frac{(\phi_j^{-1})^*\big(\omega\big|_{s_j}\big)}{\omega_{\mathbb{R}^n}}\Bigg|_{T_{jki}} d\lambda \\
&= \sum_{i=1}^{m_{jk}} \alpha_{jki} \int_{\tilde{T}_{jki}} \omega \\
&= \sum_{i=1}^{m_{jk}} \alpha_{jki} \, \lambda_\omega(\tilde{T}_{jki}) \,,
\end{aligned}$$

where $\tilde{T}_{jki} := \phi^{-1}(T_{jki}) \subset s_j$ for each $i \in \{1, \ldots, m_{jk}\}$. Then,

$$c_k = \sum_{j=1}^{m} \int_{\phi_j(s_j)} h_k^j \, g_j \, d\lambda = \sum_{j=1}^{m} \sum_{i=1}^{m_{jk}} \alpha_{jki} \, \lambda_\omega(\tilde{T}_{jki}) = \int_S \eta_k \,,$$

where $\eta_k : S \to [0, +\infty)$ is the nonnegative simple function defined by

$$\eta_k := \sum_{j=1}^{m} \sum_{i=1}^{m_{jk}} \alpha_{jki} \, \chi_{\tilde{T}_{jki}} \,.$$

Moreover, since the sets $\{\tilde{T}_{jki}\}$ are mutually disjoint for $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, m_{jk}\}$, on each $\tilde{T}_{jki}$ we have

$$\eta_k\big|_{\tilde{T}_{jki}} = \alpha_{jki} = h_k^j \circ \phi_j\big|_{\tilde{T}_{jki}} \leq (f \circ \phi_j^{-1}) \circ \phi_j\big|_{\tilde{T}_{jki}} = f\big|_{\tilde{T}_{jki}} \,,$$

while η_k is zero at the points in S that are not in $\tilde{T}_{jki}$ for any $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, m_{jk}\}$. In conclusion, $\eta_k \leq f$ on S, which implies that

$$c_k = \int_S \eta_k \in R \,.$$

Since $k \in \mathbb{N}$ was arbitrary, the sequence $\{c_k\}_{k \in \mathbb{N}}$ is in R, as claimed. It is left to the reader to show that this implies that no real number strictly smaller than $\sup Q$ can be an upper bound for R, so that $\sup R = \sup Q$.

PHEEEEEEEEEEEEEWWW! □

Definition 2.73. Suppose M is oriented, $S \subset M$ is measurable, and ω is a measurable, real nonnegative n-form on S. We then call the measure λ_ω on S provided by Proposition 2.72 (i) the *measure associated to* ω.

Remark 2.74. Using notation from Proposition 2.72, suppose $T \subset S$ is a measurable set of measure 0 (Definition 2.43). Then, if $t \subset T$ and $t \subset U$ for some positively oriented chart (U, ϕ) in M, we have

$$\int_t \omega = \int_{\phi(t)} \frac{(\phi^{-1})^*\big(\omega|_t\big)}{\omega_{\mathbb{R}^n}}\, d\lambda = 0\,,$$

since $\lambda(\phi(t)) = 0$. Thus,

$$\lambda_\omega(T) = \int_T \omega = \sup_{(t_1,\dots,t_m)\in\Upsilon(T)} \sum_{j=1}^m \int_{t_j} \omega = 0\,.$$

Proposition 2.75. *Suppose M is oriented and $S \subset M$ is measurable.*

(i) *If ω is a measurable, real nonnegative n-form on S, and if $R \subset S$ is measurable, then*

$$\int_S \chi_R\, \omega = \int_R \omega\,,$$

where $\chi_R : S \to \mathbb{R}$ denotes the characteristic function on R, that is,

$$\chi_R(p) := \begin{cases} 1 & \text{if } p \in R \\ 0 & \text{if } p \in S \setminus R \end{cases} \quad , \quad p \in S\,.$$

(ii) *If ω and τ are measurable, real nonnegative n-forms on S, and if $c \in [0, +\infty)$, then*

(a)
$$\int_S (\omega + \tau) = \int_S \omega + \int_S \tau$$

and

(b)
$$\int_S (c\,\omega) = c\int_S \omega\,.$$

(iii) *If ω and τ are measurable, real nonnegative n-forms on S such that $\omega \geq \tau$, then*

(a)
$$\int_S \omega = \int_S (\omega - \tau) + \int_S \tau$$

and

(b)
$$\int_S \omega \geq \int_S \tau\,.$$

Moreover, if $\int_S \tau < +\infty$ then

(c)
$$\int_S \omega - \int_S \tau = \int_S (\omega - \tau)\,.$$

(iv) *If ω and τ are measurable, real nonnegative n-forms on S and $\omega = \tau$ almost everywhere in S, then*

$$\int_S \omega = \int_S \tau\,.$$

(v) *If ω is a measurable, real nonnegative n-form on S, then the set $Z := \{p \in S \,|\, \omega_p = 0\}$ is measurable. If τ is another measurable, real nonnegative n-form on S such that $\tau = 0$ almost everywhere in Z, then*

$$\int_S \tau = \int_{S\setminus Z} \frac{\tau}{\omega}\, \omega = \int_{S\setminus Z} \frac{\tau}{\omega}\, d\lambda_\omega\,.$$

Proof. (i) By Proposition 2.72 (ii),

$$\int_S \chi_R\,\omega = \int_S \chi_R\,d\lambda_\omega = \lambda_\omega(R) = \int_R \omega\,.$$

(ii) (a) We first show that the result is true when $S \subset U$ for some positively oriented chart (U,ϕ) in M. If this is the case, then

$$\begin{aligned}\int_S(\omega+\tau) &= \int_{\phi(S)} \frac{(\phi^{-1})^*(\omega+\tau)}{\omega_{\mathbb{R}^n}}d\lambda\\ &= \int_{\phi(S)}\left(\frac{(\phi^{-1})^*\omega}{\omega_{\mathbb{R}^n}} + \frac{(\phi^{-1})^*\tau}{\omega_{\mathbb{R}^n}}\right)d\lambda\\ &= \int_{\phi(S)}\frac{(\phi^{-1})^*\omega}{\omega_{\mathbb{R}^n}}\,d\lambda + \int_{\phi(S)}\frac{(\phi^{-1})^*\tau}{\omega_{\mathbb{R}^n}}\,d\lambda\\ &= \int_S\omega + \int_S\tau\,.\end{aligned}$$

We now prove the general case. Write

$$\int_S(\omega+\tau) = \sup P\,,\qquad \int_S\omega = \sup Q_1\,,\qquad \int_S\tau = \sup Q_2\,,$$

for

$$\begin{aligned}P &:= \left\{\sum_{j=1}^m\int_{s_j}(\omega+\tau)\,\middle|\,(s_1,\ldots,s_m)\in\Upsilon(S)\right\},\\ Q_1 &:= \left\{\sum_{j=1}^m\int_{s_j}\omega\,\middle|\,(s_1,\ldots,s_m)\in\Upsilon(S)\right\},\\ Q_2 &:= \left\{\sum_{j=1}^m\int_{s_j}\tau\,\middle|\,(s_1,\ldots,s_m)\in\Upsilon(S)\right\}.\end{aligned}$$

We claim that
(1) For every $x\in P$ there exist $y_1\in Q_1$ and $y_2\in Q_2$ such that $x = y_1+y_2$;
(2) For every $y_1\in Q_1$ and $y_2\in Q_2$ there exists $x\in P$ such that $x\geq y_1+y_2$.
For (1), choosing $x=\sum_{j=1}^m\int_{s_j}(\omega+\tau)\in P$ for some $(s_1,\ldots,s_m)\in\Upsilon(S)$, we have

$$x = \sum_{j=1}^m\int_{s_j}(\omega+\tau) = \sum_{j=1}^m\left(\int_{s_j}\omega + \int_{s_j}\tau\right) = y_1+y_2\,,$$

where

$$y_1 := \sum_{j=1}^m\int_{s_j}\omega\in Q_1 \quad\text{and}\quad y_2 := \sum_{j=1}^m\int_{s_j}\tau\in Q_2\,.$$

To prove (2), choose elements $y_1 = \sum_{j=1}^{m_1}\int_{r_j}\omega\in Q_1$ and $y_2=\sum_{i=1}^{m_2}\int_{t_i}\tau\in Q_2$, for $(r_1,\ldots,r_{m_1}),(t_1,\ldots,t_{m_2})\in\Upsilon(S)$. For each $j\in\{1,\ldots,m_1\}$ and $i\in\{1,\ldots,m_2\}$, define

$$a_j := r_j\setminus\left(\bigcup_{k=1}^{m_2}t_k\right),\quad b_{ij} := r_j\cap t_i\,,\quad c_i := t_i\setminus\left(\bigcup_{k=1}^{m_1}r_k\right).$$

Then, the collection

$$\mathscr{C} := \{a_j\}_{j=1}^{m_1} \cup \{b_{ij}\}_{\substack{j\in\{1,\dots,m_1\}\\ i\in\{1,\dots,m_2\}}} \cup \{c_i\}_{i=1}^{m_2}$$

is in $\Upsilon(S)$. We then have

$$\begin{aligned}
P &\ni \sum_{j=1}^{m_1}\int_{a_j}(\omega+\tau) + \sum_{j=1}^{m_1}\sum_{i=1}^{m_2}\int_{b_{ij}}(\omega+\tau) + \sum_{i=1}^{m_2}\int_{c_i}(\omega+\tau)\\
&= \sum_{j=1}^{m_1}\left(\int_{a_j}\omega + \sum_{i=1}^{m_2}\int_{b_{ij}}\omega\right) + \sum_{i=1}^{m_2}\left(\int_{c_i}\tau + \sum_{j=1}^{m_1}\int_{b_{ij}}\tau\right) + \sum_{j=1}^{m_1}\int_{a_j}\tau + \sum_{i=1}^{m_2}\int_{c_i}\omega\\
&= \delta_1+\delta_2+R\,,
\end{aligned}$$

where

$$\begin{aligned}
\delta_1 &:= \sum_{j=1}^{m_1}\left(\int_{a_j}\omega + \sum_{i=1}^{m_2}\int_{b_{ij}}\omega\right),\\
\delta_2 &:= \sum_{i=1}^{m_2}\left(\int_{c_i}\tau + \sum_{j=1}^{m_1}\int_{b_{ij}}\tau\right),\\
R &= \sum_{j=1}^{m_1}\int_{a_j}\tau + \sum_{i=1}^{m_2}\int_{c_i}\omega \in [0,+\infty]\,.
\end{aligned}$$

We show that $\delta_1 = y_1$ and $\delta_2 = y_2$. For each $j \in \{1,\dots,m_1\}$, we have $r_j = a_j \cup \bigcup_{i=1}^{m_2} b_{ij}$, so by (i),

$$\begin{aligned}
\int_{a_j}\omega + \sum_{i=1}^{m_2}\int_{b_{ij}}\omega &= \int_{r_j}\chi_{a_j}\,\omega + \sum_{i=1}^{m_2}\int_{r_j}\chi_{b_{ij}}\,\omega\\
&= \int_{r_j}\left(\chi_{a_j}\,\omega + \sum_{i=1}^{m_2}\chi_{b_{ij}}\,\omega\right)\\
&= \int_{r_j}\omega\,.
\end{aligned}$$

Thus,

$$\delta_1 = \sum_{j=1}^{m_1}\left(\int_{a_j}\omega + \sum_{i=1}^{m_2}\int_{b_{ij}}\omega\right) = \sum_{j=1}^{m_1}\int_{r_j}\omega = y_1\,.$$

The proof that $\delta_2 = y_2$ is similar. Then,

$$P \ni \delta_1+\delta_2+R = y_1+y_2+R \geq y_1+y_2\,,$$

so (2) is proved. It is left to the reader to show that (1) and (2) imply that

$$\sup P = \sup Q_1 + \sup Q_2\,.$$

(b) Suppose first that $S \subset U$ for some positively oriented chart (U,ϕ) in M. Then,

$$\int_S c\,\omega = \int_{\phi(S)}\frac{(\phi^{-1})^*(c\,\omega)}{\omega_{\mathbb{R}^n}}d\lambda = \int_{\phi(S)}c\,\frac{(\phi^{-1})^*\omega}{\omega_{\mathbb{R}^n}}d\lambda = c\int_{\phi(S)}\frac{(\phi^{-1})^*\omega}{\omega_{\mathbb{R}^n}}d\lambda = c\int_S\omega\,.$$

For the general case, write

$$\int_S\omega = \sup P \quad\text{and}\quad \int_S c\,\omega = \sup Q\,,$$

for

$$P := \left\{ \sum_{j=1}^{m} \int_{s_j} \omega \,\middle|\, (s_1, \ldots, s_m) \in \Upsilon(S) \right\},$$

$$Q := \left\{ \sum_{j=1}^{m} \int_{s_j} c\,\omega \,\middle|\, (s_1, \ldots, s_m) \in \Upsilon(S) \right\}.$$

Then, we have $Q = \{c\,x \mid x \in P\}$, from which it follows that $\sup Q = c \sup P$.

(iii) (a) Since $\omega \geq \tau$, the measurable n-form $\omega - \tau$ is nonnegative on S. The result then follows from (ii)(a).

(b) Follows from (iii)(a).

(c) Follows from (iii)(a).

(iv) Since $\omega = \tau$ almost everywhere in S, there exists a measurable subset $A \subset S$ of measure 0 such that $\omega_p = \tau_p$ for all $p \in S \setminus A$. Then, the measurable nonnegative n-forms $\chi_{S\setminus A}\,\omega$ and $\chi_{S\setminus A}\,\tau$ are equal on S, so by (ii)(a) and Remark 2.74,

$$\int_S \omega = \int_S (\chi_A\,\omega + \chi_{S\setminus A}\,\omega) = \int_A \omega + \int_S \chi_{S\setminus A}\,\omega = \int_S \chi_{S\setminus A}\,\omega$$

and

$$\int_S \tau = \int_S (\chi_A\,\tau + \chi_{S\setminus A}\,\tau) = \int_A \tau + \int_S \chi_{S\setminus A}\,\tau = \int_S \chi_{S\setminus A}\,\tau\,.$$

Thus,

$$\int_S \omega = \int_S \tau\,.$$

(v) We first show that Z is measurable. Choose any chart (U, ϕ) in M. Since ω is measurable, on $U \cap S$ we have $\omega = (\omega/\omega_\phi)\,\omega_\phi$, where the function $\omega/\omega_\phi : U \cap S \to \mathbb{R}$ is measurable. Then,

$$U \cap Z = \{p \in U \cap S \mid \omega_p = 0\} = \left(\frac{\omega}{\omega_\phi}\right)^{-1}(\{0\})\,,$$

which is a measurable subset of $U \cap S$ and hence a measurable subset of M. Thus, $\phi(U \cap Z)$ is measurable in $\mathbb{R}^n$, so Z is measurable. Since $\tau = 0$ almost everywhere in Z, by (iv) we have $\int_Z \tau = \int_Z 0 = 0$, so

$$\int_S \tau = \int_S (\chi_Z\,\tau + \chi_{S\setminus Z}\,\tau) = \int_Z \tau + \int_{S\setminus Z} \tau = \int_{S\setminus Z} \frac{\tau}{\omega}\,\omega = \int_{S\setminus Z} \frac{\tau}{\omega}\,d\lambda_\omega\,.$$

$\square$

Proposition 2.76. *Suppose M is oriented and $S \subset M$ is measurable.*

(i) *If ω is an integrable measurable n-form over $\mathbb{F}$ on S, and if $R \subset S$ is measurable, then the measurable n-form $\chi_R\,\omega$ is integrable on S, the restriction $\omega|_R$ is integrable on R, and*

$$\int_S \chi_R\,\omega = \int_R \omega\,.$$

(ii) *If ω and τ are integrable measurable n-forms over $\mathbb{F}$ on S and $c \in \mathbb{F}$, then*

(a) *the n-form $\omega + \tau$ on S is integrable and*

$$\int_S (\omega + \tau) = \int_S \omega + \int_S \tau\,;$$

(b) *the n-form $c\,\omega$ on S is integrable and*

$$\int_S (c\,\omega) = c\int_S \omega\,.$$

(iii) *If ω and τ are integrable measurable real n-forms on S such that $\omega \geq \tau$, then*

$$\int_S \omega \geq \int_S \tau\,.$$

(iv) *If ω and τ are integrable measurable n-forms over $\mathbb{F}$ on S such that $\omega = \tau$ almost everywhere in S, then*

$$\int_S \omega = \int_S \tau\,.$$

Proof. (i) Suppose first that $\mathbb{F} = \mathbb{R}$. On S,

$$(\chi_R\,\omega)^+ = \chi_R\,\omega^+ \leq \omega^+ \quad \text{and} \quad (\chi_R\,\omega)^- = \chi_R\,\omega^- \leq \omega^-\,,$$

so by integrability of ω,

$$\int_S (\chi_R\,\omega)^+ \leq \int_S \omega^+ < +\infty \quad \text{and} \quad \int_S (\chi_R\,\omega)^- \leq \int_S \omega^- < +\infty.$$

Thus, $\chi_R\,\omega$ is integrable on S. Moreover, on R we have

$$\left(\omega|_R\right)^+ = \omega^+|_R \quad \text{and} \quad \left(\omega|_R\right)^- = \omega^-|_R\,,$$

so

$$\int_R \left(\omega|_R\right)^+ = \int_R \omega^+|_R = \int_S \chi_R\,\omega^+ = \int_S (\chi_R\,\omega)^+$$

and

$$\int_R \left(\omega|_R\right)^- = \int_R \omega^-|_R = \int_S \chi_R\,\omega^- = \int_S (\chi_R\,\omega)^-\,.$$

Thus, $\omega|_R$ is integrable on R and

$$\int_S \chi_R\,\omega = \int_S (\chi_R\,\omega)^+ - \int_S (\chi_R\,\omega)^- = \int_R \left(\omega|_R\right)^+ - \int_R \left(\omega|_R\right)^- = \int_R \omega|_R\,.$$

If $\mathbb{F} = \mathbb{C}$, then the measurable real n-forms $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ on S are integrable on S, and on S we have

$$\mathrm{Re}(\chi_R\,\omega) = \chi_R\,\mathrm{Re}(\omega) \quad \text{and} \quad \mathrm{Im}(\chi_R\,\omega) = \chi_R\,\mathrm{Im}(\omega)\,.$$

Thus, by the real case, $\mathrm{Re}(\chi_R\,\omega)$ and $\mathrm{Im}(\chi_R\,\omega)$ are integrable on S, which implies that $\chi_R\,\omega$ is integrable on S. Moreover, on R we have

$$\mathrm{Re}\left(\omega|_R\right) = (\mathrm{Re}(\omega))|_R \quad \text{and} \quad \mathrm{Im}\left(\omega|_R\right) = (\mathrm{Im}(\omega))|_R\,,$$

so, again by the real case, $\operatorname{Re}\left(\omega|_R\right)$ and $\operatorname{Im}\left(\omega|_R\right)$ are both integrable on R, which implies that $\omega|_R$ is integrable on R. Then,

$$\begin{aligned}\int_S \chi_R\,\omega &= \int_S \operatorname{Re}(\chi_R\,\omega) + i\int_S \operatorname{Im}(\chi_R\,\omega)\\ &= \int_S \chi_R \operatorname{Re}(\omega) + i\int_S \chi_R \operatorname{Im}(\omega)\\ &= \int_R (\operatorname{Re}(\omega))|_R + i\int_R (\operatorname{Im}(\omega))|_R\\ &= \int_R \operatorname{Re}\left(\omega|_R\right) + i\int_R \operatorname{Im}\left(\omega|_R\right)\\ &= \int_R \omega|_R\,.\end{aligned}$$

(ii) (a) Suppose first that $\mathbb{F}=\mathbb{R}$. Let $p\in S$. If $(\omega+\tau)_p \geq 0$, then

$$\begin{aligned}(\omega+\tau)_p^+ = (\omega+\tau)_p = \omega_p+\tau_p &= \omega_p^+ - \omega_p^- + \tau_p^+ - \tau_p^-\\ &\leq \omega_p^+ + \tau_p^+ = (\omega^+ + \tau^+)_p\end{aligned}$$

and

$$(\omega+\tau)_p^- = 0 \leq (\omega^- + \tau^-)_p\,,$$

while if $(\omega+\tau)_p < 0$, then

$$(\omega+\tau)_p^+ = 0 \leq (\omega^+ + \tau^+)_p$$

and

$$\begin{aligned}(\omega+\tau)_p^- = -(\omega+\tau)_p = -\omega_p-\tau_p &= -\omega_p^+ + \omega_p^- - \tau_p^+ + \tau_p^-\\ &\leq \omega_p^- + \tau_p^- = (\omega^- + \tau^-)_p\,.\end{aligned}$$

Thus, on S we have

$$(\omega+\tau)^+ \leq \omega^+ + \tau^+ \quad \text{and} \quad (\omega+\tau)^- \leq \omega^- + \tau^-\,,$$

so

$$\int_S (\omega+\tau)^+ \leq \int_S \omega^+ + \int_S \tau^+ < +\infty \quad \text{and} \quad \int_S (\omega+\tau)^- \leq \int_S \omega^- + \int_S \tau^- < +\infty\,.$$

Thus, $\omega+\tau$ is integrable. Moreover, on S,

$$(\omega+\tau)^+ - (\omega+\tau)^- = \omega+\tau = \omega^+ - \omega^- + \tau^+ - \tau^-\,,$$

so

$$\omega^+ + \tau^+ - (\omega+\tau)^+ = \omega^- + \tau^- - (\omega+\tau)^-\,.$$

Then, by Proposition 2.75,

$$\begin{aligned}\int_S \omega^+ + \int_S \tau^+ - \int_S (\omega+\tau)^+ &= \int_S (\omega^+ + \tau^+) - \int_S (\omega+\tau)^+ \\ &= \int_S (\omega^+ + \tau^+ - (\omega+\tau)^+) \\ &= \int_S (\omega^- + \tau^- - (\omega+\tau)^-) \\ &= \int_S (\omega^- + \tau^-) - \int_S (\omega+\tau)^- \\ &= \int_S \omega^- + \int_S \tau^- - \int_S (\omega+\tau)^- \,,\end{aligned}$$

so that

$$\begin{aligned}\int_S (\omega+\tau) &= \int_S (\omega+\tau)^+ - \int_S (\omega+\tau)^- \\ &= \int_S \omega^+ - \int_S \omega^- + \int_S \tau^+ - \int_S \tau^- \\ &= \int_S \omega + \int_S \tau \,.\end{aligned}$$

The proof of the case when $\mathbb{F} = \mathbb{C}$ is left to the reader.

(b) Suppose first that $\mathbb{F} = \mathbb{R}$. If $c = 0$, the result is immediate. If $c > 0$, then on S

$$(c\,\omega)^+ = c\,\omega^+ \quad \text{and} \quad (c\,\omega)^- = c\,\omega^- \,,$$

so that

$$\int_S (c\,\omega)^+ = \int_S c\,\omega^+ = c\int_S \omega^+ < +\infty$$

and

$$\int_S (c\,\omega)^- = \int_S c\,\omega^- = c\int_S \omega^- < +\infty \,.$$

Thus, $c\,\omega$ is integrable and

$$\int_S c\,\omega = \int_S (c\,\omega)^+ - \int_S (c\,\omega)^- = c\left(\int_S \omega^+ - \int_S \omega^-\right) = c\int_S \omega \,.$$

If $c < 0$, then

$$(c\,\omega)^+ = -c\,\omega^- \quad \text{and} \quad (c\,\omega)^- = -c\,\omega^+ \,,$$

so

$$\int_S (c\,\omega)^+ = \int_S -c\,\omega^- = -c\int_S \omega^- < +\infty$$

and

$$\int_S (c\,\omega)^- = \int_S -c\,\omega^+ = -c\int_S \omega^+ < +\infty \,.$$

Thus, again $c\,\omega$ is integrable and

$$\int_S c\,\omega = \int_S (c\,\omega)^+ - \int_S (c\,\omega)^- = -c\left(\int_S \omega^- - \int_S \omega^+\right) = c\int_S \omega \,.$$

If $\mathbb{F} = \mathbb{C}$, we have

$$\operatorname{Re}(c\,\omega) = \operatorname{Re}(c)\operatorname{Re}(\omega) - \operatorname{Im}(c)\operatorname{Im}(\omega)$$

and

$$\operatorname{Im}(c\,\omega) = \operatorname{Re}(c)\operatorname{Im}(\omega) + \operatorname{Im}(c)\operatorname{Re}(\omega)\,,$$

so $\operatorname{Re}(c\,\omega)$ and $\operatorname{Im}(c\,\omega)$ are integrable and

$$\begin{aligned}
\int_S c\,\omega &= \int_S \operatorname{Re}(c\,\omega) + i\int_S \operatorname{Im}(c\,\omega) \\
&= \operatorname{Re}(c)\int_S \operatorname{Re}(\omega) - \operatorname{Im}(c)\int_S \operatorname{Im}(\omega) + i\Big(\operatorname{Re}(c)\int_S \operatorname{Im}(\omega) + \operatorname{Im}(c)\int_S \operatorname{Re}(\omega)\Big) \\
&= c\left(\int_S \operatorname{Re}(\omega) + i\int_S \operatorname{Im}(\omega)\right) \\
&= c\int_S \omega\,.
\end{aligned}$$

(iii) Since $\omega \geq \tau$, we have

$$\omega^+ \geq \tau^+ \quad \text{and} \quad \omega^- \leq \tau^-\,,$$

so

$$\int_S \omega = \int_S \omega^+ - \int_S \omega^- \geq \int_S \tau^+ - \int_S \tau^- = \int_S \tau\,.$$

(iv) Suppose first that $\mathbb{F} = \mathbb{R}$. Since $\omega = \tau$ almost everywhere in S, we have $\omega^+ = \tau^+$ almost everywhere in S and $\omega^- = \tau^-$ almost everywhere in S, so

$$\int_S \omega = \int_S \omega^+ - \int_S \omega^- = \int_S \tau^+ - \int_S \tau^- = \int_S \tau\,.$$

If $\mathbb{F} = \mathbb{C}$, then $\operatorname{Re}(\omega) = \operatorname{Re}(\tau)$ almost everywhere in S and $\operatorname{Im}(\omega) = \operatorname{Im}(\tau)$ almost everywhere in S, so the result follows from the real case.

□

Lemma 2.77. *Suppose M is oriented, $S \subset M$ is a compact subset, and ω is a continuous n-form over $\mathbb{F}$ on S. Then, S is measurable and ω is measurable and integrable on S.*

Proof. Since S is compact and M is Hausdorff, S is closed and hence measurable. Moreover, since ω is continuous on S, it is measurable. Suppose first that $\mathbb{F} = \mathbb{R}$. Since ω is continuous on S, so too are ω^+ and ω^-. For each point $p \in S$, choose a positively oriented chart (U_p, ϕ_p) about p in M. Since M is locally compact, we may also choose a neighbourhood V_p of p in M such that the closure $\overline{V_p}$ of V_p in M is compact and $\overline{V_p} \subset U_p$. Since S is compact and the collection $\{V_p\}_{p\in S}$ covers S, we may find a finite subcover $\{V_{p_j}\}_{j=1}^m$. Then, $S \subset \bigcup_{j=1}^m \overline{V_{p_j}}$, so $S = \bigcup_{j=1}^m (S \cap \overline{V_{p_j}})$. This implies that on S,

$$\omega^+ \leq \sum_{j=1}^m \chi_{S\cap\overline{V_{p_j}}}\,\omega^+\,,$$

so

$$\int_S \omega^+ \leq \int_S \sum_{j=1}^m \chi_{S\cap\overline{V_{p_j}}}\,\omega^+ = \sum_{j=1}^m \int_{S\cap\overline{V_{p_j}}} \omega^+ = \sum_{j=1}^m \int_{\phi_{p_j}(S\cap\overline{V_{p_j}})} \frac{(\phi_{p_j}^{-1})^*\omega^+}{\omega_{\mathbb{R}^2}}\,d\lambda\,.$$

For each $j \in \{1,\dots,m\}$, the set $S \cap \overline{V_{p_j}}$ is compact, so $\phi_{p_j}(S \cap \overline{V_{p_j}})$ is compact in $\mathbb{R}^2$. Moreover, since ω^+ is continuous on $S \cap \overline{V_{p_j}}$, the function $((\phi_{p_j}^{-1})^*\omega^+)/\omega_{\mathbb{R}^2}$ is continuous on $\phi_{p_j}(S \cap \overline{V_{p_j}})$. In conclusion,

$$\int_{\phi_{p_j}(S\cap\overline{V_{p_j}})} \frac{(\phi_{p_j}^{-1})^*\omega^+}{\omega_{\mathbb{R}^2}}\, d\lambda < +\infty$$

for all $j \in \{1,\dots,m\}$, so $\int_S \omega^+ < +\infty$. By a similar argument, $\int_S \omega^- < +\infty$, which shows that ω is integrable on S. If $\mathbb{F} = \mathbb{C}$, then the result follows from the real case: $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ are continuous real n-forms on S and hence integrable on S, so ω is integrable on S. $\square$

Remark 2.78. Recall that if M has dimension $n \in \{1,2\}$ and $\gamma : [a,b] \to M$ is a piecewise C^1 path, then the line integral along γ of a continuous (real or complex) 1-form α on M is given by

$$\int_\gamma \alpha = \sum_{k=1}^{m} \int_{s_{k-1}}^{s_k} \alpha_{\gamma(t)}(\dot\gamma_k(t))\, dt\,,$$

where $a = s_0 < \cdots < s_m = b$ is a partition of $[a,b]$ such that for each $k \in \{1,\dots,m\}$ the restriction $\gamma_k := \gamma|_{[s_{k-1},s_k]}$ is a C^1 path. Fixing some $k \in \{1,\dots,m\}$, and letting now $\gamma_k := \gamma|_{(s_{k-1},s_k)}$, for each $s \in (s_{k-1}, s_k)$ we have

$$\begin{aligned}
(\gamma_k^*\alpha)_s &= \left[(\gamma_k^*\alpha)_s\left(\left.\frac{d}{dt}\right|_s\right)\right](dt)_s \\
&= \left[\alpha_{\gamma_k(s)}\left((\gamma_k)_*\left(\left.\frac{d}{dt}\right|_s\right)\right)\right](dt)_s \\
&= \alpha_{\gamma(s)}(\dot\gamma_k(s))(dt)_s\,.
\end{aligned}$$

Since the function $s \mapsto \alpha_{\gamma(s)}(\dot\gamma_k(s))$ on $[s_{k-1}, s_k]$ is Riemann integrable, it is also Lebesgue integrable. Then, assuming first that α is real,

$$\begin{aligned}
+\infty &> \int_{[s_{k-1},s_k]} (\alpha_{\gamma(s)}(\dot\gamma_k(s)))^{\pm}\, d\lambda(s) \\
&= \int_{(s_{k-1},s_k)} (\alpha_{\gamma(s)}(\dot\gamma_k(s)))^{\pm}\, d\lambda(s) \\
&= \int_{(s_{k-1},s_k)} (\gamma_k^*\alpha)^{\pm}\,.
\end{aligned}$$

Thus, the continuous 1-form $\gamma_k^*\alpha$ is integrable on (s_{k-1}, s_k) and

$$\int_{(s_{k-1},s_k)} \gamma_k^*\alpha = \int_{s_{k-1}}^{s_k} \alpha_{\gamma(s)}(\dot\gamma_k(s))\, ds\,.$$

If α is complex, then for each $s \in (s_{k-1}, s_k)$,

$$(\mathrm{Re}\,\gamma_k^*\alpha)_s = \mathrm{Re}\left(\alpha_{\gamma(s)}(\dot\gamma_k(s))\right)(dt)_s \quad\text{and}\quad (\mathrm{Im}\,\gamma_k^*\alpha)_s = \mathrm{Im}\left(\alpha_{\gamma(s)}(\dot\gamma_k(s))\right)(dt)_s\,.$$

Then, reasoning as in the case when α is real, we conclude that $\mathrm{Re}(\gamma_k^*\alpha)$ and $\mathrm{Im}(\gamma_k^*\alpha)$ are integrable on (s_{k-1}, s_k), so that $\gamma_k^*\alpha$ is integrable on (s_{k-1}, s_k), and

$$\int_{(s_{k-1},s_k)} \gamma_k^*\alpha = \int_{s_{k-1}}^{s_k} \alpha_{\gamma(s)}(\dot\gamma_k(s))\, ds\,.$$

In conclusion, for real or complex α, we have

$$\int_\gamma \alpha = \sum_{k=1}^m \int_{(s_{k-1},s_k)} \gamma_k^* \alpha\,.$$

Lemma 2.79. *Suppose $\gamma : [a,b] \to M$ is a piecewise C^1 path and α is a continuous 1-form over $\mathbb{F}$ on M. Suppose also that the set $N := \gamma((a,b)) \subset M$ has a 1-dimensional smooth manifold structure such that the inclusion map $\iota : N \to M$ is C^∞, and that the map $\tilde\gamma : (a,b) \to N$, $t \mapsto \gamma(t) \in N$, is a diffeomorphism. If we give N the orientation induced from the diffeomorphism $\tilde\gamma$, then the 1-form $\iota^*\alpha$ is integrable on N and*

$$\int_N \iota^*\alpha = \int_\gamma \alpha\,.$$

Proof. First note that the restriction $\gamma|_{(a,b)} : (a,b) \to M$ may be written as the composition $\iota \circ \tilde\gamma$. Then, since ι and $\tilde\gamma$ are both C^∞, the restriction $\gamma|_{(a,b)}$ is C^∞. It then follows from the fact that γ is piecewise C^1 on $[a,b]$ and C^1 on (a,b) that γ is actually C^1 on $[a,b]$ (that is, it has a C^1 extension). Then, by Remark 2.78, the continuous 1-form $\gamma|_{(a,b)}^*\alpha$ on (a,b) is integrable and

$$\int_\gamma \alpha = \int_{(a,b)} \gamma|_{(a,b)}^* \alpha\,.$$

Since the map $\tilde\gamma : (a,b) \to N$ is a diffeomorphism, the pair $(N, \tilde\gamma^{-1})$ is a chart in N, and it is positively oriented. Suppose first that $\mathbb{F} = \mathbb{R}$. Then, on N we have $\iota^*\alpha = f\,d\tilde\gamma^{-1}$ for $f := \iota^*\alpha/d\tilde\gamma^{-1} : N \to \mathbb{R}$, so

$$\int_N (\iota^*\alpha)^\pm = \int_N f^\pm\,d\tilde\gamma^{-1} = \int_{(a,b)} (f^\pm \circ \tilde\gamma)\,d\lambda = \int_{(a,b)} (f \circ \tilde\gamma)^\pm\,d\lambda\,.$$

Moreover, on (a,b) we have

$$\gamma|_{(a,b)}^*\alpha = (\iota \circ \tilde\gamma)^*\alpha = \tilde\gamma^*(\iota^*\alpha) = \tilde\gamma^*(f\,d\tilde\gamma^{-1}) = (f \circ \tilde\gamma)\,dt\,,$$

so

$$+\infty > \int_{(a,b)} (\gamma|_{(a,b)}^*\alpha)^\pm = \int_{(a,b)} (f \circ \tilde\gamma)^\pm\,d\lambda = \int_N (\iota^*\alpha)^\pm\,.$$

Thus, $\iota^*\alpha$ is integrable on N and

$$\int_N \iota^*\alpha = \int_{(a,b)} \gamma|_{(a,b)}^*\alpha = \int_\gamma \alpha\,.$$

Suppose now that $\mathbb{F} = \mathbb{C}$. Then, the real 1-forms $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ on M are continuous, so $\iota^*\,\mathrm{Re}(\alpha)$ and $\iota^*\,\mathrm{Im}(\alpha)$ are integrable on N and we have

$$\int_N \iota^*\,\mathrm{Re}(\alpha) = \int_\gamma \mathrm{Re}(\alpha) \quad \text{and} \quad \int_N \iota^*\,\mathrm{Im}(\alpha) = \int_\gamma \mathrm{Im}(\alpha)\,.$$

Moreover, on N we have $\iota^* \operatorname{Re}(\alpha) = \operatorname{Re}(\iota^*\alpha)$ and $\iota^* \operatorname{Im}(\alpha) = \operatorname{Im}(\iota^*\alpha)$, so $\iota^*\alpha$ is integrable on N and

$$\begin{aligned}
\int_N \iota^*\alpha &= \int_N \operatorname{Re}(\iota^*\alpha) + i \int_N \operatorname{Im}(\iota^*\alpha) \\
&= \int_\gamma \operatorname{Re}(\alpha) + i \int_\gamma \operatorname{Im}(\alpha) \\
&= \int_{(a,b)} \gamma|_{(a,b)}^* \operatorname{Re}(\alpha) + i \int_{(a,b)} \gamma|_{(a,b)}^* \operatorname{Im}(\alpha) \\
&= \int_{(a,b)} \operatorname{Re}(\gamma|_{(a,b)}^* \alpha) + i \int_{(a,b)} \operatorname{Im}(\gamma|_{(a,b)}^* \alpha) \\
&= \int_{(a,b)} \gamma|_{(a,b)}^* \alpha \\
&= \int_\gamma \alpha\,.
\end{aligned}$$

□

Definition 2.80. Suppose M has dimension 2 and $\Omega \subset M$ is open. Then, Ω is called *smooth*, or C^∞, if for all $p \in M$ there exists a chart $(U, \phi) = (U, x, y)$ about p in M such that

$$U \cap \Omega = \{q \in U \,|\, x(q) < 0\} = \phi^{-1}\big(\{(r,s) \in \phi(U) \,|\, r < 0\}\big)\,.$$

Proposition 2.81. *Suppose M has dimension 2 and $\Omega \subset M$ is a smooth open set. Then, the topological boundary $\partial\Omega$ of Ω is either empty or a 1-dimensional smooth submanifold of M.*

Proof. Suppose $\partial\Omega$ is nonempty. For each $p \in \partial\Omega$, we may choose a chart (U, ϕ) about p in M such that $U \cap \Omega = \{q \in U \,|\, x(q) < 0\}$. Let $A := \{q \in U \,|\, x(q) = 0\}$. We show that $U \cap \partial\Omega = A$. If $q \in A$ and $W \subset M$ is a neighbourhood of q, then $\phi(U \cap W)$ is open in $\phi(U) \subset \mathbb{R}^2$ and $\phi(q) = (0, y(q)) \in \phi(U \cap W)$, so there exists $\varepsilon \in (0, +\infty)$ such that $a := (-\varepsilon, y(q)) \in \phi(U \cap W)$. Then, $\phi^{-1}(a) \in U \cap W$ and $x(\phi^{-1}(a)) < 0$, so $\phi^{-1}(a) \in \Omega \cap W$. Thus, q is not an exterior point of Ω, and since $q \notin \Omega$, we must have $q \in \partial\Omega$. Since $q \in A \subset U$, we have $q \in U \cap \partial\Omega$, which shows that $A \subset U \cap \partial\Omega$. To show the opposite inclusion, observe that if $q \in U \setminus A$, then either $x(q) < 0$, in which case $q \in \Omega \subset (\partial\Omega)^c$, or $x(q) > 0$. If the latter holds, then denoting by $R \subset \mathbb{R}^2$ the open right half-plane, we have $\phi(q) \in \phi(U) \cap R$, so $B := \phi^{-1}(\phi(U) \cap R) \subset U$ is a neighbourhood of q such that for all $r \in B$, $x(r) > 0$. Thus, $B \subset \Omega^c$, so q is an exterior point of Ω. In conclusion, if $q \in \partial\Omega$ then we must have $q \in A$. Thus, $U \cap \partial\Omega = \{q \in U \,|\, x(q) = 0\}$. Then, since $\partial\Omega$ is closed in M, it is a smooth 1-dimensional submanifold of M. □

Lemma 2.82. *Suppose M has dimension 2 and is oriented, and let $\Omega \subset M$ be a smooth open subset such that $\partial\Omega \neq \emptyset$.*

(i) *The collection $\mathfrak{U}$ of all positively oriented charts $(U, \phi) = (U, x, y)$ in M such that $U \cap \Omega = \{q \in U \,|\, x(q) < 0\}$ covers M.*
(ii) *The collection $\mathfrak{V} := \big\{\big(U \cap \partial\Omega, y \circ \big(\iota|_{U\cap\partial\Omega}\big)\big) \,|\, (U, \phi) = (U, x, y) \in \mathfrak{U}\big\}$ of charts in $\partial\Omega$ induced from $\mathfrak{U}$, where $\iota : \partial\Omega \to M$ is the inclusion map, is an oriented atlas in $\partial\Omega$ and hence defines an orientation on $\partial\Omega$.*

Proof. (i) First note that if $(U,\phi)=(U,x,y)$ is a chart in M such that $U\cap\Omega=\{q\in U\,|\,x(q)<0\}$ and $U'\subset U$ is an open subset, then the chart $(U',\phi|_{U'})=(U',x|_{U'},y|_{U'})$ also fulfils $U'\cap\Omega=\{q\in U'\,|\,x|_{U'}(q)<0\}$. For each $p\in M$, choose a chart $(U_p,\phi_p)=(U_p,x_p,y_p)$ about p in M such that $U_p\cap\Omega=\{q\in U_p\,|\,x_p(q)<0\}$. Then, without loss of generality, we may assume that $U_p\subset V_p$ for some positively oriented chart (V_p,ψ_p) about p, and that U_p is connected. Then, the transition map

$$F:=\psi_p\circ\phi_p^{-1}:\phi_p(U_p\cap V_p)=\phi_p(U_p)\to\psi_p(U_p)=\psi_p(U_p\cap V_p)$$

has either $\mathcal{J}_F>0$ everywhere or $\mathcal{J}_F<0$ everywhere. If the latter holds, we may redefine ϕ_p by changing the sign of y_p, so that $\mathcal{J}_F>0$ everywhere in $\phi_p(U_p)$ and the redefined chart (U_p,ϕ_p) still fulfils $U_p\cap\Omega=\{q\in U_p\,|\,x_p(q)<0\}$. We show that the chart (U_p,ϕ_p) is positively oriented. If (W,γ) is any positively oriented chart in M, then the transition map

$$G:=\gamma\circ\psi_p^{-1}:\psi_p(V_p\cap W)\to\gamma(V_p\cap W)$$

has positive Jacobian determinant everywhere. Then, the restrictions

$$F|_{\phi_p(U_p\cap W)}:\phi_p(U_p\cap W)\to\psi_p(U_p\cap W)$$

and

$$G|_{\psi_p(U_p\cap W)}:\psi_p(U_p\cap W)\to\gamma(U_p\cap W)$$

have positive Jacobian determinant in their respective domains, and thus so too does the composition $G|_{\psi_p(U_p\cap W)}\circ F|_{\phi_p(U_p\cap W)}:\phi_p(U_p\cap W)\to\gamma(U_p\cap W)$, which is precisely the transition map $\gamma\circ\phi_p^{-1}$ of the charts (U_p,ϕ_p) and (W,γ). Thus, (U_p,ϕ_p) is positively oriented, so $(U_p,\phi_p)\in\mathfrak{U}$. Since $p\in U_p$, the result is proved.

(ii) Since $\mathfrak{U}$ is an atlas in M such that for all $(U,\phi)=(U,x,y)\in\mathfrak{U}$ we have $U\cap\partial\Omega=\{q\in U\,|\,x(q)=0\}$, the collection $\mathfrak{V}$ of charts in $\partial\Omega$ induced from $\mathfrak{U}$ is an atlas. We show that $\mathfrak{V}$ is oriented. Choose two charts $(U\cap\partial\Omega,\tilde{\phi})=\big(U\cap\partial\Omega,y\circ\big(\iota|_{U\cap\partial\Omega}\big)\big)$ and $(V,\tilde{\psi})=\big(V\cap\partial\Omega,\tilde{y}\circ\big(\iota|_{V\cap\partial\Omega}\big)\big)$ in $\mathfrak{V}$ induced respectively from two charts $(U,\phi)=(U,x,y)$ and $(V,\psi)=(V,\tilde{x},\tilde{y})$ in $\mathfrak{U}$. The transition map $\tilde{\psi}\circ\tilde{\phi}^{-1}$ between the charts $(U\cap\partial\Omega,\tilde{\phi})$ and $(V\cap\partial\Omega,\tilde{\psi})$ can be written as the composition

$$\pi\circ(\psi\circ\phi^{-1})\circ\tilde{\iota}:y(U\cap V\cap\partial\Omega)\to\tilde{y}(U\cap V\cap\partial\Omega)\subset\mathbb{R}\,,$$

where

$$\tilde{\iota}:y(U\cap V\cap\partial\Omega)\to\phi(U\cap V)\,,\quad t\mapsto(0,t)\,,$$

is the inclusion map, and

$$\pi:\psi(U\cap V)\to\mathbb{R}\,,\quad(0,t)\mapsto t\,,$$

is the projection map. Then, denoting by (r,s) the standard coordinates on $\mathbb{R}^2$, and letting $f:=r\circ\psi\circ\phi^{-1}$ and $g:=s\circ\psi\circ\phi^{-1}$ be the component functions of the transition map $\psi\circ\phi^{-1}:\phi(U\cap V)\to\psi(U\cap V)$, the Jacobian matrix of the transition map $\tilde{\psi}\circ\tilde{\phi}^{-1}$ at a point $t\in y(U\cap V\cap\partial\Omega)$ is the product

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial r}(0,t) & \frac{\partial f}{\partial s}(0,t)\\ \frac{\partial g}{\partial r}(0,t) & \frac{\partial g}{\partial s}(0,t)\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}\frac{\partial g}{\partial s}(0,t)\end{pmatrix}\,.$$

Thus,

$$\mathcal{J}_{\tilde{\psi}\circ\tilde{\phi}^{-1}}(t) = \frac{\partial g}{\partial s}(0,t)\,.$$

We show that $\frac{\partial g}{\partial s}(0,t) > 0$. Since $(0,t) \in \phi(U \cap V \cap \partial\Omega)$, we have $\phi^{-1}(0,t) \in U \cap V \cap \partial\Omega$, which implies that $f(0,t) = 0$. If $\{t_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}\setminus\{t\}$ such that $(0,t_k) \in \phi(U \cap V)$ for all $k \in \mathbb{N}$, and $t_k \to t$ as $k \to \infty$, then we also have $f(0,t_k) = 0$ for all $k \in \mathbb{N}$, so

$$\frac{\partial f}{\partial s}(0,t) = \lim_{k\to\infty} \frac{f(0,t_k) - f(0,t)}{t_k - t} = 0\,.$$

Thus, $\mathcal{J}_{\psi\circ\phi^{-1}}(0,t) = \frac{\partial f}{\partial r}(0,t)\cdot\frac{\partial g}{\partial s}(0,t)$. This implies that the partial derivatives $\frac{\partial f}{\partial r}(0,t)$ and $\frac{\partial g}{\partial s}(0,t)$ are either both positive or both negative, so to show that $\frac{\partial g}{\partial s}(0,t) > 0$, it suffices to show that $\frac{\partial f}{\partial r}(0,t) > 0$. If $\{c_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}\setminus\{0\}$ such that $(c_k,t) \in \phi(U\cap V)$ for all $k \in \mathbb{N}$, and $c_k \to 0$ as $k \to \infty$, then for each $k \in \mathbb{N}$ the numbers c_k and $f(c_k,t)$ are either both positive or both negative: if $c_k > 0$, then $\phi^{-1}(c_k,t) \in U \cap V \cap (\overline{\Omega})^c$, so $f(c_k,t) > 0$; while if $c_k < 0$, then $\phi^{-1}(c_k,t) \in U \cap V \cap \Omega$, so $f(c_k,t) < 0$. Thus,

$$\frac{\partial f}{\partial r}(0,t) = \lim_{k\to\infty}\frac{f(c_k,t) - f(0,t)}{c_k - 0} = \lim_{k\to\infty}\frac{f(c_k,t)}{c_k} \geq 0\,,$$

so we must have $\frac{\partial f}{\partial r}(0,t) > 0$.

□

Definition 2.83. If M has dimension 2 and is oriented, and if $\Omega \subset M$ is a smooth open subset with nonempty boundary $\partial\Omega$, we call the orientation on $\partial\Omega$ given by Lemma 2.82 the *induced orientation (from M with respect to Ω)*.

Proposition 2.84. *Let M and N be 2-dimensional smooth manifolds, and $F : M \to N$ a diffeomorphism. Suppose $\Omega \subset M$ is a smooth open set. Then,*

(i) *$F(\Omega)$ is a smooth open subset of N;*
(ii) *if M and N are oriented, F is orientation-preserving, and $\partial\Omega \neq \emptyset$, then the diffeomorphism $F : \partial\Omega \to \partial F(\Omega)$ is orientation-preserving, assuming that $\partial\Omega$ and $\partial F(\Omega)$ are given the induced orientations from M and N and with respect to Ω and $F(\Omega)$, respectively.*

Proof. (i) Denote by (r,s) the standard coordinates on $\mathbb{R}^2$. For each chart (U,ϕ) in M such that $U \cap \Omega = \{q \in U \mid (r\circ\phi)(q) < 0\}$, the chart $(F(U), \phi\circ F^{-1})$ in N fulfils $F(U) \cap F(\Omega) = \{q \in F(U) \mid (r\circ\phi\circ F^{-1})(q) < 0\}$. Since we can cover M by such charts (U,ϕ), it follows that $F(\Omega)$ is a smooth open set in N.

(ii) Let $\mathfrak{U}$ denote the atlas in M consisting of all positively oriented charts (U,ϕ) in M that fulfil $U \cap \Omega = \{q \in U \mid (r\circ\phi)(q) < 0\}$. Then, the oriented atlas

$$\mathfrak{V} := \{(U\cap\partial\Omega, s\circ\phi\circ\iota) \mid (U,\phi) \in \mathfrak{U}\}$$

in $\partial\Omega$ is in its orientation. For each $(U,\phi) \in \mathfrak{U}$, the chart $(F(U), \phi\circ F^{-1})$ in N is positively oriented and fulfils $F(U) \cap F(\Omega) = \{q \in F(U) \mid (r\circ\phi\circ F^{-1})(q) < 0\}$, so the chart $(F(U)\cap\partial F(\Omega), s\circ\phi\circ F^{-1}\circ\iota)$ in $\partial F(\Omega)$ is positively oriented. Moreover, $(F(U)\cap\partial F(\Omega), s\circ\phi\circ F^{-1}\circ\iota)$ is precisely the chart induced from $(U\cap\partial\Omega, s\circ\phi\circ\iota)$ by $F : \partial\Omega \to \partial F(\Omega)$. Thus, $\mathfrak{V}$ is an oriented atlas in the orientation of $\partial\Omega$ such that for all $(V, \psi) \in \mathfrak{V}$, the chart $(F(V), \psi \circ F^{-1})$ in $\partial F(\Omega)$ is positively oriented. It follows that $F : \partial\Omega \to \partial F(\Omega)$ is orientation-preserving.

$\square$

Theorem 2.85 (Stokes' Theorem)**.** *Suppose M is a 2-dimensional oriented smooth manifold. Let $\Omega \subset M$ be a nonempty smooth open subset, and suppose α is a C^1 1-form over $\mathbb{F}$ on M such that $\overline{\Omega} \cap \operatorname{supp} \alpha$ is compact. Denote by $\iota : \partial\Omega \to M$ the inclusion map. Then, $d\alpha$ is integrable on Ω, the pullback $\iota^*\alpha$ is integrable on $\partial\Omega$, and*

$$\int_\Omega d\alpha = \int_{\partial\Omega} \iota^*\alpha \,.$$

Proof. We first prove the theorem for the case when $\mathbb{F} = \mathbb{R}$. First note that since $\partial\Omega \subset M$ can be covered by charts $(U, \phi) = (U, x, y)$ in M such that $U \cap \partial\Omega = \{q \in U \,|\, x(q) = 0\} = \phi^{-1}(\phi(U) \cap L)$, where L denotes the y-axis in $\mathbb{R}^2$, the set $\partial\Omega$ has measure 0 in M. Since α is C^1 on M, the exterior derivative $d\alpha$ is a continuous 2-form on M, and hence measurable. We have

$$\int_{\overline{\Omega}} (d\alpha)^\pm = \int_{\partial\Omega} (d\alpha)^\pm + \int_\Omega (d\alpha)^\pm = \int_\Omega (d\alpha)^\pm \,,$$

where the notation $\pm$ means that the array of equalities holds when read only with the $+$ signs and when read only with the $-$ signs. Note also that $(\operatorname{supp} \alpha)^c \subset M$ is an open subset where $\alpha = 0$, so we also have $d\alpha = 0$ on $(\operatorname{supp} \alpha)^c$. We then have $\operatorname{supp}(d\alpha)^\pm \subset \operatorname{supp} d\alpha \subset \operatorname{supp} \alpha$. Then, letting $K := \overline{\Omega} \cap \operatorname{supp} \alpha$,

$$\int_\Omega (d\alpha)^\pm = \int_{\overline{\Omega}} (d\alpha)^\pm = \int_{\overline{\Omega} \cap (\operatorname{supp} \alpha)^c} (d\alpha)^\pm + \int_K (d\alpha)^\pm = \int_K (d\alpha)^\pm \,,$$

which is finite by Lemma 2.77. Thus, $d\alpha$ is integrable on Ω.

Let $p \in K \subset \overline{\Omega}$. If $p \in \Omega$, choose a positively oriented chart $(U_p, \phi_p) = (U_p, x_p, y_p)$ about p in M such that $U_p \cap \Omega = \{q \in U_p \,|\, x_p(q) < 0\}$ and $U_p \subset \Omega$. If $p \in \partial\Omega$, choose a positively oriented chart $(U_p, \phi_p) = (U_p, x_p, y_p)$ about p in M such that $U_p \cap \Omega = \{q \in U_p \,|\, x_p(q) < 0\}$ and $\phi_p(U_p)$ is an open disc of finite radius in $\mathbb{R}^2$. Then, the collection $\{(U_p, \phi_p)\}_{p \in K}$ has a finite subcollection that covers K, which we index by $\{(U_j, \phi_j) = (U_j, x_j, y_j)\}_{j=1}^m$. Let $W := \bigcup_{j=1}^m U_j$. Since W is a finite union of coordinate open sets in M, it is a second countable smooth manifold with the induced differentiable structure from M. Thus, there exists a smooth partition of unity $\{\rho_j\}_{j=1}^m$ on W such that for each $j \in \{1, \ldots, m\}$, $\operatorname{supp}_W \rho_j \subset U_j$, where for a function or differential form ρ on W the notation $\operatorname{supp}_W \rho$ denotes the closure *in* W of the set $\{q \in W \,|\, \rho(q) \neq 0\}$ (which may differ from the closure of this set in M). Then, on W we have $\alpha = \sum_{j=1}^m \rho_j \alpha$. We now fix some $j \in \{1, \ldots, m\}$. We have $\operatorname{supp}_W \rho_j \alpha \subset \operatorname{supp}_W \rho_j \cap \operatorname{supp} \alpha$. We consider two cases.

(i) $U_j \cap \partial\Omega = \emptyset$. We then have $U_j \subset \Omega$, so

$$\begin{aligned} S_j := \operatorname{supp}_W \rho_j \alpha &\subset \operatorname{supp}_W \rho_j \cap \operatorname{supp} \alpha \\ &\subset U_j \cap \operatorname{supp} \alpha \\ &\subset \overline{\Omega} \cap \operatorname{supp} \alpha \\ &= K \,. \end{aligned}$$

Then, since S_j is closed in W, it is closed in $K \subset W$, so S_j is compact. Moreover, the 1-form $\rho_j \alpha$ is C^1 on W, so $d(\rho_j\alpha)$ is a continuous 2-form on W (and hence measurable), and $\operatorname{supp}_W (d(\rho_j\alpha))^{\pm} \subset \operatorname{supp}_W d(\rho_j\alpha) \subset S_j \subset U_j$. Then,

$$\int_{\overline{\Omega}\cap W} (d(\rho_j\alpha))^{\pm} = \int_{U_j} (d(\rho_j\alpha))^{\pm} = \int_{S_j} (d(\rho_j\alpha))^{\pm} \,,$$

which is finite, since S_j is compact and $(d(\rho_j\alpha))^{\pm}$ is continuous. Thus, $d(\rho_j\alpha)$ is integrable on $\overline{\Omega}\cap W$. On U_j, we have $\rho_j\alpha = f^j\,dx_j + g^j\,dy_j$ for some C^1 real-valued functions f^j and g^j on U_j, so on U_j,

$$d(\rho_j\alpha) = h^j\,dx_j \wedge dy_j \quad \text{for} \quad h^j := \frac{\partial g^j}{\partial x_j} - \frac{\partial f^j}{\partial y_j}\,.$$

Then,

$$\begin{aligned} +\infty > \int_{U_j} (d(\rho_j\alpha))^{\pm} &= \int_{\phi_j(U_j)} (h^j)^{\pm} \circ \phi_j^{-1}\,d\lambda \\ &= \int_{\phi_j(U_j)} (h^j \circ \phi_j^{-1})^{\pm}\,d\lambda\,, \end{aligned}$$

so the (continuous) function $h^j \circ \phi_j^{-1} : \phi_j(U_j) \to \mathbb{R}$ is integrable, and

$$\begin{aligned} \int_{U_j} d(\rho_j\alpha) &= \int_{\phi_j(U_j)} h^j \circ \phi_j^{-1}\,d\lambda \\ &= \int_{\phi_j(U_j)} \left(\frac{\partial g^j}{\partial x_j} - \frac{\partial f^j}{\partial y_j}\right) \circ \phi_j^{-1}\,d\lambda \\ &= \int_{\phi_j(U_j)} \left(\frac{\partial g^j}{\partial x_j} \circ \phi_j^{-1} - \frac{\partial f^j}{\partial y_j} \circ \phi_j^{-1}\right) d\lambda\,. \end{aligned}$$

The continuous functions $\frac{\partial g^j}{\partial x_j} \circ \phi_j^{-1}$ and $\frac{\partial f^j}{\partial y_j} \circ \phi_j^{-1}$ vanish outside of the compact set $\phi_j(S_j)$, and thus they are integrable. Then,

$$\begin{aligned} \int_{\overline{\Omega}\cap W} d(\rho_j\alpha) &= \int_{U_j} d(\rho_j\alpha) \\ &= \int_{\phi_j(U_j)} \frac{\partial g^j}{\partial x_j} \circ \phi_j^{-1}\,d\lambda - \int_{\phi_j(U_j)} \frac{\partial f^j}{\partial y_j} \circ \phi_j^{-1}\,d\lambda \\ &= \int_{\phi_j(U_j)} \frac{\partial (g^j \circ \phi_j^{-1})}{\partial r}\,d\lambda - \int_{\phi_j(U_j)} \frac{\partial (f^j \circ \phi_j^{-1})}{\partial s}\,d\lambda\,, \end{aligned}$$

where (r, s) are the standard coordinates on $\mathbb{R}^2$. Then, since the functions $f^j \circ \phi_j^{-1}$ and $g^j \circ \phi_j^{-1}$ are C^1 and have compact support on $\phi_j(U_j)$, one may apply iterated integration (Fubini's Theorem) and the Fundamental Theorem of Calculus to conclude that

$$\int_{\phi_j(U_j)} \frac{\partial (g^j \circ \phi_j^{-1})}{\partial r}\,d\lambda = \int_{\phi_j(U_j)} \frac{\partial (f^j \circ \phi_j^{-1})}{\partial s}\,d\lambda = 0\,,$$

so that

$$\int_{\overline{\Omega}\cap W} d(\rho_j\alpha) = 0\,.$$

(ii) $U_j \cap \partial\Omega \neq \emptyset$. We have

$$T_j := \mathrm{supp}_W \rho_j\alpha \cap \overline{\Omega} \subset \mathrm{supp}\,\alpha \cap \overline{\Omega} = K\,,$$

so T_j is compact. Moreover, $\mathrm{supp}_W\, d(\rho_j\alpha) \cap \overline{\Omega} \subset T_j \subset U_j \cap \overline{\Omega}$, so

$$\int_{W\cap\overline{\Omega}} (d(\rho_j\alpha))^{\pm} = \int_{U_j\cap\overline{\Omega}} (d(\rho_j\alpha))^{\pm} = \int_{T_j} (d(\rho_j\alpha))^{\pm} < +\infty\,.$$

Thus, $d(\rho_j\alpha)$ is integrable on $W\cap\overline{\Omega}$. As before, on U_j we have $\rho_j\alpha = f^j\,dx_j + g^j\,dy_j$ for some C^1 real-valued functions f^j and g^j on U_j, and $d(\rho_j\alpha) = h^j\,dx_j \wedge dy_j$ for $h^j = \frac{\partial g^j}{\partial x_j} - \frac{\partial f^j}{\partial y_j}$. Moreover,

$$\begin{aligned} +\infty > \int_{U_j\cap\overline{\Omega}} d(\rho_j\alpha)^{\pm} &= \int_{\phi_j(U_j\cap\overline{\Omega})} (h^j)^{\pm} \circ \phi_j^{-1}\,d\lambda \\ &= \int_{\phi_j(U_j\cap\overline{\Omega})} (h^j \circ \phi_j^{-1})^{\pm}\,d\lambda\,, \end{aligned}$$

so the function $h^j \circ \phi_j^{-1} : \phi_j(U_j \cap \overline{\Omega}) \to \mathbb{R}$ is integrable, and

$$\begin{aligned} \int_{W\cap\overline{\Omega}} d(\rho_j\alpha) &= \int_{U_j\cap\overline{\Omega}} d(\rho_j\alpha) \\ &= \int_{\phi_j(U_j\cap\overline{\Omega})} h^j \circ \phi_j^{-1}\,d\lambda \\ &= \int_{\phi_j(U_j\cap\overline{\Omega})} \left(\frac{\partial g^j}{\partial x_j} \circ \phi_j^{-1} - \frac{\partial f^j}{\partial y_j} \circ \phi_j^{-1}\right) d\lambda\,. \end{aligned}$$

Since the continuous functions $\frac{\partial g^j}{\partial x_j} \circ \phi_j^{-1}, \frac{\partial f^j}{\partial y_j} \circ \phi_j^{-1} : \phi_j(U_j \cap \overline{\Omega}) \to \mathbb{R}$ vanish outside of the compact set $\phi_j(T_j)$, they are integrable, so

$$\begin{aligned} \int_{W\cap\overline{\Omega}} d(\rho_j\alpha) &= \int_{\phi_j(U_j\cap\overline{\Omega})} \frac{\partial g^j}{\partial x_j} \circ \phi_j^{-1}\,d\lambda - \int_{\phi_j(U_j\cap\overline{\Omega})} \frac{\partial f^j}{\partial y_j} \circ \phi_j^{-1}\,d\lambda \\ &= \int_{\phi_j(U_j\cap\overline{\Omega})} \frac{\partial(g^j \circ \phi_j^{-1})}{\partial r}\,d\lambda - \int_{\phi_j(U_j\cap\overline{\Omega})} \frac{\partial(f^j \circ \phi_j^{-1})}{\partial s}\,d\lambda\,. \end{aligned}$$

By our choice of the chart (U_j, ϕ_j), denoting by $\overline{H}$ the closed left half-plane in $\mathbb{R}^2$ and by L the y-axis in $\mathbb{R}^2$, we have

$$\begin{aligned} \phi_j(U_j \cap \overline{\Omega}) &= \phi_j(U_j) \cap \overline{H}\,, \\ \phi_j(U_j \cap \partial\Omega) &= \phi_j(U_j) \cap L \neq \emptyset\,. \end{aligned}$$

Moreover, $\phi_j(U_j)$ is an open disc of finite radius in $\mathbb{R}^2$, so $\phi_j(U_j) \cap L$ is an open interval (a_j, b_j) in L for some $a_j, b_j \in \mathbb{R}$ with $a_j < b_j$. Then, one may again use iterated integration and the Fundamental Theorem of Calculus to conclude that

$$\int_{\phi_j(U_j)} \frac{\partial(f^j \circ \phi_j^{-1})}{\partial s}\,d\lambda = 0$$

and

$$\int_{\phi_j(U_j)} \frac{\partial(g^j \circ \phi_j^{-1})}{\partial r}\,d\lambda = \int_{a_j}^{b_j} (g^j \circ \phi_j^{-1} \circ \tilde{\imath})(t)\,dt\,,$$

where $\tilde{\imath} : (a_j, b_j) \to \phi_j(U_j)$ is the inclusion map $t \mapsto (0, t)$.

In conclusion, for each $j \in \{1, \dots, m\}$ we have

$$\int_{\overline{\Omega} \cap W} d(\rho_j \alpha) = \begin{cases} 0 & \text{if } U_j \cap \partial\Omega = \emptyset \\ \int_{a_j}^{b_j} (g^j \circ \phi_j^{-1} \circ \tilde{\iota})(t)\, dt & \text{if } U_j \cap \partial\Omega \neq \emptyset\,. \end{cases}$$

We now consider the pullback $\iota^*\alpha$ on the 1-dimensional smooth submanifold $\partial\Omega$. Since the map $\iota : \partial\Omega \to M$ is C^∞, the 1-form $\iota^*\alpha$ on $\partial\Omega$ is C^1. Since $\iota^*\alpha$ vanishes on $\partial\Omega \cap (\operatorname{supp}\alpha)^c \supset \partial\Omega \cap W^c$, we have

$$\int_{\partial\Omega} (\iota^*\alpha)^\pm = \int_{\partial\Omega \cap W} (\iota^*\alpha)^\pm = \int_{\partial\Omega \cap \operatorname{supp}\alpha} (\iota^*\alpha)^\pm\,,$$

which is finite, since $(\iota^*\alpha)^\pm$ is continuous and $\partial\Omega \cap \operatorname{supp}\alpha$ is compact. Thus, $\iota^*\alpha$ is integrable on $\partial\Omega$. Moreover, on $\partial\Omega \cap W$ we have $\iota^*\alpha = \sum_{j=1}^m (\rho_j \circ \iota)\iota^*\alpha$. Fix $j \in \{1, \dots, m\}$, and consider the 1-form $\iota^*(\rho_j\alpha) = (\rho_j \circ \iota)\iota^*\alpha$ on $\partial\Omega \cap W$. As before, we consider the two cases $U_j \cap \partial\Omega = \emptyset$ and $U_j \cap \partial\Omega \neq \emptyset$:

(i) If $U_j \cap \partial\Omega = \emptyset$, then $\partial\Omega \cap W \subset (U_j)^c \cap W \subset (\operatorname{supp}_W \rho_j)^c \cap W$, so for all $p \in \partial\Omega \cap W$ we have $((\rho_j \circ \iota)\iota^*\alpha)_p = \rho_j(\iota(p))(\iota^*\alpha)_p = 0$. Thus,

$$\int_{\partial\Omega \cap W} (\rho_j \circ \iota)\iota^*\alpha = \int_{\partial\Omega \cap W} 0 = 0 = \int_{\overline{\Omega} \cap W} d(\rho_j\alpha)\,.$$

(ii) Suppose that $U_j \cap \partial\Omega \neq \emptyset$. Since $(\rho_j \circ \iota)\iota^*\alpha$ vanishes on

$$\partial\Omega \cap W \cap (\operatorname{supp}_W \rho_j \cap \operatorname{supp}\alpha)^c \supset \partial\Omega \cap W \cap (U_j)^c\,,$$

we have

$$\int_{\partial\Omega \cap W} ((\rho_j \circ \iota)\iota^*\alpha)^\pm = \int_{\partial\Omega \cap U_j} ((\rho_j \circ \iota)\iota^*\alpha)^\pm = \int_{\partial\Omega \cap \operatorname{supp}_W \rho_j \cap \operatorname{supp}\alpha} ((\rho_j \circ \iota)\iota^*\alpha)^\pm\,,$$

which is finite, since $((\rho_j \circ \iota)\iota^*\alpha)^\pm$ is continuous and $\partial\Omega \cap \operatorname{supp}_W \rho_j \cap \operatorname{supp}\alpha$ is compact. Thus, $(\rho_j \circ \iota)\iota^*\alpha$ is integrable on $\partial\Omega \cap W$. Since the chart $(U_j, \phi_j) = (U_j, x_j, y_j)$ was chosen to be positively oriented and to fulfil $U_j \cap \Omega = \{q \in U_j \mid x_j(q) < 0\}$, the induced chart $(U_j \cap \partial\Omega, y_j \circ \iota)$ in $\partial\Omega$ is positively oriented. Since on U_j we have $\rho_j\alpha = f^j\, dx_j + g^j\, dy_j$, on $U_j \cap \partial\Omega$

$$\begin{aligned} (\rho_j \circ \iota)\iota^*\alpha &= \iota^*(\rho_j\alpha) \\ &= \iota^*(f^j\, dx_j + g^j\, dy_j) \\ &= (f^j \circ \iota)\, d(x_j \circ \iota) + (g^j \circ \iota)\, d(y_j \circ \iota) \\ &= (g^j \circ \iota)\, d(y_j \circ \iota)\,, \end{aligned}$$

since $d(x_j \circ \iota) = 0$ on $U_j \cap \partial\Omega$. Then,

$$\begin{aligned}
\int_{\partial\Omega\cap W} (\rho_j \circ \iota)\iota^*\alpha &= \int_{\partial\Omega\cap U_j} ((\rho_j \circ \iota)\iota^*\alpha)^+ - \int_{\partial\Omega\cap U_j} ((\rho_j \circ \iota)\iota^*\alpha)^- \\
&= \int_{\partial\Omega\cap U_j} (g^j \circ \iota)^+ \, d(y_j \circ \iota) - \int_{\partial\Omega\cap U_j} (g^j \circ \iota)^- \, d(y_j \circ \iota) \\
&= \int_{y_j(\partial\Omega\cap U_j)} (g^j \circ \iota)^+ \circ (y_j \circ \iota)^{-1} \, d\lambda - \int_{y_j(\partial\Omega\cap U_j)} (g^j \circ \iota)^- \circ (y_j \circ \iota)^{-1} \, d\lambda \\
&= \int_{(a_j,b_j)} (g^j \circ \phi_j^{-1} \circ \tilde{\iota})^+ \, d\lambda - \int_{(a_j,b_j)} (g^j \circ \phi_j^{-1} \circ \tilde{\iota})^- \, d\lambda \\
&= \int_{(a_j,b_j)} (g^j \circ \phi_j^{-1} \circ \tilde{\iota}) \, d\lambda \\
&= \int_{a_j}^{b_j} (g^j \circ \phi_j^{-1} \circ \tilde{\iota})(t) \, dt \\
&= \int_{\overline{\Omega}\cap W} d(\rho_j\alpha)\,.
\end{aligned}$$

In conclusion, the following hold:

- $d\alpha$ is integrable on Ω and $\iota^*\alpha$ is integrable on $\partial\Omega$;
- we have

$$\int_\Omega d\alpha = \int_{\overline{\Omega}\cap W} d\alpha$$

  and

$$\int_{\partial\Omega} \iota^*\alpha = \int_{\partial\Omega\cap W} \iota^*\alpha\,;$$

- $d\alpha = \sum_{j=1}^m d(\rho_j\alpha)$ on $\overline{\Omega} \cap W$, and $\iota^*\alpha = \sum_{j=1}^m (\rho_j \circ \iota)\iota^*\alpha$ on $\partial\Omega \cap W$;
- for each $j \in \{1, \ldots, m\}$, $d(\rho_j\alpha)$ is integrable on $\overline{\Omega} \cap W$, $(\rho_j \circ \iota)\iota^*\alpha$ is integrable on $\partial\Omega \cap W$, and

$$\int_{\overline{\Omega}\cap W} d(\rho_j\alpha) = \int_{\partial\Omega\cap W} (\rho_j \circ \iota)\iota^*\alpha\,.$$

Then, finally (YES!),

$$\begin{aligned}
\int_\Omega d\alpha &= \int_{\overline{\Omega}\cap W} d\alpha \\
&= \int_{\overline{\Omega}\cap W} \sum_{j=1}^m d(\rho_j\alpha) \\
&= \sum_{j=1}^m \int_{\overline{\Omega}\cap W} d(\rho_j\alpha) \\
&= \sum_{j=1}^m \int_{\partial\Omega\cap W} (\rho_j\circ\iota)\iota^*\alpha \\
&= \int_{\partial\Omega\cap W} \sum_{j=1}^m (\rho_j\circ\iota)\iota^*\alpha \\
&= \int_{\partial\Omega\cap W} \iota^*\alpha \\
&= \int_{\partial\Omega} \iota^*\alpha\,.
\end{aligned}$$

For the case when $\mathbb{F}=\mathbb{C}$, observe that the real 1-forms $\operatorname{Re}(\alpha)$ and $\operatorname{Im}(\alpha)$ on M are C^1 and we have $\operatorname{supp}\operatorname{Re}(\alpha)\subset\operatorname{supp}\alpha$ and $\operatorname{supp}\operatorname{Im}(\alpha)\subset\operatorname{supp}\alpha$, so that the sets $\overline{\Omega}\cap\operatorname{supp}\operatorname{Re}(\alpha)$ and $\overline{\Omega}\cap\operatorname{supp}\operatorname{Im}(\alpha)$ are compact. Thus, by the case $\mathbb{F}=\mathbb{R}$, the 2-forms $d(\operatorname{Re}(\alpha))$ and $d(\operatorname{Im}(\alpha))$ on M are integrable over Ω, the 1-forms $\iota^*\operatorname{Re}(\alpha)$ and $\iota^*\operatorname{Im}(\alpha)$ on $\partial\Omega$ are integrable over $\partial\Omega$, and

$$\int_\Omega d(\operatorname{Re}(\alpha)) = \int_{\partial\Omega}\iota^*\operatorname{Re}(\alpha) \quad\text{and}\quad \int_\Omega d(\operatorname{Im}(\alpha)) = \int_{\partial\Omega}\iota^*\operatorname{Im}(\alpha)\,.$$

Moreover, on Ω we have $d(\operatorname{Re}(\alpha))=\operatorname{Re}(d\alpha)$ and $d(\operatorname{Im}(\alpha))=\operatorname{Im}(d\alpha)$, and on $\partial\Omega$ we have $\iota^*\operatorname{Re}(\alpha)=\operatorname{Re}(\iota^*\alpha)$ and $\iota^*\operatorname{Im}(\alpha)=\operatorname{Im}(\iota^*\alpha)$. Thus, $d\alpha$ is integrable over Ω, $\iota^*\alpha$ is integrable over $\partial\Omega$, and

$$\begin{aligned}
\int_\Omega d\alpha &= \int_\Omega \operatorname{Re}(d\alpha) + i\int_\Omega \operatorname{Im}(d\alpha) \\
&= \int_\Omega d(\operatorname{Re}(\alpha)) + i\int_\Omega d(\operatorname{Im}(\alpha)) \\
&= \int_{\partial\Omega} \iota^*\operatorname{Re}(\alpha) + i\int_{\partial\Omega} \iota^*\operatorname{Im}(\alpha) \\
&= \int_{\partial\Omega} \operatorname{Re}(\iota^*\alpha) + i\int_{\partial\Omega} \operatorname{Im}(\iota^*\alpha) \\
&= \int_{\partial\Omega} \iota^*\alpha\,.
\end{aligned}$$

$\square$

3.1. C^∞ approximation.

We recall a theorem from the theory of Lebesgue integration:

Theorem 3.1. (Dominated derivation) *Suppose X is a measure space with measure μ, let $U \subset \mathbb{R}$ be open, and let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Suppose $F : X \times U \to \mathbb{F}$ is a function fulfilling*

(i) *for each $y \in U$ the function $F_y : X \to \mathbb{F}$, $x \mapsto F(x,y)$, is integrable,*
(ii) *for each $x \in X$ the function $F_x : U \to \mathbb{F}$, $y \mapsto F(x,y)$ is differentiable,*
(iii) *there exists an integrable function $g : X \to [0,+\infty)$ such that for all $x \in X$ we have $\left|\frac{dF_x}{dy}\right| \leq g(x)$ on U.*

Then, we have

(a) *the function*
$$U \to \mathbb{F}\,, \quad y \mapsto \int_X F(x,y)\,d\mu(x)$$
is differentiable,
(b) *for each $y_0 \in U$ the function*
$$X \to \mathbb{F}\,, \quad x \mapsto \frac{dF_x}{dy}(y_0)$$
is integrable,
(c) *for all $y_0 \in U$,*
$$\frac{d}{dy}\left(\int_X F(x,y)\,d\mu(x)\right)\bigg|_{y_0} = \int_X \frac{dF_x}{dy}(y_0)\,d\mu(x)\,.$$

Definition 3.2. For a point $x \in \mathbb{R}^n$ and a nonempty subset $S \subset \mathbb{R}^n$, we define the distance between x and S by
$$\operatorname{dist}(x,S) := \inf\{|x-s| \mid s \in S\}\,.$$
It may be useful to also define
$$\operatorname{dist}(x,\emptyset) := +\infty\,.$$

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{C}$ a locally integrable function. Suppose $k : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function such that $\operatorname{supp} k \subset B(0,1)$. Fix $\delta \in (0,+\infty)$.*

(i) *The set*
$$\Omega_\delta := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x,\Omega^c) > \delta\}$$
is open and contained in Ω.
(ii) *For every $x \in \Omega_\delta$, we have $\overline{B(x,\delta)} \subset \Omega$. Moreover, the function*
$$v_{\delta,x} : \Omega \to \mathbb{C}\,, \quad y \mapsto u(y)\,k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,,$$
is integrable and
$$\begin{aligned}\int_\Omega u(y)\,k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,d\lambda(y) &= \int_{B(x,\delta)} u(y)\,k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,d\lambda(y)\\ &= \int_{B(0,1)} u(x-\delta y)k(y)\,d\lambda(y)\,.\end{aligned}$$

(iii) *The function*

$$u_\delta : \Omega_\delta \to \mathbb{C}\,, \quad x \mapsto \int_\Omega u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y)$$

is C^∞.

(iv) *The extension $\hat{u}_\delta : \Omega \to \mathbb{C}$ of u_δ to Ω by 0 is locally integrable.*

Suppose further that k is nonnegative on $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} k\, d\lambda = 1$. Then,

(v) *If u is continuous, then for any compact subset $K \subset \Omega$ we have $\hat{u}_\delta \to u$ uniformly as $\delta \to 0^+$ on K.*

(vi) *For u not necessarily continuous, $\hat{u}_\delta \to u$ in $L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$ as $\delta \to 0^+$.*

Proof. (i) If $\Omega = \mathbb{R}^n$, then we have $\Omega_\delta = \Omega$. Suppose $\Omega \neq \mathbb{R}^n$. If $x \in \Omega_\delta$ then we cannot have $x \in \Omega^c$, so $\Omega_\delta \subset \Omega$. For any $x_0 \in \Omega_\delta$ we may let $a := \operatorname{dist}(x_0, \Omega^c)$ and choose $\sigma \in (\delta, a)$. Then, for all $x \in B(x_0, a - \sigma)$ and $y \in \Omega^c$ we have

$$|x-y| \geq |x_0 - y| - |x - x_0| > a + \sigma - a = \sigma\,,$$

so

$$\operatorname{dist}(x, \Omega^c) = \inf\{|x-y| \mid y \in \Omega^c\} \geq \sigma > \delta\,,$$

which implies that $x \in \Omega_\delta$. Thus, Ω_δ is open.

(ii) Fix $x \in \Omega_\delta$. For every $y \in \Omega^c$, we have

$$|x-y| \geq \operatorname{dist}(x, \Omega^c) > \delta\,,$$

so $y \in \overline{B(x,\delta)}^c$. Thus, $\overline{B(x,\delta)} \subset \Omega$. Moreover, if y is any point in $\overline{B(x,\delta)}^c$ then $\big|(x-y)/\delta\big| > 1$, so

$$k\left(\frac{x-y}{\delta}\right) = 0\,.$$

Thus, for all $y \in \Omega$ we have

$$v_{\delta,x}(y) = u(y)\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n} = \chi_{\overline{B(x,\delta)}}(y)\, u(y)\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,.$$

Since k is bounded on $\mathbb{R}^n$ and u is locally integrable on Ω, the function

$$y \mapsto u(y)\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}$$

on $\overline{B(x,\delta)}$ is integrable, so $v_{\delta,x}$ is integrable on Ω. Moreover, applying change of variables via the diffeomorphism

$$F : B(0,1) \to B(x,\delta)\,, \quad y \mapsto -\delta y + x\,,$$

we have

$$\begin{aligned}
\int_\Omega u(y)\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\, d\lambda(y) &= \int_{\overline{B(x,\delta)}} u(y)\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\, d\lambda(y) \\
&= \int_{B(x,\delta)} u(y)\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\, d\lambda(y) \\
&= \int_{B(0,1)} u(-\delta y + x)\, k(y)\, d\lambda(y)\,.
\end{aligned}$$

(iii) First note that since $\operatorname{supp} k$ is compact and contained in $B(0,1)$, we may find $\gamma \in (0,1)$ such that $\operatorname{supp} k \subset \overline{B(0,\gamma)}$. Then, for any $x \in \Omega_\delta$ we have

$$u_\delta(x) = \int_{B(x,\delta)} u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) = \int_{\overline{B(x,\delta\gamma)}} u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y)\,.$$

We first show that u_δ is continuous on Ω_δ. Define the set

$$C := \overline{\bigcup_{x\in\Omega_\delta} \overline{B(x,\delta\gamma)}}\,.$$

Then, for any $c \in C$ there is $x \in \Omega_\delta$ and $s \in \overline{B(x,\delta\gamma)}$ such that $|c-s| < \delta(1-\gamma)$, so that

$$|c-x| \leq |c-s| + |s-x| < \delta(1-\gamma) + \delta\gamma = \delta\,.$$

Thus, $c \in B(x,\delta) \subset \Omega$, so $C \subset \Omega$. Suppose $r \in \Omega_\delta$ and $\{r_m\}_{m\in\mathbb{N}}$ is a sequence in Ω_δ converging to r. Then, the set

$$E := \overline{\left(\bigcup_{m\in\mathbb{N}} \overline{B(r_m,\delta\gamma)}\right) \cup \overline{B(r,\delta\gamma)}} \subset C$$

is compact and contained in Ω. For all $x \in \{r_m\}_{m\in\mathbb{N}} \cup \{r\}$, we have

$$u_\delta(x) = \int_E u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y)\,.$$

The functions

$$E \to \mathbb{C}\,, \quad y \mapsto u(y)\, k\left(\frac{r_m-y}{\delta}\right) \frac{1}{\delta^n}$$

for $m \in \mathbb{N}$ converge pointwise to the function

$$E \to \mathbb{C}\,, \quad y \mapsto u(y)\, k\left(\frac{r-y}{\delta}\right) \frac{1}{\delta^n}\,;$$

and choosing an upper bound $M \in (0,+\infty)$ for $|k|$ on $\mathbb{R}^n$, for all $m \in \mathbb{N}$ and $y \in E$ we have

$$\left| u(y)\, k\left(\frac{r_m-y}{\delta}\right) \frac{1}{\delta^n} \right| \leq \big|u(y)\big| \frac{M}{\delta^n}.$$

(please admire my superior skillz of deployment of the semicolon above). Then, by the Dominated Convergence Theorem,

$$u_\delta(r_m) = \int_E u(y)\, k\left(\frac{r_m-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) \to \int_E u(y)\, k\left(\frac{r-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) = u_\delta(r)$$

as $m \to \infty$. Thus, u_δ is continuous.

Let $r = (r_1,\ldots,r_n) \in \Omega_\delta$, and choose $\varepsilon \in (0,\delta(1-\gamma))$ such that $B(r,\varepsilon) \subset \Omega_\delta$. Then, for every point t in the interval $(r_1-\varepsilon, r_1+\varepsilon) \subset \mathbb{R}$, we have $s_t := (t,r_2,\ldots,r_n) \in \Omega_\delta$. Moreover, if $t \in (r_1-\varepsilon,r_1+\varepsilon)$ and $y \in \overline{B(s_t,\delta\gamma)}$ then

$$|y-r| \leq |y-s_t| + |s_t-r| < \delta\gamma + \varepsilon < \delta\gamma + \delta(1-\gamma) = \delta\,,$$

and hence $\overline{B(s_t,\delta\gamma)} \subset B(r,\delta)$ for all $t \in (r_1-\varepsilon, r_1+\varepsilon)$. Consider the function

$$G : \overline{B(r,\delta)} \times (r_1-\varepsilon, r_1+\varepsilon) \to \mathbb{C}\,, \quad (y,t) \mapsto u(y)\, k\left(\frac{s_t - y}{\delta}\right)\frac{1}{\delta^n}$$
$$= \chi_{\overline{B(s_t,\delta\gamma)}}(y) u(y)\, k\left(\frac{s_t - y}{\delta}\right)\frac{1}{\delta^n}\,.$$

We have:

(a) For all $t \in (r_1-\varepsilon, r_1+\varepsilon)$, the function $y \mapsto G(y,t)$ on $\overline{B(r,\delta)}$ is integrable.

(b) For each $y \in \overline{B(r,\delta)}$ the function $t \mapsto G(y,t)$ on $(r_1-\varepsilon, r_1+\varepsilon)$ is differentiable. Denoting by $x^1,\dots,x^n$ the standard coordinates on $\mathbb{R}^n$, for each $t_0 \in (r_1-\varepsilon, r_1+\varepsilon)$ the chain rule gives

$$\left.\frac{dG(y,t)}{dt}\right|_{t_0} = u(y)\,\frac{\partial k}{\partial x^1}\left(\frac{s_{t_0}-y}{\delta}\right)\frac{1}{\delta^{n+1}}\,.$$

(c) Choose $P \in (0,+\infty)$ with $\left|\frac{\partial k}{\partial x^1}\right| \le P$ on $\mathbb{R}^n$. Then, for all $y \in \overline{B(r,\delta)}$ and $t_0 \in (r_1-\varepsilon, r_1+\varepsilon)$ we have

$$\left|\left.\frac{dG(y,t)}{dt}\right|_{t_0}\right| = \left|u(y)\,\frac{\partial k}{\partial x^1}\left(\frac{s_{t_0}-y}{\delta}\right)\frac{1}{\delta^{n+1}}\right| \le \left|u(y)\right|\frac{P}{\delta^{n+1}}\,,$$

and $|u|\,\frac{P}{\delta^{n+1}}$ is integrable on $\overline{B(r,\delta)}$.

Then, by dominated derivation, the function

$$(r_1-\varepsilon, r_1+\varepsilon) \to \mathbb{C}\,, \quad t \mapsto \int_{\overline{B(r,\delta)}} G(y,t)\, d\lambda(y) = u_\delta(t, r_2, \dots, r_n)$$

is differentiable at r_1, so u_δ has a partial derivative with respect to x_1 at r and

$$\begin{aligned}\left.\frac{\partial u_\delta}{\partial x^1}\right|_r &= \frac{d}{dt}\left.\left(\int_{\overline{B(r,\delta)}} G(y,t)\, d\lambda(y)\right)\right|_{r_1}\\ &= \int_{\overline{B(r,\delta)}} u(y)\,\frac{\partial k}{\partial x^1}\left(\frac{r-y}{\delta}\right)\frac{1}{\delta^{n+1}}\, d\lambda(y)\\ &= \int_\Omega u(y)\,\frac{\partial k}{\partial x^1}\left(\frac{r-y}{\delta}\right)\frac{1}{\delta^{n+1}}\, d\lambda(y)\,.\end{aligned}$$

Then, letting $\ell := \frac{1}{\delta}\frac{\partial k}{\partial x^1}$ on $\mathbb{R}^n$, for all $x \in \Omega_\delta$ we have

$$\frac{\partial u_\delta}{\partial x^1}(x) = \int_\Omega u(y)\,\ell\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\, d\lambda(y)\,.$$

Since ℓ is C^∞ and $\operatorname{supp} \ell \subset \operatorname{supp} k \subset B(0,1)$, the function ℓ fulfils all our assumptions for k. Thus, we can reason for ℓ exactly as we did for k to conclude that the function $\frac{\partial u_\delta}{\partial x^1}$ on Ω_δ is continuous and also has a partial derivative with respect to x^1. Reasoning similarly for the partial derivatives of u_δ with respect to the remaining coordinates $x^2,\dots,x^n$, and applying induction on the order of the derivatives, we can conclude that u_δ is C^∞ on Ω_δ.

(iv) Let K be a compact subset of Ω. If $K \cap \Omega_\delta = \emptyset$, then $\hat{u}_\delta = 0$ on K and thus it is integrable on K. Suppose $K \cap \Omega_\delta \ne \emptyset$, and define the set

$$S := \overline{\bigcup_{x \in K\cap\Omega_\delta} \overline{B(x,\delta\gamma)}} \subset C \subset \Omega\,.$$

Since K is bounded, so too is S, so S is compact. Then, for each $x \in K$, we have

$$\left|\hat{u}_\delta(x)\right| = \begin{cases} \left| \displaystyle\int_{\overline{B(x,\delta\gamma)}} u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) \right| & \text{if } x \in K \cap \Omega_\delta\,, \\ 0 & \text{if } x \in K \setminus \Omega_\delta\,, \end{cases}$$

so that for all $x \in K$,

$$\left|\hat{u}_\delta(x)\right| \leq \int_S \left|u(y)\right| \frac{M}{\delta^n}\, d\lambda(y) =: R \in [0, +\infty)\,.$$

Thus,

$$\int_K \left|\hat{u}_\delta(x)\right|\, d\lambda(x) \leq \int_K R\, d\lambda = R\, \lambda(K) < +\infty\,.$$

For the proofs of parts (v) and (vi), fix a nonempty compact subset $K \subset \Omega$. Define $d_1 \in (0, +\infty]$ to be $+\infty$ if $\Omega^c = \emptyset$, and

$$\inf\{|x-y| \mid x \in K \text{ and } y \in \Omega^c\} \in (0, +\infty)$$

if $\Omega^c \neq \emptyset$. Choose $d_2 \in (0, d_1)$ and suppose $\delta \leq d_2$. If $\Omega^c = \emptyset$ then $K \subset \Omega = \Omega_\delta$. If $\Omega^c \neq \emptyset$, then for each $x \in K$ and $y \in \Omega^c$ we have $|x-y| \geq d_1$, so

$$\operatorname{dist}(x, \Omega^c) \geq d_1 > d_2 \geq \delta\,,$$

and hence $x \in \Omega_\delta$. Thus, $K \subset \Omega_\delta$ for all $\delta \in (0, d_2]$. In particular, $K \subset \Omega_{d_2}$, and thus the set

$$C_K := \overline{\bigcup_{x \in K} \overline{B(x, d_2\gamma)}}\,,$$

which is compact, is contained in Ω.

(v) Suppose u is continuous on Ω, and let $\varepsilon \in (0, +\infty)$. Since u is uniformly continuous on C_K, there exists $a \in (0, +\infty)$ such that for all $y_1, y_2 \in C_K$ with $|y_1 - y_2| < a$, we have $\left|u(y_1) - u(y_2)\right| < \varepsilon$. Let $d := \min\{d_2, a\} \in (0, +\infty)$, and suppose $\delta < d$. Since $\delta < d_2$, we have $K \subset \Omega_\delta$. Then, for all $x \in K$ we have

$$\begin{aligned} \left|\hat{u}_\delta(x) - u(x)\right| &= \left| \int_{\overline{B(x,\delta\gamma)}} u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) - u(x) \right| \\ &= \left| \int_{\overline{B(x,\delta\gamma)}} u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) - \int_{\overline{B(x,\delta\gamma)}} u(x)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) \right| \\ &= \left| \int_{\overline{B(x,\delta\gamma)}} (u(y) - u(x))\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) \right| \\ &\leq \int_{\overline{B(x,\delta\gamma)}} \left|u(y) - u(x)\right|\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y)\,, \end{aligned}$$

and since for all $y \in \overline{B(x,\delta\gamma)} \subset \overline{B(x, d_2\gamma)}$ we have $x, y \in C_K$ and $|y - x| \leq \delta\gamma < \delta < d \leq a$, we know that $\left|u(y) - u(x)\right| < \varepsilon$. Thus,

$$\int_{\overline{B(x,\delta\gamma)}} \left|u(y) - u(x)\right|\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) \leq \int_{\overline{B(x,\delta\gamma)}} \varepsilon\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) = \varepsilon\,.$$

In conclusion, for all $\delta \in (0, +\infty)$ such that $\delta < d$ and for all $x \in K$, we have

$$\left|\hat{u}_\delta(x) - u(x)\right| < \varepsilon\,.$$

(vi) If $\lambda(K) = 0$, then for all $\delta \in (0, +\infty)$ we have

$$\|\hat{u}_\delta - u\|_{L^1(K)} = \int_K |\hat{u}_\delta - u| \, d\lambda = 0 \,,$$

so $\|\hat{u}_\delta - u\|_{L^1(K)} \to 0$ as $\delta \to 0^+$. Suppose $\lambda(K) > 0$, and let again $\varepsilon \in (0, +\infty)$. Since u is locally integrable on Ω and $C_K \subset \Omega$ is compact, there exists a continuous function $f : \Omega \to \mathbb{C}$ with compact support such that

$$\|u - f\|_{L^1(C_K)} < \frac{\varepsilon}{3} \,.$$

It follows that

$$\|u - f\|_{L^1(K)} \leq \|u - f\|_{L^1(C_K)} < \frac{\varepsilon}{3} \,.$$

Moreover, since f is integrable (and hence locally integrable) and continuous on Ω, by (v) there exists $d \in (0, d_2]$ such that for all $\delta \in (0, +\infty)$ with $\delta < d$ and for all $x \in K$, $\left|\hat{f}_\delta(x) - f(x)\right| < \frac{\varepsilon}{3\lambda(K)}$. Fix $\delta \in (0, d)$. We have

$$\left\|\hat{f}_\delta - f\right\|_{L^1(K)} = \int_K \left|\hat{f}_\delta - f\right| \, d\lambda \leq \int_K \frac{\varepsilon}{3\lambda(K)} \, d\lambda = \frac{\varepsilon}{3} \,.$$

Since $\delta < d \leq d_2$, we know that $K \subset \Omega_\delta$. Then,

$$\begin{aligned}
\left\|\hat{u}_\delta - \hat{f}_\delta\right\|_{L^1(K)} &= \int_K \left| \int_{\overline{B(x,\delta\gamma)}} (u(y) - f(y))\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n} \, d\lambda(y) \right| d\lambda(x) \\
&\leq \int_K \left(\int_{\overline{B(x,\delta\gamma)}} \left|u(y) - f(y)\right| \, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n} \, d\lambda(y) \right) d\lambda(x) \\
&= \int_K \left(\int_{C_K} \left|u(y) - f(y)\right| \, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n} \, d\lambda(y) \right) d\lambda(x) \,.
\end{aligned}$$

We wish to show that the function

$$H : K \times C_K \to \mathbb{R} \,, \quad (x, y) \mapsto \left|u(y) - f(y)\right| \, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}$$

is measurable on $K \times C_K \subset \mathbb{R}^{2n}$. Since the function $|u - f|$ is measurable on C_K, if $U \subset \mathbb{R}$ is open then the set

$$A_U := \left\{ y \in C_K \,\middle|\, \left|u(y) - f(y)\right| \in U \right\}$$

is measurable in $C_K \subset \mathbb{R}^n$. Observe that as subsets of $K \times C_K$,

$$\left\{ (x, y) \in K \times C_K \,\middle|\, \left|u(y) - f(y)\right| \in U \right\} = K \times A_U \,.$$

Since A_U is measurable in $C_K \subset \mathbb{R}^n$, the set $K \times A_U$ is measurable in $K \times C_K \subset \mathbb{R}^{2n}$, which shows that the function

$$K \times C_K \to \mathbb{R} \,, \quad (x, y) \mapsto \left|u(y) - f(y)\right|$$

is measurable. Moreover, the function

$$K \times C_K \to \mathbb{R} \,, \quad (x, y) \mapsto k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}$$

is continuous and hence measurable. Thus, H is a product of two measurable functions and so it is itself measurable. We may then apply the Fubini-Tonelli theorem to write

$$\left\|\hat{u}_\delta - \hat{f}_\delta\right\|_{L^1(K)} = \int_{C_K}\left(\int_K \left|u(y) - f(y)\right|\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,d\lambda(x)\right)d\lambda(y)\,.$$

For each $y \in C_K$, we have

$$\begin{aligned}\int_K \left|u(y) - f(y)\right|\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,d\lambda(x) &= \left|u(y) - f(y)\right| \int_K k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,d\lambda(x)\\ &= \left|u(y) - f(y)\right| \int_{\mathbb{R}^n} \chi_K(x)\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,d\lambda(x)\\ &\le \left|u(y) - f(y)\right| \int_{\mathbb{R}^n} k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\,d\lambda(x)\\ &= \left|u(y) - f(y)\right|\,.\end{aligned}$$

Thus,

$$\left\|\hat{u}_\delta - \hat{f}_\delta\right\|_{L^1(K)} \le \int_{C_K} \left|u(y) - f(y)\right|\, d\lambda(y) = \|u - f\|_{L^1(C_K)} < \frac{\varepsilon}{3}\,.$$

In conclusion, by the triangle inequality,

$$\begin{aligned}\|\hat{u}_\delta - u\|_{L^1(K)} &\le \|u - f\|_{L^1(K)} + \left\|\hat{f}_\delta - f\right\|_{L^1(K)} + \left\|\hat{u}_\delta - \hat{f}_\delta\right\|_{L^1(K)}\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\\ &= \varepsilon\,,\end{aligned}$$

where $\delta \in (0, d)$ is arbitrary.

□

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be open, and suppose $u : \Omega \to \mathbb{C}$ is a locally integrable function such that*

$$\int_K u\,d\lambda = 0$$

for every compact subset $K \subset \Omega$. Then, $u = 0$ almost everywhere in Ω.

Proof. We first prove the statement for the case when u is real-valued. Recall that the Lebesgue measure is inner regular, that is, for any measurable set $S \subset \mathbb{R}^n$,

$$\lambda(S) = \sup\{\lambda(K)\,|\,K \text{ is a compact subset of } S\}\,.$$

Let $n \in \mathbb{N}$, and suppose $\lambda\big(u^{-1}\big((\tfrac{1}{n}, +\infty)\big)\big) > 0$. Then, there exists a compact set $K_n \subset u^{-1}\big((\tfrac{1}{n}, +\infty)\big)$ with $\lambda(K_n) > 0$. Since $u > \frac{1}{n}$ on K_n, we then have

$$\int_{K_n} u\,d\lambda \ge \int_{K_n} \frac{1}{n}\,d\lambda = \frac{1}{n}\lambda(K_n) > 0\,,$$

which is a contradiction. Thus, we must have $\lambda\big(u^{-1}\big((\tfrac{1}{n}, +\infty)\big)\big) = 0$. Similarly, if $\lambda\big(u^{-1}\big((-\infty, -\tfrac{1}{n})\big)\big) > 0$, we may find a compact set $K_{-n} \subset u^{-1}\big((-\infty, -\tfrac{1}{n})\big)$ with positive measure, and we would then obtain the contradiction

$$\int_{K_{-n}} u\,d\lambda \le \int_{K_{-n}} -\frac{1}{n}\,d\lambda = -\frac{1}{n}\lambda(K_{-n}) < 0\,.$$

This shows that for all $n \in \mathbb{N}$ we must have

$$\lambda\left(u^{-1}\left(\left(\frac{1}{n}, +\infty\right)\right)\right) = \lambda\left(u^{-1}\left(\left(-\infty, -\frac{1}{n}\right)\right)\right) = 0\,.$$

Thus, the set

$$T := u^{-1}(\mathbb{R} \setminus \{0\}) = \bigcup_{n\in\mathbb{N}} \left[u^{-1}\left(\left(\frac{1}{n}, +\infty\right)\right) \cup u^{-1}\left(\left(-\infty, -\frac{1}{n}\right)\right)\right]$$

has measure 0 and $u = 0$ on $\Omega \setminus T$. Thus, the lemma holds when u is real-valued. If u is complex-valued and its integral over any compact subset of Ω vanishes, then for all compact $K \subset \Omega$ we have

$$\int_K \mathrm{Re}(u)\, d\lambda + i \int_K \mathrm{Im}(u)\, d\lambda = \int_K u\, d\lambda = 0\,.$$

Thus, the integrals of $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ over any compact subset of Ω vanish, which implies that $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ vanish almost everywhere in Ω and hence also $u = 0$ almost everywhere in Ω. $\square$

Definition 3.5. For an open set $\Omega \subset \mathbb{R}^n$, and letting $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, we denote by $\mathcal{D}(\Omega, \mathbb{F})$ the vector space of C^∞ $\mathbb{F}$-valued functions on Ω with compact support in Ω. In this text, we will also write $\mathcal{D}(\Omega)$ to denote $\mathcal{D}(\Omega, \mathbb{C})$.

Lemma 3.6. *Suppose $\Omega \subset \mathbb{R}^n$ is open.*

(i) *If $u : \Omega \to \mathbb{C}$ is a locally integrable function such that for all $\psi \in \mathcal{D}(\Omega, \mathbb{R})$*

$$\int_\Omega \psi u\, d\lambda = 0\,,$$

then $u = 0$ almost everywhere on Ω.

(ii) *As a consequence, if $u, v \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$ fulfil*

$$\int_\Omega \psi u\, d\lambda = \int_\Omega \psi v\, d\lambda\,,$$

for all $\psi \in \mathcal{D}(\Omega, \mathbb{R})$, then $u = v$ almost everywhere in Ω.

Proof. (i) Fix a compact set $K \subset \Omega$, and let $\delta \in (0, +\infty)$. Using notation from Lemma 3.3, for every $x \in \Omega_\delta$ we have

$$u_\delta(x) = \int_\Omega u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) = 0\,,$$

since the function $y \mapsto k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n} \in \mathbb{R}$ on Ω is C^∞ with compact support. Thus, we have $\hat{u}_\delta = 0$ on Ω. Then, choosing a sequence $\{\delta_m\}_{m\in\mathbb{N}}$ in $(0, +\infty)$ converging to 0, for all $m \in \mathbb{N}$ we have

$$\begin{aligned}\left|\int_K u\, d\lambda\right| &\le \int_K |u|\ d\lambda \\ &= \int_K |u - \hat{u}_{\delta_m}|\ d\lambda\,,\end{aligned}$$

which by Lemma 3.3 (vi) converges to 0 as $m \to \infty$. Thus,

$$\int_K u\, d\lambda = 0$$

for every compact subset $K \subset \Omega$, which by Lemma 3.4 implies that $u = 0$ almost everywhere in Ω.

(ii) It follows from (i) that $u - v = 0$ almost everywhere in Ω and hence $u = v$ almost everywhere in Ω.

$\square$

3.2. **Differential Operators and Formal Adjoints.**

Let $\Omega \subset \mathbb{R}^n$ be open, and let $\mathcal{V}(\Omega)$ denote the vector space of complex-valued functions on Ω. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We define the notation

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} := \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}$$

if $\alpha \neq (0, \cdots, 0)$, and we let $\left(\frac{\partial}{\partial x}\right)^{(0,\cdots,0)}$ denote the identity operator $\mathcal{V}(\Omega) \to \mathcal{V}(\Omega)$.

For $k \in \mathbb{Z}_{\geq 0}$, we define a *linear differential operator A of order k on Ω* to be a $\mathbb{C}$-linear map of the form

$$A : C^k(\Omega) \to \mathcal{V}(\Omega)\,, \quad Af := \sum_{\substack{\alpha \in (\mathbb{Z}_{\geq 0})^n \\ |\alpha| \leq k}} a_\alpha \left(\frac{\partial}{\partial x}\right)^{\alpha} f$$

for $f \in C^k(\Omega)$, where for each $\alpha \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha| \leq k$, a_α is a complex-valued function on Ω and is called a *coefficient* of A. We may define the notation

$$\Theta_k^n := \left\{\alpha \in (\mathbb{Z}_{\geq 0})^n \,\middle|\, |\alpha| \leq k\right\},$$

and write

$$A = \sum_{\alpha \in \Theta_k^n} a_\alpha \left(\frac{\partial}{\partial x}\right)^{\alpha}.$$

If $U \subset \Omega$ is an open set, we may also denote by A the operator

$$C^k(U) \to \mathcal{V}(U)\,, \quad g \mapsto \sum_{\alpha \in \Theta_k^n} a_\alpha\big|_U \left(\frac{\partial}{\partial x}\right)^{\alpha} g$$

on U, if there is no possibility of confusion. We also define the *conjugate* of A to be the linear differential operator on Ω given by

$$\bar{A} : C^k(\Omega) \to \mathcal{V}(\Omega)\,, \quad \bar{A}f := \overline{A\overline{f}} = \sum_{\alpha \in \Theta_k^n} \overline{a_\alpha} \left(\frac{\partial}{\partial x}\right)^{\alpha} f$$

for each $f \in C^k(\Omega)$. Note that if $k = 0$, and letting $\mathbf{0} := (0, \dots, 0) \in \mathbb{R}^n$, the action of the operator A is simply multiplication by the function $a_{\mathbf{0}} : \Omega \to \mathbb{C}$, so we may also regard A as a linear map from the vector space $\mathcal{V}(\Omega)$ of complex-valued functions on Ω to itself.

Note that the set of linear differential operators of a given order $k \in \mathbb{Z}_{\geq 0}$ on Ω forms a complex vector space, which we may denote by $\mathcal{LD}^k(\Omega)$. Moreover, if $k, \ell \in \mathbb{Z}_{\geq 0}$ and $k \leq \ell$, then we may regard $\mathcal{LD}^k(\Omega)$ as a subspace of $\mathcal{LD}^\ell(\Omega)$. Then, if A and B are linear differential operators on Ω of respective orders k and ℓ, we may denote by $A + B$ the sum of A and B in $\mathcal{LD}^\ell(\Omega) = \mathcal{LD}^{\max\{k,\ell\}}(\Omega)$.

Definition 3.7. Let $\Omega \subset \mathbb{R}^n$ be open, let $k \in \mathbb{Z}_{\geq 0}$, and suppose

$$A = \sum_{\alpha \in \Theta_k^n} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha$$

is a linear differential operator of order k on Ω. Suppose further that for each $\alpha \in \Theta_k^n$, we have $a_\alpha \in C^{|\alpha|}(\Omega)$.

(i) We define the *formal transpose* of A to be the linear differential operator ${}^tA$ of degree k on Ω given by

$${}^tA : C^k(\Omega) \to \mathcal{V}(\Omega)\,, \quad f \mapsto \sum_{\alpha \in \Theta_k^n} (-1)^{|\alpha|} \left(\frac{\partial}{\partial x}\right)^\alpha (a_\alpha f)\,,$$

for $f \in C^k(\Omega)$.

(ii) The *formal adjoint* of A is defined to be $A^* := {}^t\bar{A} = \overline{{}^tA}$, that is, the linear differential operator of degree k on Ω given by

$$A^* : C^k(\Omega) \to \mathcal{V}(\Omega)\,, \quad f \mapsto \sum_{\alpha \in \Theta_k^n} (-1)^{|\alpha|} \left(\frac{\partial}{\partial x}\right)^\alpha (\overline{a_\alpha} f)$$

for each $f \in C^k(\Omega)$.

(iii) If $u, v \in L^1_{\text{loc}}(\Omega, \mathbb{C})$, we say that $v = A_{\text{distr}} u$ if for every function $\varphi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega u \cdot {}^tA\varphi \, d\lambda = \int_\Omega v\varphi \, d\lambda\,,$$

or, equivalently,

$$\int_\Omega u \cdot \overline{A^*\varphi} \, d\lambda = \int_\Omega v\bar{\varphi} \, d\lambda$$

for every function $\varphi \in \mathcal{D}(\Omega)$.

Remark 3.8. In Definition 3.7 (iii), if $w \in L^1_{\text{loc}}(\Omega, \mathbb{C})$ is another function such that $w = A_{\text{distr}} u$, then for all $\varphi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega v\varphi \, d\lambda = \int_\Omega u \cdot {}^tA\varphi \, d\lambda = \int_\Omega w\varphi \, d\lambda\,,$$

so that $v = w$ almost everywhere by Lemma 3.6 (ii).

Definition 3.9. Let $k, \ell \in \mathbb{Z}_{\geq 0}$ and suppose A and B are linear differential operators on Ω of orders k and ℓ respectively. Suppose further that the coefficients of B are C^k. We then define AB to be the linear differential operator of order $k + \ell$ on Ω given by

$$AB : C^{k+\ell}(\Omega) \to \mathcal{V}(\Omega)\,, \quad (AB)f := A(Bf)$$

for each $f \in C^{k+\ell}(\Omega)$.

Lemma 3.10. *Suppose $\Omega \subset \mathbb{R}^n$ is open, $k \in \mathbb{Z}_{\geq 0}$, and*

$$A = \sum_{\alpha \in \Theta_k^n} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha$$

is a linear differential operator of order k on Ω with C^∞ coefficients.

(i) *Let $u : \Omega \to \mathbb{C}$ be a function. If $k = 0$ and u is locally integrable (which includes the case when u is C^0), or if $k \geq 1$ and u is C^k, then Au is locally integrable on Ω and $Au = A_{\text{distr}} u$.*

(ii) *Let* $u, v \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$*. Then,* $v = A_{\mathrm{distr}} u$ *if and only if for every point* $p \in \Omega$ *there is a neighbourhood* U *of* p *in* Ω *such that* $v|_U = A_{\mathrm{distr}}\big(u|_U\big)$*.*

(iii) ${}^t({}^tA) = A$ *and* $(A^*)^* = A$*.*

Suppose now that $\ell \in \mathbb{Z}_{\geq 0}$ *and*

$$B = \sum_{\alpha \in \Theta^n_\ell} b_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha$$

is another linear differential operator on Ω*, of order* ℓ *and with* C^∞ *coefficients.*

(iv) *For every* $\zeta \in \mathbb{C}$*, we have* ${}^t(\zeta A + B) = \zeta\, {}^tA + {}^tB$ *and* $(\zeta A + B)^* = \bar{\zeta} A^* + B^*$*.*

(v) ${}^t(AB) = {}^tB\,{}^tA$ *and* $(AB)^* = B^* A^*$*.*

(vi) *Suppose* $u, v, \hat{u}, \hat{v} \in L^1_{\mathrm{loc}}(\Omega)$ *with* $v = A_{\mathrm{distr}} u$ *and* $\hat{v} = A_{\mathrm{distr}} \hat{u}$*. Then, for any* $\zeta \in \mathbb{C}$ *we have* $\zeta v + \hat{v} = A_{\mathrm{distr}}(\zeta u + \hat{u})$*, or in other words,* $A_{\mathrm{distr}}(\zeta u + \hat{u}) = \zeta A_{\mathrm{distr}} u + A_{\mathrm{distr}} \hat{u}$*.*

(vii) *Suppose* $u, v, w \in L^1_{\mathrm{loc}}(\Omega)$ *with* $v = A_{\mathrm{distr}} u$ *and* $w = B_{\mathrm{distr}} u$*. Then, for any* $\zeta \in \mathbb{C}$ *we have* $\zeta v + w = (\zeta A + B)_{\mathrm{distr}} u$*, or in other words,* $(\zeta A + B)_{\mathrm{distr}} u = \zeta A_{\mathrm{distr}} u + B_{\mathrm{distr}} u$*.*

(viii) *Suppose* $u, v, w \in L^1_{\mathrm{loc}}(\Omega)$ *with* $v = B_{\mathrm{distr}} u$*. Then,* $w = (AB)_{\mathrm{distr}} u$ *if and only if* $w = A_{\mathrm{distr}} v = A_{\mathrm{distr}}(B_{\mathrm{distr}} u)$*.*

(ix) *Suppose* $k = 1$ *and* $a_{\mathbf{0}} = 0$*, that is,* A *may be written as*

$$A = \sum_{j=1}^{n} f_j \frac{\partial}{\partial x^j}$$

for some C^∞ *complex-valued functions* $f_1, \cdots, f_n$ *on* Ω*. Suppose also that* $u, v \in L^1_{\mathrm{loc}}(\Omega)$ *with* $v = A_{\mathrm{distr}} u$*, and let* $\rho \in C^\infty(\Omega)$*. Then, we have*

$$\rho v + u A \rho = A_{\mathrm{distr}}(\rho u)\,,$$

or in other words,

$$A_{\mathrm{distr}}(\rho u) = (A_{\mathrm{distr}} \rho) u + \rho A_{\mathrm{distr}} u\,.$$

Proof. (i) Suppose first that $k = 0$ and u is locally integrable. Then, letting again $\mathbf{0} := (0, \ldots, 0) \in \mathbb{R}^n$, the function Au on Ω is the product $a_{\mathbf{0}} u$, where the function $a_{\mathbf{0}}$ is continuous on Ω. Then, if $K \subset \Omega$ is a compact set, u is integrable on K and $a_{\mathbf{0}}$ is measurable and bounded on K, and hence $a_{\mathbf{0}} u$ is integrable on K. Thus, $Au \in L^1_{\mathrm{loc}}(\Omega)$, and the fact that $Au = A_{\mathrm{distr}} u$ is immediate. Assume now that $k \geq 1$ and $u \in C^k(\Omega)$. Since the function $Au : \Omega \to \mathbb{C}$ is continuous, it is in $L^1_{\mathrm{loc}}(\Omega)$. To show that $Au = A_{\mathrm{distr}} u$, let $\varphi \in \mathcal{D}(\Omega)$. For a fixed $\alpha \in \Theta^n_k \setminus \{\mathbf{0}\}$, we may write

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \prod_{j=1}^{|\alpha|} \frac{\partial}{\partial r^j}\,,$$

where for each $j \in \{1, \ldots, |\alpha|\}$, r^j is one of the standard coordinates $x^1, \ldots, x^n$ on $\mathbb{R}^n$. If $|\alpha| = 1$, we have

$$a_\alpha \varphi \left(\frac{\partial}{\partial x}\right)^\alpha u = a_\alpha \varphi \frac{\partial u}{\partial r^1} = \frac{\partial(a_\alpha \varphi u)}{\partial r^1} - \frac{\partial(a_\alpha \varphi)}{\partial r^1} u = \frac{\partial(a_\alpha \varphi u)}{\partial r^1} - \left(\left(\frac{\partial}{\partial x}\right)^\alpha (a_\alpha \varphi)\right) u\,,$$

and for $|\alpha| = 2$, we have

$$\begin{aligned} a_\alpha\varphi\left(\frac{\partial}{\partial x}\right)^\alpha u &= a_\alpha\varphi\frac{\partial^2 u}{\partial r^1\partial r^2} \\ &= \frac{\partial}{\partial r^1}\left(a_\alpha\varphi\frac{\partial u}{\partial r^2}\right) - \frac{\partial}{\partial r^2}\left(\frac{\partial(a_\alpha\varphi)}{\partial r^1}u\right) + \left(\left(\frac{\partial}{\partial x}\right)^\alpha(a_\alpha\varphi)\right)u\,. \end{aligned}$$

If $|\alpha| \geq 3$, we may continue using the product rule repeatedly to show that

$$\begin{aligned} a_\alpha\varphi\left(\frac{\partial}{\partial x}\right)^\alpha u &= a_\alpha\varphi\left[\prod_{j=1}^{|\alpha|}\frac{\partial}{\partial r^j}\right]u \\ &= \frac{\partial}{\partial r^{|\alpha|}}\left(a_\alpha\varphi\left[\prod_{j=1}^{|\alpha|-1}\frac{\partial}{\partial r^j}\right]u\right) \\ &\quad + \sum_{\ell=1}^{|\alpha|-2}(-1)^\ell\frac{\partial}{\partial r^{|\alpha|-\ell}}\left(\left(\left[\prod_{j=|\alpha|-\ell+1}^{|\alpha|}\frac{\partial}{\partial r^j}\right](a_\alpha\varphi)\right)\left(\left[\prod_{j=1}^{|\alpha|-\ell-1}\frac{\partial}{\partial r^j}\right]u\right)\right) \\ &\quad + (-1)^{|\alpha|-1}\frac{\partial}{\partial r^1}\left(\left(\left[\prod_{j=2}^{|\alpha|}\frac{\partial}{\partial r^j}\right](a_\alpha\varphi)\right)u\right) \\ &\quad + (-1)^{|\alpha|}\left(\left(\frac{\partial}{\partial x}\right)^\alpha(a_\alpha\varphi)\right)u\,. \end{aligned}$$

Thus, allowing any value of $|\alpha| \in \mathbb{Z}_{\geq 1}$, we have

$$a_\alpha\varphi\left(\frac{\partial}{\partial x}\right)^\alpha u = \mathcal{S}_\alpha + (-1)^{|\alpha|}\left(\left(\frac{\partial}{\partial x}\right)^\alpha(a_\alpha\varphi)\right)u\,,$$

where $\mathcal{S}_\alpha$ is a sum of functions $\Omega \to \mathbb{C}$, each of which is the partial derivative $\frac{\partial\vartheta}{\partial r}$ with respect to some standard coordinate r on $\mathbb{R}^n$ of a C^1 function $\vartheta : \Omega \to \mathbb{C}$ with compact support $\operatorname{supp}\vartheta \subset \operatorname{supp}\varphi$. We may choose an open bounded rectangle $R \subset \mathbb{R}^n$ containing $\operatorname{supp}\varphi$, and write the integral over Ω of each term $\frac{\partial\vartheta}{\partial r}$ of the sum $\mathcal{S}_\alpha$ as an integral over R by restricting ϑ to R, extending it by 0 if $R \not\subset \Omega$. We may then apply Fubini's theorem to write the integral over R of each term $\frac{\partial\vartheta}{\partial r}$ of $\mathcal{S}_\alpha$ so that the innermost integral is taken with respect to the corresponding coordinate r of the outermost partial derivative, and then apply the Fundamental Theorem of Calculus to conclude that

$$\int_\Omega \mathcal{S}_\alpha\,d\lambda = 0\,,$$

so that

$$\int_\Omega a_\alpha\varphi\left(\frac{\partial}{\partial x}\right)^\alpha u\,d\lambda = \int_\Omega (-1)^{|\alpha|}\left(\left(\frac{\partial}{\partial x}\right)^\alpha(a_\alpha\varphi)\right)u\,d\lambda\,.$$

Note that the above equality also holds for $\alpha = \mathbf{0}$ and hence it holds for all $\alpha \in \Theta_k^n$. It follows that

$$\begin{aligned}
\int_\Omega (Au)\varphi\, d\lambda &= \int_\Omega \bigg(\sum_{\alpha \in \Theta_k^n} a_\alpha \Big(\frac{\partial}{\partial x} \Big)^\alpha u \bigg) \varphi\, d\lambda \\
&= \sum_{\alpha \in \Theta_k^n} \int_\Omega a_\alpha \varphi \Big(\frac{\partial}{\partial x} \Big)^\alpha u\, d\lambda \\
&= \sum_{\alpha \in \Theta_k^n} \int_\Omega (-1)^{|\alpha|} \bigg(\Big(\frac{\partial}{\partial x} \Big)^\alpha (a_\alpha \varphi) \bigg) u\, d\lambda \\
&= \int_\Omega \sum_{\alpha \in \Theta_k^n} (-1)^{|\alpha|} \bigg(\Big(\frac{\partial}{\partial x} \Big)^\alpha (a_\alpha \varphi) \bigg) u\, d\lambda \\
&= \int_\Omega u \cdot {}^t\!A\varphi\, d\lambda\,,
\end{aligned}$$

which shows that indeed $Au = A_{\text{distr}}u$.

(ii) ($\Leftarrow$) Suppose that for every point $p \in \Omega$ there exists a neighbourhood U of p in Ω such that $v|_U = A_{\text{distr}}\big(u|_U\big)$. Let $\varphi \in \mathcal{D}(\Omega)$. We may find a finite collection $\{U_j\}_{j=1}^m$ of open subsets of Ω, for $m \in \mathbb{N}$, such that $\operatorname{supp}\varphi \subset \bigcup_{j=1}^m U_j$ and for all $j \in \{1,\dots,m\}$ we have $v|_{U_j} = A_{\text{distr}}\big(u|_{U_j}\big)$. The collection $\mathcal{U} := \{U_j\}_{j=1}^m \cup \{\Omega \setminus \operatorname{supp}\varphi\}$ is then an open cover for Ω, which is second countable, and hence there exists a smooth partition of unity $\{\rho_j\}_{j=1}^{m+1}$ on Ω such that $\operatorname{supp}\rho_j \subset U_j$ for all $j \in \{1,\dots,m\}$ and $\operatorname{supp}\rho_{m+1} \subset \Omega \setminus \operatorname{supp}\varphi$. Fix $j \in \{1,\dots,m\}$. The function $\rho_j\varphi$ is smooth on Ω and $\operatorname{supp}\rho_j\varphi \subset \operatorname{supp}\rho_j \cap \operatorname{supp}\varphi$. Since $\operatorname{supp}\rho_j\varphi$ is a closed subset of the compact set $\operatorname{supp}\varphi$, it is itself compact, and we also have $\operatorname{supp}\rho_j\varphi \subset \operatorname{supp}\rho_j \subset U_j$. It follows that the restriction $\rho_j\varphi|_{U_j}$ is in $\mathcal{D}(U_j)$. Since $\rho_j\varphi$ vanishes on the open set $\Omega \setminus \operatorname{supp}\rho_j\varphi \supset \Omega \setminus U_j$, so too does ${}^t\!A(\rho_j\varphi)$, and thus

$$\begin{aligned}
\int_\Omega u \cdot {}^t\!A(\rho_j\varphi)\, d\lambda &= \int_{U_j} u|_{U_j} \cdot \big({}^t\!A(\rho_j\varphi)\big)|_{U_j}\, d\lambda \\
&= \int_{U_j} u|_{U_j} \cdot {}^t\!A(\rho_j\varphi|_{U_j})\, d\lambda \\
&= \int_{U_j} v|_{U_j} \cdot \rho_j\varphi|_{U_j}\, d\lambda \\
&= \int_\Omega v\rho_j\varphi\, d\lambda\,.
\end{aligned}$$

For a point $p \in \Omega$, if $p \in \operatorname{supp}\varphi$ then

$$\bigg(\varphi \sum_{j=1}^m \rho_j\bigg)(p) = \bigg(\varphi \sum_{j=1}^{m+1} \rho_j\bigg)(p) = \varphi(p)\,,$$

while if $p \in \Omega \setminus \operatorname{supp}\varphi$ we also have

$$\bigg(\varphi \sum_{j=1}^m \rho_j\bigg)(p) = 0 = \varphi(p)\,.$$

Thus,

$$\begin{aligned}\int_\Omega u\cdot {}^tA\varphi\,d\lambda &= \int_\Omega u\cdot {}^tA\Bigl(\varphi\sum_{j=1}^m \rho_j\Bigr)\,d\lambda\\ &= \int_\Omega \sum_{j=1}^m u\cdot {}^tA(\rho_j\varphi)\,d\lambda\\ &= \sum_{j=1}^m \int_\Omega u\cdot {}^tA(\rho_j\varphi)\,d\lambda\\ &= \sum_{j=1}^m \int_\Omega v\rho_j\varphi\,d\lambda\\ &= \int_\Omega \sum_{j=1}^m v\rho_j\varphi\,d\lambda\\ &= \int_\Omega v\varphi\,d\lambda\,,\end{aligned}$$

which shows that $v = A_{\mathrm{distr}}u$ on Ω.

($\Rightarrow$) Obviously.

(iii) We first show that ${}^t({}^tA)\varphi = A\varphi$ for all $\varphi \in \mathcal{D}(\Omega)$. Choose $\varphi \in \mathcal{D}(\Omega)$. Then, for any other $\vartheta \in \mathcal{D}(\Omega)$, by (i) we have

$$\int_\Omega \vartheta\cdot {}^t({}^tA)\varphi\,d\lambda = \int_\Omega ({}^tA\vartheta)\varphi\,d\lambda = \int_\Omega \vartheta A\varphi\,d\lambda\,,$$

which implies that

$$\int_\Omega \vartheta\bigl({}^t({}^tA)\varphi - A\varphi\bigr)\,d\lambda = 0\,.$$

Since the function ${}^t({}^tA)\varphi - A\varphi$ is locally integrable on Ω, by Lemma 3.6 (i) we have ${}^t({}^tA)\varphi - A\varphi = 0$ almost everywhere, and continuity of ${}^t({}^tA)\varphi - A\varphi$ then implies ${}^t({}^tA)\varphi - A\varphi = 0$ on Ω. To show that ${}^t({}^tA) = A$ in general, choose arbitrary $f \in C^k(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Applying again (i), we have

$$\int_\Omega f\cdot {}^t({}^t({}^tA))\varphi\,d\lambda = \int_\Omega \bigl({}^t({}^tA)f\bigr)\varphi\,d\lambda$$

and

$$\int_\Omega f\cdot {}^t({}^t({}^tA))\varphi\,d\lambda = \int_\Omega f\cdot {}^tA\varphi\,d\lambda = \int_\Omega (Af)\varphi\,d\lambda\,,$$

so that

$$\int_\Omega \bigl({}^t({}^tA)f - Af\bigr)\varphi\,d\lambda = 0\,.$$

Since the function ${}^t({}^tA)f - Af$ is continuous on Ω, it is locally integrable, so again it follows from Lemma 3.6 (i) that ${}^t({}^tA)f - Af = 0$ almost everywhere; and again by continuity, we must then have ${}^t({}^tA)f - Af = 0$ on Ω. This shows that indeed ${}^t({}^tA) = A$. The fact that $(A^*)^* = A$ follows, since

$$(A^*)^* = \overline{{}^t({}^t\overline{A})} = \overline{\overline{A}} = A\,.$$

(iv) This part is left for the reader to check.

(v) Given arbitrary $f \in C^{k+\ell}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}\int_\Omega \varphi \cdot {}^t(AB)f &= \int_\Omega f \cdot (AB)\varphi \, d\lambda \\ &= \int_\Omega f \cdot A(B\varphi)\, d\lambda \\ &= \int_\Omega ({}^tAf)B\varphi \, d\lambda \\ &= \int_\Omega {}^tB({}^tAf)\varphi \, d\lambda \\ &= \int_\Omega \big(({}^tB{}^tA)f\big)\varphi \, d\lambda\end{aligned}$$

By a similar argument as in part (iii), continuity of the function ${}^t(AB)f - ({}^tB{}^tA)f$ implies that ${}^t(AB)f - ({}^tB{}^tA)f = 0$ on Ω, and since $f \in C^{k+\ell}(\Omega)$ was arbitrary, this shows that ${}^t(AB) = {}^tB{}^tA$ as operators. We may then show that $(AB)^* = B^*A^*$ as follows: for any $f \in C^{k+\ell}(\Omega)$, we have

$$(AB)^*f = \overline{{}^t(AB)}f = \overline{{}^t(AB)\overline{f}} = \overline{({}^tB{}^tA)\overline{f}} = \overline{{}^tB({}^tA\overline{f})} = \overline{{}^tB}(\overline{{}^tA}f) = (B^*A^*)f\,.$$

Statements (vi), (vii) and (viii) are left for the reader to check.

(ix) First observe that under our requirements for A, for any $g, h \in C^1(\Omega)$ we have

$$A(gh) = (Ag)h + gAh$$

and

$${}^tAg = -g\sum_{j=1}^n \frac{\partial f_j}{\partial x^j} - Ag\,.$$

Then, for any $\varphi \in D(\Omega)$,

$$\begin{aligned}{}^tA(\rho\varphi) &= -\rho\varphi \sum_{j=1}^n \frac{\partial f_j}{\partial x^j} - A(\rho\varphi) \\ &= -\rho\varphi \sum_{j=1}^n \frac{\partial f_j}{\partial x^j} - (A\rho)\varphi - \rho(A\varphi) \\ &= \rho\,{}^tA\varphi - (A\rho)\varphi\,,\end{aligned}$$

and hence

$$u\rho\,{}^tA\varphi = u\,{}^tA(\rho\varphi) + u(A\rho)\varphi\,.$$

Since $\rho\varphi \in D(\Omega)$, we have

$$\int_\Omega u\,{}^tA(\rho\varphi)\, d\lambda = \int_\Omega v\rho\varphi\, d\lambda\,,$$

so that

$$\begin{aligned}\int_\Omega u\rho\,{}^tA\varphi\, d\lambda &= \int_\Omega v\rho\varphi\, d\lambda + \int_\Omega u(A\rho)\varphi\, d\lambda \\ &= \int_\Omega \varphi(v\rho + uA\rho)\, d\lambda\,,\end{aligned}$$

which proves the claim.

$\square$

Lemma 3.11. *Suppose $k : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function with $\operatorname{supp} k \subset B(0,1)$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\delta \in (0, +\infty)$. Define $\Omega_\delta \subset \Omega$ as in Lemma 3.3, that is,*

$$\Omega_\delta := \{x \in \mathbb{R}^n \,|\, \operatorname{dist}(x, \Omega^c) > \delta\}\,;$$

and for any $u \in L^1_{loc}(\Omega)$, define $u_\delta : \Omega_\delta \to \mathbb{C}$ also as in Lemma 3.3, that is,

$$u_\delta(x) := \int_\Omega u(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y)$$

for each $x \in \Omega_\delta$. For $\ell \in \mathbb{Z}_{\geq 0}$, let

$$A := \sum_{\alpha \in \Theta^n_\ell} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha$$

be a linear differential operator of order ℓ on $\mathbb{R}^n$ with constant coefficients. Then, if $u, v \in L^1_{loc}(\Omega)$ with $v = A_{\mathrm{distr}} u$, on Ω_δ we have

$$v_\delta = A(u_\delta)\,.$$

Proof. For each $x \in \Omega_\delta$, define

$$\psi_{\delta,x} : \Omega \to \mathbb{R}^n\,, \quad y \mapsto \frac{x-y}{\delta}$$

and

$$k_{\delta,x} := \frac{1}{\delta^n}\, k \circ \psi_{\delta,x} \in \mathcal{D}(\Omega)\,.$$

If $\ell = 0$, then for any $x \in \Omega_\delta$

$$\begin{aligned}
A(u_\delta)(x) &= \int_\Omega u(y)\, a_{\mathbf{0}}\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) \\
&= \int_\Omega u \cdot {}^t A k_{\delta,x}\, d\lambda \\
&= \int_\Omega v\, k_{\delta,x}\, d\lambda \\
&= \int_\Omega v(y)\, k\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y) \\
&= v_\delta(x)\,.
\end{aligned}$$

Suppose $\ell \geq 1$ and fix $\alpha \in \Theta^n_\ell \setminus \{\mathbf{0}\}$. We have

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \prod_{j=1}^{|\alpha|} \frac{\partial}{\partial r^j}\,,$$

where for each $j \in \{1, \ldots, |\alpha|\}$ we have $r^j = x^{p(j)}$ for some $p(j) \in \{1, \ldots, n\}$. By the proof of Lemma 3.3 (iii), we know that for any $w \in L^1_{loc}(\Omega)$ and any C^∞ function $l : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{supp} l \subset B(0,1)$, the C^∞ function

$$g : \Omega_\delta \to \mathbb{C}\,, \quad x \mapsto \int_\Omega w(y)\, l\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y)$$

fulfils

$$\frac{\partial g}{\partial r^j} = \left(x \mapsto \int_\Omega \frac{w(y)}{\delta} \frac{\partial l}{\partial r^j}\left(\frac{x-y}{\delta}\right) \frac{1}{\delta^n}\, d\lambda(y)\right)$$

for each $j \in \{1, \ldots, |\alpha|\}$. Using this fact, we may show by induction that

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} u_\delta = \left[\prod_{j=1}^{|\alpha|} \frac{\partial}{\partial r^j}\right]\left(x \mapsto \int_\Omega u(y)\, k\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\, d\lambda(y)\right)$$

$$= \left(x \mapsto \int_\Omega \frac{u(y)}{\delta^{|\alpha|}}\left(\left[\prod_{j=1}^{|\alpha|} \frac{\partial}{\partial r^j}\right]k\right)\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\, d\lambda(y)\right)$$

$$= \left(x \mapsto \int_\Omega \frac{u(y)}{\delta^{|\alpha|}}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha} k\right)\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\, d\lambda(y)\right).$$

Since this equality also holds for $\alpha = \mathbf{0}$, we have

$$A(u_\delta) = \sum_{\alpha \in \Theta_\ell^n} a_\alpha \left(\frac{\partial}{\partial x}\right)^{\alpha} u_\delta$$

$$= \left(x \mapsto \sum_{\alpha \in \Theta_\ell^n} \int_\Omega a_\alpha \frac{u(y)}{\delta^{|\alpha|}}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha} k\right)\left(\frac{x-y}{\delta}\right)\frac{1}{\delta^n}\, d\lambda(y)\right).$$

Let again $\alpha \in \Theta_\ell^n \setminus \{\mathbf{0}\}$ and

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} = \prod_{j=1}^{|\alpha|} \frac{\partial}{\partial r^j},$$

where for each $j \in \{1, \ldots, |\alpha|\}$ we have $r^j = x^{q(j)}$ for some $q(j) \in \{1, \ldots, n\}$. Observe that for any C^∞ function $l : \mathbb{R}^n \to \mathbb{R}$, we have

$$\frac{\partial}{\partial r^j}\left(\frac{1}{\delta^n}\, l \circ \psi_{\delta,x}\right) = \frac{1}{\delta^n}\left(\frac{-1}{\delta}\right)\frac{\partial l}{\partial r^j} \circ \psi_{\delta,x}.$$

for each $j \in \{1, \ldots, |\alpha|\}$. Using this, we may show by induction that

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} k_{\delta,x} = \left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{1}{\delta^n}\, k \circ \psi_{\delta,x}\right) = \frac{1}{\delta^n}\left(\frac{-1}{\delta}\right)^{|\alpha|}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha} k\right) \circ \psi_{\delta,x},$$

which also holds if $\alpha = \mathbf{0}$. Then, for any $x \in \Omega_\delta$, we have

$$\begin{aligned}
v_\delta(x) &= \int_\Omega v\, k_{\delta,x}\, d\lambda \\
&= \int_\Omega u \cdot {}^t A k_{\delta,x}\, d\lambda \\
&= \int_\Omega u \sum_{\alpha \in \Theta_\ell^n} (-1)^{|\alpha|}\left(\frac{\partial}{\partial x}\right)^{\alpha} (a_\alpha k_{\delta,x})\, d\lambda \\
&= \int_\Omega \sum_{\alpha \in \Theta_\ell^n} u\left((-1)^{|\alpha|}\, a_\alpha\, \frac{1}{\delta^n}\left(\frac{-1}{\delta}\right)^{|\alpha|}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha} k\right) \circ \psi_{\delta,x}\right) d\lambda \\
&= \sum_{\alpha \in \Theta_\ell^n} \int_\Omega \frac{u(y)}{\delta^{|\alpha|}} a_\alpha\, \frac{1}{\delta^n}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha} k\right)\left(\frac{x-y}{\delta}\right) d\lambda(y) \\
&= (A(u_\delta))(x).
\end{aligned}$$

□

For the remainder of this text, for any $z_0 \in \mathbb{C}$ and $R \in (0, +\infty]$, we define

$$\Delta(z_0; R) := \{z \in \mathbb{C} \mid |z - z_0| < R\}$$

and

$$\Delta^*(z_0; R) := \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\} .$$

Also, for $r \in [0, R)$, we define

$$\Delta(z_0; r, R) := \{z \in \mathbb{C} \mid r < |z - z_0| < R\} .$$

4.1. **Background Material on Holomorphic Functions.**
In this section, we recall some basic definitions and results about holomorphic functions that are normally studied in a first course on complex analysis. We therefore skip proofs.

Definition 4.1. Let $A \subset \mathbb{C}$. Suppose $f : A \to \mathbb{C}$ is a function defined on some neighbourhood U of a point z_0 in $\mathbb{C}$. Then, f is said to be *complex-differentiable* at z_0 if the function

$$z \mapsto \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C} ,$$

which is defined on $U \setminus \{z_0\}$, has a limit at z_0. Then, we define the *complex derivative* of f at z_0 to be the complex number

$$f'(z_0) = \frac{df}{dz}(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} .$$

Definition 4.2. Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is said to be *holomorphic on* Ω (or just *holomorphic*) if it is complex-differentiable at every point in Ω. We then define the function $f' : \Omega \to \mathbb{C}$, which maps a point $z \in \Omega$ to the derivative of f at z. We denote the set of holomorphic functions on Ω by $\mathcal{O}(\Omega)$. A function $f \in \mathcal{O}(\mathbb{C})$ is said to be *entire*.

Theorem 4.3. *A holomorphic function $f : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{C}$ is open, is continuous on Ω.*

Theorem 4.4. *Let f and g be two holomorphic functions on an open subset $\Omega \subset \mathbb{C}$. Then,*

(i) *the function $f + g$ is holomorphic on Ω, and $(f + g)' = f' + g'$;*
(ii) *the function fg is holomorphic on Ω, and $(fg)' = f'g + fg'$;*
(iii) *if g is nowhere-vanishing on Ω, then the function $\frac{f}{g}$ is holomorphic on Ω, and $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.*

Moreover, if $f \in \mathcal{O}(\Omega)$ and $g \in \mathcal{O}(\Upsilon)$, where $\Omega, \Upsilon \subset \mathbb{C}$ are open and $f(\Omega) \subset \Upsilon$, then the function $g \circ f : \Omega \to \mathbb{C}$ is holomorphic on Ω, and $(g \circ f)' = (g' \circ f)f'$.

Example 4.5. (i) A constant function mapping $\mathbb{C}$ to a fixed complex number is entire and its derivative is the zero function on $\mathbb{C}$.
(ii) For all $n \in \mathbb{N}$, the function $z \mapsto z^n$ on $\mathbb{C}$ is entire, and its derivative is the function $z \mapsto nz^{n-1}$ on $\mathbb{C}$. In particular, the identity function on $\mathbb{C}$ is entire and is derivative is the constant function 1.

(iii) It follows from (i),(ii) and Theorem 4.4 that every polynomial on $\mathbb{C}$ is entire, and that every rational function defined on some open subset $U \subset \mathbb{C}$ is holomorphic on U.

(iv) The function mapping a complex number z to its complex conjugate $\bar{z} \in \mathbb{C}$ is not holomorphic on any open subset of $\mathbb{C}$.

For the remainder of Section 4.1, we fix an open subset $\Omega \subset \mathbb{C}$ and write $C^k(\Omega)$ to denote $C^k(\Omega, \mathbb{C})$, for $k \in \mathbb{N}_0 \cup \{\infty\}$.

We define the following linear differential operators, which map $C^k(\Omega)$ to $C^{k-1}(\Omega)$ for any $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

One can check that these operators, apart from being $\mathbb{C}$-linear, fulfill the following properties:

(i)
$$\frac{\partial z}{\partial z} = 1, \quad \frac{\partial \bar{z}}{\partial z} = 0, \quad \frac{\partial z}{\partial \bar{z}} = 0, \quad \frac{\partial \bar{z}}{\partial \bar{z}} = 1,$$
where z and $\bar{z}$ are the identity and conjugate functions respectively on Ω;

(ii) (Leibniz rule) for $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ and $f, g \in C^k(\Omega)$,
$$\frac{\partial(fg)}{\partial z} = \left(\frac{\partial f}{\partial z}\right) g + f\left(\frac{\partial g}{\partial z}\right),$$
and similarly for $\frac{\partial(fg)}{\partial \bar{z}}$;

(iii) if $a, b \in \mathbb{R}$, $a < b$, $\gamma : (a, b) \to \mathbb{C}$ is real-differentiable and $f \in C^1(\Omega)$, and assuming $\gamma(a, b) \subset \Omega$, then
$$\frac{d}{dt}(f \circ \gamma) = \left(\frac{\partial f}{\partial z} \circ \gamma\right)\frac{d\gamma}{dt} + \left(\frac{\partial f}{\partial \bar{z}} \circ \gamma\right)\frac{d\bar{\gamma}}{dt}$$
on (a, b).

Suppose $f \in \mathbb{C}^1(\Omega)$, and let $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$. Then, on Ω,

$$\begin{aligned} & \frac{\partial f}{\partial \bar{z}} = 0 \\ \iff & \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0 \\ \iff & \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{aligned}$$

We will refer to the two equations on the last line as the *homogenous Cauchy-Riemann equations*.

Theorem 4.6. *A function $f : \Omega \to \mathbb{C}$ is holomorphic on Ω if and only if it is real-differentiable and $\frac{\partial f}{\partial \bar{z}} = 0$ on Ω, that is, if and only if it is real-differentiable and its partial derivatives (whose existence follows from real-differentiability) fulfil the homogeneous Cauchy-Riemann equations on Ω. Moreover, the derivative of f on Ω is given by*
$$f' = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Definition 4.7. Let $f : \Omega \to \mathbb{C}$, and suppose there is a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F' = f$ on Ω. Then, F is said to be a *primitive* for f on Ω.

4.2. $\mathbb{C}$ **as a manifold.**

We give $\mathbb{C}$ the smooth manifold structure and standard orientation of $\mathbb{R}^2$, and we let (x, y) denote the standard coordinates on $\mathbb{C}$. Since the functions $z := x + iy$ and $\bar{z} := x - iy$ on $\mathbb{C}$ are C^∞, we may take their differentials

$$dz = dx + i\,dy\,, \quad d\bar{z} = dx - i\,dy\,.$$

For each $p \in \mathbb{C}$, the elements $(dz)_p, (d\bar{z})_p \in (T_p^*\mathbb{C})_{\mathbb{C}}$ are linearly independent and hence form a basis for $(T_p^*\mathbb{C})_{\mathbb{C}}$. Then, any complex 1-form α on an open subset $\Omega \subset \mathbb{C}$ can be written as

$$\alpha = P\,dx + Q\,dy = A\,dz + B\,d\bar{z}$$

for some unique functions $P, Q, A, B : \Omega \to \mathbb{C}$, with

$$A = \frac{1}{2}(P - iQ)\,, \quad B = \frac{1}{2}(P + iQ)\,.$$

and

$$P = A + B\,, \quad Q = i(A - B)$$

Thus, for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, by Proposition 2.13, α is C^k if and only the functions P and Q are C^k, or if and only if the functions A and B are C^k.

Suppose $\Omega \subset \mathbb{C}$ is open and $\gamma : [a, b] \to \Omega$ is a C^1 path, for $a, b \in \mathbb{R}$ with $a < b$. If $\alpha = P\,dx + Q\,dy = A\,dz + B\,d\bar{z}$ is a continuous complex 1-form on Ω, for continuous functions $P, Q, A, B : \Omega \to \mathbb{C}$, then letting $u := x \circ \gamma = \operatorname{Re}\gamma$ and $v := y \circ \gamma = \operatorname{Im}\gamma$, we have

$$\begin{aligned}\int_\gamma \alpha &= \int_a^b \left(P(\gamma(s)) \frac{du}{dt}\bigg|_s + Q(\gamma(s)) \frac{dv}{dt}\bigg|_s \right) ds \\ &= \int_a^b \left(A(\gamma(s)) \frac{d\gamma}{dt}\bigg|_s + B(\gamma(s)) \frac{d\bar{\gamma}}{dt}\bigg|_s \right) ds\,.\end{aligned}$$

For each $p \in \mathbb{C}$, we have

$$(dz)_p \wedge (d\bar{z})_p = -2i\,(dx)_p \wedge (dy)_p\,,$$

so $dz \wedge d\bar{z} = -2i\,dx \wedge dy$ on $\mathbb{C}$. Then, a complex 2-form β on a subset $E \subset \mathbb{C}$ can be written as

$$\beta = f\,dx \wedge dy = f\frac{i}{2}\,dz \wedge d\bar{z}\,,$$

where $f := \beta/(dx \wedge dy) : E \to \mathbb{C}$. Moreover, for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and assuming $E \subset \mathbb{C}$ is open if $k \geq 1$, the 2-form β is C^k on E if and only if f is C^k on E, or if and only if $f\frac{i}{2}$ is C^k on E. If the set E is measurable in $\mathbb{C}$ and the function f is measurable on E, then the 2-form β is measurable on E, and

$$\int_E \beta^\pm = \int_E f^\pm\,dx \wedge dy = \int_E f^\pm\,d\lambda\,,$$

which shows that β is integrable on E if and only if f is integrable on E.

Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is a C^1 function. Then, on Ω we have

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z} = \partial f + \bar{\partial} f\,,$$

where

$$\partial f := \frac{\partial f}{\partial z}\, dz \quad \text{and} \quad \bar{\partial} f := \frac{\partial f}{\partial \bar{z}}\, d\bar{z}\,.$$

Moreover, f is holomorphic on Ω if and only if $\bar{\partial} f = 0$ on Ω, or if and only if $df = h\, dz$ on Ω for some function $h : \Omega \to \mathbb{C}$.

If $\Omega \subset \mathbb{C}$ is open and

$$\alpha = P\, dx + Q\, dy = A\, dz + B\, d\bar{z}$$

is a C^1 complex 1-form, for some C^1 functions $P, Q, A, B : \Omega \to \mathbb{C}$, on Ω we have

$$d\alpha = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

and

$$\begin{aligned} d\alpha &= dA \wedge dz + dB \wedge d\bar{z} \\ &= \left(\frac{\partial B}{\partial z} - \frac{\partial A}{\partial \bar{z}} \right) dz \wedge d\bar{z} \\ &= \partial \alpha + \bar{\partial} \alpha\,, \end{aligned}$$

where

$$\partial \alpha := \frac{\partial B}{\partial z}\, dz \wedge d\bar{z} = \partial B \wedge d\bar{z}$$

and

$$\bar{\partial} \alpha := -\frac{\partial A}{\partial \bar{z}}\, dz \wedge d\bar{z} = \bar{\partial} A \wedge dz\,.$$

4.3. **Polar Coordinates.**

For any $\theta_0 \in \mathbb{R}$, the map

$$[0, +\infty) \times [\theta_0, \theta_0 + 2\pi) \to \mathbb{R}^2\,, \quad (r, \theta) \mapsto (r \cos\theta, r \sin\theta)\,,$$

is continuous and surjective. Its restriction $(0, +\infty) \times [\theta_0, \theta_0 + 2\pi) \to \mathbb{R}^2 \setminus \{(0,0)\}$ is a bijection, while its restriction $(0, +\infty) \times (\theta_0, \theta_0 + 2\pi) \to \mathbb{R}^2 \setminus Z_{\theta_0}$, where

$$Z_{\theta_0} := \{(r \cos\theta_0, r \sin\theta_0) \in \mathbb{R}^2 \mid r \geq 0\}$$

is the closed ray emerging from the origin at an angle θ_0 with the positive x-axis, is a diffeomorphism. Denoting this diffeomorphism by F_{θ_0}, we have

$$\mathcal{J}_{F_{\theta_0}}(r, \theta) = \begin{vmatrix} \cos\theta & -r \sin\theta \\ \sin\theta & r \cos\theta \end{vmatrix} = r$$

at each $(r, \theta) \in (0, +\infty) \times (\theta_0, \theta_0 + 2\pi)$. Note that $\lambda(Z_{\theta_0}) = 0$ in $\mathbb{R}^2$. Then, by Theorem 2.44, the Fubini-Tonelli Theorem and Fubini's Theorem, if X is $\overline{\mathbb{R}}, \mathbb{R}$ or $\mathbb{C}$ and $f : \mathbb{R}^2 \to$

X is nonnegative measurable (for $X = \mathbb{R}$ or $\overline{\mathbb{R}} := [0, +\infty]$) or integrable (for $X = \mathbb{R}$ or $\mathbb{C}$), we have

$$\begin{aligned}\int_{\mathbb{R}^2} f\, d\lambda &= \int_{\mathbb{R}^2 \setminus Z_{\theta_0}} f\, d\lambda \\ &= \int_{(0,+\infty)\times(\theta_0,\theta_0+2\pi)} (f \circ F_{\theta_0}) \left| \mathcal{J}_{F_{\theta_0}} \right| \, d\lambda \\ &= \int_{(0,+\infty)} \int_{(\theta_0,\theta_0+2\pi)} f(r\cos\theta, r\sin\theta)\, r\, d\lambda(\theta)\, d\lambda(r)\,.\end{aligned}$$

Definition 4.8. For any point $(x, y) \in \mathbb{R}^2$ and $(r, \theta) \in [0, +\infty) \times \mathbb{R}$ such that $(x, y) = (r\cos\theta, r\sin\theta)$, we call (r, θ) *polar coordinates for* (x, y). For any $\theta \in \mathbb{R}$, we define

$$e^{i\theta} := \cos\theta + i\sin\theta \in \mathbb{C}\,,$$

so that if $(x, y) = x + iy$ is a point in $\mathbb{C}$ with polar coordinates (r, θ) under the identification of $\mathbb{C}$ with $\mathbb{R}^2$, we have

$$x + iy = r\cos\theta + i\, r\sin\theta = re^{i\theta}\,.$$

Suppose $\theta_0 \in \mathbb{R}$. Since $F_{\theta_0}^{-1} : \mathbb{R}^2 \setminus Z_{\theta_0} \to (0, +\infty) \times (\theta_0, \theta_0 + 2\pi)$ is a diffeomorphism, the pair $(\mathbb{R}^2 \setminus Z_{\theta_0}, F_{\theta_0}^{-1})$ is a chart on $\mathbb{R}^2$. Denoting by (s, t) the standard coordinates on $(0, +\infty) \times (\theta_0, \theta_0 + 2\pi)$, and letting $r := s \circ F_{\theta_0}^{-1}$ and $\theta := t \circ F_{\theta_0}^{-1}$, we have

$$dx \wedge dy = (\mathcal{J}_{F_{\theta_0}} \circ F_{\theta_0}^{-1})\, dr \wedge d\theta = r\, dr \wedge d\theta$$

on $\mathbb{R}^2 \setminus Z_{\theta_0}$, since F_{θ_0} is the transition map between the charts $(\mathbb{R}^2 \setminus Z_{\theta_0}, F_{\theta_0}^{-1})$ and $(\mathbb{R}^2, \mathbb{1}_{\mathbb{R}^2})$. Moreover, we have $\mathcal{J}_{F_{\theta_0}} > 0$ on $(0, +\infty) \times (\theta_0, \theta_0 + 2\pi)$, so that the atlas $\{(\mathbb{R}^2 \setminus Z_{\theta_0}, F_{\theta_0}^{-1}), (\mathbb{R}^2, \mathbb{1}_{\mathbb{R}^2})\}$ is oriented and thus $(\mathbb{R}^2 \setminus Z_{\theta_0}, F_{\theta_0}^{-1})$ is positively oriented.

Proposition 4.9. *For any $R \in (0, +\infty)$, the open disc $\Delta(0; R) \subset \mathbb{R}^2$ of radius R centred at the origin is a smooth open set in $\mathbb{R}^2$.*

Proof. Let $\theta_0 \in \mathbb{R}$. By the above discussion, $(\mathbb{R}^2 \setminus Z_{\theta_0}, F_{\theta_0}^{-1}) = (\mathbb{R}^2 \setminus Z_{\theta_0}, r, \theta)$ is a chart in $\mathbb{R}^2$, and we have

$$(\mathbb{R}^2 \setminus Z_{\theta_0}) \cap \Delta(0; R) = \{p \in \mathbb{R}^2 \setminus Z_{\theta_0} \mid r(p) < R\}\,.$$

Consider the diffeomorphism

$$G : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad (x, y) \mapsto (x - R, y)\,.$$

Restricting G to the diffeomorphism

$$(0, +\infty) \times (\theta_0, \theta_0 + 2\pi) \to (-R, +\infty) \times (\theta_0, \theta_0 + 2\pi)\,,$$

we obtain another chart $(\mathbb{R}^2 \setminus Z_{\theta_0}, G \circ F_{\theta_0}^{-1})$. Denoting by (s, t) the coordinate functions of this chart, we have

$$(\mathbb{R}^2 \setminus Z_{\theta_0}) \cap \Delta(0; R) = \{p \in \mathbb{R}^2 \setminus Z_{\theta_0} \mid s(p) < 0\}\,.$$

To show that $\Delta(0; R)$ is a smooth open set, we need to cover $\mathbb{R}^2$ by charts $(U, \phi) = (U, x_1, x_2)$ such that

$$U \cap \Delta(0; R) = \{p \in U \mid x_1(p) < 0\}\,.$$

Choosing any $\theta_1, \theta_2 \in \mathbb{R}$ that do not differ by an integer multiple of 2π, for example $\theta_1 := 0$ and $\theta_2 := \pi$, the charts $(\mathbb{R}^2 \setminus Z_{\theta_1}, G \circ F_{\theta_1}^{-1})$ and $(\mathbb{R}^2 \setminus Z_{\theta_2}, G \circ F_{\theta_2}^{-1})$ cover $\mathbb{R}^2 \setminus \{(0, 0)\}$.

To cover the origin, choose a chart $(\Delta(0,\frac{R}{2}),\phi)$ where ϕ is any diffeomorphism mapping $\Delta(0,\frac{R}{2})$ to the open left half-plane, for example G. $\square$

Remark 4.10. Referring to the proof of Proposition 4.9, note that for any $\theta_0 \in \mathbb{R}$ the chart $(\mathbb{R}^2 \setminus Z_{\theta_0}, G \circ F_{\theta_0}^{-1}) = (\mathbb{R}^2 \setminus Z_{\theta_0}, s, t)$ in $\mathbb{R}^2$ is positively oriented. Thus, if we give the boundary $\partial\Delta(0;R)$ the induced orientation from $\mathbb{R}^2$ with respect to $\Delta(0;R)$, the chart

$$(U_{\theta_0}, \phi_{\theta_0}) := ((\mathbb{R}^2 \setminus Z_{\theta_0}) \cap \partial\Delta(0;R), s \circ \iota) = (\{Re^{i\theta} \in \mathbb{R}^2 \,|\, \theta \in (\theta_0, \theta_0 + 2\pi)\}, s \circ \iota)$$

in $\partial\Delta(0;R)$ is positively oriented. As a consequence, the diffeomorphism

$$\phi_{\theta_0} = s \circ \iota : \{Re^{i\theta} \in \mathbb{R}^2 \,|\, \theta \in (\theta_0, \theta_0 + 2\pi)\} = U_{\theta_0} \to (\theta_0, \theta_0 + 2\pi)\,, \quad Re^{i\theta} \mapsto \theta$$

is orientation-preserving (since for any positively oriented chart (V,ψ) in an oriented smooth manifold M, the map $\psi : V \to \psi(V)$ is an orientation-preserving diffeomorphism). Let $\Omega \subset \mathbb{R}^2$ be an open subset such that $\partial\Delta(0;R) \subset \Omega$. Consider the map

$$\gamma : [\theta_0, \theta_0 + 2\pi] \to \Omega\,, \quad \theta \mapsto Re^{i\theta} = (R\cos\theta, R\sin\theta)\,.$$

We have:

(a) γ is a C^1 path. Indeed, the map

$$\mathbb{R} \to \Omega\,, \quad \theta \mapsto (R\cos\theta, R\sin\theta)$$

is C^∞ and restricts to γ on $[\theta_0, \theta_0 + 2\pi]$.

(b) The image $\gamma((\theta_0, \theta_0 + 2\pi)) = U_{\theta_0}$ is a 1-dimensional smooth manifold as an open subset of $\partial\Delta(0;R)$. Moreover, the inclusion map $\iota : U_{\theta_0} \to \Omega$ is C^∞, as the restriction $U_{\theta_0} \to \Omega$ of the inclusion map $\partial\Delta(0;R) \to \mathbb{R}^2$.

(c) The map

$$(\theta_0, \theta_0 + 2\pi) \to U_{\theta_0}\,, \quad \theta \mapsto \gamma(\theta) = Re^{i\theta}$$

is precisely

$$\phi_{\theta_0}^{-1} : \phi_{\theta_0}(U_{\theta_0}) = (\theta_0, \theta_0 + 2\pi) \to U_{\theta_0}$$

and hence it is a diffeomorphism. Moreover, since $\phi_{\theta_0}^{-1}$ is orientation-preserving, U_{θ_0} has the orientation induced from $(\theta_0, \theta_0 + 2\pi)$ via $\phi_{\theta_0}^{-1}$.

In conclusion, γ fulfils all the conditions in the hypothesis of Lemma 2.79, so that if α is a continuous 1-form on Ω, then $\iota^*\alpha$ is integrable on U_{θ_0} and

$$\int_\gamma \alpha = \int_{U_{\theta_0}} \iota^*\alpha\,.$$

The following proposition is a useful application of polar coordinates:

Proposition 4.11. *For $R \in (0, +\infty)$, the function $z \mapsto \frac{1}{z}$ on $\Delta^*(0;R) = \Delta(0;R) \setminus \{0\}$ is integrable. As a consequence, the function*

$$g : \mathbb{C} \to \mathbb{C}\,, \quad z \mapsto \begin{cases} 0 & \text{if } z = 0 \\ \frac{1}{z} & \text{if } z \neq 0 \end{cases}$$

is locally integrable.

Proof. Denote by f the function $z \mapsto \left|\frac{1}{z}\right|$ on $\Delta^*(0;R)$. Regarding $\Delta^*(0;R)$ to be a subset of $\mathbb{R}^2$, we have $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ for each $(x,y) \in \Delta^*(0;R)$. Let

$$A := \{(x,0) \in \mathbb{R}^2 \,|\, x \geq 0\}.$$

Since A has measure 0 in $\mathbb{R}^2$, we have

$$\int_{\Delta^*(0;R)} f \, d\lambda = \int_{\Delta^*(0;R)\setminus A} f \, d\lambda \,,$$

so we may use the diffeomorphism

$$F : (0,R) \times (0,2\pi) \to \Delta^*(0;R) \setminus A \,, \quad (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

to write

$$\begin{aligned}
\int_{\Delta^*(0;R)} f \, d\lambda &= \int_{\Delta^*(0;R)\setminus A} f \, d\lambda \\
&= \int_{(0,R)\times(0,2\pi)} (f \circ F)|J_F| \; d\lambda \\
&= \int_{(0,R)\times(0,2\pi)} \frac{1}{r} \, r \, d\lambda \\
&= 2\pi R \,.
\end{aligned}$$

Since f has finite integral over $\Delta^*(0;R)$, the function $z \mapsto \frac{1}{z}$ is integrable over this set, or in other words, g is integrable over $\Delta(0;R)$. For any $p \in \mathbb{C} \setminus \{0\}$, we may choose a bounded neighbourhood U of p such that $\overline{U} \subset \mathbb{C} \setminus \{0\}$. Then, since g is continuous on the compact set $\overline{U}$, it is integrable on $\overline{U}$ and hence on U. Thus, g is locally integrable on $\mathbb{C}$. ☐

Remark 4.12. For any $z_0 \in \mathbb{C}$, the map

$$F : \mathbb{C} \to \mathbb{C} \,, \quad z \mapsto z + z_0$$

is an orientation-preserving diffeomorphism. Thus, by Remark 2.84 and Proposition 4.9, for each $R \in (0,+\infty)$ the open set $\Delta(z_0;R) = F(\Delta(0;R)) \subset \mathbb{C}$ is smooth, and F restricts to an orientation-preserving diffeomorphism $F : \partial\Delta(0;R) \to \partial\Delta(z_0;R)$. Then, for $\theta_0 \in \mathbb{R}$ and using notation from Remark 4.10, since the chart $(U_{\theta_0}, \phi_{\theta_0})$ in $\partial\Delta(0;R)$ is positively oriented, the chart

$$(F(U_{\theta_0}), \phi_{\theta_0} \circ F^{-1}) = (\{z_0 + Re^{i\theta} \,|\, \theta \in (\theta_0, \theta_0 + 2\pi)\}, z_0 + Re^{i\theta} \mapsto \theta)$$

in $\partial\Delta(z_0;R)$ is also positively oriented.

Definition 4.13. For $z_0 \in \mathbb{C}$ and $R \in (0,+\infty)$, we call the orientation on $\partial\Delta(z_0;R)$ induced from $\mathbb{C}$ with respect to $\Delta(z_0;R)$ the *counterclockwise orientation* on $\partial\Delta(z_0;R)$. The remaining orientation on $\partial\Delta(z_0;R)$ is called the *clockwise orientation*. If nothing else is stated, we will assume $\partial\Delta(z_0;R)$ is equipped with the counterclockwise orientation.

4.4. **Local Solutions of the Cauchy-Riemann Equation.**

Lemma 4.14. *Let $\Omega \subset \mathbb{C}$ be a smooth relatively compact open subset, and suppose f is a C^1 complex-valued function on a neighbourhood W of $\overline{\Omega}$.*

(i) (Cauchy Integral Formula). *For each $z_0 \in \Omega$, we have*

$$f(z_0) = \frac{1}{2\pi i}\left(\int_{\partial\Omega} \frac{f(z)}{z - z_0}\, dz + \int_{\Omega\setminus\{z_0\}} \frac{\partial f/\partial\bar{z}}{z - z_0}\, dz \wedge d\bar{z}\right).$$

(ii) (Cauchy's Theorem).

$$\int_{\partial\Omega} f(z)\, dz + \int_{\Omega} \frac{\partial f}{\partial\bar{z}}\, dz \wedge d\bar{z} = 0\,.$$

Proof. (i) Fix $z_0 \in \Omega$, and consider the C^1 1-form

$$\alpha := -\frac{f(z)}{z - z_0}\, dz$$

on $W \setminus \{z_0\}$. Choose $N \in \mathbb{N}$ such that $\Delta(z_0; \frac{1}{N}) \subset \Omega$, and for each $n \in \mathbb{N}$, define $\varepsilon_n := \frac{1}{N+n}$ and $\Omega_n := \Omega \setminus \overline{\Delta(z_0; \varepsilon_n)}$. Then, for a fixed $n \in \mathbb{N}$, we have $\partial\Omega_n = \partial\Omega \cup \partial\Delta(z_0; \varepsilon_n)$ and $\partial\Omega \cap \partial\Delta(z_0; \varepsilon_n) = \emptyset$. Denote by (r, s) the standard coordinates in $\mathbb{R}^2$. For each $p \in \partial\Omega$, we may choose a chart (U, ϕ) about p in $\mathbb{C}$ such that $U \cap \Omega = \{q \in U \,|\, (r \circ \phi)(q) < 0\}$ and $U \subset \overline{\Delta(z_0; \varepsilon_n)}^c$, so that $U \cap \Omega \subset \overline{\Delta(z_0; \varepsilon_n)}^c \cap \Omega = \Omega_n$ and we actually have

$$U \cap \Omega_n = U \cap \Omega = \{q \in U \,|\, (r \circ \phi)(q) < 0\}\,.$$

As the exterior of a smooth open set, $\overline{\Delta(z_0; \varepsilon_n)}^c = \operatorname{ext} \Delta(z_0; \varepsilon_n)$ is also a smooth open set. Thus, for each $p \in \partial\Delta(z_0; \varepsilon_n)$ we may choose a chart (V, ψ) about p in $\mathbb{C}$ such that $V \cap \overline{\Delta(z_0; \varepsilon_n)}^c = \{q \in V \,|\, (r \circ \psi)(q) < 0\}$ and $V \subset \Delta(z_0; \frac{1}{N}) \subset \Omega$, so that $V \cap \overline{\Delta(z_0; \varepsilon_n)}^c \subset \Omega$ and we have

$$V \cap \Omega_n = V \cap \overline{\Delta(z_0; \varepsilon_n)}^c \cap \Omega = V \cap \overline{\Delta(z_0; \varepsilon_n)}^c = \{q \in V \,|\, (r \circ \psi)(q) < 0\}\,.$$

It follows that Ω_n is a smooth open set in $\mathbb{C}$ and hence in $W \setminus \{z_0\}$. Since $\overline{\Omega_n} \subset W \setminus \{z_0\}$, the closure of Ω_n in $W \setminus \{z_0\}$ is precisely $\overline{\Omega_n}$, and hence it is compact. Moreover, since $\Delta(z_0; \frac{1}{N}) \setminus \overline{\Delta(z_0, \varepsilon_n)} \subset \Omega_n$, we have $\Omega_n \neq \emptyset$. Then, since α is a C^1 1-form on $W \setminus \{z_0\}$ and $\overline{\Omega_n} \cap \operatorname{supp}_{W\setminus\{z_0\}}\alpha$ is compact, by Stokes' Theorem,

$$\int_{\Omega_n} d\alpha = \int_{\partial\Omega_n} \iota^*\alpha\,.$$

Since $\partial\Omega_n$ is the union of the disjoint sets $\partial\Omega$ and $\partial\Delta(z_0; \varepsilon_n)$,

$$\begin{aligned}\int_{\partial\Omega_n} \iota^*\alpha &= \int_{\partial\Omega} \iota^*\alpha + \int_{\partial\Delta(z_0;\varepsilon_n)} \iota^*\alpha \\ &= \int_{\partial\Omega} \left(-\frac{f(z)}{z - z_0} \circ \iota\right) d(z \circ \iota) + \int_{\partial\Delta(z_0;\varepsilon_n)} \iota^*\alpha\,.\end{aligned}$$

Letting $Z := \{z_0 + r \in \mathbb{C} \,|\, r \geq 0\}$, the map

$$F : (-\varepsilon_n, +\infty) \times (0, 2\pi) \to \mathbb{C} \setminus Z\,, \quad (r, \theta) \mapsto z_0 + (r + \varepsilon_n)e^{i\theta}$$

is a diffeomorphism, and the chart $(\mathbb{C}\setminus Z, F^{-1})$ on $\mathbb{C}$ is positively oriented and fulfils $(\mathbb{C}\setminus Z)\cap\Delta(z_0;\varepsilon_n) = \{q \in \mathbb{C}\setminus Z \,|\, (r\circ F^{-1})(q) < 0\}$. F restricts to a diffeomorphism $\tilde{F} : (-\varepsilon_n, \frac{1}{N} - \varepsilon_n) \times (0, 2\pi) \to \Delta(z_0; \frac{1}{N}) \setminus Z$, and the chart

$$(\Delta(z_0; \frac{1}{N}) \setminus Z, -\tilde{F}^{-1}) = \left(\Delta(z_0; \frac{1}{N}) \setminus Z, -r \circ \tilde{F}^{-1}, -s \circ \tilde{F}^{-1}\right)$$

is in $W \setminus \{z_0\}$, is positively oriented, and fulfils

$$\begin{aligned}\left(\Delta(z_0; \frac{1}{N}) \setminus Z\right) \cap \Omega_n &= \left(\Delta(z_0; \frac{1}{N}) \setminus Z\right) \cap \operatorname{ext}\Delta(z_0; \varepsilon_n)\\ &= \{q \in \Delta(z_0; \frac{1}{N}) \setminus Z \,|\, (-r \circ \tilde{F}^{-1})(q) < 0\}.\end{aligned}$$

Thus, we may induce from $(\Delta(z_0; \frac{1}{N})\setminus Z, -\tilde{F}^{-1})$ a positively oriented chart (U_n, ϕ_n) on $\partial\Omega_n$, given by

$$U_n := \left(\Delta(z_0; \frac{1}{N}) \setminus Z\right) \cap \partial\Omega_n = \{z_0 + \varepsilon_n e^{i\theta} \,|\, \theta \in (0, 2\pi)\},$$

$$\phi_n : U_n \to (-2\pi, 0), \quad z_0 + \varepsilon_n e^{i\theta} \mapsto -\theta.$$

Then, we have

$$\begin{aligned}\int_{\partial\Delta(z_0,\varepsilon_n)} \iota^*\alpha &= \int_{U_n} \iota^*\alpha\\ &= \int_{(-2\pi,0)} \frac{-f(z_0 + \varepsilon_n e^{-i\theta})}{\varepsilon_n e^{-i\theta}}\, d(z_0 + \varepsilon_n e^{-i\theta})\\ &= \int_{(-2\pi,0)} if(z_0 + \varepsilon_n e^{-i\theta})\, d\theta\\ &= \int_{(-2\pi,0)} if(z_0 + \varepsilon_n e^{-i\theta})\, d\lambda(\theta).\end{aligned}$$

For each $n \in \mathbb{N}$, let $g_n : (-2\pi, 0) \to \mathbb{C}$ be the function $\theta \mapsto if(z_0 + \varepsilon_n e^{-i\theta})$. For each $\theta \in (-2\pi, 0)$, we have

$$\Omega \supset z_0 + \varepsilon_n e^{-i\theta} \to z_0 \quad \text{as } n \to \infty,$$

so, by continuity of f,

$$g_n(\theta) = if(z_0 + \varepsilon_n e^{-i\theta}) \to if(z_0) \quad \text{as } n \to \infty.$$

Thus, the sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ converges pointwise to the constant function $if(z_0)$ on $(0, 2\pi)$. Since f is continuous, we have $|f| \leq P$ on $\overline{\Omega}$ for some $P \in (0, +\infty)$, so for all $n \in \mathbb{N}$ we also have $|g_n| \leq P$ on $(-2\pi, 0)$. Since the constant function P on $(-2\pi, 0)$ is integrable, by the Dominated Convergent Theorem

$$\int_{\partial\Delta(z_0,\varepsilon_n)} \iota^*\alpha = \int_{(-2\pi,0)} g_n\, d\lambda \to \int_{(-2\pi,0)} if(z_0)\, d\lambda = 2\pi i f(z_0) \quad \text{as } n \to \infty.$$

Fix again some $n \in \mathbb{N}$. On $\Omega \setminus \{z_0\}$, and hence on Ω_n, we have

$$d\alpha = \frac{\partial f/\partial\bar{z}}{z - z_0}\, dz \wedge d\bar{z} = -2i\, \frac{\partial f/\partial\bar{z}}{z - z_0}\, dx \wedge dy,$$

so
$$\int_{\Omega_n} d\alpha = \int_{\Omega_n} -2i\,\frac{\partial f/\partial \bar z}{z-z_0}\,d\lambda = \int_{\Omega\setminus\{z_0\}} \chi_{\Omega_n}(-2i)\frac{\partial f/\partial \bar z}{z-z_0}\,d\lambda\,.$$
Choose $R \in (0,+\infty)$ such that $\Omega \subset \Delta(z_0;R)$. By Proposition 4.11, the function $z \mapsto \frac{1}{z}$ is integrable on $\Delta^*(0;R)$, which implies that the function $z \mapsto \frac{1}{z-z_0}$ is integrable on $\Delta^*(z_0;R)$ and hence on $\Omega\setminus\{z_0\}$. Moreover, the function $-2i\,\frac{\partial f}{\partial \bar z}$ is continuous on $\overline{\Omega}$ and hence there exists $Q \in (0,+\infty)$ such that $\left|-2i\,\frac{\partial f}{\partial \bar z}\right| \leq Q$ on $\overline{\Omega}$. Thus, for all $n\in\mathbb{N}$ and $z \in \Omega\setminus\{z_0\}$ we have
$$\left|\chi_{\Omega_n}(-2i)\frac{\partial f/\partial \bar z}{z-z_0}\right| \leq \left|-2i\,\frac{\partial f/\partial \bar z}{z-z_0}\right| = \frac{\left|-2i\,\partial f/\partial \bar z\right|}{|z-z_0|} \leq \frac{Q}{|z-z_0|}\,.$$
Since $\frac{Q}{|z-z_0|}$ is integrable on $\Omega\setminus\{z_0\}$ and
$$\chi_{\Omega_n}(-2i)\frac{\partial f/\partial \bar z}{z-z_0} \to (-2i)\frac{\partial f/\partial \bar z}{z-z_0} \quad \text{as } n\to\infty$$
pointwise on $\Omega\setminus\{z_0\}$, we have
$$\int_{\Omega\setminus\{z_0\}} \chi_{\Omega_n}(-2i)\frac{\partial f/\partial \bar z}{z-z_0}\,d\lambda \to \int_{\Omega\setminus\{z_0\}} -2i\,\frac{\partial f/\partial \bar z}{z-z_0}\,d\lambda \quad \text{as } n\to\infty\,.$$
It follows that the 2-form $-2i\,\frac{\partial f/\partial \bar z}{z-z_0}\,dx\wedge dy = \frac{\partial f/\partial \bar z}{z-z_0}\,dz\wedge d\bar z$ is integrable on $\Omega\setminus\{z_0\}$ and
$$\int_{\Omega_n} d\alpha \to \int_{\Omega\setminus\{z_0\}} \frac{\partial f/\partial \bar z}{z-z_0}\,dz\wedge d\bar z \quad \text{as } n\to\infty\,.$$
In conclusion, taking limits on both sides of the equality
$$\int_{\Omega_n} d\alpha = \int_{\partial\Omega}\left(-\frac{f(z)}{z-z_0}\right)dz + \int_{\partial\Delta(z_0;\varepsilon_n)} \iota^*\alpha\,,$$
we obtain
$$\int_{\Omega\setminus\{z_0\}} \frac{\partial f/\partial \bar z}{z-z_0}\,dz\wedge d\bar z = \int_{\partial\Omega} -\frac{f(z)}{z-z_0}\,dz + 2\pi i f(z_0)\,,$$
or
$$f(z_0) = \frac{1}{2\pi i}\left(\int_{\partial\Omega}\frac{f(z)}{z-z_0}\,dz + \int_{\Omega\setminus\{z_0\}}\frac{\partial f/\partial \bar z}{z-z_0}\,dz\wedge d\bar z\right).$$

(ii) Follows directly from Stokes' Theorem applied to the C^1 1-form $f\,dz$ on W.

□

Lemma 4.15. (Local solution of the inhomogeneous Cauchy-Riemann equation). *Let $D := \Delta(z_0;R) \subset \mathbb{C}$ for $z_0\in\mathbb{C}$ and $R\in(0,+\infty)$. Suppose $\alpha : \partial D \to \mathbb{C}$ is a continuous function, $k \in \mathbb{Z}_{\geq 1}\cup\{\infty\}$, and β is a C^k complex-valued function on a neighbourhood W of $\overline{D}$. Then, the function*
$$f : D \to \mathbb{C}\,,\quad f(z) := \frac{1}{2\pi i}\left(\int_{\partial D}\frac{\alpha(\zeta)}{\zeta - z}\,d\zeta + \int_{D\setminus\{z\}}\frac{\beta(\zeta)}{\zeta - z}\,d\zeta\wedge d\bar\zeta\right)$$
is C^k and fulfils
$$\frac{\partial f}{\partial \bar z} = \beta$$
on D. In particular, f is holomorphic on $D\setminus \operatorname{supp}\beta$.

Proof. We first show that the function

$$f_1 : D \to \mathbb{C}\,, \quad z \mapsto \int_{\partial D} \frac{\alpha(\zeta)}{\zeta - z}\, d\zeta$$

is C^∞ and holomorphic. Using the fact that the map

$$\partial D \setminus \{z_0 + R\} = \{z_0 + Re^{i\theta} \in \partial D \,|\, \theta \in (0, 2\pi)\} \to (0, 2\pi)\,, \quad z_0 + Re^{i\theta} \mapsto \theta$$

defines a positively oriented chart on ∂D, for each $z \in D$ we have

$$f_1(z) = \int_{\partial D} \frac{\alpha(\zeta)}{\zeta - z}\, d\zeta = \int_{(0,2\pi)} \frac{\alpha(z_0 + Re^{i\theta})}{z_0 + Re^{i\theta} - z}\, iRe^{i\theta}\, d\lambda(\theta)\,.$$

Suppose $\sigma : (0, 2\pi) \to \mathbb{C}$ is a continuous bounded function and let $n \in \mathbb{N}$. Then, for any $z \in D$ the function

$$(0, 2\pi) \to \mathbb{C}\,, \quad \theta \mapsto \frac{\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^n}$$

is continuous and bounded and hence integrable. We aim to differentiate the function

$$g : D \to \mathbb{C}\,, \quad z \mapsto \int_{(0,2\pi)} \frac{\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^n}\, d\lambda(\theta)\,.$$

Choose any $w \in D$ and let $a := \mathrm{Re}(w)$ and $b := \mathrm{Im}(w)$. Let $I \subset \mathbb{R}$ be the open interval $I := \{x \in \mathbb{R} \,|\, x + ib \in D\}$, and choose $c, d \in \mathbb{R}$ such that $a \in (c, d)$ and $[c, d] \subset I$. Then, the function

$$F : (0, 2\pi) \times (c, d) \to \mathbb{C}\,, \quad (\theta, x) \mapsto \frac{\sigma(\theta)}{(z_0 + Re^{i\theta} - (x + ib))^n}$$

fulfils

(i) for all $x \in (c, d)$ the function $\theta \mapsto F(\theta, x)$ on $(0, 2\pi)$ is integrable;

(ii) for all $\theta \in (0, 2\pi)$ the function $x \mapsto F(\theta, x)$ on (c, d) is differentiable with derivative

$$x \mapsto \frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - (x + ib))^{n+1}}$$

on (c, d);

(iii) since the sets $S := \{x + ib \,|\, x \in [c, d]\}$ and ∂D are compact and disjoint, there is $P \in (0, +\infty)$ such that for all $z_1 \in S$ and $z_2 \in \partial D$ we have $|z_1 - z_2| \geq P$. Then, choosing $Q \in (0, +\infty)$ such that $|\sigma| \leq Q$ on $(0, 2\pi)$, for all $\theta \in (0, 2\pi)$ and $x \in (c, d)$ we have

$$\left| \frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - (x + ib))^{n+1}} \right| \leq \frac{n\,Q}{P^{n+1}}\,.$$

Thus, applying Theorem 3.1, we conclude that the function

$$(c, d) \to \mathbb{C}\,, \quad x \mapsto \int_{(0,2\pi)} \frac{\sigma(\theta)}{(z_0 + Re^{i\theta} - (x + ib))^n}\, d\lambda(\theta)$$

is differentiable and that

$$\begin{aligned} \frac{\partial g}{\partial x}\bigg|_{w=a+ib} &= \frac{d}{dx}\left(\int_{(0,2\pi)} \frac{\sigma(\theta)}{(z_0 + Re^{i\theta} - (x + ib))^n}\, d\lambda(\theta) \right)\bigg|_a \\ &= \int_{(0,2\pi)} \frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - (a + ib))^{n+1}}\, d\lambda(\theta)\,. \end{aligned}$$

Since $w \in D$ was arbitrary, we conclude that for $z \in D$

$$\frac{\partial g}{\partial x}(z) = \frac{\partial}{\partial x}\int_{(0,2\pi)} \frac{\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^n}\, d\lambda(\theta) = \int_{(0,2\pi)} \frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^{n+1}}\, d\lambda(\theta)\,.$$

Analogous reasoning shows that for $z \in D$

$$\frac{\partial g}{\partial y}(z) = \frac{\partial}{\partial y}\int_{(0,2\pi)} \frac{\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^n}\, d\lambda(\theta) = \int_{(0,2\pi)} \frac{in\,\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^{n+1}}\, d\lambda(\theta) = i\frac{\partial g}{\partial x}(z)\,.$$

We show that $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are continuous on D. Suppose $z \in D$ and let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in D converging to z. Define the functions

$$h_k : (0, 2\pi) \to \mathbb{C}\,, \quad h_k(\theta) := \frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - z_k)^{n+1}}$$

for each $k \in \mathbb{N}$, and

$$h : (0, 2\pi) \to \mathbb{C}\,, \quad h(\theta) := \frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^{n+1}}\,.$$

Then, $h_k \to h$ pointwise very wisely as $k \to \infty$ on $(0, 2\pi)$. We may choose $\varepsilon \in (0, +\infty)$ such that $\overline{\Delta(z;\varepsilon)} \subset D$, and $N \in \mathbb{N}$ such that for all $k \geq N$ we have $|z_k - z| < \varepsilon$. Then, the set

$$S' := \overline{\Delta(z;\varepsilon)} \cup \{z_k\}_{k\leq N} \subset D$$

is compact and $z_k \in S'$ for all $k \in \mathbb{N}$. Thus, since S' and ∂D are compact and disjoint, there exists $P' \in (0, +\infty)$ such that for all $w_1 \in S'$ and $w_2 \in \partial D$ we have $|w_1 - w_2| \geq P'$. Then, for all $k \in \mathbb{N}$ and $\theta \in (0, 2\pi)$,

$$\big|h_k(\theta)\big| = \left|\frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - z_k)^{n+1}}\right| \leq \frac{n\,Q}{(P')^{n+1}}\,.$$

Thus, by the Dominated Convergence Theorem,

$$\frac{\partial g}{\partial x}(z_k) = \int_{(0,2\pi)} \frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - z_k)^{n+1}}\, d\lambda(\theta) \to \int_{(0,2\pi)} \frac{n\,\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^{n+1}}\, d\lambda(\theta) = \frac{\partial g}{\partial x}(z)$$

as $k \to \infty$. This shows that $\frac{\partial g}{\partial x}$ is continuous on D, and hence so is $\frac{\partial g}{\partial y} = i\,\frac{\partial g}{\partial x}$. Observe also that the fact that g is real-differentiable and $\frac{\partial g}{\partial y} = i\,\frac{\partial g}{\partial x}$ implies that g is holomorphic on D.

Now, since the function $\theta \mapsto \alpha(z_0 + Re^{i\theta})\, iRe^{i\theta}$ on $(0, 2\pi)$ is continuous and bounded, the function

$$f_1 : D \to \mathbb{C}\,, \qquad z \mapsto \int_{(0,2\pi)} \frac{\alpha(z_0 + Re^{i\theta})\, iRe^{i\theta}}{z_0 + Re^{i\theta} - z}\, d\lambda(\theta)\,,$$

has continuous partial derivatives of first order, of the form

$$\frac{\partial f_1}{\partial x}(z) = \int_{(0,2\pi)} \frac{\sigma_1(\theta)}{(z_0 + Re^{i\theta} - z)^2}\, d\lambda(\theta)\,, \quad \frac{\partial f_1}{\partial y}(z) = \int_{(0,2\pi)} \frac{\sigma_2(\theta)}{(z_0 + Re^{i\theta} - z)^2}\, d\lambda(\theta)$$

for $z \in D$, where $\sigma_1, \sigma_2 : (0, 2\pi) \to \mathbb{C}$ are the continuous bounded functions defined by

$$\sigma_1(\theta) := \alpha(z_0 + Re^{i\theta})\, iRe^{i\theta}\,, \quad \sigma_2(\theta) := i\,\alpha(z_0 + Re^{i\theta})\, iRe^{i\theta}\,, \quad \theta \in (0, 2\pi)\,.$$

Moreover, if $n \in \mathbb{N}$ and f_1 has continuous partial derivatives of nth order, each of which can be written as

$$z \mapsto \int_{(0,2\pi)} \frac{\sigma(\theta)}{(z_0 + Re^{i\theta} - z)^{n+1}}\, d\lambda(\theta)\,,$$

for some continuous bounded function $\sigma : (0, 2\pi) \to \mathbb{C}$, then f_1 also has continuous partial derivatives of $(n+1)th$ order which can each be written as

$$z \mapsto \int_{(0,2\pi)} \frac{\nu(\theta)}{(z_0 + Re^{i\theta} - z)^{n+2}} \, d\lambda(\theta) \, ,$$

for some continuous bounded function $\nu : (0, 2\pi) \to \mathbb{C}$. Thus, by induction, f_1 has continuous partial derivatives of all orders on D and is therefore C^∞. Moreover, f_1 is holomorphic on D and so too are all its partial derivatives of all orders.

We now wish to show that the function

$$f_2 : D \to \mathbb{C} \, , \quad z \mapsto \int_{D\setminus\{z\}} \frac{\beta(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}$$

is C^k on D. For this, we show that for any $r \in (0, R)$, f_2 is C^k on $\Delta(z_0; r)$. Choose $r, r_1, r_2 \in (0, R)$ with $r < r_1 < r_2$. Multiplying β by a suitable C^∞ bump function, we may obtain a C^k function $\beta_1 : W \to \mathbb{C}$ that is equal to β on $\overline{\Delta(z_0; r_1)}$ and such that $\operatorname{supp}\beta_1 \subset \overline{\Delta(z_0; r_2)}$. Then, the function $\beta_2 := \beta - \beta_1$ is also C^k. For $z \in \Delta(z_0; r)$, we have

$$\begin{aligned}
f_2(z) &= \int_{D\setminus\{z\}} \frac{\beta(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta} \\
&= \int_{D\setminus\{z\}} \frac{\beta(\zeta)}{\zeta - z} (-2i) \, dx \wedge dy \\
&= \int_{D\setminus\{z\}} \frac{(\beta_1 + \beta_2)(\zeta)}{\zeta - z} (-2i) \, d\lambda(\zeta) \\
&= \int_{D\setminus\{z\}} \frac{\beta_1(\zeta)}{\zeta - z} (-2i) \, d\lambda(\zeta) + \int_{D\setminus\{z\}} \frac{\beta_2(\zeta)}{\zeta - z} (-2i) \, d\lambda(\zeta) \, .
\end{aligned}$$

Let $m \in \mathbb{Z}_{>0} \cup \{\infty\}$. Suppose $\eta : D \to \mathbb{C}$ is a C^m function with $\operatorname{supp}\eta \subset \overline{\Delta(z_0; r_2)}$. Since η is bounded on D, for any $z \in \Delta(z_0; r)$ the integral

$$\int_{D\setminus\{z\}} \frac{\eta(\zeta)}{\zeta - z} \, d\lambda(\zeta)$$

exists. Define the function

$$\ell : \Delta(z_0; r) \to \mathbb{C} \, , \quad z \mapsto \int_{D\setminus\{z\}} \frac{\eta(\zeta)}{\zeta - z} \, d\lambda(\zeta) \, .$$

Choose any $w \in \Delta(z_0; r)$ and set $a := \operatorname{Re}(w)$ and $b := \operatorname{Im}(w)$. Let $J \subset \mathbb{R}$ be the open interval

$$J := \{x \in \mathbb{R} \,|\, x + ib \in \Delta(z_0; r)\} \, .$$

For each $x \in J$, let $D_x := \Delta(z_0 - (x + ib); R)$ and $B := \Delta^*(0, R + r)$. Then, $D_x \setminus \{0\} \subset B$, and

$$\begin{aligned}
\ell(x + ib) &= \int_{D\setminus\{x+ib\}} \frac{\eta(\zeta)}{\zeta - (x + ib)} \, d\lambda(\zeta) \\
&= \int_{D_x\setminus\{0\}} \frac{\eta(\zeta + x + ib)}{\zeta} \, d\lambda(\zeta) \\
&= \int_B \chi_{D_x\setminus\{0\}}(\zeta) \, \frac{\eta(\zeta + x + ib)}{\zeta} \, d\lambda(\zeta) \, .
\end{aligned}$$

Consider the function

$$G : B \times J \to \mathbb{C}\,, \quad (\zeta, x) \mapsto \chi_{D_x \setminus \{0\}}(\zeta)\, \frac{\eta(\zeta + x + ib)}{\zeta}\,.$$

The following hold:

(i) For each $x \in J$, the function $\zeta \mapsto G(x, \zeta)$ on B is integrable.

(ii) For each $\zeta \in B$ and $x \in J$, we have

$$G(\zeta, x) = \begin{cases} \dfrac{\eta(\zeta + x + ib)}{\zeta} & \text{if } |\zeta - z_0 + x + ib| < R \\ 0 & \text{if } |\zeta - z_0 + x + ib| > r_2\,. \end{cases}$$

Fix some $\zeta \in B$ and $x_0 \in J$. If $|\zeta - z_0 + x_0 + ib| < R$, then there is some neighbourhood U of x_0 in J such that for all $x \in U$ we also have $|\zeta - z_0 + x + ib| < R$ and hence

$$G(\zeta, x) = \frac{\eta(\zeta + x + ib)}{\zeta}$$

for all $x \in U$. Thus, applying the chain rule and denoting by (s_1, s_2) the standard coordinates on $\mathbb{R}^2$, we have

$$\frac{d}{dx}\, G(\zeta, x)\bigg|_{x_0} = \frac{d}{dx}\left(\frac{\eta(\zeta + x + ib)}{\zeta}\right)\bigg|_{x_0} = \frac{1}{\zeta}\frac{\partial \eta}{\partial s_1}\bigg|_{\zeta + x_0 + ib}.$$

On the other hand, if $|\zeta - z_0 + x_0 + ib| > r_2$, there is some neighbourhood U' of x_0 in J such that for all $x \in U'$ we also have $|\zeta - z_0 + x + ib| > r_2$. Then, $G(\zeta, x) = 0$ for all $x \in U'$ and thus

$$\frac{d}{dx}\, G(\zeta, x)\bigg|_{x_0} = 0\,.$$

In conclusion, for each $\zeta \in B$ the function $x \mapsto G(\zeta, x)$ is C^1 on J and

$$\frac{d}{dx}\, G(\zeta, x)\bigg|_{x_0} = \begin{cases} \dfrac{1}{\zeta}\dfrac{\partial \eta}{\partial s_1}\bigg|_{\zeta + x_0 + ib} & \text{if } |\zeta - z_0 + x_0 + ib| < R \\ 0 & \text{if } |\zeta - z_0 + x_0 + ib| > r_2\,. \end{cases}$$

(iii) Since $\frac{\partial \eta}{\partial s_1}$ is continuous on D and $\operatorname{supp} \frac{\partial \eta}{\partial s_1} \subset \overline{\Delta(z_0; r_2)}$, we may choose $T \in (0, +\infty)$ such that $\left|\frac{\partial \eta}{\partial s_1}\right| < T$ on D. Then, for each $\zeta \in B$ and $x_0 \in J$,

$$\left|\frac{d}{dx}\, G(\zeta, x)\bigg|_{x_0}\right| \le \frac{T}{|\zeta|}\,.$$

Thus, we may apply dominated derivation to conclude that

$$\begin{aligned} \frac{d}{dx}\left(\int_{D \setminus \{x + ib\}} \frac{\eta(\zeta)}{\zeta - (x + ib)}\, d\lambda(\zeta)\right)\bigg|_a &= \frac{d}{dx}\left(\int_B \chi_{D_a \setminus \{0\}}(\zeta)\, \frac{\eta(\zeta + x + ib)}{\zeta}\, d\lambda(\zeta)\right)\bigg|_a \\ &= \int_B \chi_{D_a \setminus \{0\}}(\zeta)\, \frac{1}{\zeta}\frac{\partial \eta}{\partial s_1}\bigg|_{\zeta + a + ib} d\lambda(\zeta) \\ &= \int_{D_a \setminus \{0\}} \frac{1}{\zeta}\frac{\partial \eta}{\partial s_1}(\zeta + a + ib)\, d\lambda(\zeta) \\ &= \int_{D \setminus \{a + ib\}} \frac{1}{\zeta - (a + ib)}\frac{\partial \eta}{\partial s_1}(\zeta)\, d\lambda(\zeta)\,, \end{aligned}$$

so that

$$\frac{\partial \ell}{\partial x}(w) = \int_{D\setminus\{w\}} \frac{1}{\zeta - w}\frac{\partial \eta}{\partial s_1}(\zeta)\, d\lambda(\zeta) = \int_B \chi_{\Delta(z_0-w;R)\setminus\{0\}}(\zeta)\, \frac{1}{\zeta}\frac{\partial \eta}{\partial s_1}(\zeta + w)\, d\lambda(\zeta)\,.$$

Similar reasoning shows that

$$\frac{\partial \ell}{\partial y}(w) = \int_{D\setminus\{w\}} \frac{1}{\zeta - w}\frac{\partial \eta}{\partial s_2}(\zeta)\, d\lambda(\zeta) = \int_B \chi_{\Delta(z_0-w;R)\setminus\{0\}}(\zeta)\, \frac{1}{\zeta}\frac{\partial \eta}{\partial s_2}(\zeta + w)\, d\lambda(\zeta)\,.$$

Note that the functions $\zeta \mapsto \frac{\partial \eta}{\partial s_1}(\zeta)$ and $\zeta \mapsto \frac{\partial \eta}{\partial s_2}(\zeta)$ are C^{m-1} on D and have support in $\overline{\Delta(z_0; r_2)}$. We show that $\frac{\partial \ell}{\partial x}$ and $\frac{\partial \ell}{\partial y}$ are continuous on $\Delta(z_0; r)$. Let $z \in \Delta(z_0; r)$ and suppose $\{z_k\}_{k\in\mathbb{N}}$ is a sequence in $\Delta(z_0; r)$ converging to z. For each $k \in \mathbb{N}$, define the function

$$\begin{aligned} v_k : B \to \mathbb{C}\,, \quad v_k(\zeta) &:= \chi_{\Delta(z_0-z_k;R)\setminus\{0\}}(\zeta)\, \frac{1}{\zeta}\frac{\partial \eta}{\partial s_1}(\zeta + z_k) \\ &= \begin{cases} \dfrac{1}{\zeta}\dfrac{\partial \eta}{\partial s_1}(\zeta + z_k) & \text{if}\, |\zeta - z_0 + z_k| < R \\ 0 & \text{if}\, |\zeta - z_0 + z_k| > r_2\,, \end{cases} \end{aligned}$$

and define also

$$\begin{aligned} v : B \to \mathbb{C}\,, \quad v(\zeta) &:= \chi_{\Delta(z_0-z;R)\setminus\{0\}}(\zeta)\, \frac{1}{\zeta}\frac{\partial \eta}{\partial s_1}(\zeta + z) \\ &= \begin{cases} \dfrac{1}{\zeta}\dfrac{\partial \eta}{\partial s_1}(\zeta + z) & \text{if}\, |\zeta - z_0 + z| < R \\ 0 & \text{if}\, |\zeta - z_0 + z| > r_2\,. \end{cases} \end{aligned}$$

For each $\zeta \in B$, if $|\zeta - z_0 + z| < R$ then there is $N \in \mathbb{N}$ such that for all $k \geq N$ we also have $|\zeta - z_0 + z_k| < R$; and similarly, if $|\zeta - z_0 + z| > r_2$ then there is $N' \in \mathbb{N}$ such that for all $k \geq N'$ we also have $|\zeta - z_0 + z_k| > r_2$. As a result, $v_k \to v$ pointwise as $k \to \infty$. Moreover, for all $k \in \mathbb{N}$ and $\zeta \in B$,

$$\big|v_k(\zeta)\big| \leq \frac{T}{|\zeta|}\,.$$

We may then apply the Dominated Convergence Theorem to conclude that

$$\frac{\partial \ell}{\partial x}(z_k) = \int_B v_k\, d\lambda \to \int_B v\, d\lambda = \frac{\partial \ell}{\partial x}(z) \quad \text{as } k \to \infty\,.$$

Thus, $\frac{\partial \ell}{\partial x}$ is continuous on $\Delta(z_0; r)$, and by a similar argument, $\frac{\partial \ell}{\partial y}$ is continuous on $\Delta(z_0; r)$. We use this to show that the function

$$\vartheta_1 : \Delta(z_0; r) \to \mathbb{C}\,, \quad z \mapsto \int_{D\setminus\{z\}} \frac{(-2i)\beta_1(\zeta)}{\zeta - z}\, d\lambda(\zeta)$$

is C^k. Since the function $\zeta \mapsto (-2i)\beta_1(\zeta)$ on D is C^k and has support in $\overline{\Delta(z_0; r_2)}$, the function ϑ_1 has continuous partial derivatives of first order, of the form

$$\frac{\partial \vartheta_1}{\partial x}(z) = \int_{D\setminus\{z\}} \frac{\delta_1(\zeta)}{\zeta - z}\, d\lambda(\zeta)$$

and

$$\frac{\partial \vartheta_1}{\partial y}(z) = \int_{D\setminus\{z\}} \frac{\delta_2(\zeta)}{\zeta - z}\, d\lambda(\zeta)\,,$$

on D, where $\delta_1 := (-2i)\frac{\partial\beta_1}{\partial s_1}$ and $\delta_2 := (-2i)\frac{\partial\beta_1}{\partial s_2}$ are C^{k-1} functions $D \to \mathbb{C}$ also supported in $\overline{\Delta(z_0; r_2)}$. If $k = 1$, then we have shown ϑ_1 is C^k. Suppose $k \geq 2$ and let $m \in \{1, \dots, k-1\}$. If ϑ_1 has continuous partial derivatives of mth order, each of which can be written as

$$z \mapsto \int_{D\setminus\{z\}} \frac{\delta(\zeta)}{\zeta - z}\, d\lambda(\zeta)$$

for some C^{k-m} function $\delta : D \to \mathbb{C}$ supported in $\overline{\Delta(z_0; r_2)}$, then ϑ_1 has continuous partial derivatives of $(m+1)$th order, which can each be written as

$$z \mapsto \int_{D\setminus\{z\}} \frac{\gamma(\zeta)}{\zeta - z}\, d\lambda(\zeta)$$

for some $C^{k-(m+1)}$ function $\gamma : D \to \mathbb{C}$ supported in $\overline{\Delta(z_0; r_2)}$. Thus, ϑ_1 has continuous partial derivatives of all orders up to k and hence it is C^k on $\Delta(z_0; r)$.

It remains to show that the function

$$\vartheta_2 : \Delta(z_0; r) \to \mathbb{C}\,, \quad z \mapsto \int_{D\setminus\{z\}} \frac{(-2i)\beta_2(\zeta)}{\zeta - z}\, d\lambda(\zeta)$$

is C^k. Suppose $\varphi : D \to \mathbb{C}$ is a measurable bounded function such that $\varphi = 0$ on $\overline{\Delta(z_0; r_1)}$. Suppose $w \in \Delta(z_0; r)$ with $a := \mathrm{Re}(w)$ and $b := \mathrm{Im}(w)$, and let $J' \subset \mathbb{R}$ be the open interval $J' := \{x \in \mathbb{R} \,|\, x + ib \in \Delta(z_0; r)\}$. Let also $D' := D \setminus \{x + ib \,|\, x \in \mathbb{R}\}$. For any $n \in \mathbb{N}$, the function

$$H : D' \times J' \to \mathbb{C}\,, \quad (\zeta, x) \mapsto \frac{\varphi(\zeta)}{(\zeta - (x + ib))^n}$$

fulfils:

(i) For a fixed $x \in J'$, if $E \in (0, +\infty)$ is an upper bound for $|\varphi|$ on D, then the function $\zeta \mapsto H(\zeta, x)$ on D' is bounded by $\frac{E}{(r_1 - r)^n}$ and hence it is integrable.

(ii) For each $\zeta \in D'$ the function $x \mapsto H(\zeta, x)$ on J' is differentiable with

$$\frac{d}{dx}H(\zeta, x)\bigg|_{x_0} = \frac{n\,\varphi(\zeta)}{(\zeta - (x_0 + ib))^{n+1}}$$

for each $x_0 \in J'$.

(iii) For each $\zeta \in D'$ and $x_0 \in J'$, we have

$$\left|\frac{d}{dx}H(\zeta, x)\bigg|_{x_0}\right| \leq \frac{nE}{(r_1 - r)^{n+1}}\,.$$

Applying again dominated derivation, we conclude that

$$\frac{d}{dx}\left(\int_{D'} \frac{\varphi(\zeta)}{(\zeta - (x + ib))^n}\, d\lambda(\zeta)\right)\bigg|_a = \int_{D'} \frac{n\,\varphi(\zeta)}{(\zeta - (a + ib))^{n+1}}\, d\lambda(\zeta)\,.$$

Thus, letting

$$\Gamma : \Delta(z_0; r) \to \mathbb{C}\,, \quad z \mapsto \int_{D\setminus\{z\}} \frac{\varphi(\zeta)}{(\zeta - z)^n}\, d\lambda(\zeta)\,,$$

for $x \in J'$ we have

$$\Gamma(x + ib) = \int_{D\setminus\{x+ib\}} \frac{\varphi(\zeta)}{(\zeta - (x + ib))^n}\, d\lambda(\zeta) = \int_{D'} \frac{\varphi(\zeta)}{(\zeta - (x + ib))^n}\, d\lambda(\zeta)\,.$$

Then,
$$\frac{\partial \Gamma}{\partial x}(w) = \int_{D'} \frac{n\,\varphi(\zeta)}{(\zeta - w)^{n+1}}\,d\lambda(\zeta) = \int_{D\setminus\{w\}} \frac{n\,\varphi(\zeta)}{(\zeta - w)^{n+1}}\,d\lambda(\zeta)\,,$$
and, by a similar argument,
$$\frac{\partial \Gamma}{\partial y}(w) = \int_{D'} \frac{in\,\varphi(\zeta)}{(\zeta - w)^{n+1}}\,d\lambda(\zeta) = \int_{D\setminus\{w\}} \frac{in\,\varphi(\zeta)}{(\zeta - w)^{n+1}}\,d\lambda(\zeta) = i\frac{\partial \Gamma}{\partial x}(w).$$
As before, we show that $\frac{\partial \Gamma}{\partial x}$ and $\frac{\partial \Gamma}{\partial y}$ are continuous. Let $z \in \Delta(z_0; r)$ and let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\Delta(z_0; r)$ converging to z. Define the functions
$$u_k : D \to \mathbb{C}\,, \quad \zeta \mapsto \chi_{D\setminus\{z_k\}}(\zeta)\frac{n\,\varphi(\zeta)}{(\zeta - z_k)^{n+1}}$$
for $k \in \mathbb{N}$, and
$$u : D \to \mathbb{C}\,, \quad \zeta \mapsto \chi_{D\setminus\{z\}}(\zeta)\frac{n\,\varphi(\zeta)}{(\zeta - z)^{n+1}}\,.$$
Then, $u_k \to u$ as $k \to \infty$ pointwise on D, and for all $k \in \mathbb{N}$ we have $|u_k| \le \frac{nE}{(r_1 - r)^{n+1}}$. By the Dominated Convergence Theorem,
$$\frac{\partial \Gamma}{\partial x}(z_k) = \int_D u_k\,d\lambda \to \int_D u\,d\lambda = \frac{\partial \Gamma}{\partial x}(z)$$
as $k \to \infty$. This shows that $\frac{\partial \Gamma}{\partial x}$ and $\frac{\partial \Gamma}{\partial y}$ are continuous on $\Delta(z_0; r)$. Again, since Γ is real-differentiable and fulfils $\frac{\partial \Gamma}{\partial y} = i\frac{\partial \Gamma}{\partial x}$, it is holomorphic on $\Delta(z_0; r)$.

Then, since the function $\zeta \mapsto (-2i)\beta_2(\zeta)$ on D is continuous, bounded, and vanishes on $\overline{\Delta(z_0; r_1)}$, the function ϑ_2 has continuous partial derivatives of first order given by
$$\frac{\partial \vartheta_2}{\partial x}(z) = \int_{D\setminus\{z\}} \frac{(-2i)\beta_2(\zeta)}{(\zeta - z)^2}\,d\lambda(\zeta)$$
and
$$\frac{\partial \vartheta_2}{\partial y}(z) = \int_{D\setminus\{z\}} \frac{2\beta_2(\zeta)}{(\zeta - z)^2}\,d\lambda(\zeta)$$
for $z \in \Delta(z_0; r)$. If $n \in \mathbb{N}$ and ϑ_2 has continuous partial derivatives of nth order which can each be written as
$$z \mapsto \int_{D\setminus\{z\}} \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}\,d\lambda(\zeta)$$
for some measurable bounded function $\varphi : D \to \mathbb{C}$ that vanishes on $\overline{\Delta(z_0; r_1)}$, then ϑ_2 has continuous partial derivatives of $(n+1)$th order that can each be written as
$$z \mapsto \int_{D\setminus\{z\}} \frac{\heartsuit(\zeta)}{(\zeta - z)^{n+2}}\,d\lambda(\zeta)$$
for some measurable bounded hearty function $\heartsuit : D \to \mathbb{C}$ that vanishes on $\overline{\Delta(z_0; r_1)}$ (a function is defined to be *hearty* when the author has run out of letters in the Latin and Greek alphabets to write it). Thus, ϑ_2 is C^∞ on $\Delta(z_0; r)$, and it is also holomorphic.

We gather all the results to conclude the proof. As we showed initially, the function f_1 is C^∞ and holomorphic on D. Moreover, the function
$$\vartheta_1 : \Delta(z_0; r) \to \mathbb{C}\,, \quad z \mapsto \int_{D\setminus\{z\}} \frac{(-2i)\beta_1(\zeta)}{\zeta - z}\,d\lambda(\zeta)$$

is C^k and the function

$$\vartheta_2 : \Delta(z_0; r) \to \mathbb{C}\,, \quad z \mapsto \int_{D\setminus\{z\}} \frac{(-2i)\beta_2(\zeta)}{\zeta - z}\, d\lambda(\zeta)$$

is C^∞ and holomorphic. Then, we have:

(a) $f_2|_{\Delta(z_0;r)} = \vartheta_1 + \vartheta_2$ is C^k, and since $r \in (0, R)$ is arbitrary, f_2 is C^k on D. Thus,

$$f = \frac{1}{2\pi i}(f_1 + f_2)$$

is C^k on D.

(b) Returning to our choice of r, for all $z \in \Delta(z_0; r)$ we have

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}}(z) &= \frac{1}{2\pi i}\left(\frac{\partial f_1}{\partial \bar{z}}(z) + \frac{\partial \vartheta_1}{\partial \bar{z}}(z) + \frac{\partial \vartheta_2}{\partial \bar{z}}(z)\right)\\ &= \frac{1}{2\pi i}\frac{\partial \vartheta_1}{\partial \bar{z}}(z)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \vartheta_1}{\partial \bar{z}}(z) &= \frac{1}{2}\left(\int_{D\setminus\{z\}} \frac{1}{\zeta - z}(-2i)\frac{\partial \beta_1}{\partial s_1}(\zeta)\, d\lambda(\zeta) + i\int_{D\setminus\{z\}} \frac{1}{\zeta - z}(-2i)\frac{\partial \beta_1}{\partial s_2}(\zeta)\, d\lambda(\zeta)\right)\\ &= \frac{1}{2}\left(\int_{D\setminus\{z\}} \frac{1}{\zeta - z}\frac{\partial \beta_1}{\partial s_1}(\zeta)\, d\zeta \wedge d\bar{\zeta} + i\int_{D\setminus\{z\}} \frac{1}{\zeta - z}\frac{\partial \beta_1}{\partial s_2}(\zeta)\, d\zeta \wedge d\bar{\zeta}\right)\\ &= \int_{D\setminus\{z\}} \frac{\partial \beta_1/\partial \bar{\zeta}}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}\,.\end{aligned}$$

Since β_1 is C^k on a neighbourhood of $\overline{D}$, for each $z \in \Delta(z_0; r) \subset \Delta(z_0; r_1) \subset D$ Cauchy's Integral Formula gives

$$\int_{D\setminus\{z\}} \frac{\partial \beta_1/\partial \bar{\zeta}}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = 2\pi i \beta_1(z) - \int_{\partial D} \frac{\beta_1(\zeta)}{\zeta - z}\, d\zeta = 2\pi i \beta(z)\,,$$

so that

$$\frac{\partial f}{\partial \bar{z}}(z) = \beta(z)\,.$$

Again, since $r \in (0, R)$ was arbitrary, we have

$$\frac{\partial f}{\partial \bar{z}} = \beta$$

on D.

(c) By (b), for all $z \in D \setminus \operatorname{supp}\beta$ we have

$$\frac{\partial f}{\partial \bar{z}}(z) = 0\,,$$

and since $D \setminus \operatorname{supp}\beta$ is open and f is C^k, this implies that f is holomorphic on $D \setminus \operatorname{supp}\beta$.

□

Lemma 4.16. *Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is a C^1 holomorphic function. Then, f is C^∞ and f' is also holomorphic.*

Proof. Let $z_0 \in \Omega$ and choose $R \in (0, \infty)$ such that $D := \Delta(z_0; R)$ fulfils $\overline{D} \subset \Omega$. Since f is continuous on Ω, so too is its restriction to ∂D (that is, the pullback $\iota^* f$ by $\iota : \partial D \to \Omega$). Then, by Lemma 4.15 with $\alpha = f$ on ∂D and $\beta = 0$ on Ω, the function

$$g : D \to \mathbb{C}, \quad z \mapsto \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

is C^∞. Moreover, since f is C^1 and $\partial f / \partial \bar{z} = 0$ on Ω, Cauchy's Integral Formula gives

$$f(z) = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for all $z \in D$, which shows that f is C^∞ on D. Thus, f is C^∞ on Ω. Moreover, as we saw in the proof of Lemma 4.15, the partial derivatives $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ on D are also holomorphic, so that

$$\frac{\partial f'}{\partial \bar{z}}(z_0) = \left(\frac{\partial}{\partial \bar{z}} \frac{\partial f}{\partial x} \right)(z_0) = 2\pi i \left(\frac{\partial}{\partial \bar{z}} \frac{\partial g}{\partial x} \right)(z_0) = 0 \, .$$

Thus, f' is also holomorphic on Ω. □

Theorem 4.17. (Goursat's Theorem) *Let $S = (a, b) \times (c, d) \subset \mathbb{C}$ be an open bounded rectangle, for suitable $a, b, c, d \in \mathbb{R}$. If $f : \Omega \to \mathbb{C}$ is a complex-differentiable function on some neighbourhood Ω of $\overline{S}$, then letting*

$$\begin{aligned}
\gamma_1 : [a, b] \to \Omega \, , &\quad t \mapsto t + ic \, , \\
\gamma_2 : [c, d] \to \Omega \, , &\quad t \mapsto b + it \, , \\
\gamma_3 : [-b, -a] \to \Omega \, , &\quad t \mapsto -t + id \, , \\
\gamma_4 : [-d, -c] \to \Omega \, , &\quad t \mapsto a - it \, ,
\end{aligned}$$

which are C^1 paths whose images are respectively the bottom, right, top, and left sides of ∂S, we have

$$I_S f := \int_{\gamma_1} f \, dz + \int_{\gamma_2} f \, dz + \int_{\gamma_3} f \, dz + \int_{\gamma_4} f \, dz = 0 \, .$$

Proof. For any open bounded rectangle R in Ω such that $\overline{R} \subset \Omega$ and any continuous function $h : \Omega \to \mathbb{C}$, denote by $I_R h$ the sum of the integrals of $h \, dz$ along each of the four sides of R as defined above for S and f. Then, if $R = (a', b') \times (c', d')$, we have

$$\begin{aligned}
I_R h = &\int_{(a', b')} h(t + ic') \, d\lambda(t) + \int_{(c', d')} h(b' + it) i \, d\lambda(t) \\
&+ \int_{(-b', -a')} h(-t + id')(-1) \, d\lambda(t) + \int_{(-d', -c')} h(a' - it)(-i) \, d\lambda(t) \, ,
\end{aligned}$$

and

$$\begin{aligned}
|I_R h| &\le \int_{(a', b')} \left| h(t + ic') \right| \, d\lambda(t) + \int_{(c', d')} \left| h(b' + it) \right| \, d\lambda(t) \\
&\quad + \int_{(-b', -a')} \left| h(-t + id') \right| \, d\lambda(t) + \int_{(-d', -c')} \left| h(a' - it) \right| \, d\lambda(t) \\
&\le \left(\sup_{z \in \overline{R}} \left| h(z) \right| \right) \cdot L_R \, ,
\end{aligned}$$

where $L_R := 2(b' - a' + d' - c')$ denotes the sum of the lengths of the four sides of R. Let $S_1 := S$. For each $n \in \mathbb{N}$, starting from $n = 1$, divide S_n into a 2×2 array of four equally sized subrectangles R_1^n, R_2^n, R_3^n and R_4^n as in the figure, and define $S_{n+1} := R_j^n$ for any $j \in \{1, 2, 3, 4\}$ such that

$$\left|I_{R_j^n} f\right| = \max\left\{\left|I_{R_1^n} f\right|, \left|I_{R_2^n} f\right|, \left|I_{R_3^n} f\right|, \left|I_{R_4^n} f\right|\right\}.$$

Define also $D_n := \sup\{\, |z_1 - z_2| \mid z_1, z_2 \in \overline{S_n}\}$ (which is the length of the diagonal of $\overline{S_n}$) for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. One can check that

$$I_{S_n} f = I_{R_1^n} f + I_{R_2^n} f + I_{R_3^n} f + I_{R_4^n} f\,,$$

so that

$$|I_{S_n} f| \leq \left|I_{R_1^n} f\right| + \left|I_{R_2^n} f\right| + \left|I_{R_3^n} f\right| + \left|I_{R_4^n} f\right| \leq 4\left|I_{S_{n+1}} f\right|\,.$$

We also have $L_{S_{n+1}} = \frac{1}{2} L_{S_n}$, and $D_{n+1} = \frac{1}{2} D_n$. Then, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
|I_S f| &\leq 4^{n-1} |I_{S_n} f|\,,\\
L_{S_n} &= \frac{1}{2^{n-1}} L_S\,,\\
D_n &= \frac{1}{2^{n-1}} D_1\,.
\end{aligned}$$

Since $\{\overline{S_n}\}_{n \in \mathbb{N}}$ is a decreasing sequence of compact subsets of $\mathbb{C}$, there exists a point $z_0 \in \bigcap_{n \in \mathbb{N}} \overline{S_n} \subset \Omega$. Since f is complex-differentiable on Ω, the function

$$g : \Omega \to \mathbb{C}\,, \quad z \mapsto \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$$

is continuous, and for all $z \in \Omega$ we have $f(z) = g(z)(z - z_0) + f'(z_0)(z - z_0) + f(z_0)$. For each $n \in \mathbb{N}$, explicit computation gives

$$I_{S_n}(z \mapsto f'(z_0)(z - z_0) + f(z_0)) = 0\,,$$

so that

$$\begin{aligned}
|I_S f| &\leq 4^{n-1} |I_{S_n} f| \\
&= 4^{n-1} \left|I_{S_n}(z \mapsto g(z)(z - z_0))\right| \\
&\leq 4^{n-1} \left(\sup_{z \in \overline{S_n}} \left|g(z)(z - z_0)\right| \right) \cdot L_{S_n} \\
&\leq 4^{n-1} \left(\sup_{z \in \overline{S_n}} \left|g(z)\right| \right) \cdot D_n L_{S_n} \\
&= 4^{n-1} \left(\sup_{z \in \overline{S_n}} \left|g(z)\right| \right) \cdot \frac{1}{4^{n-1}} D_1 L_S \\
&= \left(\sup_{z \in \overline{S_n}} \left|g(z)\right| \right) \cdot D_1 L_S\,.
\end{aligned}$$

For any $\varepsilon \in (0, +\infty)$, we may find $\delta \in (0, +\infty)$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$, we have $|g(z)| < \varepsilon/(D_1 L_S)$. We may also find $N \in \mathbb{N}$ with $D_N < \delta$, so that for all $z \in \overline{S_N}$ we have $|z - z_0| < \delta$ and hence $|g(z)| < \varepsilon/(D_1 L_S)$. Then,

$$|I_S f| \leq \left(\sup_{z \in \overline{S_N}} |g(z)| \right) \cdot D_1 L_S \leq \frac{\varepsilon}{D_1 L_S} \cdot D_1 L_S = \varepsilon\,.$$

$\square$

Lemma 4.18. *Let $S \subset \mathbb{C}$ be an open bounded rectangle, and $f : S \to \mathbb{C}$ a holomorphic function. Choose any $z_0 = x_0 + iy_0 \in S$, with $x_0, y_0 \in \mathbb{R}$, and define the function*

$$F : S \to \mathbb{C}\,, \quad z \mapsto \int_{x_0}^{x} f(t + iy_0)\,dt + i \int_{y_0}^{y} f(x + it)\,dt$$

for $z \in S$ with $x := \mathrm{Re}(z)$ and $y := \mathrm{Im}(z)$. Then, F is C^1 and holomorphic with $F' = f$.

Proof. We first show that $\frac{\partial F}{\partial y} = if$. Suppose $S = (a, b) \times (c, d)$ for suitable $a, b, c, d \in \mathbb{R}$. Choose $w \in S$ and set $\alpha := \mathrm{Re}(w)$ and $\beta := \mathrm{Im}(w)$. Consider the function

$$g : S \to \mathbb{C}\,, \quad z \mapsto \int_{y_0}^{y} f(x + it)\,dt\,.$$

We have $\{y \in \mathbb{R} \,|\, \alpha + iy \in S\} = (c, d)$. If $\{y_k\}_{k \in \mathbb{N}}$ is a sequence in (β, d) converging to β, for each $k \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{g(\alpha + iy_k) - g(\alpha + i\beta)}{y_k - \beta} &= \frac{\displaystyle\int_{y_0}^{y_k} f(\alpha + it)\,dt - \int_{y_0}^{\beta} f(\alpha + it)\,dt}{y_k - \beta} \\
&= \frac{\displaystyle\int_{\beta}^{y_k} f(\alpha + it)\,dt}{y_k - \beta} \\
&= \frac{\displaystyle\int_{(0,1)} f\big(\alpha + i(t(y_k - \beta) + \beta)\big) \cdot (y_k - \beta)\,d\lambda(t)}{y_k - \beta} \\
&= \int_{(0,1)} f\big(\alpha + i(t(y_k - \beta) + \beta)\big)\,d\lambda(t)\,.
\end{aligned}$$

For all $t \in (0, 1)$, we have $f\big(\alpha + i(t(y_k - \beta) + \beta)\big) \to f(\alpha + i\beta)$ as $k \to \infty$. Moreover, there exists a compact subset $A \subset S$ such that for all $k \in \mathbb{N}$ and $t \in (0, 1)$ we have $\alpha + i(t(y_k - \beta) + \beta) \in A$. Thus, for some $P \in (0, +\infty)$ we have $\Big|f\big(\alpha + i(t(y_k - \beta) + \beta)\big)\Big| \leq P$ for all $k \in \mathbb{N}$ and $t \in (0, 1)$. We may then apply the Dominated Convergence Theorem to conclude that

$$\frac{g(\alpha + iy_k) - g(\alpha + i\beta)}{y_k - \beta} \to f(\alpha + i\beta) \quad \text{as } k \to \infty\,.$$

On the other hand, if $\{y_k\}_{k\in\mathbb{N}}$ is a sequence in (c,β) converging to β, for each $k\in\mathbb{N}$ we have

$$
\begin{aligned}
\frac{g(\alpha+iy_k)-g(\alpha+i\beta)}{y_k-\beta} &= \frac{\displaystyle\int_{y_0}^{y_k} f(\alpha+it)\,dt-\int_{y_0}^{\beta} f(\alpha+it)\,dt}{y_k-\beta} \\
&= \frac{\displaystyle\int_{y_k}^{\beta} f(\alpha+it)\,dt}{\beta-y_k} \\
&= \frac{\displaystyle\int_{(0,1)} f\big(\alpha+i(t(\beta-y_k)+y_k)\big)\cdot(\beta-y_k)\,d\lambda(t)}{\beta-y_k} \\
&= \int_{(0,1)} f\big(\alpha+i(t(\beta-y_k)+y_k)\big)\,d\lambda(t)\,,
\end{aligned}
$$

and reasoning as above, we obtain

$$\frac{g(\alpha+iy_k)-g(\alpha+i\beta)}{y_k-\beta}\to f(\alpha+i\beta)\quad \text{as } k\to\infty\,.$$

Thus,

$$\frac{\partial g}{\partial y}(w)=f(w)\,,$$

so

$$\frac{\partial F}{\partial y}(w)=if(w)\,.$$

We now show that $\frac{\partial F}{\partial x}=f$. By Goursat's Theorem (Theorem 4.17), for each $z=x+iy\in S$ we also have

$$F(z)=i\int_{y_0}^{y} f(x_0+it)\,dt+\int_{x_0}^{x} f(t+iy)\,dt\,.$$

Then, reasoning analogously, for any $w\in S$ we have

$$\frac{\partial F}{\partial x}(w)=f(w).$$

In conclusion, we have $\frac{\partial F}{\partial x}=f$ and $\frac{\partial F}{\partial y}=if$ on S. Thus, F has continuous partial derivatives of first order on S and hence it is C^1, and the fact that $\frac{\partial F}{\partial y}=i\frac{\partial F}{\partial x}$ guarantees that the Cauchy-Riemann equations are fulfilled, so that F is holomorphic. Moreover, we have

$$F'=\frac{\partial F}{\partial x}=f\,.$$

□

Theorem 4.19. *A holomorphic complex-valued function on an open subset of $\mathbb{C}$ is smooth.*

Proof. Suppose $\Omega\subset\mathbb{C}$ is open and $f:\Omega\to\mathbb{C}$ is holomorphic. For any point $w\in\Omega$, we may choose an open bounded rectangle S in $\mathbb{C}$ such that $w\in S\subset\Omega$. By Lemmas 4.16 and 4.18, there exists a holomorphic C^∞ function $F:S\to\mathbb{C}$ such that $F'=f|_S$ on S. Since $F'=\frac{\partial F}{\partial x}$ is C^∞ on S, so too is $f|_S$. □

Remark 4.20. Note that it follows from Lemma 4.16 and Theorem 4.19 that if $\Omega \subset \mathbb{C}$ is open and f is holomorphic on Ω, then $f' = \frac{df}{dz} = \frac{\partial f}{\partial x}$ is also holomorphic on Ω.

Theorem 4.21. *Suppose $\Omega \subset \mathbb{C}$ is open, $K \subset \Omega$ is compact and nonempty, and*

$$A := \sum_{\alpha \in \Theta_\ell^2} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha$$

is a linear differential operator of order $\ell \in \mathbb{Z}_{\geq 0}$ on Ω such that for all $\alpha \in \Theta_\ell^2$ we have $a_\alpha \in L^\infty_{\mathrm{loc}}(\Omega)$. Then, there exists $C = C(\Omega, K, A) \in [0, +\infty)$ such that for all $f \in \mathcal{O}(\Omega)$ and for all $p \in [1, +\infty]$,

$$\|Af\|_{L^\infty(K)} \leq C\|f\|_{L^p(\Omega)} ,$$

where we let $\|f\|_{L^p(\Omega)} := +\infty$ if $f \notin L^p(\Omega)$.

Proof. Choose an open set $U \subset \Omega$ such that $K \subset U \Subset \Omega$, and let

$$C_1 := \begin{cases} 1 & \text{if } \lambda(\overline{U}) \leq 1 \\ \lambda(\overline{U}) & \text{if } \lambda(\overline{U}) > 1 \end{cases} .$$

Fix $f \in \mathcal{O}(\Omega)$ and $p \in [1, +\infty]$, and suppose $f \in L^p(\Omega)$. Define

$$q := \begin{cases} +\infty & \text{if } p = 1 \\ \frac{p}{p-1} & \text{if } p \in (1, +\infty) \\ 1 & \text{if } p = +\infty . \end{cases}$$

Then, we have

$$\frac{1}{p} + \frac{1}{q} = 1 ,$$

with the conventions $\frac{1}{0} := +\infty$ and $\frac{1}{+\infty} := 0$. Since $\overline{U} \subset \Omega$, we have $f \in L^p(\overline{U})$; moreover, letting $g : \overline{U} \to \mathbb{C}$ denote the constant function 1 on $\overline{U}$, we have $g \in L^q(\overline{U})$. By Hölder's inequality, we know that the function $f = fg$ on $\overline{U}$ is in $L^1(\overline{U})$ and

$$\|f\|_{L^1(\overline{U})} = \|fg\|_{L^1(\overline{U})} \leq \|f\|_{L^p(\overline{U})} \|g\|_{L^q(\overline{U})} = \|f\|_{L^p(\overline{U})} \lambda(\overline{U})^{\frac{1}{q}}$$

(where $\lambda(\overline{U})^{\frac{1}{q}} = 1$ if $q = +\infty$). Then, we have

$$\|f\|_{L^1(\overline{U})} \leq C_1 \|f\|_{L^p(\overline{U})} \leq C_1 \|f\|_{L^p(\Omega)} .$$

If $f \notin L^p(\Omega)$, the above equality also holds.

Since $K \subset U$ is compact, we may choose $a \in (0, +\infty)$ such that for all $z_0 \in K$, $\Delta(z_0; a) \subset U$. Choose also $b \in (0, a) \cap (0, 3)$, and let $\rho : \Delta(0; a) \to \mathbb{R}$ be a C^∞ bump function such that $\rho = 1$ on $\overline{\Delta(0; \frac{b}{3})}$ and $\operatorname{supp}\rho \subset \overline{\Delta(0; \frac{2b}{3})}$. Define

$$M := \max_{m \in \{0, \ldots, \ell\}} \left\| \frac{\partial^m}{\partial x^m} \frac{\partial \rho}{\partial \bar{z}} \right\|_{L^\infty(\Delta(0,b))} \in (0, +\infty) .$$

Fix $z_0 \in K$, and define the functions

$$\mu_{z_0} : \Delta(z_0; a) \to \Delta(0; a) , \quad z \mapsto z - z_0$$

and

$$\rho_{z_0} := \rho \circ \mu_{z_0} : \Delta(z_0; a) \to \mathbb{R} .$$

Fix also $m \in \{0, \ldots, \ell\}$. Since the function $\rho_{z_0} \frac{d^m f}{dz^m}$ on $\Delta(z_0; a)$ is C^1, by Cauchy's Integral Formula we have

$$\frac{d^m f}{dz^m}(z_0) = \left(\rho_{z_0} \frac{d^m f}{dz^m}\right)(z_0) = \frac{1}{2\pi i}\left(\int_{\partial\Delta(z_0;b)} \frac{\left(\rho_{z_0}\frac{d^m f}{dz^m}\right)(z)}{z-z_0}\, dz + \int_{\Delta(z_0;b)\setminus\{z_0\}} \frac{\partial\left(\rho_{z_0}\frac{d^m f}{dz^m}\right)/\partial\bar{z}}{z-z_0}\, dz \wedge d\bar{z}\right).$$

Since $\mathrm{supp}\rho_{z_0} \subset \overline{\Delta(z_0; \frac{2b}{3})}$, we have

$$\int_{\partial\Delta(z_0;b)} \frac{\left(\rho_{z_0}\frac{d^m f}{dz^m}\right)(z)}{z-z_0}\, dz = 0\,.$$

Moreover,

$$\int_{\Delta(z_0;b)\setminus\{z_0\}} \frac{\partial\left(\rho_{z_0}\frac{d^m f}{dz^m}\right)/\partial\bar{z}}{z-z_0}\, dz \wedge d\bar{z} = \int_{\Delta(z_0;b)\setminus\{z_0\}} \frac{(\partial\rho_{z_0}/\partial\bar{z})\frac{d^m f}{dz^m}}{z-z_0}\,(-2i)\, d\lambda\,.$$

Define

$$\varphi_{z_0} : \Delta(z_0; b) \setminus \{z_0\} \to \mathbb{C}\,, \quad z \mapsto \frac{(\partial\rho_{z_0}/\partial\bar{z})(z)}{z-z_0}\,.$$

Since $\rho_{z_0} = 1$ on $\Delta(z_0; \frac{b}{3})$ and $\rho_{z_0} = 0$ on $\Delta(z_0; a) \setminus \overline{\Delta(z_0; \frac{2b}{3})}$, the function φ_{z_0} vanishes outside of the annulus

$$\left\{ z \in \mathbb{C} \;\middle|\; \frac{b}{3} \leq |z - z_0| \leq \frac{2b}{3} \right\},$$

and hence we have $\varphi_{z_0} \in \mathcal{D}(\Delta(z_0; b) \setminus \{z_0\})$. Since f is holomorphic, we have

$$\frac{d^m f}{dz^m} = \frac{\partial^m f}{\partial x^m}\,;$$

so, regarding $B := \frac{\partial^m}{\partial x^m}$ as a linear differential operator of order m on $\Delta(z_0; b) \setminus \{z_0\}$, and applying Lemma 3.10 (i), we have

$$\begin{aligned}
\int_{\Delta(z_0;b)\setminus\{z_0\}} \frac{(\partial\rho_{z_0}/\partial\bar{z})\frac{d^m f}{dz^m}}{z-z_0}\,(-2i)\, d\lambda &= \int_{\Delta(z_0;b)\setminus\{z_0\}} \varphi_{z_0} B(-2if)\, d\lambda \\
&= \int_{\Delta(z_0;b)\setminus\{z_0\}} (-2i)f \cdot {}^t B\varphi_{z_0}\, d\lambda \\
&= \int_{\Delta(z_0;b)\setminus\{z_0\}} (-2i)f \cdot (-1)^m \frac{\partial^m \varphi_{z_0}}{\partial x^m}\, d\lambda\,.
\end{aligned}$$

As one can check, on $\Delta(z_0; a)$, and hence on $\Delta(z_0; b) \setminus \{z_0\}$, for all $q \in \{0, \ldots, \ell\}$ we have

$$\frac{\partial^q}{\partial x^q}\frac{\partial\rho_{z_0}}{\partial\bar{z}} = \frac{\partial^q}{\partial x^q}\frac{\partial}{\partial\bar{z}}(\rho \circ \mu_{z_0}) = \left(\frac{\partial^q}{\partial x^q}\frac{\partial\rho}{\partial\bar{z}}\right) \circ \mu_{z_0}\,.$$

Then, on $\Delta(z_0; b) \setminus \{z_0\}$,

$$\begin{aligned}
\frac{\partial^m \varphi_{z_0}}{\partial x^m} &= \frac{\partial^m}{\partial x^m}\left(\frac{\partial\rho_{z_0}/\partial\bar{z}}{z-z_0}\right) \\
&= \sum_{q=0}^{m} \binom{m}{q} \left(\frac{\partial^{m-q}}{\partial x^{m-q}}\frac{\partial\rho_{z_0}}{\partial\bar{z}}\right) \frac{\partial^q}{\partial x^q}\left(\frac{1}{z-z_0}\right) \\
&= \sum_{q=0}^{m} \binom{m}{q} \left(\left(\frac{\partial^{m-q}}{\partial x^{m-q}}\frac{\partial\rho}{\partial\bar{z}}\right) \circ \mu_{z_0}\right) (-1)^q\, q\,! \frac{1}{(z-z_0)^{q+1}}\,.
\end{aligned}$$

Let now $w \in \Delta(z_0; b) \setminus \{z_0\}$. If $|w - z_0| \geq \frac{b}{3}$, using the fact that $\frac{3}{b} > 1$, we have

$$\begin{aligned}
\left|\frac{\partial^m \varphi_{z_0}}{\partial x^m}(w)\right| &= \left|\sum_{q=0}^{m} \binom{m}{q} \left(\left(\frac{\partial^{m-q}}{\partial x^{m-q}} \frac{\partial \rho}{\partial \bar{z}} \right)(w - z_0) \right) (-1)^q \, q! \, \frac{1}{(w - z_0)^{q+1}} \right| \\
&\leq \sum_{q=0}^{m} \binom{m}{q} M q! \left(\frac{3}{b}\right)^{q+1} \\
&\leq \sum_{q=0}^{\ell} \ell! M \ell! \left(\frac{3}{b}\right)^{\ell+1} \\
&= (\ell+1)(\ell!)^2 M \left(\frac{3}{b}\right)^{\ell+1} \quad ;
\end{aligned}$$

while if $|w - z_0| < \frac{b}{3}$, we have

$$\left|\frac{\partial^m \varphi_{z_0}}{\partial x^m}(w)\right| = 0 \, .$$

Thus, we have

$$\left|\frac{\partial^m \varphi_{z_0}}{\partial x^m}\right| \leq (\ell+1)(\ell!)^2 M \left(\frac{3}{b}\right)^{\ell+1} =: C_2$$

on $\Delta(z_0; b) \setminus \{z_0\}$. Gathering our results so far, we have

$$\begin{aligned}
\left|\frac{d^m f}{dz^m}(z_0)\right| &= \left|\frac{1}{2\pi i} \int_{\Delta(z_0;b)\setminus\{z_0\}} (-2i) f \cdot (-1)^m \frac{\partial^m \varphi_{z_0}}{\partial x^m} \, d\lambda \right| \\
&\leq \frac{1}{\pi} \int_{\Delta(z_0;b)\setminus\{z_0\}} \left| f \, \frac{\partial^m \varphi_{z_0}}{\partial x^m} \right| \, d\lambda \\
&\leq \frac{1}{\pi} \, C_2 \int_{\Delta(z_0;b)\setminus\{z_0\}} |f| \, d\lambda \\
&\leq \frac{1}{\pi} \, C_2 \|f\|_{L^1(\overline{U})} \\
&\leq \frac{1}{\pi} \, C_2 \, C_1 \|f\|_{L^p(\Omega)} \ ,
\end{aligned}$$

which implies that

$$\left\|\frac{d^m f}{dz^m}\right\|_{L^\infty(K)} \leq \frac{1}{\pi} \, C_2 \, C_1 \|f\|_{L^p(\Omega)} \ .$$

Now, observe that for each $\alpha = (\alpha_1, \alpha_2) \in \Theta_\ell^2$, on Ω we may write

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} f = \frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}} \frac{\partial^{\alpha_1} f}{\partial x^{\alpha_1}} = i^{\alpha_2} \frac{d^{|\alpha|} f}{dz^{|\alpha|}} \, .$$

Then, we have

$$A f = \sum_{q=0}^{\ell} b_q \frac{d^q f}{dz^q} \, ,$$

where

$$b_q := \sum_{j=0}^{q} i^j a_{(q-j,j)} \in L^\infty_{\mathrm{loc}}(\Omega)$$

for each $q \in \{0, \dots, \ell\}$. We may then choose a set $S \subset K$ of measure 0 such that on $K \setminus S$ we have

$$\left|b_q\right| \le \left\|b_q\right\|_{L^\infty(K)}$$

for all $q \in \{0, \dots, \ell\}$. Then, letting

$$C_3 := \max_{q \in \{0, \dots, \ell\}} \left\|b_q\right\|_{L^\infty(K)} ,$$

for all $z_0 \in K \setminus S$ we have

$$\begin{aligned} \left|(Af)(z_0)\right| &= \left|\sum_{q=0}^{\ell} b_q(z_0) \frac{d^q f}{dz^q}(z_0)\right| \\ &\le \sum_{q=0}^{\ell} \left\|b_q\right\|_{L^\infty(K)} \frac{1}{\pi} C_2 \, C_1 \|f\|_{L^p(\Omega)} \\ &\le (\ell+1) \frac{1}{\pi} C_3 \, C_2 \, C_1 \|f\|_{L^p(\Omega)} . \end{aligned}$$

Thus, letting $C := (\ell+1)\frac{1}{\pi} C_3 \, C_2 \, C_1 \in [0, +\infty)$, we have

$$\|Af\|_{L^\infty(K)} \le C \|f\|_{L^p(\Omega)} .$$

□

Corollary 4.22. *Suppose $\Omega \subset \mathbb{C}$ is open. Then, for every nonempty compact subset $K \subset \Omega$ there exists a constant $C = C(\Omega, K) \in [0, +\infty)$ such that for all $f \in \mathcal{O}(\Omega)$, $p \in [1, +\infty]$, and $z, w \in K$, we have*

$$\left|f(z) - f(w)\right| \le |z - w| \, C \|f\|_{L^p(\Omega)} .$$

Proof. Choose an open subset $U \subset \Omega$ such that $K \subset U \Subset \Omega$, and choose $a \in (0, +\infty)$ such that for all $z_0 \in K$ we have $\Delta(z_0; a) \subset U$. Choose also $C_1, C_2 \in [0, +\infty)$ such that for all $f \in \mathcal{O}(\Omega)$ and for all $p \in [1, +\infty]$,

$$\left\|\frac{df}{dz}\right\|_{L^\infty(\overline{U})} = \left\|\frac{\partial f}{\partial x}\right\|_{L^\infty(\overline{U})} \le C_1 \|f\|_{L^p(\Omega)}$$

and

$$\|f\|_{L^\infty(\overline{U})} \le C_2 \|f\|_{L^p(\Omega)} .$$

Fix $f \in \mathcal{O}(\Omega)$, $p \in [1, +\infty]$, and $z, w \in K$. We consider two cases:

(i) $|w - z| < a$. Choosing small enough $\varepsilon \in (0, +\infty)$, for all $t \in (-\varepsilon, 1 + \varepsilon)$ we have $z + (w - z)t \in \Delta(z; a)$, so we may define the function

$$\mu : (-\varepsilon, 1 + \varepsilon) \to \Delta(z; a) , \quad t \mapsto z + (w - z)t .$$

The composition $g := f \circ \mu : (-\varepsilon, 1 + \varepsilon) \to \mathbb{C}$ is then C^∞, and as one can check, for each $t_0 \in (-\varepsilon, 1 + \varepsilon)$ we have

$$\frac{dg}{dt}(t_0) = (w - z) \cdot \frac{df}{dz}(z + (w - z)t_0) .$$

Applying the Fundamental Theorem of Calculus, we obtain

$$\int_{[0,1]} \frac{dg}{dt} \, d\lambda = g(1) - g(0) = f(w) - f(z) ,$$

so that

$$
\begin{aligned}
\left|f(w)-f(z)\right| &= \left|\int_{[0,1]} \frac{dg}{dt}\,d\lambda\right| \\
&= \left|\int_{[0,1]} (w-z)\cdot\frac{df}{dz}(z+(w-z)t)\,d\lambda(t)\right| \\
&\leq |w-z| \int_{[0,1]} \left|\frac{df}{dz}(z+(w-z)t)\right|\,d\lambda(t) \\
&\leq |w-z| \left\|\frac{df}{dz}\right\|_{L^\infty(U)} \\
&\leq |w-z| \left\|\frac{df}{dz}\right\|_{L^\infty(\overline{U})} \\
&\leq |w-z|\, C_1 \|f\|_{L^p(\Omega)} .
\end{aligned}
$$

(ii) $|w-z| \geq a$. We have

$$
\frac{\left|f(w)-f(z)\right|}{|w-z|} \leq \frac{2\|f\|_{L^\infty(U)}}{a} \leq \frac{2\|f\|_{L^\infty(\overline{U})}}{a} \leq \frac{2C_2\|f\|_{L^p(\Omega)}}{a} .
$$

Thus, letting $C := \max\{C_1, \frac{2C_2}{a}\}$, we have

$$
\left|f(w)-f(z)\right| \leq |w-z|\, C\|f\|_{L^p(\Omega)}
$$

for all $f \in \mathcal{O}(\Omega)$, $p \in [1,+\infty]$, and $z, w \in K$. $\square$

Corollary 4.23. *Let $\Omega \subset \mathbb{C}$ be open. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of holomorphic functions on Ω, and suppose $f : \Omega \to \mathbb{C}$ is a function such that $\{f_n\}$ converges uniformly to f on compact subsets of Ω. Then, f is holomorphic and for all $m \in \mathbb{N}$ the sequence $\{f_n^{(m)}\}_{n\in\mathbb{N}}$ of mth complex derivatives converges to the mth complex derivative $f^{(m)}$ uniformly on compact subsets of Ω.*

Proof. First note that the fact that $\{f_n\}$ converges uniformly to f on compact subsets of Ω implies that f is continuous on Ω. Let $w \in \Omega$, and choose $a \in (0,+\infty)$ fulfilling $\overline{\Delta(w;a)} \subset \Omega$. Then, for each $n \in \mathbb{N}$ and $z_0 \in \Delta(w;a)$, we have

$$
f_n(z_0) = \int_{\partial\Delta(w;a)} \frac{f_n(z)}{z-z_0}\,dz = \int_{(0,2\pi)} \frac{f_n(w+ae^{i\theta})}{w+ae^{i\theta}-z_0}\, aie^{i\theta}\,d\lambda(\theta).
$$

Since f is continuous on $\partial\Delta(w;a)$, the function

$$
g : \Delta(w;a) \to \mathbb{C}, \quad z_0 \mapsto \int_{\partial\Delta(w;a)} \frac{f(z)}{z-z_0}\,dz = \int_{(0,2\pi)} \frac{f(w+ae^{i\theta})}{w+ae^{i\theta}-z_0}\, aie^{i\theta}\,d\lambda(\theta)
$$

is holomorphic. Fix $z_0 \in \Delta(w;a)$, and let $M_{z_0} := \operatorname{dist}(z_0, \partial\Delta(w;a)) \in (0,+\infty)$. Let $\varepsilon \in (0,+\infty)$, and choose $N \in \mathbb{N}$ such that for all $n \geq N$

$$
\sup_{\partial\Delta(w;a)} |f_n - f| \leq \frac{M_{z_0}\varepsilon}{2\pi a} .
$$

Then, for all $n \geq N$ we have

$$
\begin{aligned}
\left|f_n(z_0) - g(z_0)\right| &= \left|\int_{(0,2\pi)} \frac{f_n(w+ae^{i\theta})}{w+ae^{i\theta}-z_0}\, aie^{i\theta}\, d\lambda(\theta) - \int_{(0,2\pi)} \frac{f(w+ae^{i\theta})}{w+ae^{i\theta}-z_0}\, aie^{i\theta}\, d\lambda(\theta)\right| \\
&\leq \int_{(0,2\pi)} \left| aie^{i\theta} \frac{\left(f_n(w+ae^{i\theta}) - f(w+ae^{i\theta})\right)}{w+ae^{i\theta}-z_0}\right| d\lambda(\theta) \\
&\leq \int_{(0,2\pi)} a \frac{M_{z_0}\varepsilon/(2\pi a)}{M_{z_0}} d\lambda(\theta) \\
&= \varepsilon\,,
\end{aligned}
$$

which shows that $f_n(z_0) \to g(z_0)$ as $n \to \infty$. By uniqueness of the limit, we must then have $g(z_0) = f(z_0)$. Since $z_0 \in \Delta(w; a)$ was arbitrary, we have $f = g$ on $\Delta(w; a)$, so f is holomorphic on $\Delta(w; a)$. It follows that $f \in \mathcal{O}(\Omega)$.

We now show that for all $m \in \mathbb{N}$, $f_n^{(m)} \to f^{(m)}$ uniformly on compact subsets of Ω. Fix $m \in \mathbb{N}$ and let $K \subset \Omega$ be compact. Choose open subsets $U, V \subset \Omega$ fulfilling $K \subset U \Subset \Omega$ and $\overline{U} \subset V \Subset \Omega$. Choose also $C_m \in (0, +\infty)$ such that for all $g \in \mathcal{O}(V)$ we have

$$
\left\|g^{(m)}\right\|_{L^\infty(\overline{U})} = \left\|\frac{\partial^m g}{\partial x^m}\right\|_{L^\infty(\overline{U})} \leq C_m \|g\|_{L^\infty(V)}\,.
$$

Let $\varepsilon \in (0, +\infty)$ and choose $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$
\sup_{\overline{V}} |f_n - f| \leq \frac{\varepsilon}{C_m}\,.
$$

Then, for all $n \geq N$ and $z_0 \in K$ we have

$$
\begin{aligned}
\left|f_n^{(m)}(z_0) - f^{(m)}(z_0)\right| &= \left|\frac{\partial^m (f_n - f)}{\partial x^m}(z_0)\right| \\
&\leq \left\|\frac{\partial^m (f_n - f)}{\partial x^m}\right\|_{L^\infty(U)} \\
&\leq \left\|\frac{\partial^m (f_n - f)}{\partial x^m}\right\|_{L^\infty(\overline{U})} \\
&\leq C_m \|f_n - f\|_{L^\infty(V)} \\
&\leq \varepsilon\,.
\end{aligned}
$$

□

To prove the next corollary, we will apply the following theorem, which we state without proof:

Theorem 4.24. (Arzelà–Ascoli theorem). *Suppose $\Omega \subset \mathbb{C}$ is open and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of complex-valued functions on Ω such that on every compact subset of Ω the functions $\{f_n\}_{n\in\mathbb{N}}$ are uniformly bounded and equicontinuous. Then, there exists a function $f : \Omega \to \mathbb{C}$ and a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ converging to f uniformly on compact subsets of Ω.*

Corollary 4.25. (Montel's theorem). *Suppose $\Omega \subset \mathbb{C}$ is open and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then,* *there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and a function $f \in \mathcal{O}(\Omega)$ such that $\{f_{n_k}\}$ converges uniformly to f on compact subsets of Ω.*

Proof. We first show that the functions $\{f_n\}_{n\in\mathbb{N}}$ are equicontinuous on compact subsets of Ω; that is, that given a compact subset $K \subset \Omega$, for every $\varepsilon \in (0, +\infty)$ there exists $\delta \in (0, +\infty)$ such that for all $n \in \mathbb{N}$ and for all $z, w \in K$ fulfilling $|z - w| < \delta$, we have $\big|f_n(z) - f_n(w)\big| < \varepsilon$. Let $K \subset \Omega$ be compact and nonempty, and choose an open subset $U \Subset \Omega$ containing K. Choose also $M \in (0, +\infty)$ such that for all $n \in \mathbb{N}$ we have $|f_n| \leq M$ on $\overline{U}$. By Corollary 4.22, there exists $C \in (0, +\infty)$ such that for all $g \in \mathcal{O}(U)$ and for all $z, w \in K$,

$$\big|g(z) - g(w)\big| \leq |z - w|\, C \|g\|_{L^\infty(U)}\ .$$

Let $\varepsilon \in (0, +\infty)$, and let $\delta := \varepsilon/(2CM)$. Then, for all $n \in \mathbb{N}$ and for all $z, w \in K$ such that $|z - w| < \delta$, we have

$$\big|f_n(z) - f_n(w)\big| \leq |z - w|\, C \|f_n\|_{L^\infty(U)} \leq \delta\, CM = \frac{\varepsilon}{2} < \varepsilon\,.$$

Thus, the functions $\{f_n\}_{n\in\mathbb{N}}$ are equicontinuous on K. By Theorem 4.24, it follows that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and a function $f : \Omega \to \mathbb{C}$ such that $\{f_{n_k}\}$ converges uniformly to f on compact subsets on Ω. By Corollary 4.23, we have $f \in \mathcal{O}(\Omega)$, which concludes the proof. □

Lemma 4.26. *Suppose $\Omega \subset \mathbb{C}$ is open and $v : \Omega \to \mathbb{C}$ is a locally integrable function fulfilling*

$$0 = \left(\frac{\partial}{\partial \bar{z}}\right)_{\mathrm{distr}} v\,.$$

Then, there exists a function $f \in \mathcal{O}(\Omega)$ such that $v = f$ almost everywhere in Ω.

Proof. Choose a smooth nonnegative function $k : \mathbb{C} \to \mathbb{R}$ fulfilling $\operatorname{supp} k \subset \Delta(0; 1)$ and $\int_{\mathbb{C}} k\, d\lambda = 1$. Choose also $w \in \Omega$ and $a \in (0, +\infty)$ such that $\overline{\Delta(w; a)} \subset \Omega$, and let $D := \Delta(w; a)$. Since v is locally integrable on Ω, for each $\delta \in (0, +\infty)$ we may consider the function

$$v_\delta : \Omega_\delta := \{x \in \mathbb{C} \,|\, \operatorname{dist}(x, \Omega^c) > \delta\} \to \mathbb{C}\,, \quad x \mapsto \int_D v(y)\, k\!\left(\frac{x - y}{\delta}\right) \frac{1}{\delta^2}\, d\lambda(y)\,,$$

which is C^∞. Moreover, since $\partial/\partial\bar{z}$ is a linear differential operator with constant coefficients, by Lemma 3.11 we have

$$0 = 0_\delta = \frac{\partial v_\delta}{\partial \bar{z}}$$

on Ω_δ. This shows that for each $\delta \in (0, +\infty)$ the function $v_\delta : \Omega_\delta \to \mathbb{C}$ is holomorphic. Since $\overline{D}$ is compact, we may choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$ we have

$$\overline{D} \subset \Omega_{\frac{1}{N}} \subset \Omega_{\frac{1}{n}}\,.$$

We then obtain a sequence $\{v_{\frac{1}{n}}\}_{n\geq N}$ of holomorphic functions on $\Omega_{1/N}$. We wish to show that this sequence is uniformly bounded on compact subsets of $\Omega_{1/N}$. Fix then a compact set $K \subset \Omega_{1/N}$, and choose open subsets $U, V \subset \Omega_{1/N}$ fulfilling $K \subset U \Subset \Omega_{1/N}$ and $\overline{U} \subset V \Subset \Omega_{1/N}$. By Lemma 3.3 (vi), we know that $\left\|v_{\frac{1}{n}} - v\right\|_{L^1(\overline{V})} \to 0$ as $\mathbb{N}_{\geq N} \ni$

$n \to \infty$, which implies that there exists $M \in (0, \infty)$ such that for all $n \in \mathbb{N}_{\geq N}$ we have $\left\| v_{\frac{1}{n}} - v \right\|_{L^1(\overline{V})} \leq M$. We may also choose $C \in (0, +\infty)$ such that for all $g \in \mathcal{O}(V)$,

$$\|g\|_{L^\infty(\overline{U})} \leq C\|g\|_{L^1(V)} \, .$$

Then, for all $n \in \mathbb{N}_{\geq N}$ and $z_0 \in K$ we have

$$\begin{aligned}
\left| v_{\frac{1}{n}}(z_0) \right| &\leq \left\| v_{\frac{1}{n}} \right\|_{L^\infty(U)} \\
&\leq \left\| v_{\frac{1}{n}} \right\|_{L^\infty(\overline{U})} \\
&\leq C \left\| v_{\frac{1}{n}} \right\|_{L^1(V)} \\
&\leq C \left\| v_{\frac{1}{n}} \right\|_{L^1(\overline{V})} \\
&\leq C \Big(\left\| v_{\frac{1}{n}} - v \right\|_{L^1(\overline{V})} + \|v\|_{L^1(\overline{V})} \Big) \\
&\leq C \big(M + \|v\|_{L^1(\overline{V})} \big) \, ,
\end{aligned}$$

which shows that $\{v_{\frac{1}{n}}\}_{n \geq N}$ is uniformly bounded on K. By Montel's theorem (Corollary 4.25), it follows that there exists a subsequence $\{v_{\frac{1}{n_k}}\}_{k \in \mathbb{N}}$ and a function $f_D \in \mathcal{O}(\Omega_{1/N})$ such that $\{v_{\frac{1}{n_k}}\}_{k \in \mathbb{N}}$ converges to g uniformly on compact subsets of $\Omega_{1/N}$. Then, since $\overline{D} \subset \Omega_{1/N}$ is compact, for each $k \in \mathbb{N}$ we have

$$\left\| v_{\frac{1}{n_k}} - g \right\|_{L^1(\overline{D})} = \int_{\overline{D}} \left| v_{\frac{1}{n_k}} - g \right| \, d\lambda \leq \bigg(\sup_{\overline{D}} \left| v_{\frac{1}{n_k}} - g \right| \bigg) \pi a^2 \to 0$$

as $k \to \infty$. Thus, we have $v_{\frac{1}{n_k}} \to g$ in $L^1(\overline{D})$ as $k \to \infty$, and again by Lemma 3.3 (vi), we also have $v_{\frac{1}{n_k}} \to v$ in $L^1(\overline{D})$ as $k \to \infty$. It follows that $v = g$ almost everywhere in $\overline{D}$ and hence also $v = g$ almost everywhere in D.

We may find a countable open cover $\{D_m\}_{m \in \mathbb{N}}$ for Ω such that for all $m \in \mathbb{N}$ we have $D_m = \Delta(w; a)$ for some $w \in \Omega$ and $a \in (0, +\infty)$ fulfilling $\overline{\Delta(w; a)} \subset \Omega$. Then, by the above reasoning, for each $m \in \mathbb{N}$ there exists a function $f_m \in \mathcal{O}(D_m)$ such that $v = f_m$ almost everywhere on D_m, that is, there exists a measurable subset $S_m \subset D_m$ of measure 0 such that $v = f_m$ on $D_m \setminus S_m$. Suppose $m_1, m_2 \in \mathbb{N}$ and $D_{m_1} \cap D_{m_2} \neq \emptyset$. We wish to show that then $f_{m_1} = f_{m_2}$ on $D_{m_1} \cap D_{m_2}$. Fix $z_0 \in D_{m_1} \cap D_{m_2}$. If $z_0 \notin S_{m_1} \cup S_{m_2}$, then we have $f_{m_1}(z_0) = v(z_0) = f_{m_2}(z_0)$. Suppose that $z_0 \in S_{m_1} \cup S_{m_2}$, and choose $\varepsilon \in (0, \infty)$ such that $\Delta(z_0; \varepsilon) \subset D_{m_1} \cap D_{m_2}$. For each $n \in \mathbb{N}$, we cannot have $\Delta(z_0; \frac{\varepsilon}{n}) \subset S_{m_1} \cup S_{m_2}$, since $S_{m_1} \cup S_{m_2}$ has measure 0. Thus, we may choose a point $z_n \in \Delta(z_0; \frac{\varepsilon}{n}) \setminus (S_{m_1} \cup S_{m_2})$, which fulfils $f_{m_1}(z_n) = v(z_n) = f_{m_2}(z_n)$. We then obtain a sequence $\{z_n\}_{n \in \mathbb{N}} \subset D_{m_1} \cap D_{m_2}$ converging to z_0 such that for all $n \in \mathbb{N}$ we have $f_{m_1}(z_n) = f_{m_2}(z_n)$, and by continuity of f_{m_1} and f_{m_2} on $D_{m_1} \cap D_{m_2}$ it then follows that $f_{m_1}(z_0) = f_{m_2}(z_0)$. Then, the function

$$f : \Omega \to \mathbb{C} \, , \quad z \mapsto f_m(z) \quad \text{if } z \in D_m \text{ for } m \in \mathbb{N}$$

is well defined and holomorphic on Ω. Moreover, the set $\mathcal{S} := \bigcup_{m \in \mathbb{N}} S_m \subset \Omega$ has measure 0, and for each $z_0 \in \Omega \setminus \mathcal{S}$ we have $v(z_0) = f(z_0)$. This concludes the proof. $\square$

Theorem 4.27. (Regularity theorem). *Suppose $\Omega \subset \mathbb{C}$ is open and let $k \in \mathbb{Z}_{\geq 1}$. If $\beta \in C^k(\Omega)$ and $u \in L^1_{\text{loc}}(\Omega)$ are functions satisfying*

$$\beta = \left(\frac{\partial}{\partial \bar{z}}\right)_{\text{distr}} u \,,$$

then there exists a function $f \in C^k(\Omega)$ such that $u = f$ almost everywhere in Ω.

Proof. Choose $w \in \Omega$ and $a \in (0, +\infty)$ fulfilling $\overline{\Delta(w; a)} \subset \Omega$, and let $D := \Delta(w; a)$. By Lemma 4.15, there exists a function $g_D \in C^k(D)$ fulfilling $\partial g_D / \partial \bar{z} = \beta$ on D. Then, on D we have

$$\beta = \left(\frac{\partial}{\partial \bar{z}}\right)_{\text{distr}} u \qquad \text{and} \qquad \beta = \frac{\partial g_D}{\partial \bar{z}} = \left(\frac{\partial}{\partial \bar{z}}\right)_{\text{distr}} g_D \,,$$

from which it follows that

$$0 = \left(\frac{\partial}{\partial \bar{z}}\right)_{\text{distr}} (u - g_D) \,.$$

Then, by Lemma 4.26, there exists a function $h_D \in \mathcal{O}(D)$ such that $u - g_D = h_D$ almost everywhere in D. It follows that $u = g_D + h_D$ almost everywhere on D, where $g_D + h_D \in C^k(D)$.

As in the proof of Lemma 4.26, we may find a countable open cover $\{D_m\}_{m \in \mathbb{N}}$ for Ω such that for each $m \in \mathbb{N}$ we have $D_m = \Delta(w; a)$ for some $w \in \Omega$ and $a \in (0, +\infty)$ fulfilling $\overline{\Delta(w; a)} \subset \Omega$. Then, for each $m \in \mathbb{N}$ there exists a function $f_m \in C^k(D_m)$ such that $u = f_m$ almost everywhere on D_m. The proof that for each $m_1, m_2 \in \mathbb{N}$ we have $f_{m_1} = f_{m_2}$ on $D_{m_1} \cap D_{m_2}$ is exactly as in the proof of Lemma 4.26. Then, the function

$$f : \Omega \to \mathbb{C} \,, \quad z \mapsto f_m(z) \quad \text{if } z \in D_m \text{ for } m \in \mathbb{N}$$

is well defined and C^k on Ω, and we have $u = f$ almost everywhere in Ω. □

Theorem 4.28. (Mean value property). *Suppose $z_0 \in \mathbb{C}$, $R \in (0, +\infty)$, and $f \in \mathcal{O}(\Delta(z_0; R))$. Then, for all $r \in (0, R)$ we have*

$$\frac{1}{2\pi} \int_{(0,2\pi)} f(z_0 + re^{i\theta}) \, d\lambda(\theta) = f(z_0) = \frac{1}{\pi r^2} \int_{\Delta(z_0; r)} f \, d\lambda \,.$$

Proof. Fix $r \in (0, R)$. The first equality is given by Cauchy's Integral Formula:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{\partial \Delta(z_0; r)} \frac{f(z)}{z - z_0} \, dz \\ &= \frac{1}{2\pi i} \int_{(0,2\pi)} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \, ire^{i\theta} \, d\lambda(\theta) \\ &= \frac{1}{2\pi} \int_{(0,2\pi)} f(z_0 + re^{i\theta}) \, d\lambda(\theta) \,. \end{aligned}$$

For the second equality, we have

$$
\begin{aligned}
\frac{1}{\pi r^2}\int_{\Delta(z_0;r)} f\,d\lambda &= \frac{1}{\pi r^2}\int_{(0,r)} \rho\Big(\int_{(0,2\pi)} f(z_0+\rho e^{i\theta})\,d\lambda(\theta)\Big)\,d\lambda(\rho)\\
&= \frac{1}{\pi r^2}\int_{(0,r)} \rho\,2\pi f(z_0)\,d\lambda(\rho)\\
&= \frac{1}{\pi r^2}\,2\pi f(z_0)\,\frac{r^2}{2}\\
&= f(z_0)\,.
\end{aligned}
$$

□

Lemma 4.29. *Suppose $\Omega \subset \mathbb{C}$ is open and $z_0 \in \Omega$. If $f : \Omega \to \mathbb{C}$ is a continuous function that is holomorphic on $\Omega \setminus \{z_0\}$, then f is holomorphic on Ω.*

Proof. Define the function

$$
g : \Omega \to \mathbb{C}\,, \quad z \mapsto (z - z_0)f(z)\,.
$$

On $\Omega \setminus \{z_0\}$, g is holomorphic as a product of holomorphic functions. Moreover, if $\{z_n\}_{n\in\mathbb{N}}$ is a sequence in $\Omega \setminus \{z_0\}$ converging to z_0, then for each $n \in \mathbb{N}$

$$
\frac{g(z_n) - g(z_0)}{z_n - z_0} = f(z_n)\,,
$$

which converges to $f(z_0)$ as $n \to +\infty$. This shows that g is holomorphic on Ω.

Choose $a \in (0, +\infty)$ such that $\overline{\Delta(z_0; a)} \subset \Omega$. Fix $w \in \Delta(z_0; a) \setminus \{z_0\}$, and choose $r \in (0, |w - z_0|)$. Denote by A_r the open annulus

$$
A_r := \Delta(z_0; r, a) = \{z \in \mathbb{C} \,|\, r < |z - z_0| < a\}\,,
$$

which contains w. Since A_r is a relatively compact smooth open set in $\mathbb{C}$, and since f is holomorphic on a neighbourhood of the closure of A_r in $\mathbb{C}$, we have

$$
f(w) = \frac{1}{2\pi i}\int_{\partial A_r} \frac{f(z)}{z - w}\,dz\,,
$$

where ∂A_r has the smooth manifold structure and orientation induced from $\mathbb{C}$ with respect to A_r. Defining

$$
U := \{z_0 + re^{i\theta} \,|\, \theta \in (0, 2\pi)\} = \partial\Delta(z_0; r) \setminus \{z_0 + r\}
$$

and

$$
\phi : U \to (-2\pi, 0)\,, \quad z_0 + re^{i\theta} \mapsto -\theta\,,
$$

the pair (U, ϕ) is a positively oriented chart on ∂A_r. We may then write

$$
\begin{aligned}
f(w) &= \frac{1}{2\pi i}\int_{\partial\Delta(z_0;a)} \frac{f(z)}{z - w}\,dz + \frac{1}{2\pi i}\int_{\partial\Delta(z_0;r)} \frac{f(z)}{z - w}\,dz\\
&= \frac{1}{2\pi i}\int_{\partial\Delta(z_0;a)} \frac{f(z)}{z - w}\,dz + \frac{1}{2\pi i}\int_{(-2\pi,0)} \frac{f(z_0 + re^{-i\theta})}{z_0 + re^{-i\theta} - w}(-i)re^{-i\theta}\,d\lambda(\theta)\\
&= \frac{1}{2\pi i}\int_{\partial\Delta(z_0;a)} \frac{f(z)}{z - w}\,dz - \frac{1}{2\pi i}\int_{(0,2\pi)} \frac{f(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - w}\,ire^{i\theta}\,d\lambda(\theta)\,.
\end{aligned}
$$

We wish to show that

$$\frac{1}{2\pi i}\int_{(0,2\pi)} \frac{f(z_0+re^{i\theta})}{z_0+re^{i\theta}-w}\, ire^{i\theta}\, d\lambda(\theta)=0\,.$$

For this, observe that on $\Omega\setminus\{w\}$, which is a neighbourhood of $\overline{\Delta(z_0;r)}$, the function $z\mapsto \frac{g(z)}{z-w}$ is holomorphic. Then, Cauchy's Integral Formula gives

$$\frac{g(z_0)}{z_0-w}=\frac{1}{2\pi i}\int_{\partial\Delta(z_0;r)} \frac{g(z)}{(z-w)(z-z_0)}\, dz\,,$$

where the boundary $\partial\Delta(z_0;r)$ now has the counterclockwise orientation. Since $g(z_0)=0$, we then have

$$\begin{aligned}0&=\frac{1}{2\pi i}\int_{\partial\Delta(z_0;r)} \frac{g(z)}{(z-w)(z-z_0)}\, dz\\&=\frac{1}{2\pi i}\int_{\partial\Delta(z_0;r)} \frac{f(z)}{z-w}\, dz\\&=\frac{1}{2\pi i}\int_{(0,2\pi)} \frac{f(z_0+re^{i\theta})}{z_0+re^{i\theta}-w}\, ire^{i\theta}\, d\lambda(\theta)\,.\end{aligned}$$

It then follows that

$$f(w)=\frac{1}{2\pi i}\int_{\partial\Delta(z_0;a)} \frac{f(z)}{z-w}\, dz\,,$$

where $\partial\Delta(z_0;a)$ has the counterclockwise orientation and w was an arbitrary point in $\Delta(z_0;a)\setminus\{z_0\}$. Since f is continuous, the function

$$h:\Delta(z_0;a)\to\mathbb{C}\,,\quad \zeta\mapsto\frac{1}{2\pi i}\int_{\partial\Delta(z_0;a)} \frac{f(z)}{z-\zeta}\, dz$$

is holomorphic, and by the above reasoning, we have $f=h$ on $\Delta(z_0;a)\setminus\{z_0\}$. By continuity of f and h, we must then have $f=h$ on $\Delta(z_0;a)$, from which it follows that f is holomorphic on $\Delta(z_0;a)$ and hence on Ω. □

Theorem 4.30. (Riemann's extension theorem) *Suppose $\Omega\subset\mathbb{C}$ is open and $z_0\in\Omega$, and suppose $f:\Omega\to\mathbb{C}$ is a function that is holomorphic on $\Omega\setminus\{z_0\}$. If*

(i) $$\lim_{z\to z_0}(z-z_0)f(z)=0\,,$$

or if

(ii) $$f\in L^p_{\mathrm{loc}}(\Omega)\quad \textit{for some } p\in[2,+\infty],$$

then there exists a (unique) function $\hat{f}\in\mathcal{O}(\Omega)$ such that $\hat{f}=f$ on $\Omega\setminus\{z_0\}$.

Proof. Choose $a\in(0,+\infty)$ such that $\overline{\Delta(z_0;a)}\subset\Omega$, and fix $w\in\Delta(z_0;a)\setminus\{z_0\}$. As in the proof of Lemma 4.29, it follows from the fact that f is holomorphic on $\Omega\setminus\{z_0\}$ that for any $r\in(0,|w-z_0|)$ we have

$$f(w)=\frac{1}{2\pi i}\int_{\partial\Delta(z_0;a)} \frac{f(z)}{z-w}\, dz-\frac{1}{2\pi i}\int_{(0,2\pi)} \frac{f(z_0+re^{i\theta})}{z_0+re^{i\theta}-w}\, ire^{i\theta}\, d\lambda(\theta)\,,$$

where $\partial\Delta(z_0; a)$ has the counterclockwise orientation. In particular, this means that for all $r \in (0,|w - z_0|)$ we have

$$\frac{1}{2\pi i} \int_{(0,2\pi)} \frac{f(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - w} \, ire^{i\theta} \, d\lambda(\theta) = \alpha \,,$$

where

$$\alpha := \frac{1}{2\pi i} \int_{\partial\Delta(z_0;a)} \frac{f(z)}{z - w} \, dz - f(w)$$

is independent of r. We wish to show that whenever (i) or (ii) hold, we have $\alpha = 0$.

Suppose first that (i) holds, and fix $r \in (0,|w - z_0|)$. The function

$$g : \Omega \to \mathbb{C} \,, \quad z \mapsto (z - z_0) f(z)$$

is continuous on Ω and holomorphic on $\Omega \setminus \{z_0\}$, and hence by Lemma 4.29 we have $g \in \mathcal{O}(\Omega)$. Moreover, also as in the proof of Lemma 4.29, since the function $z \mapsto \frac{g(z)}{z-w}$ on $\Omega \setminus \{w\} \supset \overline{\Delta(z_0; r)}$ is holomorphic, we have

$$0 = \frac{g(z_0)}{z_0 - w} = \frac{1}{2\pi i} \int_{(0,2\pi)} \frac{f(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - w} \, ire^{i\theta} \, d\lambda(\theta) = \alpha \,.$$

Suppose now that (ii) holds. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,|w - z_0|)$ converging to 0, and fix $n \in \mathbb{N}$. Defining

$$q := \begin{cases} \dfrac{p}{p-1} \in (1, 2] & \text{if } p \in [2, +\infty) \\ 1 & \text{if } p = +\infty \end{cases} \,,$$

we have $\frac{1}{p} + \frac{1}{q} = 1$. Since the function

$$h : \Omega \setminus \{w\} \to \mathbb{C} \,, \quad z \mapsto \frac{z - z_0}{z - w}$$

is continuous, it is in $L^q(\Delta(z_0; r_n))$. Since $f \in L^p(\Delta(z_0; r_n))$, by Hölder's inequality we then have $fh \in L^1(\Delta(z_0; r_n))$ and

$$\left| \int_{\Delta(z_0;r_n)} fh \, d\lambda \right| \leq \int_{\Delta(z_0;r_n)} |fh| \, d\lambda \leq \|f\|_{L^p(\Delta(z_0;r_n))} \|h\|_{L^q(\Delta(z_0;r_n))} \,.$$

Note that

$$\begin{aligned} \int_{\Delta(z_0;r_n)} fh \, d\lambda &= \int_{\Delta(z_0;r_n)} \frac{f(z)(z - z_0)}{z - w} \, d\lambda(z) \\ &= \int_{(0,r_n)} \rho \left(\int_{(0,2\pi)} \frac{f(z_0 + \rho e^{i\theta}) \, \rho e^{i\theta}}{z_0 + \rho e^{i\theta} - w} \, d\lambda(\theta) \right) d\lambda(\rho) \\ &= \int_{(0,r_n)} \rho \, 2\pi\alpha \, d\lambda(\rho) \\ &= \pi r_n^2 \alpha \,. \end{aligned}$$

Moreover, for each $z \in \Delta(z_0; r_n)$ we have

$$|z - w| \geq |w - z_0| - |z - z_0| \geq |w - z_0| - r_n > 0 \,,$$

so that

$$\left| h(z) \right| = \left| \frac{z - z_0}{z - w} \right| \leq \frac{r_n}{|w - z_0| - r_n} \,.$$

Then,

$$
\begin{aligned}
\|h\|_{L^q(\Delta(z_0;r_n))} &= \left(\int_{\Delta(z_0;r_n)} |h|^q \, d\lambda\right)^{\frac{1}{q}} \\
&\le \left(\int_{\Delta(z_0;r_n)} \left|\frac{r_n}{|w-z_0|-r_n}\right|^q \, d\lambda\right)^{\frac{1}{q}} \\
&= \left(\pi (r_n)^2 \left|\frac{r_n}{|w-z_0|-r_n}\right|^q\right)^{\frac{1}{q}} \\
&= \pi^{\frac{1}{q}} (r_n)^{\frac{2}{q}} \frac{r_n}{|w-z_0|-r_n} \, .
\end{aligned}
$$

Then, we have

$$
\pi r_n^2 |\alpha| = \left|\int_{\Delta(z_0;r_n)} f h \, d\lambda\right| \le \|f\|_{L^p(\Delta(z_0;r_n))} \, \pi^{\frac{1}{q}} (r_n)^{\frac{2}{q}} \frac{r_n}{|w-z_0|-r_n} \, ,
$$

so that

$$
|\alpha| \le \|f\|_{L^p(\Delta(z_0;r_n))} \, \pi^{\frac{1}{q}-1} \frac{(r_n)^{\frac{2}{q}-1}}{|w-z_0|-r_n} \, .
$$

Suppose first that $p \in [2,+\infty)$. As $n \to \infty$, we then have $\frac{(r_n)^{\frac{2}{q}-1}}{|w-z_0|-r_n} \to 0$ if $q \in (1,2)$, and $\frac{(r_n)^{\frac{2}{q}-1}}{|w-z_0|-r_n} \to \frac{1}{|w-z_0|}$ if $q = 2$. Thus, since $\|f\|_{L^p(\Delta(z_0;r_n))} \to 0$, we must have $|\alpha| = 0$. If $p = +\infty$, the sequence $\{ \|f\|_{L^p(\Delta(z_0;r_n))} \}_{n \in \mathbb{N}}$ is bounded and $\frac{(r_n)^{\frac{2}{q}-1}}{|w-z_0|-r_n} = \frac{r_n}{|w-z_0|-r_n} \to 0$ as $n \to +\infty$, which also implies that $|\alpha| = 0$ (alternatively, we could use the fact that $L^\infty_{\mathrm{loc}}(\Omega) \subset L^s_{\mathrm{loc}}(\Omega)$ for all $s \in [1,+\infty]$, so that we can always assume $p \in [2,+\infty)$). This shows that we also have $\alpha = 0$ when (ii) holds.

It follows that whenever either (i) or (ii) hold, we have

$$
f(w) = \frac{1}{2\pi i} \int_{\partial\Delta(z_0;a)} \frac{f(z)}{z-w} \, dz
$$

for all $w \in \Delta(z_0;a) \setminus \{z_0\}$. Since f is continuous on $\Omega \setminus \{z_0\}$, and hence on $\partial\Delta(z_0;a)$, the function

$$
\Delta(z_0;a) \to \mathbb{C} \, , \quad \zeta \mapsto \frac{1}{2\pi i} \int_{\partial\Delta(z_0;a)} \frac{f(z)}{z-\zeta} \, dz
$$

is holomorphic. Then, the function

$$
\hat{f} : \Omega \to \mathbb{C} \, , \quad \zeta \mapsto \begin{cases} f(\zeta) & \text{if } \zeta \in \Omega \setminus \{z_0\}, \\ \dfrac{1}{2\pi i} \displaystyle\int_{\partial\Delta(z_0;a)} \frac{f(z)}{z-\zeta} \, dz & \text{if } \zeta = z_0 \end{cases}
$$

is holomorphic and fulfils $\hat{f} = f$ on $\Omega \setminus \{z_0\}$. □

4.5. **Power Series Representation and Global Solution to the Inhomogeneous Cauchy-Riemann Equation.**

Definition 4.31. (Complex power series) Let $z_0 \in \mathbb{C}$. A *(complex) power series centered* at z_0 is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n\,,$$

where $\{a_n\}_{n\in\mathbb{N}_0}$ is a collection of complex numbers. We say that the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges* at some $w \in \mathbb{C}$ if the series

$$\left\{\sum_{n=0}^{N} a_n(w-z_0)^n\right\}_{N\in\mathbb{N}_0} \subset \mathbb{C}$$

converges in $\mathbb{C}$ to some limit $\sum_{n=0}^{\infty} a_n(w-z_0)^n$. We say that a power series converges on a set $A \subset \mathbb{C}$ if it converges at all points in A. If the power series does not converge at some point $w \in \mathbb{C}$, we say it *diverges* at w.

Theorem 4.32. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series centered at $z_0 \in \mathbb{C}$, and let*

$$S := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \in [0,+\infty]\,, \quad R := \frac{1}{S}\,,$$

where $\frac{1}{S} := +\infty$ if $S = 0$, and $\frac{1}{S} := 0$ if $S = +\infty$. Then,

(i) *if $R = +\infty$, the power series converges absolutely on $\mathbb{C}$;*
(ii) *if $R \in (0,+\infty)$, the power series converges absolutely on the open disc $\{z \in \mathbb{C} \,|\, |z-z_0| < R\} = \Delta(z_0;R)$, and diverges on the set $\{z \in \mathbb{C} \,|\, |z-z_0| > R\}$;*
(iii) *if $R = 0$, then the power series converges only at z_0.*

Proof. (i) Suppose $R = +\infty$, and let $z \in \mathbb{C} \setminus \{z_0\}$. Since $S = 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{n\geq N} |a_n|^{\frac{1}{n}} < \frac{1}{2|z-z_0|}\,.$$

Then, for every $n \in \mathbb{N}_{\geq N}$ we have

$$|a_n|^{\frac{1}{n}}|z-z_0| < \frac{1}{2}\,,$$

which implies that

$$|a_n||z-z_0|^n < \frac{1}{2^n}\,.$$

Thus, for every $n \in \mathbb{N}_{\geq N}$ we have

$$\begin{aligned}
\sum_{k=0}^{n} |a_k||z-z_0|^k &= \sum_{k=0}^{N-1} |a_k||z-z_0|^k + \sum_{k=N}^{n} |a_k||z-z_0|^k \\
&\leq \sum_{k=0}^{N-1} |a_k||z-z_0|^k + \sum_{k=N}^{n} \frac{1}{2^k} \\
&\leq \sum_{k=0}^{N-1} |a_k||z-z_0|^k + 1\,.
\end{aligned}$$

It follows that the increasing sequence

$$\left\{\sum_{k=0}^{n} |a_k||z-z_0|^k\right\}_{n\in\mathbb{N}} \subset \mathbb{R}$$

is bounded and hence it converges in $\mathbb{R}$.

(ii) Since $R \in (0,+\infty)$, we must also have $S \in (0,+\infty)$. Suppose $z \in \Delta\big(z_0; \frac{1}{S}\big)$. Letting

$$a := \frac{\frac{1}{S} - |z-z_0|}{2} > 0\,,$$

we have $|z-z_0| < \frac{1}{S} - a$. Since $S = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, there exists $N \in \mathbb{N}$ such that

$$\sup_{n\geq N} |a_n|^{\frac{1}{n}} < S + aS^2\,.$$

Then, for all $n \in \mathbb{N}_{\geq N}$ we have

$$|a_n|^{\frac{1}{n}} |z-z_0| < \left(\frac{1}{S} - a\right)(S + aS^2) = 1 - a^2S^2 < 1\,,$$

so that

$$|a_n||z-z_0|^n < (1-a^2S^2)^n\,.$$

It follows that for all $n \in \mathbb{N}_{\geq N}$ we have

$$\begin{aligned}\sum_{k=0}^{n} |a_k||z-z_0|^k &= \sum_{k=0}^{N-1} |a_k||z-z_0|^k + \sum_{k=N}^{n} |a_k||z-z_0|^k \\ &< \sum_{k=0}^{N-1} |a_k||z-z_0|^k + \sum_{k=N}^{n} (1-a^2S^2)^k \\ &< \sum_{k=0}^{N-1} |a_k||z-z_0|^k + \sum_{k=N}^{\infty} (1-a^2S^2)^k \in \mathbb{R}\,.\end{aligned}$$

Thus, the increasing sequence

$$\left\{ \sum_{k=0}^{n} |a_k||z-z_0|^k \right\}_{n\in\mathbb{N}} \subset \mathbb{R}$$

is bounded and hence it converges in $\mathbb{R}$.

We now show that the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges at every point of $A := \{z \in \mathbb{C} \,|\, |z-z_0| > \frac{1}{S}\}$. Let $z \in A$, and define

$$b := \frac{|z-z_0| - \frac{1}{S}}{2} > 0\,.$$

We then have $|z-z_0| > \frac{1}{S} + b$. Since for every $n \in \mathbb{N}$ we have

$$\sup_{m\geq n} |a_m|^{\frac{1}{m}} \geq S > S - \frac{bS^2}{1+bS}\,,$$

we may find a subsequence $\left\{ \left|a_{n_k}\right|^{\frac{1}{n_k}} \right\}_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we have

$$\left|a_{n_k}\right|^{\frac{1}{n_k}} > S - \frac{bS^2}{1+bS}\,.$$

Then, for every $k \in \mathbb{N}$,

$$\left|a_{n_k}\right|^{\frac{1}{n_k}} |z-z_0| > \left(S - \frac{bS^2}{1+bS}\right)\left(\frac{1}{S} + b\right) = 1\,,$$

so that

$$\left|a_{n_k}\right| |z-z_0|^{n_k} > 1\,.$$

It follows that the sequence $\{a_n(z-z_0)^n\}_{n\in\mathbb{N}}$ does not converge to 0, which implies that the series $\sum_{n=0}^\infty a_n(z-z_0)^n$ cannot converge in $\mathbb{C}$.

(iii) Since $R=0$, we have $S=+\infty$, which means that the sequence $\left\{|a_n|^{\frac{1}{n}}\right\}_{n\in\mathbb{N}}$ is unbounded. Then, for any $z\in\mathbb{C}\setminus\{z_0\}$ we may find a subsequence $\left\{\left|a_{n_k}\right|^{\frac{1}{n_k}}\right\}_{k\in\mathbb{N}}$ such that for all $k\in\mathbb{N}$ we have $\left|a_{n_k}\right|^{\frac{1}{n_k}} > \frac{1}{|z-z_0|}$. Then, for all $k\in\mathbb{N}$,

$$\left|a_{n_k}\right|^{\frac{1}{n_k}} |z-z_0| > 1\,,$$

so that

$$\left|a_{n_k}\right| |z-z_0|^{n_k} > 1\,.$$

As before, this implies the sequence $\{a_n(z-z_0)^n\}_{n\in\mathbb{N}}$ does not converge to 0, so the series $\sum_{n=0}^\infty a_n(z-z_0)^n$ cannot converge in $\mathbb{C}$.

□

The number R in Theorem 4.32 is referred to as the *radius of convergence* of the power series. We call the set $\Delta(z_0;R)$ the *(open) disc of convergence* of the power series, where we let $\Delta(z_0;0):=\emptyset$. Note that the open disc of convergence is the largest open set on which the power series converges.

Theorem 4.33. *A complex power series $\sum_{n=0}^\infty a_n(z-z_0)^n$ with radius of convergence $R\in(0,+\infty]$ converges uniformly to its limit function*

$$f:\Delta(z_0;R)\to\mathbb{C}\,,\quad z\mapsto\sum_{n=0}^\infty a_n(z-z_0)^n$$

on compact subsets of $\Delta(z_0;R)$. Moreover, f is holomorphic on $\Delta(z_0;R)$.

Proof. For each $n\in\mathbb{N}_0$, define the function

$$g_n:\Delta(z_0;R)\to\mathbb{C}\,,\quad z\mapsto a_n(z-z_0)^n\,.$$

Then, we have $\sum_{n=0}^N g_n\to f$ pointwise on $\Delta(z_0;R)$ as $\mathbb{N}_0\ni N\to\infty$. Let $K\subset\Delta(z_0;R)$ be compact. Then, there exists $r\in(0,R)$ such that $K\subset\Delta(z_0;r)\subset\Delta(z_0;R)$. Since $z_0+r\in\Delta(z_0;R)$, the power series converges absolutely at z_0+r, that is, the series $\sum_{n=0}^\infty|a_n|\,r^n$ converges. For each $n\in\mathbb{N}_0$, and for every $w\in K$, we have

$$\left|g_n(w)\right| = |a_n||w-z_0|^n \le |a_n|\,r^n =: M_n\,.$$

Thus, since $\sum_{n=0}^\infty M_n$ converges, by the Weierstrass M-test it follows that $\sum_{n=0}^\infty g_n$ converges uniformly on K to some function $h:K\to\mathbb{C}$. We must then have $h=f|_K$, which shows that the power series converges uniformly to f on compact subsets of $\Delta(z_0;R)$. Then, by Corollary 4.23, we have $f\in\mathcal{O}(\Delta(z_0;R))$. □

Theorem 4.34. *Let $z_0\in\mathbb{C}$, and let $\sum_{n=0}^\infty a_n(z-z_0)^n$ be a complex power series with radius of convergence $R\in[0,+\infty]$.*

(i) *The power series*

$$\sum_{n=1}^\infty na_n(z-z_0)^{n-1}=\sum_{n=0}^\infty(n+1)a_{n+1}(z-z_0)^n$$

also has radius of convergence R. Moreover, if $R \in (0, +\infty]$, and letting

$$f : \Delta(z_0; R) \to \mathbb{C}\,, \quad z \mapsto \sum_{n=0}^{\infty} a_n (z - z_0)^n\,,$$

we have

$$\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} = f'(z)$$

for all $z \in \Delta(z_0; R)$.

(ii) *The power series*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} (z - z_0)^n$$

also has radius of convergence R, and if $R \in (0, +\infty]$, then defining

$$g : \Delta(z_0; R) \to \mathbb{C}\,, \quad z \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}\,,$$

we have

$$g'(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in \Delta(z_0; R)$.

Proof. (i) Let S be the radius of convergence of the power series

$$\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z - z_0)^n\,.$$

We first show that $S \leq R$. This is automatically true if $S = 0$. Suppose then that $S \in (0, +\infty]$, and let $w \in \Delta(z_0; S)$. Then, the (increasing) sequence

$$\left\{ \sum_{n=1}^{N} n |a_n| |w - z_0|^{n-1} \right\}_{N \in \mathbb{N}} \subset \mathbb{R}$$

converges to some $L \in [0, +\infty)$ as $N \to +\infty$. For each $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{n=0}^{N} |a_n| |w - z_0|^n &= |a_0| + |w - z_0| \sum_{n=1}^{N} |a_n| |w - z_0|^{n-1} \\
&\leq |a_0| + |w - z_0| \sum_{n=1}^{N} n |a_n| |w - z_0|^{n-1} \\
&\leq |a_0| + |w - z_0|\, L\,.
\end{aligned}$$

Thus, the increasing sequence

$$\left\{ \sum_{n=0}^{N} |a_n| |w - z_0|^n \right\}_{N \in \mathbb{N}} \subset \mathbb{R}$$

is bounded and hence it converges. It follows that the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at w. Thus, the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at all points of $\Delta(z_0; S)$, which implies that $R > 0$ and $\Delta(z_0; S) \subset \overline{\Delta(z_0; R)}$. Thus, we must have $S \leq R$. We now show that $R \leq S$. As before, this holds if $R = 0$. Suppose $R \in (0, +\infty]$, and define the functions

$$f : \Delta(z_0; R) \to \mathbb{C}\,, \quad z \mapsto \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and

$$f_N : \Delta(z_0; R) \to \mathbb{C}\,, \quad z \mapsto \sum_{n=0}^{N} a_n (z - z_0)^n$$

for each $N \in \mathbb{N}_0$. By Theorem 4.33, the sequence $\{f_N\}_{N \in \mathbb{N}_0} \subset \mathcal{O}(\Delta(z_0; R))$ converges to f uniformly on compact subsets of $\Delta(z_0; R)$. Then, by Corollary 4.23, the sequence $\{f'_N\}_{N \in \mathbb{N}_0}$ of complex derivatives converges uniformly on compact subsets of $\Delta(z_0; R)$ to f'. Since $f'_0 = 0$ and for each $N \in \mathbb{N}$ we have

$$f'_N = \left(z \mapsto \sum_{n=1}^{N} n a_n (z - z_0)^{n-1} = \sum_{n=0}^{N-1} (n+1) a_{n+1} (z - z_0)^n \right),$$

it follows that the power series $\sum_{n=0}^{\infty} (n+1) a_{n+1} (z - z_0)^n$ converges on $\Delta(z_0; R)$ to f'. We must then have $R \leq S$, which concludes the proof that $S = R$. Moreover, if $R \in (0, +\infty]$, then we have

$$\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z - z_0)^n = f'(z)$$

for each $z \in \Delta(z_0; R)$.

(ii) Let T be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} (z - z_0)^n\,,$$

and for each $n \in \mathbb{N}$, let

$$b_n := \frac{a_{n-1}}{n}\,.$$

By (i), the power series

$$\sum_{n=1}^{\infty} n b_n (z - z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) b_{n+1} (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

also has radius of convergence T, so we must have $T = R$. Suppose $R \in (0, +\infty]$, and let

$$g : \Delta(z_0; R) \to \mathbb{C}\,, \quad z \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} = \sum_{n=1}^{\infty} b_n (z - z_0)^n\,.$$

Then, also by (i), for all $z \in \Delta(z_0; R)$ we have

$$g'(z) = \sum_{n=1}^{\infty} n b_n (z - z_0)^{n-1} = \sum_{n=0}^{\infty} a_n (z - z_0)^n\,.$$

□

Definition 4.35. A function $f : \Omega \to \mathbb{C}$ is said to be *(complex) analytic* at a point $z_0 \in \Omega$ if there exist a neighbourhood U of z_0 in Ω and a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converging on U whose sum is equal to f on U. We say that f is *analytic* on Ω if it is analytic at all points of Ω.

We know from Theorem 4.33 that a complex analytic function $f : \Omega \to \mathbb{C}$ on an open subset $\Omega \subset \mathbb{C}$ is holomorphic on Ω. The next theorem shows that, conversely, a holomorphic function on an open subset $\Omega \subset \mathbb{C}$ is analytic on Ω.

Theorem 4.36. *Let $z_0 \in \mathbb{C}$ and $R \in (0, +\infty]$, and suppose $f : \Delta(z_0; R) \to \mathbb{C}$ is a holomorphic function. Then, the power series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

converges to f on $\Delta(z_0; R)$.

Proof. Fix $z \in \Delta(z_0; R)$, and choose $r_z \in (|z-z_0|, R)$. For each $n \in \mathbb{N}_0$, define

$$a_n(z) := \frac{1}{2\pi i}\int_{\partial\Delta(z_0;r_z)} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta = \frac{1}{2\pi}\int_{(0,2\pi)} \frac{f(z_0+r_z e^{i\theta})}{(r_z e^{i\theta})^n}\,d\lambda(\theta)$$

and

$$g_n : (0,2\pi) \to \mathbb{C}\,, \quad \theta \mapsto \frac{1}{2\pi}\frac{f(z_0+r_z e^{i\theta})}{(r_z e^{i\theta})^n}(z-z_0)^n$$

(where, throughout this proof, we let $0^0 := 1$). Define also

$$h : (0,2\pi) \to \mathbb{C}\,, \quad \theta \mapsto \frac{1}{2\pi} f(z_0+r_z e^{i\theta})\frac{r_z e^{i\theta}}{z_0+r_z e^{i\theta}-z}\,.$$

For each $\theta \in (0, 2\pi)$, we have

$$\left|\frac{z-z_0}{r_z e^{i\theta}}\right| = \frac{|z-z_0|}{r_z} \in [0,1)\,,$$

so for $N \in \mathbb{N}_0$,

$$\begin{aligned}\sum_{n=0}^{N} g_n(\theta) &= \frac{f(z_0+r_z e^{i\theta})}{2\pi}\sum_{n=0}^{N}\left(\frac{z-z_0}{r_z e^{i\theta}}\right)^n \\ &\to \frac{f(z_0+r_z e^{i\theta})}{2\pi}\cdot\frac{1}{1-\frac{z-z_0}{r_z e^{i\theta}}} = \frac{f(z_0+r_z e^{i\theta})}{2\pi}\cdot\frac{r_z e^{i\theta}}{z_0+r_z e^{i\theta}-z} = h(\theta)\end{aligned}$$

as $N \to \infty$. Moreover, letting $P \in (0, +\infty)$ be an upper bound for $|f|$ on $\partial\Delta(z_0; r_z)$, for each $n \in \mathbb{N}_0$ we have

$$|g_n| \le \frac{P}{2\pi}\left(\frac{|z-z_0|}{r_z}\right)^n =: M_n$$

on $(0, 2\pi)$. Then, by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} g_n$ converges uniformly to h on $(0, 2\pi)$. It then follows that

$$\int_{(0,2\pi)} \sum_{n=0}^{N} g_n\,d\lambda \to \int_{(0,2\pi)} h\,d\lambda$$

as $N \to +\infty$. For each $N \in \mathbb{N}_0$, we have

$$\begin{aligned}\int_{(0,2\pi)} \sum_{n=0}^{N} g_n \, d\lambda &= \sum_{n=0}^{N} \int_{(0,2\pi)} \frac{1}{2\pi} \frac{f(z_0 + r_z e^{i\theta})}{(r_z e^{i\theta})^n} (z - z_0)^n \, d\lambda(\theta) \\ &= \sum_{n=0}^{N} \left(\frac{1}{2\pi} \int_{(0,2\pi)} \frac{f(z_0 + r_z e^{i\theta})}{(r_z e^{i\theta})^n} \, d\lambda(\theta) \right) (z - z_0)^n \\ &= \sum_{n=0}^{N} a_n(z)(z - z_0)^n \,,\end{aligned}$$

and

$$\begin{aligned}\int_{(0,2\pi)} h \, d\lambda &= \int_{(0,2\pi)} \frac{1}{2\pi} f(z_0 + r_z e^{i\theta}) \frac{r_z e^{i\theta}}{z_0 + r_z e^{i\theta} - z} \, d\lambda(\theta) \\ &= \frac{1}{2\pi i} \int_{(0,2\pi)} f(z_0 + r_z e^{i\theta}) \frac{i r_z e^{i\theta}}{z_0 + r_z e^{i\theta} - z} \, d\lambda(\theta) \\ &= \frac{1}{2\pi i} \int_{\partial\Delta(z_0;r_z)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \\ &= f(z) \,.\end{aligned}$$

Thus, we have

$$\sum_{n=0}^{\infty} a_n(z)(z - z_0)^n = f(z) \,.$$

We now show that for each $n \in \mathbb{N}_0$ and for all $z \in \Delta(z_0; R)$, we have

$$a_n(z) = \frac{f^{(n)}(z_0)}{n!} \,.$$

Fix $n \in \mathbb{N}_0$, and let $z, w \in \Delta(z_0; R)$. We may assume without loss of generality that $r_z \leq r_w$. If $r_z = r_w$, then

$$a_n(z) = \frac{1}{2\pi i} \int_{\partial\Delta(z_0;r_z)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta = \frac{1}{2\pi i} \int_{\partial\Delta(z_0;r_w)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta = a_n(w) \,.$$

Suppose $r_z < r_w$. Since the function $\zeta \mapsto \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$ on $\Delta(z_0; R) \setminus \{z_0\}$ is holomorphic and $\overline{\Delta(z_0; r_z, r_w)} \subset \Delta(z_0; R) \setminus \{z_0\}$, Cauchy's Theorem gives

$$\begin{aligned}0 &= \int_{\partial\Delta(z_0;r_z,r_w)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \\ &= \int_{\partial\Delta(z_0;r_w)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta - \int_{\partial\Delta(z_0;r_z)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \\ &= 2\pi i \big(a_n(w) - a_n(z) \big) \,,\end{aligned}$$

where $\partial\Delta(z_0; r_w)$ and $\partial\Delta(z_0; r_z)$ have the counterclockwise orientation. Thus, in this case we also have $a_n(z) = a_n(w)$. It follows that $a_n(z) = a_n(w)$ for all $z, w \in \Delta(z_0; R)$, so that we may then define $a_n := a_n(z)$ for any $z \in \Delta(z_0; R)$. We then have

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = f(z)$$

for every $z \in \Delta(z_0; R)$, that is, the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to f on $\Delta(z_0; R)$. Then, applying induction on Theorem 4.34 (i), for all $m \in \mathbb{N}_0$ the power series

$$\sum_{n=0}^{\infty} \frac{(n+m)!}{n!} a_{n+m}(z-z_0)^n$$

converges on $\Delta(z_0; R)$ to $f^{(m)}$. It follows that for each $m \in \mathbb{N}_0$ we have

$$f^{(m)}(z_0) = m!\, a_m \,,$$

or

$$a_m = \frac{f^{(m)}(z_0)}{m!} \,,$$

which concludes the proof. □

Remark 4.37. It follows from Theorems 4.36 and 4.33 that if $f \in \mathcal{O}(\Delta(z_0; R))$ for $z_0 \in \mathbb{C}$ and $R \in (0, +\infty]$, then there exists a sequence $\{g_n\}_{n \in \mathbb{N}_0}$ of holomorphic functions on $\mathbb{C}$ that converges to f uniformly on compact subsets of $\Delta(z_0; R)$, namely

$$g_n : \mathbb{C} \to \mathbb{C}\,, \quad z \mapsto \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

for each $n \in \mathbb{N}_0$.

We are now ready to prove the existence of a global solution to the inhomogeneous Cauchy-Riemann equation on an open disc of radius $R \in (0, +\infty]$. An alternative proof can be found in [1]. We first prove the particular case of compact support:

Lemma 4.38. *Let $z_0 \in \mathbb{C}$ and $R \in (0, +\infty]$. Suppose $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ and $f : \Delta(z_0; R) \to \mathbb{C}$ is a C^k function with compact support. Then, the function*

$$g : \Delta(z_0; R) \to \mathbb{C}\,, \quad z \mapsto \frac{1}{2\pi i} \int_{\Delta(z_0;R)\setminus\{z\}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

is also C^k and fulfils

$$\frac{\partial g}{\partial \bar{z}} = f$$

on $\Delta(z_0; R)$.

Proof. If $R \in (0, +\infty)$, we may extend f to a C^k function on a neighbourhood of $\overline{\Delta(z_0; R)}$, by defining it to be 0 outside of $\Delta(z_0; R)$. The result then follows directly from Lemma 4.15. Suppose $R = +\infty$, and let $w \in \mathbb{C}$. Since $\operatorname{supp} f \cup \{w\}$ is compact, we may choose $r \in (0, +\infty)$ such that $\operatorname{supp} f \cup \{w\} \subset \Delta(z_0; r)$. For each $z \in \Delta(z_0; r)$, we have

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \int_{\mathbb{C}\setminus\{z\}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} \int_{\Delta(z_0;r)\setminus\{z\}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}\,. \end{aligned}$$

Since f is C^k on a neighbourhood of $\overline{\Delta(z_0; r)}$, it follows from Lemma 4.15 that the restriction $g|_{\Delta(z_0;r)}$ is C^k and fulfils

$$\frac{\partial \left(g|_{\Delta(z_0;r)} \right)}{\partial \bar{z}} = f$$

on $\Delta(z_0; r)$. Thus, every point $w \in \mathbb{C}$ has a neighbourhood where g is C^k and satisfies $\partial g/\partial \bar{z} = f$. The desired result then follows. $\square$

Theorem 4.39. (Global solution of the inhomogeneous Cauchy-Riemann equation). *Let $z_0 \in \mathbb{C}$ and $R \in (0, +\infty]$. For $k \in \mathbb{Z}_{\geq 1}$, suppose $f : \Delta(z_0; R) \to \mathbb{C}$ is a C^k function. Then, there exists another C^k function $g : \Delta(z_0; R) \to \mathbb{C}$ fulfilling*

$$\frac{\partial g}{\partial \bar{z}} = f$$

on $\Delta(z_0; R)$.

Proof. If $R \in (0, +\infty)$, choose $N \in \mathbb{N}$ with $\frac{1}{N} < R$. For each $n \in \mathbb{N}$, define

$$A_n := \begin{cases} \Delta\left(z_0; R - \dfrac{1}{N+n}\right) & \text{if } R \in (0, +\infty), \\ \Delta(z_0; n) & \text{if } R = +\infty\,. \end{cases}$$

Also, for each $n \in \mathbb{N}$ define $\rho_n : \Delta(z_0; R) \to \mathbb{C}$ to be a smooth bump function that is equal to 1 on A_{n+1} and has support in A_{n+2}, and let $F_n : \Delta(z_0; R) \to \mathbb{C}$ be a C^k function fulfilling $\partial F_n/\partial \bar{z} = \rho_n f$ on $\Delta(z_0; R)$ (Lemma 4.38). Let $G_1 := F_1$. On A_2, we have

$$\frac{\partial(F_2 - G_1)}{\partial \bar{z}} = \rho_2 f - \rho_1 f = f - f = 0\,,$$

so $F_2 - G_1 \in \mathcal{O}(A_2)$. Then, by Remark 4.37, there exists a function $h_1 \in \mathcal{O}(\mathbb{C})$ such that $\left|(F_2 - G_1) - h_1\right| < \frac{1}{2}$ on $\overline{A_1}$, which is a compact subset of A_2. Restrict h_1 to $\Delta(z_0; R)$, and define $G_2 := F_2 - h_1$. Then, G_2 is C^k on $\Delta(z_0; R)$ and $\partial G_2/\partial \bar{z} = \rho_2 f$. We repeat the process inductively: for each $n \in \mathbb{N}_{\geq 2}$, starting from $n = 2$, we have $G_n \in C^k(\Delta(z_0; R))$ and $\partial G_n/\partial \bar{z} = \rho_n f$, so on A_{n+1},

$$\frac{\partial(F_{n+1} - G_n)}{\partial \bar{z}} = \rho_{n+1} f - \rho_n f = f - f = 0\,.$$

Thus, since $F_{n+1} - G_n \in \mathcal{O}(A_{n+1})$, we may find a function $h_n \in \mathcal{O}(\mathbb{C})$ such that $\left|(F_{n+1} - G_n) - h_n\right| < \frac{1}{2^n}$ on $\overline{A_n}$. Restrict h_n to $\Delta(z_0; R)$ and define $G_{n+1} := F_{n+1} - h_n$. We then have again $G_{n+1} \in C^k(\Delta(z_0; R))$ and $\partial G_{n+1}/\partial \bar{z} = \rho_{n+1} f$.

Now, for all $m \in \mathbb{N}_{\geq 2}$, on A_1 we have $\partial(G_m - G_1)/\partial \bar{z} = 0$ and

$$|G_m - G_1| \leq \sum_{j=0}^{m-2} \left|G_{m-j} - G_{m-j-1}\right| \leq \sum_{j=0}^{m-2} \frac{1}{2^{m-j-1}} < 1\,.$$

Thus, the holomorphic functions $\{G_m - G_1\}_{m \in \mathbb{N}_{\geq 2}}$ on A_1 are uniformly bounded there. It follows that there exist a function $g_1 \in \mathcal{O}(A_1)$ and a subsequence $\{G_{m_1(k)} - G_1\}_{k \in \mathbb{N}}$, where m_1 is a strictly increasing function $\mathbb{N} \to \mathbb{N}_{\geq 2}$, such that $G_{m_1(k)} - G_1$ converges to g_1 on A_1 as $k \to \infty$. It follows that the sequence $\{G_{m_1(k)}\}_{k \in \mathbb{N}}$ converges to $G_1 + g_1$ on A_1. We use induction again. For each $n \in \mathbb{N}$ and for all $m \in \mathbb{N}_{\geq n+1}$, on A_n we have $\partial(G_m - G_n)/\partial \bar{z} = 0$ and

$$|G_m - G_n| \leq \sum_{j=0}^{m-n-1} \left|G_{m-j} - G_{m-j-1}\right| \leq \sum_{j=0}^{m-n-1} \frac{1}{2^{m-j-1}} < 1\,,$$

so the holomorphic functions $\{G_m - G_n\}_{m \in \mathbb{N}_{\geq n+1}}$ are uniformly bounded on A_n. For a fixed $n \in \mathbb{N}$, suppose we have strictly increasing functions $\{m_\ell\}_{\ell=1}^{n}$ from $\mathbb{N}$ to $\mathbb{N}_{\geq 2}$ (note that then the composition $m_1 \circ \cdots \circ m_n$ is a strictly increasing function mapping $\mathbb{N}$ to

$\mathbb{N}_{\geq n+1}$) such that the sequence of holomorphic functions $\{G_{(m_1\circ\cdots\circ m_n)(k)} - G_n\}_{k\in\mathbb{N}}$ on A_n converges to some function $g_n \in \mathcal{O}(A_n)$. Then, the sequence $\{G_{(m_1\circ\cdots\circ m_n)(k)}\}_{k\in\mathbb{N}}$ converges to $G_n + g_n$ on A_n. Since the sequence of holomorphic functions $\{G_{(m_1\circ\cdots\circ m_n)(k)} - G_{n+1}\}_{k\in\mathbb{N}_{\geq 2}}$ on A_{n+1} is uniformly bounded, it has a subsequence, which we may write $\{G_{(m_1\circ\cdots\circ m_n\circ m_{n+1})(k)} - G_{n+1}\}_{k\in\mathbb{N}}$ for some strictly increasing function $m_{n+1} : \mathbb{N} \to \mathbb{N}_{\geq 2}$, converging to some function $g_{n+1} \in \mathcal{O}(A_{n+1})$. Then, $\{G_{(m_1\circ\cdots\circ m_n\circ m_{n+1})(k)}\}_{k\in\mathbb{N}}$ converges to $G_{n+1} + g_{n+1}$ on A_{n+1}. On A_n, since $\{G_{(m_1\circ\cdots\circ m_n\circ m_{n+1})(k)}\}_{k\in\mathbb{N}}$ is a subsequence of $\{G_{(m_1\circ\cdots\circ m_n)(k)}\}_{k\in\mathbb{N}}$, we must have $G_n + g_n = G_{n+1} + g_{n+1}$. We may then define the function

$$g : \Delta(z_0; R) \to \mathbb{C}\,, \quad g|_{A_n} := G_n|_{A_n} + g_n \text{ for each } n \in \mathbb{N}\,.$$

Then, for each $n \in \mathbb{N}$, on A_n the function g is C^k and fulfils

$$\frac{\partial g}{\partial \bar{z}} = \rho_n f = f\,.$$

Since $\Delta(z_0; R) = \bigcup_{n\in\mathbb{N}} A_n$, this concludes the proof. □

Theorem 4.40. *Suppose $\Omega \subset \mathbb{C}$ is a connected open set, and suppose $f : \Omega \to \mathbb{C}$ is a holomorphic function that vanishes on a nonempty open subset U of Ω. Then, $f = 0$ on Ω.*

Proof. Define the set

$$S := \{w \in \Omega \,|\, f \text{ vanishes on a neighbourhood of } w \text{ in } \Omega\}$$

Then, S is open, and it is also nonempty, since $U \subset S$. We show that S is also closed in Ω. First note that for each $w \in S$ we have $f^{(n)}(w) = 0$ for all $n \in \mathbb{N}_0$. Suppose z_0 is a point in Ω such that there exists a sequence $\{w_k\}_{k\in\mathbb{N}}$ in S converging to z_0. Then, for each $n \in \mathbb{N}_0$ we have $0 = f^{(n)}(w_k) \to f^{(n)}(z_0)$ as $k \to \infty$, which implies that $f^{(n)}(z_0) = 0$. Choosing $R \in (0, +\infty)$ such that $\Delta(z_0; R) \subset \Omega$, Theorem 4.36 then gives $f = 0$ on $\Delta(z_0; R)$. It follows that $z_0 \in S$, which shows that S is closed in Ω. Since Ω is connected, we must then have $S = \Omega$. Thus, $f = 0$ on Ω. □

Corollary 4.41. *Suppose $\Omega \subset \mathbb{C}$ is a connected open set, and let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic function.*

(i) *For all $z_0 \in \Omega$ there exists $m \in \mathbb{N}$ such that $f^{(m)}(z_0) \neq 0$.*
(ii) *For each $z_0 \in \Omega$, there exist unique $m \in \mathbb{N}_0$ and unique $g \in \mathcal{O}(\Omega)$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for all $z \in \Omega$.*
(iii) (Identity Theorem) *The set $f^{-1}(0)$ has no limit points in Ω.*
(iv) *The sets $(\mathrm{Re}(f))^{-1}(0)$ and $(\mathrm{Im}(f))^{-1}(0)$ are nowhere dense in Ω.*

Proof. (i) Let $z_0 \in \Omega$, and suppose $f^{(m)}(z_0) = 0$ for all $m \in \mathbb{N}$. Then, choosing $R \in (0, +\infty)$ such that $\Delta(z_0; R) \subset \Omega$, by Theorem 4.36 we must have $f = f(z_0)$ on $\Delta(z_0; R)$. Then, the holomorphic function $z \mapsto f(z) - f(z_0)$ on Ω vanishes on $\Delta(z_0; R)$, which by Theorem 4.40 implies that it vanishes on Ω. It follows that $f = f(z_0)$ in Ω, which contradicts the fact that f is nonconstant.

(ii) Let $z_0 \in \Omega$. We first show that if there exist $m, n \in \mathbb{N}_0$ and $g, h \in \mathcal{O}(\Omega)$ such that $g(z_0), h(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z) = (z - z_0)^n h(z)$ for all $z \in \Omega$, then we have $n = m$ and $g = h$. If $n = m$, then we must have $g = h$ on $\Omega \setminus \{z_0\}$, so $g = h$ on Ω by continuity. If $n \neq m$, and assuming $n < m$, we have $(z - z_0)^{m-n}(z - z_0)^n g(z) = (z - z_0)^n h(z)$ for all $z \in \Omega$. It follows that

$(z-z_0)^{m-n}g(z) = h(z)$ for all $z \in \Omega$, so $h(z_0) = 0$, which is a contradiction. Thus, we must have $n = m$ and $g = h$. We now show the existence of such $m \in \mathbb{N}_0$ and $g \in \mathcal{O}(\Omega)$. If $f(z_0) \neq 0$, then letting $m := 0$ and $g := f$ the desired conditions are fulfilled. Suppose $f(z_0) = 0$, and let

$$m := \min\{n \in \mathbb{N} \mid f^{(n)}(z_0) \neq 0\}\,.$$

Choosing $R \in (0, +\infty)$ such that $\Delta(z_0; R) \subset \Omega$, for every $z \in \Delta(z_0; R)$ we have

$$\sum_{n=m}^{N} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^m \sum_{n=m}^{N} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \to f(z)$$

as $\mathbb{N}_{\geq m} \ni N \to \infty$. Then, for all $z \in \Delta(z_0; R) \setminus \{z_0\}$ we have

$$\sum_{n=m}^{N} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \to \frac{f(z)}{(z-z_0)^m}$$

as $\mathbb{N}_{\geq m} \ni N \to \infty$. It follows that the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n$$

converges on $\Delta(z_0; R)$, and it does so to $\frac{f(z)}{(z-z_0)^m}$ for each $z \in \Delta(z_0; R) \setminus \{z_0\}$. Then, defining the function

$$g : \Omega \to \mathbb{C}\,, \quad g(z) := \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n & \text{if } z \in \Delta(z_0; R) \\ \displaystyle\frac{f(z)}{(z-z_0)^m} & \text{otherwise} \end{cases}$$
$$= \begin{cases} \displaystyle\frac{f(z)}{(z-z_0)^m} & \text{if } z \in \Omega \setminus \{z_0\} \\ \displaystyle\frac{f^{(m)}(z_0)}{m!} & \text{if } z = z_0\,, \end{cases}$$

we have $g \in \mathcal{O}(\Omega)$, $g(z_0) \neq 0$ and $f(z) = (z-z_0)^m g(z)$ for all $z \in \Omega$.

(iii) Suppose there exist $z_0 \in \Omega$ and a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $f^{-1}(0) \setminus \{z_0\}$ converging to z_0. Let $g \in \mathcal{O}(\Omega)$ and $m \in \mathbb{N}_0$ such that $g(z_0) \neq 0$ and $f(z) = (z-z_0)^m g(z)$ for all $z \in \Omega$, as given by (ii). Then, for each $n \in \mathbb{N}$ we have $0 = f(w_n) = (w_n - z_0)^m g(w_n)$, which implies that $g(w_n) = 0$. It follows that $g(z_0) = 0$, which is a contradiction. Thus, the set $f^{-1}(0)$ has no limit points in Ω.

(iv) Let $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$. Suppose $u^{-1}(0)$ is not nowhere dense, that is, the closure of $u^{-1}(0)$ in Ω, which is $u^{-1}(0)$ itself, has nonempty interior in Ω. Then, there exists a nonempty open subset $U \subset \Omega$ contained in $u^{-1}(0)$. On U, we have

$$f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0\,,$$

which implies that $f^{(m)} = 0$ on U for all $m \in \mathbb{N}$. This contradicts (i), which shows that indeed $u^{-1}(0)$ must be nowhere dense in Ω. The proof that $v^{-1}(0)$ also is nowhere dense in Ω is similar.

□

Remark 4.42. Suppose f is a holomorphic function on an open subset $\Omega \subset \mathbb{C}$, not necessarily connected, and let $z_0 \in \Omega$. If f is not identically 0 on any neighbourhood of z_0 in Ω, then the set

$$\{n \in \mathbb{N}_0 \mid f^{(n)}(z_0) \neq 0\}$$

is nonempty and hence it has a minimum $m \in \mathbb{N}_0$. Then, by the proof of Corollary 4.41 (ii), defining

$$g : \Omega \to \mathbb{C}\,, \qquad g(z) := \begin{cases} \dfrac{f(z)}{(z-z_0)^m} & \text{if } z \in \Omega \\ \dfrac{f^{(m)}(z_0)}{m!} & \text{if } z = z_0, \end{cases}$$

m and g are respectively the unique number $m \in \mathbb{N}_0$ and unique function $g \in \mathcal{O}(\Omega)$ such that $g(z_0) \neq 0$ and $f(z) = (z-z_0)^m g(z)$ for all $z \in \Omega$. We then call m the *order of f at z_0*, and denote it by $\mathrm{ord}_{z_0} f$. If $\mathrm{ord}_{z_0} f \geq 1$, then we have $f(z_0) = 0$, and z_0 is called a *zero of order m* (a *simple zero* if $\mathrm{ord}_{z_0} f = 1$). If f does vanish on a neighbourhood of z_0 in Ω, then we say that f has order $\mathrm{ord}_{z_0} f = \infty$ at z_0.

Theorem 4.43. (Open Mapping Theorem). *Suppose $\Omega \subset \mathbb{C}$ is a connected open set and $f : \Omega \to \mathbb{C}$ a nonconstant holomorphic function. Then, the image $f(\Omega)$ is open in $\mathbb{C}$.*

Proof. We wish to show that for each $z_0 \in \Omega$ there exists a neighbourhood U_{z_0} of z_0 in Ω such that $f(U_{z_0})$ is open in $\mathbb{C}$. Let $z_0 \in \Omega$, and suppose first that $f'(z_0) \neq 0$. Then, letting $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$, the Jacobian determinant of f at z_0 is given by

$$\begin{vmatrix} \dfrac{\partial u}{\partial x}(z_0) & \dfrac{\partial u}{\partial y}(z_0) \\ \\ \dfrac{\partial v}{\partial x}(z_0) & \dfrac{\partial v}{\partial y}(z_0) \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x}(z_0) & -\dfrac{\partial v}{\partial x}(z_0) \\ \\ \dfrac{\partial v}{\partial x}(z_0) & \dfrac{\partial u}{\partial x}(z_0) \end{vmatrix} = \left(\frac{\partial u}{\partial x}(z_0)\right)^2 + \left(\frac{\partial v}{\partial x}(z_0)\right)^2 = \left|f'(z_0)\right|^2$$

and hence it is nonzero. It follows from the Inverse Function Theorem that there exist a neighbourhood U_{z_0} of z_0 in Ω and a neighbourhood V of $f(z_0)$ in $\mathbb{C}$ such that $f(U_{z_0}) = V$. Thus, $f(U_{z_0})$ is open in $\mathbb{C}$.

Suppose now that $f'(z_0) = 0$. By Corollary 4.41 (iii), we may find $R \in (0, +\infty)$ such that $\overline{\Delta(z_0; R)} \subset \Omega$ and for all $z \in \overline{\Delta(z_0; R)} \setminus \{z_0\}$ we have $f'(z) \neq 0$ and $f(z) \neq f(z_0)$. Then, by the above argument, for each $z \in \Delta(z_0; R) \setminus \{z_0\} = \Delta^*(z_0; R)$ we may find a neighbourhood W_z of z in $\Delta^*(z_0; R)$ such that $f(W_z)$ is open in $\mathbb{C}$. We have

$$f(\Delta^*(z_0; R)) = f\left(\bigcup_{z \in \Delta^*(z_0;R)} W_z\right) = \bigcup_{z \in \Delta^*(z_0;R)} f(W_z)\,,$$

so $f(\Delta^*(z_0; R))$ is open in $\mathbb{C}$. Moreover, since $f(\overline{\Delta(z_0; R)})$ is closed and contains $f(\Delta^*(z_0; R))$, we have

$$\begin{aligned} \partial\big(f(\Delta^*(z_0; R))\big) &\subset f(\overline{\Delta(z_0; R)}) \\ &= f(\Delta^*(z_0; R)) \cup f(\partial\Delta(z_0; R)) \cup \{f(z_0)\}\,. \end{aligned}$$

Since $f(\Delta^*(z_0; R))$ is open, we must then have

$$\partial\big(f(\Delta^*(z_0; R))\big) \subset f(\partial\Delta(z_0; R)) \cup \{f(z_0)\}\,.$$

Now, since the sets $f(\partial\Delta(z_0;R))$ and $\{f(z_0)\}$ in $\mathbb{C}$ are compact and disjoint, there exists $a \in (0,+\infty)$ such that $\Delta(f(z_0);a) \cap f(\partial\Delta(z_0;R)) = \emptyset$. It follows that

$$\Delta^*(f(z_0);a) \cap \partial\big(f(\Delta^*(z_0;R))\big) = \emptyset\,,$$

so

$$\Delta^*(f(z_0);a) \subset f(\Delta^*(z_0;R)) \cup \operatorname{ext}\left(f(\Delta^*(z_0;R))\right).$$

Since $\Delta^*(f(z_0);a)$ is connected, we must then have either $\Delta^*(f(z_0);a) \subset f(\Delta^*(z_0;R))$ or $\Delta^*(f(z_0);a) \subset \operatorname{ext}\left(f(\Delta^*(z_0;R))\right)$. Choosing a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\Delta^*(z_0;R)$ converging to z_0, the sequence $\{f(w_n)\}_{n\in\mathbb{N}} \subset f(\Delta^*(z_0;R))$ must have points in $\Delta^*(f(z_0);a)$. Thus, we must have

$$\Delta^*(f(z_0);a) \subset f(\Delta^*(z_0;R))\,,$$

so that, letting $U_{z_0} := \Delta(z_0;R)$, the image

$$\begin{aligned} f(U_{z_0}) &= f(\Delta^*(z_0;R)) \cup \{f(z_0)\} \\ &= f(\Delta^*(z_0;R)) \cup \Delta^*(f(z_0);a) \cup \{f(z_0)\} \\ &= f(\Delta^*(z_0;R)) \cup \Delta(f(z_0);a) \end{aligned}$$

is open in $\mathbb{C}$.

Since for each $z_0 \in \Omega$ there is a neighbourhood U_{z_0} of z_0 in Ω such that $f(U_{z_0})$ is open in $\mathbb{C}$, the image

$$f(\Omega) = f\left(\bigcup_{z_0\in\Omega} U_{z_0}\right) = \bigcup_{z_0\in\Omega} f(U_{z_0})$$

is open in $\mathbb{C}$. $\square$

Corollary 4.44. (Maximum Principle). *Suppose $\Omega \subset \mathbb{C}$ is a connected open set and $f : \Omega \to \mathbb{C}$ a holomorphic function. If $|f|$ attains a local maximum at some $z_0 \in \Omega$, then f is constant.*

Proof. Let $z_0 \in \Omega$, and suppose there exists a neighbourhood U of z_0 in Ω such that for all $z \in U$ we have $\big|f(z)\big| \le \big|f(z_0)\big|$. We may assume that U is connected. Then, $f(U)$ cannot contain an open disc about $f(z_0)$, so $f(U)$ is not open in $\mathbb{C}$. It follows from Theorem 4.43 that f is constant on U. Since Ω is connected, it follows that f is constant on Ω. $\square$

References

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