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PRECISE COMPUTATION OF MODULI SPACE FOR STANDARD EMBEDDING.

Universitetet i Stavanger

BY

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ABSTRACT

This thesis has undertaken the task of clearly defining the full heterotic moduli space for $E_8 \times E_8$ heterotic string theory in standard embedding while utilizing the mathematical tools offered by the deformation theory of holomorphic structure. Our approach involved simultaneous deformations of the complex structure J on a Calabi-Yau manifold X and the connection A on the bundle V. We drew inspiration from Atiya's work, which focused on the deformations of holomorphic bundles on complex manifolds. We introduced a bundle Q as an extension of the cotangent bundle $T^*(X)$ which result from the simultaneous deformation of the complex structure and the connection. We further defined an operator $\overline{D}$ on Q and demonstrated explicitly that in standard embedding the cohomology class on Q, defined as $H_{\overline{D}}(Q)$ provides the complete moduli spectrum of $E_8 \times E_8$ heterotic string theory.

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Chapter 1

INTRODUCTION

1.1 General Overview

The subject of string compactification has a long history that serves to construct four-dimensional effective theories, from smooth compactifications of for example the heterotic string. This process involves curling up six internal dimensions of the 10-dimensional string theory on the compactification geometry by choosing backgrounds of the form [1]

$$M_{10} = M_{10-n} \times X_n^{compact}. \tag{1.1}$$

There are diverse possible backgrounds that can give rise to a string compactification. While it is difficult to determine the exact number of all possible string compactifications, their primary reason is to construct four-dimensional theories that can describe the observed particles and interactions in our universe. There are many types of compactifications in the literature, including various Calabi-Yau manifolds, orbifolds, and more exotic constructions such as the Gepner models [2]. Each of these gives rise to different types of low-energy physics. It has been proven that the best compactification geometry for effective string compactification is the Calabi-Yau manifold because of its ability to preserve supersymmetry conditions on the 4-dimensional Minkowski space [3, 4]. In this thesis, we define the compactification geometry as Calabi-Yau threefold, X, where the 10-dimensional space, M_{10}, decomposes as $M_4 \times X_6$.

Many authors have established the existence of a solution of the 10-dimensional heterotic supergravity and realized a 4-dimensional Minkowski Space via the standard embedding and some other ways, for example, line bundle models [3, 5, 6, 7, 8]. The main idea behind the standard embedding in the heterotic string compactification is to select a gauge bundle on the compactification manifold that will preserve supersymmetry and maintain the consistency of the low-energy physics. Specifically, this involves taking the connection of the E_8 gauge bundle to be the spin connection of the Calabi-Yau manifold. It was quickly realized that standard embedding provides an easy route to describing viable phenomenological solutions of string theory [9, 10, 11]. Despite vast literature and substantial progress, there are many open questions in the field of string theory that accommodate new methods and aim to achieve a more effective low-energy theory.

1.2 The goal of the thesis

This thesis examines the moduli of the $E_8 \times E_8$ heterotic string model via the standard embedding while utilizing the mathematical tools offered by the deformation theory of holomorphic structure. Roughly, moduli are infinitesimal deformations of the compactification geometry, X, that leads to the M_4 spectrum by determining its massless fields through the associated cohomology classes. We draw inspiration from Atiya's work [12], where the simultaneous deformations of the complex structure and the bundle were considered. We shall explicitly show this in Chapter 6.

The main objective of this thesis is to explicitly compute the total heterotic moduli of $E_8 \times E_8$ string theory in the standard embedding. We are interested in the total moduli that result from the simultaneous deformations of the metric and the gauge connection. To achieve this, We define a bundle Q, constructed when the metric and the holomorphic structures are varied together as an extension bundle by the holomorphic cotangent bundle T^*X of a bundle E given by the short exact sequence

$$0 \longrightarrow T^{*(1,0)}X \longrightarrow Q \longrightarrow E \longrightarrow 0. \tag{1.2}$$

The bundle E is an "Atiya sequence" which is governed by the simultaneous deformations of the holomorphic structure and gauge connection. We will rely heavily on this bundle to determine the total heterotic moduli. The operator $\overline{D}$ on Q will be defined as a holomorphic structure map:

$$\overline{D} : \Omega^{(0,p)}(Q) \rightarrow \Omega^{(0,p+1)}(Q)$$

Short and long cohomological sequences associated with (1.2) will be utilized together with the holomorphicity of the bundle to compute the desired moduli. It is well known in the literature that the total moduli for $E_8 \times E_8$ heterotic string theory at the standard embedding are given as:

$$H^{(0,1)}(Q) = H^{(1,1)}(X) + H^{(2,1)}(X) + H^{(0,1)}(End(V)). \tag{1.3}$$

Though there is substantial literature for this claim, it has not been explicitly shown that this leads to total heterotic moduli with standard embedding, which is the aim of this thesis. In the upcoming section, we briefly give the outline of this thesis.

1.3 Road Map of the thesis

We will explore fascinating topics such as complex manifolds, Hodge theory, Calabi-Yau manifolds, and vector bundles, which will serve as essential tools for the final Chapter, 6. For a better understanding, a significant portion of the thesis will focus on building the necessary knowledge to perform the calculation.

To keep the text concise, some proofs will not be included, but the reader can find them in the references. In Chapter 2, will familiarize the reader with complex manifolds and their significance. We will introduce important concepts, such as the Dolbeaut cohomology, which will be used extensively in the following Chapters. In particular, complex manifolds will provide a foundation for comprehending Calabi-Yau manifolds in Chapters 4. In Chapter 3, we will explore the Hodge theory, which introduces the concept of the Hodge star this allows us to define an inner product on differential forms. We will also examine the Hodge decomposition theorem, which is an important theorem. This theorem states that when a manifold X is equipped with a Riemannian metric g, every cohomology class has a unique representative referred to as harmonic forms. In our final computation, in Chapter 6, we will utilize some of the concepts defined in this Chapter. In Chapter 4, we will delve into the Calabi-Yau manifold, which is the geometry of interest for compactification. The math tools presented in this Chapter will be utilized for the computation of the total moduli spaces in 6. We will explain the definition of the Calabi-Yau manifold, the complex structure and Kähler form deformations (regarded as the metric moduli), and the Hodge numbers, which are an essential concept in determining the spectrum of the 4-dimensional string theory.

In Chapter 5, we will explore vector bundles as a final mathematical tool for our computation. Vector bundles are one of the most critical tools in string theory. In order to reduce the dimensions of a string theory, a 10-dimensional E_8 gauge symmetry must break down to a subgroup, $G \subset E_8$, where $G \times H \subset E_8$, and H is the symmetry associated with the gauge fields over the compact dimensions [1]. Geometrically, these are considered as vector bundles on a Calabi-Yau manifold.

In Chapter 6, which is the central part of this thesis will focus exclusively on computing the total heterotic moduli space. Specifically, we will explicitly show that the total heterotic moduli for E_8 heterotic string theory at the standard embedding are given as:

$$H^{(0,1)}(Q) = H^{(1,1)}(X) + H^{(2,1)}(X) + H^{(0,1)}(End(V)). \tag{1.4}$$

Before performing explicit calculations, we will present several other essential concepts for better comprehension and insight. Finally in Chapter 7, we will provide a summary of the thesis by outlining its main points.

Please note that Chapters 2-5 will follow closely the textbooks [13] and [14]. In Chapter 6, much of the details will be drawn from [1, 15, 16].

CHAPTER 2

COMPLEX MANIFOLDS

Manifolds are one of the most important and fundamental objects in mathematics and physics. They allow more complicated structures to be expressed in an understandable way. A complex manifold is a manifold with additional structure. We assume the reader is familiar with some basics of a manifold and point-set topology which will not cover in this thesis, for more details you can refer to [13]. Roughly, a manifold is a space that resembles Euclidean space or complex space locally, but not globally. In a real manifold, a smooth manifold is a space in which some neighborhood of every point is homeomorphic to an open subset of $\mathbb{R}^n$ such that the transitions between those open sets are given by smooth functions. Similarly, a complex manifold is a space in which some neighborhood of every point is homeomorphic to an open subset of $\mathbb{C}^n$, and the transition from one coordinate system to the other is analytic [14].

Complex manifolds are essential objects in many different areas of string theory. For example, the Euclidean string worldsheet is a complex manifold. They also play a major role in understanding a Calabi-Yau manifold, which is a natural background for superstring theory. In order to lay a foundation for Calabi-Yau manifolds which we will discuss in Chapter 4, this Chapter will briefly define holomorphic functions, discuss complex manifolds by looking at their geometry and examples, and further discuss Kähler manifolds and Kähler differential geometry, which are Calabi-Yau manifolds' major ingredients. In Chapter 3, we will discuss one of the main results of complex manifolds and Kähler manifold called Hodge theory.

2.1 Complex Manifolds

A lot of details in this Chapter are based on [14] Chapter 8. We first define a holomorphic function as one major ingredient of a complex manifold.

Definition 2.1 (holomorphic (or analytic) map on $\mathbb{C}^n$)**.** *A complex-valued function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if $f = f_1 + if_2$ satisfies the Cauchy-Riemann relations for each $z^\mu = x^\mu + iy^\mu$,*

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu} \qquad\qquad \frac{\partial f_2}{\partial x^\mu} = -\frac{\partial f_1}{\partial y^\mu}. \tag{2.1}$$

A map $(f^1, ..., f^n) : \mathbb{C}^M \to \mathbb{C}^n$ is called holomorphic if each fuction $f^\lambda, (1 \leq \lambda \leq n)$ is holomorphic.

Now we have what we need to define what a complex manifold is.

2.1.1 Definitions

Definition 2.2 (complex manifold)**.** *([14], definition 8.1) X is a complex manifold if the following axioms hold,*

(i) X is a topological space.

(ii) X is provided with a family of pairs $\{U_i, \psi_i\}$

(iii) U_i is a family of open sets which covers X. The map ψ_i is a homeomorphism from U_i to an open subset U of $\mathbb{C}^m$. [Hence, X is even dimension]

(iv) Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ji} = \psi_j \circ \psi_i^{-1}$ from $\psi_i(U_i \cap U_j)$ to $\psi_j(U_i \cap U_j)$ is holomorphic.

A complex manifold is one with enough structure to define the notion of holomorphic functions $f : X \to \mathbb{C}^n$. If we demand that the transition function, $\psi_j \circ \psi_i$ satisfy the Cauchy Riemann equations, then the analytic properties of f can be studied using its coordinates representative $f \circ \psi_i^{-1}$. The local complex coordinate map $\psi_i : U_i \to \mathbb{C}^n$ with holomorphic transition functions and holomorphic maps $\psi_j \circ \psi_i$ provide X with a complex structure. The number d is called the complex dimension of X and is denoted as $dim_{\mathbb{C}} X = d$. The real dimension $2d$ is denoted either by $dim_{\mathbb{R}} X$ or simply by $dim X$. The axioms above ensure that calculus on complex manifolds can be carried out independently of the special coordinates chosen.

Remark: Given a real manifold X, we can define a two-form tensor,J, on X that satisfies $J^2 = -\mathbb{I}$. We refer to J as a complex structure. When a real manifold X is endowed with this structure, it is known as an almost complex manifold. These are complex manifolds that are not necessarily complex manifolds but possess many properties of complex manifolds

We will not cover these manifolds in this thesis but we will use the concept in some of our computations. For a better understanding of complex manifolds we will now examine a few examples.

2.1.2 Examples

Example 1 ([14], example 8.1.2). *The unit two-sphere S^2, which is the subset of $\mathbb{R}^3$ defined by*

$$x^2 + y^2 + z^2 = 1$$

is a complex manifold.

The S^2 can be parametrized by two coordinates system polar coordinates and stereo-

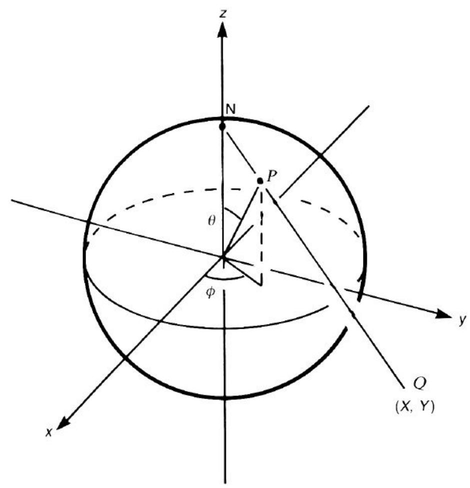

Figure 2.1: polar coordinates, (θ, ϕ), and stereographic coordinates, (X, Y), of a point p on a the sphere S^2

graphic coordinates. Polar coordinates θ and ϕ are usually defined by

$$x = sin\theta cos\phi \qquad y = sin\theta sin\phi \qquad z = cos\theta \tag{2.2}$$

where ϕ runs from 0 to 2π and θ from 0 to π. However, stereographic coordinates are defined by projection from different points on the sphere. One can use stereographic projection from the north pole onto the equatorial plane as shown in figure (2.1) of a point $p(x, y, z,) \in S^2$ with coordinates X and Y given by

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

This can be done for any point except the North Pole itself (corresponding to $z = 1$). To include the North Pole, we similarly, introduce a second chart, in which we *stereographically project from the South Pole:*

$$(U, V) = \left(\frac{x}{1+z}, -\frac{y}{1+z}\right),$$

which holds for all points in S^2 except when $z = -1$.
We can define complex coordinates:

$$Z = X + iY \qquad \overline{Z} = X - iY \qquad W = U + iV \qquad \overline{W} = U - iV$$

It can be shown that on the overlap of the two patches, the transition function is holomorphic. That is, W is a holomorphic function of Z

$$W = \frac{x - iy}{1+z} = \frac{1-z}{1+z}(X - iY) = \frac{X - iY}{X^2 + Y^2} = \frac{1}{Z}$$

Thus, S^2 is a complex manifold that is identified with the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Another important example of the complex manifold is the complex projective space $\mathbb{CP}^n$, which also serves as an example of Kähler manifolds and Calabi-Yau manifolds. These concepts will be discussed later in this Chapter and Chapter 4 respectively. Below we will discuss how the holomorphic charts are defined on $\mathbb{CP}^n$.

Example 2. *The complex projective space, $\mathbb{CP}^n$ is a complex manifold of dimension $2n$. Notice the complex projective space, $\mathbb{CP}^n$, is the set of lines through the origin in $\mathbb{C}^{n+1}$. The n-tuple $z = (z^0, \ldots\ldots\ldots, z^n) \in \mathbb{C}^{n+1}$ determines a complex line through the origin provided $z \neq 0$. The z^i are called homogeneous coordinates.*
On $\mathbb{CP}^n$, we can quotient, by the equivalence relation, by identifying

$$(z^0, \ldots\ldots, z^n) \approx \lambda(z^0, \ldots\ldots\ldots, z^n)$$

for any non-zero complex λ. The equivalence relation relates every two points on the line by rescaling. A chart U_μ is a subset of $\mathbb{C}^{n+1}/0$ defined as:

$$U_\mu = \{z^a, a = 0, ..., n|\ z^\mu \neq 0\}, \qquad\qquad \zeta^a_{[\mu]} = \frac{z^a}{z^\mu}$$

for fixed μ. These $n+1$ charts cover the entire space since the origin was left out in the complex plane. Hence, $\{U_\mu\}^n_{\mu=0}$ defines an atlas on the space. We define the inhomogeneous coordinates in a chart U_μ as $\zeta^a_{[\mu]} = \frac{z^a}{z^\mu}$. Note that ζ^a_μ are well defined on U_μ since $z^\mu \neq 0$ and they are invariant under the equivalence relation defined above. On the overlap, $U_\mu \cap U_v \neq \emptyset$, the coordinate transformation $\psi_{\mu v} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is

$$\zeta^\lambda_{[v]} \mapsto \zeta^\lambda_{[\mu]} = \frac{z^v}{z^\mu}\zeta^\lambda_{[v]}.$$

Accordingly, $\psi_{\mu v}$ is a multiplication by $\frac{z^v}{z^\mu}$, hence the transition functions are holomorphic .

After exploring the captivating examples, in the upcoming sections, we will explore further into more interesting features of complex manifolds, in particular differential forms. However, before we begin discussing differential forms on complex manifolds, it is crucial to examine a key component of the definition surrounding them – namely the tangent space. Our focus is on the complexified tangent space, which we define as the tangent space of a complex manifold. Essentially, this is just an extension of the tangent space of a real manifold. We will soon see how these real tangent spaces complexify into the tangent spaces on complex manifolds and explicitly define what constitutes a tangent space of a complex manifold. The tangent space is also an important structure in defining the almost complex structure manifold.

2.1.3 Complexification of the tangent space

The tangent space to a manifold at a particular point is essentially a vector space of derivations at the point [13]. T_pX is the closest flat approximation to X at p. As in multivariate calculus, T_pX encapsulates the slope of X at p which is identical to the first-order variations at p. As such, a convenient basis for T_pX consists of n linearly independent partial derivative operators:

$$T_PX : \left\{ \frac{\partial}{\partial x^1} \mid_p, ..., \frac{\partial}{\partial x^n} \mid_p \right\}$$

such that for every $v \in T_p(X)$ can be expressed as $v = v^\alpha \frac{\partial}{\partial x^\alpha}|_p$. Every vector space V has a dual space V^* consisting of real-valued linear maps on V which defines the dual space $T_P^* X$, with a basis dual to the one above.

$$T_p^*X : \left\{ dx^1|_p, ..., dx^n|_p \right\}$$

$dx^i \ : T_pX \to \mathbb{R}$ is a linear map with $dx^i(\frac{\partial}{\partial x^j}|_p) = \delta^i_j$ [9].

If X is a complex manifold with dimension $d = \frac{n}{2}$, we have the complexification $T_pX^{\mathbb{C}}$ of T_pX, whose elements are expressed as $Z = u + iv$, $(u, v \in T_pX)$, $T_pX^{\mathbb{C}}$ is called a complex tangent space. In terms of complex coordinates, the complex tangent space $T_pX^{\mathbb{C}}$ is spanned by $2d$ vectors

$$\left\{ \frac{\partial}{\partial z^1} \mid_p, ..., \frac{\partial}{\partial z^d} \mid_p, \frac{\partial}{\partial \bar{z}^1} \mid_p, ..., \frac{\partial}{\partial \bar{z}^d} \mid_p \right\}$$

Similarly, as with the real case, we can define the dual to $T_pX^{\mathbb{C}}$, denoted by $T_p^*X^{\mathbb{C}}$, with a basis $\left\{ dz^1 \mid_p, ...dz^d \mid_p, d\bar{z}^1 \mid_p, ..., d\bar{z}^d \mid_p \right\}$. Note that with some refinements and considerations $T_pX^{\mathbb{C}}$ can be decomposed as $T_pX^{\mathbb{C}} = T_pX^{(1,0)} \oplus T_pX^{(0,1)}$, where the latter is holomorphic and anti-holomorphic tangent space respectively. $T_pX^{(1,0)}$ is spanned by $\left\{ \frac{\partial}{\partial z^1}|_p, ..., \frac{\partial}{\partial z^d}|_p \right\}$ and $T_pX^{(0,1)}$ is spanned by $\left\{ \frac{\partial}{\partial \bar{z}^1}|_p, ..., \frac{\partial}{\partial \bar{z}^d}|_p \right\}$.

Similarly, we can write $T_p^* X^{\mathbb{C}} = T_p^* X^{(1,0)} \oplus T_p^* X^{(0,1)}$, where $T_P^* X^{(1,0)}$ and $T_P^* X^{(0,1)}$ are holomorphic and anti-holomorphic cotangent spaces spanned by $\left\{ dz^1|_p, ...dz^d|_p \right\}$ and $\left\{ d\bar{z}^1 \mid_p, ...d\bar{z}^d \mid_p \right\}$ respectively. In the next section we will explore one aspect of the tangent space namely differential forms. An important implication of this is they allow us to perform calculus on a manifold. Later in the Chapter we will see that differential forms also provide a basis for understanding cohomology classes.

2.2 Complex differential forms

Differential forms are generalizations of real-valued functions on a manifold. Instead of assigning to each point of the manifold a number, a differential k-form assigns to each point a k-covector on its tangent space. For $k = 0$ and 1, differential k-forms are functions and covector fields respectively [13]. In contrast to vector fields, which are also intrinsic to manifolds, differential forms have a far richer algebraic structure. Due to the existence of the wedge product, a grading, and the exterior derivative, the set of smooth forms on a manifold is both a graded algebra and a differential complex [13]. There are so many ways to define or introduce differential forms. However, in this section, we will state the definition in the context of a real manifold, then extend it to a complex manifold with some refinements.

2.2.1 Complexification of real differential forms

Definition 2.3 (Cotangent Space:)**.** *Let X be a smooth manifold and p a point in X. The cotangent space of X at p denoted by $T_p^*(X)$ or $T_p^* X$, is defined to be the dual* *space of the tangent space T_pX:*

$$T_p^*X = (T_pX)^v = Hom(T_pX, \mathbb{R}). \tag{2.3}$$

An element of the cotangent space T_p^*X is called a covector at p. Thus, a covector ω_p at p is a linear function $\omega_p : T_pX \to \mathbb{R}$. A covector field, a differential 1-form, or more simply a 1-form on X, is a function ω that assigns to each point p in X a covector ω_p at p. In this sense, it is dual to a vector field on X, which assigns to each point in X a tangent vector at p [13]. One generalization of this idea is to consider a q-tensor, α which is a real-valued multi-linear map from $T_pX \times T_pX \times .... \times T_pX$ (with q factors) $\alpha(v_{(1)p}, ..., v_{(q)p}) \in \mathbb{R}$ [9]. Multilinearity here means linear on each factor independently.

For the purpose of Calabi-Yau manifolds, it proves worthwhile to consider a q-form which is a more constrained generalization of a one-form. This is a special type of q-tensor that is totally antisymmetric. If ω is a q-form on X at p, then $\omega(v_{(1)p}, ..., v_{(q)p})_p \in \mathbb{R}$ with $\omega(v_{(1)p}, ..., v_{(q)p})_p = -\omega(v_{(2)p}, v_{(1)p}........., v_{(q)p})_p$. A basis for two-tensors can clearly be gotten from considering all $dx^i \otimes dx^j$, where $dx^i \otimes dx^j : T_p(X) \otimes T_p(X) \to \mathbb{R}$.

To get a basis for two-form, we can simply antisymmetrize the basis for two-tensors by defining:

$$dx^i \wedge dx^j = \frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i). \tag{2.4}$$

By construction, $dx^i \wedge dx^j$ satisfies

$$dx^i \wedge dx^j(v_{(1)}, v_{(2)}) = -dx^i \wedge dx^j(v_{(2)}, v_{(1)}). \tag{2.5}$$

Any two-form ω can be written as

$$\omega = \omega_{ij} dx^i \wedge dx^j. \tag{2.6}$$

In general, a q-form has a basis of $dx^{i_1} \wedge dx^{i_2} \wedge .... \wedge dx^{i_q}$ and;

$$dx^{i_1} \wedge dx^{i_2} \wedge .... \wedge dx^{i_q} = \frac{1}{q!} \sum Signp dx^{i_1} \otimes dx^{i_2} \otimes .... \otimes dx^{i_q} \tag{2.7}$$

Where p is a permutation of $1......q$, and $Signp$ is ± 1 depending on whether the permutation is even or odd. Then any q-form ω can be written as

$$\omega = \frac{1}{q!} \omega_{i_1,......,i_q} dx^{i_1} \wedge dx^{i_2} \wedge .... \wedge dx^{i_q}. \tag{2.8}$$

The fact that ω is a totally antisymmetric map is sometimes denoted by $\omega \in \wedge^q T^* X$, where $\wedge^q$ denotes the q^{th} antisymmetric tensor product. All of these ideas extend directly to the realm of complex manifolds, together with some refinements due to the additional structure of having local complex coordinates[9].

Definition 2.4 (complexification of differential forms)**.** *Let X be a differentiable manifold with $dim_{\mathbb{R}} X = d$. Take two q-forms $\gamma, \tau \in \Omega^q_p(X)$ at p and define a complex q-form $\omega = \gamma + i\tau$. We denote the vector space of complex q-forms at p by $\Omega^q_p(X)^{\mathbb{C}}$. Clearly $\Omega^q_p(X) \subset \Omega^q_p(X)^{\mathbb{C}}$. The conjugate of ω is $\bar{\omega} = \gamma - i\tau$. A complex q- form ω is real if $\omega = \bar{\omega}$.*

A complex q-form ω defined on a differentiable manifold X is a smooth assignment of an element of $\Omega^q_p(X)^{\mathbb{C}}$. These set of complex q-forms is denoted by $\Omega^q(X)^{\mathbb{C}}$. A complex q-form ω is uniquely decomposed as $\omega = \gamma + i\tau$, where $\gamma, \tau \in \Omega^q(X)$[14].

2.2.2 Differential forms on complex manifolds

Now we restrict ourselves to complex manifolds in which we have the decomposition $T_p^*X^{\mathbb{C}} = T_pX^{(1,0)} \otimes T_pX^{(0,1)}$.

Definition 2.5 (Complex Differential Forms:). *Let X be a complex manifold with $dim_{\mathbb{C}}X = d$. Let $\omega \in \Omega_p^q(X)^{\mathbb{C}}$, $(q \leq 2d)$ and r, s be positive intergers such that $r + s = q$. Let $v_i \in T_pX^{\mathbb{C}}, (1 \leq i \leq q)$ be vectors in either $T_pX^{(1,0)}$ or $T_pX^{(0,1)}$. If $\omega(v_1, ..., v_q) = 0$ unless r of the v_i are in $T_pX^{(1,0)}$ and s of the v_i are in $T_pX^{(0,1)}$, ω is said to be of bidegree (r, s)-form. The set of (r, s)-forms at p is denoted by $\Omega_p^{(r,s)}(X)$. If an (r, s)-form is assigned smoothly at each point of X, we have an (r, s)-form defined over X. The set of (r, s)-forms over X is denoted by $\Omega^{(r,s)}(X)$*

With these bases, in a complex manifold, (r, s)-form ω is written as:

$$\omega = \frac{1}{r!s!}\omega_{\mu_1,......,\mu_r v_1,......,v_s} dz^{\mu_1} \wedge .... \wedge dz^{\mu_r} \wedge d\bar{z}^{v_1} \wedge ...... \wedge d\bar{z}^{v_s} \tag{2.9}$$

The set $\{dz^{\mu_1} \wedge .... \wedge dz^{\mu_r} \wedge d\bar{z}^{v_1} \wedge ...... \wedge d\bar{z}^{v_s}\}$ is the basis of $\Omega_p^{(r,s)}$. The summand above is labeled by the number r and s of holomorphic and anti-holomorphic one-forms it contains respectively. Whereas in the definition $r + s = q$.

Proposition 1. *Let X be a complex manifold of $dim_{\mathbb{C}}X = d$ and ω and ζ be complex differential forms on X*

(a) If $\omega \in \Omega^{(q,r)}(X)$ then $\bar{\omega} \in \Omega^{(r,q)}(X)$.

(b) If $\omega \in \Omega^{(q,r)}(X)$ and $\zeta \in \Omega^{(q',r')}(X)$, then $\omega \wedge \zeta \in \Omega^{(q+q',r+r')}(X)$.

(c) A complex q-form ω is uniquely written as

$$\omega = \sum_{r+s=q} \omega^{(r,s)} \tag{2.10}$$

Where $\omega^{(r,s)} \in \Omega^{(r,s)}(X)$. Thus, we have the decomposition

$$\Omega^q(X)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{(r,s)}(X) \tag{2.11}$$

Hence, any q-form ω is decomposed as

$$\omega = \sum_{r+s=q} \omega^{(r,s)} = \sum_{r+s=q} \frac{1}{r!s!} \omega_{\mu_1,......,\mu_r \bar{v_1},......,\bar{v_s}} dz^{\mu_1} \wedge .... \wedge dz^{\mu_r} \wedge d\bar{z}^{v_1} \wedge ...... \wedge d\bar{z}^{v_s} \tag{2.12}$$

Where

$$\omega_{\mu_1,......,\mu_r \bar{v_1},......,\bar{v_s}} = \omega(\frac{\partial}{\partial z^{\mu_1}}, ..., \frac{\partial}{\partial z^{\mu_r}}, \frac{\partial}{\partial \bar{z}^{v_1}}, ..., \frac{\partial}{\partial \bar{z}^{v_s}}) \tag{2.13}$$

This tells us that every complex differential form can be written as a sum of real differential forms. Generally, differential forms are very important tools in differential geometry and string theory; they are basic objects in calculus on manifolds. In the upcoming section, we will look at one important application of differential forms, that motivates our discussion. The differentiation on differential forms allows us to define cohomology groups on manifolds, which are topological invariant structures. We will extensively use this in our computation analysis.

2.2.3 Exterior derivative and dolbeault cohomology

Given a real manifold X and a set of differential forms $\Omega^*(X)$, there is a natural differentiation operation that takes a q-form ω on a differentiable manifold X to a $(q+1)$-form on X i.e $d : \Omega^q(X) \to \Omega^{q+1}X$. Explicitly, in local coordinates, this map d is called exterior differentiation and is given by:

$$d :\Rightarrow \omega \to d\omega = \frac{1}{q!}\frac{\partial \omega_{i_1,\ldots\ldots,i_q}}{\partial x^{i_{q+1}}} dx^{i_{q+1}} \wedge dx^{i_1} \wedge \ldots\ldots\ldots\ldots \wedge dx^{i_q}. \tag{2.14}$$

If X is a complex manifold, there is a refinement of exterior differentiation which prove to be a central concern in this thesis. If we define $\omega = \frac{1}{r!s!}\omega_{\mu_1,\ldots\ldots,\mu_r\bar{v}_1,\ldots\ldots,\bar{v}_s} dz^{\mu_1} \wedge \ldots\ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{v_1} \wedge \ldots\ldots \wedge d\bar{z}^{v_s}$ as in previous section. We find

$$d\omega = \frac{1}{r!s!}\left(\frac{\partial}{\partial z^\lambda}\omega_{\mu_1,\ldots\ldots,\mu_r\bar{v}_1,\ldots\ldots,\bar{v}_s} dz^\lambda + \frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1,\ldots\ldots,\mu_r\bar{v}_1,\ldots\ldots,\bar{v}_s} d\bar{z}^\lambda\right) \times dz^{\mu_1} \wedge \ldots\ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{v_1} \wedge \ldots\ldots \wedge d\bar{z}^{v_s} \tag{2.15}$$

$d\omega$ is a mixture of an $(r+1, s)$-form and an $(r, s+1)$-form [9]. This form decomposes into $\Omega^{(r+1,s)} \otimes \Omega^{(r,s+1)}$ using the complex structure of X, and the equation above can be summarized by $d\omega^{(r,s)} = \partial\omega^{(r,s)} + \bar{\partial}\omega^{(r,s)}$. The real exterior differentiation d is being decomposed as $d = \partial + \bar{\partial}$, where the latter two are exterior differentiation in the holomorphic and anti-holomorphic directions respectively. The operators ∂ and $\bar{\partial}$ are called the dolbeault operators, defined as $\partial : \Omega^{(r,s)}(X) \to \Omega^{(r+1,s)}(X)$ and $\bar{\partial} : \Omega^{(r,s)}(X) \to \Omega^{(r,s+1)}$ [14].

Theorem 1. *Let X be a complex manifold and let $\omega \in \Omega^q(X)^{\mathbb{C}}$ and $\eta \in \Omega^p(X)^{\mathbb{C}}$. Then*

$$\partial\partial\omega = (\partial\bar{\partial} + \bar{\partial}\partial)\omega = \overline{\partial\partial}\omega = 0 \tag{2.16}$$

$$\partial\bar{\omega} = \overline{\bar{\partial}\omega}, \bar{\partial}\bar{\omega} = \overline{\partial\omega} \tag{2.17}$$

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^q \omega \wedge \partial\eta \tag{2.18}$$

$$\bar{\partial}(\omega \wedge \eta) = \bar{\partial}\omega \wedge \eta + (-1)^q \omega \wedge \bar{\partial}\eta \tag{2.19}$$

In fact, according to the Nivenberg-Newlander theorem, $\partial^2 = \bar{\partial}^2 = 0$ is equivalent to an almost complex structure being an integrable complex structure [17].

Definition 2.6 (Holomorphic r-forms)**.** *Let X be a complex manifold. If $\omega \in \Omega^{(r,0)}(X)$ satisfies $\bar{\partial}\omega = 0$ the r-form is called a* **holomorphic r-form**

The antisymmetry involved in the exterior differentiation ensures that $d(d\omega) = 0$ for any form ω. If ω is an (r, s)-form for which $\bar{\partial} = 0$ such an ω is called closed. Every such ω has two possibilities either being exact or not. The former suffice that ω can be expressed as $\bar{\partial}\eta$ for some $\eta \in \Omega^{(r,s-1)}(X)$. Those ω which are closed, but not exact provide non-trivial solutions to the equation $\bar{\partial}\omega = 0$ which motivates the following definition:

Definition 2.7 (Dolbeault Cohomology)**.** *[14] The $(r, s) - th$ Dolbeault cohomology group $H^{(r,s)}_{\bar{\partial}}(X)$ on a complex manifold X is the quotient space;*

$$H^{(r,s)}_{\bar{\partial}}(X, \mathbb{C}) = \left\{ \frac{\omega^{(r,s)} | \bar{\partial}\omega^{(r,s)} = 0}{\omega^{(r,s)} | \omega^{(r,s)} = \bar{\partial}\eta^{(r,s-1)}} \right\} \tag{2.20}$$

Note that, the Dolbeault cohomology can also be formulated using ∂ as shown above, i.e

$$H^{(r,s)}_{\partial}(X, \mathbb{C}) = \left\{ \frac{\omega^{(r,s)} | \partial\omega^{(r,s)} = 0}{\omega^{(r,s)} | \omega^{(r,s)} = \partial\eta^{(r-1,s)}} \right\} \tag{2.21}$$

The cohomology groups of X probe fundamental information about its geometrical structure and will play a central role in the physical analysis. In general, cohomology groups serve as a tool that "measures" the extent to which closed forms fail to be exact. Through cohomology groups and Hodge theory, we can identify whether a solution for a particular differential equation exists without necessarily solving it.

We now switch our focus to Kähler manifolds, which is the final topic of this Chapter. As previously mentioned, Kähler manifolds play a significant role in defining Calabi-Yau manifolds. Although we will not extensively discuss other properties of Kähler manifolds, such as the holonomy of Kähler manifolds, we will instead concentrate on the characteristics that lead to the formation of Calabi-Yau manifolds and other features that we will utilize in subsequent Chapters.

2.3 Kähler manifolds and Kähler geometry

To introduce the notion of a Kähler manifold, we first need to define the concept of Hermitian metric.

2.3.1 Definitions

Definition 2.8 (Hermitian Metric and Hermitian Manifold). *Let M be a complex manifold with a complex structure J, and a Riemannian metric g. g is said to be a Hermitian metric if it satisfies*

$$g(JX, JY) = g(X, Y) \tag{2.22}$$

for any two vector fields X and Y. The pair (M, g) is called the Hermitian manifold.

Theorem 2. *A complex manifold always admits a Hermitian metric.*

Definition 2.9 (Kähler form). *Let (M, g) be a Hermitian manifold. We can define a tensor field ω whose action on X, $Y \in T_pM$ is*

$$\omega_p(X, Y) = g_p(J_pX, Y) \qquad\qquad X, Y \in T_pM \tag{2.23}$$

Note that ω is anti-symmetric.

$$\omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(JX, Y) = -\omega(X, Y). \tag{2.24}$$

Hence, ω defines a two-form called the Kähler form of Hermitian metric g.

In local real coordinates, the components of ω are

$$\omega_{\mu v}(x) = J^p_\mu(x) g_{pv}(x). \tag{2.25}$$

Definition 2.10 (Kähaler Manifold). *A Kähler manifold is a Hermitian manifold (X, g) whose Kähler form ω is closed, $d\omega = 0$. The metric g is called the Kähler metric on X.*

Remark: Not all complex manifolds admit Kähler metric.

Theorem 3. *A Hermitian manifold (X, g) is a Kähler manifold if and only if the almost complex structure J satisfies*

$$\nabla_\mu J = 0 \tag{2.26}$$

where ∇_μ is the Levi-civita connection associated with g.

We will discuss this further in section 2.3.3.

2.3.2 The Kähler Potential

In complex the condition that the Kähler form ω is closed translate to the following.

$$
\begin{aligned}
d\omega \Rightarrow (\partial+\bar{\partial})ig_{\mu\bar{v}}dz^{\mu}\wedge d\bar{z}^{v} &= i\partial_{\lambda}g_{\mu\bar{v}}dz^{\lambda}\wedge d\bar{z}^{v}+i\partial_{\bar{\lambda}}g_{\mu\bar{v}}d\bar{z}\lambda\wedge dz^{\mu}\wedge d\bar{z}^{v}\\
&= \frac{1}{2}i(\partial_{\lambda}g_{\mu\bar{v}}-\partial_{\mu}g_{\lambda\bar{v}})dz^{\lambda}\wedge dz^{\mu}\wedge d\bar{z}^{v}\\
&+\frac{1}{2}i(\partial_{\bar{\lambda}}g_{\mu\bar{v}}-\partial_{\bar{v}}g_{\mu\bar{\lambda}})d\bar{z}^{\lambda}\wedge dz^{\mu}\wedge d\bar{z}^{v}=0,
\end{aligned}
$$

where we take

$$\omega = ig_{\mu\bar{v}}dz^{\mu}\wedge d\bar{z}^{v},$$

from which we find

$$\frac{\partial g_{\mu\bar{v}}}{\partial z^{\lambda}}=\frac{\partial g_{\lambda\bar{v}}}{\partial z^{\mu}} \qquad\qquad \frac{\partial g_{\mu\bar{v}}}{\partial\bar{z}^{\lambda}}=\frac{\partial g_{\mu\bar{\lambda}}}{\partial\bar{z}^{v}}. \tag{2.27}$$

If we suppose that a Hermitian metric g is given on chart U_i by

$$g_{\mu\bar{v}}=\partial_{\mu}\partial_{\bar{v}}\kappa_i \tag{2.28}$$

where $\kappa_i \in \mathrm{F}(U_i)$. This metric satisfies the condition in equation (2.27) hence, it is Kähler. Conversely, it can be shown that any Kähler metric is locally expressed as in equation (2.28). The function κ_i is a real function called the Kähler potential of a Kähler metric. It follows that $\omega = i\partial\bar{\partial}\kappa_i$ on U_i. This presents a huge simplification since a single function gives the full metric. The function κ_i does not need to be defined globally, and the Kähler potentials on various patches are related by Kähler transformations through holomorphic and antiholomorphic functions [18]. Using the Dobeault operators one can write the Kähler form as

$$\omega = 2i\partial\partial\kappa_i \tag{2.29}$$

In the following section, we will explore further Kähler geometry and we will introduce another important structure called the Ricci tensor which is one of the ingredients in the Calabi-Yau manifold definition.

2.3.3 Kähler geometry

Kähler geometry is the study of Kähler manifolds, their geometry, and topology, as well as the structures and constructions that can be applied to Kähler manifolds, including the existence of special connections such as the hermitian Yang-mills connections, [19]. Kähler manifolds possess several compatible structures that enable them to be described from different viewpoints. In this thesis, we will examine Kähler manifolds from a complex perspective, as previously defined in the earlier section. The main goal in this section is to introduce two crucial elements of Kähler geometry that are also major components of Calabi-Yau manifolds, namely the Ricci-tensor and Ricci form.

The Ricci-tensor and Ricci-form of a Kähler manifold

We will give a few definitions as a setup for the section before giving the definition of Ricci-tensor and Ricci-form.

Definition 2.11. *Let (X, g) be any Riemannian manifold. In local real coordinates x^μ, we define the Christofell symbols as [20]*

$$\Gamma^\rho_{\mu v} \equiv \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma v} + \partial_v g_{\mu\sigma} - \partial_\sigma g_{\mu v}).$$

These symbols form a component of a torsion-free connection ∇, called the Levi-Civita connection which preserves the metric, $\nabla g = 0$. Generally, the Levi-Civita connections are natural generalizations of the connection defined in the classical geometry of surfaces [20]. It follows from the result in equation (2.27) that the Kähler metric is torsion free:

$$T^{\lambda}_{\mu v} = g^{\bar{\zeta}\lambda}(\partial_\mu g_{v\bar{\zeta}} - \partial_v g_{\mu\bar{\zeta}}) = 0 \tag{2.30}$$

$$T^{\bar{\lambda}}_{\bar{\mu}\bar{v}} := g^{\bar{\lambda}\zeta}(\partial_{\bar{\mu}} g_{\bar{v}\zeta} - \partial_{\bar{v}} g_{\bar{\mu}\zeta}) = 0 \tag{2.31}$$

This condition adds an extra symmetry to the Riemann tensor

$$R^{k}_{\lambda\mu\bar{v}} = -\partial_{\bar{v}}(g^{\bar{\zeta}k}\partial_\mu g_{\lambda\bar{\zeta}}) = -\partial_{\bar{v}}(g^{\bar{\zeta}k}\partial_\lambda g_{\mu\bar{\zeta}}) = R^{k}_{\mu\lambda\bar{v}} \tag{2.32}$$

as well as the known symmetry operations [14],

$$R^{\bar{k}}_{\bar{\lambda}\bar{\mu}v} = R^{\bar{k}}_{\bar{\mu}\bar{\lambda}v}, \qquad R^{k}_{\lambda\bar{\mu}v} = R^{k}_{v\bar{\mu}\lambda}, \qquad R^{\bar{k}}_{\bar{\lambda}\mu\bar{v}} = R^{\bar{k}}_{\bar{v}\mu\bar{\lambda}} \tag{2.33}$$

Definition 2.12 (Riemann Curvature). *The curvature R of a connection ∇ is a rank $(1,3)$-tensor with components, in local coordinates*

$$R^{\sigma}_{\mu v\rho} \equiv \partial_\mu \Gamma^{\sigma}_{v\rho} - \Gamma^{\tau}_{\mu\rho}\Gamma^{\sigma}_{v\tau} - (\mu \leftrightarrow v),$$

where Γ are the connection symbols of ∇.

Definition 2.13. *The Ricci-tensor on a Riemannian manifold is defined by*

$$R_{\mu v} = R^{\rho}_{\mu\rho v} \tag{2.34}$$

The Ricci-tensor is symmetric in its indicies. On a Kähler manifold this takes a particularly simple form. The components with both holomorphic indices are zero and similar for both anti-holomorphic indices

$$R_{ab} = R_{\bar{a}\bar{b}} = 0 \tag{2.35}$$

while the non-vanishing components are

$$R_{\overline{ab}} = \partial_a \partial_{\bar{b}}[log\sqrt{det(g)}] \tag{2.36}$$

where we have used equation (2.34). From this, we can define the Ricci form $\mathfrak{R}$ as,

$$\mathfrak{R} = i\partial_{\bar{v}}\partial_{\mu}log(g)dz^{\mu} \wedge d\bar{z}^{v}. \tag{2.37}$$

As stated earlier, Ricci tensors are very crucial when studying Calabi-Yau manifolds, which we will discuss in Chapter 4. It measures the curvature of a manifold. In Calabi-Yau manifolds, the Ricci tensor is zero, which means the manifold is Ricci flat. This is a key property of Calabi-Yau manifolds, and it is necessary for them to be considered Calabi-Yau. Ricci flat manifolds are also important in physics, as they solve Einsteins' equations in a vacuum. This marks the end of our discussion on complex manifolds. However, before we conclude, we should highlight the numerous applications of complex and Kähler manifolds in the field of differential geometry and string theory. In the upcoming Chapter, we will briefly look at one of the most significant applications of complex and Kähler manifolds, namely Hodge theory. This fascinating topic introduces another kind of differential form known as harmonic forms. Toward the end of our computation, we will utilize this form to verify the exactness of certain parameters.

CHAPTER 3

HODGE THEORY

(This Chapter is based on [14], Chapter 7.9)

In the previous Chapter, we examined differential forms on both real manifolds and complex manifolds. These differential forms act as machinery for Hodge's theory. One important observation in Hodge's theory is that when a manifold X is equipped with a Riemannian metric g, every cohomology class has a canonical representative known as harmonic forms [21]. These are differential forms that vanish when subjected to the Laplacian operator of the metric which we will discuss in the Chapter. We will introduce a new operator "Hodge star" and use this to define the adjoint of the exterior derivative; $d^{\dagger}$. From this, we will define the Laplacian operator in terms of exterior derivative d and $d^{\dagger}$. Thereafter, we will look at the "Hodge decomposition theorem" which is one of the main takeaways for the Chapter.

Hodge theory further enables us to define an inner product on manifolds and plays a crucial role in relating the topology of the moduli space to the geometry of the manifold. This can be valuable in understanding the properties of heterotic string theory. This is one of the reasons we are interested in Hodge's theory. In our computation, we will use the Hodge decomposition theorem and Harmonic forms to measure which forms live in the cohomology class of interest, which we will look at later in Chapter 6.

3.1 Operations in Hodge Theory

[[14], Chapter 7]

Before we look at the details of the Chapter, recall that in Chapter 2, we introduced differential forms and the operator d known as an exterior derivative. If we consider an exterior derivative d on the manifold X, as defined in section 2.14, the operator d has an adjoint, denoted as $d^{\dagger}$ which maps p-forms to $p-1$-forms. To gain a better understanding of the meaning of this operation, it is necessary to introduce the notion of Hodge star operation, denoted as $*$. Given an m-dimensional manifold, X, one can show that $\Omega^p(X)$ is isomorphic to $\Omega^{m-p}(X)$. If X is a Riemannian manifold, we call this isomorphism $*$.

Definition 3.1. *The Hodge star is a linear map, $* : \Omega^p(X) \to \Omega^{m-p}(X)$ whose action on a basis vector of $\Omega^p(X)$ is defined by*

$$*\omega = \frac{\sqrt{|g|}}{p!(m-p)!}\omega_{\mu_1\mu_2\cdots\cdots\mu_p}\epsilon^{\mu_1\mu_2\cdots\cdots\cdots\mu_p}{}_{v_{p+1}\cdots\cdots v_m}dx^{v_{p+1}}\wedge\cdots\cdots\wedge dx^{v_m} \tag{3.1}$$

for any $\omega = \frac{1}{p!}\omega_{\mu_1\mu_2\cdots\cdots\cdots\mu_p}\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge\cdots\cdots\wedge dx^{\mu_p} \in \Omega^p(X)$. Where $\epsilon^{\mu_1\mu_2\cdots\cdots\cdots\mu_p}$ is a totally anti-symmetric tensor defined as:

$$\epsilon^{\mu_1\mu_2\cdots\cdots\cdots\mu_p} = \left\{\begin{array}{ll} +1 & if\ \ \mu_1\mu_2\cdots\cdots\cdots\mu_p\ is\ an\ even\ permutation\ of\ (12,\cdots\cdots,m), \\ -1 & if\ \ \mu_1\mu_2\cdots\cdots\cdots\mu_p\ is\ an\ odd\ permutation\ of\ (12,\cdots\cdots,m), \\ 0 & otherwise. \end{array}\right\}$$

and $m = dim(X)$.

In general case we call $\sqrt{|g|}dx^{\mu_2} \wedge\cdots\cdots\wedge dx^{\mu_p}$ a top-form. In Chapter 2, we stated that differential forms allow us to perform calculus on a manifold X. It turns out that, in order to perform an integration over an m-dimensional manifold, we require a non-vanishing m-form, called top-form. If X is orientable and equipped with a metric g then there exists a top form defined as:

$$\chi_X := \sqrt{|g|}dx^1 \wedge \cdots\cdots \wedge dx^m$$

which is invariant under a coordinate transformation. Here g denotes the determinant, $detg_{(ab)}$, while x^μ are the local coordinates of the chart (U, ϕ).

Theorem 4 ([14]Theorem 7.4)**.** *Given a Riemannian manifold, (X,g), of dimension* m *with* $\omega \in \Omega^p(X)$ *then*

$$* * \omega = (-1)^{p(m-p)}\omega.$$

The theorem explains that if we $**$ on a differential form ω, we obtain the original ω with either a plus or minus sign.

As stated earlier in the Chapter's introduction, we will now look at another important feature that allows us to define the inner product on differential forms of a manifold X. Take two forms ω and $\eta \in \Omega^p(X)$. We then define an exterior product $\omega \wedge *\eta$ that gives us an m-form as:

$$\begin{aligned}
\omega \wedge *\eta = &\frac{1}{(p!)^2}\omega_{\mu_1\cdots\mu_p}\eta_{v_1\cdots v_p}\frac{\sqrt{|g|}}{(m-p)!}\epsilon^{v_1\cdots v_p}{}_{\mu_{p+1}\cdots\mu_m} \\
&\times dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_m} \\
&= \frac{1}{p!}\sum_{\mu v}\omega_{\mu_1\cdots\mu_p}\eta^{v_1\cdots v_p}\frac{1}{p!(m-p)!}\epsilon_{v_1\cdots v_p\mu_{p+1}\cdots\mu_m} \times \epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_m}\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^m \\
&= \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\eta^{\mu_1\cdots\mu_p}\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^m.
\end{aligned}$$

This expression shows that the product is symmetric. That is:

$$\omega \wedge *\eta = \eta \wedge *\omega$$

Since $\omega \wedge *\eta$ is a top-form, its integral over a manifold X is well defined. We can then define the inner product $\langle \omega, \eta \rangle$ of two p-forms as:

$$
\begin{aligned}
\langle \omega, \eta \rangle &\equiv \int \omega \wedge *\eta \\
&= \frac{1}{p!} \int_X \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} \eta^{\mu_1 \cdots \mu_p} \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^m
\end{aligned}
$$

Notice $\omega \wedge *\eta = \eta \wedge *\omega$ guarantees that the inner product is symmetric. If (X, g) is Riemannian, the inner product is positive definite, that is $\langle \omega, \omega \rangle \geq 0$, where the equality holds only when $\omega = 0$. We will now define the adjoint operator $d^\dagger$.

Definition 3.2 ([14], Definition 7.6). *Let $d : \Omega^{p-1}(X) \to \Omega^p(X)$ be the exterior derivative operator, the adjoint exterior derivative $d^\dagger : \Omega^p(X) \to \Omega^{p-1}(X)$ is defined by*

$$d^\dagger = (-1)^{mp+m+1} * d* \tag{3.2}$$

where m is the dimension of X.

Theorem 5 ([14], Theorem 7.5). *Suppose (X, g) is a compact orientable manifold without a boundary and $\omega \in \Omega^p(X)$,$\eta \in \Omega^{p-1}(X)$. Then*

$$\langle \omega, d\eta \rangle = \langle d^\dagger \omega, \eta \rangle \tag{3.3}$$

In the previous Chapter, we showed that the exterior derivative $d^2 = 0$. It follows that the adjoint operator $d^\dagger$ squares to zero in a similar fashion. Notice

$$d^{\dagger 2} = *d * *d* \propto *d^2* = 0$$

Now, we will introduce one more operator, the Laplacian operator, before presenting the Hodge decomposition theorem, which is the final section of this Chapter.

3.2 Laplacian and Hodge decomposition

Definition 3.3 ([14], Definition 7.7)**.** *The Laplacian, Δ, is a map $\Delta : \Omega^p(X) \to \Omega^p(X)$ defined by*

$$\Delta = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d \tag{3.4}$$

Example 3 ([14], Example 5.11 and 7.16)**.** *In electromagnetism, it can be shown that the electromagnetic potential, A, is a one form denoted as $A = A_\mu dx^\mu$. If we consider an electromagnetic field tensor, which is a two-form tensor, expressed as $F = dA$. This allows us to rewrite Maxwell's equation in a different way, where the two source-less equations $\nabla.B$ and $\nabla \times E = -\frac{\partial B}{\partial t}$ reduces to the identity,*

$$dF = d^2 A = 0$$

For the other two equations, we first let ρ be the electric charge density and j the electric current density. Together ρ and j constructs the current one form $j = \eta_{\mu v} j^v dx^\mu = -\rho dt + j.dx$. Then the two remaining Maxwell equations with the source $\nabla.E = \rho$ and $\partial \times B - \frac{\partial E}{\partial t} = j$ become

$$d^\dagger F = d^\dagger dA = j.$$

We are free to change the vector potential A under gauge transformations; $A \to A + d\epsilon$. We can then always choose an A that satisfies the Lorentz condition*, $d^\dagger A = 0$. Then the equation $d^\dagger F = d^\dagger dA = j$ becomes $(dd^\dagger + d^\dagger d)A = \Delta A = j$*

Recall from the previous Chapter we defined a closed form on a differential form $\omega \in \Omega^p(X)$. In a similar fashion, ω is called coclosed-form if it satisfies $d^\dagger w = 0$. Finally, a p-form is called harmonic if $\Delta\omega = 0$. This leads us to the theorem below.

Theorem 6 ([14],Theorem 7,6). *On a Riemannian manifold, a p-form ω is called harmonic if and only if ω is closed and coclosed.*

This theorem follows from the definition of harmonic form. Notice that $\Delta\omega = 0$ implies that $d^\dagger d\omega + dd^\dagger\omega = 0$. We can consider $d^\dagger d\omega$ and $dd^\dagger\omega$ as orthogonal vectors, implying that $d^\dagger d\omega = dd^\dagger\omega = 0$, which further implies that $d\omega = 0$ and $d^\dagger\omega = 0$. Moreover, following our definition of exact forms in Chapter 2, a p-form ω is called *coexact* if it can be expressed globally as

$$\omega_p = d^\dagger\beta_{p+1} \tag{3.5}$$

where $\beta_{p+1} \in \Omega^{(p+1)}(X)$. We refer to the set of co-exact forms as $d^\dagger\Omega^{p-1}(X)$. We now finally introduce the Hodge decomposition theorem.

Theorem 7 (Hodge decomposition theorem, [14], Theorem 7.7). *Let (X, g) be a compact orientable Riemannian manifold without a boundary then the $\Omega^p(X)$ is uniquely decomposed as:*

$$\Omega^p(X) = d\Omega^{p-1}(X) \oplus d^\dagger\Omega^{p+1}(X) \oplus Harm^p(X), \tag{3.6}$$

where $Harm^p(X)$ denotes the space of harmonic forms. In particular, any $\omega \in \Omega^p(X)$ can be expressed globally as:

$$\omega = d\alpha + d^\dagger\beta + \gamma \tag{3.7}$$

where $\alpha \in \Omega^{p-1}(X)$, $\beta \in \Omega^{p+1}(X)$, and $\gamma \in Harm^p(X)$.

Remark: $Harm^p$ is denoted as $\mathcal{H}^p$ and is isomorphic to a comology class $H^p(X)$ on a manifold X, i.e $\mathcal{H}^p(X) \cong H^p(X)$.

Notice that if ω is closed β vanishes, that is

$$\omega = d\alpha + \gamma \tag{3.8}$$

which implies that $[d\alpha+\gamma] \in H^p(X, \mathbb{R})$. This tells us that there is a unique harmonic p-form representative in each cohomology class of $H^p(X, \mathbb{R})$.

If X is a complex manifold and $\omega^{(p,q)}$ is a (p,q)-form, the Hodge decomposition allows us to write:

$$\omega^{(p,q)} = \bar{\partial}\alpha^{(p,q-1)} + \bar{\partial}^{\dagger}\beta^{(p,q+1)} + \gamma^{(p,q)} \tag{3.9}$$

Where $\gamma^{(p,q)}$ is a harmonic form with respect to Laplacian

$$\Delta_{\bar{\partial}} = \bar{\partial}^{\dagger}\bar{\partial} + \bar{\partial}\bar{\partial}^{\dagger} \tag{3.10}$$

Remark: $\bar{\partial}^{\dagger}$ and $\Delta_{\bar{\partial}}$ are defined in a similar fashion as in sections 3.2 and 3.3 respectively, with $\bar{\partial}$ as defined in section 2.2.3.

Similar to the real case, if $\omega^{(p,q)}$ is $\bar{\partial}$-closed, then the above decomposition gives us a unique harmonic (p,q)-form representative for each class in $H^{(p,q)}(X, \mathbb{C})$. In a special case where X is a Kähler manifold, it can be shown that all of the Laplacians built from d, $\bar{\partial}$ and ∂ namely, Δ , $\Delta_{\bar{\partial}}$, and Δ_{∂}, are related by:

$$\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial} \tag{3.11}$$

In this case, the harmonic forms with respect to each operator are the same [9]. We define $h^{(p,q)}(X)$ to be the complex dimension of $H^{(p,q)}_{\bar{\partial}}(X)$ as in Chapter 2, which is the same as the dimension of the vector space of harmonic (p, q)-forms on X. The Hodge star operator with the obvious extension into the complex realm ensures that:

$$h^{(p,q)}(X) = h^{(m-p,m-q)}(X). \tag{3.12}$$

Using complex conjugation and Kählerity we also have

$$h^{(p,q)}(X) = h^{(q,p)}(X). \tag{3.13}$$

The Kählerity further ensures that [9]

$$H^r_d(X) = \bigoplus_{p+q=r} H^{(p,q)}_{\bar{\partial}}(X), \tag{3.14}$$

which relates the cohomology classes on a real and complex manifold X.

As already stated in the Chapter's introduction the main point of this Chapter is that every p-form $\omega \in \Omega^p(X)$ can be decomposed into three components as shown in theorem 7, which gives a representative in a respective cohomology class called a harmonic form when ω is closed. In the next Chapter, we will study Calabi-Yau manifolds as our compactification geometry, where we will explore the hodge number $h^{(p,q)}(X)$ in the context of Calabi-Yau manifolds in detail.

CHAPTER 4

CALABI-YAU MANIFOLDS

We are now ready to investigate Calabi-Yau manifolds. As already pointed out in the introduction, the Calabi-Yau manifold has proven to be the best candidate for effective string compactifications. The importance of the Calabi-Yau manifold in the field of string theory has led to its extensive study in the past decades. Since the origin of the universe, a lot of theories have been established to study the nature of the particles that make all matter and their interactions. One of the scientists' best current theories is the Standard model of particle physics, which is also regarded as a ”low-energy theory”. Compared to string theory, the model describes the most basic building blocks of the universe, and explains how quarks and leptons make up all known matter. However, despite its major contribution to theoretical physics and the literature, the theory is incompatible with the theory of general relativity. In trying to fill this gap supersymmetry comes into play. This is an extension of the Standard model that predicts partner particles for each particle in the Standard model. In string theory, supersymmetry acts as a connection between the two types of particles: bosons and fermions, Calabi-Yau manifolds preserve this property, which makes them more favourable for string compactification.

While the mathematical study of Calabi–Yau manifolds has helped us to understand compactifications of string theory, the study of string theory has led to fascinating insights into the geometry of Calabi–Yau manifolds, for example, the study of the Calabi–Yau moduli space and mirror symmetry [20]. Understanding the compactification geometry is vital for effective string compactification. Taking a Calabi-yau manifold X as our compactification geometry in this thesis, the moduli space of X

will play a major role in our computational analysis in Chapter 6. Our main focus in this Chapter is to explore some properties of Calabi–Yau manifolds, with particular emphasis on the moduli space of Calabi-Yau manifolds and Hodge numbers which we mentioned in the previous Chapter. We will end this Chapter by quickly looking into a few examples of Calabi–Yau manifolds.

4.1 Calabi-Yau Geometry

Before delving into the specifics, we will first provide a precise definition of the Calabi-Yau manifold and examine its associated geometry thereafter.

Definition 4.1. *A Calabi-Yau manifold X is a compact complex, Kähler manifold with vanishing first Chern class, which according to Yau's theorem, admits a Ricci-flat metric. i.e*

$$R_{i\bar{j}} = -\frac{\partial}{\partial \bar{z}^{\bar{j}}}\Gamma^{k}_{ik} = 0, \tag{4.1}$$

where the parameters in the equation are the same as we defined in Chapter 2.

There are many ways to define a Calabi-Yau manifold, but all the definitions are proven to be equivalent. One can also define a Calabi-Yau manifold as a compact Kähler manifold of complex dimension n that admits a nowhere vanishing holomorphic n-form, Ω [20]. Using a similar definition of differential forms as in Chapter 2. We can define Ω in local complex coordinates, z^a and $\bar{z}^a$; for $a = 1, ..., n$ as;

$$\Omega = \frac{1}{n!}\Omega_{a_1 .... a_n}(z) dz^{a_1} \wedge .... \wedge dz^{a_n} \tag{4.2}$$

where the coefficients are holomorphic functions. For the case of Calabi-Yau threefold, this is a 3-form given as

$$\Omega = \frac{1}{3!}\Omega_{abc}(z)dz^a \wedge dz^b \wedge dz^c \tag{4.3}$$

In relation to the definition of Calabi-Yau manifolds, we will state two important theorems that give some insights into the Calabi-Yau geometry.

Theorem 8. *The holomorphic Ω is closed.*

Theorem 9 (Yau)**.** *If X is a complex Kähler manifold with vanishing first Chern class and with Kähler form J, then there exists a unique Ricci-flat metric on X whose Kähler form J' is in the same cohomology class as J.*

The utility of this theorem is that it is generally quite hard to directly determine whether or not X admits a Ricci-flat metric g. In fact, to this date, no explicit Ricci-flat metrics have been discovered on any Calabi-Yau manifolds. For the proofs of these theorems, we refer the reader to [22, 20]. However, as previously stated, according to Calabi and proven by Yau, an n-dimensional complex Kähler manifold with vanishing first Chern class admits a metric with $SU(n)$ holonomy. In this thesis, we will generally take Calabi-Yau to mean holonomy being precise $SU(3)$. For clarity, holonomy is a concept in differential geometry, that measures the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported. It is normally defined on a connection of a smooth manifold. The Chern classes of X probe basic topological properties of X. Specifically, the k-th Chern class $c_k(x)$ is an element of cohomology group $H^k_d(X)$ defined from the expansion [23];

$$c(x) = 1 + \sum_j c_j(x) = det(1 + R) = 1 + trR + tr(R \wedge R - 2(trR)^2) + ... \tag{4.4}$$

where R is the matrix-valued curvature 2-form

$$R = R^k_{li\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \tag{4.5}$$

The curvature tensor R is considered as the curvature tensor of the tangent bundle TX of X. However, the Chern classes are dependent only on the basic topological properties of X, even though they are constructed from the local curvature tensor. For example, if X has a vanishing Ricci tensor, then the first Chern class, which is the trace of the curvature 2-form vanishes[23]. The Chern classes are very vital in defining Calabi-Yau manifolds. Analogously, it is easy to compute the first Chern class than determine the presence of Ricci-flat metric on a manifold X.

In the next section, we will discuss the moduli spaces of Calabi-Yau manifolds. However, it is worthwhile to recall the definitions of complex and Kähler manifolds as discussed in Chapter 2. Where we defined an n-dimensional complex manifold X, as a topological space equipped with a holomorphic atlas. We can define a complex structure J^b_a on X as mentioned in Chapter 2, section 2.1; that admits the following conditions:

$$J^b_a J^c_a = -\delta^c_a$$

$$0 = N^a_{bc} \equiv J^d_b(\partial_d J^a_c) - J^d_c(\partial_d J^a_b - \partial_b J^a_d) \tag{4.6}$$

where N^a_{bc} is the Nijenhuis tensor and is identically zero for a complex manifold. The complex structure J is a 2-form tensor that satisfies $J^2 = -\mathbb{I}$. For a Kähler manifold, given a hermitian metric g on a complex manifold X we can construct (1,1)-form ω defined as $\omega = i g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$. X is Kähler if ω is closed i.e $d\omega = 0$. The complex structure J and Kähler form ω play a crucial role in computing the moduli space of Calabi-Yau manifolds, which we are about to delve into.

4.2 Moduli space of Calabi-Yau manifolds and Hodge numbers

It is of interest to know what types of theories arise at low energies from the compactification of the heterotic theory. For instance, the compactification of six out of the ten dimensions of string theory on a Calabi-Yau manifold produces an effective four-dimensional supergravity theory at energies much smaller than the plank scale. The fields of this theory corresponds to the parameters describing possible Calabi-Yau manifold deformations. The number of parameters varies depending on the structure of the complex manifold and the parameters related to the deformations of its Kähler metric, which create the complex structure moduli space and the Kähler structure moduli space, respectively. These spaces relate to the cohomology groups of (2,1) forms and (1,1) forms. For a Calabi-Yau threefold, these cohomology classes and their dimensions determine the number of massless fields on M_4 that generate the desired spectrum. The dimension of a cohomology class $H^{(p,q)}$ represents the Hodge numbers. In this section, we will discuss the general outline of Hodge numbers on Calabi-Yau manifolds, particularly for the Calabi-Yau threefolds. Additionally, we will delve into the resulting moduli spaces from the deformations of the Calabi-Yau metric.

4.2.1 Hodge Numbers Of Calabi-Yau Manifolds

The Hodge numbers of a Calabi-Yau manifold, denoted as $h^{(p,q)}$, represent the dimensions of its Dolbeault cohomology classes, $H^{(p,q)}(X)$, as stated above [23]. All Calabi-Yau manifolds are complex and Kähler manifolds and Cohomology class on Calabi-Yau is defined as Dolbeault cohomology, as defined in Chapter 2, definition 2.7. The properties of a Kähler manifold impose certain conditions on the Calabi-

$$
d = 3 \quad \left|\quad
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
\;=\;
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\right.
$$

Figure 4.1: Hodge Diagram for Calabi-Yau manifold with $d = 3$

Yau manifold X, which allows us to modify the Hodge numbers. For instance, since the holonomy of X is $SU(d)$, one can show that $h^{(0,s)} = h^{(s,0)} = 0$ for $1 < s < d$, and $h^{(0,d)} = h^{(d,0)} = 1$. Here, $h^{(d,0)}$ denotes a holomorphic nowhere vanishing differential form of type $(d, 0)$ on X, typically referred to as Ω, as defined above. Throughout our discussions, we assume that X is connected, and for any connected manifold X, $h^{(0,0)} = 1$. Knowledge of these properties enables us to define all the Hodge numbers of a given Calabi-Yau manifold, X. For example, a Calabi-Yau threefold has a complex dimension of $d = 3$, and its Hodge numbers can be defined as shown in the Hodge diagram in figure 4.1, where the arrangement is known as the Hodge diamond. The diagram summarises all the properties of the Hodge numbers on a Calabi-Yau threefold. As stated in Chapter 3 equations (3.12) and (3.13) we note that $h^{(p,q)}$ is always equal to $h^{(q,p)}$, when using complex conjugation, and Hodge-star duality ensures that $h^{(p,q)} = h^{(n-q,n-p)}$, as shown in the right of the diagram in figure 4.1.

The weighted sum of the Hodge numbers creates the Euler characteristic, denoted by χ, which is a topological invariant. For a Calabi-Yau threefold, the Euler characteristic χ is:

$$\chi = \sum_{p,q=0}^{3} (-1)^{p+q} h^{(p,q)}. \tag{4.7}$$

As previously mentioned, $h^{(1,1)}$ and $h^{(2,1)}$ determine the dimensions of the Kähler and complex structure moduli spaces, respectively, which collectively establish the complete metric moduli space of a Calabi-Yau threefold [4]. The compactification of heterotic strings relies on smooth Calabi-Yau threefolds. To construct practical models with minimal complexity and an accurate depiction of M_4, we search for a Calabi-Yau manifold with small Hodge numbers. The majority of known Calabi-Yau manifolds with small Hodge numbers are generated as quotients of simply connected manifolds by a freely acting group [24]. Having looked at the Hodge numbers we are now ready to discuss moduli spaces.

4.2.2 Complex and Kähler moduli

As previously noted, the moduli spaces on the Calabi-Yau manifold result from the deformations of the Calabi-Yau metric, g. Essentially, this entails the deformation of the complex structure and Kähler form, classified as the Calabi-Yau metric moduli. These two sectors are independent of each other. Any infinitesimal deformation of g can be broken down into one of two index types:

$$g \to g(0) + \delta g \Rightarrow \delta g \begin{cases} \text{Kähler fluctuations, of type (1,1)} \\ \text{complex structure fluctuations, of type (2,0) + (0,2)} \end{cases} \tag{4.8}$$

For a given complex structure J on a complex manifold X, there exists a cohomology group that determines all possible infinitesimal deformations of the complex structure, denoted as $H^{(0,1)}(T^{(1,0)}(X))$, where T denotes the holomorphic tangent bundle that we will briefly discuss in the next Chapter. Under small fluctuations δJ, using the vanishing of the Niejenhuis tensor, we obtain [1]:

$$J^i_j \to J^i_j + \delta J^i_j, \quad \Rightarrow \delta J^a_b = \delta J^{\bar{a}}_{\bar{b}} = 0, \quad \bar{\partial} \delta J^a_{\bar{b}} = 0. \tag{4.9}$$

This implies that $\delta J^a_{\bar{b}} \in H^1(TX) \cong H^{(2,1)}(X)$ [1]. Similarly to the deformations of a complex structure X, we can also consider the deformations of a Kähler form ω. More precisely, for X a Kähler manifold, recall that X it is endowed with a metric $g_{i\bar{j}}dx^i \otimes d\bar{z}^{\bar{j}}$ from which we can construct the Kähler form $\omega = ig_{i\bar{j}}dx^i \otimes d\bar{z}^{\bar{j}}$ which is closed and defines the cohomology class in $H^{(1,1)}(X)$. Deformations of the Kähler structure on X refers to deformations of the Kähler metric which preserve the $(1,1)$ nature of ω, and cannot be realized by a change of coordinates on X [22]. If we deform the Kähler form ω, the fluctuations $\delta\omega$ remain closed and $\delta\omega \in H^1(T^*X) \cong H^{(1,1)}(X)$. In Chapter 6 we will demonstrate that these fluctuations indeed contribute to total moduli space in the standard embedding.

For more insights, we now give two examples of Calabi-Yau manifolds, which is our final section in this Chapter.

4.3 Examples of Calabi-Yau manifolds

Roughly, there are half a billion examples of Calabi-Yau manifolds that physicists and mathematicians have constructed so far. Almost all known Calabi-Yau manifolds are constructed using one of the following three main approaches namely; Algebraic Varieties, Fibrations, and Surgery. In this thesis, we will only focus on the Calabi-Yau manifolds built from algebraic varieties. In which we embed X in a simple (complex) geometry, an affine space $\mathbb{A}$ as the zero locus of a set of polynomials i.e

$$p_1(x_i), p_2(x_i), ...p_n(x_i) = 0. \tag{4.10}$$

Here $p_j(x_i)$'s are polynomials in complex coordinates x_i's of $\mathbb{A}$. Here the topological data of X is determined by the topology of $\mathbb{A}$ and the form of the polynomial expressions above [1]. The simplest example of such a manifold is the quintic hypersurface in $\mathbb{P}^4$ which we will discuss later in the section.

4.3.1 Single Hypersurface Calabi-Yau Manifolds

We will first consider a single hypersurface in an ambient projective space $\mathbb{P}^n$, defined by one holomorphic equation. We define X as the zero locus of polynomial

$$p(z^i) = 0 \tag{4.11}$$

of homogeneous degree m in holomorphic projective coordinates of $\mathbb{P}^n$.

Definition 4.2 (Projective Space). *Recall in Chapter 2, example 2 we defined the projective space, $\mathbb{P}^n$, as the set of lines in $\mathbb{C}^{n+1}$ through the origin which can be viewed as the set of equivalence classes of points $(z_0, z_1, ..., z_n) \neq (0, 0, ..., 0)$ in $\mathbb{C}^{n+1}$ modulo scaling by nonzero elements of $\mathbb{C}$.*

Note, the elements in $\mathbb{P}^n$ are referred as points written as $(z_0 : z_1 : .... : z_n)$. Thus

$$(z_0 : z_1 : .... : z_n) = (y_0 : y_1 : .... : y_n) \tag{4.12}$$

means there exists a nonzero $\lambda \in \mathbb{C}$ such that $(z_0, z_1, ..., z_n) = (\lambda y_0, \lambda y_1, ..., \lambda y_n)$. These $(n+1)$-tuples, well defined only up to scale, are referred to as the homogeneous coordinates of points in $\mathbb{P}^n$ [14]. It is crucial to note that the polynomial (4.10), must be homogeneous and holomorphic to get a well-defined complex manifold. If this condition fails to hold, its zero locus would not be well defined under scaling relation in $\mathbb{P}^n$.

In order for the projective hypersurface defined above to be a Calabi-Yau manifold, it needs to satisfy some more extra conditions. According to Yau's theorem, we need a Kähler manifold with vanishing first Chern class. It can be shown that any complex submanifold of a Kähler manifold (defined via holomorphic equations) is Kähler [9], however, the condition on the curvature given by $c_1(TX) = 0$ is less obvious . In order for this condition to be satisfied, we use the natural Kähler metric that comes together with $\mathbb{P}^n$, called the Fubini-Study metric:

$$g_{a\bar{b}} = \partial_a \, \partial_{\bar{b}} \log(|z_0|^2 \; + ... + |z_n|^2) \tag{4.13}$$

which be restricted to the $p(z^i) = 0$ hypersurface [1]. Note this restricted metric will not be Ricci-flat! However, it can be shown that the Ricci 2-Form has the following form:

$$R_{Fs}|_x = ((n+1) - m) \, g_{Fs} + totalderivatives. \tag{4.14}$$

(see [25]for details). So if $m = n + 1$ then $[tr(R)] = 0$, and this means $c_1(TX) = 0$. Therefore if the degree condition is met, X is a Calabi-Yau manifold! The Ricci-flat metric will not be the one we obtained above, but it will share the topological property (invariant under metric deformations) that $c_1(TX) \propto [tr(R)] = 0$ [1].

4.3.2 Quintic Calabi-Yau Manifold

As in the example above, the quintic Calabi-Yau manifold is the simplest known Calabi-Yau threefold, in which a hypersurface of degree 5 is embedded in $\mathbb{P}^4$, denoted as $\mathbb{P}^4[5]$. It can be described as the locus of the equation:

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \tag{4.15}$$

These coordinates define five complex variables in $\mathbb{C}^5$.
We can projectivise to $\mathbb{CP}^5$ by introducing inhomogeneous coordinates as in example (2). For example, in the patch where $z_5 \neq 0$, $y_k = \frac{z_k}{z_5}$, $k = 1, ..., 4$.

In these coordinates, the equation above can be written as

$$y_1^5 + y_2^5 + y_3^5 + y_4^5 = -1 \tag{4.16}$$

The solution of the equation is a complex three-dimensional compact space, and we can eliminate e.g y_4 in terms of the others. This manifold turns out to be Calabi-Yau. This can be shown by defining a holomorphic three-form

$$\Omega = \frac{1}{y_4^4} dy_1 \wedge dy_2 \wedge dy_3 \tag{4.17}$$

That this form is globally defined and nowhere vanishing is a non-trivial exercise that relies on the fact that we took polynomials of fifth-degree in equation (4.15). Again, any quintic polynomial

$$\sum_{ijklm} a_{ijklm} z_i z_j z_k z_l z_m = 0 \tag{4.18}$$

yields a Calabi-Yau threefold. The symmetric tensor a_{ijklm} contains 126 independent parameters. This can be shown using the fact that a symmetric tensor with r indices running over n values has $\begin{pmatrix} n+r-1 \\ r \end{pmatrix}$ independent components [20] where in our case $r = n = 5$ this gives us $\begin{pmatrix} 5+5-1 \\ 5 \end{pmatrix} = 126$. Notice that not all of these result in distinct defining equations. The $GL(5,\mathbb{C})$ action on $\mathbb{P}^4$ leads to $(25-1)$ coordinates redefinitions [1]. This means that $126 - (25-1)$ independent components remain after rescaling, which correspond to complex moduli. These moduli can only be observed through algebraic description. Additionally, the number of ambient spaces in which we define our polynomial corresponds to Kähler moduli. Consequently, for quintic, $h^{(2,1)}(X) = 102$ and $h^{(1,1)}(X) = 1$, then it follows from the equation

(4.7) that the Euler number for quintic is 101. Due to the higher Euler number, the quintic manifolds are not typically favored for string compactification. This is because the Euler number does not align with the number of generations in low energy theory.

In conclusion, despite the numerous number of Calabi-Yau manifolds in the literature, Calabi-Yau threefolds are perhaps the most interesting due to their connection to string theory. Moreover, a comprehensive understanding of Calabi-Yau geometry requires more advanced mathematics than what we have covered in this Chapter. For proper comprehension, we urge the reader to refer to [26, 27] for further reading.

CHAPTER 5

VECTOR BUNDLES

Bundles, generally called fiber bundles, are topological spaces that locally look like a direct product of two topological spaces [14]. Vector bundles are examples of fiber bundles, and they are essential objects for studying the properties of the space X. In algebraic geometry, vector bundles allow examining geometric properties of a space from a more algebraic perspective. Essentially, in string theory, vector bundles are one of the key ingredients for effective string compactification. For example, in heterotic string compactification, the Calabi-Yau manifold alone does not determine the heterotic string compactification [19]. In addition, one must construct a gauge bundle with a hermitian Yang-Mills connection. The connection satisfies complicated non-linear differential systems which can be lifted with some algebraic geometry criterion [19]. It suffices to construct a stable holomorphic vector bundle on X. This is the Donaldson-Vehlenbeck-Yau theorem.

The presence of such a connection breaks the 10-dimensional E_8 gauge symmetry down to a subgroup, $G \subset E_8$ where $G \times H \subset E_8$ and H is the gauge symmetry associated with the gauge fields over the compact directions [1]. In this Chapter we will provide an overview of vector bundles with one example, and how they are linked to heterotic string compactification.

5.1 Vector bundles

5.1.1 Definitions and examples

Definition 5.1 (Vector bundle)**.** *A vector bundle $E \xrightarrow{\pi} X$ is a smooth map with a fibre F, where π is surjective and every preimage, $\pi^{-1}(x)$, is isomorphic to the vector space F. In other words, a vector bundle is a fiber bundle whose fiber, F, is a vector space [14].*

Note that X can be a compact real or complex manifold and it is called a base manifold. E is the total space of the bundle. Similar to the definition of manifolds, we can cover X with open sets and local coordinates $\{U_i, \phi_i\}$, such that the map $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ called local trivialization is an isomorphism. The coordinate transformation between patches is similar[1]. We define the transition functions on $U_i \cap U_j$ as $t_{ij} = \phi_i \circ \phi_j^{-1} : U_j \times F \to U_i \times F$. Over any point $x \in U_i \cap U_j$, $t_{ij}(x)$ is a homomorphism inside the vector space F. For the purpose of this thesis in the later Chapters and other sections, we will take the base space, X to be the Calabi-Yau manifold and the fiber F will be associated with a representation of a Lie group.

The transition functions belong to $GL(k, \mathbb{R})$, since it maps a vector space onto another vector space of the same dimension isomorphically. If F is a complex vector space $\mathbb{C}^k$, the structure group is $GL(k, \mathbb{C})$. In principle, the transition functions can be elements of Lie groups G, in various representations. G is called the structure group of the bundle and the rank of the bundle $rk(V)$ is the dimension of F as a vector space [1]. In most cases we take G to be the smallest subgroup of $GL(n, \mathbb{K})$, for which the transition functions all take values in G. Here, $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$.

The vector bundle whose fiber is one-dimensional $(F = \mathbb{R}\ or\ \mathbb{C})$ is called a line bundle [14].

Example 4 (Canonical line bundle, L)**.** *Recall that an element p of $\mathbb{P}^n$ is a complex line in $\mathbb{C}^{n+1}$ through the origin. The fibre $\pi^{-1}(p)$ of L is defined to be the line in $\mathbb{C}^{n+1}$ which belongs to p. More formally, let $I^{n+1} \equiv \mathbb{CP}^n \times \mathbb{C}^{n+1}$ be a trivial bundle over $\mathbb{P}^n$. If we write an element of I^{n+1} as (p, v), $p \in \mathbb{P}$, $v \in \mathbb{C}^{n+1}$, L is defined by*

$$L \equiv \{(p, v) \in I^{n+1} \,|v = ap, a \in \mathbb{C}\} \tag{5.1}$$

The projection is $\pi : (p, v) \to p$ [14].

5.2 Section of a vector bundle

A section of a vector bundle, $E \to X$, over a base manifold X, is a map $S : X \to E$. S is a section if for every point $p \in X$, $S(p)$ is an element of the fiber $E_p = \pi^{-1}(p)$. In other words, a section assigns to each point of the base space a vector in the corresponding fiber [28]. Locally this means over each open patch in the base manifold X, there is a map $S_i : U_i \to F$ such that for any $p \in U_i$, $S_i(p)$ is a unique vector in F. These local maps glue together by the transition functions as $S_i = t_{ij}S_j$ to make a global section [14, 1].

Sections of vector bundles have many applications in different fields of Mathematics and Physics. They are greatly used in studying the notion of curvature and connection, which are essential tools in differential geometry and quantum field theory which we are interested in.

5.3 Connection and Curvature

5.3.1 Connections On Vector Bundles

Intuitively, a connection on a vector bundle can be thought of as a way to parallel transport vectors along the curves on the base manifold, similar to the tangent vectors on a manifold. The aim of this section is to introduce the concept of connection on a complex vector bundle, which is a fundamental ingredient for defining covariant derivatives and constructing gauge theories that will be discussed in the next section.

In the context of heterotic string compactifications, the connection on a complex vector bundle plays a significant role in defining the gauge field on the compactified space. A connection on a complex vector bundle over a smooth manifold X is given by a differential 1-form on X with values in the Lie algebra of the structure group of the bundle. In the specific case of heterotic string compactifications, the structure group is usually taken to be a subgroup of the gauge group $E_8 \times E_8$ or $SO(32)$ [5].

Mathematically, a connection is a differential operator that acts on the sections of the complex vector bundle.

Definition 5.2. *Let $\pi : E \to X$ be a complex vector bundle, as defined in section 5.1. A connection on E is a linear map*

$$\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E)$$

which satisfies Leibniz rule:

$$\nabla(\sigma \otimes \tau) = (\nabla\sigma) \otimes \tau + \sigma \otimes (\nabla\tau) \forall \sigma, \tau \in \Gamma(E)$$

Here $\Gamma(E)$ is the space of sections of E, and T^*X is a cotangent bundle on X. Consider the local space $U_i \times F$, with the basis $\{e^1, ..., e^p\}$ for F. Then parallel transport of e^i in the direction μ in the base manifold is defined by the connection:

$$\nabla_\mu e^i = \partial_\mu e^i + A_\mu{}^i{}_j e^j,$$

where A_μ is a one form with values in the structure group lie algebra [1]. A_μ is called the gauge potential.

5.3.2 Curvature

Definition 5.3. *[14] The curvature two-form is given by*

$$\mathcal{F} \equiv \nabla^2 = dA + A \wedge A \in \Omega^2(X) \otimes g \tag{5.2}$$

where g is the Lie-algebra of the structure group G also denoted $End(E)$. The curvature $\mathcal{F}$ is identified with the Yang-Mills field strength, which we will briefly explain in the next section. In particular, the curvature of the connection determines the gauge field strength and the topological properties of the bundle that are intricately tied to the topology of the compactified space [5].

5.4 Gauge Theory

A gauge theory is a physical theory that describes the behavior of fields that transform under certain symmetry groups, called gauge groups. For example a$U(1)$ gauge group, which is Abelian and one dimensional. The theory plays an important role in studying heterotic string compactifications [29]. The type of string theory we are interested in the $E_8 \times E_8$ heterotic string, contains two independent gauge groups each associated with one of the E_8 factors. For string compactifications to lower dimensions we define a gauge bundle with a gauge connection A, which describes how the gauge fields are twisted as they wrap around the compact dimension. The presence of such $<A> \neq 0$ breaks the 10-dimensional E_8 gauge symmetry down to a subgroup, $G \subset E_8$ where $G \times H \subset E_8$ and H is the gauge symmetry associated to the gauge fields over the compact directions [1].

Geometrically, gauge theory is typically formulated in terms of vector bundles over a compactification space X, such as the Calabi-Yau manifold [29], where the gauge fields over X are represented by the section of a vector bundle over the space and the gauge transformations are represented by certain automorphisms of the bundle. The curvature of the bundle is crucial for the theory, and it leads to the appearance of gauge fields and other geometric structures [30]. Mathematically, gauge theory involves the study of connections on vector bundles and the associated curvature forms.

There are several examples of gauge theories, and one of them is Yang-Mills theory In the Yang-Mills theory, particles are not seen as isolated objects, but rather as fields that pervade space and time. These fields interact with each other through the exchange of other particles, such as gluons or photons. The interactions between these fields are described by mathematical gauge fields. The gauge fields A_μ act the same as a connection on the vector bundle [5].

5.4.1 The Yang-Mills theory

Yang-Mills' theory rests on the idea of a Lie group. A compact Lie group G has an underlying Lie algebra $\mathfrak{g}$, whose generators T^a satisfy

$$[T^a, T^b] = iF^{abc}T^c \tag{5.3}$$

where $a, b, c = 1, ..., dimG$, F^{abc} is fully anti-symmetric. The factor i on the right ensures that the generators are hermitian. For each element of the algebra, we introduce a gauge field, A^a_μ. In general, these are packaged in the Lie algebra value gauge potential [31]

$$A_\mu = A^a_\mu T^a. \tag{5.4}$$

From the gauge potential, we construct the Lie-algebra valued field strength:

$$\mathcal{F} \equiv dA + A \wedge A = \frac{1}{2}\mathcal{F}_{\mu v} dx^\mu \wedge dx^v \tag{5.5}$$

where

$$\mathcal{F}_{\mu v} = \partial_\mu A_v - \partial_v A_\mu + [A_\mu, A_v] = F^a_{\mu v} T^a, \tag{5.6}$$

As pointed out above, in mathematical terminology, A_μ is called a connection and the field strength $F_{\mu v}$ is referred to as the curvature.
The Yang-Mills action is defined as [14];

$$S_{YM}(A) = \frac{1}{4}\int_X trF \wedge *F \tag{5.7}$$

where X, is a compact manifold endowed with a metric g and tr denotes the trace.

The classical equations of motion are derived by minimizing the action with respect to the gauge field, $\nabla \mathcal{F} = (d+A)\mathcal{F} = d_A\mathcal{F} = 0$ is a well-known identity called Bianchi identity which follows from the definition of the curvature $\mathcal{F}$ [14]. The variation of the action with respect to $\mathcal{A}_\mu$ yields the equation of motion

$$\nabla_\mu \mathcal{F}^{\mu v} = 0 \qquad or \qquad \nabla * \mathcal{F} = 0 \tag{5.8}$$

Note also that, $S_{YM}(A)$ is invariant under global gauge transformations, $\mathcal{F} \to G\mathcal{F}G^{-1}$,

$$S_{YM}(G\mathcal{F}G^{-1}) = \frac{1}{4}\int_X tr\, G\,\mathcal{F}\,G^{-1} \wedge *\, G\,\mathcal{F}\,G^{-1} = S_{YM}(\mathcal{F}). \tag{5.9}$$

Furthermore, to solve equations of motion we can consider the infinitesimal variations, $\delta S_{YM}(A)$, which yield;

$$\frac{1}{4}\int_X tr(\delta F \wedge *F + F \wedge *\delta F) \tag{5.10}$$

by applying Leibniz rule. The vanishing of $\delta S_{YM}(A)$ satisfies the equations of motion. We are looking for a gauge field A that optimizes $\delta S_{YM}(A)$. If we consider $F = dA + A \wedge A$, under small fluctuations, δF, gives;

$$d_A\delta A + \delta A \wedge A + A \wedge \delta A = d_A \delta A$$

since $A \wedge \delta A$ is antisymmetric. Where locally A resides in $\Omega^1(End(E))$. Now the equation (5.10) reduces to

$$\begin{aligned} \delta S_{YM}(A) = &\quad \frac{1}{4}\int_X tr(d_A\delta A \wedge *F + F \wedge *d_A\delta A) \\ = &\quad \frac{2}{4}\int_X tr(d_A\delta_A \wedge *F). \end{aligned}$$

Applying Stokes' theorem yields;

$$\frac{1}{2}\int_X tr(\delta A \wedge d_A * F) = 0$$

Notice here that $d_A * F$ is a Bianchi identity which is trivial, then we conclude that the Yang-Mills action S_{YM} indeed solves equations of motion. Before we go to the next section which is the last section of this Chapter we point out that the Yang-Mills theory is a broad topic in quantum field theory. It underlies the standard model of particle physics, describing both the weak and strong forces, for more infomation we urge the reader to refer to [32].

5.5 Holomorphic Bundles

For manifolds we have complex manifolds, the same as in vector bundles we have Holomorphic vector bundles. As one of the geometric properties we have looked at so far, holomorphic vector bundles are an important aspect in the study of heterotic string compactification and realistic models. They play a key role in understanding the geometric and algebraic structures of these models. In this section, we will provide an overview of holomorphic vector bundles and their properties.

Definition 5.4. *Let X be a complex manifold. A holomorphic vector bundle E over X is a topological space E, together with a projection $\pi : E \to X$, and a collection of local trivialization biholomorphism maps $\psi_i : U_i \times \mathbb{C}^n \to \pi^{-1}(U_i)$, where $U_i \subseteq X$ are open sets, such that the following compatibility condition holds on the overlaps:*

$\psi_i \circ \psi_j^{-1} = t_{ij} : U_i \cap U_j \to U_i \cap U_j$ is holomorphic for each pair of open sets (U_i, U_j) [14]. Pointwise, the transition functions t_{ij} are elements of $GL(n, \mathbb{C})$.

The collection of local trivialization maps $(\{U_i, \psi_i)\}$ is called a holomorphic atlas for the bundle. Consider a connection d_A on a vector bundle, we define a curvature $\mathcal{F}(A) = d_A^2$, for $\mathcal{F}(A) \in \Omega^2(End(E))$. Notice,

$$\begin{aligned} \mathcal{F} &= d_A^2 \\ &= (\bar{\partial}_A + \partial_A)^2 \\ &= \bar{\partial}_A^2 + \bar{\partial}_A \partial_A + \partial_A \bar{\partial}_A + \partial_A^2 \end{aligned} \tag{5.11}$$

Where, $d_A = \bar{\partial}_A + \partial_A$. It can be shown that, a vector bundle E is holomorphic if and only if $\bar{\partial}_A^2 = 0$, which implies that the curvature, $F^{(0,2)} = 0$. $\bar{\partial}_A$ also defines a cohomology $H^{(p,q)}_{\bar{\partial}_A}(E)$.

Holomorphic vector bundles provide a manifold with a finer topological structure than the generic vector bundle. The reason we are more interested in Holomorphic vector bundles is, they play a crucial role in the physics of heterotic string compactifications, where they are used to describe certain gauge fields and internal degrees of freedom [12].

In conclusion, holomorphic vector bundles provide a powerful tool for understanding the geometric and algebraic structures of realistic models under heterotic string compactification, which we shall discuss later in the next Chapter.

CHAPTER 6

MODEL BUILDING

As pointed out in the introduction, an effective compactification preserves the features that describe the 4-dimensional(M_4) Minkowski space we observe, such as chiral matter, gauge symmetries, and supersymmetry which is not observed yet. This imposes some constraints on the compactification geometry.

This Chapter is a central part of this thesis, in which we will look at some realistic heterotic string models. In particular, a model based on the $E_8 \times E_8$ heterotic string theory, which is perhaps the most studied heterotic string theory. The compactification of $E_8 \times E_8$ theory involves curling up the six dimensions of the 10-dimensional theory on a Calabi-Yau manifold[33]. Many of the constructions and computations in this Chapter will be based on the properties of the features we discussed in the previous Chapters.

In Chapter 4, we explained that, at low energies (small compared to the plank scale), the components of the 4-dimensional supergravity theory that arises after the compactification corresponds to the parameters that describe the possible deformations of the Calabi-Yau manifold, called moduli spaces. We further pointed out that the Kähler and Complex structures lead to the classification of Calabi-Yau metric moduli. It turns out that, to work out the effective physics of a string compactification, we need much more information than just the metric and its moduli.

This Chapter aims to answer the question; ”what is the total heterotic moduli space?” by computing the total moduli space for a given Calabi-Yau manifold in the case of standard embedding. In the next section, we will discuss standard embedding and show the explicit computations of its total moduli in the later sections.

6.1 Standard Embedding

In Chapter 5, we discussed how the 10-dimensional E_8 gauge symmetry breaks down to subgroup $G \subset E_8$ in the presence of the connection $<A> \neq 0$. The standard embedding is one of the important techniques in heterotic string compactification which was first introduced by [3]. It involves embedding the gauge group of the low-energy effective theory of heterotic string compactification into the structure group of the Calabi-Yau manifold.

In a grand unified theory (GUT) all the forces; the strong force, weak force, and electromagnetic force come together in a single force. This GUT gauge group contains the Standard Model gauge group, G_{SM}, which comprises the strong force, weak force, and electromagnetic force, i.e

$$G_{SM} = SU(3) \times SU(2) \times U(1)$$

At high energy, spontaneous symmetry breaking occurs, and the gauge group becomes the standard model group [34]. We are interested in string setups that have the gauge group containing the standard model. A popular one is the E_6 gauge group, where $G_{SM} \subset E_6 \subset E_8$.

Consider the $E_8 \times E_8$ heterotic string, where we take the connection for the first E_8 bundle to be the spin connection of the Calabi-Yau manifold, which we refer to as standard embedding [3]. The connection lives in the adjoint representation of E_8 under the maximal subgroup $SU(3) \times E_6 \subset E_8$. Note that for physical analysis E_8 as the adjoint representation has a 248 dimension which decomposes under $SU(3) \times E_6$ as:

$$248 = (3, 27) + (\bar{3}, \bar{27}) + (1, 78) + (8, 1) \tag{6.1}$$

where 78 and 8 are adjoint representation of E_6 and $SU(3)$ respectively, and 27 , 3 are their respective fundamental representation with their respective conjugates $\bar{3}$ and $\overline{27}$ [1].

At the level of Dolbeault cohomology, this yields;

$$H^{(0,p)}(X, V_{E_8}) = H^{(0,p)}(TX) \otimes 27 + H^{(0,p)}(T^*X) \otimes \overline{27} + H^{(0,p)}(O) \otimes 78 + H^{(0,p)}(End_o TX) \otimes 1$$

Notice the tangent space, TX, and cotangent space, T^*X, transform to fundamental representation in $SU(3)$ and its conjugate respectively. In Chapter 2, we defined dolbeault cohomology on $\bar{\partial}$. These cohomology groups are essential to understanding the reduction of the gauge sector [35]. As we shall see below, the spectrum is counted by $p = 1$.

Recall the Calabi-Yau manifold, X, lives in a special holonomy $SU(3)$, and it is endowed with a metric g. We define a connection on X as a Levi-Civita connection, ∇_a^{LC}, that resides in $\Omega^{(1,0)}(End(T^{(1,0)}(X))) \subseteq Ad(SU(3))$ as a subgroup of $End(T^{(1,0)}X)$. That is, $\nabla_a^{LC} \in \Omega^{(1,0)}(8)$. In the standard embedding, we take the gauge connection, A_a, to be the Levi-Civita connection, ∇_a^{LC}. The fluctuations of A need to reside in the adjoint representation of E_8; $[\delta A] \in H^{(0,1)}(End(E)) = H^{(0,1)}(248)$. The holomorphy-preserving deformations are then counted by cohomology, $H^{(0,1)}(248)$.

It follows from this that the desired spectrum is computed when $p = 1$. Which yields;

$$H^{(0,1)}(248) = H^{(0,1)}(8) \oplus H^{(0,1)}(T^{*(1,0)}) \otimes 27 \oplus H^{(0,1)}(T^{(1,0)}) \otimes \overline{27} \oplus H^{(0,1)}(X) \otimes 72. \quad (6.2)$$

Moreover, the cohomology groups that give rise to the spectrum we are interested in are $H^{(0,1)}(T^*X)$ and $H^{(0,1)}(TX)$. This is because on a Calabi -Yau manifold, the cohomology group $H^{(0,1)}(O)$ is trivial since $H^{(0,1)}$ vanishes on the Calabi-Yau manifold as shown in figure 4.1, and the matter field we are interested in lives in the fundamental representation of E_6 [36]. However, $H^{(0,1)}(End(TX))$ is non-trivial yet, is not charged under E_6 and does not contribute to the charged spectrum of the 4-dimensional Minkowski space.

The fluctuations of these cohomology groups correspond to the matter fields in M_4. Using complex top form and the fact that TX transform to fundamental representation in $SU(3)$, we obtain isomorphisms:

$$H^{(0,1)}(T^*X) \cong H^{(1,1)}(X)$$

and

$$H^{(0,1)}(TX) \cong H^{(2,1)}(X).$$

These give the same count as the Kähler moduli and complex moduli respectively. It is the dimensions of these cohomology groups that determine the desired spectrum. The spectrum turns out to be computed by the Euler number, χ. For Calabi-Yau threefolds, as highlighted in Chapter 4 this can be computed as:

$$\chi = \sum_{p,q=0}^{3} (-1)^{p+q} h^{(p,q)}$$

Moreover, it can be shown from figure 4.1 in Chapter 4 that $\chi = 2|(h^{(1,1)} - h^{(2,1)})|$ on Calabi-Yau threefolds. This formula is particularly useful in determining the number of chiral fermions that appear in the low-energy effective theory resulting from a heterotic string compactification on a Calabi-Yau manifold, X. Specifically, the Euler number of the Calabi-Yau manifold determines the number of generations of chiral fermions in the standard model, requiring a Calabi-Yau threefold with $\chi = 6$ to generate the desired three generations, as assumed in [3]. However, as pointed out in Chapter 4, finding a Calabi-Yau manifold with an Euler number of 6 can be challenging, so one alternative to consider is a Calabi-Yau manifold with a freely acting discrete group, G_0, which lacks fixed points resulting in singularities. These manifolds have a nice structure in which the Euler number is reduced by a factor equal to the order of G_0 when one quotients out with the respective G_0. Until now, this has been the most effective approach to finding a Calabi-Yau manifold with an Euler number of ∓ 6. It is important to note that the sign does not matter, for instance, Lütken's in his paper [24] offers good examples of freely acting Calabi-Yau manifolds with an Euler number of ∓ 6 which aligns with the number of generations in the standard model.

It is worth noting that the Euler number only provides a spectrum for standard embedding, while the index of the bundle is used for the general case. Nevertheless, we focus on the Euler number because fundamental and antifundamental terms can combine to create mass terms. For example, coupling $\alpha^{aA} \in H^{(0,1)}(T) \otimes 27$ and $\beta_{bA} \in H^{(0,1)}(T^*) \otimes \overline{27}$ results in $\alpha^{aA} \wedge \beta_{bA}$, which lies in $H^{(0,2)}$ when a and b are contracted, and is thus zero on the Calabi-Yau manifold. This is not very helpful. However, if we also have $\gamma_a^b \in H^{(0,1)}(End(T))$, coupling it with $\alpha^{aA} \wedge \beta_{bA}$ it produces a non-vanishing 3-form in $H^{(0,3)}$, $\alpha^{aA} \wedge \beta_{bA} \wedge \gamma b_a$, this is non-zero on Calabi-Yau threefold. The coulpling is known as Yukawa coupling which effectively gives a mass to all equal number of fundamental and anti-fundamentals. It is important to note that the Euler number only takes into account the massless terms that contribute to the total spectrum and leaves out all the mass terms.

We now turn to the moduli sector. We will examine the simultaneous deformations of both the metric and gauge connection. Specifically, we will explicitly calculate the complete heterotic moduli for the standard embedding scenario in which the metric and gauge connection are varied together. However, prior to this, we will provide additional insight into bundle moduli in the next section.

6.2 Bundle Moduli

In Chapter 4, Section 4.2, we discussed the fluctuations of the Calabi-Yau metric. We summarized the standard result that, in the absence of gauge fields or flux, the moduli of a Calabi-Yau manifold are parametrized by $h^{(1,1)}(X)$, Kähler moduli, and $h^{(2,1)}(X)$, complex structure moduli. Similarly to complex and Kähler moduli, the bundle moduli are derived by considering fluctuations, $A \to \delta A$. These fluctuations need to preserve the SUSY conditions, by satisfying the equations:

$$\begin{aligned} F_{\bar{a}\bar{b}} &= 0 \\ \omega \wedge \omega \wedge F &= 0 \end{aligned}$$

Remark; Supersymmetric solutions always give solutions to equations of motion. The fluctuations of such a connection,A, while the metric is held constant leads to

$$F_{\bar{a}\bar{b}} = 0 \quad \Longrightarrow \overline{D}(\delta A_{\bar{b}}) = 0$$

which implies that the bundle fluctuations preserving the holomorphic structure of

V are closed under the bundle-valued covariant operator, $\overline{D}$ [1]. Moreover, considering the fact that exact fluctuations are pure gauge, the bundle moduli can be expressed as;

$$BundleModuli \Longrightarrow H^{(0,1)}(X; End(V)) \tag{6.3}$$

$H^{0,1}(X; End(V))$ is a space of 1-forms since δA has one spacetime index over the Calabi-Yau manifold, and it is valued as adjoint in G. It has been observed in the literature that the naive massless singlets in a heterotic theory on a Calabi-Yau background can be calculated as shown in[1];

$$h^{(1,1)}(X) \; + \; h^{(2,1)}(X) \; + \; h^{(0,1)}(X; End(V)). \tag{6.4}$$

These dimensions are associated with complex structure moduli, Kähler moduli, and bundle moduli respectively, as cohomology classes. In the upcoming section, we will further look at the Simultaneous Variation of the metric and gauge connection.

6.3 Moduli Problem: Simultaneous Variation of the metric and gauge connection

In the previous section, the computation of bundle moduli assumed a constant metric, and similarly, the gauge fields were omitted when considering the metric fluctuations. However, this is not the actual case as in a true fluctuation, all fields must be varied simultaneously. This raises questions about whether the bundle and manifold can constrain each other. By examining the equations that arise from super-symmetry variations, it is evident that $F_{ab} = F_{\bar{a}\bar{b}} = 0$ and $g^{a\bar{b}}F_{a\bar{b}} = 0$ are dependent on the complex structure and Kähler structure of X, as well as on the fluctuations of the gauge connection as discussed in [1].

This brings our focus to what happens when the metric and connection are varied simultaneously. With this in mind, it is now possible to consider the simultaneous perturbations of the complex structure on X and the connection on V:

$$J = J^{(0)} + \delta J, \qquad\qquad A = A^{(0)} + \delta A$$

One way to comprehend this is by drawing inspiration from Atiyah's work [12] which focused on the deformations of a holomorphic bundle on a complex manifold. In a similar fashion, research has been conducted on the same topic within the context of heterotic theories [37]. It is beneficial to examine this work in detail as it sheds light on the complete heterotic narrative. Atiyah noticed in 1955 that not all complex structure deformations are allowed, as some might introduce a non-trivial $F^{(0,2)}$ component that could obstruct the holomorphicity of the bundle. Only the ones that satisfy

$$\bar{\partial}_A(\alpha_a A) + F_d \mu_a^d = 0, \quad and \quad \bar{\partial}\mu_a = 0 \tag{6.5}$$

are allowed. We will discuss this equation later in the upcoming section. Although such deformations can introduce $F^{(0,2)}$, they can be balanced by simultaneously deforming the bundle. To perceive these constraints more accurately, a short exact sequence vector bundle Q_1 needs to be defined

$$0 \xrightarrow{i_1} End(\mathbf{E}) \xrightarrow{i_2} Q_1 \xrightarrow{i_3} T^{(1,0)}X \xrightarrow{i_4} 0 \tag{6.6}$$

This is when the metric is fixed. From the definition of short exact sequences, it can be established that each map in the aforementioned sequence satisfies $ker(i_{n+1}) = Im(i_n)$ for all values of n. As a result, it is demonstrated that $End(E) \subset Q_1$, with the bundle Q_1 being defined as a direct sum of $End(E)$ and $T^{(1,0)}$, expressed as:

$$Q_1 = End(E) \oplus T^{(1,0)}X \tag{6.7}$$

Based on the properties of short exact sequences, we can define the connection,$\bar{\partial}_1$, on Q_1 as:

$$\begin{pmatrix} \bar{\partial}_A & \mathcal{F} \\ 0 & \bar{\partial} \end{pmatrix}$$

Here, $\bar{\partial}_A$ and $\bar{\partial}$ serve as connections on $End(E)$ and $T^{(1,0)}X$, respectively. $\mathcal{F}$ represents the curvature that resides in $End(V)$, defined as:

$$\mathcal{F} : \Omega^{(0,p)}(T^{(1,0)}) \to \Omega^{(0,p+1)}(End(E)),$$

which takes values from $\Omega^{(0,p)}(T^{(1,0))})$ to $\Omega^{(0,p+1)}(End(E))$.

From the long exact sequences in cohomology groups associated with the sequence 6.6, we can work out that:

$$H^1(X, Q_1) = H^1(End(\ v)) \oplus ker(\alpha) \tag{6.8}$$

Here, $\alpha = F^0_{a\bar{b}}$ is the Atiya class. In essence, the infinitesimal moduli are counted by $H^{(0,1)}_{\bar{\partial}}(Q_1)$. Cohomologically, this is precisely the content of equation (6.5)

As we are interested in the moduli when both the metric and the connection are varied simultaneously, we extend the above co-chain sequence with the cotangent bundle to yield:

$$0 \to \Omega^{(0,p)}(T^{*(1,0)}X) \to \Omega^{(0,p)}_{\bar{D}}(Q) \to \Omega^{(0,p)}(T^{(1,0)}X) \to 0 \tag{6.9}$$

In topological terms, we define Q to be $T^{*(1,0)} \oplus End(V) \oplus T^{(1,0)}X$. We can then define the operator on Q as:

$$\bar{D} = \begin{pmatrix} \bar{\partial} & \mathcal{H} \\ 0 & \bar{\partial}_1 \end{pmatrix} \tag{6.10}$$

which follows from the same logic as before. Here, $\mathcal{H}$ is an extension class defined as follows:

$$\mathcal{H} : \Omega^{(0,p)}(Q_1) \rightarrow \Omega^{(0,p+1)}(T^{*(1,0)}X)$$

Notice we can write $\bar{D}$ as:

$$\overline{D} = \begin{pmatrix} \bar{\partial}_A & \mathcal{H} \\ 0 & \bar{\partial} \end{pmatrix} = \begin{pmatrix} \bar{\partial} & \mathcal{F}^* & R.\nabla \\ 0 & \bar{\partial}_A & \mathcal{F} \\ 0 & 0 & \bar{\partial} \end{pmatrix} \tag{6.11}$$

$H^{(0,1)}_{\bar{D}}(Q)$ then gives the full moduli spectrum of heterotic compactifications at the standard embedding [16]. [16] provides comprehensive details on how this is computed, and further expounds upon the complexities that may exceed the parameters of this thesis. Our goal is to examine the moduli when both the metric and the connection vary simultaneously, for which we have defined an operator $\overline{D}$ on the space Q. The sequence (6.9) is the co-chain sequence associated with this analysis. In the following section, we will explore how these operators define the moduli spectrum on E_8 Heterotic string theory.

6.4 Total Heterotic Moduli: Case Of Standard Embedding

In this section, we will explicitly compute the total moduli of the E_8 heterotic string theory in the case of standard embedding and check if the simultaneous deformation of the metric and the connection gives rise to the desired spectrum. As previously defined in section 6.3, we use $\overline{D}$, as an operator on the bundle Q to find the total heterotic moduli from the simultaneous deformations of the metric and the gauge connection. We consider the action of $\overline{D}$ on some $(0,p)$-forms valued in Q_1 and the extension bundle Q. Later in the section, we will see that this action yields the equations:

$$\bar{\partial}x + R^{*}\alpha + R.\nabla\mu = 0 \tag{6.12}$$

$$\bar{\partial}_{A}\alpha + R\mu = 0 \tag{6.13}$$

$$\bar{\partial}\mu = 0, \tag{6.14}$$

Here $R^{*} = F^{*}$ and $R = F$ as we are at the standard embedding. Ensuring that these equations modulo $\overline{D}$-exact solutions result in the desired spectrum. Furthermore, we verify that under simultaneous variation of a metric and connection in standard embedding, the total moduli are given as:

$$H^{(0,1)}(Q) \cong H^{(1,1)}(X) \oplus H^{(2,1)}(X) \oplus H^{(0,1)}(End(V)), \tag{6.15}$$

as expected in the literature, which will be explicitly shown here. To interpret the equations (6.12) to (6.14) in cohomology, we consider the holomorphic bundle on a complex manifold. We are only interested in unobstructed deformations that lead to the cohomology classes of interest. These deformations must satisfy all the equations above. We first consider the equations, (6.14) and (6.13). Clearly, $\bar{\partial}\mu = 0$ implies that μ is $\bar{\partial}$-closed. We need to check if $[\mu] \in ker(R)$, which is required from the second equation:

$$dz^{\bar{a}} R_{c\bar{a}b}{}^{d} \wedge \mu^c = \bar{\partial}_A \alpha^b d. \tag{6.16}$$

If the equation (6.16) is satisfied, we can infer that $\mu_a R_{a\bar{b}} dz^{\bar{b}}$ is trivial in the cohomology, and $[\mu] \in ker(R)$.

To achieve this objective, we consider the vector bundle Q_1, in equation (6.6), as defined in section 6.3. Consider the short exact sequence as before

$$0 \xrightarrow{i_1} \Omega^{(0,p)}(End(V)) \xrightarrow{i_2} \Omega^{(0,p)}(Q_1) \xrightarrow{i_3} \Omega^{(0,p)}(T^{1,0}) \xrightarrow{i_4} 0 \tag{6.17}$$

One can then study $(0,p)$-forms valued in Q_1, which are vectors, schematically:

$$\begin{pmatrix} \alpha \\ \mu \end{pmatrix}$$

where α is a $(0,p)$-form valued in $End(V)$ and μ is a $(0,p)$-form valued in $T^{(1,0)}$. We further consider the operator $\overline{D}_1$ acting on Q_1 defined as follows:

$$\overline{D}_1 = \begin{pmatrix} \bar{\partial}_A & \mathcal{F} \\ 0 & \bar{\partial} \end{pmatrix} \tag{6.18}$$

The action of $\overline{D}_1$ on $\begin{pmatrix} \alpha \\ \mu \end{pmatrix}$ vanishes if the field deformation satisfies the required equations of motion, i.e,

$$\overline{D}_1 \begin{pmatrix} \alpha \\ \mu \end{pmatrix} = \begin{pmatrix} \bar{\partial}_A(\alpha) + \mathcal{F}(\mu) \\ \bar{\partial}(\mu) \end{pmatrix} = 0 \tag{6.19}$$

Furthermore, $\overline{D}_1^2 = 0$ if and only if $F^{(0,2)} = 0$ and its Bianchi identity, $d_A F = 0$ holds. The short exact sequence (6.17) gives a long-exact sequence in cohomology:

$$0 \to\ H^1(End(V))\ \to H^1(Q_1) \to\ H^1(T^{(1,0)})\ \xrightarrow{\mathcal{F}}\ H^{(0,2)}((End(V)) \to \cdots \tag{6.20}$$

Using the properties of exact sequences, we get:

$$H^1(Q_1) \cong H^1(End(V)) \oplus ker(\mathcal{F}) \tag{6.21}$$

where $ker(\mathcal{F}) \subseteq H^1(T^{(1,0)})$.

In general, deforming the complex structure and holomorphic bundle simultaneously gives:

$$\mu^a F_{a\bar{b}} dz^{\bar{b}} = \bar{\partial}_A \delta A, \tag{6.22}$$

which reduces to equation (6.16) in the standard embedding. This implies that in cohomology $\mathcal{F}([\mu]) = 0$, where $[\mu] \in H^{(0,1)}(T^{(1,0)})$ is a complex structure deformation. This is precisely the content of equation (6.21).

Deformations μ that do not preserve the equation (6.22) are obstructed. For such deformations, there are no corresponding bundle deformations that keep the bundle holomorphic. However, on Calabi-Yau manifolds, the bundle in question is the holomorphic tangent bundle, which is invariant under deformations. This guarantees that equation (6.13) has solutions in the standard embedding, since $V = TX$. Furthermore, this shows that $ker(\mathcal{F}) \equiv H^1(T^{(1,0)})$, hence the equation (6.21) becomes;

$$H^1(Q_1) \cong H^1(End(V)) \oplus H^1(T^{(1,0)}). \tag{6.23}$$

We will now examine the question of what value or expression of $\alpha^c{}_d$ satisfies equation (6.24):

$$dz^{\bar{a}} R_{c\bar{a}b}{}^{d} \wedge \mu^c = \bar{\partial}_A \alpha^b{}_d \tag{6.24}$$

In [16], the authors defined $\alpha^c{}_d = \nabla_d \mu^c + \alpha^c_0{}_d$ under certain manipulations. We assume $\bar{\partial}_A \alpha^c_{0d} = 0$, so that $\alpha^c_0{}_d$ is in $H^{(0,1)}(End(V))$ corresponding to a bundle modulus. Using this expression for $\alpha^c{}_d$, we explicitly check if it satisfies equation (6.24).

$$\bar{\partial}_A \alpha^c{}_d = \bar{\partial}_A(\nabla_d \mu^c + \alpha^c_0{}_d) \tag{6.25}$$

$$= \bar{\partial}_A \nabla_d \mu^c \tag{6.26}$$

$$= [\bar{\partial}_A, \nabla_d]\mu^c \tag{6.27}$$

$$= dz^{\bar{a}} R_{\bar{a}d}{}^{c}{}_{b} \wedge \mu^b \tag{6.28}$$

Noting that α_0 is $\bar{\partial}$ closed, the second term in the first part of the equation vanishes. The commutator bracket $[\bar{\partial}_A, \nabla_d]$ is the curvature, which is a Riemannian curvature with symmetry properties. Using these properties and Bianchi's identity we can derive the result that

$$R_{\bar{a}d}{}^{c}{}_{b} = R_{\bar{a}b}{}^{c}{}_{d}. \tag{6.29}$$

Therefore,

$$\Rightarrow \ \bar{\partial}_A \alpha^c{}_d = dz^{\bar{a}} R_{\bar{a}d}{}^{c}{}_{b} \wedge \mu^b = \partial z^{\bar{a}} R_{\bar{a}b}{}^{c}{}_{d} \wedge \mu^b \tag{6.30}$$

We conclude that $\alpha^c{}_d = \nabla_d \mu^c + \alpha^c_{0}{}_d$. Furthermore, the invariance of the holomorphic bundle on the Calabi-Yau manifold, which is $TX = V$ in the standard embedding, gives an important isomorphism between $Ker(\mathcal{F})$ and $H^{(0,1)}(T^{(1,0)})$, leading to a desired result.

Next, we turn on deformations of the hermitian metric and consider the extension bundle Q. Consider a short exact sequence

$$0 \to T^{*(1,0)}X \xrightarrow{\gamma} Q \xrightarrow{\pi} Q_1 \to 0 \tag{6.31}$$

Here, $T^{*(1,0)}$ denotes holomorphic co-vectors (forms). P-forms valued in Q are denoted as:

$$\delta = \begin{pmatrix} x \\ \alpha \\ \mu \end{pmatrix}$$

Consider the operator $\overline{D}$ acting on sections valued in the bundle Q, as defined in equation (6.11). Before analyzing the action of $\overline{D}$ on δ, we examine the short exact sequence defined in equation (6.9)

$$0 \to \Omega^{(0,p)}(T^{*(1,0)}) \xrightarrow{\gamma} \Omega^{(0,p)}(Q) \xrightarrow{\pi} \Omega^{(0,p)}(Q_1) \to 0$$

By similar arguments to those in equation (6.20), this yields the cohomological long exact sequence:

$$\begin{aligned} 0 \to H^0(T^{*(1,0)}) \to H^0(Q) \to H^0(Q_1) \to H^1(T^{*(1,0)}) \to H^1(Q) \\ \to H^1(Q_1) \xrightarrow{\mathcal{H}} H^2(T^{*(1,0)}) \to \cdots\cdots\cdots \end{aligned} \tag{6.32}$$

Using exactness, we can deduce that:

$$H^{(1,0)} \cong H^{*(1,0)} \oplus ker(\mathcal{H}), \tag{6.33}$$

with $Ker(\mathcal{H}) \subseteq H^{(1,0)}(Q_1)$. The sequence giving the sequance (6.20) is really:

$$
\begin{aligned}
0 \to\; & H^0(End(V)) \;\to\; H^0(Q_1) \;\to\; H^0(T^{(1,0)}) \;\to\; H^1(End(V)) \;\to\; H^1(Q_1) \\
& \to\; H^1(T^{(1,0)}) \;\xrightarrow{\mathcal{H}}\; H^2(End(V)) \;\to \cdots\cdots\cdots
\end{aligned}
\tag{6.34}
$$

Recall for compactifications in string theory, 4-dimensional supersymmetry imposes a constraint on holomorphic bundles: $g^{a\bar{b}}F_{a\bar{b}} = 0$. According to the Donaldson-Uhlenbeck-Yau theorem, such a condition is only satisfied in a holomorphic vector bundle over a compact Kähler manifold if the bundle is poly-stable. A bundle is called poly stable if it can be expressed as a combination of stable bundles [1, 38, 39]. In our analysis, we assume the bundle in question is stable, which on a Calabi-Yau manifold implies that it contains no holomorphic sections. This means that the cohomology class $H^0(T)$ is trivial. In the standard embedding scenario where $End(V) = End(T)$, this also renders $H^0(End(V))$ trivial. Consequently, the sequence (6.34) simplifies as follows.

$$
\begin{aligned}
0 \;\to\; & H^0(Q_1) \;\to\; 0 \;\to\; H^1(End(V)) \;\to\; H^1(Q_1) \\
& \to\; H^1(T^{(1,0)}) \;\xrightarrow{\mathcal{H}}\; H^2(End(V)) \;\to \cdots\cdots\cdots
\end{aligned}
$$

Therefore, we can conclude from this that $H^0(Q_1)$ is trivial, and this reduces the sequence 6.32 to:

$$
0 \;\to\; H^1(T^*) \;\to\; H^1(Q) \to\; H^1(Q_1) \xrightarrow{\mathcal{H}} H^2(T^*) \to \cdots
$$

From the properties of exact sequences, this gives us:

$$H^1(Q) \cong H^1(T^{*(1,0)}) \oplus Ker(\mathcal{H}) \tag{6.35}$$

where $Ker(\mathcal{H}) \subseteq H^1(Q_1)$.

Now, let's consider the map $\mathcal{H}$ in the following equation:

$$\mathcal{H}(\mu, \alpha) = \frac{\alpha'}{2} tr(F \wedge \alpha) - \frac{\alpha'}{2} R^c{}_d \nabla_c \mu^d. \tag{6.36}$$

To be moduli, we must have $\mathcal{H}(\mu, \alpha)$ trivial; $\mathcal{H}(\mu, \alpha) = \bar{\partial} x$. We want to verify that for any $\left[\begin{pmatrix} \alpha \\ \mu \end{pmatrix}\right] \in H^1(Q_1)$, the element $\left[\begin{pmatrix} \alpha \\ \mu \end{pmatrix}\right] \in Ker(\mathcal{H})$. Consider $\alpha^c{}_d$ as before, and taking $R = F$ we get:

$$\mathcal{H}(\mu, \alpha) = \frac{\alpha'}{2} R^c_d (\nabla_c \mu^d + \alpha_0{}^d{}_c) - \frac{\alpha'}{2} R^c{}_d \nabla_c \mu^d = \bar{\partial} x \tag{6.37}$$

$$\Rightarrow \frac{\alpha'}{2} R^c_d \nabla_c \mu^d + \frac{\alpha'}{2} R^c_d \, \alpha_0{}^d{}_c - \frac{\alpha'}{2} R^c{}_d \nabla_c \mu^d = \bar{\partial} x \tag{6.38}$$

$$\Rightarrow \frac{\alpha'}{2} R^c_d \, \alpha_0{}^d{}_c = \bar{\partial} x \tag{6.39}$$

Furthermore, We assume α_0 is closed and represents bundle moduli in $H^{(0,1)}(End(T))$. Also, let $x \in \Omega^{(1,1)}$, where x can be seen as the Kähler moduli with both closed and non-closed parts. The closed part of x is the Kähler moduli, the non-closed part are other things. By using Bianchi identity, we can see that the term $\frac{\alpha'}{2} R^c_d \, \alpha_0{}^d{}_c$ is $\bar{\partial}$ closed. We are looking for the α_0 so that this becomes exact.

One way to check this is to take the inner product with a harmonic form, such as $\bar{\chi} \in \mathcal{H}^{(1,2)}$. If the inner product is zero then $\frac{\alpha'}{2} R^c_d \, \alpha_0{}^d{}_c$ is $\bar{\partial}$ exact. This follows from the orthogonal decomposition of forms in the Hodge decomposition theorem. The exact forms are always orthogonal to harmonic forms, i.e. $\langle R^c_d\, \alpha_0{}^d{}_c, \bar{\chi}\rangle = 0$.

Let $\bar{\chi} \in \mathcal{H}^{(1,2)}(X)$ be harmonic then

$$\langle R^c_d\, \alpha_0{}^d{}_c, \bar{\chi}\rangle = \int R^c{}_d \alpha^d_0{}_c \wedge *\chi \tag{6.40}$$

where $*\chi$ is a harmonic $(2,1)$-form defined as:

$$*\chi = \frac{1}{2}\tilde{\mu}^a\, \Omega_{abc} dz^{bc} \tag{6.41}$$

Here, $*$ takes $(2,1)$-forms to (2,1)-forms. Also $\tilde{\mu}^a \in \mathcal{H}^{(0,1)}(T^{(1,0)})$. We can abuse the indices by introducing the free index "e", to yield:

$$\langle R^c_d\, \alpha_0{}^d{}_c, \bar{\chi}\rangle = \frac{1}{2}\int tr(R_e \wedge \alpha_0)\ \wedge\ \tilde{\mu_a}\ \wedge\ \Omega_{abc} dz^{bce} \tag{6.42}$$

Now we have 4-holomorphic indices which can be antisymmetrized as:

$$0 = \frac{1}{4} R_{[e}\Omega_{abc]} = (3\ R_e \Omega_{abc} - R_a \Omega_{ebc}) dz^{bce} \tag{6.43}$$

by considering all possible permutations of the indices with appropriate signs. The expression on the right-hand side arises from expanding out the antisymmetric expression and collecting terms. Using this, we find

$$\langle R^c_d\, \alpha_0{}^d{}_c, \bar{\chi}\rangle = \int tr(R_a \wedge \alpha) \wedge \tilde{\mu}^a \wedge \Omega. \tag{6.44}$$

But we already know that $R_a \tilde{\mu_a} = \bar{\partial}_A$-exact, as $\tilde{\mu_a}$ corresponds to a complex structure deformation. We can then apply Stokes' theorem to obtain:

$$\int tr(R_e \wedge \alpha_0)\ \wedge\ \tilde{\mu_a}\ \wedge\ \Omega_{abc} dz^{bce} = 0 \tag{6.45}$$

Therefore, we have shown that the inner product $\langle R^c_d\, \alpha_0{}^d{}_c, \bar{\chi}\rangle$ vanishes. Stokes' theorem relates the integral of the exterior derivative of a form over a closed domain $\mathcal{M}$ to the integral of the original form over the boundary of the domain $\partial\mathcal{M}$:

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega \tag{6.46}$$

We, therefore, conclude that $\mathcal{H}$ is trivial as a map in cohomology, that is :

$$Ker\mathcal{H} \equiv H^1(Q_1) \cong H^1(T^{(1,0)}) \oplus H^1(End(V)). \tag{6.47}$$

Therefore,

$$H^1(Q) = H^1(T^{*(1,0)} \oplus H^1(T^{(1,0)}) + H^1(End(V)) \cong H^{(1,1)}(X) \oplus H^{(2,1)}(X) + H^{(0,1)}(End(V)).$$

This proves the claim that in the standard embedding, the total heterotic moduli in $E_8 \times E_8$ heterotic string are given by, Kähler moduli, $H^{(1,1)}$, complex moduli, $H^{(2,1)}$, and bundle moduli $H^{(0,1)}(End(V))$ as claimed in the literature.

CHAPTER 7

Conclusion

String theory seeks to effectively describe the universe by understanding the spectrum that makes up its entirety. In this thesis, we focused on the moduli space of the $E_8 \times E_8$ heterotic string in the standard embedding. The concept of the moduli space has long been a central issue in heterotic string theory and is considered fundamental in determining the spectrum of M_4 in string compactifications. Moduli in string theory are classified as possible backgrounds for the string, and thus, it is important to identify which deformations can potentially lead to moduli of a given string model. While the formula for total heterotic moduli is provided in the literature, it falls short in proving the explicit result.

Our findings in Chapter 6 confirm that the calculation of massless singlets in heterotic theory on a Calabi-Yau background, in the context of standard embedding, is consistent with prior research. Where the massless singlets in a heterotic theory on a Calabi-Yau background is given as:

$$h^{(1,1)}(X) \; + \; h^{(2,1)}(X) \; + \; h^{(0,1)}(X; End(V)).$$

In which these dimensions correspond to complex structure moduli, Kähler moduli, and bundle moduli as cohomology classes. Our thesis has undertaken the task of clearly defining the full heterotic moduli space for $E_8 \times E_8$ heterotic string theory in the standard embedding. Our approach involved the simultaneous perturbations of the complex structure J on X, and the connection A on the vector bundle V. We drew inspiration from Atiya's work [12], which focused on the deformations of holomorphic bundles on complex manifolds. Using the resulting simultaneous deformations including the metric, we defined the bundle Q in equation (6.9) as an extension bundle of bundle Q_1, which is observed when the metric is fixed. The resulting bundle Q is then obtained by augmenting the cotangent bundle with the following:

$$Q = T^{*(1,0)}(X) \oplus End(V) \oplus T^{(1,0)}(X)$$

We further defined an operator $\overline{D}$ on Q and demonstrate explicitly that in the Standard embedding, the cohomology class $H^{(0,1)}_{\overline{D}}(Q)$ provides the complete moduli spectrum of heterotic compactifications.

Our computations were based on the assumption that, in the standard embedding, we consider the holomorphic tangent bundle TX as our vector bundle V, which remains holomorphic under deformations. The holomorphicity of the bundle ensures that no deformations are obstructed, thus leading to our desired result. Using the sequence defined in the equations (6.6) and (6.9), we established the bundle Q_1 and Q, respectively, as short exact sequences. We also leveraged these sequences to determine the spectrum found in a particular cohomology class. Some of the concepts used in our computation were adapted from [16]. In the final step of our computation, Hodge theory played a role as we applied the Hodge orthogonal decomposition to compute our final results, using that every element in cohomology $H^{(p,q)}_{\bar{\partial}}(X)$ has a canonical representative.

It is noteworthy to mention that some of the contents in this thesis may require additional knowledge beyond what has been covered here. For instance, a further understanding of the parameters behind the definition of the operator $\overline{D}$ on Q, which we used extensively in our computation, would be necessary. Nonetheless, based on our computations, we can confidently conclude that the total heterotic moduli for the $E_8 \times E_8$ heterotic string model in standard embedding is comprised of Kähler moduli $H^{(1,1)}(X)$, complex structure moduli $H^{(2,1)}(X)$, and bundle moduli $H^{(0,1)}(End(V))$, as claimed in the literature.

It would be intriguing to reconcile these findings with other string models. Further studies can be conducted to investigate if the results obtained in this thesis can be extended to examine other models, such as the line bundle models discussed in [6].

Bibliography

[1] Lara B Anderson and Mohsen Karkheiran. Tasi lectures on geometric tools for string compactifications. *arXiv preprint arXiv:1804.08792*, 2018.

[2] Eran Palti. The swampland: introduction and review. *Fortschritte der Physik*, 67(6):1900037, 2019.

[3] Philip Candelas, Gary T Horowitz, Andrew Strominger, and Edward Witten. Vacuum configurations for superstrings. *Nuclear Physics B*, 258:46–74, 1985.

[4] Philip Candelas and Xenia C de La Ossa. Moduli space of calabi-yau manifolds. *Nuclear physics b*, 355(2):455–481, 1991.

[5] Edward Witten. String theory dynamics in various dimensions. *Nuclear Physics B*, 443(1-2):85–126, 1995.

[6] Lara B Anderson, James Gray, Andre Lukas, and Eran Palti. Two hundred heterotic standard models on smooth calabi-yau threefolds. *Physical Review D*, 84(10):106005, 2011.

[7] Anthony Ashmore, Sebastian Dumitru, and Burt A Ovrut. Explicit soft supersymmetry breaking in the heterotic m-theory b- l mssm. *Journal of High Energy Physics*, 2021(8):1–51, 2021.

[8] Ron Donagi, Burt A Ovrut, Tony Pantev, and Daniel Waldram. Standard models from heterotic m-theory. *arXiv preprint hep-th/9912208*, 1999.

[9] Alexander Belopolsky. New geometrical approach to superstrings. *arXiv preprint hep-th/9703183*, 1997.

[10] Edward Witten. New issues in manifolds of su (3) holonomy. In *Current Physics–Sources and Comments*, volume 1, pages 195–228. Elsevier, 1988.

[11] Louis Witten and Edward Witten. Large radius expansion of superstring compactifications. *Nuclear Physics B*, 281(1-2):109–126, 1987.

[12] Michael Francis Atiyah. Complex analytic connections in fibre bundles. *Transactions of the American Mathematical Society*, 85(1):181–207, 1957.

[13] Loring W.. Tu. *An introduction to manifolds*. Springer., 2011.

[14] Mikio Nakahara. *Geometry, topology and physics*. CRC press, 2003.

[15] Xenia de la Ossa and Eirik E Svanes. Holomorphic bundles and the moduli space of n= 1 supersymmetric heterotic compactifications. *Journal of High Energy Physics*, 2014(10):1–55, 2014.

[16] Jock McOrist and Eirik Eik Svanes. Heterotic quantum cohomology. *Journal of High Energy Physics*, 2022(11):1–39, 2022.

[17] Ben McMillan. The newlander-nirenberg theorem. *Available online via: http://math. berkeley. edu/˜ bmcmilla/Talks/Newlander-Nirenberg\% 20Theorem. pdf*.

[18] Harold Erbin. Supergravity and kähler geometries. *France: Sorbonne Universités*, 2015.

[19] Volker Braun, Yang-Hui He, Burt A Ovrut, and Tony Pantev. Vector bundle extensions, sheaf cohomology, and the heterotic standard model. *Advances in Theoretical and Mathematical Physics*, 10(4):525–589, 2006.

[20] Stefan Vandoren. Lectures on riemannian geometry, part ii: Complex manifolds. *Fulltext on Stefan Vandoren's personal page*, 2008.

[21] Wikipedia. Hodge theory. [online; accessed on 10-May-2023. 2023.

[22] Paul S Aspinwall, Brian R Greene, and David R Morrison. Calabi-yau moduli space, mirror manifolds and spacetime topology change in string theory. *Nuclear Physics B*, 416(2):414–480, 1994.

[23] Challenger Mishra. *Calabi-Yau manifolds, discrete symmetries and string theory*. PhD thesis, University of Oxford, 2017.

[24] CA Lütken. A large class of calabi-yau vacua. Technical report, CM-P00055767, 1987.

[25] Anamarıa Font and Stefan Theisen. Introduction to string compactification. *Geometric and Topological Methods for Quantum Field Theory*, pages 101–181, 2005.

[26] Philip Candelas. Lectures on complex manifolds. In *Superstrings and grand unification*. 1988.

[27] Mark Gross, Daniel Huybrechts, and Dominic Joyce. *Calabi-Yau manifolds and related geometries: lectures at a summer school in Nordfjordeid, Norway, June 2001*. Springer Science & Business Media, 2012.

[28] Raoul Bott, Loring W Tu, et al. *Differential forms in algebraic topology*, vol-

ume 82. Springer, 1982.

[29] Henry Taubes. *An introduction to manifolds*, volume 1. Princeton University Press, 2011.

[30] Gang Tian. Gauge theory and calibrated geometry, i. *Annals of Mathematics*, 151(1):193–268, 2000.

[31] Arthur Jaffe and Edward Witten. Quantum yang-mills theory. *The millennium prize problems*, 1:129, 2006.

[32] David Tong. Gauge theory. *Lecture notes, DAMTP Cambridge*, 10, 2018.

[33] Florian Gmeiner, Ralph Blumenhagen, Gabriele Honecker, Dieter Lüst, and Timo Weigand. One in a billion: Mssm-like d-brane statistics. *Journal of High Energy Physics*, 2006(01):004, 2006.

[34] Alvaro De Rújula, Howard Georgi, and SL Glashow. Hadron masses in a gauge theory. *Physical Review D*, 12(1):147, 1975.

[35] Joseph Gerard Polchinski. *String theory, volume I: An introduction to the bosonic string*. Cambridge university press Cambridge, 1998.

[36] Nathan Berkovits. Perturbative super-yang-mills from the topological ads5$\times$ s5 sigma model. *Journal of High Energy Physics*, 2008(09):088, 2008.

[37] Lara B Anderson, James Gray, Andre Lukas, and Burt Ovrut. Stabilizing the complex structure in heterotic calabi-yau vacua. *Journal of High Energy Physics*, 2011(2):1–16, 2011.

[38] Simon K Donaldson. Anti self-dual yang-mills connections over complex algebraic surfaces and stable vector bundles. *Proceedings of the London Mathemat-*

ical Society, 3(1):1–26, 1985.

[39] Karen Uhlenbeck and Shing-Tung Yau. On the existence of hermitian-yang-mills connections in stable vector bundles. *Communications on Pure and Applied Mathematics*, 39(S1):S257–S293, 1986.