Chapter **1**

Virtual Sensors and Actuators

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1.1. Introduction

In the literature, fault-tolerant control (FTC) has been addressed from two families of approaches. The first one is based on redesigning the control law after the fault while the second relies on the principle of hiding the fault to the controller by activating some fault masking block that prevents the controller to be redesigned. The motivation for not redesigning the controller is due to two mean reasons. First, it allows a plug-and-play deployment of the FTC in the control loops without needing to re-tuning the controller. Second, the existing control law includes valuable implicit knowledge about the plant and the possible performance of the closed-loop system. This knowledge was acquired during the design cycle and embedded in the nominal controller.

Virtual sensors and actuator belong to the the fault masking family. The formalization of the concept of virtual sensors and virtual actuators can be found in the book of Blanke *et al*. (2016) and was originally introduced by Jan Lunze. The method was first developed for LTI systems and later extended to other type of systems, as e.g. LPV systems (Rotondo *et al*. 2014*a*), Hammerstein systems (J.H.Richter and Lunze 2010), Lure systems (Pedersen *et al*. 2016), Takagi-Sugeno systems (Rotondo *et al*. 2014*a*) or piece-wise linear systems (Richter *et al*. 2011).

Intuitively, virtual sensors uses an observer to estimate the output of the faulty sensor. The name of virtual sensors come from the fact that the faulty sensor is replaced by its estimation using the model with an observer scheme. In the literature, this type of approach is also known as soft sensor. On the other hand, virtual actuators, obtained from duality from virtual sensors, replace the faulty actuator by using changing the control action provided by the other actuators. Both virtual sensors and actuators requires enough redundant sensors or actuators to compensate for the faulty components. The principle of duality existing between virtual sensors and actuators lead to the design equations of the virtual actuator can be derived from those of the virtual sensor introducing some transpositions, similarly than between state-space controllers and observers.

This chapter introduces first the design of virtual sensors and actuators using the classical eigenvalue assignment approach, widely used for the design of controllers and observers in state space. Then, the LMI formulation of the design problem is introduced as intermediate means to approach to the non-linear systems using the LPV representation that make use of the LMI formulation applied to the polytopic representation of the LPV system.

The chapter illustrates the virtual sensor and actuator approach with a well-known case study: the two tank system. Finally, the chapter ends with the presentation of the conclusions and of some outlooks of current trends of virtual sensors and actuators.

1.2. Problem statement

Let us consider the following LTI system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \hat{\mathbf{y}}(t)) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \\ \hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t) \\ \mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t) \end{cases} \qquad [1.1]$$

which is the interconnection of a plant with internal state $\mathbf{x} \in \mathcal{R}^n$ and a state observer with internal state $\hat{\mathbf{x}} \in \mathcal{R}^n$. The interconnection consists in a flow of information from the state observer to the plant through the estimate-feedback control law that updates the control input $\mathbf{u} \in \mathcal{R}^p$, and a flow of information from the plant to the observer in terms of the innovation $\mathbf{C}\mathbf{e}(t) = \mathbf{C}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \in \mathcal{R}^m$. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ denote known matrices of appropriate dimensions, whereas $\mathbf{K}$ and $\mathbf{L}$ are the controller and observer gains, that must be designed so as to satisfy some performance requirements, which typically include asymptotical stability.

It is well known that by considering the augmented state variable $\mathbf{z}_{xe}(t) = [\mathbf{x}(t)^T, \mathbf{e}(t)^T]^T$, the system [1.1] can be expressed as the following block-triangular system:

$$\dot{\mathbf{z}}_{xe}(t) = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{O} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \mathbf{z}_{xe}(t) \qquad [1.2]$$

which allows to derive the well-known *separation principle* that justifies the separate design of $\mathbf{K}$ and $\mathbf{L}$ such that $\mathbf{A}-\mathbf{BK}$ and $\mathbf{A}-\mathbf{LC}$ are asymptotically stable.

Let us now consider the case in which the output equation in [1.1] is changed to:

$$\mathbf{y}(t) = \mathbf{C}_f\mathbf{x}(t) + \mathbf{f}_y(t) \qquad [1.3]$$

where $\mathbf{C}_f$ denotes the faulty output matrix and $\mathbf{f}_y \in \mathcal{R}^m$ represents additive sensor faults. The matrix $\mathbf{C}_f$ describes multiplicative faults, as it is obtained from $\mathbf{C}$ as:

$$\mathbf{C}_f = \mathrm{diag}(\gamma_1, \ldots, \gamma_m)\mathbf{C} \qquad [1.4]$$

with $\gamma_i \in [0, 1)$ representing the effectiveness of the i-th sensor, with $\gamma_i = 1$ corresponding to the healthy case, whereas $\gamma_i = 0$ represents its total failure.

The above case is denoted as *sensor fault case*, and has (potentially serious) consequences on the control system. In fact, the system described in terms of the augmented variable $\mathbf{z}_{xe}(t)$ becomes:

$$\dot{\mathbf{z}}_{xe}(t) = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{L}(\mathbf{C}-\mathbf{C}_f) & \mathbf{A}-\mathbf{LC} \end{bmatrix} \mathbf{z}_{xe}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{f}_y(t) \qquad [1.5]$$

which:

– is not anymore in a block-triangular form, hence the separation principle does not hold anymore, and even worse the state matrix could have non-negative eigenvalues due to the effect of $\mathbf{C}_f$, in which case the system would be unstable in spite of the asymptotically stable nominal design;

– even if asymptotical stability were to be preserved in spite of the mismatch between $\mathbf{C}_f$ and $\mathbf{C}$, the augmented system is not autonomous anymore due to the presence of $\mathbf{f}_y(t)$, so that $\mathbf{z}_{xl}(t)$ would not converge to zero as desired.

Under the assumption that estimates $\hat{\mathbf{C}}_f$ and $\hat{\mathbf{f}}_y$ of $\mathbf{C}_f$ and $\mathbf{f}_y$ are available, some FTC strategy can be employed to remove (or at least alleviate) the above problems. As discussed in the introduction, a quite straightforward solution would be to redesign the controller and observer gains so that the augmented state matrix in [1.5] is asymptotically stable, and then compensate somehow for the effect of the additive fault $\mathbf{f}_y$.

However, a more attractive philosophy to tackle the problem would be to keep the nominal controller and observer gains and add a new component to the control loop, which will be in charge of hiding the sensor faults from the controller/observer point of view. This component is named *virtual sensor*, for whose design we can state the following problem.

Problem 1 ***(Virtual sensor structure choice)*** Choose the structure of the virtual sensor:

$$\begin{cases} \dot{\mathbf{x}}_{vs}(t) = \mathbf{f}_{vs}\left(\mathbf{x}_{vs}(t), \mathbf{u}(t), \mathbf{y}(t), \hat{\mathbf{C}}_f, \hat{\mathbf{f}}_y(t)\right) \\ \mathbf{y}_{vs}(t) = \mathbf{h}_{vs}\left(\mathbf{x}_{vs}(t), \mathbf{u}(t), \mathbf{y}(t), \hat{\mathbf{C}}_f, \hat{\mathbf{f}}_y(t)\right) \end{cases} \qquad [1.6]$$

so that, if the state observer in [1.1] is fed by $\mathbf{y}_{vs}(t)$ instead of $\mathbf{y}(t)$:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\left(\mathbf{y}_{vs}(t) - \hat{\mathbf{y}}(t)\right) \qquad [1.7]$$

then the overall augmented system obtained as the interconnection of the faulty plant, the state observer and the virtual sensor tends to an autonomous system which is similar to a block-triangular system with blocks $\mathbf{A} - \mathbf{BK}$ and $\mathbf{A} - \mathbf{LC}$ on the main diagonal when $\hat{\mathbf{C}}_f \to \mathbf{C}_f$ and $\hat{\mathbf{f}}_y \to \mathbf{f}_y$.

Analogously, the plant state equation in [1.1] could be changed to:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_f\left(\mathbf{u}(t) + \mathbf{f}_u(t)\right) \qquad [1.8]$$

which is denoted as *actuator fault case*, where $\mathbf{B}_f$ denotes the faulty input matrix and $\mathbf{f}_u \in \mathcal{R}^p$ represents additive actuator faults. The matrix $\mathbf{B}_f$ describes multiplicative faults, and it is obtained from $\mathbf{B}$ as:

$$\mathbf{B}_f = \mathbf{B}\mathrm{diag}\left(\phi_1, \ldots, \phi_p\right) \qquad [1.9]$$

with $\phi_i \in [0, 1)$ representing the effectiveness of the i-th actuator, with the same interpretation as γ_i in the sensor fault case. The consequences of actuator faults on the closed-loop system are equally dangerous, as the augmented variable $\mathbf{z}_{xl}(t)$ would obey:

$$\dot{\mathbf{z}}_{xe}(t) = \begin{bmatrix} \mathbf{A} - \mathbf{B}_f\mathbf{K} & \mathbf{B}_f\mathbf{K} \\ (\mathbf{B} - \mathbf{B}_f)\mathbf{K} & \mathbf{A} - \mathbf{LC} - (\mathbf{B} - \mathbf{B}_f)\mathbf{K} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_f \\ \mathbf{B}_f \end{bmatrix} \mathbf{f}_u(t) \qquad [1.10]$$

with the same arising issues as described above for the sensor fault case, which motivates the development of a component named *virtual actuator*, that uses available estimates $\hat{\mathbf{B}}_f$ and $\hat{\mathbf{f}}_u$ of $\mathbf{B}_f$ and $\mathbf{f}_u$ to hide the faults from the controller/observer point of view.

Problem 2 ***(Virtual actuator structure choice)*** Choose the structure of the virtual actuator:

$$\begin{cases} \dot{\mathbf{x}}_{va}(t) = \mathbf{f}_{va}\left(\mathbf{x}_{va}(t), \mathbf{u}(t), \mathbf{y}(t), \hat{\mathbf{B}}_f, \hat{\mathbf{f}}_u(t)\right) \\ \mathbf{u}_{va}(t) = \mathbf{g}_{va}\left(\mathbf{x}_{va}(t), \mathbf{u}(t), \mathbf{y}(t), \hat{\mathbf{B}}_f, \hat{\mathbf{f}}_u(t)\right) \\ \mathbf{y}_{va}(t) = \mathbf{h}_{va}\left(\mathbf{x}_{va}(t), \mathbf{u}(t), \mathbf{y}(t), \hat{\mathbf{B}}_f, \hat{\mathbf{f}}_u(t)\right) \end{cases} \quad [1.11]$$

so that, if the faulty plant [1.8] is fed by $\mathbf{u}_{va}(t)$ instead of $\mathbf{u}(t)$:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_f\left(\mathbf{u}_{va}(t) + \mathbf{f}_u(t)\right) \quad [1.12]$$

and the state observer is fed by $\mathbf{y}_{va}(t)$ instead of $\mathbf{y}(t)$:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\left(\mathbf{y}_{va}(t) - \hat{\mathbf{y}}(t)\right) \quad [1.13]$$

then the overall augmented system obtained as the interconnection of the faulty plant, the state observer and the virtual actuator tends to an autonomous system which is similar to a block-triangular system with blocks $\mathbf{A} - \mathbf{BK}$ and $\mathbf{A} - \mathbf{LC}$ on the main diagonal when $\hat{\mathbf{B}}_f \to \mathbf{B}_f$ and $\hat{\mathbf{f}}_u \to \mathbf{f}_u$.

Note that solving Problems 1-2 would ensure that the nominal controller and observer gains $\mathbf{K}$ and $\mathbf{L}$ enforce asymptotical stability under fault occurrence, as long as the additional blocks on the augmented system's diagonal introduced by the virtual sensor/actuator are designed to be asymptotically stable. In the next section, it will be shown the virtual sensor/actuator introduces only one additional block, which is is independent from $\mathbf{K}$ and $\mathbf{L}$.

1.3. Virtual sensors and virtual actuators

When it comes to the virtual sensor structure, two cases can be distinguished:

1) the following rank condition holds:

$$\mathrm{rank}(\hat{\mathbf{C}}_f) = \mathrm{rank}\left(\mathbf{C}\right) \quad [1.14]$$

which means that the information lost due to the sensor faults embedded in the matrix $\hat{\mathbf{C}}_f$ can be reconstructed perfectly from the available measurements without the need of resorting to the exploitation of the system's dynamic model;

2) the above rank condition [1.6.2] does not hold, which means that the knowledge of the system's dynamic model must be used to reconstruct the information lost due to the sensor faults.

In the first case, the virtual sensor [1.6] consists of a static block, which means that the function $\mathbf{g}_{vs}(\cdot)$ does not depend on $\mathbf{x}_{vs}(t)$:

$$\mathbf{y}_{vs}(t) = \mathbf{C}\hat{\mathbf{C}}_f^{\dagger}\left(\mathbf{y}(t) - \hat{\mathbf{f}}_y(t)\right) \quad [1.15]$$

On the other hand, in the second case, the virtual sensor is a dynamic block described by the following equations:

$$\begin{cases} \dot{\mathbf{x}}_{vs}(t) = \left(\mathbf{A} - \mathbf{M}\hat{\mathbf{C}}^*\right)\mathbf{x}_{vs}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{M}\mathbf{C}\hat{\mathbf{C}}_f^{\dagger}\left(\mathbf{y}(t) - \hat{\mathbf{f}}_y(t)\right) \\ \mathbf{y}_{vs}(t) = \mathbf{C}\hat{\mathbf{C}}_f^{\dagger}\left(\mathbf{y}(t) - \hat{\mathbf{f}}_y(t)\right) + \left(\mathbf{C} - \hat{\mathbf{C}}^*\right)\mathbf{x}_{vs}(t) \end{cases} \quad [1.16]$$

where $\mathbf{M} \in \mathcal{R}^{n \times m}$ denotes the virtual sensor gain matrix, and $\hat{\mathbf{C}}^*$ is defined as:

$$\hat{\mathbf{C}}^* = \mathbf{C}\hat{\mathbf{C}}_f^{\dagger}\hat{\mathbf{C}}_f \quad [1.17]$$

First of all, note that [1.15] is a special case of [1.16]. In fact, when [1.6.2] holds, then $\hat{\mathbf{C}}^* \to \mathbf{C}$ so that $(\mathbf{C} - \hat{\mathbf{C}}^*)\mathbf{x}_{vs}(t) \to \mathbf{0}$.

Second, let us consider the overall dynamics obtained as the interconnection of [1.1] with the faulty output equation [1.3], the state observer as in [1.7] and the virtual sensor [1.16], which is described by the following system excited by the additive sensor fault estimation error $\mathbf{f}_y(t) - \hat{\mathbf{f}}_y(t)$:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \\ \dot{\mathbf{x}}_{vs}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} & \mathbf{O} \\ \mathbf{L}\mathbf{C}\hat{\mathbf{C}}_f^{\dagger}\mathbf{C}_f & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C} & \mathbf{L}\left(\mathbf{C} - \hat{\mathbf{C}}^*\right) \\ \mathbf{M}\mathbf{C}\hat{\mathbf{C}}_f^{\dagger}\mathbf{C}_f & -\mathbf{B}\mathbf{K} & \mathbf{A} - \mathbf{M}\hat{\mathbf{C}}^* \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \\ \mathbf{x}_{vs}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{O} \\ \mathbf{L}\mathbf{C}\hat{\mathbf{C}}_f^{\dagger} \\ \mathbf{M}\mathbf{C}\hat{\mathbf{C}}_f^{\dagger} \end{bmatrix} (\mathbf{f}_y(t) - \hat{\mathbf{f}}_y(t)) \quad [1.18]$$

Taking into account [1.17], [1.18] simplifies into an autonomous system when $\hat{\mathbf{C}}_f \to \mathbf{C}_f$ and $\hat{\mathbf{f}}_y \to \mathbf{f}_y$:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \\ \dot{\mathbf{x}}_{vs}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} & \mathbf{O} \\ \mathbf{L}\hat{\mathbf{C}}^* & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C} & \mathbf{L}\left(\mathbf{C} - \hat{\mathbf{C}}^*\right) \\ \mathbf{M}\hat{\mathbf{C}}^* & -\mathbf{B}\mathbf{K} & \mathbf{A} - \mathbf{M}\hat{\mathbf{C}}^* \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \\ \mathbf{x}_{vs}(t) \end{bmatrix} \quad [1.19]$$

Let us define a change of variables as follows:

$$\mathbf{z}_{vs}(t) = \begin{bmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \\ \mathbf{x}_{vs}(t) \end{bmatrix} \quad [1.20]$$

then simple calculations over [1.19] lead to:

$$\dot{\mathbf{z}}_{vs}(t) = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & -\mathbf{BK} & -\mathbf{BK} \\ \mathbf{O} & \mathbf{A}-\mathbf{LC} & (\mathbf{M}-\mathbf{L})\hat{\mathbf{C}}^{*} \\ \mathbf{O} & \mathbf{O} & \mathbf{A}-\mathbf{M}\hat{\mathbf{C}}^{*} \end{bmatrix} \mathbf{z}_{vs}(t) \quad [1.21]$$

which shows that the virtual sensor structure [1.16] solves Problem 1.

This result is of extreme theoretical importance, and constitutes the main advantage of virtual sensors with respect to other FTC techniques. It represents an extended separation principle, since it justifies a design of the virtual sensor which is independent from the controller and the observer ones. More specifically, [1.21] shows that the set of eigenvalues Λ_{vs} of the augmented system is given by:

$$\Lambda_{vs} = \mathrm{eig}(\mathbf{A}-\mathbf{BK}) \cup \mathrm{eig}(\mathbf{A}-\mathbf{LC}) \cup \mathrm{eig}(\mathbf{A}-\mathbf{M}\hat{\mathbf{C}}^{*}) \quad [1.22]$$

where $\mathrm{eig}(\mathbf{M})$ denotes the eigenvalues of matrix $\mathbf{M}$. Hence, given an asymptotically stable nominal closed-loop system, fault tolerance can be added by inserting a virtual sensor designed so that the matrix $\mathbf{A}-\mathbf{M}\hat{\mathbf{C}}^{*}$ is Hurwitz.

Similarly, two cases can be distinguished when the structure of the virtual actuator is considered:

1) the following rank condition holds:

$$\mathrm{rank}(\hat{\mathbf{B}}_f) = \mathrm{rank}(\hat{\mathbf{B}}) \quad [1.23]$$

which means that the loss of actuation due to the actuator faults embedded in the matrix $\mathbf{B}_f$ can be compensated perfectly by redistributing it over the remaining actuators without the need of considering the system's dynamics described by the state matrix $\mathbf{A}$.

2) the above rank condition [1.23] does not hold; as in the virtual sensor case, the knowledge of the matrix $\mathbf{A}$ must be exploited to achieve fault tolerance.

In the first case, the virtual actuator [1.11] consists of a static block:

$$\begin{cases} \mathbf{u}_{va}(t) = \hat{\mathbf{B}}_f^{\dagger}\mathbf{B}\mathbf{u}(t) - \hat{\mathbf{f}}_u(t) \\ \mathbf{y}_{va}(t) = \mathbf{y}(t) \end{cases} \quad [1.24]$$

whereas in the second case, it is a dynamic block described by:

$$\begin{cases} \dot{\mathbf{x}}_{va}(t) = (\mathbf{A} - \hat{\mathbf{B}}^*\mathbf{N})\mathbf{x}_{va}(t) + (\mathbf{B} - \hat{\mathbf{B}}^*)\mathbf{u}(t) \\ \mathbf{u}_{va}(t) = \hat{\mathbf{B}}_f^{\dagger}\mathbf{B}\left(\mathbf{u}(t) + \mathbf{N}\mathbf{x}_{va}(t)\right) - \hat{\mathbf{f}}_u(t) \\ \mathbf{y}_{va}(t) = \mathbf{y}(t) + \mathbf{C}\mathbf{x}_{va}(t) \end{cases} \quad [1.25]$$

where $\mathbf{N} \in \mathcal{R}^{p\times n}$ denotes the virtual actuator gain, and $\hat{\mathbf{B}}^*$ is defined as:

$$\hat{\mathbf{B}}^* = \hat{\mathbf{B}}_f\hat{\mathbf{B}}_f^{\dagger}\mathbf{B} \quad [1.26]$$

Also in this case, when the rank condition [1.23] holds, $\hat{\mathbf{B}}^* \to \mathbf{B}$ so that $(\mathbf{B} - \hat{\mathbf{B}}_f^*)\mathbf{u}(t) \to \mathbf{0}$ which means that $\mathbf{x}_{va}(t) = \mathbf{0}\,\forall t \in \mathcal{R}$ provided that $\mathbf{x}_{va}(0) = \mathbf{0}$, thus showing that [1.24] is a special case of [1.25].

Let us consider the overall dynamics obtained as the interconnection of [1.1] with the faulty state equation [1.12] and the virtual actuator [1.25], which is described by the following system excited by the additive actuator fault estimation error $\mathbf{f}_u(t) - \hat{\mathbf{f}}_u(t)$:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \\ \dot{\mathbf{x}}_{va}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}_f\hat{\mathbf{B}}_f^{\dagger}\mathbf{B}\mathbf{K} & \mathbf{B}_f\hat{\mathbf{B}}_f^{\dagger}\mathbf{B}\mathbf{N} \\ \mathbf{L}\mathbf{C} & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C} & \mathbf{L}\mathbf{C} \\ \mathbf{O} & (\hat{\mathbf{B}}^* - \mathbf{B})\mathbf{K} & \mathbf{A} - \hat{\mathbf{B}}^*\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \\ \mathbf{x}_{va}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_f \\ \mathbf{O} \\ \mathbf{O} \end{bmatrix} (\mathbf{f}_u(t) - \hat{\mathbf{f}}_u(t)) \quad [1.27]$$

Taking into account [1.26], [1.27] simplifies into an autonomous system when $\hat{\mathbf{B}}_f \to \mathbf{B}_f$ and $\hat{\mathbf{f}}_u \to \mathbf{f}_u$:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \\ \dot{\mathbf{x}}_{va}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\hat{\mathbf{B}}^*\mathbf{K} & \hat{\mathbf{B}}^*\mathbf{N} \\ \mathbf{L}\mathbf{C} & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C} & \mathbf{L}\mathbf{C} \\ \mathbf{O} & (\hat{\mathbf{B}}^* - \mathbf{B})\mathbf{K} & \mathbf{A} - \hat{\mathbf{B}}^*\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \\ \mathbf{x}_{va}(t) \end{bmatrix} \quad [1.28]$$

Let us define a change of variables as follows:

$$\mathbf{z}_{va}(t) = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \\ \mathbf{x}_{va}(t) \end{bmatrix} \quad [1.29]$$

then, we obtain:

$$\dot{\mathbf{z}}_{va}(t) = \begin{bmatrix} \mathbf{A}-\hat{\mathbf{B}}^*\mathbf{N} & \left(\hat{\mathbf{B}}^*-\mathbf{B}\right)\mathbf{K} & \left(\hat{\mathbf{B}}^*-\mathbf{B}\right)\mathbf{K} \\ \mathbf{O} & \mathbf{A}-\mathbf{BK} & -\mathbf{BK} \\ \mathbf{O} & \mathbf{O} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \mathbf{z}_{va}(t) \qquad [1.30]$$

which demonstrates that the set of eigenvalues Λ_{va} of the augmented system is given by:

$$\Lambda_{va} = \text{eig}(\mathbf{A}-\mathbf{BK}) \cup \text{eig}(\mathbf{A}-\mathbf{LC}) \cup \text{eig}(\mathbf{A}-\hat{\mathbf{B}}^*\mathbf{N}) \qquad [1.31]$$

so that fault tolerance can be added to an asymptotically stable nominal system by inserting a virtual actuator designed so that the matrix $\mathbf{A}-\hat{\mathbf{B}}^*\mathbf{N}$ is Hurwitz.

1.4. LMI-based design

Although the design of the virtual sensor and virtual actuator gains $\mathbf{M}$ and $\mathbf{N}$ can be performed by applying standard LTI design tools, such as the Ackermann formula (Ackermann 1972), in this section we will present a linear matrix inequality (LMI)-based procedure. As explained in the next section, the attractiveness of this solution is that the virtual sensor/actuator technique described so far can be extended easily to work with nonlinear systems described by some convex representation (López-Estrada *et al*. 2019), such as the linear parameter varying one (Rotondo *et al*. 2014*b*).

The LMI-based design procedure is based on the notion of quadratic stability, which is defined formally as follows.

Definition 1 ***(Quadratic stability)*** The autonomous LTI system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \qquad [1.32]$$

is said to be *quadratically stable* if there exists a Lyapunov function $V(\mathbf{x}(t)) = \mathbf{x}(t)^T\mathbf{P}\mathbf{x}(t)$, with $\mathbf{P} \succ 0$, such that $\dot{V}(\mathbf{x}(t)) < 0\,\forall \mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, i.e.:

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} \prec 0 \qquad [1.33]$$

Note that the symbols $\succ$ and $\prec$ must be interpreted in the sense of positive and negative definiteness, respectively, which means that all the eigenvalues of the matrix $\mathbf{P}$ are required to be positive and all the eigenvalues of $\mathbf{PA}+\mathbf{A}^T\mathbf{P}$ are required to be negative.

Given the above definition, we can state the virtual sensor and virtual actuator design problems as follows.

Problem 3 ***(Virtual sensor gain design)*** Given matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\hat{\mathbf{C}}^*$, $\mathbf{K}$ and $\mathbf{L}$, choose the virtual sensor gain $\mathbf{M}$ so that the autonomous system [1.21] is quadratically stable.

Problem 4 ***(Virtual actuator gain design)*** Given matrices $\mathbf{A}$, $\mathbf{B}$, $\hat{\mathbf{B}}^*$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{L}$, choose the virtual actuator gain $\mathbf{N}$ so that the autonomous system [1.30] is quadratically stable.

A key theoretical result that enables a simple solution of Problems 3-4 is the following lemma.

Lemma 1 ***(Quadratic stability of block-triangular systems)*** Consider the following autonomous system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) \qquad [1.34]$$

and assume that the subsystems $\dot{\mathbf{x}}_1(t) = \mathbf{A}_{11}\mathbf{x}_1(t)$ and $\dot{\mathbf{x}}_2(t) = \mathbf{A}_{22}\mathbf{x}_2(t)$ are quadratically stable. Then, [1.34] is quadratically stable.

Proof of Lemma 1. The quadratic stability of $\dot{\mathbf{x}}_1(t) = \mathbf{A}_{11}\mathbf{x}_1(t)$ implies the existence of a symmetric matrix $\mathbf{P}_1 \succ 0$ such that:

$$\mathbf{P}_1\mathbf{A}_{11} + \mathbf{A}_{11}^T\mathbf{P}_1 \prec 0 \qquad [1.35]$$

Similarly, the quadratic stability of $\dot{\mathbf{x}}_2(t) = \mathbf{A}_{22}\mathbf{x}_2(t)$ implies the existence of a symmetric matrix $\mathbf{P}_2 \succ 0$ such that:

$$\mathbf{P}_2\mathbf{A}_{22} + \mathbf{A}_{22}^T\mathbf{P}_2 \prec 0 \qquad [1.36]$$

In the following, it is proved that there exists $\varepsilon > 0$ such that:

$$\mathbf{P} = \begin{bmatrix} \varepsilon\mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{bmatrix} \succ 0 \qquad [1.37]$$

satisfies [1.33] with $\mathbf{A}$ as in [1.34]. In fact, taking into account [1.34], [1.33] and [1.37] lead to:

$$\begin{bmatrix} \varepsilon\left(\mathbf{P}_1\mathbf{A}_{11} + \mathbf{A}_{11}^T\mathbf{P}_1\right) & \varepsilon\mathbf{P}_1\mathbf{A}_{12} \\ \varepsilon\mathbf{A}_{12}^T\mathbf{P}_1 & \mathbf{P}_2\mathbf{A}_{22} + \mathbf{A}_{22}^T\mathbf{P}_2 \end{bmatrix} \prec 0 \qquad [1.38]$$

that, using Schur complements (Duan and Yu 2013), and by defining $\gamma = 1/\varepsilon$ is equivalent to:

$$\gamma\left(-\mathbf{P}_1\mathbf{A}_{11} - \mathbf{A}_{11}^T\mathbf{P}_1\right) + \mathbf{P}_1\mathbf{A}_{12}\left(\mathbf{P}_2\mathbf{A}_{22} + \mathbf{A}_{22}^T\mathbf{P}_2\right)^{-1}\mathbf{A}_{12}^T\mathbf{P}_1 \prec 0 \quad [1.39]$$

Since $-\mathbf{P}_1\mathbf{A}_{11} - \mathbf{A}_{11}^T\mathbf{P}_1 \succ 0$ according to [1.35], the proof is completed by showing that, given $\mathbf{Z} \succ 0$ and a matrix $\mathbf{W}$ with the same dimensions, there exists $\gamma > 0$ such that $\gamma\mathbf{Z} - \mathbf{W} \succ 0$. In fact, the matrix $\mathbf{Z}$ has some minimum singular value σ_Z such that $\sigma_Z > 0$, and $\mathbf{W}$ has some maximum singular value σ_W. Also, for any non-zero vector v:

$$\mathbf{v}^T\mathbf{Z}\mathbf{v} \geq \sigma_Z\|\mathbf{v}\|^2 \quad [1.40]$$

$$\mathbf{v}^T\mathbf{W}\mathbf{v} \leq \sigma_W\|\mathbf{v}\|^2 \quad [1.41]$$

So $\mathbf{v}^T\left(\gamma\mathbf{Z} - \mathbf{W}\right)\mathbf{v} \geq (\gamma\sigma_Z - \sigma_W)\|\mathbf{v}\|^2$ and $(\gamma\sigma_Z - \sigma_W)\|\mathbf{v}\|^2 > 0$ whenever $\gamma\sigma_Z > \sigma_W$, which shows the existence of $\gamma > 0$ such that $\gamma\mathbf{Z} - \mathbf{W} \succ 0$, thus completing the proof. □

A consequence of Lemma 1 is that, under the assumption that the subsystems:

$$\dot{\mathbf{x}}_K(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\,\mathbf{x}_K(t) \quad [1.42]$$

$$\dot{\mathbf{x}}_L(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\,\mathbf{x}_L(t) \quad [1.43]$$

are quadratically stable, Problems 3-4 are solved by choosing the gains $\mathbf{M}$ and $\mathbf{N}$ so that:

$$\dot{\mathbf{x}}_M(t) = \left(\mathbf{A} - \mathbf{M}\hat{\mathbf{C}}^*\right)\mathbf{x}_M(t) \quad [1.44]$$

or:

$$\dot{\mathbf{x}}_N(t) = \left(\mathbf{A} - \hat{\mathbf{B}}^*\mathbf{N}\right)\mathbf{x}_N(t) \quad [1.45]$$

are quadratically stable, respectively.

In the first case, the quadratic stability condition from Definition 1 involves finding a matrix $\mathbf{P} \succ 0$ such that:

$$\mathbf{P}\left(\mathbf{A} - \mathbf{M}\hat{\mathbf{C}}^*\right) + \left(\mathbf{A} - \mathbf{M}\hat{\mathbf{C}}^*\right)^T\mathbf{P} \prec 0 \quad [1.46]$$

which is a bilinear matrix inequality (BMI) due to the product between the decision variables $\mathbf{P}$ and $\mathbf{M}$. However, the change of variables $\mathbf{PM} = \mathbf{W}$ converts [1.46] into the LMI:

$$\mathbf{PA} - \mathbf{W}\hat{\mathbf{C}}^* + \left(\mathbf{PA} - \mathbf{W}\hat{\mathbf{C}}^*\right)^T \prec 0 \qquad [1.47]$$

which can be solved efficiently, so that the virtual sensor gain is computed a posteriori as $\mathbf{M} = \mathbf{P}^{-1}\mathbf{W}$.

On the other hand, the quadratic stability for the system [1.45] corresponds to finding a matrix $\mathbf{P} \succ 0$ such that:

$$\mathbf{P}\left(\mathbf{A} - \hat{\mathbf{B}}^*\mathbf{N}\right) + \left(\mathbf{A} - \hat{\mathbf{B}}^*\mathbf{N}\right)^T \mathbf{P} \prec 0 \qquad [1.48]$$

which is also a BMI that can be converted into an LMI by pre- and post-multiplying it by $\mathbf{Q} = \mathbf{P}^{-1}$ (congruence transformations do not change the signature of a quadratic form, a.k.a. Sylvester's law of inertia) so that we can write:

$$\mathbf{AQ} - \hat{\mathbf{B}}^*\mathbf{H} + \left(\mathbf{AQ} - \hat{\mathbf{B}}^*\mathbf{H}\right)^T \prec 0 \qquad [1.49]$$

with $\mathbf{H} = \mathbf{NQ}$ so that $\mathbf{N}$ can be recovered a posteriori from $\mathbf{H}$ as $\mathbf{N} = \mathbf{HQ}^{-1}$.

1.5. Additional considerations

In Section 1.3, a separate (and simpler) formulation of virtual sensors/virtual actuators has been presented. In Rotondo *et al.* (2014*b*), it was shown that a joint formulation could be provided by modifying adequately the virtual sensor [1.16], feeding it with:

$$\mathbf{y}_{va}^f(t) = \mathbf{y}(t) + \hat{\mathbf{C}}_f \mathbf{x}_{va}(t) \qquad [1.50]$$

instead of $\mathbf{y}(t)$:

$$\begin{cases} \dot{\mathbf{x}}_{vs}(t) = \left(\mathbf{A} - \mathbf{M}\hat{\mathbf{C}}^*\right)\mathbf{x}_{vs}(t) + \mathbf{Bu}(t) + \mathbf{MC}\hat{\mathbf{C}}_f^\dagger\left(\mathbf{y}_{va}^f(t) - \hat{\mathbf{f}}_y(t)\right) \\ \mathbf{y}_{vs}(t) = \mathbf{C}\hat{\mathbf{C}}_f^\dagger\left(\mathbf{y}_{va}^f(t) - \hat{\mathbf{f}}_y(t)\right) + \left(\mathbf{C} - \hat{\mathbf{C}}^*\right)\mathbf{x}_{vs}(t) \end{cases} \qquad [1.51]$$

Then, under the assumption that $\hat{\mathbf{B}}_f \rightarrow \mathbf{B}_f$, $\hat{\mathbf{C}}_f \rightarrow \mathbf{C}_f$, $\hat{\mathbf{f}}_u \rightarrow \hat{\mathbf{f}}_u$ and $\hat{\mathbf{f}}_y \rightarrow \hat{\mathbf{f}}_y$, the overall augmented system obtained as the interconnection of the observer's output equation $\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$, the control law $\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$, the faulty output equation [1.3], the faulty state equation [1.12], the state observer's equation [1.7], the virtual actuator [1.25] and the virtual sensor [1.50] fed by [1.51], is described by the following autonomous system:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \\ \dot{\mathbf{x}}_{va}(t) \\ \dot{\mathbf{x}}_{vs}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}^*\mathbf{K} & \mathbf{B}^*\mathbf{N} & \mathbf{O} \\ \mathbf{LC}^* & \mathbf{A}-\mathbf{BK}-\mathbf{LC} & \mathbf{LC}^* & \mathbf{L}\left(\mathbf{C}-\mathbf{C}^*\right) \\ \mathbf{O} & \left(\mathbf{B}^*-\mathbf{B}\right)\mathbf{K} & \mathbf{A}-\mathbf{B}^*\mathbf{N} & \mathbf{O} \\ \mathbf{MC}^* & -\mathbf{BK} & \mathbf{MC}^* & \mathbf{A}-\mathbf{MC}^* \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \\ \mathbf{x}_{va}(t) \\ \mathbf{x}_{vs}(t) \end{bmatrix} \quad [1.52]$$

which, by means of the following change of variables:

$$\mathbf{z}(t) = \begin{bmatrix} -\mathbf{I} & \mathbf{O} & -\mathbf{I} & \mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} & -\mathbf{I} \\ \mathbf{I} & \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \\ \mathbf{x}_{va}(t) \\ \mathbf{x}_{vs}(t) \end{bmatrix} \quad [1.53]$$

can be shown to be similar to:

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} \mathbf{A}-\mathbf{MC}^* & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \left(\mathbf{M}-\mathbf{L}\right)\mathbf{C}^* & \mathbf{A}-\mathbf{LC} & \mathbf{O} & \mathbf{O} \\ -\mathbf{BK} & -\mathbf{BK} & \mathbf{A}-\mathbf{BK} & \mathbf{O} \\ \left(\mathbf{B}^*-\mathbf{B}\right)\mathbf{K} & \left(\mathbf{B}^*-\mathbf{B}\right)\mathbf{K} & \left(\mathbf{B}^*-\mathbf{B}\right)\mathbf{K} & \mathbf{A}-\mathbf{B}^*\mathbf{N} \end{bmatrix} \mathbf{z}(t) \quad [1.54]$$

The block-triangularity of the augmented state matrix in [1.54] justifies the separate design of the virtual sensor and virtual actuator gains $\mathbf{M}$ and $\mathbf{N}$, respectively, following the discussion provided in the previous sections.

It should be recalled that although the asymptotical stability of $\mathbf{A}-\mathbf{BK}$ and $\mathbf{A}-\mathbf{LC}$ is sufficient to guarantee the asymptotical stability of an observer-based control system, special care should be put in the relative position of the poles of $\mathbf{A}-\mathbf{LC}$ when compared to those of $\mathbf{A}-\mathbf{BK}$.

Similarly, when one undergoes the task of designing suitable virtual sensor and virtual actuator gains $\mathbf{M}$ and $\mathbf{N}$, the existence of a hierarchy between the different poles should be taken into account. In fact, in order to minimize the performance degradation brought by the introduction of the virtual components in the loop, the poles of $\mathbf{A}-\mathbf{MC}^*$ and $\mathbf{A}-\mathbf{B}^*\mathbf{N}$ should be faster than those of $\mathbf{A}-\mathbf{BK}$. However, due to the reduced number of available sensors/actuators under fault occurrence, in practical situations some performance degradation should be accepted since the following relationship corresponds to more realistic requirements:

$$\begin{aligned} &\mathrm{Re}\left(\mathrm{eig^d}\left(\mathbf{A}-\mathbf{LC}\right)\right) < \mathrm{Re}\left(\mathrm{eig^d}\left(\mathbf{A}-\mathbf{MC}^*\right)\right) < \cdots \\ &\qquad \cdots \mathrm{Re}\left(\mathrm{eig^d}\left(\mathbf{A}-\mathbf{BK}\right)\right) < \mathrm{Re}\left(\mathrm{eig^d}\left(\mathbf{A}-\mathbf{B}^*\mathbf{N}\right)\right) \end{aligned} \quad [1.55]$$

where $\mathrm{Re}\left(\mathrm{eig}^{\mathrm{d}}(\mathbf{X})\right)$ denotes the real part of the dominant eigenvalue of the matrix $\mathbf{X}$. An LMI-based design procedure that takes into account specifications on the pole location can be obtained by modifying adequately the approach described in Section 1.5 by means of the concept of $\mathcal{D}$-stability. The interested reader is referred to Chilali and Gahinet (1996).

As mentioned at the beginning of Section 1.5, the main advantage of the LMI-based design is the fact that it enables the extension of the virtual sensor/actuator technique to LPV systems. In this case, we assume that the state-space matrices are not constant but are functions of a vector of varying parameter $\theta(t) \in \Theta$, with Θ compact set, which can be either measured or estimated online. For design purposes, a common assumption made on these functions is that they can be expressed as a convex combination of constant *vertex matrices* (the reader is referred to Rotondo, Sanchez, Nejjari and Puig (2019) for further information):

$$\begin{pmatrix} \mathbf{A}(\theta(t)) \\ \mathbf{B}(\theta(t)) \\ \mathbf{B}^*(\theta(t)) \\ \mathbf{C}(\theta(t)) \\ \mathbf{C}^*(\theta(t)) \end{pmatrix} = \sum_{i=1}^{N} \alpha_i(\theta(t)) \begin{pmatrix} \mathbf{A}_i \\ \mathbf{B}_i \\ \mathbf{B}_i^* \\ \mathbf{C}_i \\ \mathbf{C}_i^* \end{pmatrix} \quad [1.56]$$

with:

$$\sum_{i=1}^{N} \alpha_i(\theta) = 1, \quad \alpha_i(\theta) \geq 0 \quad \forall \theta \in \Theta \quad [1.57]$$

Then, the virtual sensor and virtual actuator structures in [1.16] and [1.25] are modified by allowing the gains to be functions of $\theta(t)$, as follows:

$$\begin{pmatrix} \mathbf{M}(\theta(t)) \\ \mathbf{N}(\theta(t)) \end{pmatrix} = \sum_{i=1}^{N} \alpha_i(\theta(t)) \begin{pmatrix} \mathbf{M}_i \\ \mathbf{N}_i \end{pmatrix} \quad [1.58]$$

and the design LMIs [1.47] and [1.49] are replaced by their parameter-varying versions:

$$\mathbf{PA}(\theta) - \mathbf{W}(\theta)\hat{\mathbf{C}}^*(\theta) + \left(\mathbf{PA}(\theta) - \mathbf{W}(\theta)\hat{\mathbf{C}}^*(\theta)\right)^T \prec 0, \quad \forall \theta \in \Theta \quad [1.59]$$

$$\mathbf{A}(\theta)\mathbf{Q} - \hat{\mathbf{B}}^*(\theta)\mathbf{H}(\theta) + \left(\mathbf{A}(\theta)\mathbf{Q} - \hat{\mathbf{B}}^*(\theta)\mathbf{H}(\theta)\right)^T \prec 0, \quad \forall \theta \in \Theta \quad [1.60]$$

which can be converted into a finite set of LMIs by exploiting the above assumption, i.e. Eqs. [1.56]-[1.58], following the approach described in Sala and Arino (2007).

Notably, in the particular case in which the input and output matrices are constant, i.e. only $\mathbf{A}(\cdot)$ is a function of θ, [1.59]-[1.60] reduce to:

$$\mathbf{P}\mathbf{A}_i - \mathbf{W}_i\hat{\mathbf{C}}^* + \left(\mathbf{P}\mathbf{A}_i - \mathbf{W}_i\hat{\mathbf{C}}^*\right)^T \prec 0, \qquad \forall i = 1, \ldots, N \qquad [1.61]$$

$$\mathbf{A}_i\mathbf{Q} - \hat{\mathbf{B}}^*\mathbf{H}_i + \left(\mathbf{A}_i\mathbf{Q} - \hat{\mathbf{B}}^*\mathbf{H}_i\right)^T \prec 0, \qquad \forall i = 1, \ldots, N \qquad [1.62]$$

from which, once solved, one can recover the vertex virtual sensor and virtual actuator gains as $\mathbf{M}_i = \mathbf{P}^{-1}\mathbf{W}_i$ and $\mathbf{N}_i = \mathbf{H}_i\mathbf{Q}^{-1}$, respectively.

.

1.6. Application example

To illustrate the use of virtual sensors and actuators, a well-known case study based on a modification of the well-known four tank system proposed by Johansson (2000) will be used (see Figure 1.1). But, the example has been modified to include an extra pump and valve that send water to tank 3 and 4, respectively. This modification has been introduced to add actuator redundancy that allow illustrating the virtual actuator approach.

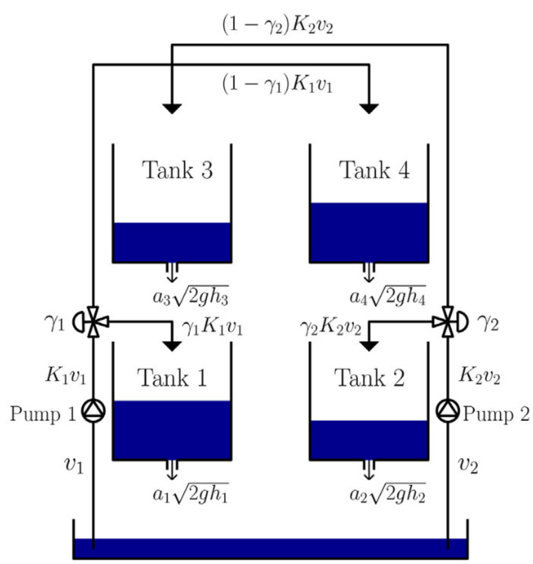

Figure 1.1. *The quadruple-tank system.*

The linearised model of this system around the operating point given by $u^0 = [3 \quad 3]^T$ and $x^0 = [12.4 \quad 12.7 \quad 1.8 \quad 1.4]^T$ is the following:

$$\dot{x} = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} & 0 \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} & \frac{\gamma_3 k_3}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 & \frac{(1-\gamma_3)k_3}{A_4} \end{bmatrix} u \qquad [1.63]$$
$$y = \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \\ 0 & 0 & k_c & 0 \\ 0 & 0 & 0 & k_c \end{bmatrix}$$

where A_i are the cross sections of the tanks, T_i are the constant times, k_i are static gains and γ_i is the degree of valve opening and k_c are the sensor gains. The values of this parameters can be found in the Table 1.1.

Table 1.1. *Model parameters.*

Parameter value	Unit
$A_1 = A_3 = 28$	cm^2
$A_2 = A_4 = 32$	cm^2
$T_1 = 62.7034, T_3 = 23.8900$	s
$T_2 = 90.3353, T_4 = 29.9930$	s
$k_1 = 3.33, k_2 = 3.35, k_3 = 3.34$	cm^3/Vs
$\gamma_1 = 0.7, \gamma_2 = 0.6, \gamma_3 = 0.65$	
$k_c = 0.5$	V/cm

To control the system, a state-feedback controller is used considering that all the states (levels) are measured. The controller has been designed using LMIs to stabilise the system around the operating point used for linearisation, being equal to

$$\mathbf{K} = \begin{bmatrix} 272.4147 & -242.6840 & 354.2708 & -697.7773 \\ 59.3260 & -54.6411 & 103.7366 & -187.5428 \\ 72.6110 & -110.9882 & 116.9174 & -249.8481 \end{bmatrix} \qquad [1.64]$$

1.6.1. *Virtual actuator*

A virtual actuator based on the LMI procedure presented in Section 1.4 is designed for the case of a fault affecting the third actuator (the additional pump added in the four-tank case study). Considering that the remaining to two actuators (pumps) are healthy, the system still is controllable. But, the rank condition [1.23] is not satisfied. As discussed, this means that just with an static virtual actuator block is not enough to mask the actuator fault, but instead a dynamic virtual actuator with structure [1.25] is required. By solving the LMI [1.49], the virtual actuator matrix is obtained

$$\mathbf{N} = \begin{bmatrix} 5.1552 & 1.1071 & 0.8158 & 1.6584 \\ -1.1474 & 5.3208 & 2.9609 & -0.2542 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [1.65]$$

Figure 1.2 presents the closed-loop response of the system when the state feedback controller [1.64] and the virtual actuator [1.25] with gain [1.65] is applied when a fault in the third actuator appears at time 10s. For comparison, the closed-loop results in the nominal case and in the faulty case without virtual actuator are also presented. The virtual actuator is activated at 15s. It can be seen that thanks to the inclusion of the virtual actuator after the fault, the behaviour of the closed-loop system recovers the nominal behaviour. However, in the case of not using the virtual actuator the closed-loop system is not able to recover the nominal behaviour.

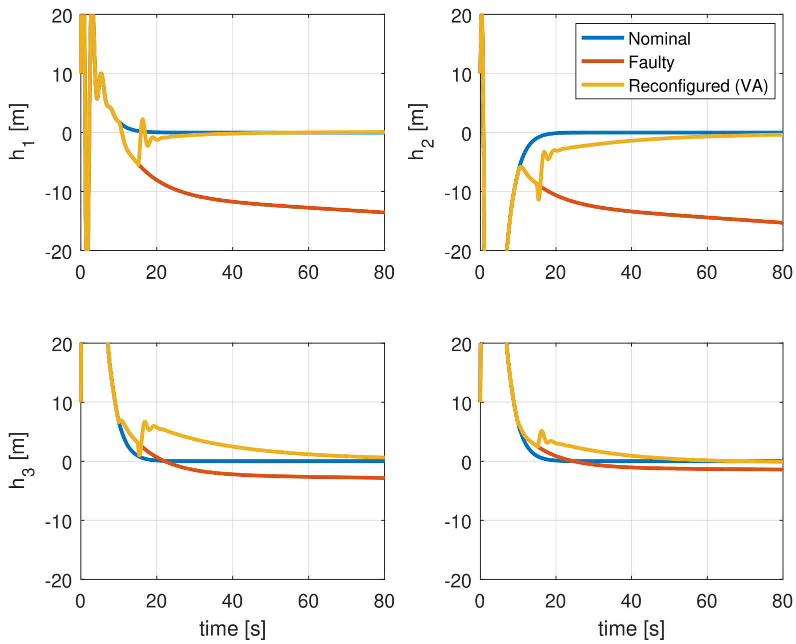

Figure 1.2. *Virtual actuator results.*

1.6.2. *Virtual sensors*

A virtual sensor based on the LMI procedure presented in Section 1.4 is designed for the case of a fault affecting the sensor that mesaures the level of the second tank. Considering that the remaining three sensors are healthy, the system still is controllable. But, the rank condition is not satisfied. As discussed, this means that just with an static virtual sensor block is not enough to mask the sensor fault, but instead a dynamic virtual sensor with structure [1.16] is required. By solving the LMI [1.47], the virtual sensor matrix is obtained

$$\mathbf{M} = \begin{bmatrix} 0.4841 & 0 & 0 \\ 0 & 0 & 0.0333 \\ 0.0419 & 0.4581 & 0 \\ 0 & 0 & 0.4667 \end{bmatrix} \qquad [1.66]$$

Figure 1.3 presents the closed-loop response of the system when the state feedback controller [1.64] and the virtual sensor [1.16] with gain [1.66] is applied when a fault in the second sensor appears at time 10s. For comparison, the closed-loop results in the nominal case and in the faulty case without virtual sensor are also presented. The virtual sensor is activated at 15s. It can be seen that thanks to the inclusion of the virtual sensor after the fault, the behaviour of the closed-loop system recovers the nominal behaviour. However, in the case of not using the virtual sensor the closed-loop system is not able to recover the nominal behaviour.

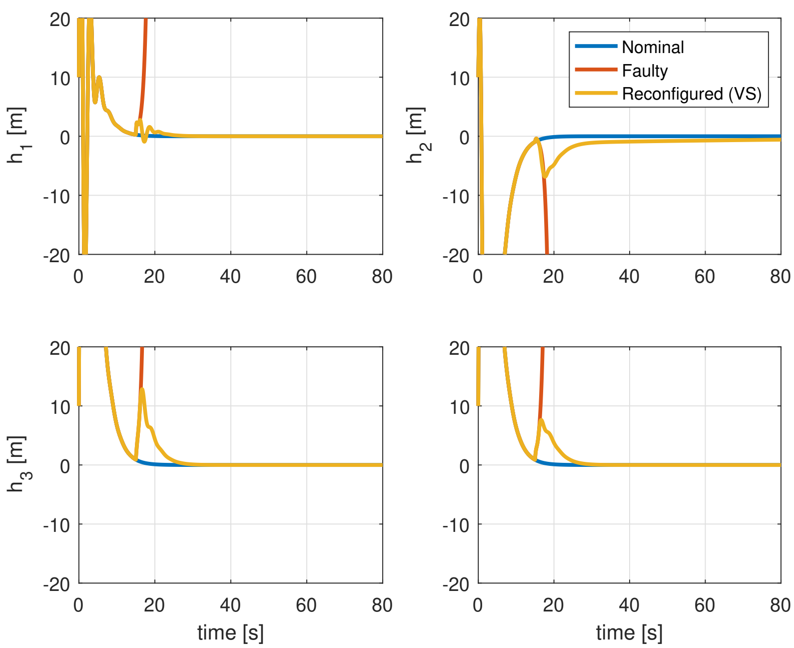

Figure 1.3. *Virtual sensors results.*

1.7. Conclusions

This chapter has presented the approach fo FTC based on the principle of fault masking introducing two well-known schemes to mask sensors faults (virtual sensor) and actuator faults (virtual actuators). Both schemes relies on the principle of hiding the fault to the controller by activating some fault masking block that prevents the controller to be redesigned. Fault masking approach to FTC has the advantage with respect to the classical approach based on redesigning the controller after the fault, that follows a plug and play approach. The fault hiding block can be included in the loop with needing to redesign the controller. Virtual actuators and sensors can be easily extended to non-linear systems via the LPV and LMI frameworks.

Currently, the research on fault masking blocks that generalizes the concept of virtual sensors and actuators is still active using the concept of reconfiguration blocks (RB), see as e.g. Bessa *et al*. (2020). RB can be generically used as virtual actuators o sensors to deal either with sensor or actuator faults.

Finally, it is worth mentioning that the recent work (Rotondo, Sánchez, Puig, Escobet and Quevedo 2019) has extended the virtual actuator technique to deal with denial-of-service (DoS) attacks. In contrast to the case of faults, when DoS attacks are considered, it is assumed that a malicious attacker has a limited energy to affect the system's actuators, causing them to operate incorrectly and potentially destabilizing the control system. In this case, the virtual actuator design can be modified to take into account the known constraints on the attacker's available energy, generally leading to milder conditions on the closed-loop virtual actuator matrix $\mathbf{A} - \mathbf{B}^*\mathbf{N}$. For instance, this matrix can be allowed to be unstable as long as the instability during the DoS attack is sufficiently compensated by the behavior of the asymptotically stable closed-loop system when the system is not under attack. The interested reader is referred to Rotondo, Sánchez, Puig, Escobet and Quevedo (2019) for more details

1.8. Bibliography

Ackermann, J. (1972), Der entwurf linearer regelungssysteme im zustandsraum, *at-Automatisierungstechnik*, 20(1-12), 297–300.

Bessa, I., Puig, V., Palhares, R. M. (2020), Ts fuzzy reconfiguration blocks for fault tolerant control of nonlinear systems, *Journal of the Franklin Institute*, 357(8), 4592 – 4623.

Blanke, M., Kinnaert, M., Lunze, J., Staroswiecki (2016), *Diagnosis and Fault-Tolerant Control*, 3rd edition, Springer Verlag.

Chilali, M., Gahinet, P. (1996), h_∞ design with pole placement constraints: an LMI approach, *IEEE Transactions on automatic control*, 41(3), 358–367.

Duan, G.-R., Yu, H.-H. (2013), *LMIs in control systems: analysis, design and applications*, CRC press.

J.H.Richter, Lunze, J. (2010), Reconfigurable control of hammerstein systems after actuator failures: stability, tracking, and performance, *International Journal of Control*, 83(8), 1612–1630.

Johansson, K. H. (2000), The quadruple-tank process: a multivariable laboratory process with an adjustable zero, *IEEE Transactions on Control Systems Technology*, 8(3), 456–465.

López-Estrada, F.-R., Rotondo, D., Valencia-Palomo, G. (2019), A review of convex approaches for control, observation and safety of linear parameter varying and takagi-sugeno systems, *Processes*, 7(11), 814.

Pedersen, A. S., Richter, J. H., Tabatabaeipour, M., Johannsson, H., Blanke, M. (2016), Fault tolerant emergency control to preserve power system stability, *Control Engineering Practice*, 53, 151 – 159.

Richter, J., Heemels, W., van de Wouw, N., Lunze, J. (2011), Reconfigurable control of piecewise affine systems with actuator and sensor faults: Stability and tracking, *Automatica*, 47(4), 678 – 691.

Rotondo, D., Nejjari, F., Puig, V. (2014*a*), A virtual actuator and sensor approach for fault tolerant control of lpv systems, *Journal of Process Control*, 24(3), 203 – 222.

Rotondo, D., Nejjari, F., Puig, V. (2014*b*), A virtual actuator and sensor approach for fault tolerant control of lpv systems, *Journal of Process Control*, 24(3), 203–222.

Rotondo, D., Sanchez, H. S., Nejjari, F., Puig, V. (2019), Analysis and design of linear parameter varying systems using lmis, *Rev. Iberoam. Autom. Inform. Ind*, 16, 1–14.

Rotondo, D., Sánchez, H. S., Puig, V., Escobet, T., Quevedo, J. (2019), A virtual actuator approach for the secure control of networked lpv systems under pulse-width modulated dos attacks, *Neurocomputing*, 365, 21–30.

Sala, A., Arino, C. (2007), Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of polya's theorem, *Fuzzy sets and systems*, 158(24), 2671–2686.

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