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The Unicity of a Blow-down.

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Abstract

In algebraic geometry, projective varieties are classified up to isomorphism. This involves classifying the varieties up to birational equivalence and classifying nonsingular varieties in each equivalence class up to isomorphism. Singular projective varieties are modified to less singular or non-singular ones by blowing up the singularities. A blow up map contracts or blows down an exceptional divisor to a curve or a point. In surfaces, such maps exist and are unique up to isomorphism and by the Castelnuovo contraction criterion, any curve that can be blown down is a -1 curve. In higher dimensional varieties, contraction morphisms are uniquely determined by the extremal rays which they contract. In this thesis, we will present a result due to Lascu [1] on the uniqueness of a blow-down. Precisely, Lascu shows that any birational morphism $f : X \longrightarrow S$ that contracts a divisor $D \subset X$ to a subvariety $Y \subset S$ is a blow-down if and only if S is a nonsingular variety, D is a closed nonsingular divisor.

Dedication

I dedicate this thesis to my supervisor Professor Martin Gulbrandsen and other professors in the IMF Dept. at the University of Stavanger. Thank you a million times for making me enjoy this adventure to the end.

Contents

List of Figures

Chapter 1

Introduction

This chapter consists of what this text is all about. Worthy mentioning is the main problem of classification of varieties in algebraic geometry, its subproblems, some of the ways by which the problem has been solved and the advancements led by them. Then finally, we will highlight a crucial result by Lascu [1] in his paper on ”Sous-variétés régulièrement contractibles d’une variété algébrique” in which the uniqueness of a blow-down is proved.

1.1 The Classification of Algebraic Varieties

In all branches of mathematics, the classification of objects studied is of ultimate importance. The problem arises naturally for a better understanding of the subjects. This at its best has been a yardstick for determining and checking progress in the fields as it has fueled up a good amount of research work. Algebraic geometry is not an exception. While the main objects of study in it are algebraic varieties, the classification problem of the varieties calls for having them classified up to isomorphism, which is a weaker version of equality and equivalence.

Often a problem is connected to other subproblems. There may be a chain or a collection of them branching from the main problem. Progress towards the classification problem in algebraic geometry has resulted from the same trajectory. The problem is divided into the following parts.

1.1.1 Birational Classification.

In this part, all smooth projective varieties are classified up to birational equivalence. This is the main goal of birational geometry.

In birational geometry, birationality of varieties is equivalent to the isomorphism of their function fields. Hence, the problem of classifying varieties up to birational equivalence is equivalent to that of classifying the corresponding function fields up to isomorphism.

1.1.2 Classification of Smooth Varieties up to Isomorphism

In this part, the goal is to have in each birational equivalence class a subset of "nice" varieties such as nonsingular projective varieties and have them classified up to isomorphism.

1.1.3 A Minimal Model

In each equivalence class, we finally determine if there is a unique simple variety which is birational to the original variety [1]. One way of solving this part of the problem is by modifying varieties to make them smooth using several appropriate operations. One of the operations used involves resolving singularities of the varieties by **blowing them up**. This means that even singular varieties can be classified. In fact, Hironaka [4] on resolution of singularities proved that every variety is birational to a smooth projective variety. It is this part of the classification problem that brought forth the **Minimal Model Program** (MMP) for surfaces studied by the Italian School of Algebraic Geometry (1890-1910). This program involves picking a smooth projective surface X and have all -1 curves E on X (if they exist) contracted to obtain another smooth projective variety Y called a **minimal model** (see, e.g., Beauville [5],II.15 for an introduction). It turns out that if Y exists, it is unique and is either a minimal surface $\mathbb{P}^2$ or it is birational to a product $\mathbb{P}^1 \times C$ for some curve C, i.e, a ruled surface of the curve C.

1.2 Uniqueness of a Blow-down.

In this thesis, the focus is mainly on the birational classification of nonsingular varieties. Given a surface S, we can form a nonsingular or a less singular surface $\widetilde{S}$ by blowing up a point P on S. This gives a birational morphism $\pi : \widetilde{S} \to S$ which is not an isomorphism except outside E and P. π is a very good example of such maps called **blow-downs** as it contracts or blows down the exceptional divisor E to the point P.
We will study similar blow-downs for higher dimensional smooth projective varieties. Unlike for smooth surfaces where such morphisms contract -1 curves

[1] This refers to a canonical representative in a given class.

to points, in higher dimensions, they contract divisors to other subvarieties. In fact, by the Castelnuovo Contraction theorem (see, e.g., Hartshorne [6, Ch. V, p. 414]), every curve on a smooth surface that can be contracted is a -1 curve. For higher dimensional smooth projective varieties, we will provide a construction of extremal rays and divisorial contractions from Mori theory which generalize the notions of -1 curves and contraction morphisms that contract divisors respectively. This will be preceded by the purity theorem of the exceptional locus that indicates that only divisors can be contracted following the argument by Shafarevich [7, Ch. II.4.4, Theorem 2]. Finally, we will present Lascu's result on the uniqueness of a blow-down up to isomorphism. Precisely, we will answer the following.

Question Given a variety X with a divisor $D \subset X$ and a birational morphism $f : X \longrightarrow S$ to another variety S which contracts D to a subvariety $Y \subset S$ and a blow-up $\pi : \widetilde{S} \to S$ of S along Y. Is f a blow-up isomorphic to π along $f(D)$? Lascu's result shows that f is a blow-down under the following assumptions.

- D is a closed nonsingular divisor and
- S is a nonsingular variety.

The following is the outline of the proceeding chapters in this text.

- In Chapter 2, we discuss the background material and the main theorems therein motivated by examples. Worthy mentioning are the rational maps which find their use in the classification of varieties making algebraic geometry more rigid than differential geometry or topology. These enable us to define birational maps which appear throughout this text.
- In Chapter 3, we discuss a crucial operation used in birational geometry to modify varieties by making them smooth or less singular. In particular, we provide the construction of a blow-up with concrete examples in which it finds its use in resolving singularities.
- In Chapter 4, we discuss the birational geometry of surfaces. In particular, we discuss birational maps of surfaces and their factorization through blow-ups and blow-downs in the Zariski's factorization theorem. This factorization is another key ingredient in this text. We will also give the construction of the universal property of a blow-up.
- In Chapter 5, we give a construction of a blow-down in higher dimensional varieties and then present a result due Lascu in [1] in which it is shown that a blow-down is unique up to isomorphism.

Chapter 2

Preliminaries

As mentioned in the outline, this chapter will discuss the machinery that will be relevant throughout this text. We present definitions, main theorems and provide examples to motivate them.

2.1 Rational Maps

A study of algebraic varieties up to isomorphism can be done by understanding regular functions on them. To study varieties up to birational equivalence, a larger class of functions called **rational functions** is needed so that the notion of isomorphism is weaker. Rational functions are special cases of **rational maps**. These are generalizations of morphisms which are everywhere-defined maps between algebraic varieties. Even though much of the work in algebraic geometry is done by studying regular functions on varieties, no such non-constant functions can be defined globally on complete varieties. This is why "nice" rational maps are used. These are defined on small sets or only on dense open subsets which inherit the structure of the varieties so that the morphisms are well-defined.

Definition 2.1.1 (Rational map.)**.** *Let X and Y be algebraic varieties. A rational map from X to Y is an equivalence class of such pairs as (U, f) where U is a dense open subset of X and $f : U \longrightarrow Y$ is a morphism. The pairs (U, f) and (V, g) are equivalent if they agree on a non-empty open (hence dense) subset $W \subset U \cap V$.*

Rational maps are usually denoted as $f : X \dashrightarrow Y$[1]. The map f is defined at $x \in X$ if $x \in U'$ with (U', f') in the equivalence class.

[1] The map is not an actual mapping, that's why we use a broken arrow to denote it

The following are some of the examples of rational maps.

Example 2.1.1. *If X,Y are varieties. Any morphism $f : X \to Y$ is a rational map. Indeed, let $U = X$.*

Example 2.1.2. *Consider a variety X defined by $X = V(xz - yw) \subset \mathbb{P}^3$ and an affine line $\mathbb{A}^1$. Lets consider a map*

$$\phi : X \to \mathbb{A}^1, (x : y : z : w) \mapsto \quad \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ \frac{w}{z} & \text{if } z \neq 0. \end{cases} \tag{2.1}$$

ϕ is a rational map with pairs $(U, f = \frac{x}{y})$ and$(V, g = \frac{w}{z})$ where $U = X \setminus V(y)$ and $V = X \setminus V(z)$ are non-empty open subsets of X. $f = g$ on $W = X \setminus V(y,z)$.

2.1.1 Rational Functions

Recall that a **regular function** is defined as follows.

Definition 2.1.2. *Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. A function $f : X \to k$ is said to be regular at a point P if there exists an open neighbourhood $U \subset X$ of P and homogeneous polynomials $g,h \in k[x_0,\cdots,x]$ of the same degree such that $h \neq 0$ on U and $f = \frac{g}{h}$. f is regular on X if it is regular at every point of X.*

We can define a **rational function** as follows.

Definition 2.1.3 (Rational function)**.** *A rational function on an algebraic variety X is a rational map $X \dashrightarrow \mathbb{A}^1$.*

The set of all rational functions on X is called the function field of X denoted by $K(X)$. The functions in $K(X)$ can be added and multiplied to give other rational functions, with addition and multiplication of the functions defined as follows.

- **Addition** $(U, f) + (V, g) = (U \cap V, f + g)$.
- **Multiplication** $(U, f) \times (V, g) = (U \cap V, fg)$.

This enables $K(X)$ to be a ring.

Proposition 2.1.1. *$K(X)$ is a field.*

Proof. Any rational function $(U, f) \in K(X)$, has $(U \setminus V(f), 1/f)$ which is its multiplicative inverse. $\square$

2.2 Local Rings

We now define rings of functions associated with any variety.

Definition 2.2.1. *Let X be a variety. We denote by $\mathcal{O}(X)$ the ring of all regular functions on X. Given a point p on X, there is an open neighbourhood W of p. The ring denoted by $\mathcal{O}_{X,p}$ is a ring of germs of regular functions on W called the **local ring**.*

An element of $\mathcal{O}_{X,p}$ is a pair (U, f) where U is an open dense subset of X containing p and f is a regular function on U. Similar to how rational maps are defined in Definition 2.1.1, two pairs (U, f) and (V, g) in $\mathcal{O}_{X,p}$ are equivalent if f and g agree on some open dense subset $W \subset U \cap V$; hence on all of $U \cap V$ as W is dense. Thus, we can also define the local ring of the variety X at the point p as follows.

Definition 2.2.2. *The local ring of X at the point $p \in X$ is the ring*

$$\mathcal{O}_{X,p} = \{f \in k(X) : f \text{ is regular at } p\} \tag{2.2}$$

and by the following proposition, $\mathcal{O}_{X,p}$ is indeed a **local ring**.

Proposition 2.2.1. *The ring $\mathcal{O}_{X,p}$ is a local ring.*

Proof. Consider a homomorphism of rings given by

$$\phi : \mathcal{O}_{X,p} \to k, f \mapsto f(p) \tag{2.3}$$

This is clearly a surjective map as any constant in k is an image of the constant function in $\mathcal{O}_{X,p}$. The kernel of ϕ is the set $\mathfrak{m}_p = \{f \in \mathcal{O}_{X,p} : f(p) = 0\}$. Hence, by the first isomorphism theorem of rings,

$$\frac{\mathcal{O}_{X,p}}{\mathfrak{m}_p} \cong k \tag{2.4}$$

which implies that $\mathfrak{m}_p$ is a maximal ideal. In fact, $\mathfrak{m}_p$ is the unique maximal ideal. Indeed, every element $g = \frac{a}{b} \in \mathcal{O}_{X,p} \setminus \mathfrak{m}_p$ is a unit since $a(p)$ and $b(p)$ are non- zero. Hence, at p, $\frac{1}{g} = \frac{b}{a}$ is defined. □

2.2.1 The Structure Sheaf of a Variety

Not only can a ring be defined at a point and a variety. It can also be defined on open subsets of X. The local ring $\mathcal{O}_{X,p} \subset K(X)$ as defined in 2.2.2 is a subring of $K(X)$. Hence, it can be constructed by the following localization

$$\mathcal{O}_{X,p} = k[X]_{\mathfrak{m}_p} \tag{2.5}$$

where $k[X]$ is the coordinate ring of X.
We can define a structure sheaf $\mathcal{O}_V$ as follows.

Definition 2.2.3. *For every non-empty $V \subset X$, we can define a ring $\mathscr{O}_X(V)$ as*

$$\mathscr{O}(V) = \mathscr{O}_X(V) := \{f \in K(X) \colon f \text{ is regular on } V\} \tag{2.6}$$

$\mathscr{O}_X(V)$ is a ring and a k-algebra. The **structure sheaf** $\mathscr{O}_V$ is a set of rings $\mathscr{O}_X(V)$ together with the natural homomorphisms induced by the restrictions induced by the inclusions of open sets. The local ring $\mathscr{O}_{X,p}$ is called the **stalk of the structure sheaf** at the point p with its elements as germs of functions. The following formulations consequently arise from $\mathscr{O}(V)$.

- When $V = X$, the ring $\mathscr{O}_X(X) = K(X)$.
- Considering $\mathscr{O}(V)$ as a subset of $k(X)$, we have

$$\mathscr{O}(V) = \bigcap_{p \in V} \mathscr{O}_{X,p} \tag{2.7}$$

2.3 Birationality

Having defined rational maps, to define birational equivalence, we need to determine how two rational maps can be composed (if possible). Composition of a map with its inverse is a quick way of checking equivalence or isomorphism. The quick problem arising in composing rational maps is that they are not actual maps. Moreover, in general they are not surjective. Suppose $g : X \dashrightarrow Y$ and $h : Y \dashrightarrow X$ are two rational maps between varieties X and Y. The composition $h \circ g$ is defined if the image of g is not (entirely) contained in the locus of points where h is undefined. Consider the following example.

Example 2.3.1. *Suppose the rational maps $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are defined by $(x : y : z) \mapsto (x : y : 0)$ and $(x : y : z) \mapsto (0 : 0 : z)$ respectively.*
The composition $\phi \circ \psi$ is not defined as the image of ψ consists of the points $(0 : 0 : z)$ where ϕ is not defined. Similarly, $\psi \circ \phi$ is undefined as the image of ϕ consists of the points $(x : y : 0)$ where ψ is not defined.

Partly, the problem has been resolved in the way rational maps have been defined in 2.1.1. Indeed, if two rational maps/pairs (U, f) and (V, g) are to be equivalent if they agree on a non-empty dense open subset $W = f^{-1}(V)$, the composition of f and g becomes the equivalence class $(W, g \circ f)$.

2.3.1 Dominant Rational Maps

Far much similar to surjective maps in the category of projective spaces are rational maps called **dominant maps**. These maps enable us to establish invertibility of rational maps on dense open subsets. We define a dominant rational map as follows.

Definition 2.3.1 (Dominant Rational Maps)**.** *Let X and Y be any algebraic varieties. A rational map $f : X \dashrightarrow X'$ is said to be dominant if its image on its domain is a zariski dense open subset of X'.*

This is an analogous to a surjective map. Indeed, having f dominant $\Rightarrow$ $f(U) \subset Y$ is dense in X'. Hence, we can now be able to compose dominant rational maps due to the following lemma.

Lemma 2.3.1. *Given two rational maps $g : X \dashrightarrow X'$ and $h : X' \dashrightarrow Z$ between projective varieties. If g is dominant, then $f \circ g : X \dashrightarrow Z$ is a rational map.*

Dominant rational maps induce a contravariant k-algebra homomorphism on the function fields of the varieties called a **pullback map** by the following proposition.

Proposition 2.3.1 (Pullback of a rational map)**.** *Given any irreducible varieties X and Y. A dominant rational map $\phi : X \dashrightarrow Y$ induces a* ***pullback map*** *of function fields*

$$\phi^* : K(Y) \to K(X), f \mapsto \phi^*(f) = f \circ \phi. \tag{2.8}$$

This is a well-defined map.

Thus, there exists in a diagram as follows.

$$\begin{tikzcd} X \arrow[rr, dotted, "\phi(dominant)"] \arrow[dr, "\phi^*(f)"'] & & Y \arrow[dl, "f"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

Proof. Given a rational function $f \in K(Y)$, then f is a function $f : Y \dashrightarrow \mathbb{A}^1$. However, the map $\phi : X \dashrightarrow Y$ is a dominant rational map. Then by lemma 2.3.1, the map $\phi^*(f) = f \circ \phi$ is also a rational function given by $\phi^* f : X \dashrightarrow \mathbb{A}^1$. Thus, $\phi^* f \in K(X)$ is well-defined. However, it can happen that $\phi^*(b) = 0$ for some $b \in K(Y)$ with $b \neq 0$. In this case, we cannot define $\phi^*(\frac{a}{b})$ to be $\frac{\phi^*(a)}{\phi^*(b)}$. This is why dominant rational maps are crucial. $\square$

Dominant rational maps therefore induce a functorial assignment/property of the pullback map.

2.3.2 Birational Maps

The function field $K(X)$ captures the properties of an irreducible variety X. However, this is not precisely but only up to dominant rational maps. Consequently, an equivalence relation (relatively weaker than isomorphism) among varieties can henceforth be given by defining **birational maps**. This enables us to define invertibility of rational maps on dense open subsets.

Definition 2.3.2 (Birational Maps)**.** *Let X and Y be any algebraic varieties. A (dominant) rational map $\phi : X \dashrightarrow Y$ is said to be* ***birational*** *(or a birational equivalence) if there exists a rational (dominant) map $\psi : Y \dashrightarrow X$ such that $\phi \circ \psi = Id_X$ and $\psi \circ \phi = Id_Y$ where Id_X and Id_Y are also rational maps as well.*

Definition 2.3.3. *Two varieties X and Y are said to be birational or birationally equivalent if there exists a birational equivalence $\phi : X \dashrightarrow Y$.*

The following theorem gives other characterizations of a birational map (see e.g, Cutkosky [8], Hulek [9], Milne [10] for an introduction).

Theorem 2.3.2. *For a dominant rational map $\phi : X \dashrightarrow Y$. The following statements are equivalent.*

i. *ϕ is birational.*

ii. *$\phi^* : K(Y) \to K(X)$ is an isomorphism.*

iii. *There exist non-empty open subsets $U \subset X$ and $V \subset Y$ such that U and V are isomorphic.*

Proof. *i.* $\Rightarrow$ *ii*. With ϕ birational, the inverse dominant rational map $\phi^{-1} : Y \dashrightarrow X$ induces the k-algebra homomorphism $(\phi^{-1})^* : K(Y) \to K(X)$ which is the inverse of the pullback map ϕ^* induced by ϕ. Hence, ϕ^* is an isomorphism.
ii. $\Rightarrow$ *i*. Having an isomorphism $\phi^* : K(Y) \to K(X) \Rightarrow$ there is another pullback map $\psi^* : K(X) \to K(Y)$ such that $\phi^* \circ \psi^* = Id_{K(X)}$ and $\psi^* \circ \phi^* = Id_{K(Y)}$. However, ϕ^* is induced by a dominant rational map $\phi : X \to Y$ so that for any rational map $f \in K(Y)$, $\phi^*(f) = f \circ \phi$. We also have it that $\psi^* = (\phi^*)^{-1}$. We claim that $\psi = \phi^{-1}$. However, ψ^* is induced by a dominant rational map $\psi : Y \to X$ so for any rational map $g \in K(X)$, $\psi^*(g) = g \circ \psi$. Now, $Id_{K(X)}(g) = \phi^* \circ \psi^*(g) = \phi^*(g \circ \psi) = (g \circ \psi) \circ \phi = g \circ (\psi \circ \phi) = (\psi \circ \phi)^*(g)$. Thus, $Id_X = \psi \circ \phi \Rightarrow \psi = \phi^{-1}$ as required. Hence, ϕ is birational.
i. $\Rightarrow$ *iii*. This follows from the following lemma.

Lemma 2.3.3. *Let X be an irreducible algebraic variety. If $\emptyset \neq U \subset X$ is an open subset of X, then $K(U) \cong K(X)$.*

Proof. Pick any rational function $f \in K(U)$. This is a function defined on an open subset $V \subset U$. Thus, $f \in \mathscr{O}_U(V) \Rightarrow f \in \mathscr{O}_X(V) \Rightarrow f \in K(X)$. Thus, we have established that $\mathscr{O}_U(V) \subset \mathscr{O}_X(V)$. Similarly, having $g \in \mathscr{O}_X(V) \Rightarrow g|_{U \cap V} \Rightarrow g \in \mathscr{O}_U(U \cap V)$. Thus, $\mathscr{O}_X(V) \subset \mathscr{O}_U(V)$. Thus, we have established that $K(U) \cong K(X)$. □

Now, since X is an irreducible variety, any nonempty $U \subset X$ has $K(U) \cong K(X)$. Similarly, since Y is an irreducible variety, any open $\emptyset \neq V \subset Y$ has $K(V) \cong K(Y)$. Since, X and Y are birational, by ii., we have it that $K(X) \cong K(Y)$. Thus, $K(U) \cong K(X) \cong K(Y) \cong K(V)$ implies $K(U) \cong K(V) \Rightarrow U \cong V$.
ii. $\Rightarrow$ *iii.* and *iii.* $\Rightarrow$ *ii.* Can in a similar way be deduced from remark 2.3.3.

□

The following are some of the examples of birational maps.

Example 2.3.2. *Any isomorphism of varieties is a birational equivalence.*

Example 2.3.3. *Consider a cusp in $\mathbb{A}^2$ defined by $C = V(y^2 - x^3)$ and as shown in Figure 2.3. The map*

$$f : \mathbb{A}^1 \to C, t \mapsto (t^2, t^3) \tag{2.9}$$

is a regular map since the components of $f(t)$ are both polynomials. It also has an inverse map

$$g : C \to \mathbb{A}^1, (x, y) \mapsto \frac{y}{x} \tag{2.10}$$

which is a rational map outside the vanishing locus of x, $D(x)$. This shows that f is not an isomorphism. However, the restriction map

$$f|_{D(x)} : \mathbb{A}^1 \setminus \{0\} \to C \setminus \{0\}, t \mapsto (t^2, t^3) \tag{2.11}$$

is an isomorphism on Zariski open subsets. Hence, by Theorem 2.3.2(iii), f is a birational map.

Example 2.3.4. [2] *Consider the cubic curve $C \subset \mathbb{P}^2$ given by the equation $ZY^2 = X^3 + X^2Z$. The projection map π_p from the point $p = [0 : 0 : 1]$ gives a birational isomorphism* [3] *of C with $\mathbb{P}^1$.*

Proof. This is one of the simplest (nontrivial) examples of a birational isomorphism. Let the projection map from the point p be the map

$$\pi_p : C \setminus p \to \mathbb{P}^1$$

[2] This is Exercise 7.12 in Harris [11, pg. 79]

[3] In [11], birational isomorphism is used interchangeably with birational equivalence.

defined by $[X:Y:Z] \mapsto [X:Y]$. This defines a rational map $\pi : C \dashrightarrow \mathbb{P}^1$. However, it is worth noting that a line in $\mathbb{P}^2$ through p intersects C at exactly one point where $X = Y = 0$ with the Z coordinate free,i.e,$Z \neq 0$. Thus, the map is one-to-one. It has a rational inverse π^{-1} with the Z coordinate given by $Z = \frac{X^3}{Y^2 - X^2}$ Thus, π^{-1} is the given by

$$\pi^{-1} : \mathbb{P}^1 \dashrightarrow C$$

defined by: $[X:Y] \mapsto (X(Y^2 - X^2) : Y(Y^2 - X^2) : X^3)$. The compositions maps of π and π^{-1} are given by:

$$\begin{aligned}
\pi \circ \pi^{-1}(X:Y) &= \pi(X(Y^2 - X^2) : Y(Y^2 - X^2) : X^3) \\
&= (X(Y^2 - X^2) : Y(Y^2 - X^2)) \\
&= (X:Y) \\
&= Id_{\mathbb{P}^1} \\
\pi^{-1} \circ \pi(X:Y:Z) &= \pi^{-1}(X:Y) \\
&= (X(Y^2 - X^2) : Y(Y^2 - X^2) : X^3) \\
&= (X:Y:\frac{X^3}{Y^2 - X^2}) \\
&= (X:Y:Z) \\
&= Id_{C \setminus p}.
\end{aligned}$$

Hence, the projection map π_p indeed gives a birational isomorphism. □

2.4 Nonsingularity

Nonsingularity as a notion in algebraic geometry also appears in complex manifold theory. In algebraic geometry, analysis of singular varieties in characteristic zero is one of the important areas. Such varieties arise all the time. In particular, they lead to problems that require techniques for **resolution of singularities**[4] one of which is **blowing up** whose construction will be discussed in Chapter 3.3.
Considering the affine case, a singular point of a variety X is a bad point and at the same time special. In a geometric sense, we will note that a singular point has the following consequences on a tangent space:

- The tangent space (considered as the zero set of linear forms of a polynomial defining a variety) is not defined.

[4] Discussed in 3.2

- The tangent space has the wrong dimension (different from the one expected).

We firstly define a tangent space.

2.4.1 The Tangent Space

We begin by defining the linear approximation of a polynomial function.

Definition 2.4.1 (The Linear Approximation)**.** *Let f be a polynomial in $k[x_1,\dots,x_n]$ and $p=(p_1,p_2,\cdots,p_n)\in\mathbb{A}^n$. The linear approximation $L_p(f)$ of $f(x)$ in the neighbourhood of p is given by*

$$L_p(f)=\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(p)(x_j-p_j)\in k[x_1,\dots,x_n].$$

This is a linear form. We define the tangent space of a variety extrinsically as follows.

Definition 2.4.2 (Extrinsic definition)**.** *Let $X\subset\mathbb{A}^n$ be an affine variety and p be a point on X. Suppose $I(X)$ is the ideal generated by the polynomials $f_1,f_2,\cdots,f_r$. The tangent space T_pX to X at the point p is defined as*

$$T_pX=V(L_p(f_1),L_p(f_2),\dots,L_p(f_r))\subset\mathbb{A}^n.$$

The intrinsic definition of the tangent space is due to the following construction. The point $p=(p_1,p_2,\dots p_n)\in\mathbb{A}^n$ is in 1-1 correspondence with the maximal ideal $\mathfrak{m}_p=(x_1-p_1,x_2-p_2,\dots x_n-p_n)\subset k[x_1,\dots,x_n]$. We can define a linear map

$$\theta:k[x_1,\dots,x_n]\to k^n, f\mapsto\Big(\frac{\partial f}{\partial x_1}(p),\frac{\partial f}{\partial x_2}(p),\dots\frac{\partial f}{\partial x_n}(p)\Big)\tag{2.12}$$

The map has the following.

$$\theta(x_1-p_1)=(1,0,\dots,0)\tag{2.13}$$
$$\theta(x_2-p_2)=(0,1,\dots,0)\tag{2.14}$$
$$\vdots\tag{2.15}$$
$$\theta(x_n-p_n)=(0,0,\dots,1)\tag{2.16}$$

so that $\theta(x_i-p_i)$ forms the basis of k^n as a vector space. We can also verify that $\theta(\mathfrak{m}_p^2)=0$ so that $\mathfrak{m}_p^2$ is the kernel of θ. Hence, by the First Isomorphism Theorem, θ induces an isomorphism

$$\theta':\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}\to im(\theta)=\Big(\frac{\partial f}{\partial x_1}(p),\frac{\partial f}{\partial x_2}(p),\dots\frac{\partial f}{\partial x_n}(p)\Big)\cong k^n.\tag{2.17}$$

This enables the tangent space to have a structure of a vector space with the origin at p. Hence, we can intrinsically define the tangent space as follows.

Definition 2.4.3 (Intrinsic Definition). *Suppose that X is a quasi-projective variety and $p \in X$. The tangent space to X at the point p is the k-vector space $T_p(X)$ defined by*

$$T_p(X) = Hom_k\left(\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}, k^n\right) \tag{2.18}$$

where

$$Hom_k\left(\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}, k^n\right) = \{\psi \in Hom_k\big(\mathfrak{m}_p, k^n\big) : \psi(\mathfrak{m}_p^2) = 0\} \tag{2.19}$$

and $\mathfrak{m}_p$ is the maximal ideal of $\mathscr{O}_{X,p}$

This construction leads to the following definition of the dimension of the variety X.

Definition 2.4.4. *Let X be an irreducible variety and p be a point on X. The dimension of X is given by*

$$dim X = min\{dim T_p(X) : p \in X\}$$

2.4.2 Smooth and Singular Points

We now define a singular point as follows.

Definition 2.4.5. *Let X be an algebraic variety. A point $p \in X$ is said to be nonsingular if $dim T_pX = dim X$. Otherwise, $p \in X$ is singular. X is said to be nonsingular if all points of X are nonsingular points.*

In general, given a hypersurface defined by $f(x_1, x_2, \cdots, x_n) = 0$, singular points are attained where all the partial derivatives of f simultaneously vanish. Thus, we can also define a singular point of a variety as follows.

Definition 2.4.6 (The Jacobian Criterion). *Let $X \subset \mathbb{A}^n$ be an algebraic variety defined by the ideal $I = (f_1, f_2, \ldots, f_n)$. The Jacobian at the point $p \in X$ is given by $J_p = \left[\frac{\partial f_i}{\partial x_j}(p)\right]$. X is nonsingular at the point p if the rank of $J_p = n - r$, where $r = dim X$. If X is nonsingular at all points p, it is said to be nonsingular. Otherwise, it is singular.*

The set of all singular points on the variety is denoted $Sing(X)$.
Consider the following examples.

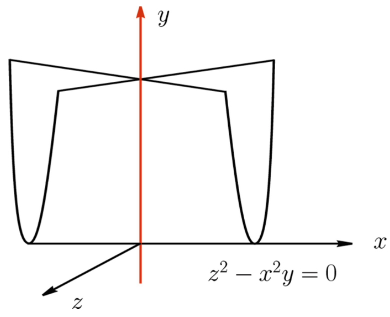

Figure 2.1: Whitney Umbrella. Source: Neumann [2, Ch 5,pg 115]

Example 2.4.1. *Consider the whitney umbrella in Figure 2.1*
$X : \left\{z^2 - x^2y = 0\right\} \subset \mathbb{A}^3$. *Let* $f(x,y,x) = z^2 - x^2y$. *At the point* $p = (a,b,c)$, *the Jacobian is given by:*

$$\begin{aligned} J_p &= \begin{bmatrix} \frac{\partial f}{\partial x}(a) & \frac{\partial f}{\partial y}(b) & \frac{\partial f}{\partial z}(c) \end{bmatrix} \\ &= \begin{bmatrix} -2ab & -a^2 & 2c \end{bmatrix} \end{aligned}$$

As long as a,b,c *are simultaneously zero, the rank of* $J_p = 0$ *which is less than* $n - r = 3 - 2 = 1$. *Hence, the variety is singular at the origin. In addition, along the handle of the umbrella* $(x = z = 0)$, *the variety* X *is also singular.*

Example 2.4.2. *Consider a parabola* $X : \left\{y - x^2\right\} \subset \mathbb{A}^2$ *as shown in Figure 2.2. At the point* $p = (a,b)$, *the jacobian is*

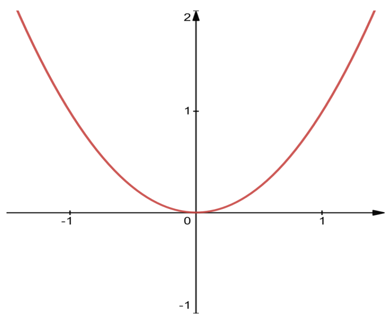

Figure 2.2: Parabola

$$\begin{aligned} J_p &= \begin{bmatrix} \frac{\partial f}{\partial x}(a) & \frac{\partial f}{\partial y}(b) \end{bmatrix} \\ &= \begin{bmatrix} -2a & 1 \end{bmatrix} \end{aligned}$$

At all points $p = (a,b)$, the rank of the Jacobian is 1 which is equal to 1 $(n-r = 2-1 = 1)$ as a parabola is an open 1-dimensional surface. Hence, a parabola X is a nonsingular variety.

Example 2.4.3. *Let X be a cusp defined by $y^2 - x^3 = 0$ and shown in Figure 2.3 .*

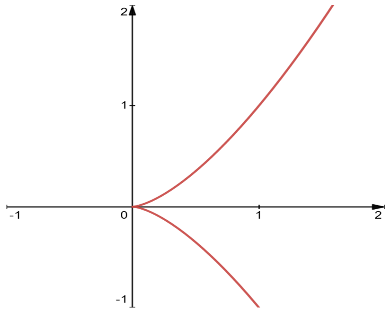

Figure 2.3: A cusp symmetric about the x-axis

This can be parameterised by $X = \{(t^2,t^3) : t \in k\}$ so that any point $p \in X$ can be written as $p = (t^2,t^3)$.
The linear approximation to X at p is the polynomial given by

$$\begin{aligned} L_pX &= -3x^2|_p(x-p_1) + 2y|_p(y-p_2) \\ &= -3t^4(x-t^2) + 2t^3(y-t^3). \end{aligned}$$

We have two consequences from the obtained form.

- *When $t \neq 0$, L_pX takes the form of the differential as we expect it to be.*
- *When $t = 0$, $L_pX = 0$. By Definition 2.4.2, the tangent space $T_pX = V(0) = \mathbb{A}^2$. Hence, the dimension of T_pX, $dim T_pX = dim \mathbb{A}^2 = 2$. By Definition 2.4.4, the cusp is a nonsingular variety since $dim X = dim\, T_pX = 2$.*

Even though nonsingularity has been defined as in Definition 2.4.6, it is still not clear if the definition holds for different generators of the ideal I. Moreover, the definition depends on the embedding of the variety. Zariski [12] proved that nonsingularity can be defined in terms of **regular local rings**[5] as follows.

Definition 2.4.7. *Let X be a variety. X is nonsingular at a point $p \in X$ if the local ring $\mathscr{O}_{X,p}$ is a regular local ring. X is nonsingular if it is nonsingular at every point.*

[5]While defined by Wolfgang Krull in 1937, early works of regular local rings appear in Zariski's work in 1940.

Chapter 3

Birational Geometry

3.1 Background

Birational geometry is a variant of algebraic geometry in which the goal is to classify nonsingular varieties up to birational equivalence. As provided in Theorem 2.3.2, this involves determining when two nonsingular varieties are isomorphic outside lower-dimensional subsets. Even though polynomial functions are supposedly nice candidates in the study of varieties up to isomorphism, for birational equivalence, rational functions as developed in 3.7 are used. Birational geometry has the so-called minimal model program at its core. This aims at classifying algebraic varieties up to birational isomorphism by identifying "nice" elements in each birational class and then classifying such elements. In dimension one, the problem is easy as a nice element in a birational class is simply a smooth and projective curve. In higher dimensions, the problem is complex. This is because there are infinitely many such elements in each class, so picking a representative is a very challenging problem.

3.2 Resolution of Singularities

Towards the classification of varieties up to birational equivalence, a good subset of a given equivalence class is that of nonsingular varieties. It is natural to ask about what can be done with singular varieties so that all the varieties (both singular and nonsingular) can be classified. This leads to the problem of resolution of singularities. In birational geometry and algebraic geometry at large, this involves asking whether any variety X has a resolution defined as follows.

Definition 3.2.1. *Suppose X is a quasi-projective variety. A resolution of singularities of X is a closed subvariety Y of $X \times \mathbb{P}^n$ (for some n) such that the projection $\pi : Y \to X$ is birational and Y is nonsingular.*

The problem of resolution of singularities is to prove the following theorem (see, e.g., Lipman [13] for an introduction).

Theorem 3.2.1. *For any algebraic variety X over an algebraically closed field k, there exists a resolution as defined in 3.2.1.*

In 1964, Hironaka [4] solved the problem for varieties of any dimension over fields of characteristic 0. For curves, this was proved by Noether [14] over the complex field $\mathbb{C}$. For surfaces, the first proof was provided by Abyankar [15]. For arbitrary positive characteristic p and higher dimensions, the problem has been solved in the works of Abhyankar [16], Albanese [17], Cossart [18], Bravo [19], Hauser [20] among others.

3.3 Blowing up

In this section, we will dicusss about a blow-up. Blow-ups with a careful choice of the center are the main tool for resolving singularities [1]. In Example 2.3.3, we had an example of a birational map which is not an isomorphism. A blow-up is another simplest example of such maps. It plays a crucial role in studying rational maps; but in essence, blowing up a variety modifies it to a birational one. We firstly discuss the graph of regular maps.

3.3.1 The Graph of a Regular Map.

Definition 3.3.1. *Let X and Y be any varieties and $\psi : X \to Y$ a regular map. The graph of ψ is the set*

$$\Gamma_\psi = \{(x, \psi(x)) \in X \times Y : x \in X\} \subset X \times Y.$$

An immediate consequence of the graph as defined in 3.3.1 is in the following proposition.

[1] To Raoul Bott – with great respect. ”At that time, blow-ups were the poor man’s tool to resolve singularities.” [21]

Proposition 3.3.1. *The graph Γ_ψ is closed in $X \times Y$.*

Proof. Having X and Y as varieties implies that $X \times Y$ is also a variety. The same follows even when the varieties X and Y are irreducible varieties.
By the universal property of products of varieties which are prevarieties, there exists a regular map

$$\psi \times id : X \times Y \to Y \times Y, (x,y) \mapsto (\psi(x), y) \tag{3.1}$$

However, Y is a variety (separable as affine scheme), hence the diagonal of Y given by

$$\triangle_Y = \{(y,y) : y \in Y\}$$

is closed. This implies that the graph Γ_ψ is also closed since

$$\begin{aligned} \Gamma_\psi &= \{(x, \psi(x)) \in X \times Y : x \in X\} \\ &= (\psi \times id)^{-1}(\triangle_Y) \end{aligned}$$

and (ψ, id) is continuous.

□

The product of the varieties X and Y given by $X \times Y$ is also a variety. While the product $X \times Y$ is irreducible if both X and Y are irreducible, the graph Γ_ψ is irreducible iff X is irreducible. The next theorem shows that the projection of Γ_ψ to X is a birational map.

Proposition 3.3.2. *[8, Ch. 5, pg. 107, Prop. 5.9] Let X and Y be irreducible varieties and Γ_ψ be the graph of a dominant rational map $\psi : X \dashrightarrow Y$ between them. The projection map $\pi_1 : \Gamma_\psi \to X$ is a birational map. If U is a non-empty open subset of X on which ψ is regular, then the restriction*

$$\pi : \Gamma_{\psi|U} = \Gamma_\psi \cap (U \times Y) \to U \tag{3.2}$$

is an isomorphism.

Proof. Following the argument presented by Cutkosky [8] with some details added, we prove the theorem as follows. Let U be an open dense subset of X on which ψ is regular.
The graph Γ_ψ can also be defined as the closure of the image of $\psi|_U : U \to Y$ in $X \times Y$ and it is also independent of the choice of the open subsets U.
However, Γ_ψ is also irreducible since Y is irreducible. Hence, Γ_ψ is also a variety.
Γ_ψ can also be viewed as the image of the product map $(id, \psi) : X \to X \times Y$ which is regular since its components i and ψ are regular. But (i, ψ) is the inverse of π_1. Hence, π is an isomorphism and eventually a birational map. □

3.3.2 Intuition behind Blowing up.

Having defined the graph of a rational map, we can now construct the blow-up. We start by providing the intuition behind the process of blowing up. This construction closely follows the presentation as provided by Gathmann [22] and Hacon [23]

Global Picture

Consider the projection map

$$\pi : \mathbb{P}^2 \to \mathbb{P}^1, (x : y : z) \mapsto (y : z). \tag{3.3}$$

This is a well-defined map except at the point $p = (1 : 0 : 0)$.
The graph $\Gamma_\pi \subset \mathbb{P}^2 \times \mathbb{P}^1$ is undefined at p. Lets consider π defined inside $\mathbb{P}^2 \setminus \{p\} \times \mathbb{P}^1$.
A point $(x : y : z; u : v) \in \Gamma_\pi \subset \mathbb{P}^2 \times \mathbb{P}^1$ satisfies the homogeneous polynomial equation given by $yv = zu$ whose points also lie inside the closure $\overline{\Gamma_\pi}$. In construction, the following questions arise naturally:

1. What is the map $\pi_1 : \overline{\Gamma_\pi} \to \mathbb{P}^2$?

2. What are the fibres of the map?

From Proposition **??**, the answer to question 1 is that the map is a birational map. For question 2, we have the following cases:

- When $(x : y : z) \neq (1 : 0 : 0)$, the image of p under the map π given by $(y_0 : y_1)$ is a unique point in $\mathbb{P}^1$. Hence, the preimage $(x : y : z; y_0 : y_1)$ of p under π_1 is a unique point in $\overline{\Gamma_\pi}$.

- When $(x : y : z) = (1 : 0 : 0)$, no restriction can be made on the image point $(y_0 : y_1)$ of π. This implies that the fibre $\pi_1^{-1}(p)$ is the entire projective line $\mathbb{P}^1$. Thus, the closure $\overline{\Gamma_\pi}$ isomorphic to $\mathbb{P}^2$ everywhere except at the point $p = (1 : 0 : 0)$. In this way, the point p is said to have been blown up.

Local Picture

We can also think about the blow-up as follows.
Consider all the pairs $(P; L) \in \mathbb{A}^2 \times \mathbb{P}^1$ such that P is a point on the line L passing throng the origin. When P is the origin, L can be any line. When P is not the origin, the line L is uniquely determined.

We firstly define the blow-up from the construction of the graph in 3.3.1.

3.3.3 The Closure of the Graph.

Definition 3.3.2. *Let $X \subset \mathbb{A}^n$ be an affine variety. For some given polynomial functions $f_1, \cdots, f_r \in A(X)$ on X, the open subset of X be $U = X \setminus V(f_1, \cdots, f_r)$. Let*

$$f : U \to \mathbb{P}^{r-1}, x \mapsto f(x) = \big(f_1(x) : f_2(x) \cdots : f_r(x)\big) \tag{3.4}$$

be a rational map. The graph of f is the set

$$\Gamma_f = \{(x, f(x)) : x \in U\} \subset U \times \mathbb{P}^{r-1}. \tag{3.5}$$

*closed in $U \times \mathbb{P}^{r-1}$ by Proposition 3.3.1. The closure of Γ_f in $X \times \mathbb{P}^{r-1}$ is called the **blow-up** of X at the coordinate functions $f_1, f_2, \cdots, f_r$. This is usually denoted by $\widetilde{X}$.*

By Proposition 3.3.2 , there is a birational map $\pi : \widetilde{X} \to X$. We sometimes refer to π as the blow-up.

3.3.4 Blowing up $\mathbb{A}^n$ at a Point

We adapt from the global construction above to define the blow-up of $\mathbb{A}^n$ at the origin as follows.

Definition 3.3.3. *The blow-up of $\mathbb{A}^n$ at the origin $O = (0, 0, \cdots, 0) \in \mathbb{A}^n$ is the set*

$$\begin{aligned} B_O(\mathbb{A}^n) &= \{(x, L) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x \in L \in\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \\ &= \{((x_1, x_2, \cdots, x_n); [y_1 : y_2 : \cdots : y_n]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x_i y_j = x_j y_i \quad i, j = 1, 2, \cdots, n\} \end{aligned}$$

By Proposition 3.3.2, there exists a commutative diagram as follows.

$$\begin{tikzcd} B_O(\mathbb{A}^n) \arrow[r, hook] \arrow[dr, "\phi"'] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d, "\pi"] \\ & \mathbb{A}^n \end{tikzcd}$$

Before we provide examples on the computations of the blow-up of $\mathbb{A}^n$, we provide a similar construction in the projective space. This is a convenient setting in this text.

3.3.5 Blowing up the Projective Space $\mathbb{P}^n$

We define the blow-up of $\mathbb{P}^n$ as follows.

Definition 3.3.4. *Let $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ be two projective spaces with homogeneous coordinates given by $u = (u_0 : u_1 : \cdots : u_n)$ and $v = (v_1 : v_2 : \cdots : v_n)$ respectively. The blow-up of $\mathbb{P}^n$ is the closed subvariety*

$$\widetilde{\mathbb{P}^n} = \{(u;v) \in \mathbb{P}^n \times \mathbb{P}^{n-1} : u_i v_j = u_j v_i \quad \forall\, i,j = 1,2,\cdots,n\} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}. \quad (3.6)$$

In the same way as in 3.3.3, the projection map

$$\pi : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n, [u;v] \mapsto u \quad (3.7)$$

is sometimes referred to as the blow-up of $\mathbb{P}^n$. The inverse rational map of π in 3.7 has a fundamental point at $P = (1:0:0:\cdot:0) \in \mathbb{P}^n$. Suppose P is the center of the blow-up of $\mathbb{P}^n$, the following occurs for the fibre of P under π.

a. If $(u_0 : u_1 : \cdots : u_n) \neq P$, the equations in 3.6 imply that $(u_0 : u_1 : \cdots : u_n) = (v_1 : v_2 : \cdots : v_n)$ so that $\pi^{-1} : \mathbb{P}^n \setminus \{p\} \to \widetilde{\mathbb{P}^n}, [u_0 : u_1 : \cdots : u_n] \mapsto [(u_0 : u_1 : \cdots : u_n);(v_1 : v_2 : \cdots : v_n)]$ is the inverse morphism of π.

b. When $(u_0 : u_1 : \cdots : u_n) = P$, the fibre $\pi^{-1}(P) = \{P\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$. Thus the equations in 3.6 are satisfied by any values v_i and π is an isomorphism between $\mathbb{P}^n \setminus \{P\}$ and $\widetilde{\mathbb{P}^n} \setminus \pi^{-1}(P)$

The following theorem entails the behavior of the preimage of the blow-up of a projective subvariety.

Theorem 3.3.1. *Let $X \subset \mathbb{P}^n$ be a quasi projective subvariety and $\pi : \Gamma \to \mathbb{P}^n$ the blow-up of $X \subset \mathbb{P}^n$ centered at P. Suppose X has a nonsingular closure $\overline{X} \neq \mathbb{P}^n$. Then the preimage of X under π is reducible consisting of components;*

$$\pi^{-1}(X) = \underbrace{\left(\{P\} \times \mathbb{P}^{n-1}\right)}_{\pi^{-1}(\{P\})} \cup Y. \quad (3.8)$$

The first component, $E = \pi^{-1}(\{P\}) \cong \mathbb{P}^{n-1}$, is called the ***exceptional divisor*** *and the second, Y, is the* ***strict transform****. Restricting π to Y gives a regular map $\pi : Y \to X$ which is an isomorphism outside P, hence a birational map. Thus, blow-ups can also be obtained by computing strict transforms.*

The following examples illustrate how a blow-up of a variety at a point is computed from a strict transform.

Example 3.3.1 (Two crossing lines.). *Consider the variety $X \subset \mathbb{A}^2$ defined by $x^2 - y^2 = 0$ and shown in Figure 3.1. Let $f(x,y) = x^2 - y^2$. The partial*

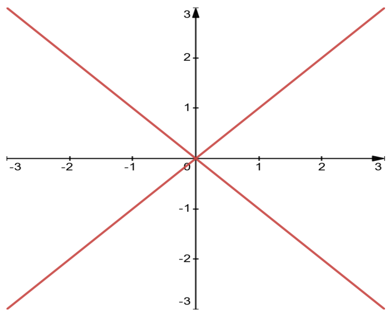

Figure 3.1: Two crossing lines

derivatives of f with respect to x and y are:

$$\frac{\partial f}{\partial x} = 2x$$
$$\frac{\partial f}{\partial y} = -2y$$

respectively. Both of them vanish at the origin, $O = (0,0) \in X$. O is therefore a singular point of X. Hence, X is singular.
Blowing up $\mathbb{A}^2$ at the origin gives the following.
Let x, y be the coordinates of $\mathbb{A}^2$ and X, Y be the homogeneous coordinates of $\mathbb{P}^1$. The blow-up of $\mathbb{A}^2$ at the origin is the set

$$\widetilde{\mathbb{A}^2} = B_o(\mathbb{A}^2) = \{(x,y;X:Y) \in \mathbb{A}^2 \times \mathbb{P}^1 : xY = yX\} \subset \mathbb{A}^2 \times \mathbb{P}^1. \tag{3.9}$$

Since X and Y are homogeneous coordinates, $\widetilde{\mathbb{A}^2}$ can be covered by the following two affine charts:

1. *Let U be the chart defined by $X \neq 0 \Rightarrow X = 1$ up to scaling.*
   In U, the equation $xY = yX$ in 3.9 becomes $y = xY$.
   Hence, the equation defining X becomes

   $$x^2 - y^2 = x^2 - x^2Y^2 = 0$$
   $$x^2(1 - Y^2) = 0$$

   so that either $x^2 = 0$ or $(1 - Y^2) = 0$. The total inverse image of X is degenerate, thus it is the union of two components namely, the exceptional curve E defined by $x = 0$ and the strict transform defined by $1 - Y^2 = 0$. The strict transform gives the blow-up of X. We notice that even though X is singular, blowing up gives a nonsingular variety as shown in Figure 3.2.

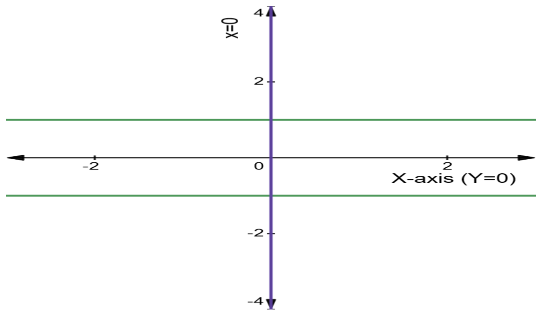

Figure 3.2: The blow-up of $x^2 - y^2 = 0$ at (0,0) in the chart U.

2. *Let V be the chart defined by $Y \neq 0 \Rightarrow Y = 1$ up to scaling.*
   $xY = yX$ becomes $yX = x$. Hence, X has

$$x^2 - y^2 = y^2X^2 - y^2 = 0$$
$$y^2(X^2 - 1) = 0$$

   Thus, we have the inverse image of X composed of two components. One is defined by $y = 0$, the exceptional curve, and the other component $X^2 - 1 = 0$ which is the blow-up of X. This is as shown in Figure 3.3.

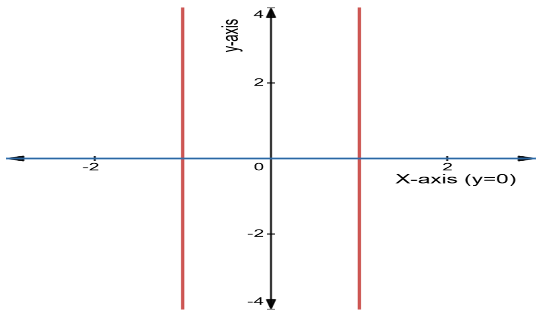

Figure 3.3: The blow-up of $x^2 - y^2 = 0$ at (0,0) in the chart $V \neq 0$.

The two blow-ups induce projection maps from the respective chart to $\mathbb{A}^2$ as shown in Figure 3.4. The projection maps are defined as

$$\pi|_U : U \to \mathbb{A}^2 \tag{3.10}$$

defined by $(x,Y) \mapsto (x,xY)$ and

$$\pi|_V : V \to \mathbb{A}^2 \tag{3.11}$$

defined by $(X,y) \mapsto (yX,y)$.

$$\begin{array}{ccc} \widetilde{\mathbb{A}^2} = U \cup V & \qquad & \widetilde{\mathbb{A}^2} = U \cup V \\ \Big\downarrow{\scriptstyle \pi|_U} & & \Big\downarrow{\scriptstyle \pi|_V} \\ \mathbb{A}^2 & & \mathbb{A}^2 \end{array}$$

Figure 3.4: Projections of charts U and V to $\mathbb{A}^2$

The preimages of the origin under the projections $\pi|_U$ *and* $\pi|_V$ *are given by* $\pi|_U^{-1}(0,0) = V(x) \cong \mathbb{A}^1$ *and* $\pi|_V^{-1}(0,0) = V(y) \cong \mathbb{A}^1$ *respectively. These fibres cover the exceptional divisor with the latter being isomorphic to the projective line* $\mathbb{P}^1$.

Example 3.3.2 (The union of two parabolas.)**.** *Let C be a variety defined by* $X = V(y^2 - x^4) \subset \mathbb{A}^2$ *as shown in Figure 3.5*

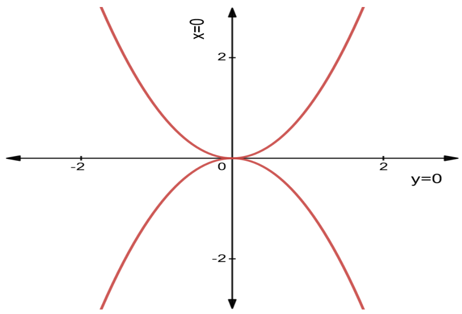

Figure 3.5: The union of two parabolas

Blowing up $\mathbb{A}^2$ *at the origin gives the following.*

$$\widetilde{\mathbb{A}^2} = \{(x,y;X,Y) \in \mathbb{A}^2 \times \mathbb{P}^1 : xY = yX\} \subset \mathbb{A}^2 \times \mathbb{P}^1; \tag{3.12}$$

where x, y *are the coordinates for* $\mathbb{A}^2$ *and* X, Y *are homogeneous coordinates for* $\mathbb{P}^1$.
$\widetilde{\mathbb{A}^2}$ *can then be covered by the following two affine patches.*

1. *A patch* U *defined by* $X \neq 0 \Rightarrow X = 1$*. When* $X = 1$*, a point* $(x,y;X:Y)$ *in* $\widetilde{\mathbb{A}^2}$ *satisfies* $y = xY$*. So that*

$$y^2 - x^4 = x^2Y^2 - x^4 = 0$$
$$x^2(Y^2 - x^2) = 0$$

which is a union of two components as shown in Figure 3.6. One is defined by $x^2 = 0$. This is the exceptional curve. Another component is defined by $Y^2 - x^2 = 0$ which is the strict transform of C. This component also has a

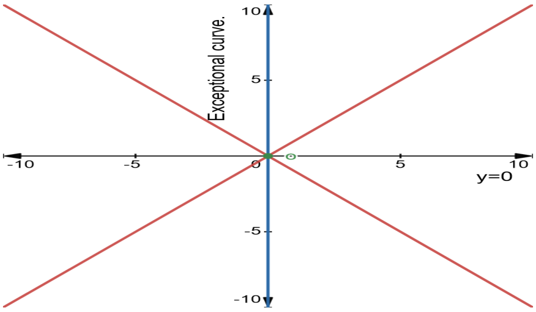

Figure 3.6: The blow-up of $x^2 - y^4 = 0$ at (0,0) in the chart U

singular point at the origin. Blowing it up at the origin leads to the same scenario as we had in Example 3.3.1.

2. *A patch V defined by $Y \neq 0 \Rightarrow Y = 1$. When $Y = 1$, a point $(x,y;X:Y)$ in $\widetilde{\mathbb{A}^2}$ satisfies $yX = x$. We then have*

$$y^2 - x^4 = y^2 - y^4X^4 = 0$$
$$y^2(1 - y^2X^4) = 0$$

which comprises of the exceptional curve $E = V(y^2)$ and the strict transform defined by $1 - y^2X^4 = 0$ which is a nonsingular cross-section of a helix surface as shown in Figure 3.7.

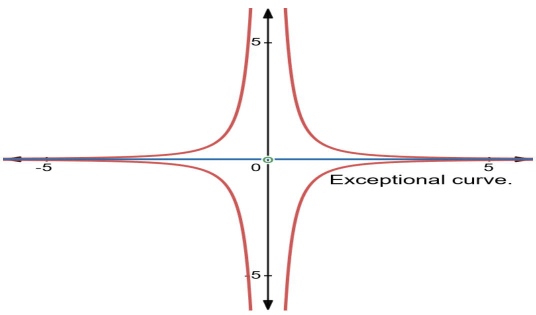

Figure 3.7: The blow-up of $x^2 - y^4 = 0$ at (0,0) in the chart V

In cases as in Example 3.3.2, more than one blow-up is required to resolve singularities. This will enable us to factor a rational map through a blow-up or sequence of them.

3.3.6 Blowing up along a Subvariety

So far we have seen blow-ups of varieties at a point. It is also possible to blow up along a subvariety. In practice, as we will notice in Example 3.3.3, blowing up a variety at a point which lies on a curve of singular points does not resolve them. In this section, we will discuss blow-ups along subvarieties. This will enable us to blow up varieties along a curve.

Definition 3.3.5. *Suppose X is an affine variety and $I \subset K[X]$ is a non-zero ideal. Suppose that $I = (f_0, f_1, \cdots, f_r)$. Let*

$$\Lambda : X \to \mathbb{P}^r, \Lambda \mapsto (f_0 : f_1 : \cdots : f_r) \tag{3.13}$$

be a rational map. The blow-up of I is $B(I) = \Gamma_\Lambda \subset X \times \mathbb{P}^r$, with the projection $\pi : B(I) \to X$.

If the ideal $I = I(Y)$ is an ideal of a subvariety $Y \subset X$, then $\pi : B(I) \to X$ is called the blow-up of X along Y or simply the blow-up of Y.
The blow-up as defined in 3.3.5 does not depend on the generators of the ideal I as verified in the following proposition.

Proposition 3.3.3. *[8, Ch. 6, Proposition 6.1] The blow-up of an affine variety X at the regular functions $f_1, f_2, \cdots, f_r$ depends only on the ideal $I = (f_1, f_2, \cdots, f_r)$. Thus, suppose that $g_1, g_2, \cdots, g_s$ is another set of regular functions generating an ideal J such that $I = (f_1, f_2, \cdots, f_r) = J = (g_1, g_2, \cdots, g_s)$, and $\pi : \widetilde{X} \to X$ and $\pi' : \widetilde{X'} \to X$ are their respective blow-ups. Thus, there exists a commutative diagram as shown in Figure 3.8*

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & \widetilde{X'} \arrow[dl, "\pi'"] \\
& X &
\end{tikzcd}$$

Figure 3.8: Commutative diagram showing that the blow-up of X along Y is well-defined.

where ψ is an isomorphism

Proof. The proof of this proposition closely follows the argument presented by Gathmann [22, Ch.4, Lemma 4.3.5] with some details added. We provide the proof as follows.

The assumption of the equality of ideals I and J implies that the components f_i and g_j can be expressed as linear combinations of each other as follows.

$$f_i = \sum_{j=1}^{s} h_{i,j} g_j \, \forall \, i = 1, 2, \cdots, r \tag{3.14}$$

$$g_j = \sum_{k=1}^{r} m_{j,k} f_k \, \forall \, j = 1, 2, \cdots, s \tag{3.15}$$

for some polynomials $h_{i,j}, m_{j,k}$.
However, the blow-ups $\widetilde{X}$ and $\widetilde{X'}$ are zariski closed subsets of $X \times \mathbb{P}^{r-1}$ and $X \times \mathbb{P}^{s-1}$ respectively. Hence, we can claim that the isomorphism $\psi : \widetilde{X} \to \widetilde{X'}$ can be defined by

$$(x, f) \mapsto (x, g) \tag{3.16}$$

$$= \big(x; \sum_{k=1}^{r} m_{1,k}(x) f_k : \cdots : \sum_{k=1}^{r} m_{s,k}(x) f_k\big). \tag{3.17}$$

Indeed, the regular functions of g are not simultaneously 0. Hence, the equations $f_i g_j = f_j g_i$ are satisfied on the open subset
$U = X \setminus V(f_1, f_2, \cdots, f_r) \subset \widetilde{X} \subset X \times \mathbb{P}^r$. Thus ,a regular function
$f_i = \sum_{i,j} h_{i,j} m_{j,k} f_k$ in $f_1, f_2, \cdots, f_r$ on U implies that $g_j = \sum_{i,j} h_{i,j} m_{j,k} g_k$ in $g_1, g_2, \cdots, g_s$ on U and henceforth on its closure.
Having $g_j = \sum_{k=1}^{r} m_{j,k} f_k = 0$ for all j implies $f_i = \sum_{j=1}^{s} h_{i,j} g_j = 0$ for all i. This contradicts the assumption that both f_i and g_j are defined on U.
For all points $(x, f) \in \widetilde{X}$, the image $F(x, f)$ lies inside $\widetilde{X'}$. Indeed,

$$\psi(x, f) = (x; \sum_{k=1}^{r} m_{1,k}(x) f_k : \sum_{k=1}^{r} m_{2,k}(x) f_k : \cdots : \sum_{k=1}^{r} m_{s,k}(x) f_k) \tag{3.18}$$

$$= (x; g_1(x) : g_2(x) : \cdots : g_s(x)) \in \widetilde{X'} \text{ on U.} \tag{3.19}$$

Hence, this also holds on the closure $\tilde{X}$.
We can use a similar construction to obtain the inverse regular map ψ^{-1} due to symmetry. □

Consider the following examples.

Example 3.3.3. *Consider the Whitney Umbrella X defined in Example 2.4.1 by $X = V(z^2 - x^2 y)$ as shown in Figure 3.9. The whole axis $x = z = 0$ has singular points.*

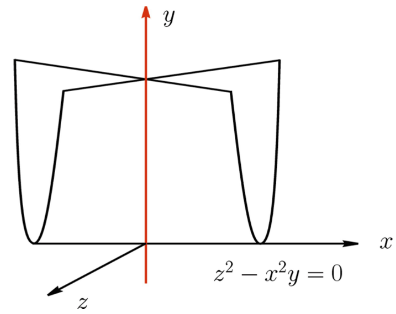

Figure 3.9: Whitney Umbrella

Lets consider blowing up the umbrella along the handle which is a subvariety defined by the ideal $I = (x,z)$. The blow-up of X along the ideal I is

$$B_I(X) = \{(x,y,z,R:S) \in \mathbb{A}^3 \times \mathbb{P}^1 : xS = zR\} \subset \mathbb{A}^3 \times \mathbb{P}^1 \tag{3.20}$$

where R,S are homogeneous coordinates of $\mathbb{P}^1$. $B_I(X)$ can be covered by the following two charts.

1. *An affine chart defined by $R = 1$.*
   This implies that $z = xS$ so that

   $$\begin{aligned} z^2 - x^2y &= x^2S^2 - x^2y = 0 \\ &\Rightarrow x^2(S^2 - y) = 0. \end{aligned}$$

   This is a union of an exceptional curve defined by $x = 0$ and the nonsingular strict transform defined by $S^2 - y = 0$.

2. *An affine chart defined by $S = 1$.*
   When $S = 1$, we have $x = zR$ so that

   $$z^2 - x^2y = z^2 - z^2R^2y = 0 \tag{3.21}$$
   $$\Rightarrow z^2(1 - R^2y) = 0 \tag{3.22}$$

   giving a nonsingular variety $1 - R^2y = 0$ as the strict transform.
   Thus, blowing up the Whitney umbrella along the subvariety gives a nonsingular variety.

Example 3.3.4. *Consider an affine 3-fold defined by $X = V(xy - zw) \subset \mathbb{A}^4$. The surface is a cone over the product $\mathbb{P}^1 \times \mathbb{P}^1$ when it is embedded in $\mathbb{P}^3$ the segre embedding.*
Blowing it up along the subvariety $x = z = 0$ gives the following.

$$B_I(X) = \{(x,y,z,w;R:S) \in \mathbb{A}^4 \times \mathbb{P}^1 : xS = zR\} \subset \mathbb{A}^4 \times \mathbb{P}^1 \tag{3.23}$$

where x, y, z, w are the affine coordinates of $\mathbb{A}^4$ and R,S are the homogeneous coordinates of $\mathbb{P}^1$.
This can be covered by 2 affine charts.

1. *A chart defined by $R \neq 0$*
   $xS = zR$ becomes $z = xS$ so that

$$\begin{aligned} xy - zw &= xy - xSw = 0 \\ &\Rightarrow x(y - wS) = 0 \end{aligned}$$

   with the exceptional curve defined by $x = 0$ and the nonsingular strict transform given by $y - wS = 0$.

2. *A chart defined by $S \neq 0$*
   $xS = zR$ becomes $zR = x$ so that

$$\begin{aligned} xy - zw &= yzR - zw = 0 \\ &\Rightarrow z(yR - w) = 0 \end{aligned}$$

   with the exceptional curve defined by $z = 0$ and the nonsingular strict transform given by $yR - w = 0$.

Chapter 4

Birational Maps of Surfaces

Having defined birational maps in Section 2.3, in this chapter we will discuss the birational maps of surfaces. A surface in this chapter and throughout this thesis refers to a projective algebraic variety $S \subset \mathbb{P}^n$ of dimension two, and divisors on surfaces are curves. The dimension of S refers to the transcendence degree of the function field $K(S)$ over k. We start with how birational maps relate to the properties of blow-ups of projective algebraic surfaces.

4.1 Blow-ups of Surfaces

Let S be a surface and P a point on S. We can obtain another surface $\widetilde{S}$ by blowing up S at the point P. By Proposition 3.3.2, there exists a blow-up map $\pi : \widetilde{S} \to S$ centered at P, which is unique up to isomorphism and π satisfies the following.

a. The fibre $E = \pi^{-1}(\{P\}) \subset \widetilde{S}$ is an exceptional divisor.

b. π is an isomorphism over $\widetilde{S} \setminus E$ and $S \setminus \{P\}$.

The blow-up π can also be constructed as follows. Consider an open neighbourhood U of P in S with local equations x, y at P. The preimage $\pi^{-1}(U)$ is a subvariety of $U \times \mathbb{P}^1$ defined by $xY = yX$ where X, Y are homogeneous coordinates of $\mathbb{P}^1$. Thus, similarily, we can define the blow-up of U as

$$\widetilde{U} = \{(x,y;X:Y) \in U \times \mathbb{P}^1 | xY = yX\}$$

Consequently, we can clearly define the birational map $\pi|_U : \widetilde{U} \to U$, which is a projection onto the first factor. We can also notice that if $(x,y) \neq (0,0)$, then $\pi|_U^{-1}((x,y)) = \{(x,y)\} \times \mathbb{P}^1$. Thus, $\pi|_U^{-1}(P) = \{P\} \times \mathbb{P}^1 \cong \mathbb{P}^1$.

Hence,

$$\pi|_U : \widetilde{U} \setminus \pi|_U^{-1}(P) \to U \setminus \{P\} \tag{4.1}$$

is an isomorphism and the exceptional curve $E = \pi|_U^{-1}(P) \cong \mathbb{P}^1$ is contracted by $\pi|_U$ to the point P. By gluing S and $\widetilde{U}$ along $S \setminus \{P\}$ and $U \setminus \{P\}$, we can obtain another surface $\widetilde{S}$ with a blow-up map $\pi : \widetilde{S} \to S$ which gives an isomorphism over $S \setminus \{P\}$ and $\widetilde{S} \setminus \pi^{-1}(P)$. The process of passing from $\widetilde{S}$ to S is called **blowing down**, since π blows down the exceptional curve E to the point P. Not only can we blow up a point P on a surface S, we can also blow up an irreducible curve C on S. We define the strict transform of C under the blow-up of S at P as follows.

Definition 4.1.1. *Let $\pi : \widetilde{S} \to S$ be the blow-up of S at the point P. Let C be a curve in S passing through P. The closure of $\pi^{-1}(C \setminus \{P\})$ is an irreducible curve $\widetilde{C}$ in $\widetilde{S}$ which we call the strict transform of C under π.*

The following lemma immediately follows from the definition of the strict transform of C.

Lemma 4.1.1. *. Let C be an irreducible curve on S that passes through P with multiplicity m. Then*

$$\pi^* C = \widetilde{C} + mE \tag{4.2}$$

where E is the exceptional curve in $\widetilde{S}$.

Proof. See for instance Beauville [5, Ch. II, Theorem II.17], Perego [24, Ch. 3, Theorem 3.1.20]. □

4.2 Factorization of Birational Maps.

4.2.1 Overview

The problem of the factorization of birational maps of nonsingular projective varieties has been studied for sometime. This is related to one of the main achievements of the 19th century Italian School of Algebraic geometry which was a complete understanding of the birational geometry of surfaces. This included the study of minimal models. Until now, factorization of birational morphisms and maps by blow-ups along smooth subvarieties remains of fundamental importance in birational geometry. The problem was explicitly stated in the work of Zariski [12] in which he proved that every birational morphism between smooth surfaces is a composition of blow-ups at closed points. It is claimed that Zariski and other members of the Italian school contemplated on the question in higher dimensions without stating the results until the work of Hironaka [4] on resolution of singularities was known in 1964. The theorem of factorization of birational maps of varieties has two versions namely; the strong factorization conjecture and the weak factorization conjecture. The weak factorization conjecture extends to the Zariski's factorization theorem which is most relevant in this thesis. Before we discuss the theorem due to Zariski, we provide a discussion on the strong and weak factorization conjectures and some of the developments in solving them.

4.2.2 The Weak Factorization Conjecture

The weak factorization conjecture is stated in the work of [25] as follows.

Conjecture 4.2.1 (Weak Factorization). *Let $\phi : X \dashrightarrow Y$ be a birational map between smooth complete varieties over an algebraically closed field of characteristic zero. Let U be an open subset of X where ϕ is an isomorphism. Then there exists a diagram as follows. where*

$$X = X_0 \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_{n-1}}{\dashrightarrow} X_{n-1} \overset{f_n}{\dashrightarrow} X_n = Y$$

Figure 4.1: Diagram of the Weak Factorization Conjecture.

1. *$\phi = f_n \circ f_{n-1} \circ \cdots f_2 \circ f_1$.*
2. *f_i are isomorphisms on U, and*
3. *Each X_i is a smooth variety and f_i is a blow-up or a blow-down at smooth centers disjoint from U.*

The conjecture was raised by Oda in [26] and it gave rise to the Oda's weak conjecture for toric varieties. Some of the early works to prove this were those Abramovich [27]; and Wlodarczyk [25] in arbitrary dimensions in which he used fan theory. In Wlodarczykh's work, varieties are replaced by simplicial fans and the blow-ups are transformations that change every cone of a fan containing a ray to some set of convex hulls with faces not containing the rays. It is also worth mentioning that the weak factorization conjecture holds in all dimensions.

4.2.3 The Strong Factorization Conjecture

The strong factorization conjecture is stated as follows.

Conjecture 4.2.2 (**Strong Factorization**). *Let ϕ be defined as in Conjecture 4.2.1. Then there exists a commutative diagram as follows.*

$$\begin{tikzcd} & Z \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ X \arrow[rr, dashed, "\phi"] & & Y \end{tikzcd}$$

Figure 4.2: Diagram of the Strong Factorization Conjecture.

The question of strong factorization was stated by Hironaka [4] in 1964. The existence of this factorization conjecture is an open problem in dimension 3 and higher. The counterexamples were constructed in the works of Hironaka [28, unpublished][1], Shannon [29] and Sally [30]. However, the local version of the strong factorization which replaces varieties with local rings dominated by a valuation on their common function fields has been proved in dimension 3 by Christensen [31] for some valuations in his PhD thesis, Cutkosky in [32] and [33]. For arbitrary dimensions, one of the known results is by Karu [34] in which he proved the conjecture for toric varieties.

—

4.2.4 The Zariski's Factorization Theorem

In this chapter, we will use the weak factorization conjecture as provided in Conjecture 4.2.1 since it is known to hold in all dimensions hence for surfaces which are at the center of our discussion in this chapter. Moreover, the weak factorization finds its use in extending the Zariski's factorization theorem which we discuss in this section. It is this theorem that enables us to factor any birational map of surfaces through blow-ups and blow-downs. Following Hartshorne [6, Ch. V, Proposition 5.3], we state the theorem as follows.

Theorem 4.2.1. ***Zariski's Factorization Theorem.***
Let $f : X' \to X$ be a birational morphism of nonsingular surfaces, and suppose that the rational map f^{-1} has a fundamental point at $P \in X$. Then f factors through a blow-up $\pi : \widetilde{X} \to X$ centered at P. Thus, there exists a commutative diagram as follows.

[1] This is from Hironaka's PhD thesis. An example can be found in Hartshorne [6, Appendix B, Example 3.4.1]

$$\begin{tikzcd}
X' \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft"] & \widetilde{X} \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

Figure 4.3: Commutative Diagram of the Zariski Factorization Theorem

Proof. This theorem is the same as the universal property[2] in Beauville [5, Ch. II, Proposion II.8] and our proof closely follows from his argument with some details added.

Let g be a birational transformation defined by the composition $g = \pi^{-1} \circ f$. We know that having such a map as g from X' to $\widetilde{X}$ means to give an open subset $V \subset X'$ and a morphism $\phi : V \to \widetilde{X}$ (representing g as a rational map) which induces an isomorphism of the function fields $K(V)$ ($\cong K(X')$ when X' is irreducible) and $K(\widetilde{X})$. Let $\Gamma^0_\phi \subset V \times \widetilde{X}$ be the graph of ϕ and $\Gamma_\phi \subset X' \times \widetilde{X}$ be the closure of Γ^0_ϕ. To prove the theorem, it suffices to check that g is a morphism, i.e., it has no fundamental points. However, π^{-1} has P as its only fundamental point and f is a morphism. Then if g has a fundamental point say P', it must be contained in $f^{-1}(P)$.
But we know that the blow-up π is an isomorphism over $X \setminus P$ and $\widetilde{X} \setminus \pi^{-1}(P)$ which are open subsets. Let $V = X' \setminus f^{-1}(P)$ and let $M = X \setminus \{P\}$ be an open subset where π^{-1} is defined. This means that we can have a map $\pi^{-1}|_M : M \to \widetilde{X}$ which is simply a restriction of π^{-1} on M. Let $\Gamma^0_{\pi^{-1}|_M} \subset M \times \widetilde{X}$ be the graph of $\pi^{-1}|_M$ and let $\Gamma_{\pi^{-1}} \subset X \times \widetilde{X}$ be the closure of $\Gamma^0_{\pi^{-1}|_M}$. Then there exists a diagram as follows.

[2] We stick to calling it the Zariski's factorization theorem in order to distinguish it from the universal property we discuss in Section 4.3.2

$$\begin{tikzcd}
 & V\times\widetilde{X}\supset\Gamma^0_\phi \arrow[rr, "f\times id"] \arrow[dl, hook', "\iota\times id"'] \arrow[dd, "p_1"] & & \Gamma^0_{\pi^{-1}|_M}\subset M\times\widetilde{X} \arrow[dl, hook'] \arrow[dd, "p_1"] \\
X'\times\widetilde{X}\supset\Gamma_\phi \arrow[rr] \arrow[dd] & & \Gamma_{\pi^{-1}|_M}\subset X\times\widetilde{X} \arrow[dd, "p_1"] & \\
 & V \arrow[rr] \arrow[dl, hook'] & & M \arrow[dl, hook'] \\
X' \arrow[rr, "f"] & & X &
\end{tikzcd}$$

Since the open subsets V and M are irreducible, the graphs Γ^0_ϕ and $\Gamma^0_{\pi^{-1}|_M}$ are also irreducible. Hence, the maps $f\times id$ are also maps to the closures of the graphs. The projection of the graph $\Gamma_{\pi^{-1}|_M}$ to $\widetilde{X}$ takes the fibre $p_1^{-1}(P)$ to E, hence the total transform $g(P') = \pi^{-1}(f(P')) = \pi^{-1}(P) = E$. On the other hand, g^{-1} has finitely many fundamental points all of which lie inside $\widetilde{X}$. Hence, we can pick a point $Q\in E$ where g^{-1} is not defined. This implies that $g^{-1}(Q) = P'$. We want to show that this needs not to be the case.

We have the inclusions: $P\in X$, $Q\in E\subset\widetilde{X}$ and $P'\in V\subset X'$. The varieties X', X and $\widetilde{X}$ have the pairs X' and X birational via f and X and $\widetilde{X}$ birational via π. By Theorem 2.3.2, the function fields of the corresponding pairs are isomorphic. Let K be the fraction field of the function fields $K(\widetilde{X})$, $K(X)$ and $K(X')$. Then the local rings $\mathscr{O}_Q$, $\mathscr{O}_P$ and $\mathscr{O}_{P'}$ can be viewed as subrings of K with $\mathscr{O}_Q$ dominating $\mathscr{O}_P$ and $\mathscr{O}_{P'}$. Indeed, a regular function $\alpha\in\mathscr{O}_{E,P'}$ has $\alpha\in\mathscr{O}_{E,Q}$ and if $\alpha\in\mathfrak{m}_{P'}$ then

$$\begin{aligned}
\alpha(P') = \frac{u(P')}{v(P')} &= 0 \quad \text{where } u,v\in\mathscr{O}_{\widetilde{X},P'} \text{ with } v(P')\neq 0\\
&\Rightarrow u(P') = 0\\
&\Rightarrow u(Q) = 0\\
&\Rightarrow \alpha\in\mathfrak{m}_Q.
\end{aligned}$$

Thus, $\mathscr{O}_{P'}\cap\mathfrak{m}_Q = \mathfrak{m}_{P'}$. Similarly, it holds for $\mathscr{O}_P$. Now, suppose $X\subset\mathbb{A}^2$. Then the blow-up of $\mathbb{A}^2$ at $P\in X$ is defined by the equations $xu = yt$ with $(x,y)\in\mathbb{A}^2$ and $(t:u)\in\mathbb{P}^1$.

Locally, around $P\in X$, we can have local coordinates x and y and an open subset U_1 containing P whose fiber $\pi^{-1}(U_1) = V(yt-xu)\subset U_1\times\mathbb{P}^1$ can be defined by $t\neq 0$. Then any point on E is defined by $y=0$. Hence, Q is defined by $t=0$, $y=0$ and $u\neq 0$. Thus, $E = V(y)$. But $E = g(P')$.

Then any regular function $\alpha \in \mathfrak{m}_{P'}$ has $\alpha \in (y)$. We also have it that P is the fundamental point of π^{-1}, then it is also a fundamental point of g. But P' in $f^{-1}(P)$ is a fundamental point of g. P being the fundamental point of f^{-1} implies that there is a curve C containing P' such that $C = f^{-1}(P)$ by the Zariski Connectedness Theorem. We can then say that C is defined by a single local equation say z in $\mathscr{O}_{P'}$. Hence, under the pullback map f^*, x and y map into $f^*(x) = x = az$ and $f^*(y) = y = bz$ respectively for some $a, b \in \mathscr{O}_{P'}$. But we also have seen that any $\alpha \in \mathscr{O}_Q$ is generated by y and is contained in $\mathfrak{m}_Q$ but not in $\mathfrak{m}_Q^2$. Hence, its image in $\mathscr{O}_{P'}$ also lies in $\mathfrak{m}_{P'}$ but not in $\mathfrak{m}_{P'}^2$. However, z already vanishes at P'. This means b must be a unit in $\mathscr{O}_{P'}$, so that $t = \frac{x}{y} = \frac{a}{b}$ lies in $\mathscr{O}_{P'}$. Thus, $t \in (y)$ which contradicts the fact that $Q \in E$ is cut out by $t = 0$ and $y = 0$. □

The Zariski's factorization theorem enables us to use blow-ups to study rational maps. This is so since by definition, for any blow-up $\widetilde{X} = Bl_Y(X) \to X$ of the variety X along a subvariety Y, there exists a rational map $\phi : X \dashrightarrow \mathbb{P}^n$. A blow-up of a smooth projective surface $X \subset \mathbb{P}^n$ at a point P can be defined as the image of the composition of $X \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{n+2}{2}-1} \dashrightarrow \mathbb{P}^{\binom{n}{2}-2}$ the 2-uple embedding. Thus, the composition is via such a map as ϕ. However, much as this map may not be regular, it can be extended to a regular map on all of $Bl_Y(X)$.

The following examples illustrate how birational maps of surfaces can be factored through blow-ups and blow-downs in the sense of the Zariski's factorization theorem in Theorem 4.2.1.

Example 4.2.1. ***The blow-up of*** $\mathbb{P}^2$
Consider a cone $X = V(xy - wz) \subset \mathbb{A}^4$. *Embedding* X *in* $\mathbb{P}^3$ *gives a nonsingular quadric surface. It is in fact the image of the segre embedding given by*

$$\Sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, ([x : y]; [w : z]) \mapsto [xw : xz : wy : yz]. \tag{4.3}$$

We show that X *embedded in* $\mathbb{P}^3$ *is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$ *due to the following proposition.*

Proposition 4.2.1. *A nonsingular quadric surface* $X \subset \mathbb{P}^3$ *is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Lets assume that our quadric surface is $X = V(z_0z_3 - z_1z_2) \subset \mathbb{P}^3$. It suffices to define two morphisms $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to X$ and $\psi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ such that $\phi \circ \psi = id_X$ and $\psi \circ \phi = id_{\mathbb{P}^1 \times \mathbb{P}^1}$ to show that $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let the two morphisms be

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to X$$

defined by

$$\big([x_0 : x_1], [y_0 : y_1]\big) \mapsto \big[x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1\big]$$

and

$$\psi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

defined by

$$\big[z_0 : z_1 : z_2 : z_3\big] \mapsto \begin{cases} \big(\big[z_0 : z_1\big], \big[z_0 : z_2\big]\big) & \text{if } z_0 \neq 0; \\ \big(\big[z_0 : z_1\big], \big[z_1 : z_3\big]\big) & \text{if } z_1 \neq 0; \\ \big(\big[z_2 : z_3\big], \big[z_0 : z_2\big]\big) & \text{if } z_2 \neq 0; \\ \big(\big[z_2 : z_3\big], \big[z_1 : z_3\big]\big) & \text{if } z_3 \neq 0. \end{cases}$$

Composing the two morphisms, we have the following:

$$\begin{aligned} (\phi \circ \psi)\big[z_0 : z_1 : z_2 : z_3\big] &= \phi\big(\big[z_0 : z_1\big], \big[z_1 : z_3\big]\big) \quad \text{if } z_1 \neq 0 \\ &= \big[z_0z_1 : z_0z_3 : z_1^2 : z_1z_3\big] \\ &= \big[z_0z_1 : z_1z_2 : z_1^2 : z_1z_3\big] \\ &= \big[z_0 : z_2 : z_1 : z_3\big] \\ &= id_X \\ (\psi \circ \phi)\big([x_0 : x_1], [y_0 : y_1]\big) &= \psi\big(\big[x_0y_0 : x_0y_1\big]; \big[x_1y_0 : x_1y_1\big]\big) \\ &= \big[x_0y_0 : x_0y_1\big]; \big[x_0y_1 : x_1y_1\big] \\ &= \big[y_0 : y_1\big]; \big[x_0 : x_1\big] \\ &= \big([x_0 : x_1], [y_0 : y_1]\big) \\ &= id_{\mathbb{P}^1 \times \mathbb{P}^1} \end{aligned}$$

Thus, the morphisms ψ and ϕ are inverses of each other. Hence,
$X \cong \mathbb{P}^1 \times \mathbb{P}^1$. □

Consider a point $a = (p,q) \in \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and a rational map

$$\pi_a : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2 \tag{4.4}$$

which maps a point $a \in \mathbb{P}^1 \times \mathbb{P}^1$ to the line $ab \in \mathbb{P}^2$ where we identify $\mathbb{P}^2$ as a set of lines in $\mathbb{P}^3$ through a as shown in Figure 4.4.
We wish to blow up $\mathbb{P}^2$. However, the following result shows that blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at a point gives a nice result about the blow-up of $\mathbb{P}^2$. We will view $\mathbb{P}^1 \times \mathbb{P}^1$ as the nonsingular quadric X.

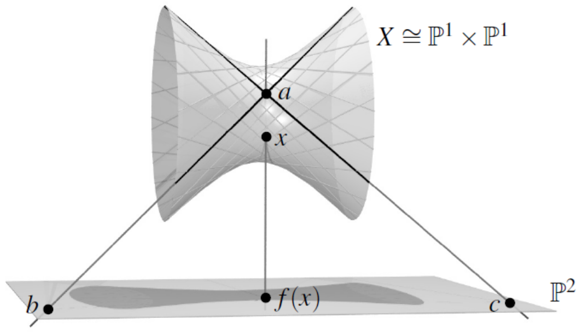

Figure 4.4: The Projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto $\mathbb{P}^2$.

Proposition 4.2.2. *The blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point is isomorphic to the blow-up of $\mathbb{P}^2$ at two points.*

Proof. Consider the quadric surface $X \subset \mathbb{P}^3$ and a projection of X from the point $a = [0:0:0:1]$ to $\mathbb{P}^2$ given by

$$\pi_a : X \setminus \{a\} \to \mathbb{P}^2, [z_0 : z_1 : z_2 : z_3] \mapsto [z_0 : z_1 : z_2]. \tag{4.5}$$

Let $V \subset X$ be an open subset defined by $V = \{[z_0 : z_1 : z_2 : z_3] : z_1 \neq 0\}$. V isomorphic to $\mathbb{A}^2$ via the map $f : V \to \mathbb{A}^2, [z_0 : z_1 : z_2 : z_3] \mapsto (z_0, z_3)$ with an inverse map $f^{-1} : \mathbb{A}^2 \to V, (z_0, z_3) \mapsto [z_0 : 1 : z_0 z_3 : z_3]$. However, $\mathbb{A}^2$ is an open subset of $\mathbb{P}^2$. Hence, by Theorem 2.3.2, $\pi_a : X \to \mathbb{P}^2$ is a birational map. Moreover, the map π_a has an inverse rational map π_a^{-1} given by

$$\pi_a^{-1} : \mathbb{P}^2 \dashrightarrow X, [z_0 : z_1 : z_2] \mapsto \left[z_0 : z_1 : z_2 : \frac{z_1 z_2}{z_0}\right] = [z_0^2 : z_0 z_1 : z_0 z_2 : z_1 z_2] \quad \text{for } z_0 \neq 0. \tag{4.6}$$

This gives the following compositions.

$$\begin{aligned} \pi_a \circ \pi_a^{-1}[z_0 : z_1 : z_2] &= \pi_a[z_0^2 : z_0 z_1 : z_0 z_2 : z_1 z_2] \\ &= [z_0^2 : z_0 z_1 : z_0 z_2] \\ &= [z_0 : z_1 : z_2] \\ &= id_X \end{aligned}$$

$$\begin{aligned}
\pi_a^{-1} \circ \pi_a[z_0 : z_1 : z_2 : z_3] &= \pi_a^{-1}[z_0 : z_1 : z_2] \\
&= [z_0^2 : z_0 z_1 : z_0 z_2 : z_1 z_2] \\
&= \left[z_0 : z_1 : z_2 : \frac{z_1 z_2}{z_0}\right] \\
&= \left[z_0 : z_1 : z_2 : z_3\right] \\
&= id_{\mathbb{P}^2}
\end{aligned}$$

Let $\Gamma \subset X \times \mathbb{P}^2$ be the graph of the birational map π. By definition, the graph Γ is given by

$$\Gamma = \{(z, \pi(z)) : z \in X\} = \{\,[\underbrace{[z_0 : z_1 : z_2 : z_3]}_{z}; \underbrace{[z_0 : z_1 : z_2]}_{\pi(z)}]\,\} \tag{4.7}$$

We claim that Γ is the blow-up of X at the point a, $Bl_a(\mathbb{P}^3)$.

Proof. Let $\widetilde{X} = Bl_a(\mathbb{P}^3)$. By Definition 3.3.4,

$$\widetilde{X} = \{[z_0 : z_1 : z_2 : z_3; X_0 : X_1 : X_2] : z_0 X_1 = z_1 X_0, z_0 X_2 = z_2 X_0, z_1 X_2 = z_2 X_1, z_0 z_3 = z_1 z_2\} \subset X \times \mathbb{P}^2. \tag{4.8}$$

where X_0, X_1, X_2 are homogeneous coordinates in $\mathbb{P}^2$.
To show that the graph $\Gamma \subset \widetilde{X}$, let $[z_0 : z_1 : z_2 : z_3; z_0 : z_1 : z_2] \in \Gamma$. Then all equations in 4.8 are satisfied by $z_0 z_3 = z_1 z_2$ having $X_0 = z_0, X_1 = z_1, X_2 = z_2$.
To show that $\widetilde{X} \subset \Gamma$, let $y = [z_0 : z_1 : z_2 : z_3; X : Y : Z] \in \widetilde{X}$. By definition, y satisfies the equations $z_0 X_1 = z_1 X_0, z_0 X_2 = z_2 X_0, z_1 X_2 = z_2 X_1, z_0 z_3 = z_1 z_2$. On three charts defined by $z_0 \neq 0, z_1 \neq 0, z_2 \neq 0$, we have the following:

- When $z_0 \neq 0$, we may assume that $z_0 = 1$ up to scaling. Then $y = [1 : z_1 : z_2 : z_3; X_0 : X_1 : X_2]$ and the equations $X_1 = z_1 X_0, X_2 = z_2 X_0, z_1 X_2 = z_2 X_1, z_3 = z_1 z_2$ are satisfied by y. Hence, y has $y = [1 : z_1 : z_2 : z_1 z_2; X_0 : z_1 X_0 : z_2 X_0] = [1 : z_1 : z_2 : z_1 z_2; 1 : z_1 : z_2]$ so that $z_0 = X_0, z_1 = X_1, z_2 = X_2$ which implies that $y \in \Gamma$.

- When $z_1 \neq 0$, we may assume that $z_1 = 1$ up to scaling. Then $y = [z_0 : 1 : z_2 : z_3; X_0 : X_1 : X_2]$ and the equations $z_0 X_1 = X_0, z_0 X_2 = z_2 X_0, X_2 = z_2 X_1, z_0 z_3 = z_2$ are satisfied by y. Hence, y has $y = [z_0 : 1 : z_0 z_3 : z_3; z_0 X_1 : X_1 : z_2 X_1] = [z_0 : 1 : z_0 z_3 : z_3; z_0 : 1 : z_2]$ so that $z_0 = X_0, z_1 = X_1, z_2 = X_2$ which implies that $y \in \Gamma$.

- When $z_2 \neq 0$, we may assume that $z_2 = 1$ up to scaling. Then $y = [z_0 : z_1 : 1 : z_3; X_0 : X_1 : X_2]$ and the equations $z_0 Y = z_1 X, z_0 Z = X, z_1 Z = Y, z_0 z_3 = z_1$ are satisfied by y.

Hence, y has
$y = [z_0 : z_0z_3 : 1 : z_3; z_0X_2 : z_1X_2 : X_2] = [z_0 : z_0z_3 : 1 : z_3; z_0 : z_1 : 1]$ so that $z_0 = X_0, z_1 = X_1, z_2 = X_2$ which implies that $y \in \Gamma$.

Thus, we have obtained the blow-up of X at the fundamental point a. This is equivalent to blowing up the variety X at the coordinate functions z_0, z_1, z_2.

□

We now blow up $\mathbb{P}^2$ as follows. Let $b = [0 : 1 : 0], c = [0 : 0 : 1] \in \mathbb{P}^2$ and let $\widetilde{\mathbb{P}^2}$ be the blow-up of $\mathbb{P}^2$ along a subvariety generated by the ideal $I = (z_0^2, z_0z_1, z_0z_2, z_1z_2)$ such that $\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^3$.
By Proposition 3.3.3, the blow-up along a subvariety does not depend on the functions generating the ideal defining the subvariety. It depends on the ideals generated by the function.
Consider the ideals of b and c given by $I(b) = (z_0, z_2)$ and $I(c) = (z_0, z_1)$. Then for $b \in \mathbb{P}^2$, there exists an open neighbourhood V_1 of b defined by $z_1 \neq 0$. In this neighbourhood, the regular functions generating I generate an ideal $I_2 = (z_0^2, z_0, z_0z_2, z_2)$ which simplifies to $I_2 = (z_0, z_2) = I(b)$ since $z_0^2 \in (z_0, z_2)$. There also exists an open neighbourhood V_2 of c defined by $z_2 \neq 0$. On V_2, the functions generating I generate the ideal $I_3 = (z_0^2, z_0z_1, z_0, z_1) = (z_0, z_1) = I(c)$. This clearly shows that $\widetilde{\mathbb{P}^2}$ is the blow-up of $\mathbb{P}^2$ at two points b and c or along two subvarieties defined by the ideals $I(b)$ and $I(c)$.
Now, having obtained the blow-ups of X and $\mathbb{P}^2$, we need to verify if there exists an isomorphism

$$f : \widetilde{X} \subset \mathbb{P}^3 \times \mathbb{P}^2 \to \widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^3. \tag{4.9}$$

Let f be defined by $[z_0 : z_1 : z_2 : z_3; w_0 : w_1 : w_2] \mapsto [w_0 : w_1 : w_2; z_0 : z_1 : z_2 : z_3]$. Outside the exceptional locus inside $\widetilde{X}$, $z_0z_3 = z_1z_2$ and $[z_0 : z_1 : z_2] = [w_0 : w_1 : w_2]$ so that in a smaller complement of the open subset defined by the non-vanishing of z_1, the image of f satisfies $[z_0 : z_1 : z_2 : z_3] = [z_0z_1 : z_1^2 : z_0z_3 : z_1z_3] = [w_0w_1 : w_1^2 : w_0w_3 : w_1w_3]$ for it to be in $\widetilde{\mathbb{P}^2}$. Similarly, for the inverse map f^{-1}, outside the exceptional locus in $\widetilde{\mathbb{P}^2}$, $[z_0 : z_1 : z_2 : z_3] = [w_0w_1 : w_1^2 : w_0w_3 : w_1w_3]$ so that $z_0z_3 = z_1z_2$ and $[z_0 : z_1 : z_2] = [w_0 : w_1 : w_2]$ whenever $w_1 \neq 0$. Thus, we have a commutative diagram as follows.

$$\begin{tikzcd}
& \widetilde{X} \cong \widetilde{\mathbb{P}^2} \arrow[dl] \arrow[dr] & \\
X \cong \mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr, dashed] & & \mathbb{P}^2
\end{tikzcd}$$

Figure 4.5: The Blow-up of $\mathbb{P}^2$.

□

Example 4.2.2. ***The Cremona Transformation.***
Consider the map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$[x : y : z] \mapsto \Big[\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\Big] = [yz : xz : xy]. \tag{4.10}$$

This is clearly a rational map whose inverse ϕ^{-1} is itself. This is why it is an example of an involution (hence a birational map). The map is called the ***Cremona transformation/standard quadratic transformation****.*
ϕ is not defined at points $(x : y : z)$ where two coordinates vanish simultaneously. Thus, the map is defined outside such points as $p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0)$ and $p_3 = 0 : 0 : 1$.
It therefore remains to check if $p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0)$ and $p_3 = (0 : 0 : 1)$ are fundamental points. This can be done by looking inside its graph, whose closure is the blow-up by the construction in section 3.3.3. This will also give us a good picture of what gets blown up and blown down.
Blowing up $\mathbb{P}^2$ at the three points gives the following:

$$\widetilde{\mathbb{P}^2} = \{(x : y : z; y_0 : y_1 : y_2) \in \mathbb{P}^2 \times \mathbb{P}^2 : xy_1 = yy_0, xy_2 = zy_0, yy_2 = zy_1\} \subset \mathbb{P}^2 \times \mathbb{P}^2. \tag{4.11}$$

where y_0, y_1, y_2 are homogeneous coordinates corresponding to the respective coordinates of the image of ϕ.
$\widetilde{\mathbb{P}^2}$ is covered by the following three open charts.

1. *Let U_1 be an open patch of $\widetilde{\mathbb{P}^2}$ defined by $y_0 \neq 0$.*
   $y_0 \neq 0$ implies that $yz \neq 0$. Thus, neither y nor z is zero. And the equations in 4.11 become $y = xy_1$ and $z = xy_2$.
   Projecting $\widetilde{\mathbb{P}^2}$ onto the first gives a birational map given by

$$\Pi|_{U_1} : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2, (x : y : z; yz : xz : xy) \mapsto (x : y : z). \tag{4.12}$$

The inverse image of p_1 under Π is then $\Pi^{-1}(p_1) = (1:0:0;0:y_1:y_2)$ with an exceptional divisor E_{p_1} which is a cartier divisor cut out by $x=0$ and the strict transform given by $\widetilde{L_1} = (0:y_1:y_2) = (0:y:z)$.
Similarly, projecting U_1 onto the second factor gives the map

$$\Pi'|_{U_1} : U_1 \to \mathbb{P}^2, (x:y:z;yz:xz:xy) \mapsto (yz:xz:xy) \tag{4.13}$$

so that the fibre of q_1 under Π' is $\Pi'|_{U_1}^{-1}(q_1) = (0:y:z;1:0:0)$ with the exceptional locus $E_{q_1} = (0:y:z) = \widetilde{L_1}$. Thus, $\widetilde{L_1}$ gets blown down to the point q_1. Indeed, $\phi(0:y:z) = (1:0:0) = q_1$.
Under the projection maps $\Pi|_{U_1}$ and $\Pi'|_{U_1}$, we can also compute the fibres of p_2 and p_3 in a similar way done for p_1. It is however evident that the strict transforms obtained by blowing up p_1, p_2 and p_3 get blown down to the points q_1, q_2 and q_3 respectively.

2. *By the symmetry of the Cremona transformation, the same applies for the remaining two open affine patches: U_2 defined by $y_1 \neq 0$ and U_3 defined by $y_2 \neq 0$.*
   Thus, there exists a commutative diagram for the factorization of ϕ into a sequence of blow-ups and blow-downs, as follows.

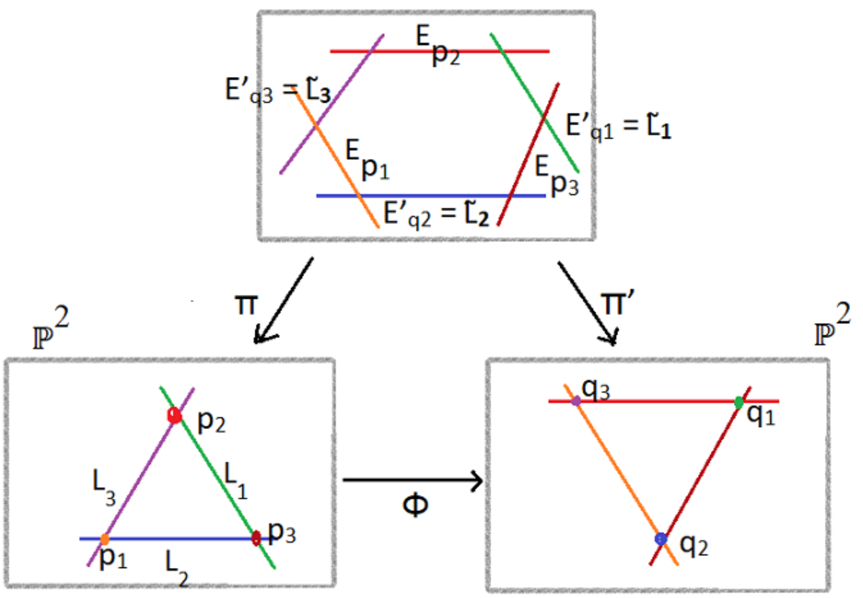

Figure 4.6: Factorization of the Standard Cremona Transformation.

In light of the Zariski's factorization theorem, birational maps of nonsingular surfaces can always be factored through blow-ups and blow-downs with smooth centers. However, for a possibility of the factorization of a birational map in the sense of Theorem 4.2.1, we do not only require the assumption of smoothness on the surfaces and the center(s) of the blow-up(s) or blow-down(s). Based on a result by Sally [30], we also require that the preimage of the center of the blow-up or blow-down be defined by an invertible ideal. The following section is devoted to developing this property in the universal property of a blow-up.

4.3 Blowing up via the Universal Property.

In this section, we aim at extending the notion of blow-ups of ideals to the general notion of blowing up **ideal sheaves** or sheaves of ideals. Sheaves of ideals globalize the notion of ideals of closed subvarieties. We will firstly develop the construction of an ideal sheaf. This will enable us to precisely define the **universal property of a blow-up**.

4.3.1 Ideal Sheaves.

Definition 4.3.1 (Sheaf of Rings of Regular Functions.)**.** *Let X be a variety. For an open subset $U \subset X$, a sheaf of regular functions on U is the set*

$$\mathscr{O}_X(U) = \{f : U \to k : f \text{ is regular}\}$$

This forms a sheaf of rings of regular functions.

We can also define a sheaf of rings of regular functions on a closed subset $Y \subset X$ as follows.

Definition 4.3.2. *Let X be an algebraic variety over k and Y be a closed subset of X. Let $U \subset X$ be an open subset of Y. A sheaf on Y can be defined by*

$$\mathscr{O}_Y(U) = \mathscr{O}_Y(Y \cap U)$$

We define an ideal sheaf as follows.

Definition 4.3.3 (Ideal Sheaf). *Let X be any variety and Y be a closed subvariety of Y. The inclusion $Y \subset X$ induces a map*

$$\iota^{\#} : \mathscr{O}_X \to \iota_* \mathscr{O}_Y. \tag{4.14}$$

This is a homomorphism of sheaves of rings by restriction of regular functions, and it is onto on open affine sets. The kernel sheaf of the map 4.14 is called the ideal sheaf on Y and is usually denoted by $\mathscr{I}_Y$.

This means that the map 4.14 induces a short exact sequence on X given by

$$0 \to \mathscr{I}_Y \hookrightarrow \mathscr{O}_X \twoheadrightarrow \iota_* \mathscr{O}_Y \to 0 \tag{4.15}$$

so that by the First Isomorphism Theorem for rings,

$$\iota_* \mathscr{O}_Y \cong \left(\frac{\mathscr{O}_X}{\mathscr{I}_Y} \right) \tag{4.16}$$

Hence, with $Y \subset X$, the ideal sheaf on Y can also be defined on U by

$$\mathscr{I}_Y(U) = \{ f \in \mathscr{O}_X(U) | f(p) = 0 \quad \forall \quad p \in Y \cap U \} \tag{4.17}$$

If X is a scheme and Y is a subscheme, 4.15 corresponds to the short exact sequence of quasi-coherent sheaves on X.
We now define an inverse image ideal as follows.

Definition 4.3.4 (Inverse Image Ideal Sheaf). *Let $\phi : X' \to X$ be a morphism of varieties (or schemes) and $\mathscr{I} \subseteq \mathscr{O}_X$ be a sheaf of ideals on X. The inverse image ideal sheaf $\mathscr{I}' \subseteq \mathscr{O}_X$ can be defined as follows:*
Consider ϕ as a continuous map of topological spaces X' and X and let $\phi^{-1}\mathscr{I}$ be the inverse image of the sheaf $\mathscr{I}$ on X. $\phi^{-1}\mathscr{I}$ is then a subsheaf of $\phi^{-1}(\mathscr{O}_X)$ whose action on $\mathscr{O}_{X'}$ makes it an $\phi^{-1}\mathscr{O}_X$-algebra. Now, there is a natural homomorphism of sheaves of rings on X' given by

$$\phi^{-1}(\mathscr{O}_X) \to \mathscr{O}_{X'}.$$

We can then define the inverse image ideal sheaf $\mathscr{I}'$ to be the ideal sheaf in $\mathscr{O}_{X'}$ generated by the image of $\phi^{-1}\mathscr{I}$. We denote $\mathscr{I}'$ by $\phi^{-1}\mathscr{I} \cdot \mathscr{O}_{X'}$ or simply $\mathscr{I} \cdot \mathscr{O}_{X'}$. This sheaf of ideal matches exactly with the image of the pull-back $\phi^* I$ under the natural morphism given by

$$\phi^{-1}\mathscr{I} \otimes_{\phi^{-1}\mathscr{O}_X} \mathscr{O}_{X'} \to \phi^{-1}\mathscr{O}_X \otimes_{\phi^{-1}\mathscr{O}_X} \mathscr{O}_{X'} \cong \mathscr{O}_{X'} \tag{4.18}$$

4.3.2 The Universal Property of a Blow-up.

Theorem 4.3.1 (The Universal Property of Blowing Up). *Suppose that X is a quasi-projective variety and $\mathscr{I}$ is an ideal sheaf on X. Let $\pi : \widetilde{X} \to X$ be the blow-up of $\mathscr{I}$. Suppose that $\phi : X' \to X$ is any morphism of quasi-projective varieties such that $\mathscr{I} \cdot \mathscr{O}_{X'}$ is a non-zero locally principal ideal sheaf. Then there exists a unique morphism $g : X' \to \widetilde{X}$ that factors ϕ so that the following diagram commutes.*

$$\begin{tikzcd} X' \arrow[rr, "g"] \arrow[dr, "\phi"'] & & \widetilde{X} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Proof. See for instance Cutkosky [8, Ch. 12, Theorem 12.5]. □

It is worthy stating that the non-zero locally principal ideal $\mathscr{I} \cdot \mathscr{O}_{X'}$ is an example of an invertible sheaf. In the language of schemes, this implies having the exceptional divisor on $\widetilde{X'}$ as a cartier divisor, as it is cut out by the equation that generates $\mathscr{I} \cdot \mathscr{O}_{X'}$.
Surprisingly, the Universal Property of the blow-up in Theorem 4.3.1 looks somewhat similar to the Zariski Factorization Theorem in Theorem 4.2.1. However, the latter implies the former when the blow-up has a point as its center. Of the two, the Zariski Factorization theorem is stronger as it makes use of the Zariski's Main Theorem.

Lets get back to the main question in this thesis.
Question
Given a variety X with a closed nonsingular divisor $D \subset X$ and a birational morphism $f : X \longrightarrow S$ to another variety S which **contracts** D to a subvariety Y. Is f a blow-up?.

Having established the universal property of the blow-up, we could have used it to prove that f is a blow-up if we assumed that the preimage of $f(D)$ under the blow-up of S along Y is a cartier divisor. However, this is not so. This leads us to the following chapter, which aims at presenting a result due to Lascu [1] in which the conditions under which f is a blow-up are constructed without needing the cartier assumption priori.

Chapter 5

Contraction Morphisms

One peculiar fact of a blow-up as constructed in Section 3.3 is that it is a birational map that is not an isomorphism. This is because the inverse rational map π^{-1} fails to be regular at the center of the blow-up. In this chapter, we will discuss similar birational maps called **contraction morphisms**.

5.1 Blow-downs and (-1) Curves

Let S be a nonsingular projective surface and p a point in S. Let $\pi : \widetilde{S} \to S$ be a blow-up of S at the point p. Then the exceptional $E \subset \widetilde{S}$ has $E^2 = -1$. Such curves are said to be -1 curves. In general, a -1 curve can be defined as follows.

Definition 5.1.1. *Let S be a smooth surface. A (-1) curve is a smooth rational curve $C \subset S$ with $C^2 = -1$.*

Exceptional divisors $E \cong \mathbb{P}^1$ resulting from the process of blowing up varieties at points or along subvarieties are the basic examples of (-1) curves. If in addition, a -1 curve C has $C \cong \mathbb{P}^1$, it is said to be an **exceptional curve of the 1st kind**.

5.2 Enriques-Castelnuovo Contraction Theorem.

Having observed that exceptional divisors arising from blow-ups are -1 curves of the first kind, it is natural to ask if every exceptional divisor of the first kind is a preimage of the center of a blow-up. This section discusses a generalization that every curve on a surface that can be blown down or contracted is a -1 curve. Before we state the main theorem, we define a contraction of surfaces as follows.

Definition 5.2.1. *Suppose X is a normal projective surface and $\mathscr{C} = \{C_1, C_2, \cdots, C_n\}$ is a finite set of closed curves on X.*

A contraction of $\mathscr{C}$ is a regular birational map $\phi : X \to Y$ such that Y is normal, there is a point $q \in Y$ such that $\phi(C_i) = q$ for all i, and $\phi : X \setminus \mathscr{C} \to Y \setminus \{q\}$ is an isomorphism.

If the map ϕ defined above exists, it is unique up to isomorphism due to the following lemma.

Lemma 5.2.1. *Let X be a normal projective variety and $\mathscr{C}$ a finite set of closed curves on X. Let ϕ be a contraction as defined in 5.2.1. If such a map ϕ exists, it is unique.*

Proof. See for instance Cutkosky [8, Ch. 20, pg. 383]. □

We now state the main result relating contraction morphisms on surfaces and exceptional divisors of the first kind as follows.

Theorem 5.2.2 (Castelnuovo Contraction Theorem.). *Suppose that S is a nonsingular projective surface and E is an exceptional divisor of the first kind. Then there exists a birational map $\phi : S \to T$ where T is a nonsingular projective surface such that $\phi(E) = p$ is a point on T and $\psi : S \to T$ is isomorphic to the blow-up $\pi : \widetilde{T} \to T$ of p.*

Proof. See for instance Beauville [5, Ch. II, Theorem II.17], Cutkosky [8, Ch. 20, Theorem 20.10], Hartshorne [6, Ch. V, Theorem 5.7] for a proof. □

A crucial result obtained in the 20th century was on the role of the canonical class in the classification process. The result is related to the following lemma.

Lemma 5.2.3. *Let X be a smooth projective surface, $C \subset X$ an irreducible curve with $C^2 = -1$. Then C is a smooth rational curve if and only if $K_X \cdot C = -1$. In other words, C is a smooth rational curve if K_X is not nef.*

A smooth rational -1 curve $C \subset X$ has $K_X \cdot C = -1$. This shows that if the canonical class is nef, then the surface has no -1 curves. A minimal model is distinguished by the lack of -1 curves; so that the classical MMP for surfaces is as follows.

- Step 1: Pick a smooth projective surface X.
- Step 2: If K_X is nef, then stop.
- Step 3: If not, then there exist -1 curve $C \subset X$ with $K_X \cdot C < 0$ and a birational morphism $f : X \to X_1$ contracting C to a point in X_1 where X_1 is also a smooth surface.
- Step 4: Repeat the process by replacing X with X_1.

After finitely many steps, we get a smooth surface Y with no -1 curves. If Y exists, it is unique and it is either $\mathbb{P}^2$ or a ruled surface over some curve.

5.3 Generalization of Contraction Morphisms.

Having constructed contraction morphisms in the case of surfaces and their uniqueness, it is appropriate to have a generalization of such in higher dimensions. Checking the uniqueness of such contraction morphisms will enable us to characterize the blow-downs. In the following section, we discuss a way to generalize contraction morphisms before presenting a result due to Lascu [1] in which the uniqueness of a blow-down is proved.

5.3.1 The Mori Cone Theorem

Having looked at the MMP in the surface case; in dimension > 2, the theory of the MMP is involving. Nevertheless, in the works of Fano, Iskovskikh, Litaka, Ueno, Shokurov, Reid, etc, it was suggested that the MMP for higher dimensional varieties similar to that of surfaces can be obtained. However, one of the obtacles to solving this problem was that it was unclear how -1 curves and even the birational morphisms to contract them could be generalized. The key result solving this problem was by Shigefumi Mori who identified that extremal rays can be contracted. This section devotes to discussing the Mori cone theorem in which extremal rays appear. We follow the construction of the cone theorem as presented in the works of Andreatta [35], Corti [36], Debarre [3] and Kuronya [37]. We start by defining a 1-cycle which will help us to define a closed cone of curves called the **Kleiman-Mori cone**.

Definition 5.3.1 (1-Cycle)**.** *Let X be a smooth projective variety. A 1-cycle*

$$D = \sum_i a_i C_i \tag{5.1}$$

over X is a finite linear combination of proper integral curves with $a_i \in \mathbb{Z}$ (or $\mathbb{Q}$ or $\mathbb{R}$).
A 1-cycle D is called effective if $a_i > 0$ for all i's. Two 1-cycles D, D' are called numerically equivalent if

$$C \cdot D = C \cdot D' \tag{5.2}$$

for all Cartier divisors C on X. The set of all equivalence classes of 1-cycles with real coefficients modulo numerical equivalence is a real vector space denoted by $N_1(X)$. We denote the numerical equivalence class of a 1-cycle D by $[D]$.

Definition 5.3.2. *Let X be a smooth projective variety. An element $H \in N_1(X)$ is called numerically eventually free or numerically effective, in short nef, if $H \cdot C \geq 0$ for every curve $C \subset X$.*

We define the Mori cone as follows.

Definition 5.3.3. *The Mori cone of closed curves*
Let X be a smooth projective variety. We define a cone of curves on X as

$$NE(X) = \Big\{ \sum_i a_i[C_i] : C_i \subset X \textit{ is an integral proper curve}, 0 \leq a_i \in \mathbb{Q} \Big\} \quad (5.3)$$

$$\overline{NE(X)} = \textit{the closure of } NE(X) \textit{ in } N_1(X) \quad (5.4)$$

The cone $\overline{NE(X)}$ is called the Kleiman-Mori cone.

Having defined the cone $\overline{NE(X)}$, it is not difficult to verify that it is a convex cone in the sense of convex geometry. Hence, we can define an extremal face on it as done on any cone in $\mathbb{R}^n$ as follows.

Definition 5.3.4. *Let $M \subset \mathbb{R}^n$ be a cone with vertex at the origin. A subcone $F \subset M$ is called an extremal face of M if for any $x, y \in C$, $x + y \in F$ implies that $x, y \in F$. If $\dim F = 1$, F is said to be an extremal ray.*

This enables us to state the cone theorem. We state the theorem as follows.

Theorem 5.3.1. *Let X be a smooth projective variety. Then*

1. *There exist countably many rational curves C_i on X such that $0 < -K_X \cdot C_i \leq \dim X + 1$, then*

$$\overline{NE(X)} = \overline{NE(X)}_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]. \quad (5.5)$$

2. *For any positive real number ε and any ample divisor H,*

$$\overline{NE(X)} = \overline{NE(X)}_{K_X + \varepsilon H \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]. \quad (5.6)$$

Having defined extremal rays, the assertions in the cone theorem can be easily interpreted. The first assertion says that in the complementary half space to the closed half of $N_1(X)$ where intersection with K_X is non-negative, the cone is spanned by a countable collection of extremal rays whose degree is bounded by the dimension of X. The second assertion suggests that away from the hyperplane $\big\{H : K_X \cdot H = 0\big\}$, the extremal rays cannot accumulate. The assertions can be illustrated geometrically in a diagram as follows.

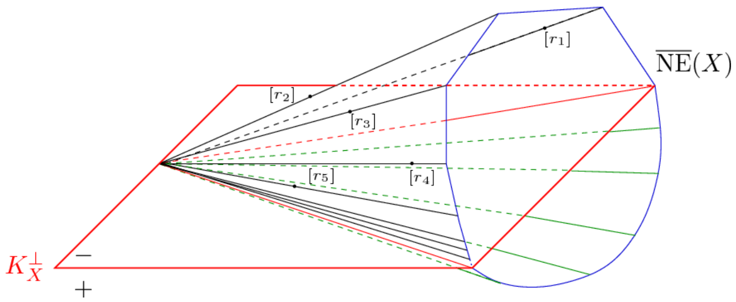

Figure 5.1: Illustration of the Mori Cone Theorem. Source: Debarre [3, Theorem 6.1]

What relates extremal rays to birational morphisms that are meant to contract them is the contraction theorem. We state the theorem as follows.

Theorem 5.3.2 (Contraction theorem)**.** *Let X be a projective variety and $F \subset \overline{NE(X)}$ be an extremal face of the cone of curves on which K_X is negative. Then there exists a morphism $cont_F : X \to Y$ called the contraction of F if the following conditions are satisfied.*

1. *$(cont_F)_* \mathscr{O}_X = \mathscr{O}_Y$. Thus, $cont_F$ has connected fibres.*

2. *If C is an irreducible curve in X, then $cont_F(C)$ is a point if and only if $[C] \in F$.*

The following proposition suggests that for projective varieties of dimension greater than 2, there are three options for *$cont_F$* if it exists.

Proposition 5.3.1. *Let X be a $Q-$factorial normal projective variety, $cont_R : X \to Y$ the contraction corresponding to an extremal ray $R \in NE(X)$. Then we have the following options.*

- *$cont_R$ is a fibre type contraction if $\dim X > \dim Y$;*

- *$cont_R$ is a divisorial contraction if it birational and the exceptional locus E is a prime divisor;*

- *$cont_R$ is a small contraction if it is birational and $codim_X E > 2$.*

In this thesis, the contraction morphisms we use are divisorial contractions. This is firstly due to their correspondence to blow-ups of points in the surface tutorial case. Furthermore, they are a generalization of all blow-ups since any blow-up of a smooth variety along a smooth center is a divisorial contraction.

5.3.2 Regular Contraction Morphisms

In this section, we discuss about regular contraction morphisms and a proof of their uniqueness. However, since the main object of this thesis is to present the result by Lascu in [1], we discuss the notion of a contraction morphism as presented in his paper. We start by defining a regular contraction as follows.

Definition 5.3.5. *Let U and X be any projective varieties. A regular contraction of a variety U along a closed subset $V \subset U$ is defined in [1] as a birational morphism $\phi : U \to X$ between the varieties such that*

1. *ϕ is proper* [21].
2. *$Y = \phi(V)$ is nonsingular and X is nonsingular at every point in Y.*
3. *$dimY < dimV$ (i.e. The dimension reduces).*
4. *ϕ is biregular over $U \setminus V$.*

In this case, a subvariety $V \subset U$ is said to have been contracted onto a subvariety $Y \subset X$ by ϕ.

5.3.3 Purity of the Exceptional Locus.

In a blow-up, we have noticed that a codimension 1 subvariety E is contracted to the center of the blow-up. The following lemma due to Lascu [1] shows that the same property holds in a case where a regular contraction exists. Precisely, every component of the exceptional locus that is contracted by ϕ is of codimension 1.

Lemma 5.3.3. *Let $\phi : U \to X$ be a regular contraction as defined in 5.3.5 and $S(\phi)$ the closed subset of U of points where ϕ is not biregular. Let V be an irreducible component of $S(\phi)$ such that the local ring $\mathscr{O}_{X,Y}$ of $Y = \overline{\phi(V)}$ in X is factorial. Then $codim_U V = 1$.*

Proof. This lemma is a slight variant of Shafarevich [7, Ch.II.44, Theorem 2] and the Van der Waerden purity theorem in Groethendieck [38, EGA IV.21.12]. However, our proof closely follows an argument presented by Shafarevich with some details added.

[21] This is synonymous to completeness, i.e., with ”no missing points”.

Idea of the proof We will first define a regular function on U. If the function locally vanishes on V, then V is of codimension 1.

With the inclusions $V \subset U$ and $Y \subset X$, ϕ induces an inclusion of the local rings $\mathscr{O}_{X,Y} \subset \mathscr{O}_{U,V}$. Since this inclusion is injective, there exists a regular function $\alpha \in \mathscr{O}_{U,V}$ such that $\alpha \notin \mathscr{O}_{X,Y}$. Let $\alpha = \frac{a}{b}$ with $a, b \in \mathscr{O}_{X,Y}$ coprime in $\mathscr{O}_{X,Y}$ Having ϕ not biregular on $S(\phi)$ implies $\phi(p)$ is a curve for $p \in S(\phi)$. We can replace U by an affine neighbourhood of x and thus, assume that U is affine. Suppose $U \subset \mathbb{A}^N$ with coordinate functions $t_1, t_2, \cdots, t_N$ so that we can have the following diagram

$$\begin{tikzcd}
V \arrow[d] & \subset S(\phi) & \subset U \arrow[r, hook] \arrow[d, "\phi"] & \mathbb{A}^N \arrow[r] & k \\
Y = \phi(V) & \subset & X \arrow[ur, dashed] \arrow[urr, dashed, "\alpha"'] & &
\end{tikzcd}$$

and $\alpha = \phi^{-1}$ is the map given by $t_i = \alpha_i$ for $i = 1, 2, \cdots, N$ and $\alpha_i = \alpha^*(t_i)$. Having ϕ not regular at x, α is not regular at $y = \phi^{-1}$. Thus, at least one of the coordinate functions α_i is not regular at x. Suppose this is α_1 so that $\alpha_1 \notin \mathscr{O}_{V,y}$.

Then α_1 can be expressed as $\alpha_1 = \frac{a}{b}$ with $a, b \in \mathscr{O}_{X,Y}$ and $b(x) = 0$. From the diagram, we have

$$\begin{aligned}
\phi^*(\alpha_1) = \frac{\phi^*(a)}{\phi^*(b)} &= x_1 \\
\Rightarrow \phi^*(a) &= \phi^*(b) x_1
\end{aligned}$$

On $y \in V$, $\phi^*(b)(y) = b(x) = 0$. Thus, $y \in V(\phi^*(b))$. Setting $V = V(\phi^*(b))$, we have it that $Codim_U V = 1$. Hence, $a, b \in \mathscr{O}_{X,Y}$ have

$$bx_1 = a$$

from which

$$(a,b)\mathscr{O}_{U,V} = b\mathscr{O}_{U,V}.$$

Hence, a, b are not coprime. $\square$

It is worth stressing that for the purity theorem to hold, U has to be smooth and the center of the blow-up should also be smooth. The following example illustrates a situation where the assumptions are violated.

Example 5.3.1. [1] *Let $S = V(xy - zw) \subset \mathbb{A}^4$ be an affine 3-fold. The variety is singular at the point $P = (0,0,0,0)$, the origin. Let S_1 be defined by the ideal $I_1 = (x,z)$ and S_2 be defined by the ideal $I_2 = (y,w)$. These are both subvarieties of S.*
In this example, we will blow up $\mathbb{A}^4$ at the origin and then along S_1. However, as we had in Example 4.2.1, S is a quadric surface when embedded in $\mathbb{P}^3$. Hence, S in $\mathbb{A}^4$ is a cone over a product $\mathbb{P}^1 \times \mathbb{P}^1$ of two copies of $\mathbb{P}^1$ with a singular point at the origin.
Let $u = (x,y,z,w)$ be a point in $\mathbb{A}^4$ and $v = (X : Y : Z : W)$ be a point in $\mathbb{P}^3$. The blow-up of $\mathbb{A}^4$ at P is given by

$$\widetilde{\mathbb{A}^4} = \{(u;v) \in \mathbb{A}^4 \times \mathbb{P}^3 : xY = yX, xZ = zX, xW = wX, yZ = zY, yW = wY, zW = wZ, xy = zw, XY = ZW\} \subset \tag{5.7}$$

This can be covered by the following four affine open charts.

1. *U_1 defined by $X \neq 0$.*
2. *U_2 defined by $Y \neq 0$.*
3. *U_3 defined by $Z \neq 0$.*
4. *U_4 defined by $W \neq 0$.*

Lets consider what happens in the chart U_1.
In U_1, points in 5.7 satisfy the equations $xY = y, xZ = z, xW = w, XY = ZW$.
Then,

$$\begin{aligned} xy - zw &= x(xY) - (xZ)(xW) \\ &= x^2(Y - ZW) = 0 \end{aligned}$$

Projecting U_1 onto the first factor gives the birational map

$$\pi|_{U_1} : U_1 \subset B_P(\mathbb{A}^4) \to \mathbb{A}^4, \quad (x,Y,Z,W) \mapsto (x, ZW, xZ, xW)$$

so that the fibre of the origin, $\pi^{-1}|_{U_1}(P)$ is the variety defined by the equation $XY = ZW$. This is a quadric in $\mathbb{P}^3$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and of a dimension equal to 2.
Lets also blow up X along the subvariety S_1.
The blow-up

$$B_{I_1}(\mathbb{A}^4) = \{(x,y,z,w; X : Y) \in \mathbb{A}^4 \times \mathbb{P}^3 : xY = zX\} \subset \mathbb{A}^4 \times \mathbb{P}^1. \tag{5.8}$$

This can be covered by the following two affine open charts.

[1] This is Exercise 6.16 in [8]

1. V_1 *defined by* $X \neq 0$.
2. V_2 *defined by* $Y \neq 0$.

Lets consider what happens in the chart V_1.
In V_1*, points in the blow-up satisfy the equation* $xY = z$.
Then,

$$\begin{aligned} xy - zw &= xy - (xY)w \\ &= x(y - wY) = 0 \end{aligned}$$

which is the union of two components; the exceptional divisor defined by the equation $x = 0$ *and the strict transform given by* $y - wY = 0$.
This induces a natural projection onto the first factor given by

$$\pi'|_{V_1} : V_1 \subset B_I(\mathbb{A}^4) \to \mathbb{A}^4, \quad (x, y, w, Y) \mapsto (x, wY, xZ, xW)$$

so that the fibre of the center of the blow-up, $\pi'|_{U_1}^{-1}(V(x,z)) = V(x) = E$ *is a cartier divisor isomorphic to* $\mathbb{P}^1$*. Under the same projection map,* $\pi'|_{V_1}$*, the fibre of the point P,* $\pi'|_{V_1}^{-1}(P) = V(x,Y)$ *which is of dimension equal to 2.*
This clearly shows that the map there exists a map

$$g : B_p(\mathbb{A}^4) \to B_{I_1}(\mathbb{A}^4) \tag{5.9}$$

whose restriction to the exceptional divisor $g|_{E_1}$ *induces the projection onto the first factor given by* $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.
This can be summarized as shown in Figure 5.2. Figure 5.2 clearly shows that the purity theorem fails in singular varieties. Indeed, blowing up S_1 *gives the exceptional locus* $E_1 \cong \mathbb{P}^1 \subset X_1$ *that is of codimension 2. This is contrary to our expectation that every blow-up gives an exceptional divisor. This is so since at the singular point, a cartier divisor and a weyl divisor differ. Furthermore, blowing up the origin P gives* $E_P \cong \mathbb{P}^1 \times \mathbb{P}^1$ *as the exceptional divisor. However, this is cartier divisor everywhere except at the singular point. Furthermore, the divisor* $E_P \cong \mathbb{P}^1 \times \mathbb{P}^1$ *does not get contracted to the point P but a curve in it does.*

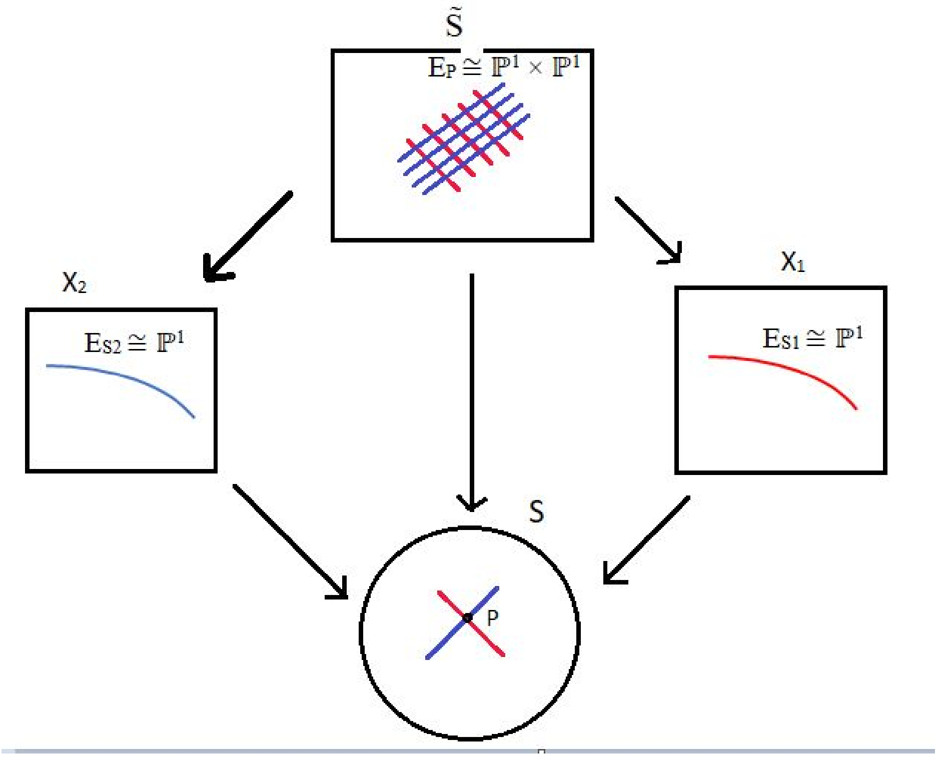

Figure 5.2: The blow-up of the affine cone.

5.3.4 Uniqueness of a Contraction Morphism.

Now we can provide a proof of the property that necessitates a contraction as defined in 5.3.5 to be unique if it exists.

Theorem 5.3.4. *[1] Let $\phi : U \to X$ be a regular contraction of U along a subvariety V. We assume that U is normal at each point in V and the ideal sheaf of V in $\mathscr{O}_U$ is inverible. If $\pi : X' \to X$ is blowing up of X along $Y = \phi(V)$, then U is canonically isomorphic to X' relative to X.*

Proof. In Lemma 5.3.3, we have proved that V is a hypersurface and of codimension 1. Assuming U to be normal and $V \subset U$ being a codimension 1 subvariety necessitates the existence of an affine open subset $T \subset U$ that has $T \cap V \neq \emptyset$ such that the ideal of $T \cap V$ in $K(W)$ is principal so that the following diagram commutes.

$$\begin{tikzcd} & X' \arrow[d, "\pi"] \\ U \arrow[ur, "\chi"] \arrow[r, "\phi"] & X \end{tikzcd}$$

We will prove the theorem by assuming the following properties.

1. For the order functions ν_Y and ν_V defined on the local regular rings $\mathscr{O}_{X,Y}, \mathscr{O}_{U,V}$ in $K(X)$ and $K(U)$ respectively, we have $\nu_Y = \nu_V \circ \phi^*$ where $\phi^* : K(X) \to K(U)$ is an isormophism induced by ϕ.

2. $\mathscr{I}\mathscr{O}_U$ is invertible, with $\mathscr{I}$ is the ideal of Y in $\mathscr{O}_X$.

The technical heart of the proof of the theorem requires proving property 1. However, by assuming it to be true, we can prove property 2 and then finally show that 1 and 2 imply the theorem. We prove property 2 by following the argument in [1] with some details added as follows.

Proof. of 2.
Let $u \in U$ and $x = \phi(u)$. To show that $\mathscr{I}\mathscr{O}_U$ is invertible in $\mathscr{O}_X$, we will show that locally, $\mathscr{I}_x\mathscr{O}_{U,u}$ is principal in $\mathscr{O}_{U,u}$. Assume that $u \in V$. x need not necessarily be in Y. Since U is normal, it has a codimension 1 component V by Theorem 2.3.1. This implies that there is an affine open subset M containing u such that an open neighbourhood $T = V \cap M$ is cut out by a single equation. T therefore gives a prime ideal of V in $\mathscr{O}_{U,u}$. Let ω be the generator of the ideal. For some regular functions a, b coprime in $\mathscr{O}_{X,x}$, ω can be written as

$$\omega = \frac{\phi^*(a)}{\phi^*(b)} \tag{5.10}$$

This means that in T, V is cut out by ω. Thus, ω is a prime element/uniformizer. Hence, the valuation $\nu_V(\omega) = 1$. This further implies that any regular function a can be uniquely expressed as

$$a = s\omega^n \text{ where } s \text{ is a unique unit in } \mathscr{O}_{U,u}. \tag{5.11}$$

However, by property 1,

$$\nu_V(\omega) = \nu_Y(a) - \nu_Y(b) \Rightarrow n = \nu_Y(a) > 0. \tag{5.12}$$

Hence over T, nV is cut out by $\phi^*(a)$. Consequently, for every $\zeta \in \mathscr{I}_x^n$, $\frac{\phi^*(\zeta)}{\phi^*(a)} \geq 0$ over T. Thus, $\phi^*(\zeta) \in \phi^*(a)\mathscr{O}_{U,u}$. Hence,

$$\mathscr{I}_x^n\mathscr{O}_{U,u} = \phi^*(a)\mathscr{O}_{U,u} \tag{5.13}$$

is generated by $\phi^*(a)$. Geometrically, working in a nonsingular setting, $\mathscr{I}_x^n \mathscr{O}_{U,u}$ being a principal ideal implies that $\mathscr{I}_x \mathscr{O}_{U,u}$ is also a principal ideal which in turn is invertible. $\square$

We wind up the proof of the theorem as follows.
By the Universal property of the blow-up in Theorem 4.3.1, the birational map ϕ factors through π as $\phi = \pi \circ \chi$ where $\chi : U \to X'$ is a birational map. By property 1, the preimage $Y' = \pi^{-1}(Y)$ is biregular to V via χ. Thus, χ is biregular in codimension 1 by Theorem 2.3.1. However, ϕ is proper so that χ is surjective. Hence, χ is an isomorphism. $\square$

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