FACULTY OF SCIENCE AND TECHNOLOGY

MASTER'S THESIS

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Author: Hanne Posti Finnestad	
Supervisor at UiS: David Ploog External supervisor(s):	
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Introduction

Combinatorics is a branch of mathematics that is not explicitly in the curriculum for 1st to 10th grade in Norwegian schools. This doesn't mean that combinatorial tasks are not used in Norwegian classrooms, just that it depends on the teacher and the textbook used by the school. There are several fun combinatorial tasks that circulate on the internet, like "what animals have 12 legs combined?" and others, easily found by for example exploring the combinatorics tag on mattelist.no.

Although combinatorics isn't in the curriculum, using combinatorial tasks can be a way to incorporate several skills at once. I am interested in exploring this with my 1st graders. Though it might seem a bit complicated to do this with 1st graders, one must remember that they are the ones who get the most out of the counting portion of the task. For higher levels, you'd like to avoid counting manually, and therefore you'd search for a formula that does the job for you. For very young students, counting is a skill that is developing, which makes this sort of tasks ideal for them. The basic skills like counting, sorting, adding and subtracting that we use when solving combinatorial tasks, these first steps, aren't considered to be "mathematical" enough to stop there, we usually go further. For young students, there is no need to go much further, as they aren't ready to comprehend it just yet. Using combinatorial tasks with this age group is beneficial because they get to practice the basic skills that is on the curriculum, and they are introduced to tasks that have more than one answer, which will eventually boost their problem-solving skills. I am a 1st grade teacher currently and the students are constantly surprising me with how much they are actually capable of, given the right framework.

Combinatorics

Combinatorics is defined as the study of "the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties" (Weisstein, n.d.). Combinatorial tasks will always have something to be combined in a certain way, depending on a few factors, and the solution has several degrees of difficulty, which makes them perfect for differentiation. What is meant by this is that, if we have combinatorial task that involves combining clothing tops and bottoms, we can simply count the combinations and stop there, or we could go much further. As a teacher, there are many directions you can lead students in. If you were to give this task and tell them to combine 3 tops with 3 bottoms, everyone would be able to arrive to the first stop, which is that there 9 ways to combine them. We can use a tree diagram to easily see this, like in figure 1.

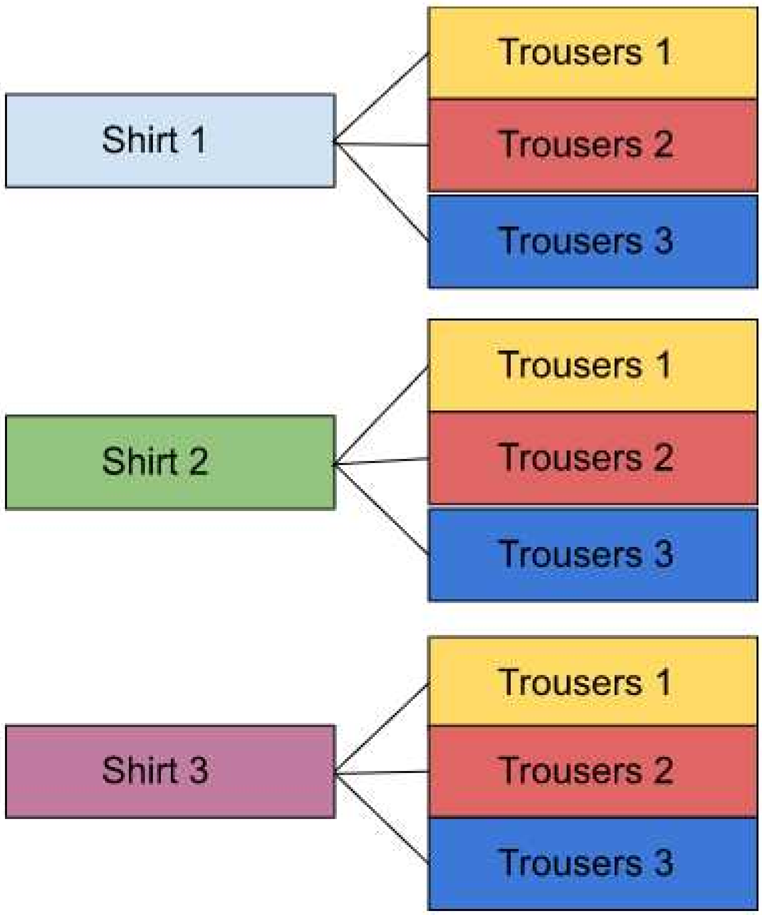


Figure 1

Now, to differentiate, you can alter the task a bit by changing the number of each item, introduce another item etc., and some students, depending on their age and ability, can start to generalise and think of a formula. If they arrive at a formula that works, you can alter the task again and ask how it affects the formula.

Different types of combinations

There are 4 types of combinations of objects, firstly determined by whether the order of object matters or not. If the order does not matter, it is just called a combination, while an ordered combination is called a permutation. A combination of objects could be differently coloured balls in some container, vegetables in a salad bowl, etc, where the placement of these objects doesn't matter. A permutation of objects could be a phone number, a license plate number, a password, etc, where the order is essential. Next, we must determine whether in our combination or permutation we allow repetitions or not. If we are talking about a license plate number, we would consider this a permutation where we allow repetition, since for example RR 55555 could exist. If we are talking about some sort of competition like a 5k race, we would not allow repetition because the same person can't finish 1st *and* 2nd place. An example of a combination where we would not allow repetition could be some sort of name draw, since each name would only be put in once and the number of names to draw would decrease each time you draw. For the last type of combination, a combination where we do allow repetition, we could consider randomly drawing coins from a wallet, since these could all have the same value.

These 4 types of combinations have different formulas, as seen in table 1.

	Repetition allowed	**Repetition not allowed**
Order matters	n^k [1]	$n^{\underline{k}}$ [2]
Order doesn't matter	$\frac{n^{\overline{k}}}{k!} = \left(\!\binom{n}{k}\!\right)$ [3]	$\frac{n^{\underline{k}}}{k!} = \binom{n}{k}$ [4]

Table 1

The variables n and k can be explained in a few different ways, you could say that n is the number of options for a certain choice, and k is the number of times you make that choice. When using the formulas in table 1, and especially when teaching students how to use these formulas, we must be able to dissect the problem and find out how many choices we have, how many times are we making that choice, is repetition allowed or not, does order matter etc. You may encounter a problem that will need a variation of these formulas or maybe a combination of them.

Dissecting the formulas

The formulas in table 1 are more intuitive than they might seem, and I think a lot of students that have a grasp on basic concepts in mathematics would be able to find them on their own, perhaps with some guidance. Below is a basic explanation of each formula, and how some of them could be invented on the spot.

With formula [1], we have n options and choose k times. We allow repetition, which means the n stays the same no matter how many times we choose something. A 4-digit code could be an example of such a permutation with repetition. 4432 is an entirely different code to 2344. In our example, k is the number of digits, so k=4, and n is the number of options for these digits, aka the numbers from 0-9; n=10. For the first digit, we have 10 options. For the second digit, we have 10 options for each of the 10 numbers digit 1 could be, so there are a 100 different combinations for these 2. For the 3rd digit, we have 10 options for each of the 100 combinations for 1 and 2, which means there are 1000 combinations for 3 digits, and finally 10 000 options for a 4-digit code. From here it's easy to see why the formula is n to the power of k, because you multipy n with itself k times.

Looking at formula [2], we encounter a formula that looks very similar to formula [1], but has what is called a falling factorial, denoted $n^{\underline{k}} = \frac{x!}{(x-n)!}$ (Graham, 1994, preface: p. x). This means that we get a polynomial equation with k terms like this: $n^{\underline{k}} = n(n-1)(n-2)(n-3)(n-4)$ ... Though one might not be familiar with the falling factorial notation, most would be able to see that we get n multiplied with itself k times, only with n decreasing with 1 for each time. If we have 50 contestants racing a 5k, there are 50 options for the 1st position, but now this person can't occupy any other positions, so we get 50-1=49 options for the 2nd position, and so on.

Skipping ahead a bit, in formula [4], order doesn't matter and we do not allow repetition, which means we get 1 less option for each choice like in formula [2] which also looks similar, but since order doesn't matter we divide this number by the number of times we make our choice. As we also see in table 1, formula [4] can also be written as n choose k, because it tells us the number of ways to choose k objects from a set of n objects. Now, formula [3] looks quite similar to formula [4], the difference is that instead of a falling factorial, [3] has a rising factorial, denoted $n^{\overline{k}} = \frac{\Gamma(x+n)}{\Gamma(x)}$ (Graham, 1994, preface: p. x). A rising factorial gives us a polynomial equation with k terms like this: $n^{\overline{k}} = n(n+1)(n+2)(n+3)(n+4)$ ... It's easy to remember that this formula looks like [3], only with a rising factorial instead of a falling one. For future reference, this formula can also be written like $\binom{n+k-1}{n}$.

It could also be visualised by considering a set like $\{a, b, c, d\}$. If we choose 2 from this set of 4 (k=2, n= 4), the 4 different types of combinations would look like table [2].

	Repetition allowed	Repetition not allowed
Order matters	aa – ab – ac – ad ba – bb – bc – bd ca – cb – cc – cd da – db – dc – dd	– ab – ac – ad ba – – bc – bd ca – cb – – cd da – db – dc –
Order doesn't matter	aa – ab – ac – ad – bb – bc – bd – cc – cd – dd	– ab – ac – ad – bc – bd – cd

Table 2

Combining coins

Learning to add and subtract using coins is a very important skill that 1st graders start learning quickly after starting school. Introducing coins that have different values is confusing at first, since counting has been pretty heavily on the agenda every math class so far. Suddenly having 3 coins doesn't necessarily mean you will have 3 kr. As with most things, this is obvious when the teacher explains it, but is harder in praxis. The students know that a 10 kr coin is worth more than a 1 kr coin, but it is usually harder to visualise this when you are presented with only one coin, and especially so if you need to find the sum of different coins combined.

The book Concrete Mathematics introduces a combinatorics problem that uses coins, "How many ways are there to pay 50 cents?" (Graham et al., 1994, p. 327), using generating functions to solve it. They find that there are 50 ways to make 50 cents when you have pennies, nickels, dimes, quarters and half-dollars. The interesting part is not the counting of combinations, this can easily be found if you have enough time on your hands to count manually, but that they used generating functions to solve it and prove it at the same time. I am interested in a similar problem, more geared towards our learning goal for this semester.

It is easier to instruct students to find out what different values you can make using 4 coins, where they would only need to count and add the values together and then count how many different values they get, than it is to expect them to master pre-algebra. It would be good practice for pre-algebraic skills, but not something that I would expect most of them to know. However, this is a great idea if some students need advancement.

If I ask my students to find out what values you can possibly make when you draw 4 random coins, and the available coins are 1 kr, 5 kr, and 10 kr, I would like to know how many combinations there are.

Counting

As with any problem, one would usually start with a small example and count the combinations manually. We do this to be able to check that we are using the correct formula. A strategic approach is obviously the best approach, this way you can be sure you haven't missed any combinations. If we have 3 different coins and we draw 3, there are 10 possible combinations, and with 3 different coins and 4 draws, there are 15 possible combinations. Now, since this is a combinatorics problem, we can categorize the problem to find a formula that can help us. In our problem, order doesn't matter because we only care about the combined value, and we do allow repetitions because we want to include the combinations where all coins are the same type of coin. This means that formula [3] should be fitting. When we input our n and k, we find that this formula gives us the correct answer.

Generating functions

It's easy to use the formulas to find the number of combinations or permutations, but what if we didn't have these formulas? Even adding one more draw or option to our problem would complicate the manual counting process. As mentioned, they used generating functions to solve and prove the coin problem in Concrete Mathematics, which I will utilize in this problem as well. Generating functions are a way to deal with number sequences via manipulation of infinite series that "generate" the sequences (Graham, 1994, p. 320). Number sequences can be described using a closed form function, or by recurrence relations, but this is not always possible or they don't provide the information we need. What we do is consider the number sequence as the coefficients in a power series f(x), where f(x) is the generating function of the sequence.

We are interested in them because finding the generating function means finding the corresponding coefficient, which will tell us the number of times some combination occurs. For all the combinatorial tasks we have mentioned so far, we could use generating functions to solve them, and they would yield the formulas in table [1] or some variation.

The sum of all the ways one can pay some amount with only 1 kr coins is

$$E(z) = z + z^2 + z^3 + z^4 + \cdots$$

And when we include 5 kr coins as well, it gives us

$$F(z) = E + 5E + 5^2E + 5^3E + \cdots$$

$$= (5 + 5^2 + 5^3 + \cdots)E$$

Finally (in our case) we include 10 kr coins and get the infinite sum

$$T(z) = (10 + 10^2 + 10^3 + \cdots)F$$

In the book, the goal is to find the number of terms that are worth 50 cents exactly. We want to find the number of terms that uses 4 coins exactly, so either something like z^4 or $z^2 5^2$, where the sum of the exponents is equal to 4. In the original example, they use z^5, z^1, z^{15} etc to signify the value of the coins, but for our goal the value doesn't matter yet, meaning we can refer to them all as z. This means 5^4 can now also be referred to as z^4, and $z^2 5^2 = z^2 z^2 = z^4$. When we have found our generating function, we will find our answer in this corresponding coefficient.

Instead of the coefficients being different because they have different values, they will in our case be the same. So we end up with the geometric series

$$E = \frac{1}{1 - z}$$

$$F = \frac{1}{1 - z}E$$

$$T = \frac{1}{1 - z}F$$

From this we get the closed form of our generating function $\frac{1}{(1-z)^k}$, where k is the number of different coins. In our case we get $\frac{1}{(1-z)^3}$.

Looking at just the closed form of E, we should immediately recognise its Taylor series expansion as $\sum_{n=0}^{\infty} z^n$. When we then need to expand $\frac{1}{(1-z)^3}$, we can use Newton's binomial theorem; $(x + 1)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i$ where r is a real number that is a non-negative integer and -1<x-1 (Guichard, section 3.1). We first swap r for -3, giving us $(x + 1)^{-3}$=$\sum_{i=0}^{\infty} \binom{2+i}{2} (-x)^i$, and then

we switch out x with -z, giving us $(1-z)^{-3}$=$\sum_{i=0}^{\infty}\binom{2+i}{2}(z)^i$. If we instead write $\binom{3+n-1}{3-1}$, it starts to look like formula [3], which can be written on the form $\binom{k+n-1}{n}$ or $\binom{k+n-1}{k-1}$. In fact, this is the coefficient of $\frac{1}{(1-z)^n}$. We get the generating function with the coefficient:

$$\sum_{n\geq 0}\binom{k+n-1}{n}z^n \rightarrow \sum_{n\geq 0}\binom{3+n-1}{n}z^n = 1+\ 3z+\ 6z^2+10z^3+15z^4+21z^5+\cdots$$

Since we want to find the number of terms that uses 4 coins exactly, we could either use the formula for the coefficient or look at the coefficient before z^4 in the terms above. We already know that when k=3 and n=3, there are 10 possible combinations, this is also confirmed by the sequence above.

Execution

"If you have 4 coins in your wallet, but you don't know what type of coins they are (except that they can't be 20 kr), what are the possible values you can have?" is how I phrased the question to my first graders. This is an abstract task at first glance and can seem intimidating. We had previously (that same day) done an example with 3 coins together, so they felt prepared, nonetheless. I gave them a sheet of paper with 2 cells per line, one big enough to place the coins in, and one big enough to write the value. I had them put the coins in the bigger cell, write the combined value, and draw the coins so they wouldn't forget what combinations they had already done. They were also given a plate with 4 coins of each possible value (1 kr, 5 kr, 10 kr). If students were unsure where to start, I told them to close their eyes and draw 4 coins at random. If they got something they had already gotten, they could choose which coins to switch out so that the outcome would be different. I expected them to find different values, such that we hopefully could find all the possible values together. When the lesson neared its end, I asked the students to bring their sheet of paper and asked them to give me some possible values. To incorporate skills other than counting and adding, I had them place the values we found from lowest to highest, discussing this while we did it. "What is the lowest value you can get, and why?", "which is the highest?", "how can we be sure that we have found all the values?" etc. Another interesting pit stop is discussing when all the coins are equal, as this is an early meeting with multiplication. It made for a very engaging lesson, even though they probably don't understand that there is a pattern to how many combinations or permutations there are for a given counting problem. With such a young group, it doesn't make sense to go into the mathematics of how we can know for sure, what is the formula, how can we generalise etc., but they are curious enough to wonder the same things. It isn't long until we can dive into these things too. Combinatorics tasks are excellent for young students, since they are learning a lot of strategies that will come in handy later. They are learning to work strategically, be attentive, add and subtract (with a purpose), and finally how to problem solve using the skills they already know. Also, they are introduced to mathematical tasks that don't have a singular answer, like they are used to. This makes for a more meaningful conversation as a class in the aftermath of the task, since more people can contribute.

Other uses for combinatorics

Combinatorics doesn't have to be limited to math either, you could incorporate it in other subjects to make it interdisciplinary. When working with CVC (consonant-vowel-consonant) words for example, you could ask how many words they can make with a certain number of consonants and vowels. In Norwegian, we don't really focus on CVC words, but it's possible to still do this or to alter it to 1-syllable words (CV) or 2-syllable words (CV-CV). It's a personal choice whether you limit the vowels and consonants so that all the possible words exist in real life or if it's not that important. This could be appropriate even for the upper grades that may be familiar with the 4 formulas, because they need to establish which one to use, what the variables are and if they need to alter it. Order does matter because **rat** and **tar** and two completely different words. Whether we allow repetition or not is a personal choice, allowing it would mean to accept word like **mam** or **did**. Let's say we did this and we had some magnetic letters, say M, L, T, S, O, I, A. So we have 4 consonants and 3 vowels. The vowel has a set place, which means that whatever number of combinations there are for the consonants times the number of available vowels, will equal the final number of combinations of both vowels and consonants. The 4 consonants can occupy 2 slots, and in our case we do not allow repetition because we only have one of each magnetic letter. We look into formula [2] and find that the number of combinations for the consonants is 12. When we include the vowels, we get 3 times 12, which equals 36. For the vowels, we used the same formula, but since k is 1, we just get 3. Had we looked at 2-syllable words (CV-CV), the vowels would also be able to occupy two slots, which would give us 6 combinations for the vowels, 6 times 12 equals 72, 72 combinations for a 2-syllable word with 4 consonants and 3 vowels. If this had been an interdisciplinary task, students could now be asked to sort the words by real words and made-up words.

Polyominoes

Polyominoes are "shapes made by connecting certain numbers of equal-sized squares, each joined together with at least one other square along an edge" (Golomb, 1994, p. 3), and are part of a branch of mathematics called combinatorial geometry. The smallest polyomino is a monomino, consisting of only one square. Since there is only one way to arrange a single square, there is only one monomino. We are dealing with so-called "free" polyominoes, meaning we consider all rotations, translations and their compositions of a shape to be identical.

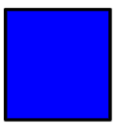


Figure 2

People are more familiar with the different polyominoes than they might think. Dominoes, polyominoes that consist of two squares, are used in the popular game with the same name. There is only one type of domino, since the only other way to place the squares while still connected to each other will result in the same shape, just rotated.

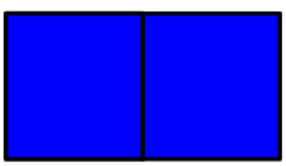


Figure 3

Any number of squares have a corresponding set of polyominoes, but polyomino puzzles and problems usually involve dominoes, trominoes (fig 4), tetrominoes (fig 5), pentominoes (fig 6) and maybe hexominoes.

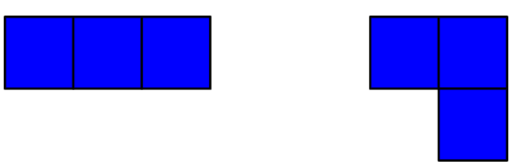


Figure 4

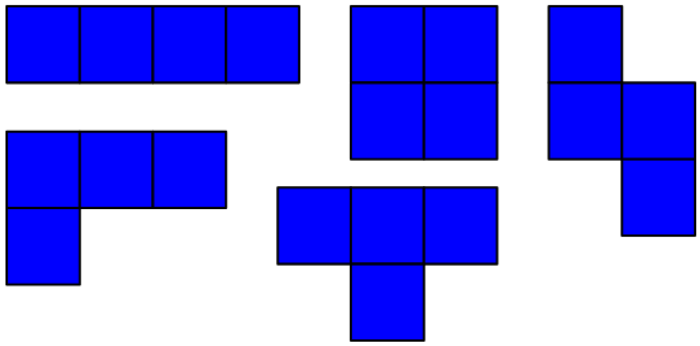


Figure 5

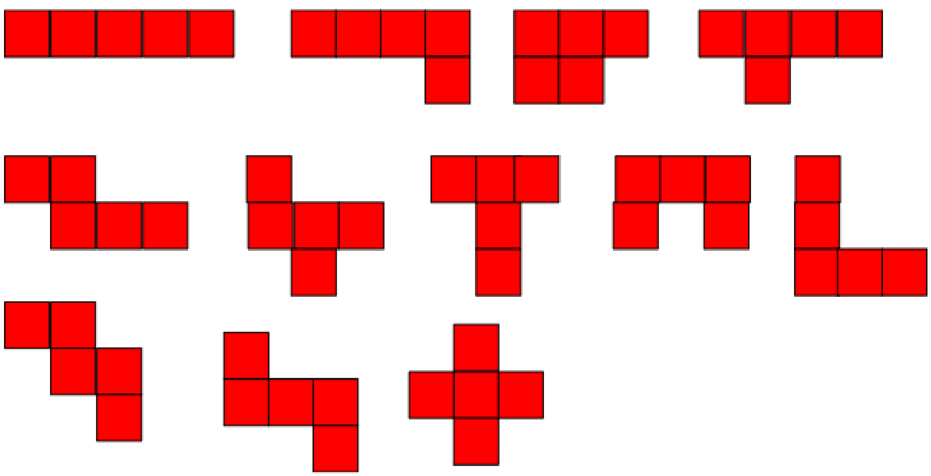


Figure 6

The number of polyominoes get increasingly higher along with the number of squares. We can see a selection in table [3].

Number of squares *n*	**Name**	**Number of free polyominoes**
1	Monomino	1
2	Domino	1
3	Tromino	2
4	Tetromino	5
5	Pentomino	12
6	Hexomino	35
7	Heptomino	108

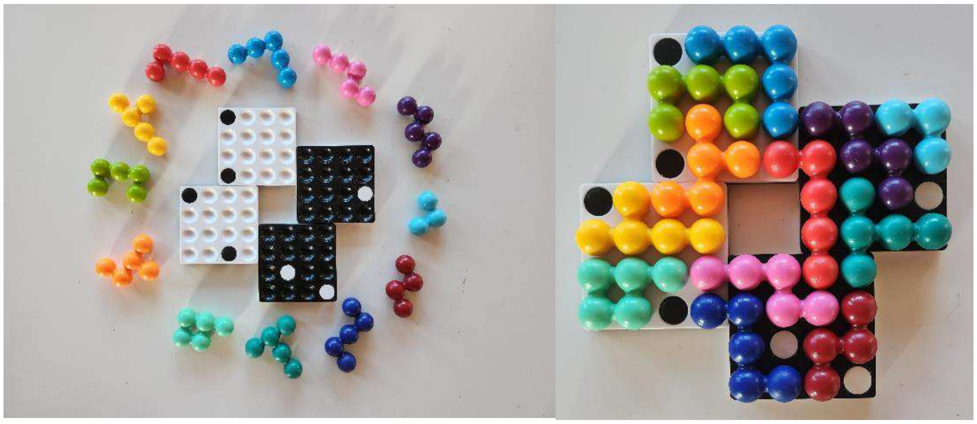
Table 3

We see that when n is higher than 6, the number of polyominoes goes above 100 and get hard to deal with. For our purposes, we will only consider the free polyominoes, meaning polyominoes that are free to rotate and flip.

There have been many attempts to find a formula that tells us the number of polyominoes for n squares, but we only have approximations that give us an upper bound. One such approximation will be discussed in a later section.

Quadrillion

There are many polyomino puzzles and tasks, and polyominoes possess many properties that allow them to easily turn into games. One such polyomino-based game is “Quadrillion” by SMART games. In this game, there are four 4x4 square tiles with grooves that make up the gameboard, and 12 polyomino puzzle pieces. There are 7 spots in total where a piece cannot be placed. The tiles can also be configurated a number of ways. You can click the tiles into place using the magnets on the sides and make a number of shapes, you can flip the tiles over to the contrasting side, and you can rotate the tiles. This results in an astonishing number of ways to make a gameboard that we will look at later on. The puzzle pieces are all shaped differently, and the objective of the game is to arrange the tiles in some way and place the puzzle pieces onto the game board so that all the grooves are covered. The pamphlet has many challenges for the player to try with varying degrees of difficulty, some where you are supposed to arrange some of the pieces before you start. The game is both fun and challenging for numerous age groups. The game itself is not necessarily combinatorial in nature, it is a puzzle game where you must be either skilled, very lucky or persevering. Some configurations have 1 unique solution, while others have very many.



What makes this game interesting in a combinatorial sense is the tiles that make up the gameboard. The tiles can be flipped over to reveal either a black side or a white side that are different from each other, they can be clicked together with magnets to each other by either a half side or a whole side. They can also be rotated, meaning there are many possible configurations for the gameboard. The pamphlet lists some of the shapes that are allowed, these are seen in fig 7.

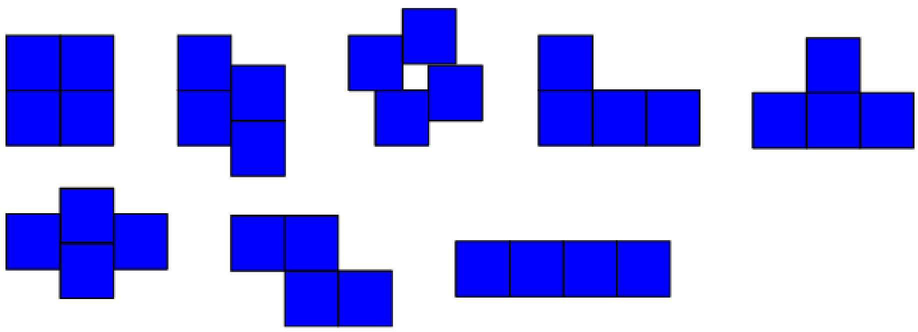


Figure 7

I am interested in finding the total number of unique configurations of the gameboard, regardless of whether they have a solution or not, as this is not necessarily our objective. There isn't an obvious reason why these configurations wouldn't have a solution, so we treat them like they do. It's clearly stated in the rulebook that "any of the configurations (see illustration on the right) with grids side by side are allowed" (SMART GAMES, 2013, p. 1). This is a non-trivial statement, to state this so confidently must mean there is either a mathematical proof or they ran them all through a computer. I'm more inclined to believe the latter since it is less laborious and more convenient. It also sparks curiosity whether they have found all the possible shapes the gameboard can have, tested them all, and found that there are some configurations for which there is no solution with the provided game pieces. If not, does this mean that there are solutions for all the shapes we can find? Or maybe there are one or two additional shapes for which there are solution for every possible rotation, flip and placement of the 4 game pieces?

Counting configurations of the game board

We already know there are quite a few ways to change up the gameboard, since we have 4 unique tiles that can be flipped, rotated and moved along whichever edge, as long as the tiles stay together. It isn't a simple combinatorics problem where you have 4 items that can be combined in a number of ways, though this is part of the final equation. After a few attempts at counting every configuration of the tiles and then multiplying to take flipping and rotating into account, it was clear that we were overcounting. To fix this, I would either need to find out how many configurations had been counted more times than one and subtract them, or to find a new strategy.

The new strategy was much more systematic, where instead of counting every time a tile is placed in a new way, you count all the unique shapes made by the tiles. We regard rotated or mirrored shapes as not unique and will therefore not include them in our counting. At this point, we don't treat them like unique tiles, and we don't take flipping or rotating into account yet. To improve the system even more, we divide into categories based on how many tiles are touching each other, or how many "neighbours" each tile has. For example, the shapes in fig 8 would belong in the same category, 1-2-2-1, since two of the tiles have only one neighbour, while the other two have two.

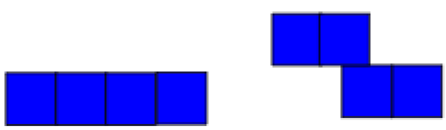


Figure 8

When we now count and record these into their respective categories, it also makes it easier to see if we have counted some configurations twice. A mirror image of one of our unique shapes must also have the same amount of neighbours. We are also more likely to observe a pattern when categorising it this way, which will help us when generalising.

As with the polyominoes, these shapes that can be made with the tiles will be considered identical if they are a reflection or rotation of a shape we have already counted. This is because we are interested in creating a unique gameboard.

2-2-2-2

There are only 3 possibilities in this group:

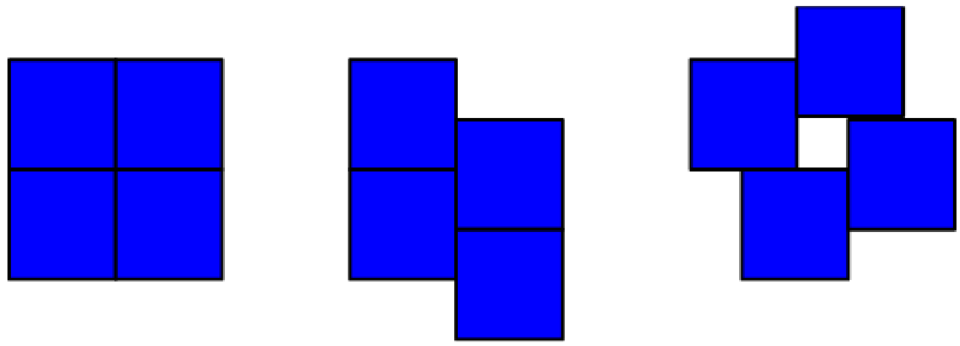
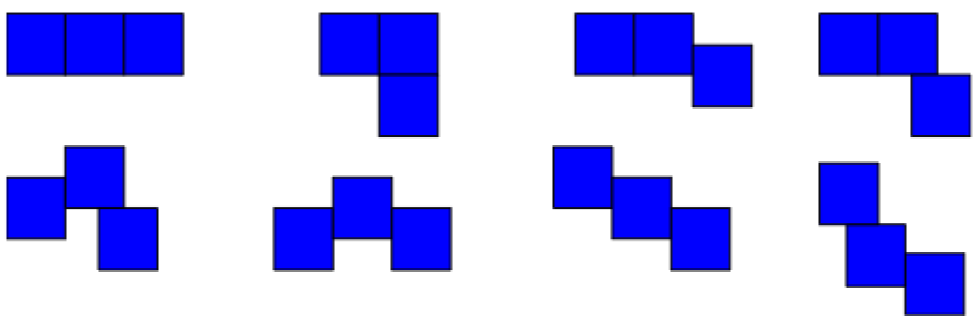


Figure 9

1-2-2-1

This is the biggest group, which is why we first look at how many ways there are to make a unique shape with 3 static tiles while still able to maintain this neighbour ratio (1-2-1), onto which we add the 4th tile.



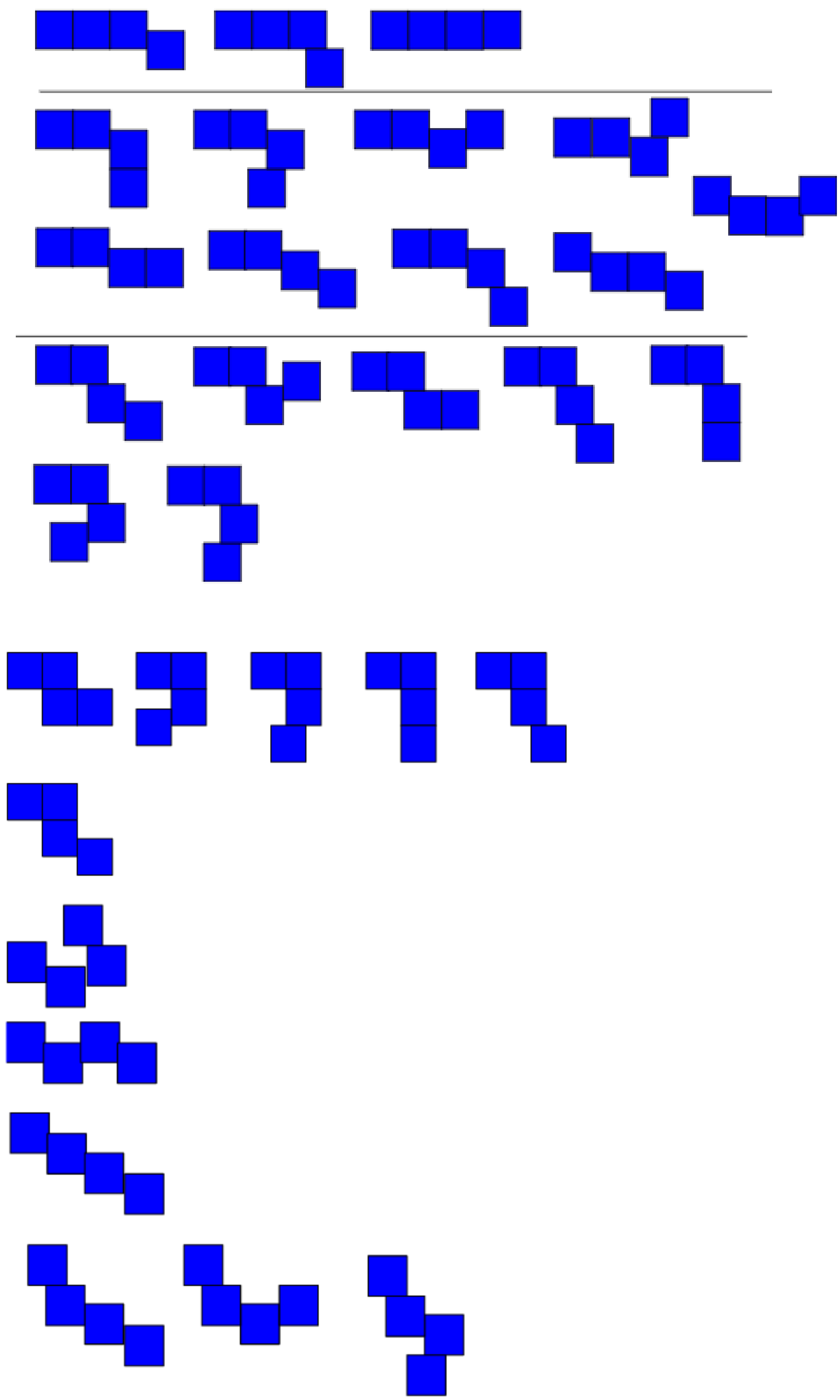


Figure 10

1-3-1-1

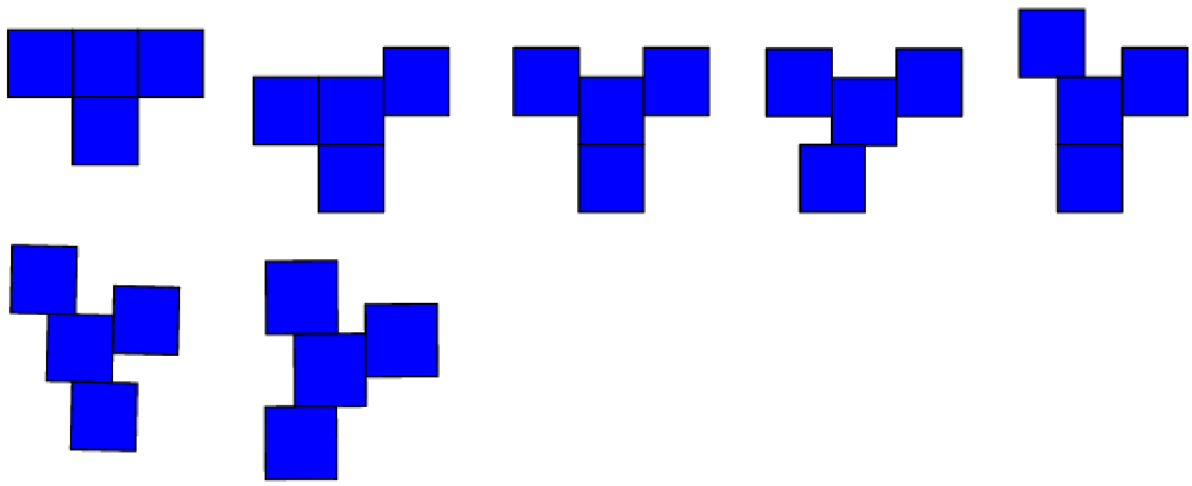


Figure 11

2-3-3-2

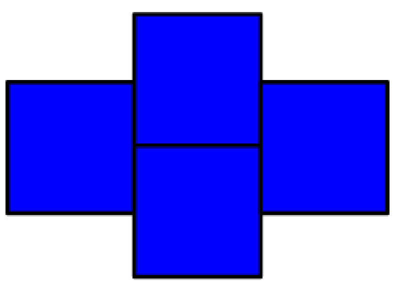


Figure 12

1-2-2-3

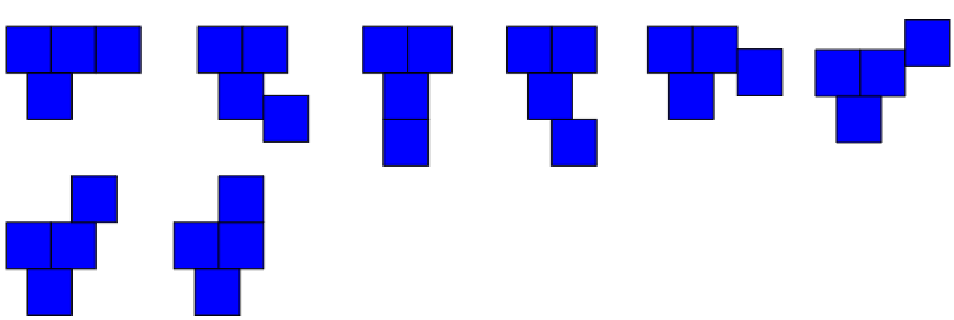


Figure 13

In our case, we find 50 unique shapes. After taking on the manual labour that is counting the shapes, it's easy to take into account the factors that result in a unique gameboard configuration, like placement (since we now consider the tiles to be unique), rotation, and which side is facing up. There are some shapes whose mirror image will be unique, due to the tiles themselves having "dot placements" in different places. If the shapes have a trivial mirror symmetry, they will contribute with a factor of 1 and will not be counted. We estimate the number of shapes where the mirror image will give us a unique gameboard at 42. This gives us (50+42) in our equation.

We have found (50+42) ways to arrange the tiles in a unique shape. We have 4 unique tiles that can be placed in 4 different slots, depending on the shape, and since they are unique, the order does matter. We now need to use a combination of the formulas [1]-[4] to find the final number of gameboard configurations. We have 4 unique tiles, each of which can be flipped to the other side, and because all sides are unique it means we allow all the tiles to be the same colour and that order does matter. This means we need to use formula [1]. We have 4 tiles and there are 2 sides to each tile, which means there are $2^4=16$ ways to combine the tiles. When we introduce rotation, we see that each tile can be rotated 4 times. This means we use formula [1] and get $4^4=256$ ways we can rotate our 4 tiles. What we mean by this is that we may have sort of original position of the tiles in regard to rotation, and there are 255 ways to alter this original position.

Lastly, we use formula [2] to find how many ways there are to place each tile into a slot. Here, a slot is referring to a vacant space in one of the gameboard shapes we have found. There are 4 tiles and 4 slots, order obviously does matter, and we do not allow repetition since the tiles are unique. This leads us to $\frac{4!}{(4-4)!} = 4! = 24$. But again, we encounter an overcounting problem, to do with symmetry. Now, if we use one of our symmetric shapes and state that all placements of the tiles will result in a unique gameboard, we'd be wrong. Let's look at an example, where I have used different colours to differentiate between the unique tiles.

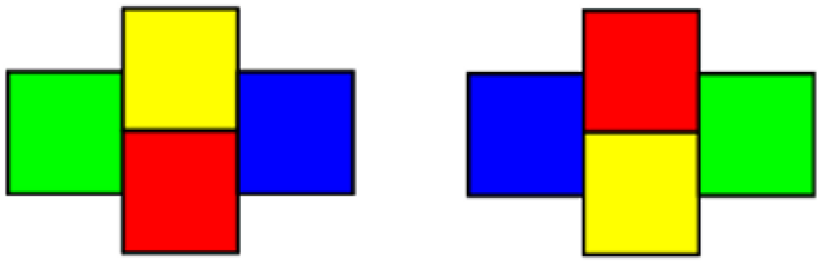


Figure 14

The first gameboard is the same as the second, only the second is rotated. Yet we have counted both of them as unique gameboards. This problem occurs with all the gameboard shapes that have rotational symmetry. The number of rotational symmetries will vary, this board has 2, while some might have 4. If we find the total number of rotational symmetries for all the boards and divide the final number of configurations with this, we will find the number of unique gameboard configurations.

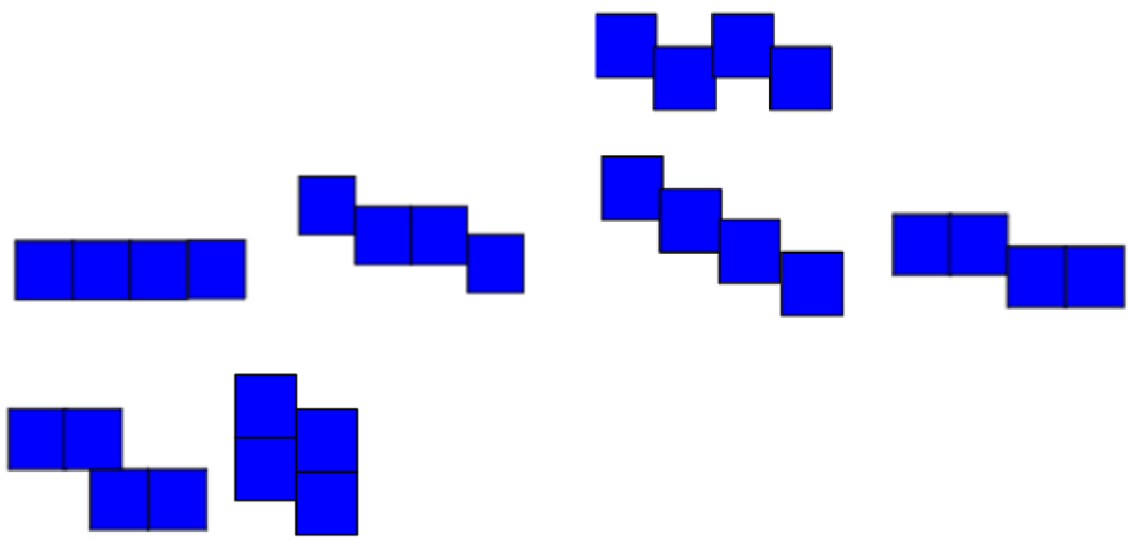


Figure 15

Figure 14 has 2 rotational symmetries, as do all the boards in figure 15. The boards in figure 16 has 4 rotational symmetries.

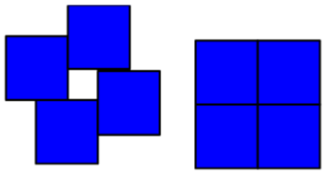


Figure 16

This means we must divide our number with 24.

We now take everything into consideration.

$$\frac{(50+42)\cdot 4^4 \cdot 2^4 \cdot 4!}{24} = 376\,832$$

There are 376 832 ways to make a totally unique gameboard using 4 distinct tiles. Although it is entirely possible to run all these configurations through a computer program to see if there is a possible solution, it is a large enough number to assume that they probably haven't been. Another interesting result we can get from all the information we've gathered, is that we can also calculate how many ways we can vary one gameboard shape just by flipping, rotating and different tile order. If we have one of the shapes with rotational symmetry, we must of course divide by the rotational symmetries. If we have an arbitrary gameboard shape, we take the number of possible combinations when we account for flipping, rotation and placement, and we arrive at $4^4 \cdot 2^4 \cdot 4! = 98304$ ways. Though I could make an educated guess as to which extra shape of our 50 could possibly also have a solution for every variation, I do not have a computer program to do this and it is quite laborious to try finding a solution for all 98304 variations.

The main reason this is so complicated, is because we are able to connect tiles by half edges, not only whole ones. If we look at the shapes we are able to make with 4 tiles when only allowed to connect by a whole edge, we get these:

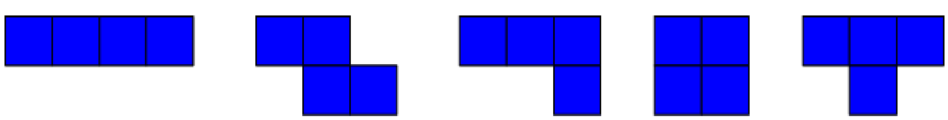


Figure 17

The 5 free tetrominoes. Not surprising, considering this is the very definition of a polyomino. If we had 5 tiles instead of 4, we would have the 12 free pentominoes, and so on. This counting has already been done for us, though there is no exact equation for it. Since there is no exact equation that gives the number of polyominoes for n squares, we won't find one when we further complicate things by being able to connect via a half-edge either. These "half-polyominoes" can be reminiscent of the pseudo- and quasi- polyominoes, both of which have quite interesting properties (Golomb, 1994, p. 85).

Approximating a formula

Though there is no formula for the number n-ominoes P(n), there are several approximations that give us a ballpark. One of these is the inequality $P(n+1) < (2n+1)P(n)$, based on the observation that one can at most place the extra square 2n+1 places (Golomb, 1994, p. 78). This gives us an upper bound on just how large the number of n-ominoes could be and is much more general as it doesn't depend on us knowing what the previous term is. The inequality $P(n+1) < (2n+1)P(n)$ gives us factorial-like bounds, let's insert some numbers for n so we can compare. Inserting 4, 5 and 6 for n gives us $P(5) < 45, P(6) < 156, P(7) < 525$. With the actual number of polyominoes for n=4, 5, 6 from table [3] being 12, 35 and 108, respectively. This approximation is far from close, but not too bad either. Another approximation for the upper bound of P(n) that has a simpler expression is the inequality $P(n) < \frac{(2n)!}{2^n n!}$, which is not dependent on us knowing the previous term (Golomb, 1994, p. 78). The simplicity and practicality of this upper bound is appealing, but the drawback here is that it gives a much larger upper bound, for example $P(5) < 945$, as opposed to 45 like in the other approximation.

Since my quest to find all the unique shapes the gameboard can have is essentially finding all the half-polyominoes for 4 squares, there must be a similar formula that can approximate this for n tiles. If we make a new table with the number of half-polyominoes for 1, 2, 3, and 4 squares, we can compare this with some of the numbers from table [3] that shows the numbers of polyominoes for n squares.

	1 square	2 squares	3 squares	4 squares
Number of polyominoes	1	1	2	5
Number of half-polyominoes	1	2	8	50

Table 4

If we try a similar approach, we don't have nearly as much data to make a very accurate approximation. Hopefully, however, we can approximate to a degree where we can be pretty confident that the number of half-polyominoes will not exceed our approximation. When placing an extra square on the half-polyominoes, some shapes have more options than others. It would only make sense for the half-polyomino approximation to be larger than the approximation for the polyominoes, since there are more options. In fact, for each option for placement in the polyomino-case, there can be at most two extra placement options in the half-polyomino case, since the extra square can move with a half-line in two directions. This would mean that a rough approximation could be $P_h(n+1) < 3(2n+1)P_h(n)$. It's already obvious that this will overestimate severely, but the point is to find an upper bound for which we are confident the actual number won't exceed. Using this approximation to approximate for $P_h(4)$, we get 168, compared to 50, which we found in an earlier section. This isn't too bad of an approximation. For $P_h(5)$, we get 1350.

Polyominoes in Quadrillion

We see that polyominoes can appear in the gameboard, as we can place the 4 tiles in ways that form the 5 free tetrominoes. There are also 12 game pieces, these are rounded to fit into the grooves, but in this illustration they will be straight-edged:

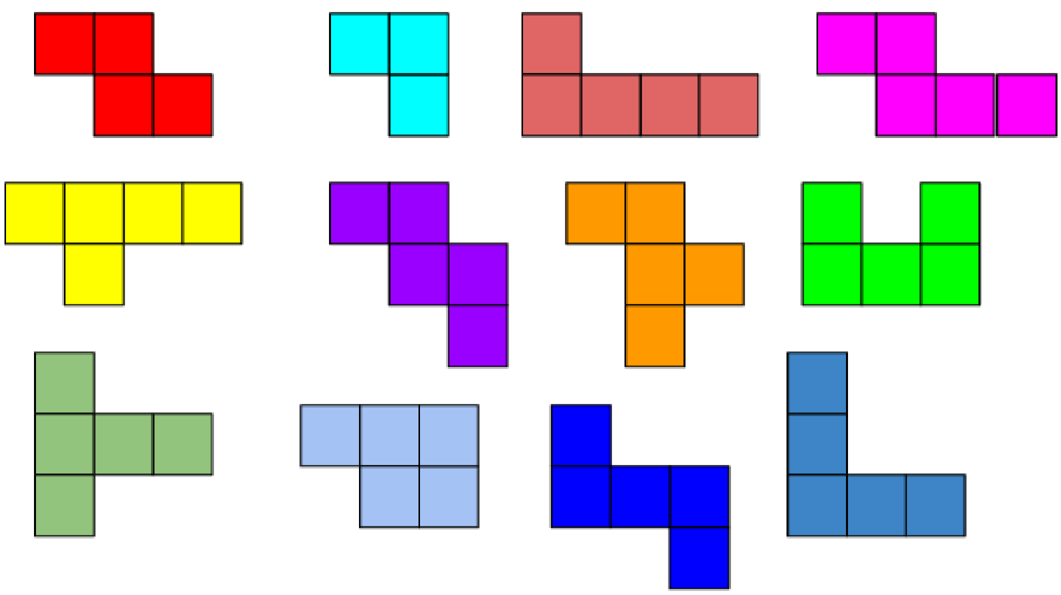


Figure 18

These are all a type of polyomino, in fact they are all pentominoes apart from the two first ones. From left to right, we have the skew tetromino, the right tromino, and the L, N, Y, W, F, U, T, P, Z, V pentominoes. There are 12 pentominoes in total, but the X and I pentomino are not game pieces in Quadrillion. Supposedly, this combination of polyominoes will solve any configuration of the tiles, as long as it's one of the allowed shapes stated in the rulebook. As mentioned before, there is no reason to believe these game pieces won't also work with other shapes; I found a solution for many of the alternative shapes that aren't in the rulebook, but since they haven't all been tested we can't guarantee that there will be a solution. Pentominoes have many fascinating qualities, so it's no wonder many of them are game pieces here.

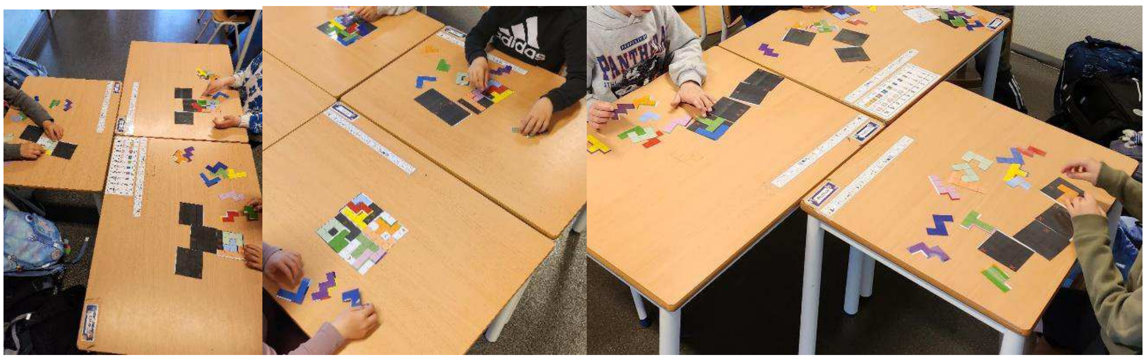
Execution

I wanted to explore using polyominoes in math class, to see if it could be resourceful for the students. The sort of puzzle problems we encounter when looking at polyominoes may be regarded by some as purely recreational mathematics, which I disagree with. When looking at the age group, they are still developing the basic skills and operations that are needed to solve these tasks. For this particular lesson, I wanted to not only imagine how I would change and differentiate the activities for older vs younger students, but actually try it out on both a 6th grade class and my own students in 1st grade. I had some alternative activities that were a bit easier for the 1st graders to try if they found it hard, but they were very resilient and were able to do the same activities as the 6th graders except for the last few steps. I didn't aim for this lesson to be very coherent, but for it to be a fun exploration of combinatorial geometry through polyominoes.

First, I did a quick introduction to the topic: what is the definition of a polyomino? I didn't want to use more than 5 minutes explaining before they try things out for themselves. Simply: "polyominoes are geometric shapes that are made by connecting a number of squares together, edge to edge." Then I show them the only domino that exist, and we agree that there is only one, because every other configuration will just be a repetition. I then show them the different trominoes, and we agree what configurations are unique and which ones are not. The students should now be ready to find the tetrominoes on their own. The 6th graders did this by drawing in their math notebook, while the 1st graders were put in small groups and were given building blocks. Discovering these sparked a genuine reaction from the 6th grade students, who recognised these shapes almost immediately. Most knew them from tetris, and some from other mobile puzzle games, and they realised then why tetris is called tetris. After finding all the tetrominoes, I tasked them with finding all the pentominoes. Most of the 6th graders and a few of the 1st graders used the tetrominoes and added a square to them in different places to find the pentominoes.

After a quick summary, where we agreed on how many pentominoes there are and what they looked like, we moved over to discuss pentominoes further. The 6th graders were more curious and asked many questions which lead to a longer discussion, while the 1st graders were anxious to get their next task. Since I had more time with the 6th graders, and they seemed interested enough, I decided to have them make a larger version of one of the pentominoes out of nine of the pentominoes. They found this very fun, and I heard them discussing circumference, area, and odd and even numbers.

For the "main activity", I introduced the rules of quadrillion and showed the variations you could do to the gameboard, and showcasing the game pieces (which they recognised). The start of the task was the same for both the 6th graders and the 1st graders. They were told to pick one tile and find which game pieces could fill that tile. This served as a good warm-up for the older students and gave the 1st graders a chance to test out which ones wouldn't work and why. If they had a tile that had 2 spots that couldn't be filled, there would be 14 open spots. If they tried to put 3 pentominoes in there, they would never fit. For some, this took a long time to find out. Most of them would just try every game piece possible until they found a solution. With a nudge in the right direction, some came to the conclusion that this wouldn't work, and neither would using the tromino. This meant that the tetromino would be essential to solving all single-tile puzzles with the exception of the tile with only 1 spot missing. When I then told them they could progress to 2 tiles, they utilized this knowledge and started counting how many open spots there were. Most 1st graders stuck to 1 or 2 tiles, while around 5 of them tried 4 tiles as well. Mind you, it doesn't state in the rulebook that there is a solution for 1, 2 or 3 tiles, but we didn't run into any problems assuming this would work. All the 6th graders moved on progressively from 1 tile up to 4 tiles, and some were frustrated to find that having found the solution to 1 tile would be little to no help to them when searching for the solution to 2 tiles. After a long period of playful exploration, both the 6th graders and 1st graders were given the option to play against eachother. They ditched the tromino and tetromino, and we brought back the I and X pentomino we used earlier in the pentomino task. They agreed on a gameboard, did rock, paper, scissors to find which one should start, and distributed the game pieces evenly. They intermittently place game pieces on the board, and the goal is to be the last one to be able to place a game piece on the board. Though the game-portion of this activity weighs more heavily than the mathematical thinking-portion, the students must still be strategic and try to obstruct their opponent without obstructing themselves. This activity was popular with both age groups, but the 1st graders probably learnt the most from it, as they would really try to stretch their brain to think several steps ahead.



I had a good experience doing this lesson with both age groups, both of which are very eager to learn in the first place, but were extra eager to try something a bit different. It was nice to see everyone engaged in the same activity, even students who are usually more apprehensive when it comes to math. Combinatorial tasks and combinatorial geometry puzzles have a low threshold, meaning it doesn't take much for anyone regardless of age and ability to at least attempt to solve them. Especially for the Quadrillion portion of the lesson, where there wasn't a clear divide between the stronger and weaker students when it came to solving the boards. Partly because they had different gameboards, and partly because some got lucky with their first placement. Of course, as with all puzzles and brain teasers, these activities brought on both genuine joy and excitement and also a fair bit of frustration if they *almost* got it right. Which of course means they had to start again from scratch. I was pleased, if they didn't care and weren't trying, they wouldn't be frustrated.

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