Game Theoretic Modeling of Economic Systems Involving Digital Currencies

by

Guizhou Wang

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University of Stavanger
NO-4036 Stavanger
NORWAY
www.uis.no

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Acknowledgments

Time has swiftly passed, and I am on the verge of completing my doctoral journey. Reflecting on my journey, it feels like I relocated to this beautiful country just yesterday.

I dedicated myself to preparing application materials, undergoing meticulous preparations, and undertaking the International English Language Testing System examination while working at a commercial bank in Beijing, China. Those were busy and unforgettable days, filled with ceaseless effort and unwavering commitment.

Fortuitously, this journey led me to Norway, where I encountered my doctoral supervisor, Professor Kjell Hausken. His persistence, expertise, encouragement, and scholarly insights have been instrumental in shaping my research endeavors. I am honored to have the opportunity to study and reside abroad.

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Professor Hausken’s advice has significantly contributed to my academic development. The fruits of his influence are evident in my work in the following ways:

- Clarity of expression: The importance of crafting clear, concise, and precise sentences has proven instrumental in formulating articulate comments and expressing personal opinions.
- Systematic thinking: The emphasis on systematic thinking has been pivotal. Approaching any modification or refinement within the

research framework systematically ensures a comprehensive and cohesive approach to each adjustment.

- Maintaining competitiveness: Acknowledging the dynamic nature of our academic landscape, my supervisor has underscored the need to remain competitive. Recognizing that continuous progress is essential in an ever-evolving environment, I have been encouraged not to rest on my laurels.
- Physical stamina, endurance, persistence, and staying power: The cultivation of stamina through regular physical exercise is related to the understanding that a robust body contributes to being an effective researcher. I have been reminded of the significance of maintaining physical well-being throughout the demanding journey of research. The key to success in the Ph.D. journey lies in maintaining persistent effort.
- Working efficiently: One needs to strike a balance between explorative and effective. The explorative phase proves valuable when seeking innovative ideas, yet a transition to a more effective work mode becomes imperative for task completion. Procrastination is detrimental, and being efficient is a good way to avoid procrastination.

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1 Introduction

1.1 Background

In modern society, money and inflation are foundational to economic stability, wealth management, investment, trading, financial innovation, and wellbeing.

Individuals' awareness and understanding of these concepts are crucial for informed financial decisions, portfolio management, the maintenance of purchasing power, and the security of a stable financial future.

The emergence of cryptocurrencies, notably Bitcoin, and the emergency of central bank digital currencies (CBDCs) have ushered in a new era characterized by novel dynamics, challenges, and opportunities (Badea & Mungiu-Pupăzan, 2021; Böhme et al., 2015; Ferrari et al., 2022; Tercero-Lucas, 2023). Hence, market participants, e.g. investors, traders, assets holders, need to stay informed about new currencies since these innovations have the potential to reshape the financial landscape, impacting individuals and societies in various ways.

Bitcoin was created by an unknown entity under the pseudonym Satoshi Nakamoto. Bitcoin runs on a peer-to-peer network without backing from physical assets, governments, central banks, or other central authorities (Nakamoto, 2008; Schilling & Uhlig, 2019). Bitcoin's uniqueness lies in its decentralized nature, limited supply (21 million), transparency, global accessibility, permissionless transactions, and security features. These characteristics challenge traditional notions of fiat money, where centralized authorities control issuance and regulation, and financial intermediaries are pivotal in facilitating transactions (Zeng et al., 2020). Therefore, Bitcoin offers an alternative paradigm that empowers individuals and challenges the status quo of the existing financial system (Levulytė & Šapkauskienė, 2021; Senner & Sornette, 2019).

Bitcoin is a precursor to a myriad of cryptocurrencies. As of February 8, 2024, 12,647 cryptocurrencies were enumerated on https://www.coingecko.com/, with a market capitalization of $1.79 trillion. Topics related to cryptocurrencies are various. For instance, inflation expectations and cryptocurrencies (Cong, Ghosh, et al., 2023), token economy (Cong, Li, & Wang, 2021; Cong et al., 2022), financial characteristics of cryptocurrencies (Charfeddine et al., 2020), cryptocurrencies market and trading (Ahmed et al., 2023; Cong, Karolyi, et al., 2021; Cong, Li, et al., 2023; Le Tran & Leirvik, 2020; Liu et al., 2022; Xu & Livshits, 2019), price and volatility (Bouri et al., 2017; Leirvik, 2022; Pichl & Kaizoji, 2017), decentralized finance (Perez et al., 2021; Xu & Vadgama, 2022; Xu et al., 2022), taxation (Cong, Landsman, et al., 2023; Wang & Hausken, 2021b), adoption (Hinzen et al., 2022), security (Pagnotta, 2022), financial inclusion (Ozili, 2022b), mining and footprint (Jiang et al., 2021; Platt et al., 2021; Richardson & Xu, 2020; Sarkodie et al., 2022), monetary policy (Karau, 2023), etc.

The foundational component of Bitcoin is the blockchain, a decentralized ledger that verifies and records transactions across numerous nodes. Bitcoin nodes are devices running Bitcoin software that maintain the network by validating transactions and blocks. All transactions backed up to the genesis block are recorded and secured in blocks, linked together using cryptography (Swan, 2015; Xu et al., 2019; Zheng et al., 2018).

Bhimani et al. (2021) investigate the impact of blockchain adoption in developed countries relative to developing countries. Blockchain technology has diverse applications, including finance, supply chain management, healthcare, education, and insurance. For example, in finance, blockchain enhances transparency and security in transactions and potentially removes financial intermediaries (Ali et al., 2020; Tapscott & Tapscott, 2017; Williamson, 2022). Supply chain management benefits from improved traceability and authenticity verification (Cole et al., 2019; Moosavi et al., 2021). In healthcare, blockchain ensures secure and interoperable health data sharing, while for the government, it facilitates transparent and tamper-proof record-keeping (Hasselgren et al., 2020). Other fields leveraging blockchain include identity verification, intellectual property protection, and decentralized applications, such as decentralized finance (DeFi), Web 3.0, GameFi, and non-fungible tokens (NFT), that run on blockchain networks, offering diverse and innovative solutions to longstanding challenges in different industries.

The rise of these digital currencies has created a dynamic and evolving landscape that underscores the imperative for a nuanced comprehension of the intricate strategic interactions embedded within economic systems. The introduction of CBDCs, representing digital forms of sovereign currencies, poses both challenges and opportunities in the realms of monetary policy, financial stability, financial intermediaries, security, cross-border payments and privacy (Allen et al., 2020; Andolfatto, 2021; Mancini-Griffoli et al., 2018; Prasad, 2023; Wang & Gao, 2024). Simultaneously, the decentralized nature of Bitcoin challenges traditional notions of fiat money. It presents an alternative some individuals perceive as a store of value and a safeguard against inflation. This transformative trajectory underscores the imperative for policymakers, economists, and stakeholders to navigate the multifaceted implications of digital currencies, steering toward a resilient and adaptive economic framework.

Most central banks worldwide recognize the transformative potential of digital currencies and have embarked on exploring national digital currencies, known as CBDCs (Adrian & Mancini-Griffoli, 2021; Chiu et al., 2023; Raskin & Yermack, 2016).

In contrast to the decentralized nature of Bitcoin, CBDCs are digital currencies issued and overseen by central authorities. CBDCs employ new technologies (e.g. blockchain and directed acyclic graphs) to facilitate secure and transparent transactions. CBDCs are commonly accepted as a new type of fiat money (Allen et al., 2022; Cunha et al., 2021). This central regulatory control introduces a nuanced dynamic because it enables governments and central banks to influence monetary policy, financial stability, and economic mechanisms while leveraging blockchain technology's efficiency and security features. The motivation behind CBDCs is multifaceted and includes the enhancement of monetary policy tools, responses to the challenges and threats from cryptocurrencies, the facilitation of financial inclusion, the improvement of transactions and payments, and adaptation to the evolving digital age (Ahmat & Bashir, 2017; Allen et al., 2022; Atici, 2018; Banet & Lebeau, 2022; Calle & Eidan, 2020; Davoodalhosseini, 2022; Kim & Kwon, 2019, 2023; Kumhof & Noone, 2021; Lee et al., 2021; Opare & Kim, 2020; Ozili, 2022a).

The development and potential adoption of CBDCs introduce a new layer of complexity to economic systems. The unfolding dynamics prompt considerations of the potential impact of digital currencies on individual financial behavior, institutional investments, monetary policies, credit supply, financial stability, and the broader economic ecosystem. They also question the coexistence and competition between centrally issued digital currencies and decentralized alternatives such as Bitcoin, the future of currency systems, and regulatory frameworks. This interaction forms a critical aspect of the strategic landscape that requires comprehensive analysis and modeling.

The advantages of fiat currencies include the following characteristics:

- A variable supply of fiat currencies enables the funding of large projects (e.g. Roosevelt's New Deal).
- Central banks' fiscal policies may stabilize the economy, for example, by smoothing out bubbles, incentivizing spending, and avoiding recessions.
- A 2% inflation target may incentivize spending and investing, preventing people from storing too many fiat currencies.

- Central banks' objectives (e.g. financial stability and controlling inflation, unemployment, interest, and exchange rates) may benefit society by supporting economic growth and job creation.
- Fiat currencies comprise over 99% of the world's transaction volume. The daily trading volume in fiat currency exchanges (aka forex), reaches $5 trillion, whereas the daily trading volume for cryptocurrencies seldom exceeds $500 billion (dydx, 2024) .
- Due to centuries of inertia, most people and institutions trust fiat currencies more than cryptocurrencies.
- Fiat currencies are mandatory in most countries, while cryptocurrencies are legal tender only in two countries (i.e. El Salvador and the Central African Republic). Bitcoin is currently illegal in several countries.
- Fiat currencies are mandatory for paying taxes and are the only available or feasible currency in most countries.

The disadvantages of fiat currencies are listed below:

- Fiat currencies experience inflation (e.g. $1 in 1923 is worth $0.06 in 2023; (Calculator, 2023).
- Centralizations of fiat currencies enable blocking or constraining accounts.
- Fiat currencies are prone to excluding marginal groups (e.g. homeless people and those in remote areas).
- Fiat currency transactions may not be completely traceable (e.g. cash transaction records are hard to trace).
- Fiat currency transactions are not necessarily final.
- Fiat currencies lack 24/7/365 transaction availability.
- Fiat currency transactions are time-consuming, especially over holidays and for cross-border payments.

The emergence of digital currencies presents a spectrum of challenges and opportunities. One concern lies in the relationship and connection between the adoption of digital currencies and economic systems' overall financial stability, with the inherent volatility of cryptocurrencies posing a distinct risk. The strategic decisions undertaken by various stakeholders, such as households, enterprises, commercial banks, central banks, governments, and countries, are determinants shaping the trajectory of economic systems. Therefore, considering the adoption, regulation, and use of digital currencies requires a thorough analysis of their implications for monetary policy, the stability of financial institutions, and potential alterations in the dynamics of cross-border transactions. Furthermore, exploring the role of digital currencies in financial inclusion, privacy considerations, and broader socioeconomic implications amplifies the complexity of this evolving narrative. As economic agents navigate this dynamic intersection, strategic foresight, informed decision-making, and adaptive policies emerge as indispensable elements in charting a course that optimally balances the challenges and opportunities of the digital age.

This dissertation includes 13 articles divided into four categories. These are game theory and security (one article), currency evolution and competition (five articles), digital currencies households, central banks, governments, and monetary policy (four articles), and interest rate modeling (three articles), see Figure 1.

Figure 1. The four categories of the 13 articles in the dissertation.

1.2 Objectives and research questions

Understanding the game theoretic underpinnings of these strategic interactions is essential for unraveling the complexities introduced by digital currencies. This dissertation analyzes how diverse stakeholders within the economic ecosystem strategically respond to introducing and integrating digital currencies and how these strategic decisions impact market dynamics. The objective is to furnish valuable insights for a spectrum of participants, including individuals, enterprises, policymakers, economists, researchers, and other stakeholders immersed in navigating the dynamic landscape of digital finance.

The primary goal of this research is to develop a game theoretic model that captures the dynamics of economic systems involving digital currencies, including CBDCs, Bitcoin, and other cryptocurrencies. How the strategic choices made by various economic players impact the evolution of digital currencies is central to this focus. Questions considered in the articles include individual taxation choices in digital currencies, monetary policy within digital currencies, fiat money printing and inflation, and competition between fiat money (exemplified by CBDCs) and hard money (exemplified by Bitcoin).

This dissertation seeks to unravel the strategic interactions of diverse economic players. It aims to highlight how the introduction and adoption of digital currencies impact economic agents' strategic choices, thereby elucidating economic systems' trajectories on the micro and macro levels.

This dissertation's research questions are as follows:

1. Currency evolution and players dynamics:

- How do the volume fractions of a national currency (e.g. CBDC) and a global currency (e.g. Bitcoin) and the fractions of the three players—conventionalists, pioneers, and criminals—evolve?

- How do currencies' various characteristics relate to supply, ownership, decentralization, regulation, transaction confirmation, geographical extension, backing, convenience, confidentiality, transaction efficiency, financial stability, and security impact the player's choice?

2. Currency competition and dynamics:

- How does a variable currency (e.g. fiat) compete with a fixed currency (e.g. Bitcoin), with a particular focus on the dynamics of currency supply?
- How do factors such as inflation/deflation affect the currency competition?

3. CBDC interest rates and economic impact:

- How do positive and negative CBDC interest rates impact household production, consumption, CBDC holding, non-CBDC holding, and utility?
- How does a household strategically allocate its resources per the central bank interest rate policy, and what are the resulting implications for economic activities?
- How does a household earn utility and allocate monetary energy between consumption, CBDC holding, and non-CBDC holding depending on the interest rate of the CBDC chosen by the central bank and the non-CBDC interest rate (both of which may be positive or negative) as well as various preferences, transaction efficiencies, and other factors?
- What are the implications for consumption, CBDC holding, non-CBDC holding, and overall utility in different interest rate scenarios?

4. Taxation choices in dual-currency economies:

- What are the household and the government's taxation choices in a two-currency economy (e.g. a national currency such as a CBDC and a global currency such as Bitcoin)?
- How does the household choose fractions of two currencies to determine the tax evasion probability for each currency?
- How does the government choose its probability of detecting and prosecuting tax evasion, the tax rate, and the penalty factor imposed on each household when tax evasion is successfully detected and prosecuted in each currency?

5. Strategies of banks and agents in hard and fiat money economies:

- How do various financial activities such as borrowing, lending, buying, and selling impact the utilities of the bank and three types of agents: borrowers and buyers, sellers, and nontraders??
- How banks and these agents choose their strategies include whether borrowers prefer to borrow hard and fiat money from banks to buy other assets from sellers, whether sellers want to sell?
- How nontraders are impacted by financial activities in hard and fiat money economies?

6. Zero-day attacks and stockpiling over the two periods:

- How does attacking player 1 allocate resources between the immediate zero-day attack in period 1 and stockpiling for attack in period 2?
- How does the defender defend in both periods, and how do the players' strategic choices in both periods depend on the model characteristics (i.e. player 1's available resources, the contest intensities in both periods, the zero-day appreciation factor from period 1 to period 2, and both players' unit costs of effort, asset valuations, and time discount factors)?

7. Interest rate modeling:

- How to combine the Taylor (1993) rule, the quantity equation (Friedman, 1970) and the Phillips (1958) curve into interest rate models?
- How to incorporate the deviations in money supply, money velocity, and the unemployment rate into interest rate models?
- How to deal with the scaling issues in interest rate models?

1.3 Scientific approach

This section overviews the scientific approach this dissertation employs.

When individuals and businesses engage in competitive or collaborative pursuits, they essentially enter a “game” wherein their decisions impact and are impacted by the choices of others. This dynamic gives rise to strategic choices, and economists use game theory to comprehend and analyze these strategic choices. Game theory is a vital tool for researchers, offering a profound understanding of economic interactions.

Game theory analysis involves at least two players, each with multiple strategies. Each player’s payoff is contingent on the choices of all participants (Fudenberg & Tirole, 1991; Osborne, 2004; Roth, 2002). Game theory mirrors the complexities of real-world decision-making involving interactions among individuals, groups, firms, organizations, and countries.

This dissertation explores multiple participants, such as individuals and households, firms, commercial banks, central banks, governments, and countries. Each participant in the system possesses a strategy set that encompasses setting interest rates and engaging in lending, borrowing, production, consumption, investment, import, export, defaulting, and imposing penalties for default. Given the intricate nature of real-world decisions involving interaction among various entities, game theory is a powerful approach for exploring these research topics.

Furthermore, game theory is a valuable complement to qualitative methods in economics and finance, such as econometrics and bibliometrics. Econometrics is a robust economics methodology that enables researchers to empirically investigate economic relationships, test hypotheses, and contribute to a deeper understanding of the complex dynamics inherent in economic systems. Bibliometric analysis is a powerful instrument for analyzing existing research (Chen, 2017; Donthu et al., 2021). It is widely employed to provide a deeper understanding of the evolution of the intellectual framework and emerging trends within a research field (Hallinger, 2022; Wang & Hausken, 2024; Wang et al., 2021; Yu et al., 2023).

1.4 Dissertation structure

This dissertation is divided into two main parts. Part I introduces background information. Part II presents 13 articles in four categories: Game theory and security (one article), currency evolution and competition (five articles), digital currencies households, central banks, governments, and monetary policy (four articles), and interest rate modeling (three articles). The data sources of this dissertation are explained in detail in corresponding articles within Section 2 (if applicable).

The remainder of Part I is organized as follows: Section 2 offers an introduction to the fundamentals of digital currency, Section 3 contextualizes the topics of the dissertation within a broader framework, and Section 4 outlines general directions for further work inspired by the research presented.

2 Fundamentals of Digital Currencies

2.1 Definitions of money and currencies

Money is a medium of exchange, a unit of account, a store of value, and, occasionally, a standard of deferred payment (Belk & Wallendorf, 1990; Furnham & Lewis, 1986). Money is widely accepted in transactions involving goods, services, and debt settlement. It serves as a unit of account, providing a common measure for valuing different goods and services. The main difference between money and currency is that money is a store of value, whereas currency is not (Battilossi et al., 2020; Goodhart, 1998; Schumpeter, 1991).

Currencies are specific types of money issued by governments or monetary authorities typically associated with a particular country or region (Cohen, 2013; Gilbert & Helleiner, 1999; Larue, 2020; Lavoie, 2022). Currencies exist physically and digitally, facilitating trade and economic activities within their respective jurisdictions. They represent value and are essential for conducting transactions in the modern financial system (Caruana, 2016; Eichengreen et al., 2018; Iancu et al., 2022). Currencies may include traditional forms like paper money, coins, banknotes, and electronic forms such as bank card balances. Currency is known as legal tender because the government approves it. Hence, it can be used domestically anywhere as a payment method. Currency is also commonly used in international transactions and trade between countries. The most used currencies for international payments are the US dollar, the euro, the British pound sterling, the Japanese yen, and the Chinese renminbi. Figure 2 displays the most used currencies in the world for international payments per the Society for Worldwide Interbank Financial Telecommunications from January 2019 to July 2023, based on share in total transaction value.

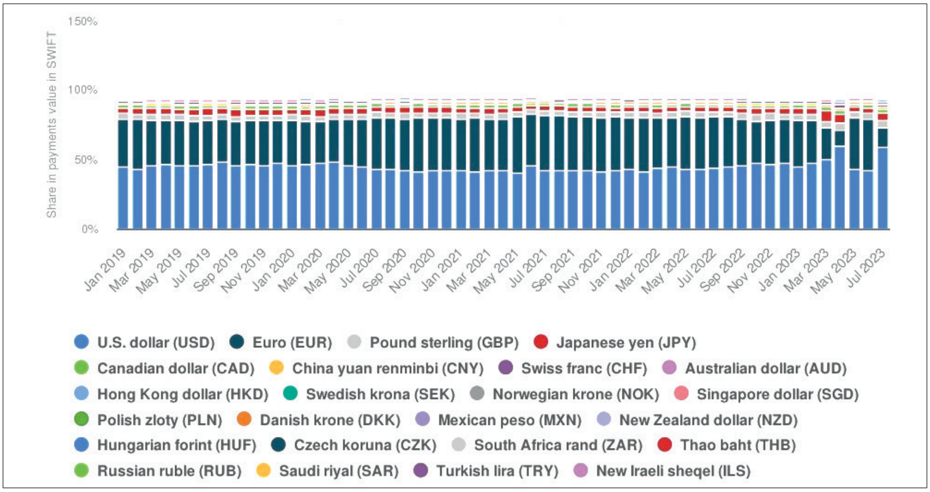


Figure 2. Monthly international payments currency share in SWIFT January 2019–July 2023 (Statista, 2023).

Table 1 presents the most traded currencies by the global foreign exchange market turnover per the Bank for International Settlements (2022). As two currencies are involved in each transaction, the sum of percentage in each currency is 200%. The US dollar (USD) holds the top position, constituting 88.5% of the daily trading volume, underscoring its widespread use and importance in international trade and finance. The euro (EUR) is the second most traded currency, with a proportion of 30.5%, indicating its significant role in global foreign exchange transactions. Other major currencies in the list include the Japanese yen (JPY), the British pound sterling (GBP), and the Chinese renminbi (CNY), each contributing a notable share to the overall market turnover. The results are similar to Figure 2.

Rank	Currency	Percentage of daily volume
1	US dollar	88.5%
2	Euro	30.5%
3	Japanese yen	16.7%
4	British pound sterling	12.9%
5	Renminbi	7.0%
6	Australian dollar	6.4%
7	Canadian dollar	6.2%

8	Swiss franc	5.2%
9	Hong Kong dollar	2.6%
10	Singapore dollar	2.4%

Table 1. Most traded currencies by foreign exchange market turnover (Bank for International Settlements, 2022).

Money can be any type of commodity, such as gold or silver. Money has intrinsic value and possesses certain features such as divisibility, durability, and portability that currency does not have (Camera et al., 2004; Fieleke, 1992; McKinnon, 1979; Velde, 1998). Bitcoin offers a decentralized and digital alternative to traditional monetary systems (Hendrickson et al., 2016; Huberman et al., 2021; Pagnotta, 2022; Weber, 2016; Xu et al., 2023).

2.2 Asset classification

This section presents a way to classify assets. The 11 mutually exclusive and jointly exhaustive assets are fiat money, cryptocurrencies, anti-inflationary investments, NFTs, bonds, stocks, other financial assets, real estate, physical assets, illegal assets, and other assets.

- Fiat money encompasses a broad category, including traditional currencies, physical coins, and CBDCs. Currently, CBDCs have been launched by 11 countries (i.e. Jamaica, Nigeria, the Bahamas, and eight Caribbean Island nations—Anguilla, Saint Kitts and Nevis, Antigua and Barbuda, Montserrat, Dominica, Saint Lucia, Saint Vincent and the Grenadines, and Grenada).
- Cryptocurrencies, such as Bitcoin, Ethereum, and Dogecoin, represent a distinct digital asset class. Cryptocurrencies has the potential to become the 12th sector of the Standard & Poor's 500 Index.
- Anti-inflationary investments like gold, silver, fine art, and limited-edition collectibles provide alternative avenues for wealth preservation. NFTs, exemplified by BRC-420, Rune Stone, NodeMonkes, Bitcoin Frogs, Bitcoin Puppets, BoredApes,

Cryptopunks, Pudgy Penguins, Azuki, and Mad Lads, showcase unique digital assets with ownership recorded on a blockchain.

- Bonds offer interest to bondholders as fixed-income instruments issued by governments or firms.
- Stocks, representing company ownership, for example, in Apple, Alphabet, and Tesla, may yield dividends as a form of return on investment.
- Other financial assets include exchange-traded funds (ETFs), mutual funds, and financial derivatives like futures, options, and swaps.
- Real estate comprises residential, commercial, industrial, raw land, and special-use properties but is distinct from physical assets.
- Physical assets include machinery, inventory, office and warehouse supplies, vehicles, and computers.
- Illegal assets involve certain drugs and funds associated with money laundering, terrorist financing, bribery and corruption, tax evasion, illegal gambling, Ponzi schemes and are considered outside the legal financial system.
- Other assets encompass a diverse range, including computer software, licenses, trademarks, patents, films, copyrights, import quotas, reputation, and design. Essentially, other assets encompass anything not explicitly covered by the previously mentioned categories, reflecting the multifaceted nature of the contemporary asset landscape.

2.3 Central bank digital currencies

CBDCs are digital currencies issued and regulated by central banks (Bordo & Levin, 2017; Mancini-Griffoli et al., 2018). CBDCs exist electronically and utilize technologies such as blockchain and directed acyclic graphs. The key features of CBDCs include the following:

- Issued by central banks: CBDCs are issued and regulated by each country’s central bank.

- Legal tender: CBDCs are recognized as legal tender for transactions and payments within the country or central regions, similar to physical banknotes and coins.
- Digital form: CBDCs are stored and transacted electronically, often through digital wallets and payment systems.
- Backed by the government: CBDCs are typically backed by governments and considered central bank liabilities.
- Controlled supply: Central banks and governments control the issuance and supply of CBDCs.
- Regulatory compliance: CBDCs are subject to each country's regulatory framework, ensuring compliance with financial laws and regulations.

An early definition of a CBDC was developed by Bjerg (2017) based on a report on cryptocurrencies published in 2015 by the Committee on Payments and Market Infrastructures (CPMI, 2015). He included the characteristics of being universally accessible in addition to electronic and central-bank-issued in defining the new concept of central bank digital currency (see Figure 3).

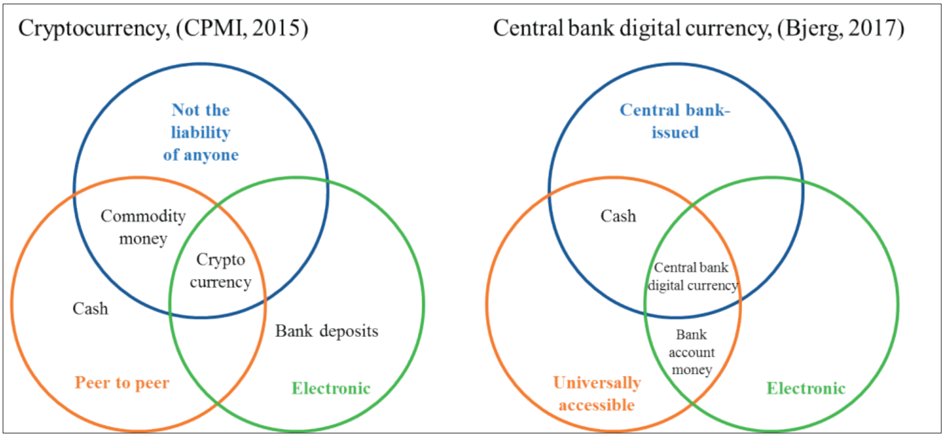


Figure 3. Cryptocurrency versus CBDC (Bjerg, 2017).

Furthermore, Bech and Garratt (2017) presented a taxonomy of money based on four key properties: the issuer (the central bank or other), the form (electronic or physical), the accessibility (universal or limited), and the transfer mechanism (centralized or decentralized). Under this framework, they defined a CBDC as an electronic form of central bank money that can be exchanged in a decentralized way. In addition, they identify two types of CBDC: retail and wholesale. Retail means a widely available, consumer-facing payment instrument targeted at retail transactions, whereas wholesale refers to a restricted-access, digital settlement token for wholesale payment applications (see Figure 4).

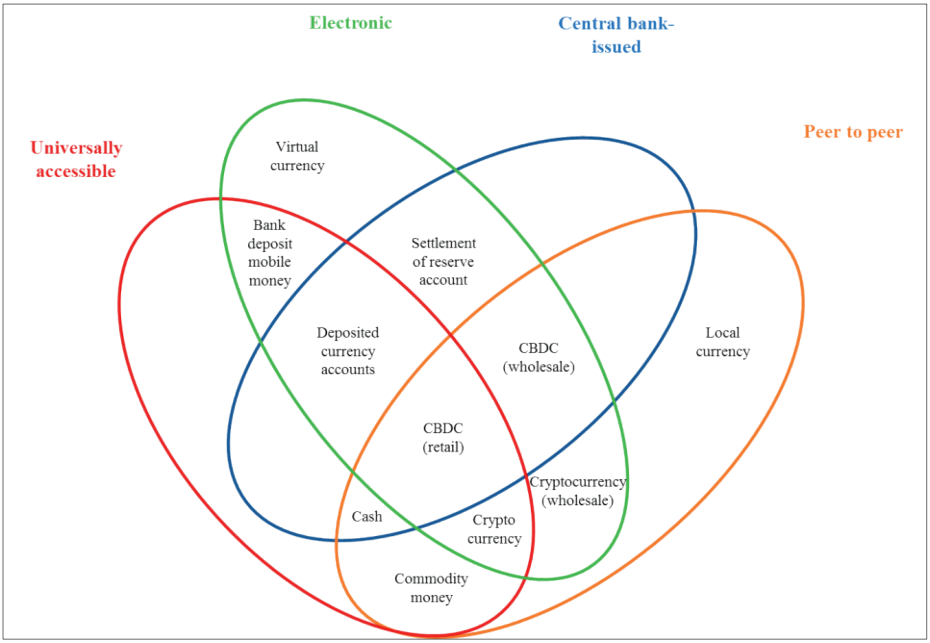


Figure 4. A taxonomy of money and currency (Bech & Garratt, 2017).

CBDCs are commonly divided into several types depending on their designs, architectures, and operational modes. Retail and wholesale CBDCs are classified based on payment types (Auer & Böhme, 2020; Cunha et al., 2021). The former is designed for the general public and daily transactions, while the latter is intended for financial institutions (e.g. commercial banks) and the settlement of transactions for large amounts. Regarding operational modes, CBDCs can be direct, indirect, or hybrid (Auer et al., 2020). A direct CBDC is issued to individuals directly by a central bank, which records all transactions, so no intermediary is needed. An indirect CBDC is first exchanged between the central bank and intermediaries, such as commercial banks and other financial institutions. The public can access the CBDC via these institutions. The central bank manages wholesale transactions and payments.

A hybrid CBDC combines a direct and indirect CBDC. Individuals can access a hybrid through intermediaries and the central bank. Intermediaries handle individual transactions and payments, and the central bank updates and retains transaction records and individual balances. Regarding system architecture, CBDCs are classified as account-based and token-based (Garratt et al., 2020; Ozili, 2023). The latter uses tokens in digital wallets and focuses on transaction validity, while the former opens a digital currency account and focuses on user identification. From an interest-rule perspective, CBDCs are interest-bearing and noninterest-bearing (Agur et al., 2022; Syarifuddin & Bakhtiar, 2022). A depositor can keep an interest-bearing CBDC and earn interest but cannot earn interest on a noninterest-bearing CBDC. Figure 5 exhibits the classification of CBDCs.

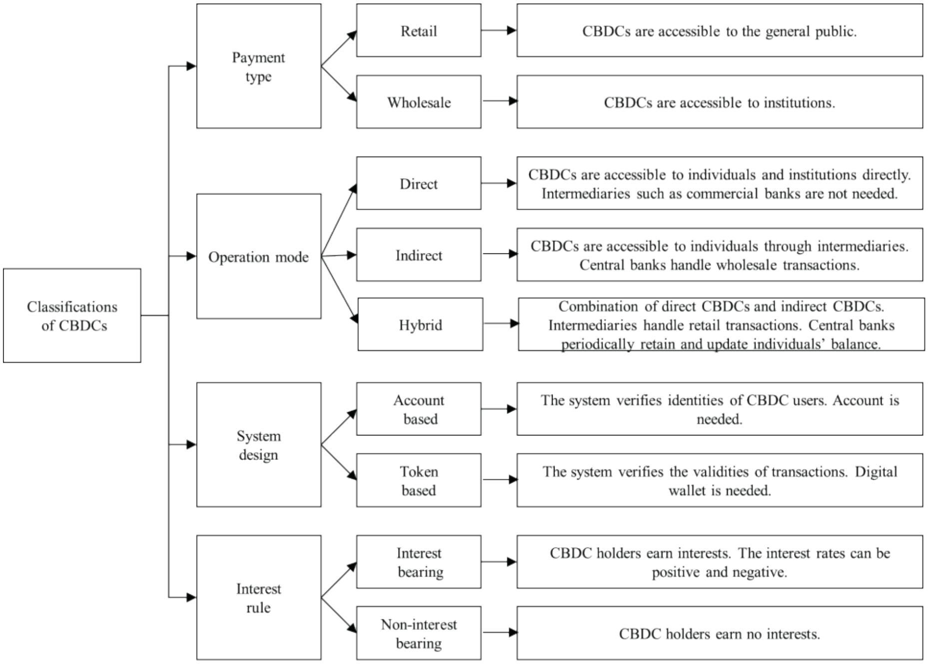


Figure 5. Classification of CBDCs.

Subsequently, Figure 6 compares the number of countries and currency unions exploring, testing, and implementing CBDCs in April 2021 and June 2023. The data indicates a growing global interest in CBDCs over the past two years.

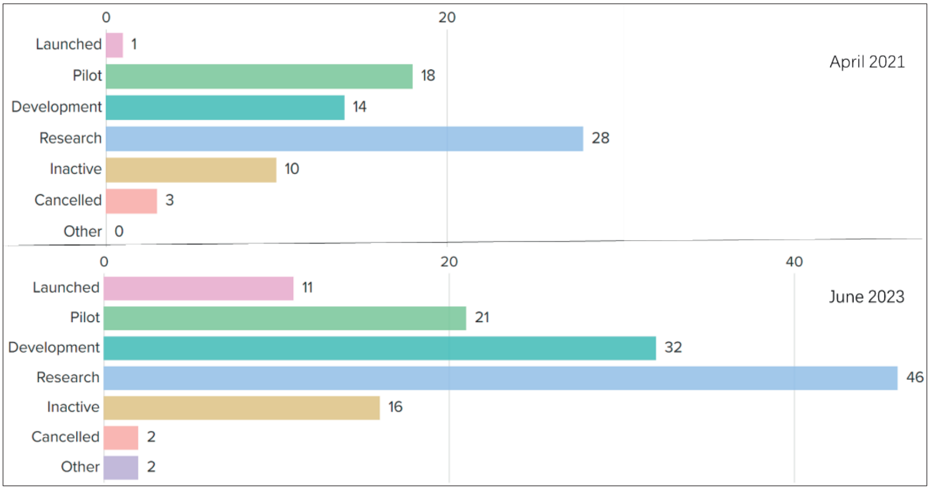


Figure 6. The number of countries and currency unions exploring CBDC in April 2021 and June 2023 (Atlantic Council, 2023).

The US Federal Reserve and the Biden Administration have expressed keen interest in developing a digital dollar (Reserve, 2022). Federal Reserve Banks, such as those in New York and Boston, are actively exploring CBDC prototypes designed for wholesale and retail purposes.

China initiated the Digital Currency Electronic Payments project in 2017, naming it e-CNY in 2021. The People's Bank of China (PBOC), China's central bank, undertakes this digital currency initiative. The primary goal of the digital currency electronic payment project is to create a digital version of the official Chinese currency, the renminbi (RMB), commonly known as the yuan. By October 2021, 123 million individual and 9.2 million corporate wallets had been established, facilitating a transaction volume of 142 million and a transaction value denominated in renminbi (The People's Bank of China, 2021; Xu, 2022). In January 2023, China incorporated the e-CNY into its currency circulation assessments, with the e-CNY constituting 0.13% of the cash and reserves held by the central bank (Amitoj Singh, 2023).

The European Central Bank is poised to commence the preparation phase in November 2023, concentrating on advancing the groundwork for developing the digital euro. This phase involves establishing the fundamentals for a prospective digital euro, involving tasks such as finalizing a rulebook and selecting providers to develop the necessary platform and infrastructure (The European Central Bank, 2023).

Norway has investigated CBDCs since 2016. In April 2021, the Central Bank of Norway announced a plan for technical testing over the next two years (The Central Bank of Norway, 2021). The Norwegian CBDC experimentation concluded in June 2023 (Helge Syrstad, 2023). The Central Bank of Norway disclosed findings from the fourth phase of its CBDC experiments on December 18, 2023, concluding that a retail CBDC is not currently necessary. Instead, it focuses on a wholesale CBDC for interbank settlement of tokenized deposits. The fifth phase, concluding in 2025, aims to provide the central bank with the necessary insights to decide on the potential launch of a CBDC.

Figure 7 displays the status of CBDC development worldwide. As of October 24, 2023, 130 countries, representing 98% of global GDP, were investigating CBDCs (Atlantic Council, 2023). Among G20 countries, 19 (except Argentina) are in advanced stages of CBDC development, with nine already in the pilot phase (i.e. Australia, China, India, Japan, South Korea, Russia, Saudi Arabia, South Africa, and Turkey).

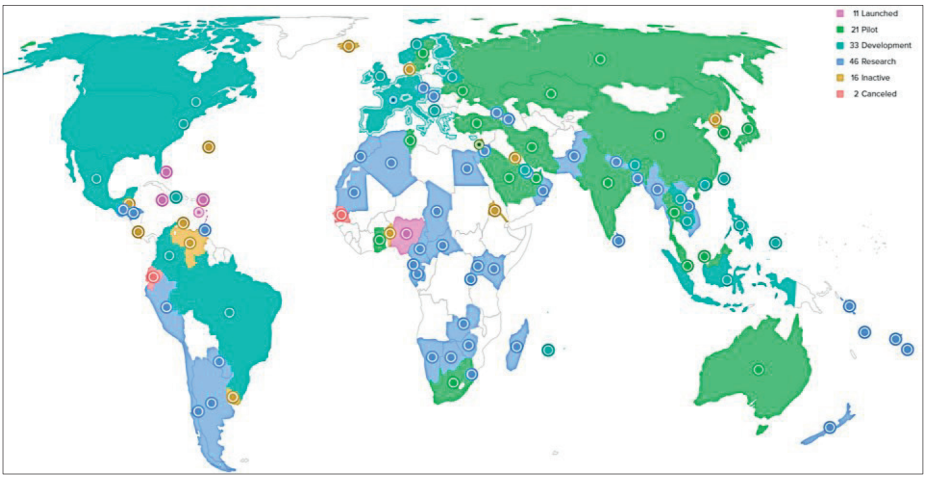


Figure 7. Status of CBDC development by country (Atlantic Council, 2023).

2.4 Bitcoin and cryptocurrencies

The Bitcoin white paper was published on October 31, 2008, by Nakamoto (2008). The genesis block, symbolizing the birth of Bitcoin, was successfully mined on January 3, 2009, at 18:15:05 UTC (Coordinated Universal Time). Within the realm of cryptocurrencies, Bitcoin is widely accepted as the most secure and decentralized cryptocurrency (Hameed, 2019). It has experienced enduring growth in both value and popularity. Bitcoin uses the proof-of-work method to secure the distributed ledger and is a peer-to-peer network without central authorities. Nobody controls the entire Bitcoin network, so everyone can join the Bitcoin network via the internet, with every transaction permanently verified and recorded. Halaburda et al. (2022) summarize the microeconomics of Bitcoin.

Bitcoin mining involves specialized devices, such as application-specific integrated circuit (ASIC) miners, to validate transactions on Bitcoin's blockchain. Bitcoin mining aims to create a new block for the blockchain by solving a complex mathematical problem. The hash of a block's header must be lower than or equal to a target value set by the network's difficulty level. Miners cannot predict which nonce (number used once) will result in a valid hash, so they go through various nonce values in combination with the block's other data. A nonce is a 32-bit (4-byte) field within a block's header, a random number that Bitcoin miners try to find to mine a new block in the blockchain and receive a block reward for their efforts. Härdle et al. (2020) provide an overview of the blockchain technology underpinning cryptocurrencies.

Miners repeatedly hash the block header using the SHA-256 algorithm, adjusting the nonce with each attempt. This process is computationally intensive and requires significant computational power. The first miner to find a nonce that satisfies the difficulty requirement broadcasts the newly mined block to the network, and the block is added to the blockchain. This process, known as proof-of-work, is how miners compete to add a new block to the blockchain and receive the associated reward. Thus, the nonce is crucial in creating a valid block hash during this computational process.

Confirmation occurs when a miner successfully creates a valid hash, discovering a new block. Upon acceptance by the entire Bitcoin network, the miner is rewarded with newly minted bitcoins. The transactions awaiting confirmation are included in this new block, becoming part of the blockchain. Figure 8 shows a simplified Bitcoin blockchain structure with four transactions.

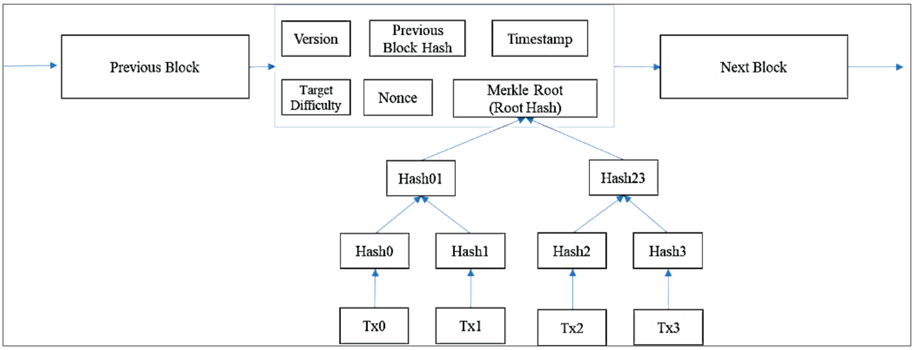


Figure 8. A simplified Bitcoin blockchain structure with four transactions in a block.

Figure 9 summarizes Bitcoin block 829,515, see https://bitaps.com/829515 for more detailed information.

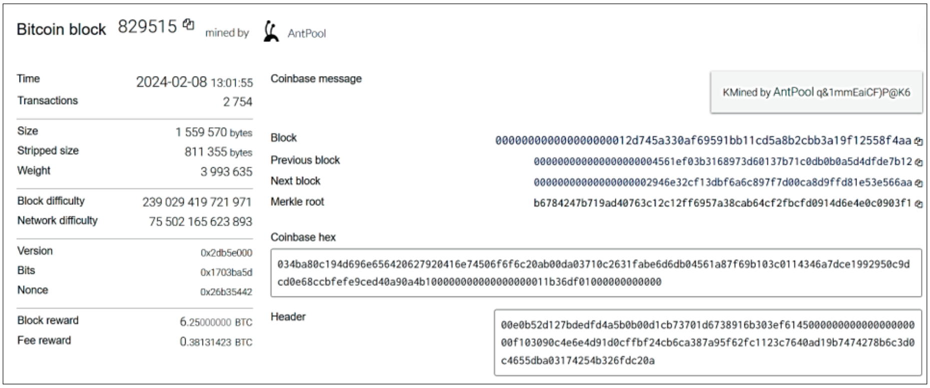


Figure 9. Summary of Bitcoin block 829,515 (Bitaps, 2024).

A Bitcoin node validates, relays, and stores transactions in each block. Nodes broadcast transactions to other nodes and miners. Miners then group these transactions into blocks and add them to the blockchain after validating them. Nodes receive and share these blocks, ensuring miners adhere to network rules. When a node obtains a new transaction or block, it relays it to peers, ensuring synchronization among all nodes and miners to maintain identical blockchains. Figure 10 shows the interactions between Bitcoin nodes and miners.

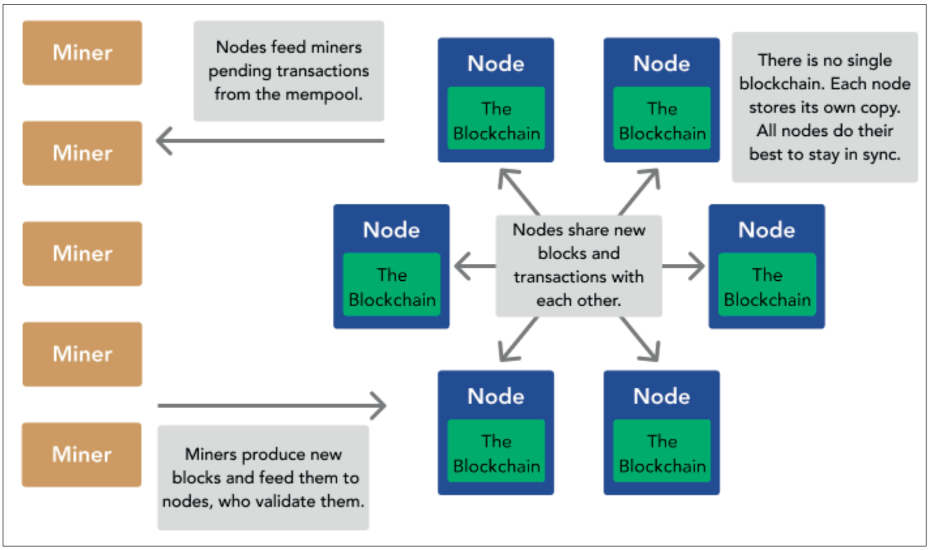


Figure 10. Bitcoin network interactions between nodes and miners (River, 2023).

Bitcoin is increasingly considered to have value (Kelleher, 2021). When the global currency is conceptualized as a cryptocurrency like Bitcoin, which currently allows five to seven transactions per second, Layer 2 is a promising solution for scaling, for example, using the lightning network, where transactions are faster, less costly, and more readily confirmed (Frankenfield, 2021). The lightning network introduces off-ledger transactions and disintermediates central institutions such as banks. Off-ledger transactions are updated on the main blockchain on base Layer 1 only when two parties open and close a payment channel on the lightning network (Poon & Dryja, 2016).

El Salvador adopted Bitcoin as legal tender on September 7, 2021, and the Central African Republic followed suit on April 27, 2022, highlighting its expanding acceptance on a global scale. Notably, prominent public companies, including MicroStrategy, Tesla, Coinbase Global, Inc., and Marathon Digital Holdings, are actively investing in and holding Bitcoin, showcasing a growing trend of institutional adoption (Buybitcoinworldwide, 2023).

The Securities and Exchange Commission (SEC) approved 11 Bitcoin exchange-traded funds (ETFs) based on Bitcoin's real-time spot price for listing and trading on the US stock exchanges on January 10, 2024 (Gensler, 2024). The 11 approved spot Bitcoin ETFs are Grayscale Bitcoin Trust (GBTC), ARK 21shares Bitcoin ETF (ARKB), Franklin Bitcoin ETF (EZBC), Invesco Galaxy Bitcoin ETF (BTCO), Fidelity Wise Origin Bitcoin Fund (FBTC), Vaneck Bitcoin Trust (HODL), Wisdomtree Bitcoin Fund (BTCW), Ishares Bitcoin Trust (IBIT), Bitwise Bitcoin Trust (BITB), Valkyrie Bitcoin Fund (BRRR), Hashdex Bitcoin ETF (DEFI). Table 2 summarizes the 11 Bitcoin ETFs.

Issuing company	Ticker	Assets under management	Management fee	Fee waiver
Grayscale	GTBC	$20 billon	1.5%	N/A
VanEck	HODL	$134 million	0.21%	N/A

Fidelity	FBTC	$2.7 billon	0.39%	N/A
21 Shares & ARK	ARKB	$717 million	0.25%	No fees for the first 6 months OR first $1 billon of inflows
Blackrock	IBTC	$3.3 billon	0.25%	0.12% for the first 6 months OR first $5 billon on inflows
Invesco & Galaxy Digital	BTCW	$306 million	0.25%	No fees for the first 6 months OR first $5 billon of inflows
Hashdex	DEFI	$5 million	0.9%	N/A
Bitwise	BITB	$672 million	0.20%	No fees for the first 6 months OR first $5 billon of inflows
Wisdomtree	BTCW	$15 million	0.3%	N/A
Valkyrie	BRRR	$114 million	0.49%	N/A
Franklin Templeton	EZBC	$64 million	0.19%	No fees for the first $10 billon of inflows

Table 2. Summary of the 11 Bitcoin ETFs as of February 8, 2024.

Bitcoin ETFs provide individual and institutional investors with exposure to Bitcoin globally without the need to buy Bitcoin directly.

Bitcoin is the first application of blockchain technology. The main features of blockchain include distribution, immutability, security, programmability, decentralization, and transparency with pseudonymity (Figure 11).

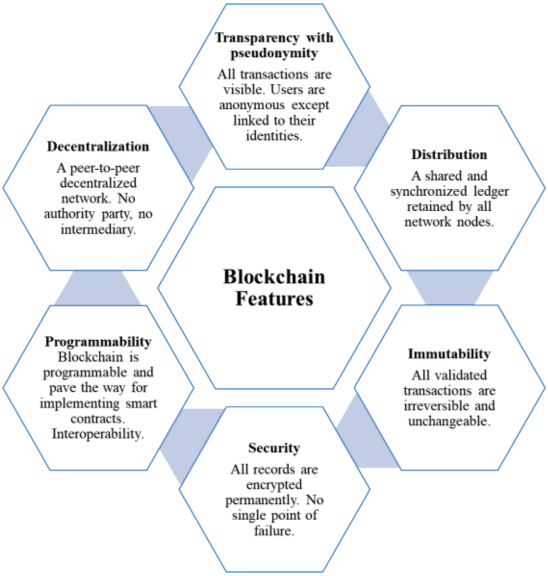


Figure 11. Summary of blockchain features.

Bitcoin and blockchain have demonstrated potential for changing the financial system due to their ability to provide decentralized, secure, and transparent systems (Biais et al., 2023). Their transformative capabilities extend beyond conventional financial paradigms and are driving the emergence of decentralized finance (Carapella et al., 2022). Other well-known cryptocurrencies include Ethereum, Solana, Dogecoin, Pepe coin, Dogwifhat, Bittensor, Chainlink, BNB coin, and Matic. Table 3 compares Bitcoin and fiat money.

Aspect	Bitcoin	Fiat money
Operation time	Open 24/7, 365 days a year	Closed on holidays
Transaction fee	Low fees with flexible transaction fees determined by miners and users. The current average transaction fee is c.a. $0.78 (Bitinfocharts, 2023).	Users pay fees to banks. Cross-border payments are costly, ranging from 1.5% to 7.5% of the transaction value (Morar, 2023).
Transaction	The speed depends on Bitcoin	Transfers may not go

speed	network congestion, normally ca. 10–15 minutes (Hari & Sai, 2015).	through on bank holidays, while international transfers are slow, ranging from one to five working days (Keating, 2023).
Transparency	All transactions are transparently recorded on the blockchain and visible to all.	Banks do not normally disclose their opaque financial transactions to the public.
Trust	Blockchain and cryptographic techniques	Trust in central authorities
Know your customer (KYC)	No KYC requirements means anyone can join the Bitcoin network.	Requires KYC procedures where users must provide personal information to create bank accounts.
Financial inclusion	A universal accessibility internet connection is required.	Limited inclusion, especially in remote areas without bank branches
Privacy	Anonymous, except when an address is linked to KYC exchange accounts.	User information is owned and managed by the bank.
Security	Cryptographic techniques secure Bitcoin with no single point of failure.	Passwords secure accounts.
Authorized party or intermediary	No authorized or intermediary parties.	Centralized control by banks that serve as intermediaries.
Control of assets	Full control	Banks have full control over the users' bank accounts (e.g. they can lock them).

Table 3. Comparison of Bitcoin and fiat money.

Discussions surrounding the regulatory framework for cryptocurrencies continue, and the situation is improving. For example, the European Union is positioned to become the first major jurisdiction to implement a dedicated and comprehensive crypto law. The Markets in Crypto Assets Regulation (MiCA), scheduled to be enforced in 2024 (European Securities and Markets Authority, 2023), signifies a groundbreaking development, offering legal clarity, addressing compliance challenges, and carrying global implications. Table 4 summarizes the regulations on cryptocurrency in the US, EU, UK, China, Canada, Singapore, and Austria.

Country or region	Regulator	Regulation framework	Main points
US	The Securities and Exchange Commission, The Commodity Futures Trading Commission		Still working toward creating an efficient set of digital asset regulations.
EU	The European Securities and Markets Authority	Markets in Crypto Assets Regulation	Identifies three categories of crypto assets: Asset-referenced tokens, electronic money tokens, and other crypto assets not covered by existing EU legislation (including utility tokens), with projected implementation in 2024.
UK	Financial Conduct Authority	Consultation paper	Firms promoting crypto assets in the UK must be authorized or registered by the Financial Conduct Authority or have their marketing approved by an authorized firm.
China	The National Financial Regulatory Administration		Legally banned crypto-asset activity in 2021. Chinese citizens are technically permitted to hold crypto assets.
Canada	Canadian Securities Administrators	Primarily under securities law, the Proceeds of Crime and Terrorist	Digital assets such as cryptocurrencies are treated as securities in Canada. Require companies transacting in cryptocurrency to keep records of all cross-border transactions, report suspicious activity, and register with

		Financing Act	local regulators.
Singapore	The Monetary Authority of Singapore	The Payment Services Act	Aims to ensure consumer protection, maintain financial stability, and guard against money laundering and terrorism financing risks. Crypto exchanges are required to be registered and hold a license to operate in Singapore. Continuously working to improve the regulatory framework of digital payment token services.
Austria	The Australian Securities and Investments Commission, The Australian Treasury	A range of laws, Australia's financial services, and anti-money laundering and counter-terrorism financing regime	Plans to release draft legislation covering licensing and custody rules for crypto-asset providers by 2024. Cryptocurrencies are legal in Australia and treated as property. Trading, spending, receiving, and storing cryptocurrency are all legally permissible activities.

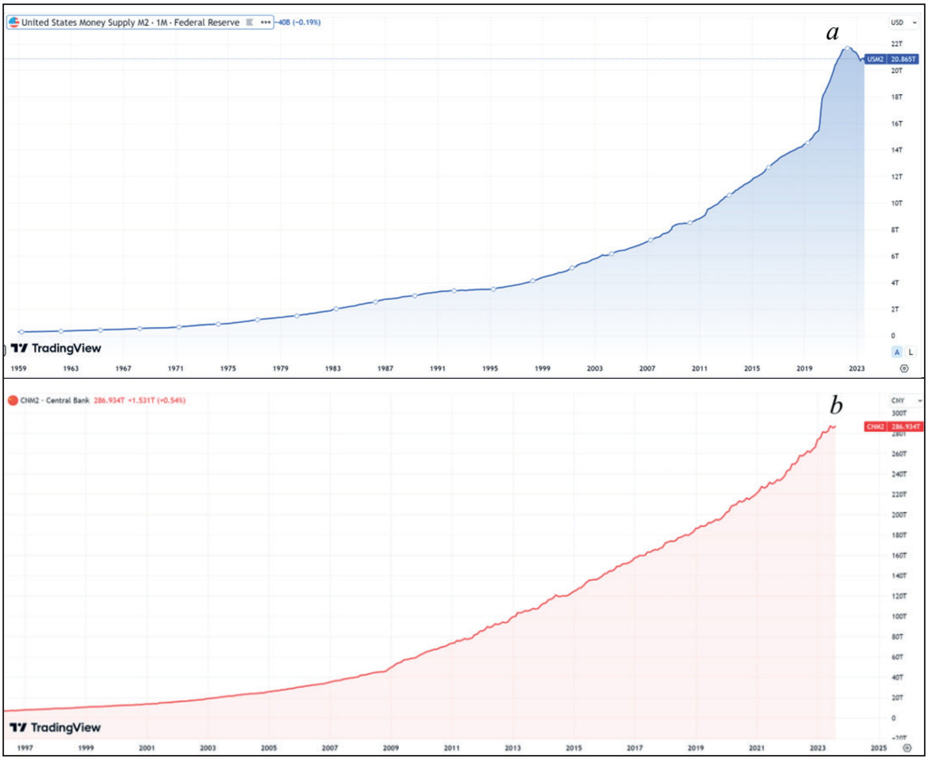
Table 4. Overview of cryptocurrency regulations in US, EU, UK, China, Canada, Singapore, and Austria.

2.5 Fiat money supply and inflation

Historically, central banks worldwide tend to print fiat money. Sometimes, they may decrease the fiat money supply to reduce a high inflation rate. Central banks use a combination of tools to control the fiat money supply, such as open market operations, adjustments to interest rates and reserve requirements, and forward guidance for printing or withdrawing fiat money from the economy.

Figure 12 depicts the historical fiat money supply represented by M2 in the US, China, and EU. M2 is a monetary aggregate representing a broader measure of the money supply within an economy (Investopedia, 2023). It is widely used by economists, researchers, and policymakers to understand and analyze the overall money supply in an economy. M2 provides a more comprehensive view than M1, which only includes the most liquid forms of money (Parhizgari & Nguyen, 2011).

As of October 2023, the M2 money supply in the US was $20.865 trillion, in China ￥286.934 trillion (equivalent to $39.2 trillion at an exchange rate of 0.14), and in the EU €19.032 trillion (equivalent to $20.06 trillion at an exchange rate of 1.05). Figure 12 illustrates that the fiat money supply has historically increased extensively. Panel a: Historical M2 money supply trend in the US, Panel b: Historical M2 money supply trend in China, and Panel c: Historical M2 money supply trend in the EU.



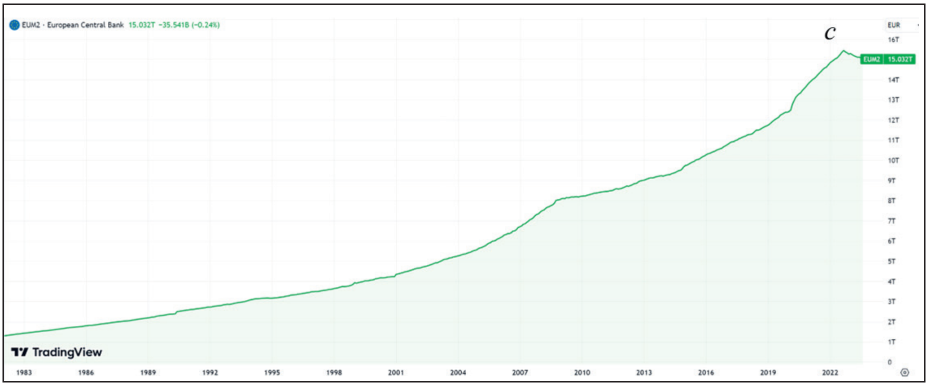


Figure 12. The historical M2 money supply trends in the US (Panel a), China (Panel b), and the EU (Panel c) (TradingView, 2023).

Inflation is a general increase in the economy's overall price level of goods and services (Board of Governors of the Federal Reserve System, 2021). Various economic theories attempt to explain the causes of inflation, such as the cost-push theory (Batten, 1981), the demand-pull theory (Holzman, 1960), the expectation theory (Carlson & Parkin, 1975), and the quantity theory of money (Friedman, 1989; Lucas, 1980).

Generally, a prevailing perspective holds that an increase in the money supply contributes to inflation. Hence, inflation is caused when an economy's money supply grows faster than the economy's ability to produce goods and services (The Federal Reserve Bank of St. Louis, 2023).

The fiat system is characterized by inflation. Figure 13 shows the inflation rate by country in 2022. The global average inflation rate was 13.9% in 2022. Zimbabwe had the highest inflation rate in the world at 284.94%, followed by Venezuela at 210%, Sudan at 154.91%, Turkey at 73.13%, and Argentina at 72.37%.

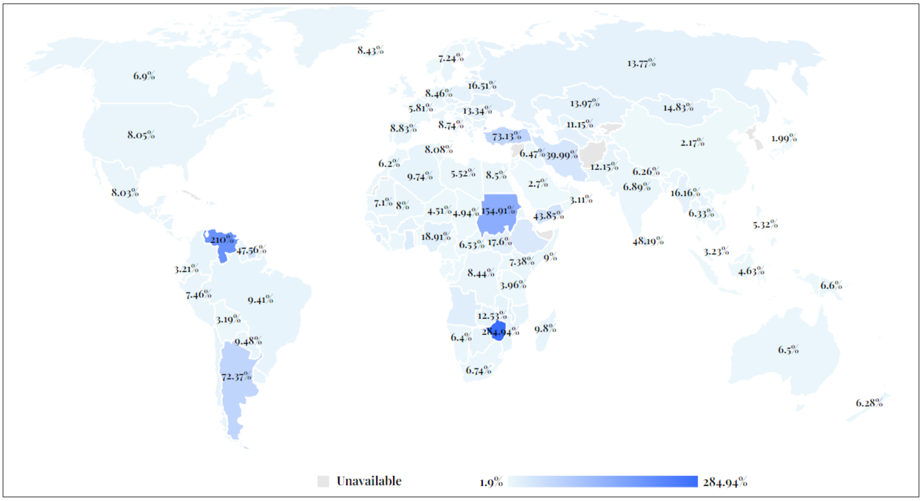


Figure 13. Inflation rate (%) by country in 2022 (Wisevoter, 2023).

Table 5 compares Bitcoin to the world's top 20 largest fiat currencies by market cap as of February 8, 2024.

Rank	Name	Price	Market Cap	Circulating Supply	Max Supply
1	Chinese Yuan (CNY)	316 sats	926,147,102 BTC	292,270,000,000,000 CNY	Unlimited
2	United States Dollar (USD)	2,253 sats	869,747,261 BTC	38,597,643,970,000 USD	Unlimited
3	Icelandic Krona (ISK)	16 sats	451,883,549 BTC	2,760,185,000,000,000 ISK	Unlimited
4	Euro (EUR)	2,428 sats	362,627,307 BTC	14,935,209,000,000 EUR	Unlimited
5	Japanese Yen (JPY)	15 sats	243,395,017 BTC	1,599,534,000,000,000 JPY	Unlimited
6	Pound Sterling (GBP)	2,845 sats	100,280,417 BTC	3,524,371,000,000 GBP	Unlimited
7	South Korean Won (KRW)	1 sats	89,609,196 BTC	5,283,931,000,000,000 KRW	Unlimited
8	Indian Rupee (INR)	27 sats	65,715,707 BTC	242,093,000,000,000 INR	Unlimited

9	Canadian Dollar (CAD)	1,673 sats	60,030,968 BTC	3,587,269,000,000 CAD	Unlimited
10	Hong Kong Dollar (HKD)	288 sats	49,665,800 BTC	17,234,053,000,000 HKD	Unlimited
11	Brazilian Real (BRL)	453 sats	48,538,158 BTC	10,707,230,000,000 BRL	Unlimited
12	Australian Dollar (AUD)	1,469 sats	43,583,367 BTC	2,966,000,000,000 AUD	Unlimited
13	New Taiwan Dollar (TWD)	71 sats	43,476,892 BTC	60,548,592,000,000 TWD	Unlimited
14	Swiss Franc (CHF)	2,577 sats	29,287,379 BTC	1,136,215,000,000 CHF	Unlimited
15	Russian Ruble (RUB)	24 sats	24,488,863 BTC	98,385,000,000,000 RUB	Unlimited
16	Mexican Peso (MXN)	132 sats	21,380,666 BTC	16,187,436,797,000 MXN	Unlimited
17	Bitcoin (BTC)	100,000,000 sats	19,607,828 BTC	19,621,506 BTC	21,000,000 BTC
18	Thai Baht (THB)	63 sats	16,137,291 BTC	25,497,000,000,000 THB	Unlimited
19	Saudi Riyal (SAR)	600 sats	16,135,110 BTC	2,685,343,000,000 SAR	Unlimited
20	United Arab Emirates Dirham (AED)	613 sats	14,654,185 BTC	2,388,647,000,000 AED	Unlimited

Table 5. Bitcoin compared to the top 20 largest fiat currencies in the world by market cap (CoinMarketCap, 2024).

3 Research Areas and Problems

3.1 Game theory and security

Article 1, in Category 1, discusses security in the production and deployment of zero-day exploits.

Article 1 introduces a two-period game theory model. Player 1 chooses to produce zero-day exploits immediately or stockpile for future deployment in period 2. Player 2 defends its asset in the two periods. The article illuminates how the two players balance exerting effort in the two periods. The analysis considers asset valuations, asset growth, time discounting, and contest intensities. Based on a no-stockpiling benchmark, 18 parameter values are varied to understand the zero-day phenomenon over the two periods.

Article 1 explores 11 possible solutions based on various settings, such as stockpiling, budget utilization, attacking in Period 2, and deterring Player 2 from defending. Player 1's stockpiling choice depends on unit effort cost, Period 1's advantage, and zero-day appreciation. Contest intensity impacts players' efforts, leading to withdrawal if negative expected utility occurs over the two periods.

The time discount factor affects efforts and expected utilities, with lower efforts when the factor decreases. The model validates intuitive results, such as players exerting more effort if it is cheaper, valuing the asset more, or having a higher growth factor. Inverse U-shaped efforts are observed, with fierce competition when advantageous and decreasing efforts in cases of extreme advantages or disadvantages.

3.2 Currency evolution and competition

Articles 2 to 6 in Category 2 discuss the following topics:

- The decision-making dynamics of conventionalists, pioneers, and criminals in choosing between a national currency and a global currency
- The evolution and competition of fixed-supply and variable-supply currencies
- The interplay between borrowers and buyers, sellers, nontraders, and banks in a hard and fiat money economic system
- Bitcoin price evolution uses differential equation growth models incorporating oscillation and lengthening cycles.

Article 2 categorizes players into three types: Conventionalists, pioneers, and criminals. Conventionalists follow traditional finance and prefer national currency. Pioneers tend to break from tradition (early adopters), and criminals prefer not to get caught. They both tend to prefer global currencies.

Article 2 examines how conventionalists, pioneers, and criminals choose between a national currency (e.g. a CBDC) and a global currency (e.g. Bitcoin), considering specific characteristics. All players have Cobb-Douglas utilities with one output elasticity for each of the two currencies, comprised of backing, convenience, confidentiality, transaction efficiency, financial stability, and security. Players choose the fraction of transaction volumes in each currency and player type accordingly for maximum expected utility.

The expected utility for each player follows an inversely U-shaped curve based on the transaction volume fraction skewed toward national currency for conventionalists and global currency for pioneers and criminals. The society's expected utility is a weighted sum of each player's expected utility, considering the fraction of each player type.

The replicator equation illustrates the evolution of player types and their currency choice over time. It shows dominance shifts based on players' expected utility preferences. Fifteen parameter values are varied to illustrate sensitivity. When conventionalists become extinct, pioneers and criminals directly compete. Players make strategic choices considering factors like criminal transaction fractions, detection probability, and scaling exponents for expected utility. Conventionalists become extinct when criminals benefit more from criminal behavior and when the parameter values in the conventionalists' expected utility are unfavorable, leading to competition between pioneers and criminals.

Article 3 investigates the competition between a fixed-supply currency (e.g. Bitcoin) and a variable-supply currency (e.g. a fiat currency such as a CBDC). Two kinds of players support fixed- and variable-supply currency differently and choose their volume fractions of transactions in each currency. Each player's utility depends on how that player supports that currency, the transaction volume fractions of all players' (of both kinds) transactions in that currency, and the fractions of players of the same kind. The player's utility in transacting in a variable-supply currency involves two ratios. The first is the initial money supply plus the cumulative money printing or withdrawal divided by the initial money supply. The second is the inverse of cumulative inflation or deflation. Currency backing considers factors such as financial stability, transaction efficiency, convenience, confidentiality, and security (Wang & Hausken, 2021a).

Three replicator equations are analyzed. Two illustrate each player's volume fraction of transactions in each currency over time, while the third shows the evolution of the fraction of each kind of player over time.

Players are inclined to prefer the variable-supply currency with a high weight assigned to the money supply relative to inflation. Conversely, a low weight assigned to the money supply relative to inflation induces players to be more inclined to prefer the fixed-supply currency. Transaction preferences between the two currencies may exhibit inverse U or U shapes before converging. Players choose their player kind and may opt for the kind with the highest support for a given currency. Players may align with the kind that supports a currency with exceptionally high support. If a player's utility is proportional to the fraction of the same kind of players, a higher impact for one kind leads to a preference for that kind.

Article 4 relates to Article 3 and analyzes how a player's low, high, increasing, and decreasing support for fixed-supply currencies relative to variable-supply currencies impacts currency choices. A currency's support by the player relies on factors such as backing, convenience, confidentiality, transaction efficiency, financial stability, and security.

A player's high weight assigned to money printing causes the quick dominance of variable-supply currency, with low support for fixed-supply currency. A player placing high weight on the fixed-supply currency may cause a temporary decrease in the fraction that eventually increases, except in cases of very high support for the fixed-supply currency. Very low support for the fixed-supply currency causes the variable-supply currency fraction to approach 1. High player support for the fixed-supply currency may temporarily increase the fraction, but it eventually decreases, especially when the player highly supports it.

With a high player weight assigned to money printing and low but linearly increasing support for the fixed-supply currency, the variable-supply currency fraction approaches 1 quickly. High and linearly increasing support may temporarily increase and eventually decrease the fraction. With a high weight on money printing, linearly decreasing support for the fixed-supply currency may temporarily decrease the fraction and then increase it toward 1. Low weight on money printing may cause the fraction to increase with low and decreasing support and decrease with slightly higher and decreasing support.

Article 5 examines the interactions between borrowers and buyers, sellers, nontraders, and banks in an economy with hard and fiat money. Hard money, approximated by Bitcoin, assumes the infeasibility of printing, withdrawal, inflation, and deflation. In contrast, fiat money, controlled by the bank, may undergo inflation or deflation through printing or withdrawal. Hence, hard money has a fixed supply, while fiat money has a variable supply.

Article 5 focuses on one unitary bank and multiple agents, comparing their utilities in Periods 1 and 2. The agents choose their actions, such as borrowing, buying, selling, and lending, to maximize utilities. Period 1 serves as a benchmark with no fiat money printing or withdrawal, maintaining a stable fiat money supply. In Period 2, the bank prints fiat money to lend to the borrower or buyer, causing corresponding inflation or deflation. Interactions and impacts on the bank and agents are explored by varying 64 parameters relative to a benchmark.

Fiat money printing benefits the borrower or buyer and, if not excessive, harms the seller and nontraders due to inflation costs. Nontraders are unaffected in a hard-money economy but vulnerable in a fiat economy with money printing. Increasing nontraders decreases inflation, benefiting the seller, nontraders, and the bank but harming the borrower or buyer.

In a hard-money economy with borrowing, neither inflation nor deflation occurs. Nontraders holding hard money and other assets remain unaffected. Borrowers and buyers, sellers, and banks experience the impact of portfolio changes between hard money, fiat money, other assets, loans, and associated interest rates.

Borrowers and buyers benefit from buying assets with borrowed money if it values other assets more than loan interest and benefits from inflation. Sellers benefit from selling assets for hard and fiat money if they value money more than the assets. Excessive lending harms the bank. Banks prefer a balanced portfolio between money holdings and lending to avoid excessive inflation.

Article 6 explores the Bitcoin price evolution using five growth models: conventional logistics, Gompertz, a charged capacitor, a combination of logistics with charged capacitor growth, and a combination of Gompertz and charged capacitor growth. These five growth models are estimated and compared using Bitcoin empirics. The models are further enhanced with oscillation and damped lengthening cycles for realistic predictions of the Bitcoin price evolution, such as the future bull market maxima and the future bear market minima.

Two Bitcoin carrying capacities are analyzed and explored, corresponding to the market capitalization of gold, which is $10 trillion, and the market capitalization as 50 times the market cap of gold.

The analysis utilizes the least squares and weighted least squares methods to estimate parameters against empirical data from July 23, 2010, to June 21, 2021. The parameters related to sine oscillations, including cycle length and degree of lengthening, are estimated using the historical three bull market maxima and three bear market minima.

The results indicate that Gompertz growth fits the damped oscillations and lengthening cycles well and tracks the early data better with the weighted least squares method. The combination of Gompertz and charged capacitor growth tracks the early data even better. Logistic growth is too slow to track the early data. The combination of logistic growth with charged capacitor growth partly tracks the early data. Pure-charged capacitor growth is unrealistic. Five future Bitcoin bull market local price maxima and the bear market local price minima are estimated under two different Bitcoin carrying capacities.

3.3 Digital currencies, households, central banks, governments and monetary policy

Articles 7 to 10 in Category 3 discuss the following topics:

- The interplay between households and governments in currency holdings and taxation
- CBDCs, other digital currencies, and negative interest rates

- The interplay between the central bank choosing interest rates and households choosing resource allocation in CBDCs and non-CBDCs.

Article 7 introduces a game involving a government and a representative household holding two currencies and taxation choices. The analysis explicitly focuses on a national currency, like a CBDC, and a global currency, like Bitcoin. The national currency is the most used by citizens for transactions such as purchasing and selling goods and services, paying taxes, and saving for retirement. Meanwhile, the global currency has limited usage within a nation. However, it may offer other opportunities such as tax evasion, user autonomy, discretion, peer-to-peer focus, no banking fees, payment on the black market, criminal activities, and potential returns.

The household's three strategic choices are as follows. First, the fraction of its holdings held in the national currency, causing the remaining fraction to be held in the global currency. Second and third are the tax evasion probabilities on the national and global currencies. The government's six strategic choices are the probabilities of detecting and prosecuting tax evasion, the tax rates, and the penalty factors on the national currency and global currency. Both the household and the government aim to maximize their expected Cobb-Douglas utilities.

The household prefers low tax rates, while the government balances household support with the need for income through taxation and penalties. The household's strategic choices align closely with its currency output elasticities and the government's preferences for taxation and penalties. A high output elasticity for the national currency prompts the government to impose higher taxes. The household's global currency tax evasion probability increases with the government's output elasticity for the global currency.

The household's tax evasion probability on the national (global) currency decreases (increases) if the government values taxation and penalties on the national (global) currency. The results are illustrated through numerical variations of eight parameter values relative to a benchmark.

Article 8 introduces a game between a central bank (accounting for the government's interest) choosing the CBDC interest rate and a representative household choosing consumption, holding a CBDC, or holding a non-CBDC, focusing on the impact of negative interest rates. The emergence of a CBDC facilitates negative interest rates, encouraging consumption over saving.

The household allocates resources to consumption, holding a CBDC, and holding a non-CBDC. The central bank controls a CBDC. A non-CBDC can be any asset not issued or controlled by the central bank. The central bank sets its interest rate for the CBDC. Holdings in both a CBDC and a non-CBDC can have positive or negative interest rates.

The central bank adopts a more negative interest rate under several conditions, such as increased household output elasticity for consumption, decreased household output elasticity for holding a CBDC, increased CBDC and non-CBDC transaction efficiencies, decreased household transaction efficiency for consumption, increased household scaling of the transaction cost, decreased scaling parameter for the central bank's profit per household, reduced household monetary energy, and decreased non-CBDC interest rate. The numerical illustrations are explored by varying nine parameter values relative to a benchmark.

Article 9 introduces a two-period decision model between a central bank and a representative household. The central bank follows the Taylor (1993) rule to set positive or negative interest rates. The household allocates resources into production, consumption, and holdings in CBDCs and non-CBDCs, guided by a Cobb-Douglas utility with elasticities. The CBDC interest rate, the non-CBDC interest rate, and transaction efficiency impact the household's utility.

The central bank chooses a negative interest rate when the household holds far more CBDCs than non-CBDCs, discouraging excessive savings in CBDCs. Increasing the non-CBDC interest rate causes the household to hold more non-CBDCs and fewer CBDCs, prompting the central bank to increase its CBDC interest rate to compete with non-CBDCs.

Increasing the household's transaction efficiencies for CBDCs and non-CBDCs causes the central bank to increase its CBDC interest rate. Decreasing the real interest rate (i.e. the nominal interest rate adjusted for inflation), the inflation rate, the household's potential production, or the weight assigned to inflation in the Taylor (1993) rule or increasing the target inflation rate or the household production parameter causes lower and eventually negative CBDC interest rates, which induces the household to decrease its CBDC holdings and increase its non-CBDC holdings, production, and consumption.

Positive production shocks lead to lower CBDC interest rates, causing the household to hold fewer CBDCs, consume more, and earn lower utility. Positive inflation shocks result in higher CBDC interest rates, increasing the household's CBDC holdings but reducing its production, consumption, and non-CBDC holdings.

Positive CBDC interest rate shocks cause the household to hold more CBDCs and fewer non-CBDCs, and conversely, positive non-CBDC interest rate shocks cause the household to hold more CBDCs and fewer non-CBDCs. Positive shocks to the real interest rate cause higher CBDC interest rates. The findings are determined analytically and illustrated numerically with variations in 19 parameter values relative to a benchmark.

Article 10 provides empirics for the model introduced in Article 9, comparing data from the United States, China, and Russia. The article also explores the implications of hypothetically higher inflation rates for these three countries. The analysis suggests that in 2021 and 2022, the

United States should have chosen a 7.56% CBDC interest rate rather than the low 0.125% CBDC interest rate to address its high October 2021 inflation rate of 6.2%. This finding aligns with the observed trend of increasing interest rates post-2022. The US Federal Reserve maintained a target range for the federal funds rate at a 22-year high of 5.25%–5.5% in its October 2023 meeting.

Conversely, the model suggests China should adopt a modest 2.99% CBDC interest rate instead of the empirical 3.85%. This adjustment is to reduce household savings in CBDC. The model recommends a 6.82% CBDC interest rate for Russia, slightly higher than the empirical 6.75%. The model further predicts that a negative CBDC interest rate is advisable when inflation and real interest rates are low but the inflation target is high.

3.4 Interest rate modeling

Articles 11 to 13 in Category 4 discuss extended interest rate modeling, considering terms beyond the Taylor (1993) rule.

Article 11 introduces an extended interest rate model, combining the Taylor (1993) rule, the quantity equation (Friedman, 1970), and the Phillips (1958) curve. The article examines how deviations in inflation rate, real GDP, money supply, money velocity, and unemployment rate interact with the interest rate.

Using the US empirics, i.e. monthly data from January 1, 1959 to March 31, 2022, the Pearson correlation analysis reveals positive correlations between the interest rate and the deviations in the inflation rate, real GDP, money supply, money velocity, and the unemployment rate. Regression analysis confirms statistically positive interactions between the interest rate, the deviations in the inflation rate, and the real GDP, aligning with the Taylor (1993) rule. Additionally, the interest rate increases with deviations from the unemployment rate, consistent with the Phillips

(1958) curve. Overall, the deviations in inflation rate, money supply, money velocity, and unemployment rate serve as effective indicators for the interest rate, providing a more realistic explanation than the Taylor (1993) rule.

Article 12 builds on the analysis in Article 11 by extending the Taylor (1993) rule to include additional variables: money supply, money velocity, and the unemployment rate. It introduces and estimates five parameters: The weights assigned to the deviations in the inflation rate, real GDP, money supply, money velocity, and the unemployment rate.

Optimal parameter values are estimated using the monthly US data from January 1, 1959 to March 31, 2022. In contrast to the Taylor (1993) rule with only two parameters (i.e. the weights assigned to the deviations in real GDP and inflation rate), the optimal parameter values assign a relatively high weight to the deviation in the unemployment rate and moderate weights to the deviations in the inflation rate, the real GDP, money supply, and money velocity. Various combinations of parameter values are tested and analyzed.

Article 13 relates to Articles 11 and 12 by scaling the terms in interest rate modeling. Specifically, it introduces the scaling and extension of the Taylor (1993) interest rate rule from four terms to seven terms. The three additional terms are the deviations in money supply, money velocity, and the unemployment rate. The four original terms are the inflation rate, the equilibrium real interest rate, the deviation in the inflation rate, and the deviation in real GDP.

The seven combinations of the Taylor (1993) rule, the quantity equation (Friedman, 1970), and the Phillips (1958) curve with scaling yield significantly improved results compared to the unscaled Taylor (1993, 1999) rules. The Phillips (1958) curve stands out as the best regarding the squared differences between the empirical interest rate and the theoretical interest rates when selecting one rule with scaling. Combining the Taylor (1993) rule and the Phillips (1958) curve emerges as the best when choosing two rules with scaling.

3.5 Mapping the core elements of the 13 articles

Table 6 presents the knowledge framing, research questions, and methods in the 13 articles.

Article number	Article title	Knowledge framing	Research questions	Methods
1	A Two-Period Game Theoretic Model of Zero-Day Attacks with Stockpiling	Security, zero-day attacks, defense, and attack	How do the defender and attacker strike balances between how to exert efforts in zero-day exploits over the two periods?	Game theory, analytical analysis, numerical analysis
2	Conventionalists, Pioneers and Criminals Choosing Between a National Currency and a Global Currency	Currency utility elasticities, currency competition	How do the players choose between two currencies, and how do the fractions of the three player types evolve over time?	Conceptual model, replicator equation, numerical analysis
3	The Evolution of Fixed-Supply and Variable-Supply Currencies	Currency competition, money printing/withdrawal	How do the two currencies compete, given that the two kinds of players support the currencies differently? How does the player’s volume fraction of transactions in each currency evolve over time?	Analytic model, empirical analysis, replicator equation, numerical analysis
4	Competition Between Variable-Supply and Fixed-Supply Currencies	Currency competition, currency support	How does the player’s low, high, or increasing and decreasing support of currencies impact the currency competition?	Analytic model, empirical analysis, replicator equation, numerical analysis

5	Hard Money and Fiat Money in an Inflationary World	Hard economy and fiat economy, borrowing, buying and selling, lending	How do the choices of the borrower and buyer, seller, nontraders, and the bank in a hard and fiat economy impact each other?	Analytic model, numerical analysis
6	A Bitcoin Price Prediction Model Assuming Oscillatory Growth and Lengthening Cycles	The Bitcoin price evolution, market prediction	How are the five growth models integrated with oscillation and lengthening cycles to capture the Bitcoin price evolution?	Growth models, lengthening cycles, oscillatory growth, empirical analysis
7	Governmental Taxation of Households Choosing Between a National Currency and a Cryptocurrency	Taxation choices given two currencies, tax evasion, detecting and prosecuting tax evasion and penalty	How do the household and government make strategic taxation choices with two available currencies?	Game theory, analytical analysis, numerical analysis
8	A Game Between Central Banks and Households Involving Central Bank Digital Currencies, Other Digital Currencies and Negative Interest Rates	Negative interest rates with CBDCs, household resources allocation, transaction efficiencies and costs	How do the household and the bank strategically interact when the household chooses holdings in CBDC, non-CBDC, and consumption, and the central bank chooses CBDC interest rates?	Game theory, analytical analysis, numerical analysis
9	A Two-Period Decision Model for Central Bank Digital Currencies and Households	Household resources allocation, the Taylor (1993) rule, CBDC interest rates	How does the household choose production, consumption, holdings in CBDCs and non-CBDCs? How does the bank choose CBDC interest rates over the two periods?	Decision model, analytical analysis, numerical analysis
10	Comparative Analysis of Households and Digital Currencies	Decision model application, CBDC interest	How do the empirics from the US, China, and Russia differ using the model in Article 9?	Empirical analysis, numerical analysis

	for the US, China, and Russia	rates empirics, economic shocks		
11	Interest Rates, the Taylor Rule, the Quantity Equation, and the Phillips Curve	Interest rates model combining the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve	How do deviations in the inflation rate, real GDP, money supply, money velocity, and the unemployment rate interact with the interest rate?	Econometric model statistical analysis, empirical analysis
12	Modeling Which Factors Impact Interest Rates	The various combinations of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve	What are the optimal weights assigned to the deviations in inflation rate, real GDP, money supply, money velocity, and unemployment rate in generalized interest rate models?	Statistical analysis, empirical analysis
13	A Generalized Interest Rates Model with Scaling	Interest rates model with scaling	How does scaling impact the interest rate models?	Statistical analysis, empirical analysis

Table 6. Mapping the core elements in the 13 articles.

Table 7 presents the main findings in Article 1 (Category 1: Game theory and security).

Article number	Article title	Main findings
1	A Two-Period Game Theoretic Model of Zero-Day Attacks	• Player 1 stockpiles zero-day exploits in period 1 if the effort cost in period 1 is lower than in period 2, if the effort cost in period 2 is higher than in period 1, and if the appreciation factor of zero-day exploits from period 1 to period 2 is above 1. • Increased contest intensity in period 1 leads to fierce competition and decreased expected utilities for both players until player 1 reaches its budget constraint.

	with Stockpiling	• Increased contest intensity in period 2 leads to increased efforts by both players until they reach zero expected utilities, assuming two players are equally advantaged in terms of unit effort cost. • A player exerts more effort if its unit costs of effort is cheaper, if it values the asset more, if the asset has a higher growth factor, and if the asset added in period 2 is more valuable. • The players compete most fiercely when equally advantaged in terms of unit effort cost and decrease efforts when too advantaged or too disadvantaged.

Table 7. Main findings in Article 1.

Table 8 presents the main findings in Articles 2-6 (Category 2: Currency evolution and competition).

Article number	Article title	Main findings
2	Conventionalists, Pioneers and Criminals Choosing Between a National Currency and a Global Currency	• Each player's expected utility is inversely U-shaped in the volume fraction of transactions in the national and global currencies, skewed toward national currency for conventionalists and global currency for pioneers and criminals. • If conventionalists become extinct, pioneers and criminals compete directly. • Conventionalists become extinct when criminals gain more from criminal behavior, which happens when the scaling proportionality parameters for criminal expected utility increase. Also, if the parameter values in conventionalists' expected utility are unfavorable, it leads to competition between pioneers and criminals, further contributing to the extinction of conventionalists. • The fraction of criminal players decreases as the probability of detection and prosecution increases.
3	The Evolution of Fixed-Supply and Variable-Supply Currencies	• Two kinds of players support fixed- and variable-supply currency differently and choose their volume fractions of transactions in each currency. • A player's utility in a currency is assumed to be proportional to its support for that currency, the volume fraction of all players' transactions, and the fraction of players of the same kind. • A player's utility in a variable-supply currency is additionally assumed to be proportional to a Cobb-Douglas

		utility, considering factors like initial money supply, cumulative money printing/withdrawal, and inflation/deflation. • The high weight assigned to the money supply relative to inflation induces each player to prefer the variable-supply currency. Each player's volume fraction of transactions in each currency can be U-shaped or inversely U-shaped when applying replicator dynamics before converging exclusively toward preferring one or the other currency. • A player may choose to support a specific currency if the player's support for that currency is especially high. When a player's utility of transacting in a given currency is proportional to the fraction of players of the same kind as the given player, and the proportional impact is higher for one kind of player than the other kind of player, the player tends to prefer to be the kind of player that support the given currency.
4	Competition Between Variable-Supply and Fixed-Supply Currencies	• $1 in 2022 buys 1.22% of what it would buy in 1695. • A player's utility increases with a high weight assigned to money printing/withdrawal and increases less or decreases overall with a high weight assigned to inflation/deflation. • Assume a high weight assigned to money printing. Then, a player's low support of the fixed-supply currency causes the fraction of transactions in the variable-supply currency to approach 1 quickly. With a higher weight assigned to the fixed-supply currency, the fraction may temporarily decrease but will eventually increase, except for very high support for the fixed-supply currency. • Assume a low weight assigned to money printing. Then, a player's low support of the fixed-supply currency still causes the fraction of transactions in the variable-supply currency to approach 1. With a high weight assigned to the fixed-supply currency, the fraction may temporarily increase but will eventually decrease. • Assume increasing support for the fixed-supply currency. Then, if a player assigns high weight to money printing, the fraction of transactions in the variable-supply currency may increase temporarily but will eventually decrease. • Assume decreasing support for the fixed-supply currency. Then, if a player assigns high weight to money printing, the fraction of transactions in the variable-supply currency may decrease temporarily but will eventually increase.
5	Hard Money and Fiat Money	• Fiat money printing benefits the borrower/buyer (preferring inflation) and the bank (if not excessive) but hurts the seller and nontraders.

	in an Inflationary World	• The seller and nontraders bear the costs of inflation. The seller and nontraders prefer a hard money economy or a fiat money economy where the bank withdraws money to ensure deflation. • More nontraders decrease inflation as money printing is distributed across more agents, benefiting the seller, nontraders, and the bank but hurting the borrower/buyer. • In a hard money economy, the bank cannot transfer inflation costs to agents. Borrowing and lending in a hard money economy results in neither inflation nor deflation. • Excessive lending does not benefit the bank, borrowers/buyers or sellers who prefer a balanced portfolio between money holdings and lending that earns interest payments from the borrower/buyer.
6	A Bitcoin Price Prediction Model Assuming Oscillatory Growth and Lengthening Cycles	• Gompertz's growth of the Bitcoin price combined with charged capacitor growth tracks the empirical data well. • Logistic growth is too slow to track the early data. Charged capacitor growth is not realistic. Logistic growth combined with charged capacitor growth somewhat tracks the early data. • Five future bull market maxima and bear market minima are predicted based on the models, with the Bitcoin carrying capacities assumed to match gold at $10 trillion and 50 times the gold market cap. • Short-term traders should consider large standard deviations for stop-loss orders. Long-term investors can compare Bitcoin price growth with competing asset classes. All market participants must consider Bitcoin's volatility and potential growth.

Table 8. Main findings in Articles 2-6.

Table 9 presents the main findings in Articles 7-10 (Category 3: Digital currencies households, central banks, governments, and monetary policy).

Article number	Article title	Main findings
7	Governmental Taxation of Households Choosing Between a	• The household chooses the fraction of a national currency and a global currency it holds and the probability of tax evasion on each currency. The government chooses the tax rate, the probability of detecting tax evasion, and the penalty factors for national and global currencies.

	National Currency and a Cryptocurrency	• The household's tax evasion probabilities on both currencies increase in the government's Cobb-Douglas output elasticity for the national currency. • The household's fraction of the national currency, the government's monitoring probability of the national currency, and the penalty factor imposed on the global currency increase the household's Cobb-Douglas output elasticity for the national currency. • The government's taxation rates on both currencies decrease the output elasticity of the national currency. • High output elasticity for the national currency eventually induces the government to tax the national currency more than the global currency. • High output elasticity for the national currency leads to higher taxation. The household's tax evasion probability depends on the government's output elasticity for each currency. • The household is less (more) likely to tax evade on the national (global) currency if the government values taxation and penalties on tax evasion on the national (global) currency.
8	A Game Between Central Banks and Households Involving Central Bank Digital Currencies, Other Digital Currencies, and Negative Interest Rates	• As the household's output elasticity for consumption increases, it consumes more and holds less non-CBDC, and the CBDC interest rate decreases and becomes negative. • As the household's output elasticity for holding CBDC increases, it holds more CBDC and less non-CBDC. The central bank eventually imposes a positive CBDC interest rate on the household since it identifies partly with the household that substitutes holding non-CBDC with holding CBDC. • The household's consumption, CBDC holding, and non-CBDC holding are affected by the transaction efficiency for CBDC relative to non-CBDC. Increasing both the CBDC and non-CBDC transaction efficiencies eventually induces the central bank to choose a negative interest rate. • An increase in the non-CBDC interest rate induces the central bank to competitively raise the CBDC interest rate to retain the household's holding of CBDC. • The central bank chooses a more negative interest rate on CBDC when the household's output elasticity for consumption increases, the household's output elasticity for holding CBDC decreases, the CBDC and non-CBDC transaction efficiencies increase, the household's transaction efficiency for consumption decreases, the household's scaling of the transaction cost increases, the scaling

		parameter for the central bank's profit per household decreases, the household's monetary energy decreases, and the non-CBDC interest rate decreases.
9	A Two-Period Decision Model for Central Bank Digital Currencies and Households	• The central bank chooses a negative CBDC interest rate. The grey zones correspond to the Period 1 game only, when the representative household holds far more CBDCs than non-CBDCs. • Increasing non-CBDC interest rates prompts the central bank to raise the CBDC interest rate to compete and retain CBDC holding by the representative household. • Increasing transaction efficiencies for CBDCs and non-CBDCs cause the central bank to increase its CBDC interest rate to support the household's CBDC holdings and compete with non-CBDCs. • Positive shocks to household production cause lower CBDC interest rates, leading to less CBDC holding by the representative household but increased household production and consumption. • Positive inflation shocks increase the household's CBDC holding due to higher CBDC interest rate while the household's production, consumption, and non-CBDC holding decrease. • Positive shocks to the CBDC and non-CBDC interest rates increase the household's holdings of CBDC and non-CBDC, leading to reduced production and consumption but higher overall utility. • Positive shocks to the real interest rate on CBDC cause higher CBDC interest rate. • The central bank may choose negative CBDC interest rate when the household holds far more CBDC than non-CBDC, causing low inflation rate low real interest rate low household's potential production, low weight assigned to inflation in the Taylor (1993) rule, high target inflation rate, and high household's production parameter. • A negative CBDC interest rate usually causes the household to decrease its CBDC holding and increase its non-CBDC holding, production and consumption.
10	Comparative Analysis of Households and Digital Currencies for the US,	• For the US, with a high empirical inflation rate (6.2%) compared to the target inflation rate (2%), the model suggests a substantially higher CBDC interest rate (7.56%) than the empirical interest rate (0.125%) to suppress the inflation of 6.2% in October 2021. • For China, with a low empirical inflation rate (2.419%) below the target inflation rate (3%), the model suggests a lower CBDC interest rate (2.99%) than the empirical

	China, and Russia	interest rate (3.85%) to increase the inflation of 2.419% in 2021. • Russia's strategy falls between that of the US and China. With an inflation rate (3.382%) below the inflation target (4%), the model suggests a slightly higher CBDC interest rate (6.82%) than the empirical interest rate (6.75%). • The model predicts that the central bank should choose a negative CBDC interest rate when the inflation and real interest rates are low, and the inflation target is high. • An extremely high inflation rate increases the CBDC interest rate significantly, making production and consumption nearly impossible unless the real interest rate is extremely negative.

Table 9. Main findings in Articles 7-10.

Table 10 presents the main findings in Articles 11-13 (Category 4: interest rate modeling).

Article number	Article title	Main findings
11	Interest Rates, the Taylor Rule, the Quantity Equation, and the Phillips Curve	• The interest rate and deviations in inflation rate, real GDP, money supply, money velocity, and unemployment rate are positively correlated. • The ranking of correlation coefficients with the interest rates are (from high to low) the deviation in the inflation rate, the money velocity, the deviation in the unemployment rate, the deviation in real GDP, and the deviation in the money supply. • Regression analysis shows positive interactions between the interest rate and the deviation in the inflation rate, the deviation in real GDP, the money supply, the money velocity, and the deviation in unemployment rate. • The interest rate increases with the deviation in the unemployment rate, aligning with the Phillips (1958) curve. • Money velocity and the deviations in inflation rate, money supply, and unemployment rate are good indicators for the interest rate.
12	Modeling Which Factors Impact Interest Rates	• The model introduces and estimates five parameters, i.e. the weights assigned to the deviations in the inflation rate, real GDP, money supply, money velocity, and unemployment rate. • The optimal parameter values assign a relatively high weight to the deviation in the unemployment rate and

		moderate weights to the deviations in the inflation rate, real GDP, money supply, and money velocity. • Optimal parameter values show lower weights for the deviation in inflation rate and the deviation in the real GDP compared to the Taylor (1993) rule. • The generalized equation fits the US empirical interest rate better than the Taylor (1993) rule, resulting in a notable decrease of 42.95% in the corresponding sum of squares.
13	A Generalized Interest Rates Model with Scaling	• All seven combinations of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve with scaling give substantially better results than both Taylor (1993, 1999) rules without scaling. • The Phillips (1958) curve is the best when choosing only one rule with scaling. • Combining the Taylor (1993) rule and the Phillips (1958) curve is best when choosing among two rules with scaling. • Among the seven terms (i.e. the inflation rate, the equilibrium real interest rate, the deviation in the inflation rate, the deviation in real GDP, the deviation in money supply, the deviation in money velocity, and the deviation in the unemployment rate), the inflation rate is the most explanatory, impacting interest rates positively. • Equilibrium real interest rate and deviation in inflation rate are also significant, affecting interest rates negatively. • Deviation in money velocity is more explanatory than money supply, impacting interest rates positively. • Deviations in the real GDP, the unemployment rate, and the money supply impact interest rates positively with varying degrees of significance.

Table 10. Main findings in Articles 10-13.

4 Future Work

4.1 Attacks, conflicts, security, safety, and game theory

Based on the insights from Article 1, one potential avenue for future research involves involving more players and examining the impact of external interference from governmental and non-governmental entities. Additionally, investigating the regulations and technological advancements in zero-day exploits may reveal valuable insights.

The estimation of parameter values can be refined by applying real-world instances of zero-day attacks to improve the accuracy of the research. Validation from both current and historical records provides empirical support. Future studies may consider increasing complexity and extending the scope beyond two time periods to enhance the analytical framework.

4.2 The conception of subelasticities for currencies

Exploring several future directions can enhance the conception model of subelasticities in national and global currencies, for example

- evaluate empirical evidence for the six output subelasticities associated with national and global currencies. Examine their relevance and consider the possibility of identifying additional subelasticities or focusing on a subset. Assess empirical support for the volume fractions players choose for national and global currencies.
- explore alternative models for players' expected utilities, incorporating different risk attitudes and modeling more than three types of players.

- extend the analysis to include more than one global currency, delving into the competition between multiple national and global currencies to highlight the complex dynamics in the currency ecosystem.
- assess the impact of currency competition on the economic system and aspects such as inflation rate, interest rate, and fiscal policy, expanding the scope of the research to provide a broader perspective. Incorporate additional players (e.g. governments) to account for external influences on the currency competition to offer a comprehensive understanding of the ecosystem and its dynamics.

4.3 Expanding perspectives in currency competition research

Future research on the competition and evolution of variable-supply and fixed-supply currencies can expand the analytical framework by encompassing various currency characteristics, players' risk attitudes and preferences, perspectives from private and public entities, empirical data, and the impact of factors like regulation and taxation.

4.4 Enhancing the models to analyze dynamics in hard and fiat money economies

Future research may

- explore alternative utility functions to overcome the limitations associated with Cobb-Douglas utility in Article 5,
- adopt a game theoretic approach to analyze the equilibrium between the bank and agents,
- explore model extensions related to hard money, such as scenarios where the burning of hard money causes a decreased supply,

- reduce the number of nontraders and assume that a [0,1] continuum represents each buyer, seller, and nontrader,
- formulate a representative agent's problem for each type, combine models, and incorporate more structure on the preferences and constraints of the agents' problem, such as a Lagos-Wright monetary model, a money-in-utility function mode, and a cash-in-advance-constraint model (Benigno et al., 2022), and
- enhance inflation modeling by incorporating factors like money velocity, production, transaction efficiency, demand, and supply shocks.

4.5 Exploring beyond Bitcoin and enhancing models in cryptocurrency market analysis

The five models introduced in Article 6 can be improved by integrating other established models, such as the stock-to-flow model, machine learning techniques, neural networks, deep learning methodologies, and econometric approaches, thereby contributing to a comprehensive modeling framework to capture the evolution of cryptocurrencies.

Future research may

- investigate the various aspects of Bitcoin, such as trading volume, mining difficulty, hash rate, network value to transactions, average transaction volume, gas fee, NFT transaction volume, active and new addresses, on-chain transaction volume, electricity consumption, renewable energy adoption, institutional investor involvement, and the connection with other cryptocurrencies like Ethereum and traditional financial assets like bonds and stocks, and
- delve into the regulatory landscape, examining regulations, policies, and attitudes across different countries can provide valuable insights into cryptocurrency markets.

4.6 Investigating dynamics in household and government strategic choices across national and global currencies

Future research may

- delve into scenarios with more than two currencies and introduce additional players, including firms, multiple governments across different countries, central banks, commercial banks, and international financial institutions,
- explore alternatives to expected utilities, considering backing, convenience, confidentiality, transaction efficiency, financial stability, and security, and
- introduce a multiple-time-periods game, along with different interest rate settings.

4.7 Exploring the diverse dimensions in strategic decision-making associated with CBDCs

Future research can incorporate more asset types, such as gold, bonds, stocks, and multiple CBDCs. This exploration may account for additional players by, for example, distinguishing between central banks and governments, commercial banks, firms, financial institutions, and households with different characteristics and risk attitudes that can be incorporated into the analysis.

Players' Cobb-Douglas utilities can be enriched to include privacy, convenience, security, and tax considerations. The strategy sets of players can be extended to allow for a more inclusive decision-making framework, such as enabling each household to choose production and leisure in addition to consumption.

4.8 Negative interest rates and players' resource allocation

Future research can introduce additional players like governments, commercial banks, and firms to enrich the model in Article 9. Research can expand the household's Cobb-Douglas utility by incorporating convenience, taxes, and preferences, thereby providing a more realistic representation of decision-making processes. The model in Article 9 can be adjusted to analyze scenarios where households, central banks, and other players make strategic decisions simultaneously or sequentially over one or multiple periods.

4.9 Exploring comprehensive interest rate models

Future research may

- explore how the weights assigned to the terms in Articles 11, 12, and 13 evolve over time, moving beyond the assumption of constant weights in the Taylor (1993) rule,
- investigate incorporating the concept of "interest smoothing" by incorporating additional lagged variables into the models and examining non-lagged variables,
- consider forward-looking approaches, as proposed by Conrad and Eife (2012), that can address the limitations of backward-looking models,
- investigate the influence of economic crises, fiscal deficits, global interest rates, and financial variables (e.g. house prices, stock prices, leverage, oil, and commodity prices) on interest rates while systematically exploring the impact paths of the terms in Articles 11, 12, and 13 on each other and the interest rate,
- compare empirical findings across different geographical regions, considering monetary policy changes over time and Evaluate alternative methods for estimating key economic

parameters (e.g. real GDP gap, long-term equilibrium real interest rate) and incorporate time series approaches and broader financial theories into interest rate analysis,
- explore interest rate models encompassing hard and soft money, such as Bitcoin and fiat money (e.g. a CBDC), and
- analyze the interplay between various players (e.g. households, firms, commercial banks, governments, and countries) in hard and fiat money economies.

4.10 Digital technologies and business analytics

Future research can explore the impact of emerging technologies (e.g. blockchain and cryptocurrencies) on business analytics. For example, the role of blockchain in ensuring data sharing, integrity, and security and how NFTs contribute to business activities and market strategies can be investigated.

4.11 Security and privacy in emerging technologies

Future research may

- explore the dynamic security and privacy landscape arising from emerging technologies like blockchain and cryptocurrencies,
- investigate the development of strategies to protect sensitive information in the context of digital currencies and digitalization,
- examine the regulatory environment surrounding these emerging technologies, assess the effectiveness of current policies, and suggest enhancements to ensure compliance with evolving security and privacy requirements, and
- consider how to balance innovation and security to foster the responsible adoption of digital currencies while addressing money laundering, fraud, and financial stability concerns.

4.12 Digital currencies, decentralized finance, and financial inclusion

Future research can investigate the promising potential of digital currencies and DeFi for enhancing financial inclusion for those currently underserved and unbanked worldwide. Examples of research questions include the following:

- What socioeconomic and technological factors are shaping the adoption of digital currencies?
- What are the regulatory challenges for digital currencies and DeFi, and how can these innovative financial technologies be integrated into existing or new regulatory frameworks?
- How does centralized finance compete with decentralized finance?

4.13 The economic system within Bitcoin and CBDCs

Future research can delve into the intricate dynamics of the economic systems within Bitcoin and CBDCs by, for example, investigating the broader impact of Bitcoin on traditional financial systems such as monetary policy, inflation, and fiat money printing; exploring the regulatory challenges associated with Bitcoin and CBDCs; exploring user behavior and adoption patterns for Bitcoin and CBDCs; and examining the competition between Bitcoin and CBDCs.

4.14 Environmental sustainability in digital currencies

Future research can assess the environmental sustainability of Bitcoin mining. Strategies can be investigated to enhance the environmental sustainability of Bitcoin mining, and technologies and practices that reduce energy consumption and the carbon footprint associated with

Bitcoin mining can be explored. The feasibility and impact of integrating renewable energy sources into Bitcoin mining can also be examined.

A comparative analysis of the environmental sustainability of Bitcoin and CBDCs can be useful. Factors such as energy consumption, resource utilization, and long-term ecological consequences can be tested to provide insights into the overall environmental impact of digital currencies. Strategies for raising public awareness and educating stakeholders about the environmental impact of digital currencies can be proposed.

4.15 Beyond applications in financial markets

Blockchain extends beyond finance, permeating diverse sectors, including management, governance, supply chain, gaming, metaverses, education, and healthcare. Blockchain has the potential to transform the way entities are structured and governed. For example, smart contracts can automate and enforce agreements, reducing reliance on intermediaries and enhancing the efficiency of decision-making processes.

Broader research topics include exploring interactions between Bitcoin and democracy, digitalization and innovation, blockchain, and organizational governance.

5 List of Articles

Articles included in this dissertation

Game theory and security

1. Wang, G., Welburn, J.W. and Hausken, K. (2020), "A Two-Period Game Theoretic Model of Zero-Day Attacks with Stockpiling," Games 11, 4, 1–26, Article number 64

Currency evolution and competition

2. Wang, G. and Hausken, K. (2022), "Conventionalists, Pioneers and Criminals Choosing Between a National Currency and a Global Currency," Journal of Banking and Financial Economics 2, 16, 104–133
3. Wang, G. and Hausken, K. (2022), "The Evolution of Fixed-Supply and Variable-Supply Currencies," Humanities & Social Sciences Communications 9, 137
4. Wang, G. and Hausken, K. (2022), "Competition Between Variable-Supply and Fixed-Supply Currencies," Economies 10, 11, 270
5. Wang, G. and Hausken, K. (2023), "Hard Money and Fiat Money in an Inflationary World," Research in International Business and Finance, 67, 102115
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Digital currencies, households, central banks, governments, and monetary policy

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23. Wang, G. and Hausken, K. (2023), “Regression Analysis of Factors Impacting Interest Rates,” International Journal of Financial Engineering, 2350052, 1–20
24. Wang, G. and Hausken, K. (2024), “Unraveling the Global Landscape of Bitcoin Research: Insights from Bibliometric Analysis,” Technology Analysis & Strategic Management, 230693, 1–18

Summary of the articles

Publication number	Journal title
1	Games
2	Journal of Banking and Financial Economics
3	Humanities & Social Sciences Communications (Springer Nature)
4	Economies
5	Research in International Business and Finance
6	Cogent Economics and Finance
7	Games
8	Cogent Economics and Finance
9	International Journal of Finance & Banking Studies
10	International Journal of Finance & Banking Studies
11	Eurasian Journal of Economics and Finance
12	Journal of Central Banking Theory and Practice
13	International Journal of Economics and Financial Issues
14	System Engineering Theory and Practice (Chinese Social Sciences Citation Index)
15	Journal of Diabetes
16	Journal of Systems Science and Information
17	Financial Regulation Research (Chinese Social Sciences Citation Index)
18	SAGE Research Methods Business
19	Journal of Finance and Investment Analysis
20	International Business Research
21	Applied Economics and Finance
22	Exploration of Financial Theory (China National Knowledge Infrastructure Journal Indexing)
23	International Journal of Financial Engineering
24	Technology Analysis & Strategic Management

Table 11. Summary of the articles.

Terminology

Bitcoin is a decentralized peer-to-peer network. All transactions are verified and recorded in a public ledger called blockchain. Bitcoin was invented by (Nakamoto, 2008). Decentralization means no central authority controls the Bitcoin network like a government or bank. Anyone can join the Bitcoin network using the internet. Bitcoin transactions are secure and transparent. They can be made directly between users without the need for intermediaries. The supply of Bitcoin is limited to 21 million. At the time of writing, the Bitcoin network block height is 827,994. See https://www.blockchain.com/explorer/blocks/btc for more information on the Bitcoin network blocks.

Bitcoin mining uses powerful computers or devices to solve complex mathematical puzzles. When a miner successfully solves a puzzle (i.e. finds a cryptographic solution that matches specific criteria), the miner owns the right to add a new block to the blockchain and earns new Bitcoin as a reward. The miner also secures the transaction feed in that block. Essentially, miners worldwide run devices (e.g. ASIC machines) to generate tons of hashes. Transactions are confirmed when miners generate a valid hash that produces a new block. These miners commit substantial electricity, time, and resources to mining Bitcoin. This proof-of-work mechanism keeps the network secure and adds to Bitcoin's value. The Bitcoin network automatically adjusts the mining difficulty level to maintain a consistent block time (ca. 10 minutes). Bitcoin mining difficulty has been increasing extensively over the years, and miners need powerful devices or join mining pools to stay competitive. The hash rate of the Bitcoin network is accessible at https://www.blockchain.com/explorer/ charts/hash-rate.

Bitcoin nodes are typically computers that run the Bitcoin software. Bitcoin nodes help relay transactions, verify the blockchain, and ensure consensus among participants in the Bitcoin network. Nodes communicate with each other to ensure that all transactions follow the rules of the Bitcoin protocol. Running a node contributes to the network's security and integrity. Bitcoin nodes can be divided into full nodes and pruned nodes. A full node maintains a copy of the entire blockchain, while a pruned node stores and relays only a copy of recent blocks limited to the computer space. Bitcoin nodes per country can be tracked at https://bitnodes.io/nodes/all/countries/1d/.

Central bank digital currency (CBDC) is a country's official currency issued and regulated by the central bank. CBDCs typically leverage emerging technologies such as blockchains. The up-to-date information on CBDCs worldwide can be found at https://www.atlanticcouncil.org/cbdctracker/.

Charged capacitor growth is the increase in voltage across the capacitor over time as it charges. A capacitor is a passive electronic component that stores electrical energy in an electric field. It involves a time-dependent increase in charge, influenced by the time constant and the opposing potential difference across the capacitor plates.

The Cobb-Douglas utility function is a mathematical representation widely used in microeconomics to model player preferences or the satisfaction derived from consuming different goods and services named after Cobb and Douglas (1928). The Cobb-Douglas utility function takes the form $U(x, y) = x^{\alpha} y^{\beta}$, where U represents the total utility, x and y are the quantities of two goods being consumed, and α and β are positive constants that represent the elasticity of substitution between the two goods. The Cobb-Douglas utility function has several interesting properties, such as constant relative risk aversion, which means that the ratio of the marginal utilities of the goods is constant. This function is widely used in economic models to analyze player behavior and production functions.

Conventionalists are players who tend to do what is traditional and historically common.

Criminals are players who conduct criminal activities and prefer not to be caught.

Currency is money issued by governments or monetary authorities (Cohen, 2013). Currency is typically associated with a particular country or region and serves as a means of exchanging commodities and services.

Decentralized finance (DeFi) offers financial services and applications built on blockchain technology. DeFi operates decentralized using smart contracts on the blockchain and encompasses various financial activities, including lending, borrowing, trading, and more, without the need for traditional intermediaries. Users typically interact with smart contracts through decentralized applications to access financial services.

Fiat money is government-issued currency declared legal tender (Ritter, 1995) that is commonly not backed by valuable commodities but by government credit. Governments or central banks determine the supply of fiat money, which is typically paper money.

Game theory analyzes strategic interactions among players. Players make decisions interdependently. This interdependence causes each player to consider the other player's possible strategies to formulate the given player's strategy. A solution to a game describes the optimal strategies of the players responding to the other players' optimal strategies.

Gompertz growth is a mathematical equation that describes the growth of a population over time. It assumes that the growth rate decays exponentially as the population approaches its maximum. The equation is named after Benjamin Gompertz, who introduced the growth function in 1833 (Gompertz, 1833).

Hard money is currency backed by or comprised of valuable commodities or backed by emerging technology such as blockchain.

Examples of hard money include Bitcoin and gold-backed currencies. The supply of hard money is relatively fixed.

A hash is a mathematical function converting input into an encrypted output of fixed length. Bitcoin uses SHA-256, which stands for Secure Hash Algorithm 256-bit. It is a cryptographic hash function that takes an input (or message) and produces a fixed-size 256-bit (32-byte) hash value.

Inflation targeting is a typical monetary policy where central banks follow a specific target for the annual inflation rate (Mishkin & Schmidt-Hebbel, 2001). The typical inflation rate is 2%.

Inflation is a general increase in the prices of goods and services in an economy (Frisch, 1977), which is usually measured using the consumer price index. When inflation occurs, each currency unit buys fewer goods and services than before.

The least squares method is used in statistics and econometrics to find the best-fitting curve for a set of data points. The idea is to minimize the sum of the square difference between the observed data points and the points predicted by the model.

Legal tender is currency acceptable for transactions and debt payment (Greco, 2001) and is recognized by law to settle debts, meet financial obligations, and make payments for services and goods.

Lengthening cycles occur when the duration of cycles, such as economic cycles, become longer over time. In the context of economic cycles, periods of expansion and contraction in the economy take more time to complete. For example, in economic terms, lengthening cycles suggest that the time between one economic recession and the next increases. Various factors, including changes in economic policies, technological advancements, global economic conditions, conflicts, wars, and epidemics, can influence these cycles.

Logistic growth is a concept in population ecology where a population's per capita growth rate decreases as it approaches carrying capacity, resulting in an S-shaped curve (Tsoularis & Wallace, 2002). It contrasts with exponential growth (Stango & Zinman, 2009), which produces a J-shaped curve without considering limiting factors. Logistic growth reflects the real-world scenario where resources become limited as a population expands, leading to a gradual slowdown in growth.

The M1 money supply is an economy's most liquid form of money. It includes physical currency (coins and paper money) outside of the private banking system, the amount of demand deposits, travel checks and other checkable deposits, and most savings accounts. These assets can be quickly converted into cash.

The M2 money supply is a monetary aggregate representing a broader measure of the money supply within an economy. In addition to the components of M1, M2 includes other less liquid assets that are still relatively easily convertible to cash, including money market accounts, retail money market mutual funds, and small-debt time deposits (certificates of deposit of under $100,000).

Money is a commodity accepted by general consent as a medium of economic exchange (Laidler, 1969), a unit of account, a store of value, and, occasionally, a standard of deferred payment.

The nominal interest rate is the stated interest rate on a financial product, such as a loan or investment, without adjusting for inflation or deflation. It represents the raw percentage return or cost of borrowing money over a specified period before considering the impact of changes in the purchasing power of money due to inflation or deflation.

Non-fungible tokens (NFTs) are digital assets using blockchain technology to represent ownership or proof of authenticity of a unique item or piece of content. The ownership of an NFT is recorded in the blockchain and can only be transferred by the owner. NFTs typically reference digital files such as artwork, photos, videos, and audio.

Oscillatory growth is a pattern of expansion and contraction or periodic fluctuations in the growth of a system or a quantity over time. In various fields, such as biology, economics, and mathematics, oscillatory growth indicates a repeating pattern of increases and decreases rather than steady or linear growth. For example, in population dynamics, oscillatory growth might describe a population that experiences cycles of increases followed by declines and then repeats the pattern. In economic terms, oscillatory growth is observed in business cycles where contractions and vice versa follow periods of economic expansion.

The Phillips curve is an inverse relationship between inflation and unemployment. Phillips observed that unemployment is high when inflation is low and vice versa (Phillips, 1958). This empirical relationship suggests that policymakers can choose between inflation and unemployment.

Pioneers are players who tend to break from tradition as the early adopters of new or emerging things.

The quantity equation is a concept in monetary economics expressing the relationship between the money supply, money velocity, price level, and the level of real output. The quantity equation $MV = PT$ relates the price level and the quantity of money (Friedman, 1989), where M is the quantity of money, V is the velocity of circulation, P is the price level, and T is the volume of transactions. The quantity equation is the basis for the quantity theory of money.

The real equilibrium interest rate, often referred to as the natural interest rate, is the theoretical interest rate at which the supply of savings equals the demand for investment in an economy, resulting in stable economic conditions. This interest rate prevails when inflation is steady, and the economy operates at full employment. The real equilibrium interest rate is adjusted for inflation, measuring the actual cost of borrowing or the real return on investments. When the actual interest rate is below the real equilibrium rate, it may stimulate economic activity, but if it is above, it can potentially slow economic growth.

The real interest rate is the nominal interest rate adjusted for inflation. It reflects the actual purchasing power and the true return on an investment after accounting for the impact of inflation. The relationship between real and nominal interest rates and the expected inflation rate is given by the Fisher equation $1 + i = (1 + r)(1 + \pi_e)$, where i denotes nominal interest rate, r denotes real interest rate, and π_e denotes the expected inflation rate. If the inflation rate and the nominal interest are relatively low, the Fisher equation can be approximated by $r = i - \pi_e$.

The replicator equation is a mathematical concept used in evolutionary game theory to model the dynamics of strategies in a population of individuals or entities engaged in a repeated or continuous game. It describes how the frequencies of different strategies change over time based on their relative success in the game (Schuster & Sigmund, 1983). Strategies with higher payoffs tend to increase in frequency, while those with lower payoffs decrease. It is a fundamental tool for studying the evolution of strategies in competitive scenarios.

The Taylor rule is the guideline for a central bank to manipulate interest rates. It states that a central bank should set its benchmark interest rate based on the deviation of actual inflation from the target inflation rate and the output gap (the difference between actual GDP and potential GDP). The Taylor rule suggests that the central bank should raise interest rates if inflation is above the target or the economy is overheating (i.e. a positive output gap; (Taylor, 1993). Conversely, the central bank should lower interest rates if inflation is below the target or the economy is in a recession (i.e. a negative output gap).

The weighted least squares method is a generalization of least squares and linear regression in which knowledge of the unequal variance of observations (heteroscedasticity) is incorporated into the regression. It assigns different weights to the data points based on their level of precision or reliability. The idea is to give more importance or weight to observations with lower variance or higher weight to observations with higher variance or lower reliability.

Zero-day exploits are a previously unknown vulnerability (Bilge & Dumitraş, 2012) that can be highly effective and pose a significant threat because no protections are in place when the vulnerability is exploited. Cybercriminals often use zero-day exploits to launch attacks on systems, compromise security, and potentially gain unauthorized access to sensitive information. Discovering and reporting zero-day vulnerabilities are crucial for developers and security experts to create patches and updates to mitigate the associated risks.

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Part II Articles

1. Wang, G., Welburn, J.W. and Hausken, K. (2020), "A Two-Period Game Theoretic Model of Zero-Day Attacks with Stockpiling," Games 11, 4, 1-26, Article number 64

games

Article

A Two-Period Game Theoretic Model of Zero-Day Attacks with Stockpiling

Guizhou Wang [1], **Jonathan W. Welburn** [2] **and Kjell Hausken** [1,*]

[1] Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway; pobewang@outlook.com
[2] RAND Corporation, National Security Research Division, 1776 Main St., Santa Monica, CA 90401, USA; jwelburn@rand.org
* Correspondence: kjell.hausken@uis.no; Tel.: +47-51-831-632; Fax: +47-51-831-550

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Abstract: In a two-period game, Player 1 produces zero-day exploits for immediate deployment or stockpiles for future deployment. In Period 2, Player 1 produces zero-day exploits for immediate deployment, supplemented by stockpiled zero-day exploits from Period 1. Player 2 defends in both periods. The article illuminates how players strike balances between how to exert efforts in the two periods, depending on asset valuations, asset growth, time discounting, and contest intensities, and when it is worthwhile for Player 1 to stockpile. Eighteen parameter values are altered to illustrate sensitivity. Player 1 stockpiles when its unit effort cost of developing zero-day capabilities is lower in Period 1 than in Period 2, in which case it may accept negative expected utility in Period 1 and when its zero-day appreciation factor of stockpiled zero-day exploits from Period 1 to Period 2 increases above one. When the contest intensity in Period 2 increases, the players compete more fiercely with each other in both periods, but the players only compete more fiercely in Period 1 if the contest intensity in Period 1 increases.

Keywords: game; cybersecurity; zero-days; stockpiling; production; attack; defense

1. Introduction

1.1. Background

Zero-day attacks are becoming increasingly common. The most well-known attack, utilizing the Stuxnet worm to exploit four zero-day vulnerabilities, is probably the 2010 attack on the Natanz nuclear facility in Iran [1]. A so-called zero-day vulnerability means that a defender's vulnerability in its computer system is known to the defender for zero days before it is discovered, most commonly through an attack. Zero-day attacks require resources and are challenging to produce. Once produced, the next challenge is whether to deploy them immediately or stockpile them for deployment at some suitable future point in time. Stockpiling can be useful for a player in providing security in the knowledge that threats posed by an opposing player can be ameliorated or eliminated. A more recent zero-day attack targeted Microsoft Windows in Eastern Europe in June 2019 [2]. The exploit abused a local privilege escalation vulnerability in Microsoft Windows pertaining to the NULL pointer dereference in the win32k.sys component (a NULL pointer dereference is an error causing a segmentation fault, which occurs when a program tries to read or write to memory with a NULL pointer). For other recent zero-day attacks, see PhishProtection [3].

1.2. Contribution

This article intends to capture the general aspects of this phenomenon, which are that a defender has an asset it seeks to defend, while the attacker seeks to attack the asset over two periods—by attacking and stockpiling in Period 1, and attacking and utilizing the stockpile in Period 2. A variety of reasons and justifications for stockpiling are illustrated. A two-player two-period game is analyzed. Player 1 is equipped with resources in Period 1, which can be utilized for producing zero-day exploits for immediate deployment in Period 1 or stockpiled for future deployment in Period 2. Player 2 defends against the attack in Period 1. Zero-day exploits may become more valuable if the stakes involved in their deployment increase, but this also entails the risk of becoming obsolete, e.g., if knowledge of their content leaks. We thus assume that Player 1′s stockpiled zero-day exploits may appreciate or depreciate in value from Period 1 to Period 2, i.e., the stockpiled zero-day exploits may become more or less valuable. Such changes in value may be due to technological, economic, or societal factors, market conditions, or the players' preferences. In Period 2, Player 1 produces new zero-day exploits for immediate deployment in Period 2 and also deploys its stockpiled zero-day exploits. In Period 2, the defender defends against the attack, i.e., against both the zero-day exploits produced by Player 1 in Period 2 and the appreciated or depreciated zero-day exploits stockpiled from Period 1 to Period 2. The presence of Period 2 enables Player 1 to strike a balance between whether or not to stockpile in Period 1, and both players strike balances between how to exert efforts in both periods.

The research questions are how the attacking Player 1 allocates its resources between immediate zero-day attack in Period 1 and stockpiling for attack in Period 2, how the defender defends in both periods, and how the players' strategic choices in both periods depend on the model characteristics, i.e., Player 1′s available resources, the contest intensities in both periods, the zero-day appreciation factor from Period 1 to Period 2, and both players' unit costs of effort, asset valuations, and time discount factors. Players in a cyberwar are always in a contest, regardless of the extent to which they understand the particulars of the contest, which justifies the use of the widely applied contest success function. The model in this article is applicable beyond zero-day vulnerabilities, assuming one attacking player and one defending player over two periods, where the attacking player can stockpile its capabilities from Period 1 to Period 2.

1.3. Literature

Aside from Hausken and Welburn [4] and, in part, Chen et al. [5], considered in Section 1.3.1, the literature has not directly considered the research questions in this article but has instead focused on various indirectly linked research questions, as shown in the subsequent subsections below. The literature on zero-day attacks is mostly concerned with detecting, mitigating, understanding, and simulating zero-day attacks. Most of the articles below have been identified by searching for the two words "zero-day" on the Web of Science database for the most recent years. Regarding zero-day vulnerabilities and their exploits, see Ablon and Bogart [6].

1.3.1. Game Theoretic Analyses

In earlier research, Hausken and Welburn [4] considered a one-period game theoretic model of zero-day cyber exploits, incorporating the benefit of stockpiling into the same period as when production and zero-day attack are determined. They found, for example, that decreasing Cobb Douglas output elasticity for a player's stockpiling causes its attack to increase and its expected utility to eventually reach a maximum, while the opposing player's expected utility reaches a minimum. Chen et al. [5] analyzed whether two countries should disclose or not disclose to the vendor the hardware/software vulnerabilities they discover in a repeated game. Disclosing may benefit the country if it gets exposed by the vulnerability. Not disclosing may benefit the country's defense given that the other country does not discover the vulnerability and is exposed by it. They develop an algorithm and find that countries benefit from discovering vulnerabilities quickly and from incurring low costs of developing exploits.

1.3.2. Detection, Prioritization, Ranking, and Classification

Singh et al. [7] realized the challenge in defending against zero-day attacks. They proposed a framework for detection and prioritization based on likelihood by identifying the zero-day attack path and ranking the severity of the vulnerability. [8] developed a detection model for crypto-ransomware zero-day attacks. The model is based on an anomaly-based estimator, which suffers from high rates of false alarms, supplemented by behaviorally-based classifiers. Venkatraman and Alazab [9] reviewed existing visualization techniques for zero-day malware and designed a visualization using a similarity matrix method for classifying malware.

1.3.3. Detection and Identification by Applying Probability Theory and Statistics

Sun et al. [10] acknowledged the information asymmetry between attackers and defenders and applied Bayesian networks for identifying zero-day attack paths probabilistically; this is intended to be superior to targeting individual zero-day exploits. Parrend et al. [11] presented a framework for characterizing zero-day attacks and multistep attacks and relevant countermeasures. They applied rule-based and outlier-detection-based statistical solutions and machine learning, which detects behavioral anomalies and tracks event sequences. Singh et al. [12] proposed a hybrid layered architecture framework for real-time zero-day attack detection based on statistics, signatures, and behavior techniques.

1.3.4. Detection Applying Learning

Kim et al. [13] proposed a method to detect zero-day malware. The method generates fake malware and learns to distinguish it from real malware. A deep autoencoder extracts appropriate features and stabilizes the generative adversarial network training. Gupta and Rani [14] observed that zero-day malware grows exponentially in terms of volume, variety, and velocity. They proposed a big data framework with scalable architecture and machine learning for detection.

1.3.5. Mitigation, Robustness, Recovery, and Simulation

Sharma et al. [15] presented a consensus framework for mitigating zero-day attacks, incorporating context behavior, an alert message protocol, and critical data-sharing protocol for reliable communication. Haider et al. [16] applied data sets based on the Windows Operating System to evaluate the robustness of host-based intrusion detection systems to zero-day and stealth attacks. Tran et al. [17] implemented an epidemiological model to combat zero-day attacks. They proposed a dynamic recovery model to combat the simulated attack and minimize disruptions. Tidy et al. [18] simulate previous and hypothetical zero-day worm epidemiology scenarios, accounting for susceptible populous and stealth-like behavior on the dynamic, heterogeneous internet.

1.3.6. Filtering, Protocol Context, Honeypots, and Signatures

Chowdhury et al. [19] proposed a multilayer hybrid strategy for zero-day filtering of phishing emails by using training data collected during an earlier time span. Duessel et al. [20] incorporated protocol context into payload-based anomaly detection of zero-day attacks, integrating syntactic and sequential features of payloads, thus proceeding beyond analyzing plain byte sequences. Chamotra et al. [21] suggested baselining high-interaction honeypots, i.e., identifying and whitelisting legitimate system activities in the honeypot attack surface. Subsequently, captured zero-day attacks are mapped to the vulnerabilities exposed by the honeypot. Afek et al. [22] presented a tool for extracting zero-day signatures for high-volume attacks, intended to detect and stop unknown attacks.

1.3.7. Cyber Security

More generally, for cybersecurity, Baliga et al. [23] identified opportunities for cyber deterrence with detection and the potential to undermine deterrence. Edwards et al. [24] considered a game theoretic model of blame, with an attacker and a defender, involving attribution, attack tolerance, and peace stability. Welburn et al. [25] found that although a cybersecurity defender prefers not to signal truthfully, the defender can enhance deterrence through signaling, which has implications for cyber deterrence policies. Nagurney and Shukla [26] considered three models for cybersecurity investment involving noncooperation, the Nash bargaining theory with information sharing, and system optimization with cooperation.

1.3.8. Information Security

Within information security, game theoretic research has focused on data survivability versus security in information systems [27], substitution and interdependence [28–30], returns on information security investment [31,32], and information sharing to prevent attacks [33–37]. See Do et al. [38], Hausken and Levitin [39], and Roy et al. [40] for reviews on game theoretic cybersecurity research.

1.4. Article Organization

Section 2 presents the model. Section 3 analyzes the model. Section 4 illustrates the solution. Section 5 discusses the results. Section 6 concludes.

2. The Model

Consider two players in a simultaneous move two-period game.

2.1. Period 1

Assume that Player 1 in Period 1 gets cyber resources R_{11} (e.g., capital, manpower, competence) from a national budget, which is allocated to develop zero-day exploits (zero-days, for short) Z_{11} deployed in Period 1 to exploit zero-day vulnerabilities for Player 2 at unit cost b_{11} and develop zero-day exploits S_1 stockpiled for use in Period 2 at unit cost b_{11}. The Nomenclature is shown before the reference list. Player 1's upper constraint R_{11} for resource allocation in Period 1 is

$$R_{11} \geq b_{11}Z_{11} + b_{11}S_1 = R_{11b} \tag{1}$$

where R_{11b} is the actual amount of resources used by Player 1 in Period 1. Player 2 exerts defense effort D_{21} in Period 1 at unit cost a_{21} to defend its asset, which it values as V_2 and Player 1 values as V_1. Figure 1 illustrates Period 1.

We apply the widely used ratio form contest success function [41], which is a plausible and widely used method for assessing two opposing players' success. See Hausken and Levitin [42], Hausken [43], and Congleton et al. [44] for the use of the contest success function. In Period 1, Player 1's expected contest success is p_{11} and Player 2's expected contest success is p_{21}, i.e.,

$$p_{11} = \frac{Z_{11}^v}{Z_{11}^v + D_{21}^v}, p_{21} = \frac{D_{21}^v}{Z_{11}^v + D_{21}^v} \tag{2}$$

where v, $v \geq 0$, is the contest intensity in Period 1. Expected contest success is usually interpreted as a probability between 0 and 1. It can also be interpreted as a guaranteed fraction of an asset one competes to obtain, which presumes that the asset is divisible. When $v = 0$, the contest is egalitarian, and efforts do not matter. When $v = 1$, efforts matter proportionally. When $v = \infty$, "winner-takes-all," so that exerting slightly more effort than one's opponent guarantees contest success. When $0 < v < 1$, a disproportional advantage exists of investing less than one's opponent. When $v > 1$, a disproportional advantage exists of investing more than one's opponent. In Equation (2), the ratios have a sum of two efforts in the denominator and one of the efforts in the numerator. That gives a number between zero and one, which specifies contest success.

With these assumptions, Player i's expected utility in Period 1 is

$$\begin{aligned} U_{11} &= p_{11}V_1 - b_{11}Z_{11} - b_{11}S_1 = \frac{Z_{11}^v}{Z_{11}^v + D_{21}^v}V_1 - b_{11}Z_{11} - b_{11}S_1, \\ U_{21} &= p_{21}V_2 - a_{21}D_{21} = \frac{D_{21}^v}{Z_{11}^v + D_{21}^v}V_2 - a_{21}D_{21} \end{aligned} \tag{3}$$

where Equations (1) and (2) have been inserted. Player 1's two free-choice variables in Period 1 are Z_{11} and S_1, constrained by Equation (1). Player 1 obtains no utility in Period 1 for allocating S_1 to stockpiling. Player 2's one free-choice variable in Period 1 is D_{21}, constrained by $D_{21} \geq 0$.

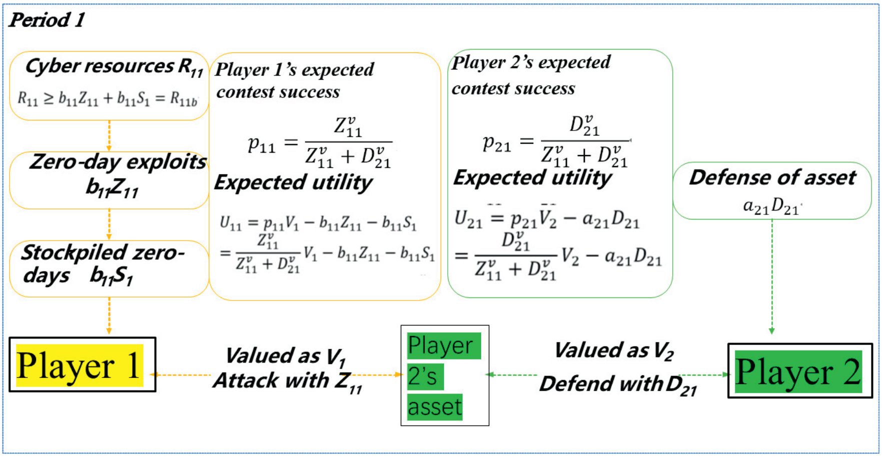

Figure 1. Illustrating Period 1.

2.2. Period 2

Figure 2 illustrates Period 2.

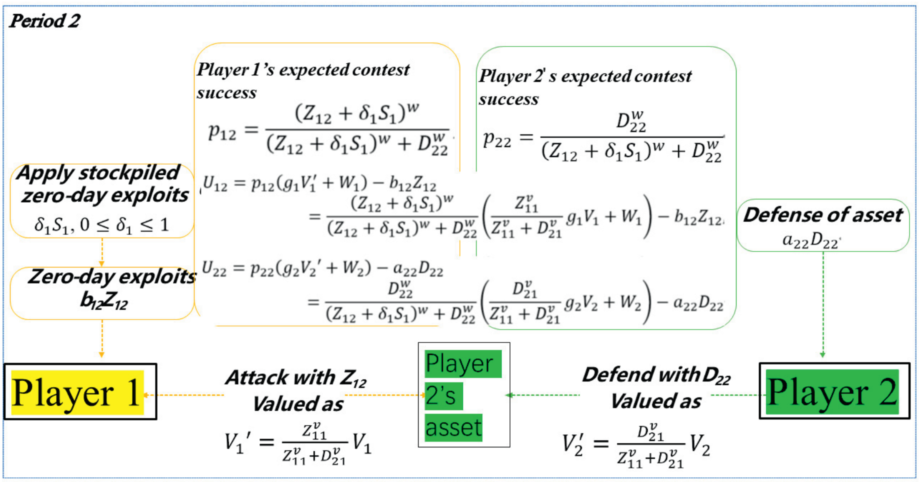

Figure 2. Illustrating Period 2.

In Period 2, Player 1 applies its stockpiled zero-day exploits S_1 from Period 1, if it has stockpiled. Additionally, in Period 2, Player 1 exerts effort Z_{12} at unit cost b_{12} to develop zero-day exploits, against which Player 2 exerts defense effort D_{22} at unit cost a_{22}. More specifically, assume that Player 1 in Period 2 applies its stockpiled zero-day exploits S_1 from Period 1, either keeping its same value with no appreciation if $\delta_1 = 1$, appreciating in value if $\delta_1 > 1$, or depreciating in value if $0 \leq \delta_1 \leq 1$. Appreciation of zero-day exploits over time occurs if technical, economic, or cultural circumstances change, making zero-day exploits more useful. In contrast, depreciation occurs if some aspects of the zero-day exploits leak or somehow becomes known or if technological or other developments make zero-day exploits less valuable over time. For example, increased competence may enable defenders against zero-day exploits to defend better, even though the nature of the zero-day exploit is unknown. 100% depreciation is expressed as $\delta_1 = 0$.

Player 1 in Period 2 exerts effort Z_{12} at unit cost b_{12} to develop zero-day exploits deployed in Period 2 to exploit zero-day vulnerabilities for Player 2. Player 2 exerts defense effort D_{22} in Period 2 at unit cost a_{22} to defend its asset, which it values as $V_2' = \frac{D_{21}^v}{Z_{11}^v + D_{21}^v} V_2$ and Player 1 values as $V_1{}' = \frac{Z_{11}^v}{Z_{11}^v + D_{21}^v} V_1$. In Period 2, Player 1's expected contest success is p_{21} and Player 2's expected contest success is p_{22}, i.e.,

$$p_{12} = \frac{(Z_{12} + \delta_1 S_1)^w}{(Z_{12} + \delta_1 S_1)^w + D_{22}^w}, p_{22} = \frac{D_{22}^w}{(Z_{12} + \delta_1 S_1)^w + D_{22}^w} \tag{4}$$

where w, $w \geq 0$, is the contest intensity in Period 2, with the same interpretation as v for Period 1, and S_1 is determined by (1).

Assume that Player 2's asset, valued as V_i by Player i, $i = 1, 2$, grows with a growth factor g_i from Period 1 to Period 2; $g_i \geq 0$, with an interpretation similar to that of δ_1 for Player 1's stockpiling S_1. That is, an asset with value V_i grows if $g_i > 1$, keeps its value if $g_i = 1$, and loses value if $0 \leq g_i < 1$. Furthermore, assume that Player 2 in Period 2 gets injected with a new fresh asset valued as W_i by Player i, $i = 1, 2$. With these assumptions, Player i's expected utility in Period 2 is

$$\begin{aligned} U_{12} &= p_{12}\left(g_1 V_1' + W_1\right) - b_{12} Z_{12} = \tfrac{(Z_{12}+\delta_1 S_1)^w}{(Z_{12}+\delta_1 S_1)^w + D_{22}^w}\left(\tfrac{Z_{11}^v}{Z_{11}^v + D_{21}^v} g_1 V_1 + W_1\right) - b_{12} Z_{12}, \\ U_{22} &= p_{22}\left(g_2 V_2{}' + W_2\right) - a_{22} D_{22} = \tfrac{D_{22}^w}{(Z_{12}+\delta_1 S_1)^w + D_{22}^w}\left(\tfrac{D_{21}^v}{Z_{11}^v + D_{21}^v} g_2 V_2 + W_2\right) - a_{22} D_{22} \end{aligned} \tag{5}$$

Player 1's one free-choice variable in Period 2 is Z_{12}, constrained by $Z_{12} \geq 0$. Player 2's one free-choice variable in Period 2 is D_{22}, constrained by $D_{21} \geq 0$.

For the two-period game as a whole, with time discount factor β_i, $0 \leq \beta_i \leq 1$, Player i's expected utility over the two periods is

$$U_1 = Max(0, U_{11} + \beta_1 U_{12}), U_2 = U_{21} + \beta_2 U_{22} \tag{6}$$

The Max function is used for Player 1 since Player 1 will not use its entire budget R_{11} if that causes negative expected utility U_1.

3. Solving the Model

In Section 3.1.1, the game is solved with backward induction starting in Period 2. In Section 3.1.1, Period 1 is solved. Thereafter, various corner solutions have been determined. The 11 solutions in Table 1 have been identified for the game. All the solutions except Solution 9 have positive efforts $Z_{11} \geq 0$ and $D_{21} \geq 0$ in Period 1, which is the nature of the ratio form contest success function in (2) and (3), with simultaneous moves in Period 1. That is, a player may decrease its effort arbitrarily close to zero, but not to zero. In Solution 9, Player 1 withdraws to avoid negative expected utility, i.e., to ensure $U_1 \geq 0$.

Table 1. Characteristics of the 11 solutions. $Z_{11} \geq 0$ and $D_{21} \geq 0$ in Period 1 in all the solutions.

Sol.	Stockpiling	Budget Constraint	Period 2	Description	Section
1	$S_1 = 0$	$R_{11} \geq R_{11b}$	$Z_{12} \geq 0, D_{22} \geq 0$	Player 1 neither stockpiles nor utilizes entire budget	Section 3.1.2
2	$S_1 \geq 0$	$R_{11b} = R_{11}$	$Z_{12} \geq 0, D_{22} \geq 0$	Player 1 stockpiles and utilizes entire budget	Section 3.1.2
3	$S_1 = 0$	$R_{11b} = R_{11}$	$Z_{12} \geq 0, D_{22} \geq 0$	Player 1 does not stockpile and utilizes entire budget	Section 3.1.3
4	$S_1 \geq 0$	$R_{11} \geq R_{11b}$	$Z_{12} = D_{22} = 0$	Player 2 is deterred; Player 1 is superior	Section 3.2.1
5	$S_1 \geq 0$	$R_{11b} = R_{11}$	$Z_{12} = D_{22} = 0$	Player 2 is deterred; Player 1 utilizes entire budget	Section 3.2.2
6	$S_1 \geq 0$	$R_{11b} = R_{11}$	$Z_{12} = 0, D_{22} \geq 0$	$\frac{\partial U_1}{\partial S_1} = 0, Z_{11} = \frac{R_{11} - b_{11} S_1}{b_{11}}$, Player 2 is not deterred	Section 3.2.3
7	$S_1 \geq 0$	$R_{11b} = R_{11}$	$Z_{12} = 0, D_{22} \geq 0$	$\frac{\partial U_1}{\partial Z_{11}} = 0, S_1 = \frac{R_{11} - b_{11} Z_{11}}{b_{11}}$, Player 2 is not deterred	Section 3.2.3
8	$S_1 \geq 0$	$R_{11b} \geq R_{11}$	$Z_{12} = 0, D_{22} \geq 0$	Player 2 is not deterred, though Player 1 is superior	Section 3.2.3
9	$S_1 = 0$	$R_{11} \geq R_{11b}$	$Z_{11} = 0, D_{22} \geq 0$	Player 1 withdraws to ensure $U_1 \geq 0$	Section 3.3
10	$S_1 = 0$	$R_{11} = R_{11b}$	$Z_{11} = D_{21}, Z_{12} = D_{22}$	Equally matched players; $U_1 = U_2 = 0$	Section 3.4
11	$S_1 = 0$	$R_{11b} \geq R_{11}$	$Z_{12} = D_{22} = 0$	Player 2 is deterred; Player 1 does not stockpile	Section 3.5

3.1. *Solutions 1, 2, 3 ($Z_{12} \geq 0$, $D_{22} \geq 0$, $S_1 \geq 0$)*

3.1.1. Solving Period 2

Differentiating Player i's expected utility U_{i2} in (5) in Period 2 with respect to its one free-choice variable, i.e., Z_{12} for Player 1 and D_{22} for Player 2, and equating it with zero, gives the first-order conditions

$$\begin{aligned} \frac{\partial U_{12}}{\partial Z_{12}} &= \frac{w D_{22}^w P_{11} (Z_{12} + \delta_1 S_1)^{w-1}}{(Z_{11}^v + D_{21}^v)\left((Z_{12} + \delta_1 S_1)^w + D_{22}^w\right)^2} - b_{12} = 0, \\ \frac{\partial U_{22}}{\partial D_{22}} &= \frac{w D_{22}^{w-1} Q_{21} (Z_{12} + \delta_1 S_1)^w}{(Z_{11}^v + D_{21}^v)\left((Z_{12} + \delta_1 S_1)^w + D_{22}^w\right)^2} - a_{22} = 0, \\ P_{11} &\equiv W_1 D_{21}^v + (g_1 V_1 + W_1) Z_{11}^v,\ Q_{21} \equiv W_2 Z_{11}^v + (g_2 V_2 + W_2) D_{21}^v \end{aligned} \tag{7}$$

which are solved to yield

$$Z_{12} = \frac{a_{22}/Q_{21}}{b_{12}/P_{11}} D_{22} - \delta_1 S_1, D_{22} = \frac{w Q_{21} A}{a_{22}\left(Z_{11}^v + D_{21}^v\right)(1 + A)^2}, A \equiv \left(\frac{a_{22}/Q_{21}}{b_{12}/P_{11}}\right)^w \tag{8}$$

The second-order conditions are

$$\begin{aligned} \frac{\partial^2 U_{12}}{\partial Z_{12}^2} &= -\frac{w D_{22}^w P_{11} (Z_{12} + \delta_1 S_1)^{w-2}\left((1+w)(Z_{12} + \delta_1 S_1) + (1-w) D_{22}^w\right)}{(Z_{11}^v + D_{21}^v)\left((Z_{12} + \delta_1 S_1)^w + D_{22}^w\right)^3}, \\ \frac{\partial^2 U_{22}}{\partial D_{22}^2} &= -\frac{w D_{22}^{w-2} Q_{21} (Z_{12} + \delta_1 S_1)^w\left((1-w)(Z_{12} + \delta_1 S_1) + (1+w) D_{22}^w\right)}{(Z_{11}^v + D_{21}^v)\left((Z_{12} + \delta_1 S_1)^w + D_{22}^w\right)^3} \end{aligned} \tag{9}$$

which are satisfied as negative when

$$\begin{aligned} (1+w)(Z_{12} + \delta_1 S_1) + (1-w) D_{22}^w \geq 0, \\ (1-w)(Z_{12} + \delta_1 S_1) + (1+w) D_{22}^w \geq 0 \end{aligned} \tag{10}$$

3.1.2. Solving Period 1

Inserting Equations (8) and (3) into Player i's expected utility in Equation (6) over the two periods gives

$$\begin{aligned} U_1 &= \frac{Z_{11}^v V_1}{Z_{11}^v + D_{21}^v} - b_{11} Z_{11} - b_{11} S_1 + \frac{\beta_1 A}{1+A}\left(\frac{Z_{11}^v}{Z_{11}^v + D_{21}^v} g_1 V_1 + W_1\right) - \frac{\beta_1 w P_{11} A}{\left(Z_{11}^v + D_{21}^v\right)(1+A)^2} + \beta_1 b_{12} \delta_1 S_1, \\ U_2 &= \frac{D_{21}^v V_2}{Z_{11}^v + D_{21}^v} - a_{21} D_{21} + \frac{\beta_2}{1+A}\left(\frac{D_{21}^v}{Z_{11}^v + D_{21}^v} g_2 V_2 + W_2\right) - \frac{\beta_2 w Q_{21} A}{\left(Z_{11}^v + D_{21}^v\right)(1+A)^2} \end{aligned} \tag{11}$$

which is rewritten as

$$\begin{aligned} U_1 &= \frac{Z_{11}^v V_1}{Z_{11}^v + D_{21}^v} - b_{11} Z_{11} + \frac{\beta_1 P_{11}(A+1-w)A}{\left(Z_{11}^v + D_{21}^v\right)(1+A)^2} - (b_{11} - \beta_1 b_{12} \delta_1) S_1, \\ U_2 &= \frac{D_{21}^v V_2}{Z_{11}^v + D_{21}^v} - a_{21} D_{21} + \frac{\beta_2 Q_{21}(1+(1-w)A)}{\left(Z_{11}^v + D_{21}^v\right)(1+A)^2} \end{aligned} \tag{12}$$

which has three unknown variables: S_1, Z_{11}, and D_{21}. Using (12), Player 1's optimal stockpiling is

$$S_1 = \begin{cases} Min\left(\frac{D_{22}\ a_{22}/Q_{21}}{\delta_1\ b_{12}/P_{11}}, \frac{R_{11} - b_{11} Z_{11}}{b_{11}}\right) if\ b_{11} \leq \beta_1 b_{12} \delta_1 \\ 0\ otherwise, \end{cases} \tag{13}$$

where $\frac{D_{22} a_{22}/Q_{21}}{\delta_1 b_{12}/P_{11}}$ according to (8) is the amount of stockpiling S_1 that causes zero effort Z_{12} for Player 1 in Period 2, and $\frac{R_{11} - b_{11} Z_{11}}{b_{11}}$ according to (1) is the maximum stockpiling S_1 permitted by Player 1's budget constraint R_{11}. Player 1 chooses the lowest of these two values since excessive stockpiling S_1 in Period 1, which cannot be utilized in Period 2, is not preferable, since Player 1 cannot exceed its budget constraint R_{11}. We refer to $S_1 = 0$ in (13) when $b_{11} > \beta_1 b_{12} \delta_1$ and $R_{11} \geq R_{11\text{b}}$ as Solution 1. If $b_{11} > \beta_1 b_{12} \delta_1$, Player 1 does not stockpile in Period 1, i.e., $S_1 = 0$, since its unit cost b_{11} of stockpiling exceeds the product of Player 1's unit cost b_{12} of exerting effort Z_{12} in Period 2, Player 1's time discount factor β_1, and Player 1's zero-day appreciation factor δ_1 from Period 1 to Period 2. We refer to $S_1 = \frac{R_{11} - b_{11} Z_{11}}{b_{11}}$ in (13) when $b_{11} \leq \beta_1 b_{12} \delta_1$ and $R_{11} = R_{11\text{b}}$ as Solution 2. Then, Player 1 chooses Z_{11}, optimally, and applies its remaining budget to stockpile $S_1 \geq 0$.

Differentiating each player's expected utility in (12) with respect to the two remaining free-choice variables, i.e., Z_{11} for Player 1 and D_{21} for Player 2, and equating it with zero, gives the first-order conditions

$$\begin{aligned} \frac{\partial U_1}{\partial Z_{11}} &= \frac{D_{21}^v v Z_{11}^{v-1}\left(A g_2 P_{11} V_2 w (B - Cw) \beta_1 + Q_{21} V_1 \left(B^3 + A g_1 \left(B^2 - Cw^2\right) \beta_1\right)\right)}{B^3 Q_{21}\left(Z_{11}^v + D_{21}^v\right)^2} - b_{11} = 0, \\ \frac{\partial U_2}{\partial D_{21}} &= \frac{D_{21}^{v-1} v Z_{11}^v \left(A g_1 Q_{21} V_1 w (B + Cw) \beta_2 + P_{11} V_2 \left(B^3 + g_2 \left(B^2 + CAw^2\right) \beta_2\right)\right)}{B^3 P_{11}\left(Z_{11}^v + D_{21}^v\right)^2} - a_{21} = 0, \\ & B \equiv 1 + A, C \equiv 1 - A \end{aligned} \tag{14}$$

which are cumbersome to analyze analytically. Hence, we solve (14) numerically for Z_{11} and D_{21} and use (13) to determine S_1, which are both inserted into (8) to determine the free-choice variables Z_{12} and D_{22} in Period 2. We finally insert the result into (12) to determine the players' expected utilities U_1 and U_2 over the two time periods.

3.1.3. Solution 3 ($Z_{11} = R_{11}/b_{11}$)

Inserting $Z_{11} = R_{11}/b_{11}$ into (1) causes zero stockpiling, $S_1 = 0$. Thus, Player 1 in Period 1 allocates all its resources to exploit zero-day vulnerabilities for Player 2 and has no resources to stockpile zero-day exploits for use in Period 2. The solution follows from solving the second first-order condition in (14) when $Z_{11} = R_{11}/b_{11}$ and applying $Z_{11} = R_{11}/b_{11}$ instead of the first first-order condition in (14).

3.2. Solutions 4–8 ($Z_{12} = 0$, $D_{22} \geq 0$, $R_{11} \geq R_{11b}$)

When $Z_{12} = 0$, Player 1 exerts no effort to develop zero-day capabilities in Period 2; instead, it relies on the stockpiling S_1 from Period 1 to attack Player 2. Solving Player 2's first-order condition in (7) when $Z_{12} = 0$ gives

$$D_{22}^{w} - \sqrt{D_{22}^{w-1}}\sqrt{\frac{wQ_{21}(\delta_1 S_1)^{w}}{a_{22}\left(Z_{11}^{v} + D_{21}^{v}\right)}} + (\delta_1 S_1)^{w} = 0 \tag{15}$$

which is not analytically solvable for general w (since w appears multiplicatively under a root sign, appears as an exponent with two different bases, appears as an exponent under a root sign and without a root sign, and appears as an exponent $w-1$ under a root sign), but is, for $w = 1$, conveniently solved to

$$D_{22} = \begin{cases} \left(\sqrt{\frac{Q_{21}}{a_{22}\left(Z_{11}^{v}+D_{21}^{v}\right)}} - \sqrt{\delta_1 S_1}\right)\sqrt{\delta_1 S_1}\ if\ \frac{Q_{21}}{a_{22}\left(Z_{11}^{v}+D_{21}^{v}\right)} > \delta_1 S_1 \\ 0\ otherwise. \end{cases} \tag{16}$$

Inserting $Z_{12} = 0$, $w = 1$, and (3) into Player i's expected utility in (6) gives

$$\begin{aligned} U_1 &= \frac{Z_{11}^{v} V_1}{Z_{11}^{v}+D_{21}^{v}} - b_{11}Z_{11} - b_{11}S_1 + \beta_1 \frac{\delta_1 S_1}{\delta_1 S_1 + D_{22}}\left(\frac{Z_{11}^{v}}{Z_{11}^{v}+D_{21}^{v}} g_1 V_1 + W_1\right) \\ U_2 &= \frac{D_{21}^{v} V_2}{Z_{11}^{v}+D_{21}^{v}} - a_{21}D_{21} + \beta_2\left(\frac{D_{22}}{\delta_1 S_1 + D_{22}}\left(\frac{D_{21}^{v}}{Z_{11}^{v}+D_{21}^{v}} g_2 V_2 + W_2\right) - a_{22}D_{22}\right) \end{aligned} \tag{17}$$

where D_{22} follows from (16). Differentiating U_1 in (17) with respect to S_1 and equating with zero gives

$$\frac{\partial U_1}{\partial S_1} = \frac{\beta_1\sqrt{\delta_1}\sqrt{a_{22}}P_{11}}{2\sqrt{S_1}\sqrt{Z_{11}^{v} + D_{21}^{v}}\sqrt{Q_{21}}} - b_{11} = 0 \Rightarrow S_1 = \frac{\beta_1^2 \delta_1 a_{22} P_{11}^2}{4b_{11}^2\left(Z_{11}^{v} + D_{21}^{v}\right)Q_{21}} \tag{18}$$

The two remaining unknown variables Z_{11} and D_{21} in (17) are determined by solving $\frac{\partial U_1}{\partial Z_{11}} = 0$ and $\frac{\partial U_2}{\partial D_{21}} = 0$ together with (18) for Period 1.

3.2.1. Solution 4 ($Z_{12} = D_{22} = 0$, $R_{11} \geq R_{11\text{b}}$)

When $\frac{Q_{21}}{a_{22}\left(Z_{11}^{v}+D_{21}^{v}\right)} \leq \delta_1 S_1$ in (16), Player 2 is deterred from exerting effort in Period 2, i.e., $D_{22} = 0$. Then, Player 1 wins the Period 2 contest since $S_1 > 0$. Inserting $Z_{12} = D_{22} = 0$, $w = 1$, and (3) into Player i's expected utility in (6) gives

$$\begin{aligned} U_1 &= \frac{Z_{11}^{v} V_1}{Z_{11}^{v}+D_{21}^{v}} - b_{11}Z_{11} - b_{11}S_1 + \beta_1\left(\frac{Z_{11}^{v}}{Z_{11}^{v}+D_{21}^{v}} g_1 V_1 + W_1\right), \\ U_2 &= \frac{D_{21}^{v} V_2}{Z_{11}^{v}+D_{21}^{v}} - a_{21}D_{21} \end{aligned} \tag{19}$$

Differentiating (19) to determine the optimal efforts Z_{11} and D_{21} for Players 1 and 2, respectively, and equating with 0 gives

$$\begin{aligned} \frac{\partial U_1}{\partial Z_{11}} &= \frac{vV_1 Z_{11}^{v-1} D_{21}^{v}(1+\beta_1 g_1)}{\left(Z_{11}^{v}+D_{21}^{v}\right)^2} - b_{11} = 0, \\ \frac{\partial U_2}{\partial D_{21}} &= \frac{vD_{21}^{v-1} Z_{11}^{v} V_2}{\left(Z_{11}^{v}+D_{21}^{v}\right)^2} - a_{21} = 0 \end{aligned} \tag{20}$$

which are solved to yield

$$Z_{11} = \frac{a_{21}/V_2}{b_{11}/V_1(1+\beta_1 g_1)} D_{21}, D_{21} = \frac{vV_2\left(\frac{a_{21}/V_2}{b_{11}/V_1(1+\beta_1 g_1)}\right)^v}{a_{21}\left(1+\left(\frac{a_{21}/V_2}{b_{11}/V_1(1+\beta_1 g_1)}\right)^v\right)^2} \tag{21}$$

The second-order conditions are

$$\begin{aligned}\frac{\partial^2 U_1}{\partial Z_{11}^2} &= -\frac{vV_1 D_{21}^v Z_{11}^{v-2}(1+\beta_1 g_1)\left((1+v)Z_{11}^v+(1-v)D_{21}^v\right)}{\left(Z_{11}^v+D_{21}^v\right)^3},\\ \frac{\partial^2 U_2}{\partial D_{21}^2} &= -\frac{vV_2 D_{21}^{v-2} Z_{11}^v\left((1-v)Z_{11}^v+(1+v)D_{21}^v\right)}{\left(Z_{11}^v+D_{21}^v\right)^3}\end{aligned} \tag{22}$$

which are satisfied as negative when

$$\begin{aligned}(1+v)Z_{11}^v + (1-v)D_{21}^v \geq 0,\\ (1-v)Z_{11}^v + (1+v)D_{21}^v \geq 0\end{aligned} \tag{23}$$

To deter Player 2 in Period 2, Player 1 must choose sufficiently large stockpiling S_1 to make Player 2 indifferent between exerting and not exerting effort D_{22} in Period 2. Inserting $Z_{12} = D_{22} = 0$ and $w = 1$ into (3), that implies

$$\begin{aligned}\frac{D_{22}}{\delta_1 S_1 + D_{22}}\left(\frac{D_{21}^v}{Z_{11}^v+D_{21}^v} g_2 V_2 + W_2\right) - a_{22} D_{22} = 0 \textit{ when } D_{22} = 0\\ \Leftrightarrow S_1 = \frac{1}{\delta_1 a_{22}}\left(\frac{D_{21}^v g_2 V_2}{Z_{11}^v+D_{21}^v} + W_2\right)\end{aligned} \tag{24}$$

where Z_{11} and D_{21} in (17) are determined in (21).

3.2.2. Solution 5 ($Z_{12} = D_{22} = 0, R_{11} = R_{11b}$)

The solution for Z_{11}, D_{21}, and S_1 in (17) and (24) presupposes that the budget constraint $R_{11} \geq b_{11}Z_{11} + b_{11}S_1 = R_{11b}$ in (1) is not exceeded. If it is exceeded, Player 1 must decrease either the effort Z_{11} or the stockpiling S_1 that deters Player 2 in Period 2. Let us analyze the event that Player 1 chooses stockpiling S_1 to deter, as in (24), and uses the budget constraint R_{11} in (1) to determine Z_{11} (which is then lower than the optimal Z_{11} with no budget constraint in (17)). Applying $\frac{\partial U_2}{\partial D_{21}} = 0$ in (20), S_1 in (24), and the budget constraint in (1) gives the three equations

$$\frac{vD_{21}^{v-1} Z_{11}^v V_2}{\left(Z_{11}^v + D_{21}^v\right)^2} = a_{21}, S_1 = \frac{1}{\delta_1 a_{22}}\left(\frac{D_{21}^v g_2 V_2}{Z_{11}^v + D_{21}^v} + W_2\right), b_{11}Z_{11} + b_{11}S_1 = R_{11}, \tag{25}$$

which are numerically solvable for Z_{11}, D_{21}, and S_1.

3.2.3. Solutions 6–8 ($Z_{12} = 0, D_{22} \geq 0, R_{11} = R_{11b}$)

If Player 1 chooses effort $Z_{12} = 0$ in Period 2 and Player 1's budget constraint $R_{11} = R_{11b}$ prevents sufficient stockpiling S_1 to deter Player 2 in Period 2, Player 2 will choose positive effort $D_{22} \geq 0$ in Period 2. Then, (16) applies for D_{22} and (17) applies for U_1 and U_2. Solution 6 follows from solving $\frac{\partial U_2}{\partial D_{21}} = 0$ in (17) together with S_1 in (18) and the budget constraint $Z_{11} = \frac{R_{11} - b_{11}S_1}{b_{11}}$. Solution 7 follows from solving $\frac{\partial U_1}{\partial Z_{11}} = 0$ and $\frac{\partial U_2}{\partial D_{21}} = 0$ in (17) together with the budget constraint $S_1 = \frac{R_{11} - b_{11}Z_{11}}{b_{11}}$. Solution 8, in which Player 1 does not utilize its entire budget $R_{11} \geq R_{11b}$, follows from solving $\frac{\partial U_1}{\partial Z_{11}} = 0$ and $\frac{\partial U_2}{\partial D_{21}} = 0$ in (17) together with S_1 in (18). Solution 8 has not been demonstrated in practice. It is distinguished from Solutions 6 and 7 in that Player 1 does not utilize its entire budget $R_{11} \geq R_{11b}$, while still not deterring Player 2. It is also distinguished from Solutions 4 and 5, where Player 2 is indeed deterred, either by the player being superior (Solution 4) or by Player 1 utilizing its entire budget $R_{11} \geq R_{11b}$.

3.3. *Solution 9* ($S_1 = Z_{11} = 0$)

Player 1's budget constraint $R_{11} \geq b_{11}Z_{11} + b_{11}S_1$ in (1) may prevent Player 1 from an optimal exertion of efforts. Hence, we require that Player 1 should always receive positive expected utility $U_1 \geq 0$ and otherwise assume that Player 1 chooses zero efforts $Z_{11} = Z_{12} = 0$ in both periods and that Player 2 keeps its asset by exerting arbitrarily small defense efforts $D_{21} = D_{22} = \epsilon > 0$, where ϵ is arbitrarily small but strictly positive. Inserting into (3), (5) and (6), the players' expected utilities are thus $U_1 = U_{11} = U_{12} = 0$, $U_{21} = V_2$, $U_{22} = g_2V_2 + W_2$, $U_2 = V_2 + \beta_2 g_2 V_2 + W_2$.

3.4. *Solution 10* ($S_1 = 0$, $Z_{11} = R_{11}/b_{11} = D_{21}$)

A solution is possible, where the players are equally matched (equally advantaged) and Player 1 chooses Period 1 effort $Z_{11} = R_{11}/b_{11} = D_{21}$, which equals Player 2's Period 1 effort D_{21}. Furthermore, if the players are equally matched in Period 2 and exert equal and high Period 2 efforts $Z_{12} = D_{22}$, a solution can emerge where they both receive zero expected utilities since their efforts in both periods outweigh the benefits they receive from the asset values, i.e., $U_1 = U_{11} = U_{12} = U_2 = U_{21} = U_{22} = 0$.

3.5. *Solution 11* ($Z_{12} = D_{22} = S_1 = 0$)

When Player 2 is deterred in Period 2, $D_{22} = 0$, and Player 1 does not stockpile in Period 1, $S_1 = 0$, what remains for Period 1 is for Player 1 to choose effort Z_{11} and Player 2 to choose effort D_{21}. In order to deter Player 2 in Period 1, so that Player 2 chooses zero effort $D_{21} = 0$, (19) for Player 2 implies

$$U_2 = \frac{D_{21}^v V_2}{Z_{11}^v + D_{21}^v} - a_{21}D_{21} \leq 0 \Leftrightarrow Z_{11} \geq \left(\frac{D_{21}^{v-1}(V_2 - a_{21}D_{21})}{a_{21}}\right)^{1/v} \tag{26}$$

Equation (26) needs to be analyzed for each combination of parameter values to determine whether Player 1's budget R_{11} enables it to choose Z_{11}/b_{11} to deter Player 2 so that $D_{21} = 0$ or whether deterrence is impossible. Solution 11 has not been demonstrated in practice. It is distinguished from Solutions 4 and 5, where Player 2 is also deterred, $D_{22} = 0$, in Period 2, but Player 1 stockpiles $S_1 \geq 0$.

4. Illustrating the Solution

Figure 3 illustrates the solution, i.e., the efforts Z_{11}, D_{21}, Z_{12}, D_{22}, stockpiling S_1, the actual amount R_{11b} (dependent variable) of resources used by Player 1 in Period 1, and the expected utilities U_1, U_2, U_{11}, U_{21}, U_{12}, U_{22} for Players 1 and 2 with the 16 benchmark parameter values $R_{11} = a_{2j} = b_{1j} = g_i = v = w = \delta_1 = \beta_i = 1$, $V_i = 2$, $W_i = 0$, $i, j = 1, 2$. We have chosen unitary parameter values whenever possible. We also plot as functions of $a_{21} = a_{22}$ and $b_{11} = b_{12}$. In each of the 16 + 2 = 18 double panels, one parameter value varies, while the other parameter values are kept at their benchmarks. The upper part of each panel shows which solution is plotted for the various ranges along the horizontal axis. The benchmark solution (which is Solution 1) is $Z_{11} = D_{21} = R_{11b} = 0.875$, $Z_{12} = D_{22} = 0.25$, $S_1 = 0$, $U_1 = U_2 = 0.375$, $U_{11} = U_{21} = 0.125$, $U_{12} = U_{22} = 0.25$.

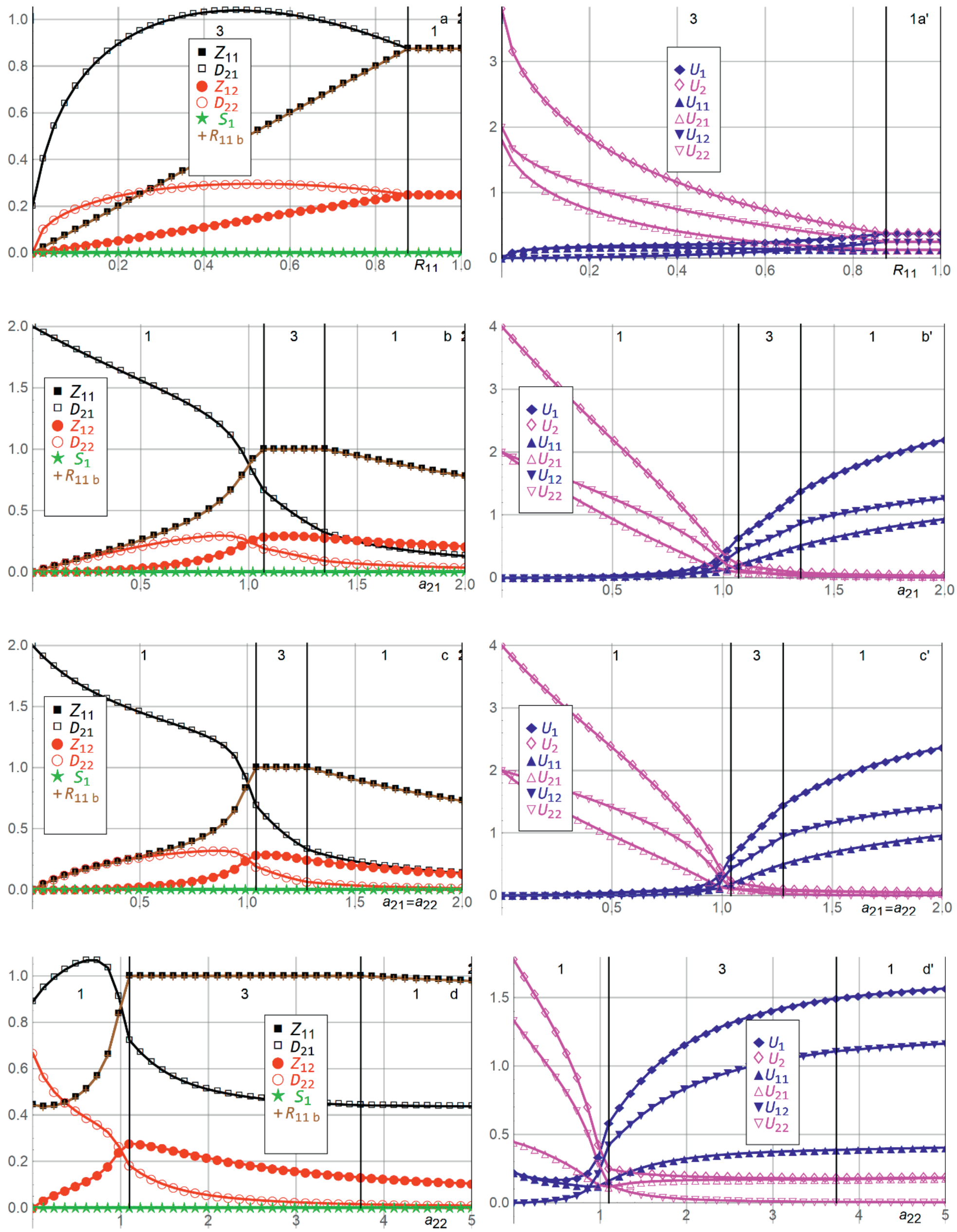

Figure 3. *Cont.*

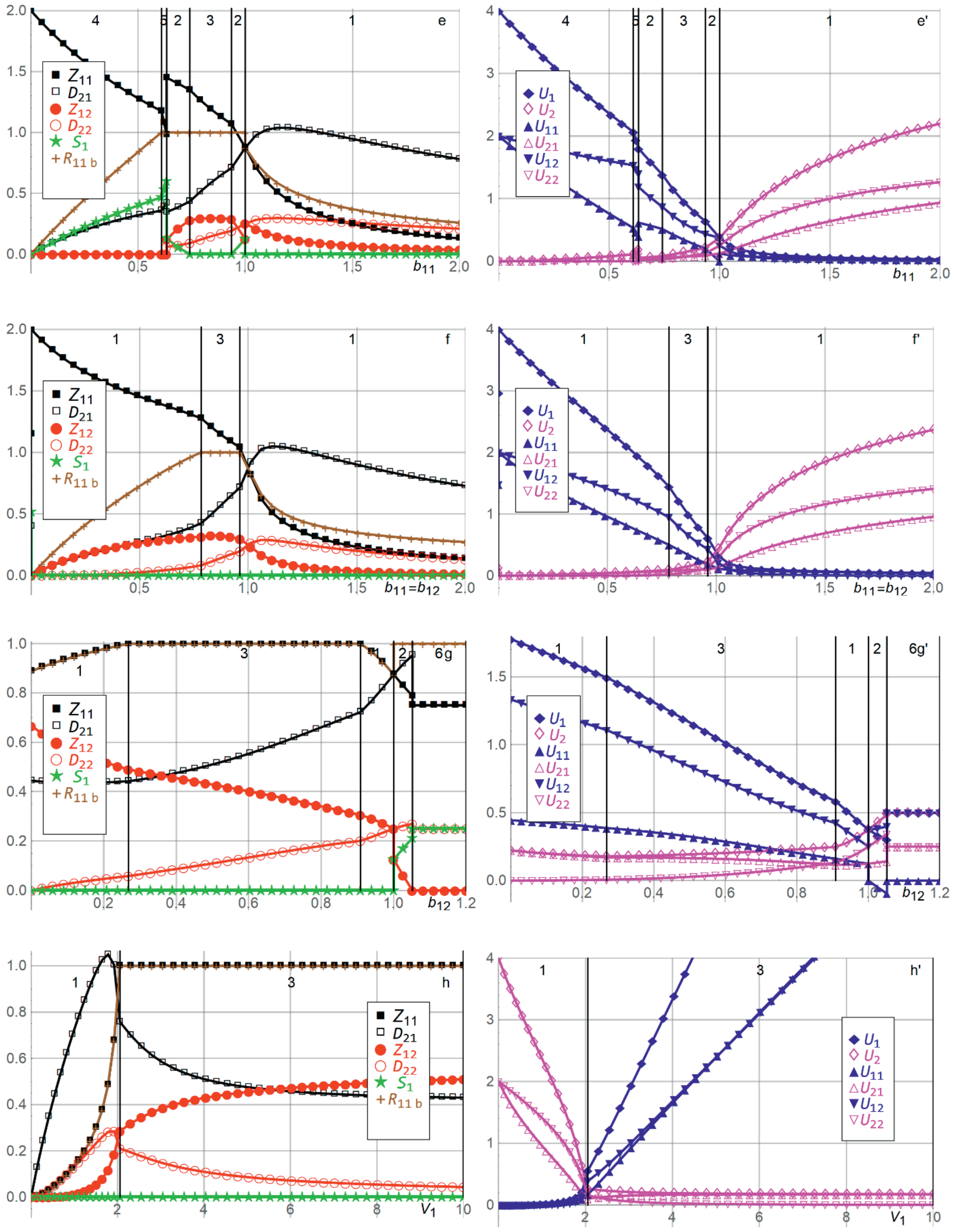

Figure 3. *Cont.*

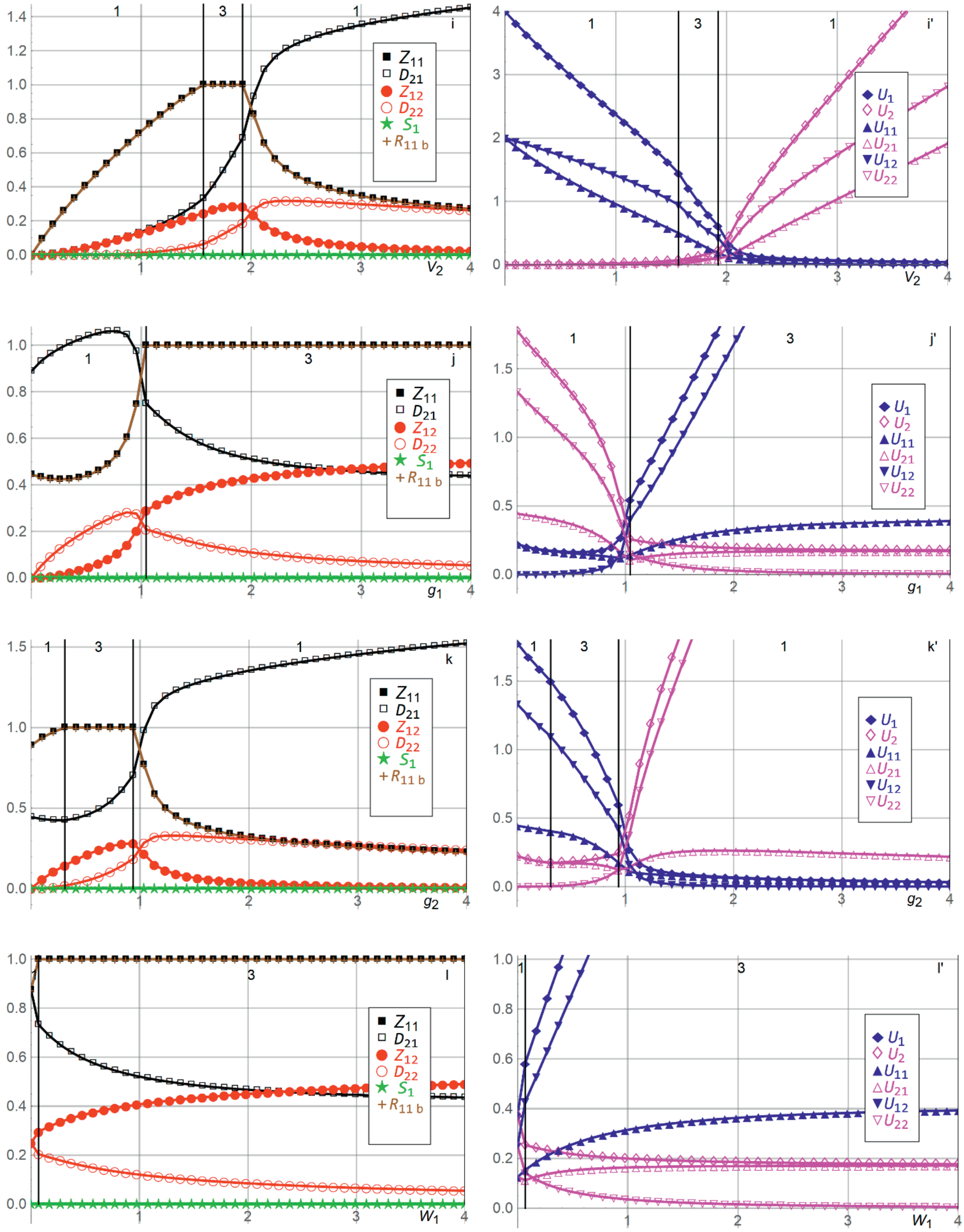

Figure 3. *Cont.*

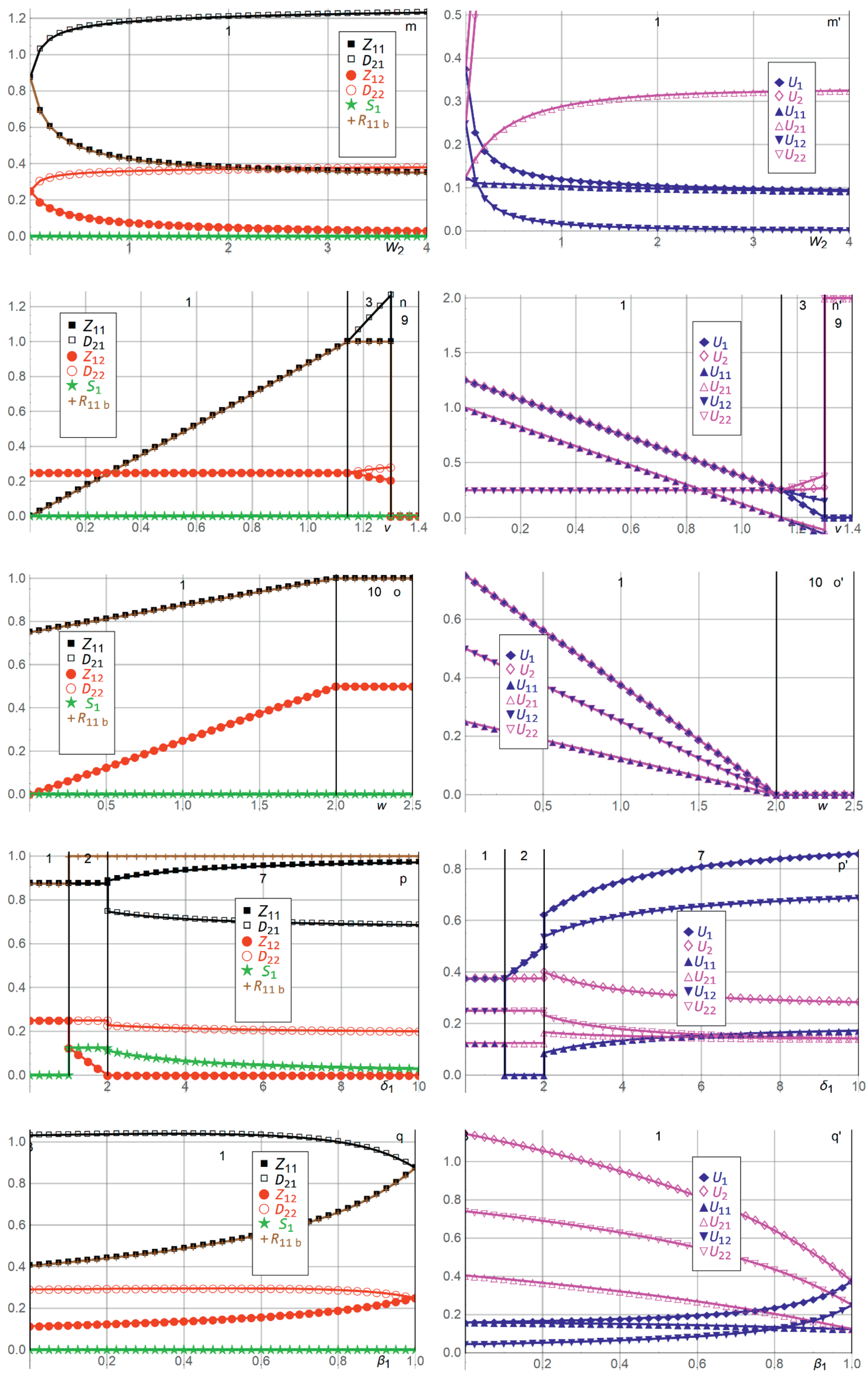

Figure 3. *Cont.*

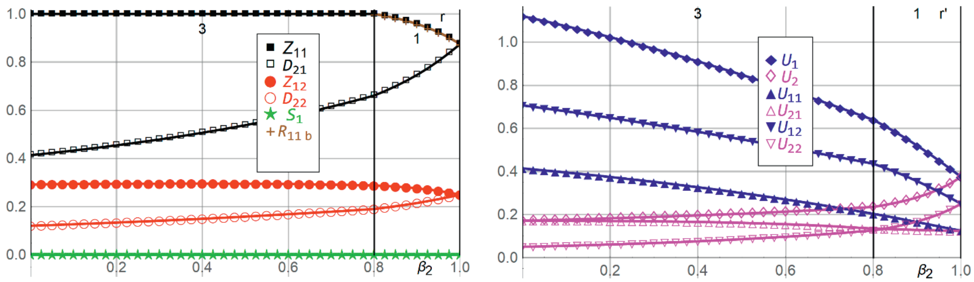

Figure 3. Efforts Z_{11}, D_{21}, Z_{12}, D_{22}, stockpiling S_1, used resources R_{11b}, and expected utilities U_1, U_2, U_{11},U_{21}, U_{12}, U_{22} for Players 1 and 2 as functions of R_{11}, $a_{2j}, b_{1j}, g_i, v, w, \delta_1, \beta_i$, V_i, W_i, $i, j = 1, 2$, relative to the benchmark parameter values $R_{11} = a_{2j} = b_{1j} = g_i = v = w = \delta_1 = \beta_i = 1$, $V_i = 2$, $W_i = 0$, $i, j = 1, 2$. See Table 2 and the text for an explanation of the 18 panels a,a′ to r,r′.

In Figure 3a,a′, when Player 1's budget constraint R_{11} exceeds the amount R_{11b} of resources used at benchmark $R_{11b} = 0.875$, all variables remain at their benchmarks, as functions of R_{11}, since Player 1 is not constrained in any way. In contrast, as R_{11} decreases below $R_{11b} = 0.875$, Player 1 is constrained in its effort $Z_{11} = R_{11}/b_{11}$, which decreases linearly to $Z_{11} = 0$ as R_{11} decreases to $R_{11} = 0$. Player 2's Period 1 defense effort D_{21} is inverse U-shaped in R_{11} since Player 1 first seeks to gain competitive advantage against Player 2 by competing more fiercely as R_{11} decreases below $R_{11b} = 0.875$. After D_{21} reaches a maximum, it decreases as Player 2 becomes more advantaged and succeeds with lower effort D_{21} due to Player 1's decreasing budget R_{11}. Hence, as R_{11} decreases, Player 1's expected utilities U_1,U_{11},U_{12} decrease and Player 2's expected utilities U_2,U_{21},U_{22} increase.

In Figure 3b,b′, as Player 2's unit effort cost a_{21} of defense in Period 1 increases above $a_{21} = 1$, the disadvantaged Player 2's efforts D_{21} and D_{22} in both periods and its expected utilities U_2,U_{21},U_{22} decrease. Player 1's efforts Z_{11} and Z_{12} in both periods are inverse U-shaped in a_{21}. Initially, as a_{21} increases above $a_{21} = 1$, Player 1 increases Z_{11} and Z_{12} to compete more successfully with Player 2. As a_{21} increases further, Player 1 decreases its efforts Z_{11} and Z_{12} due to strength and being advantaged, as Z_{11} and Z_{12} are less needed to compete successfully with Player 2. As a_{21} increases above $a_{21} = 1$, Player 1's expected utilities U_1,U_{11},U_{12} thus increase. For the range $1.07 \leq a_{21} \leq 1.35$, Player 1 reaches its budget constraint $R_{11} = 1$ due to competing fiercely with Player 2 (and being neither strongly advantaged nor strongly disadvantaged), causing maximum Period 1 effort $Z_{11} = 1$, which depresses Player 1's expected utility U_1 and increases Player 2's expected utility U_2 slightly, relative to no budget constraint. In contrast, as a_{21} decreases below $a_{21} = 1$, the advantaged Player 2 increases its Period 1 defense effort D_{21}, while Player 1 decreases its efforts Z_{11} and Z_{12} in both periods. Player 2's defense effort D_{22} in period 2 is inverse U-shaped for the same reason as above. As a_{21} approaches $a_{21} = 0$, less need exists for the advantaged Player 2 to exert effort D_{22} in Period 2, and the asset fought over is less valuable since most of the value was distributes in Period 1. Hence, as a_{21} decreases below $a_{21} = 1$, Player 2's expected utilities U_2,U_{21},U_{22} increase, and Player 1's expected utilities U_1,U_{11},U_{12} decrease. Player 1 does not stockpile $S_1 = 0$ since its efforts Z_{11} and Z_{12} are equally costly in both periods, its zero-day appreciation factor from Period 1 to Period 2 equals $\delta_1 = 1$, and its time discount factor equals $\beta_1 = 1$.

In Figure 3c,c′, Player 2's unit defense costs are assumed equal $a_{21} = a_{22}$ in both periods. Player 1 is budget constrained when $1.04 \leq a_{21} \leq 1.28$. Panel c,c′ is qualitatively similar to Panel b,b′. The main differences are that Player 2 becomes more disadvantaged when $a_{21} = a_{22}$ increases above $a_{21} = a_{22} = 1$ and more advantaged when $a_{21} = a_{22}$ decreases below $a_{21} = a_{22} = 1$ compared with Panel b,b′, where only a_{21} varies. Hence, for example, when $a_{21} = a_{22} > 1$, the two inverse-U shapes for Z_{11} and Z_{12} are narrower in Panel c,c′ than in Panel d,d′.

In Figure 3d,d′, Player 2's unit effort cost a_{22} of defense in Period 2 varies, causing results qualitatively similar to Panels b,b′ and c,c′. The main differences are that Player 2 prefers being disadvantaged in Period 2, with high a_{22} in Panel d,d′, rather than being disadvantaged in Period 1, with high a_{21} in Panel b,b′, and that Player 2 prefers being advantaged in Period 1 with low a_{21} in Panel b,b′ rather than being advantaged in Period 2 with high a_{22} in Panel b,b′. That is, Player 2 prefers to be advantaged in the important Period 1. If Player 2 is to be disadvantaged, it prefers to be so in the less important Period 2, where a less valuable asset is at stake. Player 1 is budget-constrained when $1.10 \leq a_{21} \leq 3.73$. The reason for the larger range of being budget-constrained (compared with Panels b,b′ and c,c′) is that when Player 1 is disadvantaged with a large unit effort cost $a_{21} \geq 1 = a_{11}$ in Period 2, which constrains its Period 2 effort Z_{12}, it becomes more important for Player 1 to compete as fiercely as possible with Player 2 in Period 1, utilizing the cheaper Period 1 effort Z_{11}.

In Figure 3e,e′, as Player 1's unit effort cost b_{11} of developing zero-day capabilities in Period 1 increases above $b_{11} = 1$, stockpiling $S_1 = 0$ continues not to occur in Solution 1 and exerting effort Z_{12} in Period 2 at unit cost $b_{12} = 1$ is cheaper. Player 1's efforts Z_{11} and Z_{12} in both periods decrease as b_{11} increases since Player 1 becomes more disadvantaged, cannot justify the costly efforts, and receives lower expected utilities U_1,U_{11},U_{12}. Player 2's defense efforts D_{21} and D_{22} in the two periods are inverse U-shaped as b_{11} increases above $b_{11} = 1$, which is common in such situations. That is, for intermediate b_{11} above $b_{11} = 1$, the players are similarly advantaged and Player 2 exerts high efforts D_{21} and D_{22}. As b_{11} increases, Player 2 becomes more advantaged and decreases D_{21} and D_{22} due to strength since high expected utilities U_2,U_{21},U_{22} are obtained even with low efforts. As b_{11} decreases, Player 2 becomes more disadvantaged and decreases D_{21} and D_{22} due to weakness, earning lower expected utilities U_2,U_{21},U_{22}. In contrast, as b_{11} decreases below $b_{11} = 1$, Player 1 stockpiles $S_1 \geq 0$ when the budget R_{11} permits it and it is beneficial. More specifically, decreasing b_{11} marginally below $b_{11} = 1$ causes Player 1 to replace a maximum part of its Period 2 effort Z_{12} with stockpiling $S_1 \geq 0$ until its budget $R_{11} = 1$ is reached, causing Z_{12} and S_1 to be discontinuous through $b_{11} = 1$ and causing Solution 2. As b_{11} decreases below $b_{11} = 0.94$, Solution 3 emerges. Player 1's unit efforts cost b_{11} is then so low that it chooses maximum Period 1 effort $Z_{11} = R_{11}/b_{11}$, as permitted by the budget $R_{11} = 1$, and zero stockpiling $S_1 = 0$. This continues with increasing expected utilities U_1,U_{11},U_{12} for Player 1 and decreasing expected utilities U_2,U_{21},U_{22} for Player 2, until $b_{11} = 0.74$, where Solution 2 again emerges. The reason is that for $b_{11} < 0.74$, Player 1 is sufficiently advantaged compared with Player 2, does not need to increase its Period 1 effort Z_{11} further, and prefers instead to stockpile to become more competitive in Period 2. Hence, as b_{11} decreases from $b_{11} = 0.74$ to $b_{11} = 0.63$, Player 1's Period 2 effort Z_{12} decreases as it is cost effectively replaced with stockpiling $S_1 \geq 0$. As b_{11} decreases below $b_{11} = 0.63$, Solution 5 emerges, where, interestingly, Player 1 stockpiles sufficiently with $S_1 \geq 0$ in Period 1 to deter Player 2 from defending in Period 2, i.e., $D_{22} = 0$. Player 1 exerts no effort $Z_{12} = 0$ in Period 2 (at unit cost b_{12}) since stockpiling $S_1 \geq 0$ at unit cost $b_{11} < 0.63$ is more cost effective. To accomplish the substantial stockpiling $S_1 \geq 0$ required to deter Player 2 in Period 2, Player 1 must decrease its Period 1 effort $Z_{11} = \frac{R_{11} - b_{11} S_1}{b_{11}}$ substantially below its effort Z_{11} chosen when $b_{11} < 0.63$, as required by its budget constraint $R_{11} = 1$. As b_{11} decreases below $b_{11} = 0.63$, within Solution 5, Player 1 can gradually afford to increase its Period 1 effort Z_{11}, enabling more successful competition with Player 2 in Period 1, and thus less stockpiling $S_1 \geq 0$ is required to deter Player 2 in Period 2. This process continues until $b_{11} < 0.61$, where Solution 4 emerges. In Solution 4, Player 1 is so superior that it does not need to utilize its entire budget $R_{11} = 1$. Its low unit effort cost $b_{11} < 0.61$ in Period 1 enables it to stockpile $S_1 \geq 0$ sufficiently to deter Player 2 in Period 2 and to sufficiently avoid having to exert effort in Period 2, i.e., $Z_{12} = 0$. Furthermore, as b_{11} decreases below $b_{11} = 0.61$, Player 1 competes increasingly successfully through increasing effort Z_{11} with Player 2 in Period 1, which enables decreased stockpiling $S_1 \geq 0$, increased expected utilities U_1,U_{11},U_{12} for Player 1, and decreased expected utilities U_2,U_{21},U_{22} for Player 2.

In Figure 3f,f′, Player 1's unit effort costs of developing zero-day capabilities are assumed to be equal $b_{11} = b_{12}$ in both periods. Since Player 1's zero-days do not appreciate, $\delta_1 = 1$, and Player 1 does not discount time, $\beta_1 = 0$, Player 1 does not need to stockpile, i.e., $S_1 = 0$ throughout. As $b_{11} = b_{12}$ increases above $b_{11} = b_{12} = 1$, the players' Period 1 efforts Z_{11} and D_{21} are qualitatively similar to

Panel e,e′, i.e., decreasing for Player 1 and inverse U-shaped for Player 2. In Period 2, Player 1 is more disadvantaged in Panel f,f′ than in Panel e,e′ since its unit effort cost b_{12} is higher (no longer $b_{12} = 1$). Thus Player 1′s Period 2 effort Z_{12} decreases more quickly towards zero than in Panel e,e′, enabling the advantaged Player 2 to also decrease its Period 2 defense effort D_{22} towards zero more quickly than in Panel e,e′. In contrast, as $b_{11} = b_{12}$ decreases below $b_{11} = b_{12} = 1$, Solution 2 with stockpiling does not arise as in Panel e,e′. Instead, Solution 1 continues to operate with increased Period 1 and Period 2 efforts Z_{11} and Z_{12} for Player 1 and decreased Period 1 and Period 2 efforts D_{21} and D_{22} for Player 2. This continues until $b_{11} = b_{12} = 0.96$, when Player 1 reaches its budget constraint $R_{11} = 1$ and Solution 3 emerges, as in Panel e,e′. Solution 3 is maintained, with increasing advantage for Player 1, until $b_{11} = b_{12} = 0.78$ when Player 1 is so advantaged that it does not need to utilize its entire budget $R_{11} = 1$. Instead, Solution 1 emerges for $b_{11} = b_{12} < 0.78$, where all the four efforts Z_{11}, Z_{12}, D_{21}, D_{22} are positive since stockpiling $S_1 \geq 0$ does not occur, which would deter Player 2 in Period 2, as in Panel e,e′. As $b_{11} = b_{12}$ decreases, Player 1′s Period 1 effort Z_{11} increases since the unit effort cost decreases, while Player 1′s Period 2 effort Z_{12} decreases due to Player 1′s advantage and less of Player 2′s asset left to compete in Period 2.

In Figure 3g,g′, as Player 1′s unit effort cost b_{12} of developing zero-day capabilities in Period 2 increases above $b_{12} = 1$, to the disadvantage of Player 1, stockpiling $S_1 \geq 0$ emerges in Solution 2 since Player 1′s Period 2 effort Z_{12} becomes increasingly expensive and reaches $Z_{12} = 0$ when $b_{12} > 1.05$. As b_{12} increases from $b_{12} = 1$ to $b_{12} = 1.05$, Player 1 accepts negative expected utility U_{11} in Period 1 in order to earn increasing positive expected utility U_{12} in Period 2. As b_{12} increases above $b_{12} = 1.05$, Player 1 exerts zero effort $Z_{12} = 0$ in Period 2, stockpiles optimally $S_1 \geq 0$, and chooses its Period 1 effort $Z_{11} = \frac{R_{11}-b_{11}S_1}{b_{11}}$ in Solution 6 to satisfy the budget constraint $R_{11} = 1$. Player 1 thus offsets its increasing unit effort cost $b_{12} > 1.05$ by stockpiling $S_1 \geq 0$ in Period 1. In contrast, as b_{12} decreases below $b_{12} = 1$, stockpiling $S_1 = 0$ continues not to occur in Solution 1 since exerting effort Z_{12} in Period 2 at unit cost $b_{12} = 1$ is cheaper. Player 1′s efforts Z_{11} and Z_{12} in both periods increase as b_{12} decreases since Player 1 becomes more advantaged and receives higher expected utilities U_1, U_{11},U_{12}. Player 2′s defense efforts D_{21} and D_{22} in the two periods decrease as b_{12} decreases below $b_{12} = 1$ since Player 2 becomes more disadvantaged and receives lower expected utilities U_2, U_{21}, U_{22}. This continues until $b_{12} = 0.91$, when Player 1′s Period 1 effort Z_{11} at unit cost $b_{11} = 1$ becomes too costly, Player 1 reaches its budget constraint $R_{11} = 1$, and Solution 3 emerges. Solution 3 is maintained as b_{12} decreases to $b_{12} = 0.27$, enabling Player 1 to increase its Period 2 effort Z_{12} and earn higher expected utilities U_1,U_{11},U_{12}. Player 2′s defense efforts D_{21} and D_{22} in the two periods decrease as b_{12} decreases below $b_{12} = 1$, earning lower expected utility U_2. As b_{12} decreases below $b_{12} = 0.27$, Player 1′s Period 2 effort Z_{12} becomes so high and cheap that Player 1 can rely on competing successfully with Player 2 in Period 2. Thus, Player 1 no longer needs to exert high Period 1 effort Z_{11} and no longer needs to apply its entire budget $R_{11} = 1$. Thus, Solution 1 re-emerges with higher expected utility U_1 to Player 1. Interestingly, Player 2 also receives higher expected utility U_2 as b_{12} decreases towards $b_{12} = 0$ since Player 1 still has the unit effort cost $b_{11} = 1$ of its Period 1 effort Z_{11}, and, thus, to some extent, Player 2 competes somewhat successfully with Player 1 in Period 1.

In Figure 3h,h′, when Player 1′s valuation V_1 of Player 2′s asset increases above the benchmark $V_1 = 2$, Player 1′s Period 1 effort Z_{11} increases rapidly from the benchmark $Z_{11} = 0.875$ and reaches the budget constraint $Z_{11} = R_{11} = 1$ when $V_1 > 2.06$. That causes a transition from Solution 1 to Solution 3. As V_1 increases, Player 2′s Period 1 effort D_{21} decreases, $\lim\limits_{V_1 \longrightarrow \infty} D_{21} = 0.41$, determined numerically. That is, although Player 1′s valuation V_1 increases arbitrarily, Player 2′s valuation remains at the benchmark $V_1 = 2$, causing Player 2 to compete to defend its asset in Period 1. In Period 2, this changes. As V_1 increases, Player 1 exerts increasing effort Z_{12}, $\lim\limits_{V_1 \longrightarrow \infty} Z_{12} = 0.59$, while Player 2 exerts decreasing effort D_{22}, $\lim\limits_{V_1 \longrightarrow \infty} D_{22} = 0$. As V_1 increases, Player 1 receives increasing expected utilities U_1,U_{11},U_{12}, $\lim\limits_{V_1 \longrightarrow \infty} U_1 = \lim\limits_{V_1 \longrightarrow \infty} U_{11} = \lim\limits_{V_1 \longrightarrow \infty} U_{12} = \infty$, while Player 2′s expected utility U_2 decreases, $\lim\limits_{V_1 \longrightarrow \infty} U_2 = \lim\limits_{V_1 \longrightarrow \infty} U_{21} = 0.17$, $\lim\limits_{V_1 \longrightarrow \infty} U_{22} = 0$. In contrast, as V_1 decreases below the benchmark $V_1 = 2$, the results are qualitatively similar to Player 1's budget R_{11}, decreasing below the benchmark $R_{11} = 0.785$ in Panel a,a′. That is, Player 1 exerts lower efforts Z_{11} and Z_{12} and receives lower expected utilities U_1,U_{11},U_{12}, while Player 2's efforts are inverse U-shaped and it receives increasing expected utilities U_2,U_{21},U_{22}.

In Figure 3i,i′, when Player 2's valuation V_2 of its own asset increases above the benchmark $V_2 = 2$, Player 2 exerts concavely increasing Period 1 defense effort D_{21} for its more valuable asset, $\lim_{V_2 \longrightarrow \infty} D_{21} = 2.00$. Player 2's Period 2 defense effort D_{22} is inverse U-shaped, as it first competes more fiercely with Player 1 and eventually decreases D_{22} due to being advantaged $\lim_{V_2 \longrightarrow \infty} D_{22} = 0$. Player 2's expected utilities U_2,U_{21},U_{22} thus increase, $\lim_{V_2 \longrightarrow \infty} U_2 = \lim_{V_2 \longrightarrow \infty} U_{21} = \lim_{V_2 \longrightarrow \infty} U_{22} = \infty$. Player 1 responds by decreasing its efforts Z_{11} and Z_{12} in both periods, $\lim_{V_2 \longrightarrow \infty} Z_{11} = \lim_{V_2 \longrightarrow \infty} Z_{12} = 0$, receiving decreasing expected utilities U_1,U_{11},U_{12}, $\lim_{V_2 \longrightarrow \infty} U_1 = \lim_{V_2 \longrightarrow \infty} U_{11} = \lim_{V_2 \longrightarrow \infty} U_{12} = 0$. In contrast, as V_2 decreases below the benchmark $V_2 = 2$, Player 1's Period 1 effort Z_{11} increases rapidly from the benchmark $Z_{11} = 0.875$ and reaches the budget constraint $Z_{11} = R_{11} = 1$ when $V_2 < 1.92$. That causes a transition from Solution 1 to Solution 3, but in the opposite direction compared with Panel h,h′. As V_2 decreases, Player 2's Period 1 effort D_{21} decreases convexly until $V_2 < 1.57$, causing a transition back to Solution 1 since the advantaged Player 1 no longer needs to utilize its entire budget $R_{11} = 1$. Thus, Player 1's Period 1 effort Z_{11} decreases. As V_2 decreases below the benchmark $V_2 = 2$, Player 1's Period 2 effort Z_{11} is inverse U-shaped, causing increasing expected utilities U_1,U_{11},U_{12}, while both efforts D_{21} and D_{22} by Player 2 decrease, causing decreasing expected utilities U_2,U_{21},U_{22}.

In Figure 3j,j′, when Player 1's growth factor g_1 of asset V_1 from Period 1 to Period 2 increases above the benchmark $g_1 = 1$, Player 1's Period 1 effort Z_{11} increases rapidly from the benchmark $Z_{11} = 0.875$, as in Panel h,h′, and reaches the budget constraint $Z_{11} = R_{11} = 1$ when $g_1 > 1.04$. That causes a transition from Solution 1 to Solution 3. As g_1 increases, the results are qualitatively similar to V_1 increasing in Panel h,h′, since Player 1's period 1 effort Z_{11} is locked to the budget constraint $Z_{11} = R_{11}/b_{11}$. The difference is that Player 1's Period 1 expected utility U_{11} does not approach infinity, since the growth factor g_1 is confined to Period 2, and, instead, approaches a constant concavely, $\lim_{g_1 \longrightarrow \infty} U_{11} = 0.41$. The other limit values are as in Panel h,h′, i.e., $\lim_{g_1 \longrightarrow \infty} D_{21} = 0.41$, $\lim_{g_1 \longrightarrow \infty} Z_{12} = 0.59$, $\lim_{g_1 \longrightarrow \infty} D_{22} = 0$, $\lim_{g_1 \longrightarrow \infty} U_1 = \lim_{g_1 \longrightarrow \infty} U_{12} = \infty$, $\lim_{g_1 \longrightarrow \infty} U_2 = \lim_{g_1 \longrightarrow \infty} U_{21} = 0.17$, $\lim_{g_1 \longrightarrow \infty} U_{22} = 0$. In contrast, as g_1 decreases below the benchmark $g_1 = 1$, Player 1 decreases its Period 2 effort Z_{12} since the asset has less value in Period 2, receiving decreasing expected utility U_{12} in Period 2. Both efforts D_{21} and D_{22} by Player 2 are inverse U-shaped, as in Panel h,h′, when the asset value V_1 decreases below the benchmark $V_1 = 2$. Player 1's Period 1 effort is slightly U-shaped since the asset still has value V_1 for Player 1 in Period 1. As g_1 decreases, Player 2's expected utilities U_2,U_{21},U_{22} increase, while Player 1's expected utilities U_1 and U_{11} are U-shaped. This latter remarkable result is caused by Player 1 focusing more explicitly on Period 1 when the growth factor g_1 is very low, while Player 2 focuses on both periods and strikes a balance between them.

In Figure 3k,k′, when Player 2's growth factor g_2 of asset V_2 from Period 1 to Period 2 increases above the benchmark $g_2 = 1$, Player 2's Period 1 effort D_{21} increases rapidly from the benchmark $D_{21} = 0.875$, as in Panel i,i′. Although growth g_2 does not manifest until Period 2, Player 2 competes fiercely in Period 1, knowing that what it can protect in Period 1 grows in Period 2. Thus, Player 2 exerts concavely increasing Period 1 defense effort D_{21}, $\lim_{g_2 \longrightarrow \infty} D_{21} = 2.00$. As g_2 increases, the results are qualitatively similar to V_2 increasing in Panel i,i′. The difference is that Player 2's Period 1 expected utility U_{21} does not approach infinity, since the growth factor g_2 is confined to Period 2. Instead, it is inverse U-shaped and approaches zero, $\lim_{g_2 \longrightarrow \infty} U_{21} = 0$. The other limit values are as in Panel i,i′, i.e., $\lim_{g_2 \longrightarrow \infty} U_2 = \lim_{g_2 \longrightarrow \infty} U_{22} = \infty$, $\lim_{g_2 \longrightarrow \infty} D_{22} = \lim_{g_2 \longrightarrow \infty} Z_{11} = \lim_{g_2 \longrightarrow \infty} Z_{12} = \lim_{g_2 \longrightarrow \infty} U_1 = \lim_{g_2 \longrightarrow \infty} U_{11} = \lim_{g_2 \longrightarrow \infty} U_{12} = 0$. In contrast, as g_2 decreases below the benchmark $g_2 = 1$, Player 2's Period 1 effort is slightly U-shaped since the asset still has value V_2 for Player 2 in Period 1. Solution 3 arises when

$0.31 \leq g_2 \leq 0.94$. Player 2 decreases its Period 2 effort D_{22} since the asset has less value in Period 2, receiving decreasing expected utility U_{22} in Period 2. Both efforts Z_{11} and Z_{12} by Player 1 are inverse U-shaped, as in Panel i,i′, when the asset value V_2 decreases below the benchmark $V_2 = 2$. As g_2 decreases, Player 1's expected utilities U_1,U_{11},U_{12} increase, while Player 2's expected utilities U_2 and U_{21} are U-shaped. This latter remarkable result is caused by Player 2 focusing more explicitly on Period 1, when the growth factor g_2 is very low, while Player 1 focuses on both periods and strikes a balance between them.

In Figure 3l,l′, when Player 1's valuation W_1 of Player 2's asset acquired in Period 2 increases above the benchmark $W_1 = 0$, Player 1's Period 1 effort Z_{11} quickly increases to its budget constraint $Z_{11} = R_{11}/b_{11}$, causing transition from Solution 1 to Solution 3 when $W_1 = 0.07$. Player 1's Period 1 expected utility U_{11} is thus constrained, increasing concavely to $\lim_{W_1 \longrightarrow \infty} U_{11} = 0.41$. Player 1's Period 2 effort Z_{12} increases concavely, $\lim_{W_1 \longrightarrow \infty} Z_{12} = 0.59$, and its expected utilities U_1 and U_{12} increase without bounds, $\lim_{W_1 \longrightarrow \infty} U_1 = \lim_{W_1 \longrightarrow \infty} U_{12} = \infty$. In contrast, Player 2's defense efforts D_{21} and D_{22} in the two periods and its expected utilities U_2 and U_{22} decrease convexly, $\lim_{W_1 \longrightarrow \infty} D_{21} = 0.41$, $\lim_{W_1 \longrightarrow \infty} D_{22} = 0$, $\lim_{W_1 \longrightarrow \infty} U_2 = 0.17$, $\lim_{W_1 \longrightarrow \infty} U_{22} = 0$. Player 2's Period 1 expected utility U_{21} increases concavely, $\lim_{W_1 \longrightarrow \infty} U_{21} = 0.17$, since Player 1 is budget-constrained in Period 1 and strongly focuses instead on Period 2 as W_1 increases.

In Figure 3m,m′, when Player 2's valuation W_2 of its own asset acquired in Period 2 increases above the benchmark $W_2 = 0$, Player 2's Period 1 defense effort D_{21} and expected utility U_{21} increase concavely, $\lim_{W_2 \longrightarrow \infty} D_{21} = 1.28$, $\lim_{W_2 \longrightarrow \infty} U_{21} = 0.32$. Player 1's Period 1 effort Z_{11} and expected utilities U_1 and U_{11} decrease concavely, $\lim_{W_2 \longrightarrow \infty} Z_{11} = 0.32$, $\lim_{W_2 \longrightarrow \infty} U_1 = \lim_{W_2 \longrightarrow \infty} U_{11} = 0.08$. Player 2's Period 2 defense effort D_{21} also increases concavely, $\lim_{W_2 \longrightarrow \infty} D_{22} = 0.4$, and Player 2's expected utilities U_2 and U_{22} increase without bounds, $\lim_{W_2 \longrightarrow \infty} U_2 = \lim_{W_2 \longrightarrow \infty} U_{22} = 0.08$. Player 1's Period 2 effort Z_{12} and expected utility U_{12} decrease convexly, $\lim_{W_2 \longrightarrow \infty} Z_{12} = \lim_{W_2 \longrightarrow \infty} U_{12} = 0$.

In Figure 3n,n′, when the contest intensity v in Period 1 increases above the benchmark $v = 1$, the players compete more fiercely with each other in Period 1, receiving decreasing expected utilities U_1,U_{11},U_2,U_{21} until Player 1 reaches its budget constraint $Z_{11} = R_{11}/b_{11} = 1$ when $v > 1.14$. When $v > 1.14$, which gives a transition from Solution 1 to Solution 3, Player 2 competes even more fiercely with increasing Period 1 defense effort D_{21} while accepting negative Period 1 expected utility U_2. Player 1's Period 1 expected utility U_{11} is even more negative. When $v > 1.14$, the advantaged Player 2 exerts slightly increasing Period 2 effort D_{22}, while Player 1 exerts decreasing effort Z_{12}. That continues until $v > 1.30$, when Player 1 starts to receive negative expected utility $U_1 < 0$ over the two periods, which is unacceptable for Player 1. Hence Solution 9 emerges, where Player 1 withdraws from both periods and receives zero expected utilities $Z_{11} = Z_{12} = U_1 = U_{11} = U_{12} = 0$. When $v > 1.30$, Player 2 exerts a arbitrarily small positive effort and keeps its asset, i.e., $D_{21} = D_{22} = \epsilon > 0$, where ϵ is arbitrarily small but positive, and receives expected utilities $U_2 = U_{21} = 2$, $U_2 = 4$. In contrast, as v decreases below the benchmark $v = 1$, both players exert lower Period 1 efforts Z_{11} and D_{21} and eventually zero effort $Z_{11} = D_{21} = 0$ at the limit for an egalitarian contest $v = 0$, where efforts do not matter. Concomitantly, both players' expected utilities U_1,U_{11},U_2,U_{21} increase. The players' Period 2 efforts and expected utilities are constant at $Z_{11} = D_{21} = U_{12} = U_{22} = 0.25$.

In Figure 3o,o′, when the contest intensity w in Period 2 increases from $w = 0$ (egalitarian contest) through to the benchmark $w = 1$ and to $w = 2$, the players' Period 2 efforts Z_{12} and D_{22} increase from $Z_{12} = D_{22} = 0$ through $Z_{12} = D_{22} = 0.25$, and to $Z_{12} = D_{22} = 0.5$. Simultaneously, the players' Period 1 efforts Z_{11} and D_{21} increase from $Z_{11} = D_{21} = 0.75$, when $w = 0$ (no egalitarian contest in Period 1), through the benchmark $Z_{11} = D_{21} = 0.875$, and to $Z_{11} = D_{21} = 1$ when $w = 2$. These increases in the efforts Z_{12}, D_{22}, Z_{11}, D_{21} depress the players' expected utilities U_1,U_{11},U_{12},U_2,U_{21},U_{22}, all of which decrease after reaching $U_1 = U_{11} = U_{12} = U_2 = U_{21} = U_{22} = 0$ when $w = 2$. When $w > 2$, causing transition from Solution 1 to Solution 10, we assume that the players choose the equilibrium, where they both exert the $w = 2$ efforts $Z_{12} = D_{22} = 0.5$ and $Z_{11} = D_{21} = 1$ and receive zero expected utilities $U_1 = U_{11} = U_{12} = U_2 = U_{21} = U_{22} = 0$. Increasing the Period 2 contest intensity w is quite costly for equally matched (equally advantaged) players.

In Figure 3p,p′, when Player 1's zero-day appreciation factor δ_1 of stockpiled zero-day exploits S_1 from Period 1 to Period 2 increases above the benchmark $\delta_1 = 1$, causing transition from Solution 1 to Solution 2 in Table 1, Player 1 immediately utilizes its entire Period 1 budget $R_{11} = 1$, allocating $S_1 = \frac{R_{11}-b_{11}Z_{11}}{b_{11}} = 0.125$ to stockpiling, $Z_{11} = 0.875$ to the Period 1 attack, and $Z_{12} = 0.125$ to the Period 2 attack. Hence, Player 1 cuts the Period 2 attack in half, from the benchmark $Z_{12} = 0.25$ to $Z_{12} = 0.125$, utilizing stockpiling $S_1 = 0.125$ from Period 1 instead as δ_1 increases above $\delta_1 = 1$. As δ_1 increases above $\delta_1 = 1$, Player 1 keeps its stockpiling at $S_1 = 0.125$, as permitted by its budget constraint $R_{11} = 1$, but decreases its Period 2 attack Z_{12} linearly since stockpiling at S_1 gets multiplied with the increasing δ_1 (see $\delta_1 S_1$ in (5)). Player 1's expected utilities U_1 and U_2 increase, while its Period 1 expected utility is zero, $U_{11} = 0$, since its stockpiling S_1 gives a cost in Period 1 and a benefit in Period 2. Player 2's expected utilities U_2,U_{21},U_{22} remain at their benchmarks when $1 \le \delta_1 \le 2$ since Player 1's allocation from Z_{12} to S_1 is all that happens when $1 \le \delta_1 \le 2$. As δ_1 increases above $\delta_1 = 2$, Player 1's Period 2 attack Z_{12} decreases to $Z_{12} = 0$, as it gets entirely replaced by stockpiling S_1. That causes transition from Solution 2 to Solution 7 in Table 1. As δ_1 increases above $\delta_1 = 2$, Player 1 decreases its stockpiling S_1, $\lim_{\delta_1 \longrightarrow \infty} S_1 = 0$, which continues to impact Period 2 due to $\delta_1 S_1$ in (5). That enables Player 1 to increase its Period 1 attack Z_{11}, within its budget $R_{11} = 1$, $\lim_{\delta_1 \longrightarrow \infty} Z_{11} = 1$. Thus, Player 2 decreases its defense in both periods, $\lim_{\delta_1 \longrightarrow \infty} D_{21} = 0.66$, $\lim_{\delta_1 \longrightarrow \infty} D_{22} = 0.19$. Thus, Player 1's expected utilities U_1,U_{11},U_{12} increase concavely, $\lim_{\delta_1 \longrightarrow \infty} U_1 = 0.948$, $\lim_{\delta_1 \longrightarrow \infty} U_{11} = 0.203$, $\lim_{\delta_1 \longrightarrow \infty} U_{12} = 0.745$, while Player 2's expected utilities U_2,U_{21},U_{22} decrease convexly, $\lim_{\delta_1 \longrightarrow \infty} U_2 = 0.25$, $\lim_{\delta_1 \longrightarrow \infty} U_{21} = 0.13$, $\lim_{\delta_1 \longrightarrow \infty} U_{22} = 0.12$. In contrast, when δ_1 is less than 1, i.e., $0 \le \delta_1 \le 1$, which means depreciation, then Player 1 refrains from stockpiling, S_1. Hence, all variables are constant at their benchmark values as functions of δ_1 when $0 \le \delta_1 \le 1$.

In Figure 3q,q′, as Player 1's time discount factor β_1 decreases below the benchmark $\beta_1 = 1$, so that Player 1 assigns less weight to the future Period 2, Player 1 exerts decreasing efforts Z_{11} and Z_{12} in both periods, receiving decreasing expected utilities U_1 and U_{12} but increasing expected utility U_{11} in Period 1, which is more important than Period 2 for Player 1, while Player 2 assigns equal importance to both periods. As β_1 decreases, Player 2 exerts increasing defense efforts D_{12} and D_{22} in both periods, which eventually decrease slightly, causing inverse U-shapes as β_1 approaches $\beta_1 = 0$. As β_1 decreases, Player 2 becomes more competitive due to weighing both periods equally and receiving increasing expected utilities U_2,U_{21},U_{22}. When $\beta_1 < 1$, Player 1 assigns less weight to Period 2 than Period 1, causing zero stockpiling $S_1 = 0$.

In Figure 3r,r′, as Player 2's time discount factor β_2 decreases below the benchmark $\beta_2 = 1$, so that Player 2 assigns less weight to the future Period 2, Player 2 exerts decreasing defense efforts and D_{22} in both periods, receiving decreasing expected utilities U_2 and U_{22} but increasing expected utility U_{21} in Period 1, which is more important than Period 2 for Player 2, while Player 1 assigns equal importance to both periods. As β_2 decreases, Player 1 exerts increasing efforts Z_{11} and Z_{12} in both periods, becoming more competitive due to weighing both periods equally and receiving increasing expected utilities U_1,U_{11},U_{12}. As β_2 decreases below $\beta_2 = 0.80$, Player 1 reaches its budget constraint, which constricts its Period 1 effort $Z_{11} = R_{11}/b_{11} = 1$, causing a transition from Solution 1 to Solution 3.

5. Discussion

Table 2 presents the key findings from Section 4, including the three situations where Player 1 stockpiles in Panels e,e′, g,g′, and p,p′.

Table 2. Key findings from Section 4, including the three situations where Player 1 stockpiles in Panels e,e′, g,g′, and p,p′.

Panel	Parameter(s)	Key Findings
a,a′	R_{11}	As Player 1's available resources R_{11} in Period 1 decrease, its efforts in both periods decrease, while Player 2's efforts in both periods are inverse U-shaped. Player 2 transitions from being inferior when Player 1 is resourceful to being competitive when the players are equally matched and being superior when Player 1 lacks resources.
b,b′	a_{21}	As Player 2's unit effort cost a_{21} of defense in Period 1 increases, its efforts decrease, while Player 1's efforts are inverse U-shaped and resource-constrained. As a_{21} decreases, Player 2's Period 1 effort increases, while its Period 2 effort is inverse U-shaped, and Player 1's efforts decrease.
c,c′	$a_{21} = a_{22}$	As Player 2's unit defense costs $a_{21} = a_{22}$ in both periods increase (decrease), Player 2 becomes more disadvantaged (advantaged) than when only its unit effort cost a_{21} of defense in Period 1 increases (decreases).
d,d′	a_{22}	If Player 2 can choose, it prefers being disadvantaged in Period 2 with high unit effort cost a_{22}, when a less valuable asset is at stake, rather than being disadvantaged in the more important Period 1 with high unit effort cost a_{21}. Similarly, Player 2 prefers being advantaged in the more important Period 1 with low unit effort cost a_{21}, rather than being advantaged in Period 2 with high a_{22}.
e,e′	b_{11}	Player 1 may stockpile when its unit effort cost b_{11} of developing zero-day capabilities in Period 1 decreases, through three phases, below that of Period 2. First, Player 1 stockpiles as permitted by the budget and cuts back on the Period 2 effort. Second, Player 1 utilizes its entire budget in Period 1 without stockpiling, to exploit its advantage competitively over Player 2. Third, Player 1 eventually does not need to utilize its entire budget, attacks optimally in Period 1, and stockpiles sufficiently in Period 1 to deter Player 2 from defending in Period 2.
f,f′	$b_{11} = b_{12}$	As Player 1's unit effort costs $b_{11} = b_{12}$ of developing zero-day capabilities increase equally in both periods, Player 1 does not stockpile and becomes more disadvantaged than when only one unit effort cost increases. As $b_{11} = b_{12}$ decrease, Player 1 becomes more advantaged than when only one unit effort cost decreases.
g,g′	b_{12}	As Player 1's unit effort cost b_{12} of developing zero-day capabilities in Period 2 increases above that of Period 1, Player 1 stockpiles more to exploit the advantage of the cheaper unit effort cost in Period 1, decreases the efforts in both periods, and accepts negative expected utility in Period 1 to ensure higher expected utility in Period 2. This continues until Player 1 can no longer afford to exert effort in Period 2. Player 1 instead focuses on Period 1 and stockpiles optimally for Period 2, as permitted by the budget constraint.
h,h′	V_1	As Player 1's valuation V_1 of Player 2's asset increases, Player 1 exerts higher efforts and eventually becomes resource-constrained, while Player 2 exerts lower efforts. As V_1 decreases, Player 1 exerts lower efforts and Player 2's efforts are inverse U-shaped.
i,i′	V_2	As Player 2's valuation V_2 of its own asset increases, Player 2 exerts concavely increasing Period 1 defense effort and inverse U-shaped Period 2 effort, while Player 1's efforts decrease. As V_2 decreases, Player 2's efforts decrease, while Player 1's efforts are inverse U-shaped and resource-constrained.
j,j′	g_1	As Player 1's growth factor g_1 of asset V_1 from Period 1 to Period 2 increases, Player 1's efforts increase, subject to the resource constraint, while Player 2's efforts decrease. As V_1 decreases, Player 1's efforts decrease overall, while Player 2's efforts are inverse U-shaped.
k,k′	g_2	As Player 2's growth factor g_2 of asset V_2 from Period 1 to Period 2 increases, Player 2's Period 1 effort increases, its Period 2 effort is inverse U-shaped, and Player 1's efforts decrease. As V_2 decreases, Player 2's efforts decrease overall, while Player 1's efforts are inverse U-shaped and resource-constrained.
l,l′	W_1	As Player 1's valuation W_1 of Player 2's asset, acquired in Period 2, increases, Player 1's efforts increase, subject to the budget constraint, while Player 2's efforts decrease.
m,m′	W_2	As Player 2's valuation W_2 of its own asset acquired in Period 2 increases, Player 2's efforts increase concavely, while Player 1's efforts decrease convexly.
n,n′	v	As the contest intensity v in Period 1 increases, both players' Period 1 efforts increase due to more fierce competition, until Player 1 reaches its budget constraint, after which Player 2 benefits. As v decreases, both players' Period 1 efforts decrease, causing higher expected utilities.
o,o′	w	As the contest intensity w in Period 2 increases, both players' efforts in both periods increase until the fiercer competition causes zero expected utilities to both players, assuming they are equally matched.
p,p′	δ_1	As Player 1's zero-day appreciation factor δ_1 of stockpiled zero-day exploits from Period 1 to Period 2 increases above one, Player 1 immediately utilizes its entire Period 1 budget to attack and stockpile, cutting back on its Period 2 attack. This continues until Player 1's stockpiling is so large that the Period 2 attack is no longer cost effective. Thereafter, Player 1 decreases its stockpiling (due to its appreciation) and increases its Period 1 attack, while Player 2 decreases its defense in both periods.
q,q′	β_1	As Player 1's time discount factor β_1 decreases, so that Player 1 assigns less weight to the future Period 2, Player 1's efforts decrease, causing lower expected utilities, while Player 2's efforts increase overall, causing higher expected utilities.
r,r′	β_2	As Player 2's time discount factor β_2 decreases, so that Player 2 assigns less weight to the future Period 2, Player 2's efforts decrease, causing lower expected utilities, while Player 1's efforts increase, subject to the budget constraint, causing higher expected utilities.

6. Conclusions

The article presents a two-player two-period game between players producing zero-day exploits for immediate deployment in Period 1 or stockpiles for future deployment in Period 2. In Period 2, Player 1 produces zero-day exploits for immediate deployment, supplemented by stockpiled zero-day exploits from Period 1. Player 2 defends its asset against the attack in both periods. The analysis implies 11 solutions, where Player 1 may or may not stockpile, may or may not utilize its entire budget, may or may not attack in Period 2, and may or may not deter Player 2 from defending in Period 2. Relative to a benchmark solution with no stockpiling, 18 parameter values are altered to understand the nature of the zero-day phenomenon over two periods. Both players strike balances between how to exert efforts over the two periods, while Player 1 additionally decides whether to stockpile.

Player 1 may stockpile in three situations. First, as Player 1's unit effort cost of developing zero-day capabilities in Period 1 decreases below that of Period 2, it may exploit the Period 1 advantage for stockpiling and deployment in Period 2. Second, when Player 1's unit effort cost of developing zero-day capabilities in Period 2 increases above that of Period 1, it may similarly exploit the Period 1 advantage for stockpiling, potentially even accepting negative expected utility in Period 1 in order to benefit from subsequent deployment in Period 2. Third, when Player 1's zero-day appreciation factor of stockpiled zero-day exploits from Period 1 to Period 2 increases above one, it stockpiles for utilization in Period 2 until no additional Period 2 attack is required.

When the contest intensity in Period 1 increases, the players compete more fiercely with each other in Period 1, receiving decreasing expected utilities, until Player 1 reaches its budget constraint. Thereafter, Player 2 competes more fiercely, and both players receive negative Period 1 expected utilities. This continues until Player 1 receives negative expected utility over both periods, causing it to withdraw, while Player 2 keeps its asset. When the contest intensity in Period 2 increases, all efforts increase until both players receive zero expected utilities, assuming that they are equally advantaged.

If a player's time discount factor decreases, the player exerts lower efforts in both periods and receives lower expected utilities except in Period 1. The other player exerts higher efforts overall. The model confirms many intuitive results. For example, a player exerts more effort if it is cheaper, if it values the asset more, if the asset has a higher growth factor, and if the asset added in Period 2 is more valuable. If a player's unit effort costs increase (decrease) equally as much in both periods, the player becomes more disadvantaged (advantaged) than if the unit effort cost in only one period increases (decreases). The phenomenon of inversely U-shaped efforts is documented extensively. Typically, a player competes most fiercely when equally advantaged compared with the other player and decreases its efforts due to cost-effectiveness when too advantaged (due to superiority) or too disadvantaged (due to inferiority).

Future research should include more players, outside interference from governments and nongovernment bodies, regulation, and supervision and account for technological developments of the various aspects of zero-day exploits. The parameter values should be estimated by considering zero-day attacks that have occurred. Empirical support should be provided from contemporary and historical records. More complexity and more than two time periods may also be incorporated.

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Nomenclature

Parameters

R_{11}	Player 1's cyber resources in Period 1, $R_{11} \geq 0$
a_{2j}	Player 2's unit effort cost of defense in Period j, $j = 1, 2$, $a_{2j} \geq 0$
b_{1j}	Player 1's unit effort cost of developing zero-day capabilities in Period j, $j = 1, 2$, $b_{1j} \geq 0$
V_i	Player i's valuation of Player 2's asset, $V_i \geq 0$
g_i	Growth factor of asset V_i from Period 1 to Period 2, $g_i \geq 0$
W_i	Player i's valuation of Player 2's asset acquired in Period 2, $W_i \geq 0$
v	Contest intensity in Period 1, $v \geq 0$
w	Contest intensity in Period 2, $w \geq 0$

δ_1	Player 1's zero-day appreciation factor of stockpiled zero-day exploits S_1 from Period 1 to Period 2, $\delta_1 \geq 0$
β_i	Player i's time discount factor, $0 \leq \beta_i \leq 1$
Strategic Choice Variables	
Z_{11}	Player 1's effort to develop zero-day capabilities in Period 1, $Z_{11} \geq 0$
D_{21}	Player 2's defense effort in Period 1, $D_{21} \geq 0$
Z_{12}	Player 1's effort to develop zero-day capabilities in Period 2, $Z_{12} \geq 0$
D_{22}	Player 2's defense effort in Period 2, $D_{22} \geq 0$
Dependent Variables	
S_1	Player 1's stockpiling of zero-day exploits in Period 1 for use in Period 2, $S_1 \geq 0$
p_{ij}	Player i's expected contest success in Period j, $i, j = 1, 2$, $0 \leq p_{ij} \leq 1$
U_{ij}	Player i's expected utility in Period j, $i, j = 1, 2$
U_i	Player i's expected utility over both time periods, $i = 1, 2$
$R_{11b} = b_{11}Z_{11} + b_{11}S_1 \leq R_{11}$	The actual amount of resources used by Player 1 in Period 1

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Conventionalists, Pioneers and Criminals Choosing Between a National Currency and a Global Currency

Guizhou Wang
Faculty of Science and Technology
University of Stavanger
guizhou.wang@uis.no

Kjell Hausken[1]
Faculty of Science and Technology
University of Stavanger
kjell.hausken@uis.no
https://orcid.org/0000-0001-7319-3876

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ABSTRACT

The article analyzes how conventionalists, pioneers and criminals choose between a national currency (e.g. a central bank digital currency) and a global currency (e.g. a cryptocurrency such as Bitcoin) that both have specific characteristics in an economy. Conventionalists favor what is traditional and historically common. They tend to prefer the national currency. Pioneers (early adopters) tend to break away from tradition, and criminals prefer not to get caught. They both tend to prefer the global currency. Each player has a Cobb-Douglas utility with one output elasticity for each of the two currencies, comprised of backing, convenience, confidentiality, transaction efficiency, financial stability, and security. The replicator equation is used to illustrate the evolution of the fractions of the three kinds of players through time, and how they choose among the two currencies. Each player's expected utility is inverse U-shaped in the volume fraction of transactions in each currency, skewed towards the national currency for conventionalists, and towards the global currency for pioneers and criminals. Conventionalists on the one hand typically compete against pioneers and criminals on the other hand. Fifteen parameter values are altered to illustrate sensitivity. For parameter values where conventionalists go extinct, pioneers and criminals compete directly with each other. Players choose volume fractions of each currency and which kind of player to be. Conventionalists go extinct when criminals gain more from criminal behavior, and when the parameter values in the conventionalists' expected utility are unfavorable, causing competition between pioneers and criminals.

JEL Classification: C60; E50

Keywords: Bitcoin, digital currencies, currency competition, money, evolution, replicator dynamics, cryptocurrencies, central bank digital currencies.

[1] Corresponding author: Kjell Hausken – Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway, kjell.hausken@uis.no, Tel.: +47 51831632, Fax: +47 51831550.

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1. INTRODUCTION

1.1. Background

This article considers a national currency operational within a country, and a global currency operational within the same country and also outside the country. We do not model the characteristics of more than one country, but do model the characteristics of the global currency assumed operational beyond the country under analysis. We require the two currencies to operate as media of exchange (means of payment). We do not specify whether the two currencies are non-digital or digital, paper currencies combined with physical coins, etc. The comparison of a national currency and a global currency has become more relevant with the emergence of digital currencies. At the time of writing this article most countries still allow paper currencies. For some countries most transactions are digital, conducted e.g. through debit and credit cards, electronic funds transfers, etc. We expect currencies to become increasingly digital in the future, to transform the financial system in ways that are still unclear, but with more competitors. Most central banks are in the process of launching CBDCs (central bank digital currencies), e.g. the People's Bank of China, the European Central Bank, the Bank of England, and the US Federal Reserve. The transformation is partly impacted by the emergence of blockchain technology and the cryptocurrency Bitcoin, with a genesis block mined[2] on January 3, 2009 at 18:15:05 UTC. Bitcoin is increasingly considered to have value (Kelleher, 2021). On November 22, 2021, 14,641 cryptocurrencies contribute to a marketcap of $2.5 trillion. Among these, 1,039 are coins (not tokens) which are our main interest in this article (coinmarket.com).

When the global currency is conceptualized as a cryptocurrency such as Bitcoin, which allows 5–7 transactions per second, we account for the presence of Layer 2 solutions for scaling such as the lightning network where transactions are faster, less costly and more readily confirmed (Frankenfield, 2021).[3] The lightning network introduces off-ledger transactions, and disintermediates central institutions such as banks. The off-ledger transactions are updated on the main blockchain on the base Layer 1 only when two parties open and close a payment channel on the lightning network. Two examples of Bitcoin payments on the lightning network are the El Salvador Chivo wallet, which on October 16, 2021 recorded 24,076 remittance requests, which added up to $3,069,761.05 in one day (Sarkar, 2021), and Twitter tipping applying various third party operators such as the Strike Bitcoin lightning wallet service (Rodriguez, 2021). El Salvador's acceptance of Bitcoin as legal tender, and Tesla's on-and-off acceptance of Bitcoin for car payments (Zainab Hussain & Balu, 2021) means that goods and services in principle can be priced in Bitcoin. Hence, to the extent the global currency is a cryptocurrency combined with a Layer 2 solution, the global currency functions as a medium of exchange and a unit of account. It may also function as a store of value and a standard of deferred payments, which are beyond the scope of this article.

A plethora of different kinds of digital currencies emerge, tentatively classified into CBDCs, cryptocurrencies, digital currencies issued by private companies such as Meta's Diem, which is a stablecoin, digital currencies issued by political jurisdictions such as Miami's MiamiCoin, etc. As digital currencies become more common, these can be expected to compete with each other and with non-digital currencies. Hence it becomes relevant to assess which factors affect the market share of each currency over time, the implications of different market shares, and which

[2] Mining is how new Bitcoins enter circulation and how transactions are confirmed by the network on the blockchain ledger. Bitcoins are awarded through mining to the first computer to solve mathematical problems to verify blocks of transactions, applying hardware and energy known as "proof of work" (Hong, 2021).

[3] The Bitcoin base Layer 1 requires "proof of work" to ensure decentralization, which costs energy. See Willms (2021) regarding energy consumption. Bitcoin mining enables locating stranded energy sources, favorable technology, politically favorable jurisdictions, and financially favorable circumstances; grows its network optimally, and operates optimally through space and time. Layer 2 usually does not require proof, which causes more centralization.

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kinds of users apply the various currencies. Each currency's market share may depend on various factors such as backing, convenience, confidentiality, transaction efficiency, financial stability, and security, as perceived by users, contributors, regulators, governments, etc., and as elaborated upon in this article.

Competition between currencies implies different market shares for the various currencies. The implications of changes in the shares of the various currencies, from an economic point of view, are that the various actors involved in the various currencies benefit differently and incur different costs depending on the success of each currency. Examples of actors are currency producers, users, borrowers, lenders, stakers, and miners.

For example, central banks and their associated governments can expect to benefit from the success of CBDCs. Users may benefit if the CBDC is stable with low transaction costs, but may experience a cost if they value privacy and all their transactions get centrally recorded. The success of a cryptocurrency such as Bitcoin can be expected to benefit libertarians and actors preferring decentralized currencies less controlled by central actors, and not to benefit middlemen such as banks and others enabling, facilitating and negotiating transactions. The success of Meta's Diem can be expected to benefit Meta's stakeholders and users. The success of Miami's MiamiCoin can be expected to benefit Miami.

1.2. Contribution

This article considers an economy with a national currency and a global currency. The national currency offers the most common usage, such as buying goods, paying taxes, etc. A global currency may offer more limited usage, e.g. for buying goods and paying taxes, but may offer other opportunities such as tax evasion, user autonomy, etc. Three kinds of players are assumed, i.e. conventionalists, pioneers, and criminals. These are believed, first, to represent all societal players and, second, to have different preferences for the national currency and a global currency. Conventionalists favor what is traditional and historically common, which is often the national currency. Pioneers (early adopters) tend to depart from tradition and search for new ways of transacting, which may involve a global currency. Criminals search for currencies ensuring that they do not get detected and caught, which may also involve a global currency. Conventionalists typically compete against pioneers and criminals. When conditions for conventionalists are unfavorable causing their extinction, pioneers and criminals compete more directly with each other. All the three kinds of players can in principle choose some degree of criminal behavior, but criminals are assumed to have preferences explicitly focused on criminal behavior. The three groups are assumed to be mutually exclusive and jointly exhaustive to represent all possible kinds of market participants. If a player is empirically determined to fall somewhere between two kinds of players, a choice has to be made one way or the other. A player can over time choose to change from being of one kind to being of another kind.

Each player has a Cobb-Douglas utility with one output elasticity for each of the two currencies, split into backing, convenience, confidentiality, transaction efficiency, financial stability, and security, as perceived by the player. Factors such as usability and technological potential are assumed present in most of these six subelasticities, perhaps especially in convenience and transaction efficiency.[4] These six subelasticities are assumed to comprise the main concerns relevant for each player's preferences regarding which of two currencies to choose. Each player makes two strategic simultaneous choices to maximize its expected utility which is shown to be inverse U-shaped in the volume fraction of transactions in each currency. The first choice is the volume fraction of its transactions in each currency. This choice depends on what kind of player the player is, but does not depend on how many players exist of this player's kind, and hence does

[4] A factor such as investment profitability is more relevant for the function of a cryptocurrency as a store of value rather than a medium of exchange and a unit of account.

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not depend on time. Each player's second choice is which kind of player it should be at each point in time. Hence this second choice depends on time, through replicator dynamics.

Applying replicator dynamics, the research questions are how the volume fractions of the two currencies and the fractions of the three kinds of players evolve through time, and are sensitive to various characteristics. A further research question is to determine society's expected utility to account for welfare at the societal level. Scenarios are illustrated where the output elasticities and other characteristics cause some of the three kinds of players to become dominant or inferior over time. For the stationary solution after sufficiently much time has elapsed, sensitivity analysis is conducted to show how the fractions of the three kinds of players depend on variation in parameter values relative to a benchmark. Applying credible specific functional forms, an exact analytical solution is produced for the fraction of each player's transactions in the national currency , and replicator dynamics becomes applicable to determine the fractions of how the three kinds of players evolve.[5]

The world population is 7.9 billion, of which 74% is above 15 years old (Szmigiera, 2021) and 66.8% is above 20 years old (Ang, 2021). Assume that 69.7% is above 18 years old, i.e. 5.5 billion. The World Bank (2017) estimates that 1.7 billion adults lack a bank account, which is subtracted from 5.5 billion to give 3.8 billion adults with a bank account. Howarth (2021) estimates 300 million cryptocurrency users on October 25, 2021, i.e. 5.5% of adults and 7.9% of adults with a bank account. The authors expect these percentages to increase in the future. Without knowing which digital currencies may succeed as global currencies, the authors believe that players may increasingly sort themselves into conventionalists, pioneers, and criminals.

1.3. Literature

Limited literature exists on this topic. The following literature review is intended to cover and extend beyond this article's topic, usefully divided into four groups as an overview, i.e. competition between fiat currencies and cryptocurrencies, CBDC and cryptocurrencies, the cryptocurrency market, and game theoretic analyses.

1.3.1. Competition between fiat currencies and cryptocurrencies

The following articles that have been identified are the closest relative to the current article and somehow consider competition between fiat currencies and cryptocurrencies, with various implications. Schilling and Uhlig (2019) enable agents to choose between two kinds of currencies, i.e. a cryptocurrency and a fiat currency. They explore how asymmetry in transaction costs and exchange fees decreases currency substitution. This exploration corresponds to the generally different transaction efficiencies considered for the national and global currencies in the current article. For payments of certain goods, cryptocurrencies are more suitable or cost less than fiat money, due to censorship resistance, tax evasion and anonymity. However, exchanging cryptocurrencies to fiat money is costly, and some goods are more easily purchased using fiat money. The condition under which agents are indifferent between purchasing with Bitcoin or US dollars depends on the amount of the value-added tax and transaction fees to miners. These assessments correspond to some extent to different backing, convenience, confidentiality, financial stability, and security for the national and global currencies in the current article.

Fernández-Villaverde and Sanches (2019) build a model of competition among privately issued fiat currencies. Based on the Lagos-Wright environment, they identify a price stable equilibrium for multiple currencies, comparable to two coexisting currencies in the current article,

[5] In return for sacrificing generality, a successful specification through functional forms demonstrates internal consistency and is illuminating. For example, the Cobb-Douglas function has enhanced our understanding of consumer preferences. Functional forms facilitate determining ranges of parameter values within which solutions are possible.

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and various less desirable equilibria. In the current article society's expected utility is a weighted sum, by the fraction of players of each kind, of each player's expected utility.

Almosova (2018) extends her model by assuming that the circulation of private currencies involves costs, i.e. verification of transactions, mining costs, etc. She points out that cryptocurrency competition will not cause price stability. But when the costs of private currency circulation are sufficiently low, competition will impose a downward pressure on the inflation of the public currency.

Rahman (2018) applies the Friedman rule to investigate the implications of digital and fiat currency competition for monetary policy. He finds that a monetary equilibrium with a purely private arrangement of digital currencies cannot deliver a socially efficient allocation. Rahman's (2018) article is linked to the current article, which considers society's expected utility as a weighted sum of the three kinds of players' expected utilities.

Benigno, Schilling, and Uhlig (2019) consider a two-country economy with complete markets, two national currencies and a global cryptocurrency. They propose that the deviation from interest rate equality implies the risk of approaching the zero lower bound or the abandonment of the national currency, which they call Crypto-Enforced Monetary Policy Synchronization (CEMPS). Consequently, the impossibility of simultaneously ensuring a fixed exchange rate, free capital flows and an independent monetary policy (the classic Impossible Trinity) becomes even less reconcilable.

Verdier (2021) examines how issuing a digital currency impacts competition in the deposit and lending markets. She assumes that a digital currency can be issued or managed by a central bank, a regulated entity, or a non-bank operator, and that a digital currency issued by a non-bank operator does not enable offering loans to individuals. This assumption gradually seems ready for revision as decentralized finance increasingly allows loans, e.g. of cryptocurrencies, to individuals. Verdier (2021) assumes that depositors decide how much money to store in a bank account or in a digital currency account. Thus, issuing a digital currency generates a crowding-out effect on commercial deposits. The author concludes that the lending rate of banks increases when a digital currency crowds out a higher amount of bank deposits.

1.3.2. CBDCs and cryptocurrencies

The following articles that have been identified are the closest relative to the current article and compare CBDCs and cryptocurrencies, where we interpret CBDC as the national currency and cryptocurrencies as the global currency. Caginalp and Caginalp (2019) determine Nash equilibria for how players divide their assets between a home currency and a cryptocurrency, similarly to the focus in the current article. Additionally they assume that the government seizes fractions of the players' assets with certain probabilities.

Blakstad and Allen (2018) review opportunities for central banks and individuals presented by cryptocurrencies for central banks and individuals, together with the risks. They assess possible impacts on financial systems and structures which may challenge CBDC issuance.

Masciandaro (2018) proposes a function of a store of information for cryptocurrencies and central bank digital currencies as new media of payments emerge over the next years, supplementing a medium of exchange and a store of value. Thus, the evolution of the different media of payments may depend on individual preferences.

Benigno (2021) points out that the presence of multiple currencies can jeopardize the primary function of central banking. In addition, in a world of multiple competing currencies issued by profit-maximizing agents, the nominal interest rate and inflation are both determined by structural factors, i.e. the intertemporal discount factor, the exit rate and the fixed cost of entry, and are thus not subject to manipulation.

Asimakopoulos, Lorusso, and Ravazzolo (2019) present a Dynamic Stochastic General Equilibrium (DSGE) model to evaluate the economic repercussions of cryptocurrencies. They

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estimate the model with Bayesian techniques. They document a sturdy substitution effect between the real balances of government currency and cryptocurrencies, in response to technology, preferences and monetary policy shocks. Similarly, the current article shows how the three kinds of players strike balances between the two currencies.

1.3.3. The cryptocurrency market

The following articles analyze multiple currencies in the cryptocurrency market, which relates to the current article since the two currencies may also be two cryptocurrencies which evolve over time with fluctuating volume fractions of transactions. ElBahrawy, Alessandretti, Kandler, Pastor-Satorras, and Baronchelli (2017) assess the evolutionary dynamics of the cryptocurrency market. They illustrate the fluctuating market shares of 1,469 cryptocurrencies between April 2013 and May 2017, akin to fluctuations.

Caporale, Gil-Alana, and Plastun (2018) implement a rescaled range analysis and a fractional integration method to analyze the persistence in the cryptocurrency market. They identify a positive correlation between cryptocurrencies' past and future values.

ElBahrawy, Alessandretti, and Baronchelli (2019) investigate the relationship between online attention to digital currencies on Wikipedia and market dynamics across multiple digital currencies.

White (2014) points out, based on empirical observation, that as a first-mover monopolist in the market for cryptocurrencies, Bitcoin is surrounded by effective competitors. The introduction of various altcoins, if successful, decreases Bitcoin's market share. The current article similarly shows how the market share of two currencies may change over time.

Sapkota and Grobys (2021) analyze the top ten cryptocurrencies ranked by market capitalization in 2016–2018. They find that the submarket equilibria of privacy coins and the submarket equilibria of non-privacy coins are unrelated. This contrasts with the current article where players strike balances between which currencies to choose, and what kind of player to be.

Milunovich (2018) applies Granger causality tests to five popular cryptocurrencies and six major asset classes. He estimates weak connectedness between the two groups and strong connectedness within each group. A few exceptions exist. Out of 80 cross-pairs, six statistically significant relations are shown from non-digital to digital assets (e.g. from Monero to US$), and two statistically significant relations are shown from digital to non-digital assets (e.g. from the SPGSCI commodity index to Litecoin).

Gandal and Halaburda (2016) explore how network effects impact competition in the cryptocurrency market. They identify no winner-take-all effects in the early stages, but strong network effects and winner-take-all dynamics more recently. Similarly, the current article shows how two currencies and three kinds of players may coexist, and also that one kind of players, e.g. conventionalists, may go extinct.

1.3.4. Game theoretic analyses

The following articles are game theoretic analyses, which are linked to this group since the three kinds of players, while choosing among two currencies, interact with each other through time modeled by game theory and replicator dynamics. Imhof and Nowak (2006) propose that a frequency dependent, stochastic Wright-Fisher process can be used to describe the evolutionary game dynamics in finite populations to determine which of two strategies survives. This article similarly determines how the fractions of the three kinds of players, and the volume fraction of transactions in each currency, evolve over time.

Lewenberg, Bachrach, Sompolinsky, Zohar, and Rosenschein (2015) develop a cooperative game theoretic model to explore the dynamics of pooled Bitcoin mining and rewards. They show that it is difficult or even impossible to distribute rewards in a stable way. Players are always

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incentivized to switch between pools. This is partly linked to the current article where players switch between which of three kinds of players to be, and which volume fraction of transactions in each currency to choose.

1.4. Article Organization

Section 2 presents the model. Section 3 analyzes the model. Section 4 explains the implications of the results. Section 5 concludes.

2. THE MODEL

2.1. Nomenclature

Parameters

j Currency of kind j, $j = n, g$
n National currency
g Global currency
i Player of kind i, $i = x, y, z$
x Conventionalist player
y Pioneer player
z Criminal player
b_{ij} Output subelasticity for backing of currency j at time t as perceived by player i, $b_{ij} \geq 0$
c_{ij} Output subelasticity for convenience of currency j at time t as perceived by player i, $c_{ij} \geq 0$
d_{ij} Output subelasticity for confidentiality of currency j at time t as perceived by player i, $d_{ij} \geq 0$
e_{ij} Output subelasticity for transactional efficiency for currency j at time t as perceived by player i, $e_{ij} \geq 0$
f_{ij} Output subelasticity for financial stability of currency j at time t as perceived by player i, $f_{ij} \geq 0$
s_{ij} Output subelasticity for security of currency j at time t as perceived by player i, $s_{ij} \geq 0$
w_i Fraction of player i's transactions which is criminal, $0 \leq w_i \leq 1$
k_i Scaling exponent for what player i retains after criminal behavior, $k_i \geq 0$
ω_i Probability that the government detects and prosecutes player i's criminal behavior, $0 \leq \omega_i \leq 1$
m_i Scaling exponent for how player i gets increased/decreased expected utility, $-\infty \leq m_i \leq \infty$
μ_i Scaling proportionality parameter for how player i gets increased expected utility, $\mu_i \geq 0$
α_i Parameter for the rapidity of change or sensitivity of the replicator equation, $\alpha_i > 0$
t Time, $t \geq 0$

Free choice variables

p_i Volume fraction of player i's transactions in currency n, $0 \leq p_i \leq 1$, $i = x, y, z$
$1-p_i$ Volume fraction of player i's transactions in currency g, $0 \leq 1 - p_i \leq 1$
p Volume fraction of all players' transactions in currency n, $0 \leq p \leq 1$
$1-p$ Volume fraction of all players' transactions in currency g, $0 \leq 1 - p \leq 1$
q_i Fraction of players of kind i, $0 \leq q_i \leq 1$, $i = x, y, z$, $q_x + q_y + q_z = 1$
q_x Fraction of conventionalists
q_y Fraction of pioneers
q_z Fraction of criminals, $q_z = 1 - q_x - q_y$

Dependent variables

$U_i(p_i, q_i)$ Player i's expected utility, $i = x, y, z$
U Society's expected utility

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2.2. Two Currencies *n* and *g*

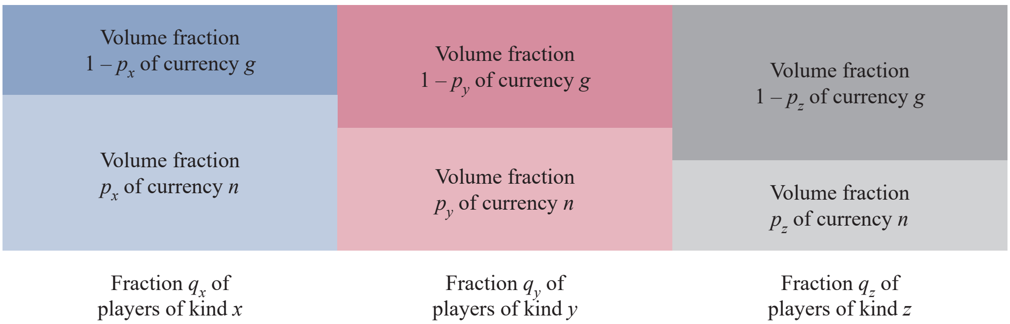
Consider an economy with two available currencies. The first currency n is national and offers the most common usage, and especially legal usage, within the economy. Examples of usage are to make various purchases or pay taxes. For simplicity, we can think of this currency as a CBDC (central bank digital currency). The second currency g is a global currency which on the one hand offers more limited usage (e.g. cannot be used for all kinds of purchases), but on the other hand offers other opportunities, e.g. tax evasion, payment on the black market, user autonomy, discretion, peer-to-peer focus, no banking fees, low transaction fees. For simplicity, we can think of this currency as a cryptocurrency such as Bitcoin or Monero, a privately issued currency such as Meta's Diem, or some future hypothetical currency operating globally.

2.3. Three Kinds of Players *x*, *y*, *z*

Assume three kinds of players which we can think of as households, referred to as player i, $i = x, y, z$. We can think of the three kinds of players as conventionalists, pioneers and criminals, respectively. Conventionalists tend to do what is traditional and historically common, and tend to prefer the national currency n more than the global currency g. Pioneers (early adopters) tend to break away from tradition and prefer the global currency g more than the national currency n. Criminals prefer not to get caught and tend to prefer the global currency g more than the national currency n if the global currency g offers confidentiality and user autonomy, e.g. through a privacy coin such as Monero. Assume that q_i, $0 \le q_i \le 1$ is the fraction of players of kind i. We assume that q_x is the fraction of conventionalists, that q_y is the fraction of pioneers, and that $q_z = 1 - q_x - q_y$ is the fraction of criminals. As time progresses, what used to be conventional may become old-fashioned, and what pioneers do may become conventional. Hence q_x and q_y may change over time. All players of the same kind i are equivalent. Player i (i.e. player of kind i) conducts a volume fraction p_i, $0 \le p_i \le 1$ of its transactions in currency n, and the remaining volume fraction $1 - p_i$ of its transactions in currency g, as shown in Figure 1 which assumes $p_x > p_y > p_z$, but generally $0 \le p_i \le 1$, $i = x, y, z$.

Figure 1

Three kinds of players. Player i (i.e. player of kind i), $i = x, y, z$, conducts a volume fraction p_i of its transactions in currency n, and the remaining volume fraction $1 - p_i$ of its transactions in currency g, $0 \le q_i \le 1$, $q_x + q_y + q_z = 1$. The illustration assumes $p_x > p_y > p_z$, but generally $0 \le p_i \le 1$, $i = x, y, z$.



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2.4. Volume Fraction p of All Players' Transactions in Currency n

The volume fraction p of all players' transactions in currency n is the weighted sum of each player i's volume fraction in currency n, weighted by the fraction of each kind of player i, $i = x, y, z$, i.e.

$$p = \sum_{i = x,y,z} p_i q_i. \tag{1}$$

2.5. Cobb-Douglas Utility With Two Output Elasticities

Assume that player i has a risk-neutral Cobb-Douglas utility in net terms, hereafter referred to as utility, described by

$$U_{iCD}(p_i) = p_i^{b_{in}+c_{in}+d_{in}+e_{in}+f_{in}+s_{in}}(1-p_i)^{b_{ig}+c_{ig}+d_{ig}+e_{ig}+f_{ig}+s_{ig}} \tag{2}$$

with one output elasticity $b_{in} + c_{in} + d_{in} + e_{in} + f_{in} + s_{in}$ for the national currency n, and one corresponding output elasticity $b_{ig} + c_{ig} + d_{ig} + e_{ig} + f_{ig} + s_{ig}$ for the global currency g. Player i's Cobb-Douglas utility $U_{iCD}(p_i)$ in (2) is multiplied with a penalty described in the next section 2.6 if player i's criminal behavior is detected and prosecuted by the government, and multiplied with the impact of the fractions q_x, q_y, q_z of the three kinds of players in the subsequent section 2.7. When $S = b_{in} + c_{in} + d_{in} + e_{in} + f_{in} + s_{in} + b_{ig} + c_{ig} + d_{ig} + e_{ig} + f_{ig} + s_{ig} = 1$, $S > 1$, $S < 1$, (2) expresses constant, increasing, and decreasing returns to scale, respectively. The 12 output subelasticities a_{ij}, $a_{ij} = b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, s_{ij}$ in (2), for currency j, $j = n, g$, at time t as perceived by player i, $i = x, y, z$, are as follows:

First, b_{ij} expresses how currency j has various forms of backing from actors, systems or characteristics that users of currency j respect and trust, as perceived by player i. Examples of backing for currency j are central banks for CBDCs, and various decentralized characteristics such as a distributed ledger technology for cryptocurrencies. The variable b_{ij} is not objective, but depends on player i's subjective judgment. The parameter b_{ij} expresses the weighted average backing of currency j by its users, i.e. within each of the three kinds x, y, z of players. For example, legitimate lawful users preferring transparency and allegiance to a certain country, may back the CBDC (central bank digital currency) of that country, which may be currency n, whereas illegitimate users may not back that currency, but back the global currency g instead. Criminal users may, for example, back a privacy cryptocurrency such as Monero, which may also be backed by many legitimate users. Currently, after the gold standard collapse (June 5, 1933 in the US), no fiat currency is backed by gold. The extent to which a player backs currency j may depend on a variety of factors. For example, a central bank may back its CBDC in the hope of obtaining a broader tax base, reduced tax evasion, a backstop to the private sector which may fail, and enhanced financial inclusion.

Second, c_{ij} expresses the convenience of using currency j as perceived by player i. One example of convenience is ease of use, e.g. few and easily comprehensible operations when purchasing at the supermarket or online, when transferring funds nationally or globally, or when incurring and paying back a loan. Other or related examples are how electronic wallets operate, how transfers between one's own and other wallets operate, and how offline transactions are processed when offline and getting back online. Furthermore, for some digital currencies users may not need to open a bank account with required identifications, but may instead install a digital currency wallet, and transact and pay via a digital currency address.

Third, d_{ij} expresses the confidentiality of using currency j, as perceived by player i, which expresses well-known balances to be struck between privacy, availability or accessibility for

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oneself and various other players, and discrimination. For example, privacy cryptocurrencies such as Monero, Dash, and Zcash[6] offer enhanced privacy for users since transactions are harder to track, which also may make it harder to rectify, correct, or reverse undesirable transactions. For example, paying ransom money in Monero may preserve the anonymity of the recipient and the provider, but may make it harder for law enforcement to reverse or prosecute the transaction. A CBDC, properly designed, may offer confidentiality for player i with respect to many other players if the central bank can be trusted, but may not offer confidentiality for player i if the central bank cannot be trusted, or a court orders the confidentiality to be broken. The output subelasticity d_{ij} thus also expresses discrimination regarding in what sense and for whom and towards whom confidentiality is honored.

Fourth, e_{ij} expresses the transaction efficiency of currency j, as perceived by player i, operationalized as low cost, fast speed, affordability, and finality. Fast speed refers to how quickly the transaction is executed, which for cryptocurrencies is impacted by how many confirmations are needed for execution and how quickly the miners can mine blocks. Wire transfers have historically had a certain speed, and may be held up over weekends. Affordability refers to a fee or cost of executing the transaction, which is usually positively correlated with how quickly the transaction is executed. Finality refers to the extent to which the transaction is final, or can somehow be reversed or negotiated. Cryptocurrency transactions are usually irreversible, which is the common logic of smart contracts on the blockchain. Non-cryptocurrency transactions, exemplified by traditional wire transfers are usually reversible, e.g. if a court of law determines that the transaction was illegal. Costs of transactions have historically varied substantially across different kinds of transactions. Affordability may depend on size, recipient, sender, whether the transaction is recurring, etc. Costs may range from the common no costs, e.g. for grocery purchases, to high costs for international money transfers. Costs of transacting cryptocurrencies have usually been low, and often beneficial when transacting high amounts, with variation across different cryptocurrencies. Speed of transfers also vary. At the time of writing, the speed of CBDC transactions is unknown. For Bitcoin the average time for mining one block is 10 minutes. For two confirmations, the transaction may take 20 minutes. The initiator of a cryptocurrency transaction is usually requested to specify a transaction fee (e.g., low, medium, high), which impacts how quickly it gets processed by the miners. For Ethereum the average time for mining one block is 10–15 seconds, which may cause one transaction after two confirmations to require 20–30 seconds. In 2019 Bitcoin processes ca 4.6 transactions per second, while Visa processes ca 1700 transactions per second. The lightning network may speed up the transaction time for Bitcoin. Credit card transactions typically require around 48 hours to settle. The finality of transactions also pertains to efficiency. Some cryptocurrency exchanges may require three confirmations, six confirmations for large transactions, and 60 confirmations for very large transactions. Different central banks may develop different procedures for finality and confirmations depending on the characteristics of transactions, senders, recipients, etc., which impacts the efficiency e_{ij}.

Fifth, f_{ij} expresses the financial stability of currency j, as perceived by player i. The financial stability of the national currency n depends on the conditions in the given country. A variety of indicators exist for the financial stability of countries and currencies. Some currencies such as the Swiss franc, the Japanese yen, and the Norwegian krone are relatively stable (Protska, 2021b), while some, such as the Venezuelan bolivar, the Iranian ria and the Vietnamese dong (Protska, 2021a) can be more unstable than many cryptocurrencies. For CBDCs the central bank adjusts interest rates (which can be negative for digital currencies), and can be expected to be able to adjust a variety of factors to adjust the financial stability of currency j, within the constraints of the country's conditions. One hypothetical possibility is to adjust the tax rate for households or individuals depending on their characteristics (e.g. in understanding with tax authorities and

[6] https://www.investopedia.com/tech/five-most-private-cryptocurrencies/, retrieved November 22, 2021.

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others) to ensure financial stability. Fast response time when faced with crises, and activities to curtail or prevent money laundering and terrorist financing may impact the financial stability of currency j. Most cryptocurrencies, and especially altcoins, have traditionally varied substantially in value, caused partly by their novelty and limited usage, but also by the absence of a governing authority. One exception is stablecoins, e.g. Tether, USD Coin, TrueUSD, Dai, Paxos Standard, Binance USD, which have the stated purpose of being stable in some sense. The top ten list of countries adopting Bitcoin typically contains countries in the western world, but also countries which struggle to ensure financial stability, e.g. Venezuela (Lanz, 2020).

Sixth, s_{ij} expresses the security of currency j, as perceived by player i. A variety of security possibilities exist for digital currencies, see e.g. Allen et al. (2020) and Kiff et al. (2020). The security of the blockchain supporting Bitcoin has not collapsed since the first block was mined on January 3, 2009 at 18:15:05, although controversies and forks have occurred. Considering that 7,594 cryptocurrencies exist (https://coinmarketcap.com), 51% attacks are relatively rare.[7]

Each of the two output elasticities consists of six summed subelasticities as expressed above. Each of the six output subelasticities for the national currency n is of the form $p_i^{a_{in}}$, where p_i is the volume fraction of player i's transactions in the national currency n. Each of the six corresponding output subelasticities for the global currency g is of the form $(1-p_i)^{a_{ig}}$, where $1-p_i$ is the volume fraction of player i's transactions in the global currency g. The parameter a_{ij}, $a_{ij} = b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, s_{ij}$ is the output subelasticity in the Cobb-Douglas function, $0 \le a_{ij} \le 1$, which is a characteristic of currency j, $j = n, g$, as perceived by player i. The output subelasticity a_{ij} may sometimes be objectively specified, and may occasionally be mutually agreed upon by the players x, y, z, allowing the removal of the subscript i from a_{ij}. Since objective specification, and mutual agreement, may not be generally possible, and player i may perceive the output subelasticity a_{ij} subjectively, we keep the subscript i on a_{ij}.

2.6. Detection and Prosecution of Criminal Behavior

Examples of criminal behavior are tax evasion, money laundering, theft, terrorist financing, corruption, and financial crimes. Although we expect criminals to be more criminal than conventionalists and pioneers, all these three kinds of players can in principle engage in criminal behavior, through both the national currency n and the global currency g. This reflects that in our societies no groups of citizens can be expected to be 100% non-criminal. We thus assume that a fraction w_i, $0 \le w_i \le 1$ of player i's transactions is criminal and is detected and prosecuted by the government with probability ω_i, $0 \le \omega_i \le 1$. The product $\omega_i w_i$ multiplies player i's fraction w_i of criminal behavior with its detection and prosecution probability ω_i. Hence $1 - \omega_i w_i$ expresses the joint probability of neither engaging in criminal behavior nor being detected and prosecuted. We introduce a scaling exponent k_i, $k_i \ge 0$, on the fraction w_i and express player i's expected utility as

$$U_{iC} = 1 - \omega_i w_i^{k_i} \tag{3}$$

which is a fraction between 0 and 1. When $k_i = 1$, player i's expected utility U_{iC} decreases linearly in the fraction w_i of player i's transactions which is criminal. When $k_i > 1$, U_{iC} decreases concavely in w_i, which economically means that a higher fraction w_i (compared with when $k_i = 1$) of player i's criminal transactions is needed in order to decrease player i's expected utility U_{iC}. In contrast, when $0 < k_i < 1$, U_{iC} decreases convexly in w_i, which economically means that a lower fraction w_i (compared with when $k_i = 1$) of player i's criminal transactions is sufficient in order to decrease

[7] The most well-known 51% attacks among cryptocurrencies occurred for Verge, Ethereum Classic, Bitcoin Gold, Feathercoin, and Vertcoin (Attah, 2019). A 51% attack means that a majority of miners impact mining to their advantage, including preventing other miners from completing blocks, and channeling funds from each block to themselves. Changing historical blocks is difficult due to the hard coding of past transactions into the Bitcoin software.

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player i's expected utility U_{iC}. When $k_i = 1$, $U_{iC} = 1 - \omega_i$ is independent of w_i. Player i's expected utility U_{iC} in (3) expresses what is probabilistically retained for potential criminal behavior, and is multiplied with player i's Cobb-Douglas utility $U_{iCD}(p_i)$ in (2) to determine what player i keeps of its utility when accounting for criminal behavior being probabilistically detected and prosecuted.

2.7. How a Fraction q_i of Players of Kind i Impacts Expected Utilities

Players of kind i may get increased or decreased expected utility if their fraction q_i increases or decreases. We operationalize this with the term $1+\mu_i q_i^{m_i}$, where μ_i, $\mu_i \geq 0$ is a scaling proportionality parameter, and m_i is a scaling exponent. The term $1+\mu_i q_i^{m_i}$ is multiplied with the Cobb-Douglas utility and what is probabilistically retained for potential criminal behavior.

Conventionalists prefer to do what others do and what is common, which gives them increased expected utility. Hence conventionalists get increased expected utility if the fraction q_x of conventionalists increases, i.e. $m_x \geq 0$. The positive exponent m_x scales the strength of how conventionalists get multiplicatively increased expected utility when the fraction q_x increases.

In contrast, pioneers prefer to do what others do not do, what is uncommon, and what breaks ground beyond what is conventional, which gives them increased expected utility. When pioneers become a majority, they are no longer pioneers, but conventionalists. Hence pioneers get decreased expected utility if the fraction q_y of pioneers increases, i.e. $m_y \leq 0$. The negative exponent m_y scales the strength of how pioneers get multiplicatively decreased expected utility when the fraction q_y increases.

Criminals focus on what is criminally lucrative, what they can get away with, and what does not get detected and prosecuted. Whether what they do is common or uncommon may be irrelevant. What criminals have in common with pioneers is that they prefer to be few so that they can operate under the radar. As criminals become more numerous, the benefits for each in most stable and relatively lawful societies can be expected to decrease since they compete with each other, and non-criminals adapt to defending against them. Exceptions, such as the Italian mafia in Italy, or the cartels in Colombia, operate according to another logic not considered in this article, where subsections of societies follow different norms. At the extreme, a society with only criminals will not function since everyone will prey on everyone causing breakdown. Hence criminals, just as pioneers, get decreased expected utility if the fraction q_z of criminals increases, i.e. $m_z \leq 0$. The negative exponent m_z scales the strength of how criminals get multiplicatively decreased expected utility when the fraction q_z increases.

The three paragraphs above enable us to operationalize player i's expected utility as

$$U_{iF}(q_i) = 1 + \mu_i q_i^{m_i} \tag{4}$$

which is multiplied with player i's Cobb-Douglas utility $U_{iCD}(p_i)$ in (2) and player i's expected utility U_{iC} in (3). When $m_i = 1$, player i's expected utility $U_{iF}(q_i)$ increases linearly in the fraction q_i of players of kind i. When $m_i > 1$, $U_{iF}(q_i)$ increases convexly in q_i, which economically means that a higher fraction q_i (compared with when $m_i = 1$) of players of kind i is needed in order to increase player i's expected utility $U_{iF}(q_i)$. In contrast, when $0 < m_i < 1$, $U_{iF}(q_i)$ increases concavely in q_i, which economically means that a lower fraction q_i (compared with when $m_i = 1$) of players of kind i is sufficient in order to increase player i's expected utility $U_{iF}(q_i)$. When $m_i = 0$, $U_{iF}(q_i) = 1 + \mu_i$ is independent of q_i.

Equation (4) means that player i's expected utility $U_{iF}(q_i)$ depends explicitly on the fraction q_i of players of kind i, $i = x, y, z$, which is a measure of the number of players of kind i. This dependence of $U_{iF}(q_i)$ on q_i implicitly means that $U_{iF}(q_i)$ depends on the fraction $1 - q_i$ of players which is not of kind i, since $q_x + q_y + q_z = 1$. That is, more players of one kind mean fewer players of the two other kinds. In the next section 3 on the replicator equation the interdependence of

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the numbers of players of each kind, and thus the interaction between the three kinds of players, becomes clearer.

2.8. The Players' Expected Utilities

This section combines multiplicatively player *i*'s expected utilities $U_{iCD}(p_i)$ in (2), U_{iC} in (3), and $U_{iF}(q_i)$ in (4), which gives player *i*'s expected utility

$$\begin{aligned} U_i &= U_i(p_i,q_i) = U_{iCD}(p_i)U_{iC}U_{iF}(q_i) \\ &= p_i^{b_{in}+c_{in}+d_{in}+e_{in}+f_{in}+s_{in}}(1-p_i)^{b_{ig}+c_{ig}+d_{ig}+e_{ig}+f_{ig}+s_{ig}}\left(1-\omega_i w_i^{k_i}\right)\left(1+\mu_i q_i^{m_i}\right). \end{aligned} \tag{5}$$

Equation (5) assumes that player *i* is risk neutral and abstracts away other factors such as player *i*'s consumption preferences concerning goods, and player *i*'s preference for work versus leisure, which are beyond the scope of this article. Such factors are to some extent implicitly or indirectly present in (5). For example, player *i*'s convenience c_{ij} of using currency *j* and transaction efficiency e_{ij} of currency *j* may play different roles for different goods, and may impact player *i*'s preference for work versus leisure.

2.9. Society's Expected Utility

Society's expected utility $U(p_x, p_y, p_z, q_x, q_y)$ is the weighted sum of each player's expected utility $U_i(p_i, q_i)$, weighted by the fraction of players of kind *i*, *i* = *x*, *y*, *z*, i.e.

$$U = U(p_x,p_y,p_z,q_x,q_y) = \sum_{i=x,y,z} q_i U_i(p_i,q_i),\ q_z = 1-q_x-q_y. \tag{6}$$

2.10. The Players' Strategic Choices

Assume that player *i* at time *t* makes two strategic simultaneous choices to maximize its expected utility $U_i(p_i, q_i)$ in (5). First, it chooses its volume fraction p_i of its transactions in currency *n*, causing the remaining volume fraction $1 - p_i$ of its transactions to be in currency *g*. Player *i*'s choice of p_i to maximize $U_i(p_i, q_i)$ in (5) does not depend on time *t*, and does not depend on the fraction q_i of player *i* in the population, since $1+\mu_i q_i^{m_i}$ appears proportionally in (5), without impacting the shape of $U_i(p_i, q_i)$ as a function of p_i, and without impacting which value of p_i causes $U_i(p_i, q_i)$ to have its maximum. Hence no dynamic considerations for player *i*'s choice of volume fraction p_i of its transactions in currency *n* are needed. Second, player *i* chooses which kind *i* of player it should be, *i* = *x*, *y*, *z*. That choice depends strongly on time *t*, as described by the replicator equation in the next section. When player *i* switches from being of one kind to another kind, *i* = *x*, *y*, *z*, its first choice of the optimal volume fraction p_i of its transactions in currency *n* also changes. In other words, as long as player *i* remains of a specific kind, its optimal volume fraction p_i does not depend on time *t*, which reflects real life, but if it switches to be of another kind according to the replicator equation described in the next section, then it also changes its optimal volume fraction p_i at time *t* to what is optimal for this new kind *i*, *i* = *x*, *y*, *z*.

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2.11. The Replicator Equation

To determine the evolution of the fraction q_i of players of kind i, $i = x, y, z$, we consider the replicator equation (Taylor & Jonker, 1978; Weibull, 1997)

$$\frac{\partial q_i}{\partial t} = \alpha_i q_i \left(U_i(p_i,q_i) - U(p_x,p_y,p_z,q_x,q_y)\right)$$
$$\Leftrightarrow \begin{bmatrix} \frac{\partial q_x}{\partial t} \\ \frac{\partial q_y}{\partial t} \end{bmatrix} = \begin{bmatrix} \alpha_x\left(U_x(p_x,q_x) - U(p_x,p_y,p_z,q_x,q_y)\right) & 0 \\ 0 & \alpha_y\left(U_y(p_y,q_y) - U(p_x,p_y,p_z,q_x,q_y)\right) \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} \tag{7}$$

where α_i, $\alpha_i > 0$, is the rapidity of change or sensitivity of the process. The process is stable when α_i is intermediate. If α_i is high, the process changes rapidly. If α_i is low, a negligible change occurs. The right hand side of (7) multiplies the fraction q_i of players of kind i with the difference $U_i(p_i, q_i) - U$ between player i's expected utility $U_i(p_i, q_i)$ and the average expected utility U of the three kinds $i = x, y, z$ of players. If the right hand side of (7) is positive (negative), player i's expected utility $U_i(p_i, q_i)$ is higher (lower) than the average expected utility U, which causes the fraction q_i of players of kind i to increase (decrease).

The economic interpretation of (7) is that the three kinds of players over time continuously move towards becoming the kind of player where the expected utility U_i, i.e. U_x, U_y, U_z, is highest. In doing so, player i accounts for both the income effect (i.e., the absolute value of player i's expected utility U_i) and the substitution effect (i.e., which kind of player is optimal for player i to be or become). As a player changes from being of one kind to becoming of another kind, the fraction q_i of players of kind i, i.e. the fractions q_x, q_y, $q_z = 1 - q_x - q_y$, change. The prominent presence of q_i in (7) on the left hand side, multiplicatively on the right hand side, and in $U_i(p_i, q_i)$ and $U(p_x, p_y, p_z, q_x, q_y)$, means that the replicator equation is quite sensitive to changes in q_i. The expected utilities $U_i(p_i, q_i)$ and $U(p_x, p_y, p_z, q_x, q_y)$ also depend on the volume fractions p_i and $1 - p_i$ of player i's transactions in the currencies n and g, respectively. Hence the replicator equation reflects how the three kinds of players perceive the two currencies n and g as they choose which kind of player they want to be to maximize their expected utility $U_i(p_i, q_i)$.

The limiting behavior (the evolutionary outcome) of the replicator equation in (7) is a Nash equilibrium. We determine a pure-strategy Nash equilibrium where each player i, $i = x, y, z$, maximizes its expected utility $U_i(p_i, q_i)$. This equilibrium is a set of strategies q_i^* for the three players, $i = x, y, z$, such that

$$U_i(p_i,q_i^*) \geq U_i(p_i,q_i) \forall 0 \leq q_i \leq 1,\ i = x, y, z;\ q_z = 1 - q_x - q_y. \tag{8}$$

For research on the equilibrium properties of replicator dynamics see (Duong & Han, 2020) and the references therein.

If $\alpha_i\left(U_i(p_i,q_i) - U(p_x,p_y,p_z,q_x,q_y)\right)$ in (7) had been constant, (7) would have been a linear time-invariant system for which well-known techniques illustrated by Khalil (2002, p. 46), or Laplace and Fourier transforms, are applicable. Since $\alpha_i\left(U_i(p_i,q_i) - U(p_x,p_y,p_z,q_x,q_y)\right)$ is not constant, (7) is a time-variant system which is more challenging to analyze theoretically. We thus proceed over to the next sections to analyze (7) with simulations.

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3. ANALYZING THE MODEL

3.1. Analyzing As a Function of p_i When q_i Is Exogenously Fixed

This section assumes that the fraction q_i of players of kind i is fixed, and analyzes how player i chooses its volume fraction p_i of currency n, implying volume fraction $1 - p_i$ for currency g. Differentiating player i's expected utility $U_i(p_i, q_i)$ in (5) with respect to p_i and equating with zero gives

$$\frac{\partial U_i(p_i,q_i)}{\partial p_i} = \left(\frac{b_{in}+c_{in}+d_{in}+e_{in}+f_{in}+s_{in}}{p_i} - \frac{b_{ig}+c_{ig}+d_{ig}+e_{ig}+f_{ig}+s_{ig}}{1-p_i}\right)p_i^{b_{in}+c_{in}+d_{in}+e_{in}+f_{in}+s_{in}}(1 - p_i)^{b_{ig}+c_{ig}+d_{ig}+e_{ig}+f_{ig}+s_{ig}}\left(1-\omega_i w_i^{k_i}\right)\left(1+\mu_i q_i^{m_i}\right) = 0 \tag{9}$$

which is solved to yield

$$p_i = p_{iopt} = \frac{b_{in}+c_{in}+d_{in}+e_{in}+f_{in}+s_{in}}{b_{in}+c_{in}+d_{in}+e_{in}+f_{in}+s_{in}+b_{ig}+c_{ig}+d_{ig}+e_{ig}+f_{ig}+s_{ig}}. \tag{10}$$

Property 1. $\partial p_{iopt}/\partial a_{in} \geq 0$, $\partial p_{iopt}/\partial a_{ig} \leq 0$, $a_{ij} = b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, s_{ij}, j = n, g$.

Proof. Follows from differentiating (10).

Property 1 states that the optimal fraction p_{iopt} of player i's transactions in currency n increases in the six subelasticities a_{in} for currency n, and decreases in the six subelasticities a_{ig} for currency g.

Inserting $p_i = p_{iopt}$ into the second order derivative gives

$$\left.\frac{\partial^2 U_i(p_i,q_i)}{\partial p_i^2}\right|_{p_i=p_{iopt}} = -\left(b_{ig}+c_{ig}+d_{ig}+e_{ig}+f_{ig}+s_{ig}\right)p_{iopt}^{b_{in}+c_{in}+d_{in}+e_{in}+f_{in}+s_{in}-1}(1 - p_{iopt})^{b_{ig}+c_{ig}+d_{ig}+e_{ig}+f_{ig}+s_{ig}-2}\left(1-\omega_i w_i^{k_i}\right)\left(1+\mu_i q_i^{m_i}\right) < 0 \tag{11}$$

which is satisfied as negative, and hence $p_i = p_{iopt}$ is a maximum.

To illustrate the model, the following plausible benchmark parameter values are chosen. If the 12 output subelasticities a_{ij}, $a_{ij} = b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, s_{ij}$, for player i, $i = x, y, z$, for currency j, $j = n, g$, were to be given equal weight, assuming constant returns to scale as specified after (2), each output subelasticity would get weight $a_{ij} = x, y, z = 1/12$.[8] Table 1a shows 36 output subelasticities a_{ij}, which all satisfy the requirement $a_{ij} \geq 0$, for player i, $i = x, y, z$, for currency j, $j = n, g$.

[8] Since we have no evidence to justify increasing or decreasing returns to scale, we make the simplest and common assumption of constant returns to scale.

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Table 1
Output subelasticities a_{ij} in three panels a,b,c for currency j, $j = n, g$, as perceived by player i, $i = x, y, z$.

Player i	$i = x$		$i = y$		$i = z$	
Currency j	$j = n$	$j = g$	$j = n$	$j = g$	$j = n$	$j = g$
Panel a						
b_{ij}	1/4	0	0	1/4	0	1/12
c_{ij}	1/12	0	0	1/12	0	1/12
d_{ij}	1/12	1/12	1/12	1/12	1/12	1/4
e_{ij}	1/12	1/12	1/12	1/12	1/12	1/12
f_{ij}	1/12	1/12	1/12	1/12	1/12	1/12
s_{ij}	1/12	1/12	1/12	1/12	1/12	1/12
Panel b						
b_{ij}	1/3	0	0	1/3	0	1/12
c_{ij}	1/12	0	0	1/12	0	1/12
d_{ij}	1/12	0	0	1/12	0	1/3
e_{ij}	1/12	1/12	1/12	1/12	1/12	1/12
f_{ij}	1/12	1/12	1/12	1/12	1/12	1/12
s_{ij}	1/12	1/12	1/12	1/12	1/12	1/12
Panel c						
b_{ij}	1/2	0	0	1/2	0	1/12
c_{ij}	1/12	0	0	1/12	0	1/12
d_{ij}	1/12	0	0	1/12	0	1/2
e_{ij}	1/12	0	0	1/12	0	1/12
f_{ij}	1/12	0	0	1/12	0	1/12
s_{ij}	1/12	1/12	1/12	1/12	1/12	1/12

Table 1a assumes that player x as a conventionalist prefers at least output subelasticity $a_{ij} = 1/12$ for all the six output subelasticities backing, convenience, confidentiality, transaction efficiency, stability, and security for the national currency n, and three times higher output subelasticity $b_{xn} = 1/4$ for the backing of currency n, which it respects and trusts, and justifies player x as a conventionalist. Table 1a further assumes that player x prefers at most output subelasticity $a_{ij} = 1/12$ for the six output subelasticities for the global currency g, and zero output subelasticity for the backing $b_{xg} = 0$ and convenience $c_{xg} = 0$ of currency g, which also justifies player x as a conventionalist. Table 1a assumes that player y as a pioneer has the opposite preference of player x, i.e. at least output subelasticity $a_{ij} = 1/12$ for all the six output subelasticities for the global currency g, and three times higher output subelasticity $b_{yg} = 1/4$ for the backing of currency g, at most output subelasticity $a_{ij} = 1/12$ for the six output subelasticities for the national currency n, and zero output subelasticity for the backing $b_{yn} = 0$ and convenience $c_{yn} = 0$ of currency n. Table 1a assumes that player z as a criminal has the same preference as the pioneer player y, except that its three times higher preference is for output subelasticity $d_{zg} = 1/4$ for the confidentiality of currency g. Hence it prefers output subelasticity $b_{zg} = 1/12$ for the backing of currency g.

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Table 1b assumes that the three kinds of players have higher preferences $b_{xn} = b_{yg} = d_{zg} = 1/3$ for their preferred output subelasticities, i.e. backing of currencies n and g for players x and y, and confidentiality of currency g for player z. They compensate for these higher preferences by having no preferences $d_{xg} = d_{yn} = d_{zn} = 0$ for confidentiality, i.e. of currency g for player x and of currency n for players y and z.

Table 1c assumes that the three kinds of players have even higher preferences $b_{xn} = b_{yg} = d_{zg} = 1/2$ for their preferred output subelasticities, i.e. backing of currencies n and g for players x and y, and confidentiality of currency g for player z. They compensate for these higher preferences by having no preferences $e_{xg} = e_{yn} = e_{zn} = f_{xg} = f_{yn} = f_{zn} = 0$ for transaction efficiency and financial stability, i.e. of currency g for player x and of currency n for players y and z. We alternate between applying Table 1 panels a, b, c, and combinations of these for players x, y, z, as our benchmark, as we proceed.

The benchmark furthermore assumes that the conventionalist player x and pioneer player y choose a zero fraction $w_i = 0$ of its transactions to be criminal, $i = x, y$, which may be a good approximation for many countries, while the criminal player z chooses a positive fraction $w_z = 0.5$ of its transactions to be criminal, assumed as a focal intermediate between $w_z = 0.5$ and $w_z = 1$. The government is assumed to detect and prosecute criminal behavior with probability $\omega_i = 0.5$, also assumed as a focal intermediate between $w_z = 0.5$ and $w_z = 1$. We assume scaling exponent $k_i = 1$ for what player i retains after criminal behavior, which in (3) means that player i's expected utility decreases linearly in the fraction w_i of player i's transactions which is criminal. The authors believe that a linear decrease is more plausible than a convex or concave decrease. Unitary values, also assumed below to the extent possible, are assumed plausible focal points when no particular evidence seems suitable for non-unitary values.

The scaling exponent for how player i gets increased or decreased expected utility depending on the fraction q_i of players of kind i is assumed to be positive and unitary, $m_x = 1$, for conventionalists, and negative and unitary, $m_y = m_z = -1$, for pioneers and criminals.

The scaling proportionality parameter μ_i for how player i gets increased or decreased expected utility depending on the fraction q_i of players of kind i, $i = x, y, z$, impacts the analysis crucially. We assume the unitary $\mu_x = 1$ as a benchmark for conventionalists, which in (4) causes $U_{xF}(q_x)$ to vary between $U_{xF}(q_x) = 1$ when $q_x = 0$ and $U_{xF}(q_x) = 2$ when $q_x = 1$. For pioneers and criminals we assume $\mu_i < 1$, since $U_{iF}(q_i)$ in (4) varies between $U_{iF}(q_i) = \infty$ when $q_i = 0$ and $U_{iF}(q_i) = 1 + \mu_i$ when $q_i = 1$, $i = x, y$, since $m_y = m_z = -1$. More specifically, we assume the five times lower $\mu_y = 0.2$ for pioneers and the ten times lower $\mu_z = 0.1$ for criminals.

In this section, where the fraction q_i of players of kind i is exogenous, we assume equally large fractions $q_i = 1/3$ of the three kinds of players, $i = x, y, z$, thus not giving eminence to one kind of player over another kind. The values $q_i = 1/3$ are needed to determine player i's expected utility $U_i(p_i, q_i)$ in (5), due to the last proportional term $1 + \mu_i q_i^{m_i}$, but do not impact the shape of $U_i(p_i, q_i)$ as a function of p_i and for which value of p_i that $U_i(p_i, q_i)$ has its maximum.

Figure 2 applies the above benchmark, including the exogenous $q_i = 1/3$, and plots player i's expected utility U_i in (5) and society's expected utility U in (6) as functions of player i's volume fraction p_i of currency n, $i = x, y, z$. The Mathematica software (www.wolfram.com) is used for plotting. Panel k assumes the output subelasticities a_{ij} in Table 1k, k = a, b, c. The two dashed vertical lines in each panel show the values of p_i where at least one expected utility U_i has its maximum value, i.e. $p_x = 2/3$ and $p_y = p_z = 1/3$ in panel a, $p_x = 3/4$ and $p_y = p_z = 1/4$ in panel b, and $p_x = 11/12$ and $p_y = p_z = 1/12$ in panel c. In panel a, society's expected utility U reaches its maximum at $p_i = 4/9$ which is the weighted sum of the p_i's across the three kinds of players. If the weights change from q_i =1/3, e.g. such that q_z increases and q_x and q_y decrease, the value p_i changes from $p_i = 4/9 \approx 0.44$ towards $p_i = 2/3$. In panels b and c, society's expected utility U reaches their maxima at $p_i = 5/12 \approx 0.42$ and $p_i = 9/25 = 0.36$, calculated analogously.

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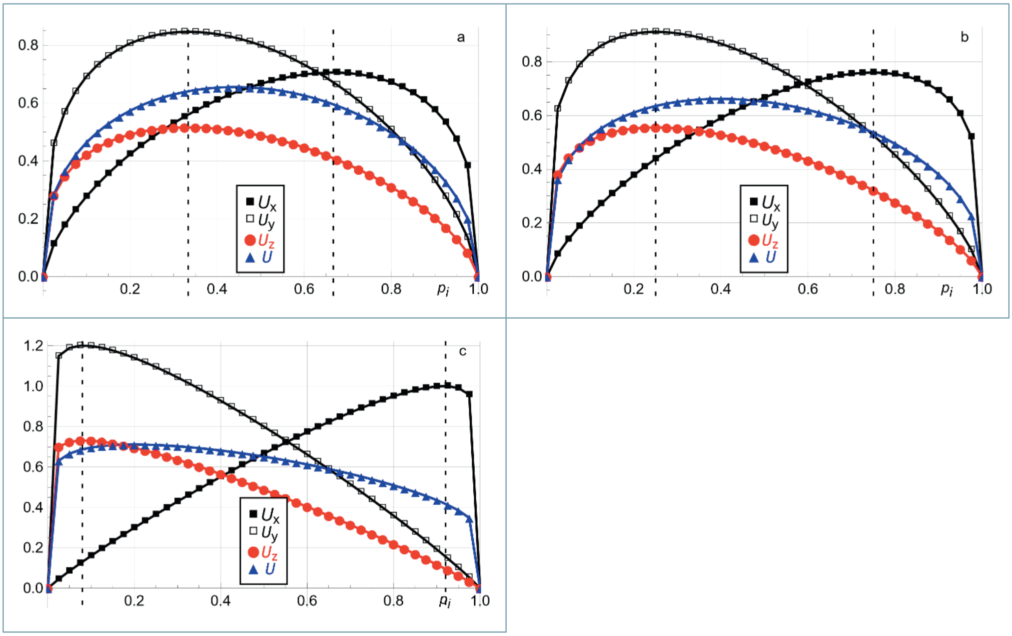
Figure 2
Player i's expected utility U_i as a function of its volume fraction p_i of currency n when $q_i = 1/3$, $i = x, y, z$. Panel k assumes the output subelasticities a_{ij} in Table 1k, k = a, b, c.



In all the three panels in Figure 2 the conventionalist player x's inverse U-shaped expected utility U_x is skewed towards the right since it values the national currency n more than the global currency g. When the volume fraction p_x of the conventionalist player x's transactions in the national currency n is low, the conventionalist player x's expected utility U_x is intuitively low. As the fraction p_x increases, its expected utility U_x increases to its maximum when $p_x = 2/3$, $p_x = 3/4$, $p_x = 11/12$, in panels a, b, c, and thereafter decreases, as player x also assigns some, although low, output subelasticities to currency g.

In contrast, in all the three panels in Figure 2 the pioneer player y's and criminal player z's inverse U-shaped expected utilities U_i are skewed towards the left since they value the global currency g more than the national currency n, and thus prefer $p_i < 1/2$. As the fraction p_i increases, its expected utility U_i increases to its maximum when $p_i = 1/3$, $p_i = 1/4$, $p_i = 1/12$, in panels a, b, c, respectively, $i = x, y$. As p_i increases further, U_i decreases. The criminal's expected utility U_z is lower than the pioneer's expected utility U_y since its fraction $w_z = 0.5$ of transactions is criminal, detected and prosecuted by the government with probability $\omega_i = 0.5$.

3.2. Analysis Applying the Replicator Equation

This section applies the replicator equation in (7) to determine the fraction q_i of players of kind i endogenously, while player i determines the volume fraction p_i of currency n by maximizing its expected utility U_i in (5), $i = x, y, z$. Figure 3 applies the output subelasticities in Table 1 and the benchmark parameter values in section 3.1, i.e. $w_x = w_y = 0$, $w_z = 0.5$, $\omega_i = 0.5$, $k_i = 1$, $m_x = 1$, $m_y = m_z = -1$, $\mu_x = 1$, $\mu_y = 0.2$, $\mu_z = 0.1$, $i = x, y, z$. Player i chooses its volume fraction p_i of currency n optimally to maximize its expected utility U_i, $i = x, y, z$. Assuming rapidity $\alpha_i = 1$ of change or sensitivity of the replicator equation, $i = x, y, z$, (7) is used to determine the fraction q_i of players of kind i, $i = x, y, z$. Figure 3 plots these fractions q_x, q_y, $q_z = 1 - q_x - q_y$, and the volume fraction p of all players' transactions in the national currency n from (1), as functions of time t.

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Figure 3
Fraction q_i of players of kind i, $i = x, y, z$, and the volume fraction p of all players' transactions in currency n, as a function of time t for the benchmark parameter values in Table 1, $w_x = w_y = 0$, $w_z = 0.5$, $\omega_i = 0.5$, $k_i = 1$, $m_x = 1$, $m_y = m_z = -1$, $\mu_x = 1$, $\mu_y = 0.2$, $\mu_z = 0.1$, $\alpha_i = 1$, $i = x, y, z$. Panel a: Table 1a. Panel b: Table 1b. Panel c: Table 1c. Panel d: Table 1a for player x and Table 1c for players y and z. Panel e: Table 1c for player x and Table 1a for players y and z.

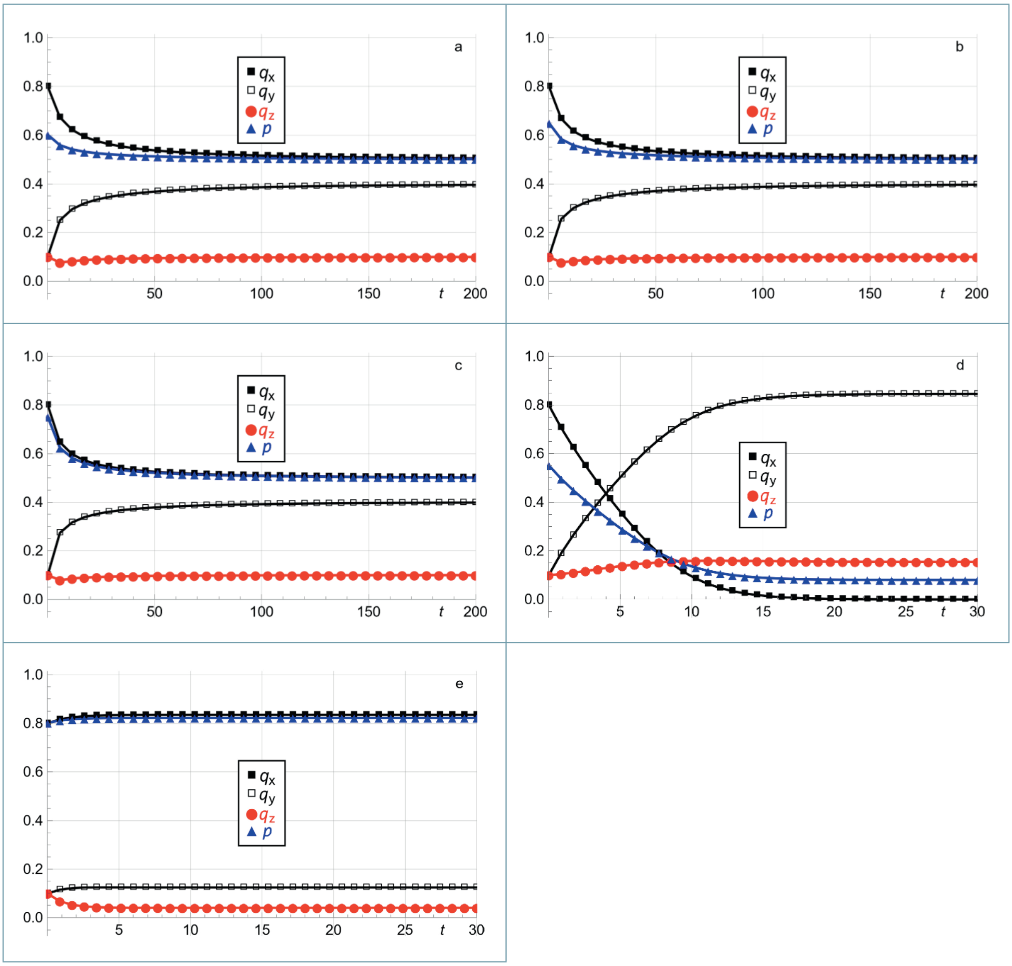

Figure 3 assumes initial conditions at time $t = 0$ equal to $q_x(0) = 0.8$ and $q_y(0) = q_z(0) = 0.1$, which means that conventionalists initially are in the majority at 80%, while pioneers and criminals are in the minority, each at 10%.

Figure 3a assumes the 36 output subelasticities in Table 1a, which according to Figure 2a gives the optimal volume fractions $p_x = 2/3$ for conventionalists and $p_y = p_z = 1/3$ for pioneers and criminals, for player i's transactions in currency n. The fraction q_x of conventionalists decreases convexly from $q_x(0) = 0.8$ to $\lim_{t\to\infty} q_x = 0.5$, hereafter referred to as the stationary solution, after sufficiently much time t has elapsed. All limit values are determined numerically. The fraction q_y of pioneers increases concavely from $q_y(0) = 0.1$ to $\lim_{t\to\infty} q_y = 0.4$. The fraction q_z of criminals first decreases marginally and briefly from $q_z(0) = 0.1$, as the fraction q_y of pioneers increases rapidly. Thereafter q_z increases concavely back up towards $\lim_{t\to\infty} q_z = 0.1$. Hence the volume fraction p of all players' transactions in the national currency n decreases towards $\lim_{t\to\infty} p = 0.5$.

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Figure 3b assumes the 36 output subelasticities in Table 1b, which according to Figure 2b gives the higher optimal volume fractions $p_x = 0.75$ for conventionalists and the lower $p_y = p_z = 0.25$ for pioneers and criminals, for player i's transactions in currency n. The evolution of the fractions q_x, q_y, q_z is qualitatively similar to Figure 3a, with the same limit values $\lim_{t\to\infty} q_x = \lim_{t\to\infty} p = 0.5$, $\lim_{t\to\infty} q_y = 0.4$, $\lim_{t\to\infty} q_z = 0.1$. The reason for the similar result is that the increase in the optimum from $p_x = 2/3$ to $p_x = 3/4$ for conventionalists equals the decrease in the optimum from $p_y = p_z = 1/3$ to $p_y = p_z = 1/4$ for pioneers and criminals. These changes are in the opposite direction and equal $3/4 – 2/3 = 1/3 – 1/4 = 1/12$. Furthermore, at the limit when $t \to \infty$, the fraction q_x of conventionalists equals the sum of the fractions q_y and q_z of pioneers and criminals, i.e. $\lim_{t\to\infty} q_x = 0.5 = \lim_{t\to\infty} q_y = 0.4 + \lim_{t\to\infty} q_z = 0.1$, which means that the impact in the opposite direction when determining q_x, q_y, q_z in (7) is equally strong.

Figure 3c assumes the 36 output subelasticities in Table 1c, which according to Figure 2c gives the higher optimal volume fractions $p_x = 0.92$ for conventionalists and the lower $p_y = p_z = 0.08$ for pioneers and criminals, for player i's transactions in currency n. Also here the evolution of the fractions q_x, q_y, q_z is qualitatively similar to Figure 3a and Figure 3b, with the same limit values $\lim_{t\to\infty} q_x = \lim_{t\to\infty} p = 0.5$, $\lim_{t\to\infty} q_y = 0.4$, $\lim_{t\to\infty} q_z = 0.1$. The reason for the similar result is again that the increase in the optimum from $p_x = 2/3$ to $p_x = 11/12$ for conventionalists equals the decrease in the optimum from $p_y = p_z = 1/3$ to $p_y = p_z = 0.08$ for pioneers and criminals. These changes are in the opposite direction and equal $11/12 – 2/3 = 1/3 – 1/12 = 1/4$. At the limit when $t \to \infty$, the fraction q_x of conventionalists equals the sum of the fractions q_y and q_z of pioneers and criminals, i.e. $\lim_{t\to\infty} q_x = 0.5 = \lim_{t\to\infty} q_y + \lim_{t\to\infty} q_z$, which means that the impact in the opposite direction when determining q_x, q_y, q_z in (7) is equally strong.

To illustrate results different from Figure 3a, b, c, we consider two extreme combinations of output subelasticities from Table 1, one favoring pioneers and criminals, and one favoring conventionalists. Figure 3d assumes the 12 output subelasticities in Table 1a for the conventionalist player x, which gives the minimum optimal volume fraction $p_x = 2/3$, and assumes the 24 output subelasticities in Table 1c for the pioneer and criminal players y and z, which gives the minimum optimal volume fractions $p_y = p_z = 1/12$. That both $p_x = 2/3$ and $p_y = p_z = 1/12$ are minimum optimum values for the respective players, among the alternatives in Table 1, chosen by the three kinds of players maximizing their expected utilities U_x, U_y, U_z in (5), means that all the three kinds of players choose currency n with minimum volume fractions p_x, p_y, p_z. That favors pioneers and criminals, who to a lower extent back and favor currency n. Consequently, the fractions q_y and q_z of pioneers and criminals increase concavely and quickly from $q_y(0) = q_z(0) = 0.1$ toward $\lim_{t\to\infty} q_y = 0.85$ and $\lim_{t\to\infty} q_z = 0.15$, while the fraction q_x of conventionalist decreases convexly and quickly from $q_x(0) = 0.8$ toward $\lim_{t\to\infty} q_x = 0$, thus going extinct. This shows how a change in the output subelasticities among the alternatives in Table 1 may tilt the balance from emphasis on the national currency n towards emphasis on the global currency g. Hence the volume fraction p of all players' transactions in the national currency n decreases towards $\lim_{t\to\infty} p = 1/12$.

Figure 3e assumes the 12 output subelasticities in Table 1c for the conventionalist player x, which gives the maximum optimal volume fraction $p_x = 11/12$, and assumes the 24 output subelasticities in Table 1a for the pioneer and criminal players y and z, which gives the maximum optimal volume fractions $p_y = p_z = 1/3$. That both $p_x = 11/12$ and $p_y = p_z = 1/3$ are maximum optimum values for the respective players, among the alternatives in Table 1, means that all the three kinds of players choose currency n with maximum volume fractions p_x, p_y, p_z. That favors conventionalists, who to a higher extent back and favor currency n. Consequently, the fraction q_x of conventionalists increases concavely, quickly and marginally from $q_x(0) = 0.8$ toward $\lim_{t\to\infty} q_x = 0.835$. The fraction q_y of pioneers increases concavely, quickly and marginally from $q_y(0) = 0.1$ toward $\lim_{t\to\infty} q_y = 0.125$. The fraction q_z of criminals decreases convexly and quickly from $q_z(0) = 0.1$ toward $\lim_{t\to\infty} q_z = 0.040$. This shows how a different change in the output subelasticities among the alternatives in Table 1 may preserve the emphasis on the

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national currency n, rather than tilting the balance towards the global currency g. The volume fraction p of all players' transactions in the national currency n increases marginally towards $\lim_{t\to\infty} p = 0.820$.

3.3. Sensitivity Analysis

The previous section 3.2 implies a stationary solution after sufficiently much time t has elapsed, i.e. at the limit when $t \to \infty$. This section 3.3 determines the sensitivity of that stationary solution relative to the output subelasticities in Table 1b and the 15 benchmark parameter values in section 3.1, i.e. $w_x = w_y = 0$, $w_z = 0.5$, $\omega_i = 0.5$, $k_i = 1$, $m_x = 1$, $m_y = m_z = -1$, $\mu_x = 1$, $\mu_y = 0.2$, $\mu_z = 0.1$, $i = x, y, z$. We choose Table 1b which has intermediate, compared with Table 1 panels a and c, optimal volume fractions $p_x = 0.75$ for conventionalists and $p_y = p_z = 0.25$ for pioneers and criminals, for player i's transactions in currency n. In Figure 4 each of the 15 parameter values is altered from its benchmark, while the other 14 parameter values are kept at their benchmarks.

Figure 4
Fraction q_i of players of kind i, $i = x, y, z$, as a function of the 15 parameters w_x, w_y, w_z, ω_i, k_i, m_x, m_y, m_z, μ_x, μ_y, μ_z, relative to the benchmark parameter values in Table 1b, $w_x = w_y = 0$, $w_z = 0.5$, $\omega_i = 0.5$, $k_i = 1$, $m_x = 1$, $m_y = m_z = -1$, $\mu_x = 1$, $\mu_y = 0.2$, $\mu_z = 0.1$, $i = x, y, z$, assuming the stationary solution, i.e. after sufficiently much time t has elapsed, in section 3.2.

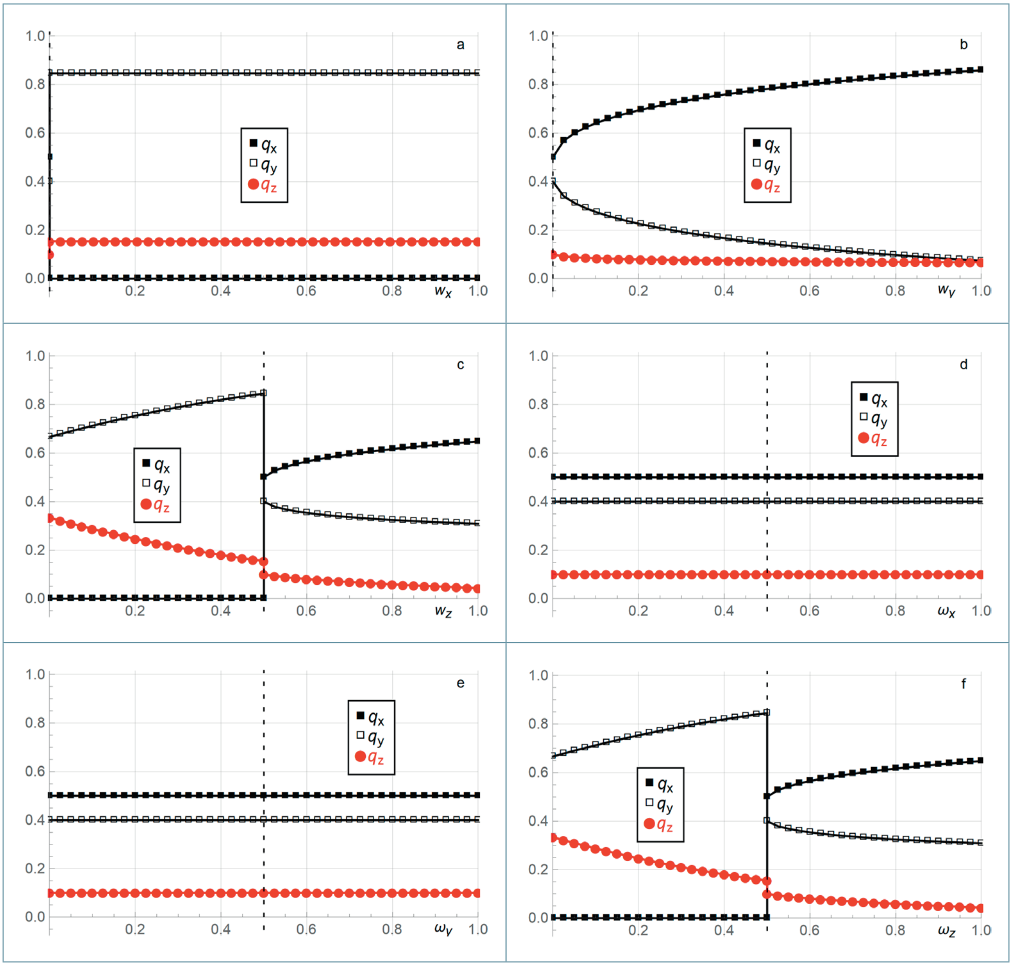

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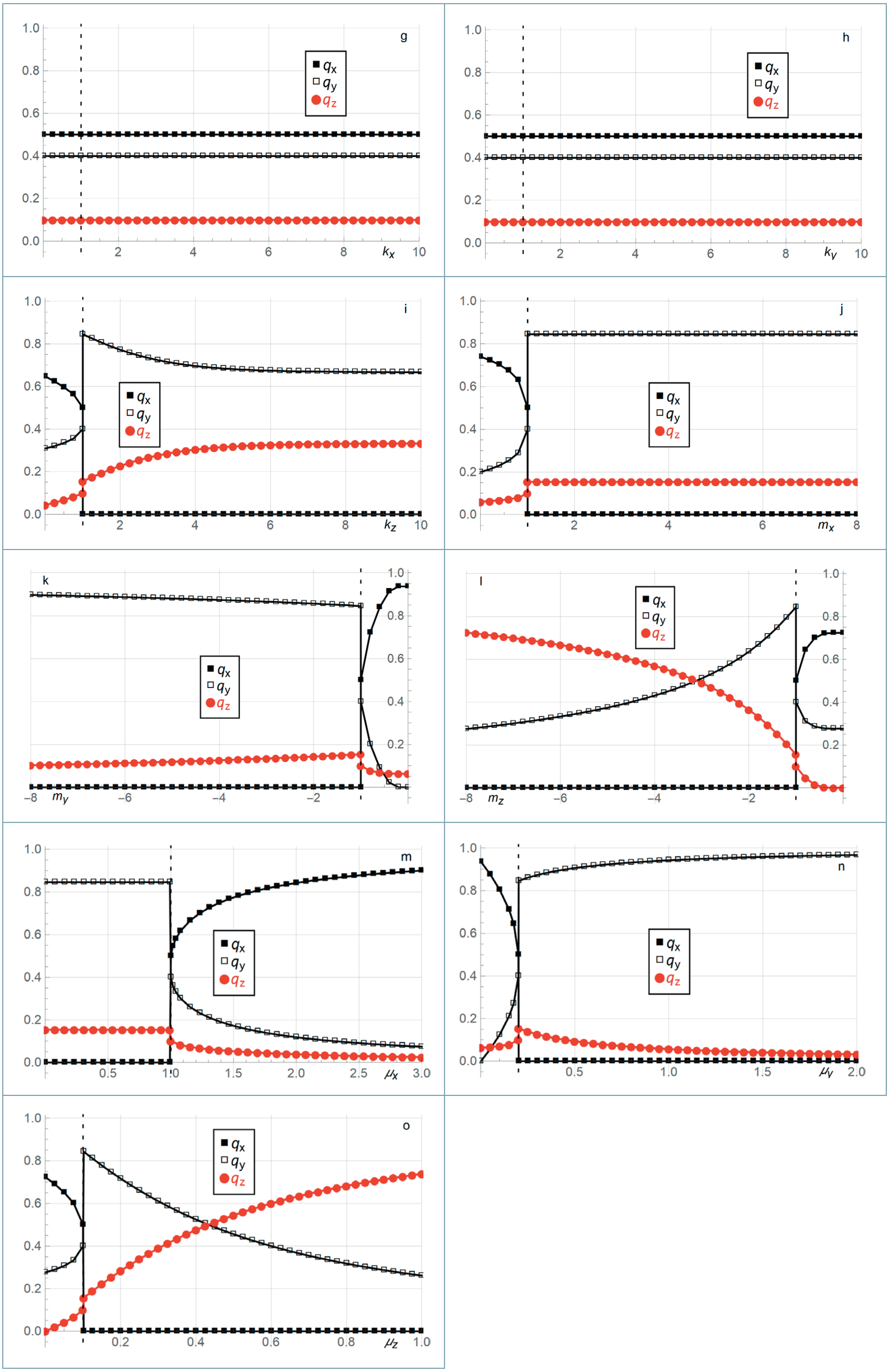
Figure 4 (cont.)



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In our benchmark from the previous section 3.2, Figure 3b based on Table 1b determines the stationary solution $\lim_{t\to\infty} q_x = 0.5$ for conventionalists, $\lim_{t\to\infty} q_y = 0.4$ for pioneers, and $\lim_{t\to\infty} q_z = 0.1$ for criminals, after sufficiently much time t has elapsed, depicted with a dashed vertical line in the 15 panels in Figure 4. As each parameter value varies, the stationary solution, hereafter for simplicity referred to as q_x, q_y, q_z, varies from $q_x = 0.5$, $q_y = 0.4$, $q_z = 0.1$ to some other values.

In Figure 4a, as the fraction w_x of conventionalists' transactions which is criminal increases above the benchmark $w_x = 0$, causing conventionalists to risk detection and prosecution if transacting criminally, the fraction q_x of conventionalists decreases from $q_x = 0.5$ to $q_y = 0$, which means extinction, due to lower expected utility. Pioneers and criminals benefit from increasing w_x. As w_x increases above $w_x = 0$, the fraction q_x of pioneers increases from $q_y = 0.4$ to $q_y = 0.85$, and the fraction q_z of criminals increases from $q_z = 0.1$ to $q_z = 0.15$, due to higher expected utilities. The fractions q_x, q_y, q_z, remain constant for $0 < w_x \leq 1$ since w_x impacts only conventionalists' expected utility, and not pioneers' and criminals' expected utilities.

In Figure 4b, as the fraction w_y of pioneers' transactions which is criminal increases above the benchmark $w_y = 0$, causing pioneers to risk detection and prosecution if transacting criminally, the fraction q_y of pioneers decreases convexly from $q_y = 0.4$ to $q_y = 0.07$ when $w_y = 1$, while the fraction q_z of criminals decreases marginally and convexly from $q_z = 0.1$ to $q_z = 0.07$ when $w_y = 1$. Conventionalists benefit from increasing w_y. As w_y increases above $w_y = 0$, the fraction q_x of conventionalists increases concavely from $q_x = 0.5$ to $q_x = 0.86$ when $w_y = 1$.

In Figure 4c, as the fraction w_z of criminals' transactions which is criminal increases above the benchmark $w_z = 0.5$, the fraction q_z of criminals decreases convexly from $q_z = 0.1$ to $q_z = 0.04$ when $w_z = 1$, while the fraction q_y of pioneers decreases convexly from $q_y = 0.4$ to $q_y = 0.31$ when $w_y = 1$. That is because criminals and pioneers do not benefit when they or their criminal transactions become more numerous, cf (4) when $m_y = m_z = -1$ and $m_x = 1$. Conventionalists benefit from increasing w_z, while criminals and pioneers do not. As w_z increases above $w_z = 0.5$, the fraction q_x of conventionalists increases concavely from $q_x = 0.5$ to $q_x = 0.65$ when $w_z = 1$. In contrast, as w_z decreases below $w_z = 0.5$, criminals benefit from their criminal transactions becoming less numerous. That causes the expected utility U_x for conventionalists to be lower than U_y and U_z for pioneers and criminals, $U_x < U_y$ and $U_x < U_z$, regardless of the fraction q_x of conventionalists. That is economically detrimental for conventionalists. In such circumstances no one wants to be a conventionalist. Hence $q_x = 0$ when $w_z < 0.5$. That gives a sudden downward jump in q_x, and hence upward jumps in q_y and q_z as all the three kinds of players adapt to the disappearance of conventionalists who cannot justify their low expected utility U_x. Hence, when $w_z < 0.5$, the replicator equation in (7) strikes a balance between the fractions q_y and q_z of pioneers and criminals, which are $q_y = 0.85$ and $q_z = 0.15$ when $w_z = 0.5 - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small but positive, thus excluding conventionalists. As w_z decreases below $w_z = 0.5$, the fraction q_z of criminals increases convexly from $q_z = 0.15$ to $q_z = 0.33$ when $w_z = 0$, while the fraction q_z of pioneers decreases concavely from $q_y = 0.85$ to $q_y = 0.67$ when $w_z = 0$.

In Figure 4d, as the probability ω_x that the government detects and prosecutes conventionalists' criminal behavior changes from the benchmark $\omega_x = 0.5$, the fractions $q_x = 0.5$, $q_y = 0.4$, $q_z = 0.1$ of conventionalists, pioneers and criminals remain constant and unchanged since ω_x in (5) is multiplied with the benchmark fraction $w_x = 0$ of conventionalists' transactions which is criminal. Since $w_x = 0$, ω_x has no impact.

In Figure 4e, analogously, as the probability ω_y that the government detects and prosecutes pioneers' criminal behavior changes from the benchmark $\omega_y = 0.5$, the fractions $q_x = 0.5$, $q_y = 0.4$, $q_z = 0.1$ of conventionalists, pioneers and criminals remain constant and unchanged since ω_y in (5) is multiplied with the benchmark fraction $w_y = 0$ of pioneers' transactions which is criminal. Since $w_y = 0$, ω_y has no impact.

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Figure 4f, where the probability ω_z that the government detects and prosecutes the criminals' criminal behavior varies, is equivalent to Figure 4c since $k_z = 1$ in (5), and thus varying ω_z has the same impact as varying the fraction w_z of the criminals' transactions which is criminal, acknowledging that both parameters are restricted to the same interval, $0 \leq \omega_z, w_z \leq 1$ and have the same benchmark values $\omega_z = w_z = 0.5$. As in Figure 4c, as $w_z < 0.5$ so that the fraction w_z of the criminals' transactions which is criminal decreases below the benchmark $w_z = 0.5$, conventionalists cannot justify their existence due to their low utility $U_x < U_y$ and $U_x < U_z$, and hence $q_x = 0$.

In Figure 4g, as the scaling exponent k_x for what conventionalists retain after criminal behavior changes from the benchmark $k_x = 1$, the fractions $q_x = 0.5$, $q_y = 0.4$, $q_z = 0.1$ of conventionalists, pioneers and criminals remain constant and unchanged since k_x in (5) is an exponent where the base $w_x = 0$ of the conventionalists' transactions which is criminal. Since $w_x = 0$, k_x has no impact.

In Figure 4h, as the scaling exponent k_y for what pioneers retain after criminal behavior changes from the benchmark $k_y = 1$, the fractions $q_x = 0.5$, $q_y = 0.4$, $q_z = 0.1$ of conventionalists, pioneers and criminals remain constant and unchanged since k_y in (5) is an exponent with base $w_y = 0$ which expresses the fraction of the pioneers' transactions which is criminal. That is, since $w_y = 0$, k_y has no impact.

In Figure 4i, as the scaling exponent k_z for what criminals retain after criminal behavior increases above the benchmark $k_z = 1$, the expected utility U_x for conventionalists becomes lower than U_y and U_z for pioneers and criminals, regardless of the fraction q_x of conventionalists, and hence $q_x = 0$ when $k_z > 1$. Hence conventionalists cannot justify their existence due to $U_x < U_y$ and $U_x < U_z$, just as when $w_z < 0.5$ in Figure 4c and Figure 4f. That causes the replicator equation in (7) to strike a balance between the fractions q_y and q_z of pioneers and criminals. As k_z increases, the fraction q_y of pioneers increases from $q_y = 0.4$ when $k_z = 1$ to $q_y = 0.85$ when $k_z > 1$, and thereafter decreases convexly towards the same value as when $w_z = 0$ in Figure 4c, or when $\omega_z = 0$ in Figure 4f, i.e. $\lim_{\substack{t \to \infty, \\ k_z \to \infty}} q_y = 0.67$. The fraction q_z of criminals increases from $q_z = 0.1$ when $k_z = 1$ to $q_z = 0.15$ when $k_z > 1$, due to the disappearance of conventionalists, and thereafter increases concavely, due to successful competition with pioneers as k_z increases, eventually reaching the same value as when $w_z = 0$ in Figure 4c, or when $\omega_z = 0$ in Figure 4f, in accordance with the term $\omega_z w_z^{k_z}$ in (5), $\lim_{\substack{t \to \infty, \\ k_z \to \infty}} q_z = 0.33$. In contrast, as k_z decreases below $k_z = 1$, the fraction q_x of conventionalists increases concavely, competing successfully against pioneers and criminals, eventually reaching $q_z = 0.65$ when $k_z = 0$. As k_z decreases below $k_z = 1$, the fractions q_y and q_z of pioneers and criminals decrease convexly towards $q_y = 0.31$ and $q_z = 0.04$ when $k_z = 0$.

In Figure 4j, as the scaling exponent m_x for how conventionalists get increased (since $m_x \geq 0$) expected utility increases above the benchmark $m_x = 1$, the expected utility U_x for conventionalists becomes lower than U_y and U_z for pioneers and criminals, regardless of the fraction q_x of conventionalists, and hence $q_x = 0$ when $m_x = 1$. Hence conventionalists cannot justify their existence, just as when $w_z < 0.5$ in Figure 4c and Figure 4f and $k_z > 1$ in Figure 4i. This follows mathematically from (5) where $q_x^{m_x}$ decreases as m_x increases when $0 < q_x < 1$. That causes the replicator equation in (7) to strike a balance between the fractions q_y and q_z of pioneers and criminals. Since m_x does not impact that balance, the fractions q_y and q_z of pioneers and criminals are constant at $q_y = 0.95$ and $q_z = 0.15$ when $m_x > 1$. In contrast, as m_x decreases below $m_x = 1$, the fraction q_x of conventionalists increases concavely, competing successfully against pioneers and criminals, eventually reaching $q_x = 0.74$ when $m_x = 0$. This also follows mathematically from (5) where $q_x^{m_x}$ increases as m_x decreases when $0 < q_x < 1$. As m_x decreases below $m_x = 1$, the fractions q_y and q_z of pioneers and criminals decrease convexly, eventually reaching, $q_y = 0.2$ and $q_z = 0.06$ when $m_x = 0$.

In Figure 4k, as the scaling exponent m_y for how pioneers get decreased (since $m_y \leq 0$) expected utility increases above the benchmark $m_y = -1$, the fraction q_y of pioneers decreases convexly, eventually going extinct, i.e. $q_y = 0$ when $m_y = 0$. This follows mathematically from

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(5) where $q_y^{m_y}$ decreases as m_y increases when $0 < q_y < 1$. As m_y increases above $m_y = -1$, the fraction q_x of conventionalists increases concavely, competing successfully with pioneers and criminals, eventually reaching $q_x = 0.94$ when $m_y = 0$, while the fraction q_z of criminals decreases convexly, eventually reaching $q_z = 0.06$ when $m_y = 0$. In contrast, as m_y decreases below $m_y = -1$, the expected utility U_x for conventionalists is lower than U_y and U_z for pioneers and criminals, regardless of the fraction q_x of conventionalists, and hence $q_x = 0$ when $m_y < -1$. Conventionalists then vanish, as in several of the panels above. That causes the replicator equation in (7) to strike a balance between the fractions q_y and q_z of pioneers and criminals, which are $q_y = 0.85$ and $q_z = 0.15$ when $m_y = -1 - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small but positive. As m_y decreases below $m_y = -1 - \varepsilon$, the fraction q_y of pioneers increases concavely, eventually outcompeting criminals, i.e. $\lim_{t\to\infty, m_y\to-\infty} q_y = 1$, while the fraction q_z of criminals decreases convexly, eventually going extinct, i.e. $\lim_{t\to\infty, m_y\to-\infty} q_z = 0$. This follows mathematically from (5) where $q_y^{m_y}$ increases without bounds as m_y decreases towards minus infinity when $0 < q_y < 1$.

In Figure 4l, as the scaling exponent m_z for how criminals get decreased (since $m_z \leq 0$) expected utility increases above the benchmark $m_z = -1$, the fraction q_z of criminals decreases convexly, eventually going extinct, i.e. $q_z = 0$ when $m_z = 0$. This follows mathematically from (5) where $q_z^{m_z}$ decreases as m_z increases when $0 < q_z < 1$. As m_z increases above $m_z = -1$, the fraction q_x of conventionalists increases concavely, competing successfully with pioneers and criminals, eventually reaching $q_x = 0.72$ when $m_z = 0$, while the fraction q_y of pioneers decreases convexly, eventually reaching $q_y = 0.28$ when $m_z = 0$. In contrast, as m_z decreases below $m_z = -1$, the expected utility U_x for conventionalists is lower than U_y and U_z for pioneers and criminals, regardless of the fraction q_x of conventionalists, and hence $q_x = 0$ when $m_z < -1$. Conventionalists then vanish, as in several of the panels above. That causes the replicator equation in (7) to strike a balance between the fractions q_y and q_z of pioneers and criminals, which are $q_y = 0.85$ and $q_z = 0.15$ when $m_z = -1 - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small but positive. As m_z decreases below $m_z = -1 - \varepsilon$, the fraction q_z of criminals increases concavely, eventually outcompeting pioneers, i.e. $\lim_{t\to\infty, m_z\to-\infty} q_z = 1$, while the fraction q_y of pioneers decreases convexly, eventually going extinct, i.e. $\lim_{t\to\infty, m_z\to-\infty} q_y = 0$. This follows mathematically from (5) where $q_z^{m_z}$ increases without bounds as m_z decreases towards minus infinity when $0 < q_z < 1$.

In Figure 4m, as the scaling proportionality parameter μ_x for how conventionalists get increased (since $m_x = 1$) expected utility increases above the benchmark $\mu_x = 1$, the fraction q_x of conventionalists increases concavely, eventually outcompeting pioneers and criminals, i.e. $\lim_{t\to\infty, \mu_x\to\infty} q_x = 1$. Thus the fractions q_y and q_z decrease concavely, $\lim_{t\to\infty, \mu_x\to\infty} q_y = \lim_{t\to\infty, \mu_x\to\infty} q_z = 0$. In contrast, as μ_x decreases below $\mu_x = 1$, the expected utility U_x for conventionalists is lower than U_y and U_z for pioneers and criminals, regardless of the fraction q_x of conventionalists, and hence $q_x = 0$ when $\mu_x < 1$. Conventionalists then vanish, as in several of the panels above. That causes the replicator equation in (7) to strike a balance between the fractions q_y and q_z of pioneers and criminals, which are $q_y = 0.85$ and $q_z = 0.15$ when $\mu_x < 1$.

In Figure 4n, as the scaling proportionality parameter μ_y for how pioneers get decreased (since $m_y = -1$) expected utility increases above the benchmark $\mu_y = 0.2$, the expected utility U_x for conventionalists becomes lower than U_y and U_z for pioneers and criminals, regardless of the fraction q_x of conventionalists, and hence $q_x = 0$ when $\mu_y > 0.2$. Conventionalists then vanish, as in several of the panels above. That causes the replicator equation in (7) to strike a balance between the fractions q_y and q_z of pioneers and criminals. As μ_y increases, the fraction q_y of pioneers increases from $q_y = 0.4$ when $\mu_y = 0.2$ to $q_y = 0.85$ when $\mu_y > 0.2$, and thereafter increases concavely, eventually outcompeting criminals, $\lim_{t\to\infty, \mu_y\to\infty} q_y = 1$. The fraction q_z of criminals increases

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from $q_z = 0.1$ when $\mu_y = 0.2$ to $q_z = 0.15$ when $\mu_y > 0.2$, due to the disappearance of conventionalists, and thereafter decreases convexly, due to unsuccessful competition with pioneers, eventually going extinct, $\lim_{t\to\infty,\ \mu_y\to\infty} q_z = 0$. In contrast, as μ_y decreases below $\mu_y = 0.2$, the fraction q_x of conventionalists increases concavely, competing successfully against pioneers and criminals, eventually reaching $q_y = 0.94$ when $\mu_y = 0$. As μ_y decreases below $\mu_y = 0.2$, the fractions q_y and q_z of pioneers and criminals decrease convexly, pioneers eventually going extinct, $q_y = 0$ when $\mu_y = 0$, while criminals enjoy some presence, i.e. $q_z = 0.06$ when $\mu_y = 0$.

In Figure 4o, as the scaling proportionality parameter μ_z for how criminals get decreased (since $m_z = -1$) expected utility increases above the benchmark $\mu_z = 0.1$, the expected utility U_x for conventionalists becomes lower than U_y and U_z for pioneers and criminals, regardless of the fraction q_x of conventionalists, and hence $q_x = 0$ when $\mu_z > 0.1$. Conventionalists then vanish, as in several of the panels above. That causes the replicator equation in (7) to strike a balance between the fractions q_y and q_z of pioneers and criminals. As μ_z increases, the fraction q_y of pioneers increases from $q_y = 0.4$ when $\mu_z = 0.1$ to $q_y = 0.85$ when $\mu_z > 0.1$, and thereafter decreases convexly, eventually being outcompeted by criminals and going extinct, $\lim_{t\to\infty,\ \mu_z\to\infty} q_y = 0$. The fraction q_z of criminals increases from $q_z = 0.1$ when $\mu_z = 0.1$ to $q_z = 0.15$ when $\mu_z > 0.1$, due to the disappearance of conventionalists, and thereafter increases concavely, due to successful competition with pioneers, eventually becoming dominant and excluding pioneers, $\lim_{t\to\infty,\ \mu_z\to\infty} q_z = 1$. In contrast, as μ_z decreases below $\mu_z = 0.1$, the fraction q_x of conventionalists increases concavely, competing successfully against pioneers and criminals, eventually reaching $q_z = 0.72$ when $\mu_z = 0$. As μ_z decreases below $\mu_z = 0.1$, the fractions q_y and q_z of pioneers and criminals decrease convexly, criminals eventually going extinct, $q_z = 0$ when $\mu_z = 0$, while pioneers are present at $q_y = 0.28$ when $\mu_z = 0$.

4. EXPLAINING THE IMPLICATIONS OF THE RESULTS

With the emergence of new currencies, each player's first choice of which volume fractions of its transactions should be in the national currency and the global currency can be expected to become more significant. The player's choice impacts both its utility, society's utility, which currencies gain traction, and which institutions and parts of society benefit from which currencies gain traction. These factors in turn can be expected to impact finance, business, markets and probably monetary policy, especially if no single currency is or becomes dominant within a given country.

Each player's second choice of whether to be a conventionalist, pioneer or criminal also impacts its utility, and impacts how society becomes composed of these three kinds of players. If conventionalists become less numerous, as illustrated for several combinations of parameter values in the previous section, society may evolve to become less conventional, with competition between pioneers and criminals.

The finding that each player's expected utility is inverse U-shaped as a function of the volume fraction of its transactions in each currency challenges each player to assess its identity as a conventionalist, pioneer or criminal. Each player is furthermore challenged to determine the impact of the subelasticities labeled as backing, convenience, confidentiality, transaction efficiency, financial stability, and security on in its Cobb-Douglas expected utility for the two currencies. This amounts to determining whether the inverse U-shape is skewed with a maximum towards the left or the right, and hence which currency should be chosen for the highest fraction of transactions, which may give fluctuations in currency markets.

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5. CONCLUSION

This article analyzes conventionalists, pioneers and criminals choosing between a national currency, e.g. a CBDC (central bank digital currency) or another currency common within a nation, and a global currency, e.g. Bitcoin or Meta's Diem, which may have limited usage within a nation (e.g. for purchases and tax payments), but may offer other possibilities such as application across nations and user autonomy. Conventionalists tend to prefer the national currency, pioneers (early adopters) tend to prefer the global currency, and criminals tend to prefer the global currency if it contributes (e.g. through confidentiality) to not getting caught.

Each player has a Cobb-Douglas utility with one output elasticity for each of the two currencies. Each output elasticity is comprised of six subelasticities, i.e. which kind of backing a currency has from trustworthy actors or systems (e.g. central banks for CBDCs and distributed ledger technology for cryptocurrencies), convenience (e.g. user friendliness), confidentiality (balancing privacy, availability, accessibility, and discrimination), transaction efficiency (low cost, fast speed, affordability, finality), financial stability (e.g. resilience during crises and shocks), and security (e.g. whether funds are safe and not subject to 51% attacks). Each player's expected utility is expanded to account negatively for detection and prosecution of criminal behavior, and accounts for the fractions of the three kinds of players. Conventionalists benefit from the presence of many conventionalists. Pioneers and criminals benefit from the presence of few pioneers and criminals, respectively.

Each player makes two strategic choices to maximize its expected utility, i.e. which volume fraction of its transactions should be in the national currency (causing the remaining fraction to be in the global currency), and what kind of player it should be, i.e. a conventionalist, pioneer or criminal. The first choice becomes increasingly relevant in today's world as we expect players to have easier access to more than one currency. Hence the market share of two currencies may change over time, as illustrated in this article. The first choice depends on which kind of player the player is, but does not depend on the number of players of this kind, and hence does not depend on time. Each player's second choice is what kind of player it should be through time. Hence this second choice depends on time, through replicator dynamics.

Each player's expected utility is inverse U-shaped as a function of the volume fraction of its transactions in the national currency. Hence each player prefers not to rely exclusively on one currency. The expected utility is skewed towards the right (high fraction) for conventionalists, who prefer the national currency, and more so if the conventionalists' six output subelasticities for the national currency are high. The expected utility is skewed towards the left (low fraction) for pioneers and criminals, who prefer the global currency, and more so if the pioneers' and criminals' six output subelasticities for the global currency are high. Three examples are considered for the degree of skewness towards the right and left. Today's financial system increasingly seems to require players to assess whether the various available currencies are characterized by inverse U-shaped expected utilities skewed towards the right or the left. Players more able to assess these inverse U-shapes as functions of volume fractions, and more able to assess whether they are conventionalists, pioneers and criminals, can expect to earn higher expected utilities. Society's expected utility is the weighted sum of each player's expected utility weighted by the fraction of players of each kind.

The replicator equation is used to illustrate the evolution of the fractions of the three kinds of players through time, assuming initial conditions with conventionalists in the majority, and pioneers and criminals in the minority. We illustrate how conventionalists may become more dominant and criminals less dominant through time if all the three kinds of players' expected utilities are skewed towards the right (i.e. prefer the national currency). In contrast, pioneers and criminals may become more dominant and conventionalists may go extinct if all the three kinds of players' expected utilities are skewed towards the left (i.e. prefer the global currency).

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Considering the stationary solution after sufficiently much time has elapsed, the model's sensitivity with respect to 15 parameter values is analyzed. The analysis shows that, typically, conventionalists (which prefer to be in the majority) tend to compete against pioneers and criminals (which prefer to be in the minority). Hence if a change in a parameter value causes the fraction of conventionalists to increase (decrease), the fractions of both pioneers and criminals may decrease (increase). The exception is, of course, when conventionalists are extinct, which is caused by their expected utility being too low, in which case pioneers and criminals compete directly with each other, so an increasing (decreasing) fraction of pioneers causes a decreasing (increasing) fraction of criminals.

As the fraction of a player's transactions which is criminal, or the probability that the government detects and prosecutes the player's criminal behavior, increases, the fraction of that kind of players in the population decreases, causing the fraction of at least one of the other kinds of players to increase. Each player thus responds to incentives, ceasing to be a kind of player with many criminal transactions, and ceasing criminal transactions if these are detected and prosecuted.

As the scaling exponent for what criminals retain after criminal behavior increases, their fraction in the population increases. That also causes the fraction of pioneers to increase, and the fraction of conventionalists to decrease, except when conventionalists are extinct, which occurs when the scaling exponent is high, in which case the fraction of pioneers decreases due to competition with criminals.

As the positive scaling exponent for how the conventionalists get increased expected utility increases, their expected utility decreases causing their fraction in the population to decrease and eventually go extinct. That causes the fractions of pioneers and criminals to increase. As the negative scaling exponents for how pioneers and criminals get decreased expected utilities increase, their expected utilities decrease causing their fractions in the population to decrease and eventually go extinct. That causes the fraction of conventionalists to transition from extinction to increase. This illustrates how economic incentives for conventionalists can make them more numerous.

As the scaling proportionality parameter for how conventionalists get increased expected utility increases, their fraction increases, as they respond to economic incentives, causing the fractions of pioneers and criminals to decrease. As the scaling proportionality parameters for how pioneers and criminals get increased expected utility increase, both their fractions increase, also responding to economic incentives, causing the fraction of conventionalists to decrease. Eventually, conventionalists go extinct, causing more pioneers and fewer criminals if the pioneers' scaling proportionality parameter increases, and more criminals and fewer pioneers if the criminals' scaling proportionality parameter increases.

Future research should compile and assess empirical support for the six kinds of output subelasticities for national and global currencies, the relevance of each output subelasticity, whether other output subelasticities can be envisioned, or whether the focus should be on fewer output subelasticities. Such empirical support should be assessed against which volume fractions players choose for national and global currencies, and which fractions of players choose to be conventionalists, pioneers, and criminals. These assessments should be made over various time periods to determine which factors impact which national and global currencies spread and become dominant, and which currencies decline in relevance and go extinct. For a more extensive dynamic analysis, the parameters such as the 12 output subelasticities may be allowed to depend on time. Various alternatives to the players' expected utilities may be evaluated, with different risk attitudes, and more than three kinds of players may be modeled. Each kind may have different time horizons and different exchange and trading strategies, e.g. many exchanges per day versus few exchanges per decade. More than one national currency may be analyzed, with competition between multiple national and global currencies which may be generalized to national and global assets (e.g. cryptoassets). The impact of competition on inflation, interest rates, etc., may be assessed, and other players such as regulators and governments may be incorporated.

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The evolution of fixed-supply and variable-supply currencies

Guizhou Wang[1,2] & Kjell Hausken[1,2✉]

Competition is analyzed between a fixed-supply currency (e.g. Bitcoin) and a variable-supply currency (e.g. a fiat currency). Two kinds of players support the currencies differently and choose their volume fractions of transactions in each currency. The variable-supply currency enables money printing/withdrawal and inflation/deflation, which counteract each other in each player's utility. The exponentially increasing 1959–2021 US M2 money supply and the positive inflation cause this utility to increase over time with high weight assigned to money printing/withdrawal, and decrease otherwise. Three replicator equations determine each player's volume fraction of transactions in each currency, and which kind of player each player prefers to be. High weight assigned to money supply relative to inflation induces players to prefer the variable-supply currency. A player's utility of transacting in each currency is proportional to the player's support of that currency, the volume fraction of all players' transactions in that currency, and the fraction of players of the same kind as the given player. A player's utility of transacting in the variable-supply currency is additionally proportional to two ratios. The first is the initial money supply plus the accumulative money printing/withdrawal divided by the initial money supply. The second is the inverse of the accumulative inflation/deflation. The players' fractions of transactions in each currency may be inverse U shaped or U shaped before typically converging towards preferring one or the other currency. If each player can choose which kind of player to be, it may choose to be the kind with the highest support of a given currency. If a player's utility of transacting in a given currency depends more on the fraction of players being of one kind than the other kind, the player prefers to be of the first kind, thus assigning less weight to its support of that currency and the volume fractions of transactions in that currency.

[1] Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway. [2] These authors contributed equally: Guizhou Wang, Kjell Hausken. ✉email: kjell.hausken@uis.no

Introduction

Background. Humans have used cash currencies for 40,000 years, which evolved from natural objects to coins to paper to digital versions (Kusimba, 2017). The Mesopotamian shekel emerged nearly 5000 years ago, and silver and gold mints emerged in Asia Minor 650–600 B.C, expanding to lead and copper coins in the first millennium A.D. Currencies commonly have a central authority and usually emerged for certain geographic areas and nations. Sometimes the expansion is global, e.g. as a world reserve currency. Fiat currencies have more recently expanded to also become digital. Digital currencies such as Bitcoin have no central authority and easily expand globally. Nakamoto (2008) shows how a decentralized currency such as Bitcoin can be built on a blockchain. He applies the proof of work technology to secure the ledger and avoid the double spending problem. Today 17,834 cryptocurrencies exist with a market cap of $1.8 trillion (https://coinmarketcap.com/, retrieved February 26, 2022). These vary substantially regarding fixed versus variable supply, consensus mechanisms (e.g. proof of stake), degree of decentralization, ownership, regulation, confirmation of transactions, etc. New digital currencies suggest competition between these and conventional currencies. Understanding this competition can be expected to be essential in the coming years.

Contribution. This article's purpose, motivation, objectives, research hypotheses, and research questions are as follows: First, competition between one fixed-supply and one variable-supply currency is analyzed to determine the evolutionary dynamics of each currency and which currency survives. Second, each player maximizes its utility by choosing which volume fraction of transactions to conduct in each currency, and which of two kinds of player to be, depending on various preferences. Third, the variable-supply currency enables money printing/withdrawal which impacts inflation/deflation which impacts each player's utility and strategic choices and thus how each currency evolves.

Being a certain kind of player means supporting one or the other currency to a certain extent. Such support is expressed by a currency's backing, convenience, confidentiality, transaction efficiency, financial stability, and security. A player's utility of transacting in the fixed-supply currency depends on the player's support of that currency, the volume fraction of all players' transactions in that currency, and the fraction of players of the same kind as the given player. A player's utility of transacting in the variable-supply currency depends on the same kinds of factors, and additionally depends on the variable money supply and inflation/deflation. That latter dependence is expressed on the Cobb Douglas form multiplying two ratios, i.e. the initial supply plus the accumulative money printing/withdrawal divided by the initial supply, and the inverse of accumulative inflation/deflation. If both ratios are valued equally and multiply to 1, money printing/withdrawal and inflation/deflation counteract each other. A product higher (lower) than 1 suggests higher (lower) weight to money printing/withdrawal.

Fixed-supply currencies have been historically uncommon. Gold viewed as a currency (Mitchell, 2021) is the best example, with 1.5% additional gold mined in 2020 (197,576 metric tons has been mined (gold.org, 2022). 3030 metric tons were produced in 2020 (Basov, 2022)). As a comparison, as of January 2022, 18.9 million Bitcoin out of 21 million coins have been mined, i.e. 90% (Hayes, 2022). The process will continue at a decreasing speed until approximately 2140. Both gold and Bitcoin are durable and fungible (Learn, 2021). Gold has more established history, with more entrenchment in cultures, central banks, and institutions, but falls short of Bitcoin on portability, divisibility, censorship resistance, verifiability, and scarcity (Ikkurty, 2019).

Whereas fixed-supply currencies eliminate inflation/deflation caused by money printing/withdrawal, variable-supply currencies do not. Variable-supply currencies offer added flexibility and possibilities not possible for fixed-supply currencies, e.g. funding wars and critical events, and Roosewelt's 1933–1939 New Deal for economic recovery. Money printing during such events suggests subsequent contraction to avoid inflation. Many economies have not exhibited the sufficient fiscal discipline. Even a traditionally fiscally responsible economy like the US has experienced that $1 in 2022 buys 1.22% of what it would buy in 1695.

Using the 1959–2021 US M2 money supply and inflation data, we show how a player's utility of exchanging in the fixed-supply currency is constant over time. The player's utility of exchanging in the variable-supply currency increases over time if more weight is assigned to money printing/withdrawal, and otherwise decreases over time.

One replicator equation expresses each kind of player's transaction volume in each currency. A third replicator equation expresses how each player prefers to be of one or the other kind. Each player's fractions of transactions in each currency may be inverse U shaped or U shaped before converging towards preferring one or the other currency, depending on the player's support of each currency. If a player can choose which kind of player to be, thus changing its support for a certain currency, it may choose to be of the kind which supports a certain currency highly. If a player is additionally impacted by how many players exist of each kind, it may choose to be of the kind that is most common.

Understanding how players choose between competing currencies is useful for consumers, traders, policy makers, regulators, institutions designing and issuing currencies, and institutions adjusting and impacting money supply and inflation/deflation.

Literature. Four groups of literature have been identified, i.e. competition between fiat currencies and cryptocurrencies, central bank digital currencies and cryptocurrencies, the cryptocurrency market, and game theoretic analyses.

Competition between fiat currencies and cryptocurrencies. Schilling and Uhlig (2019) evaluate how agents choose between a cryptocurrency and a fiat currency. Cryptocurrencies may enable tax evasion, anonymity, and censorship resistance, impacted by transaction fees to miners. Fiat currencies are currently useful for most purchases, impacted by value-added-taxes. They argue that substitution decreases as the asymmetry in exchange fees and transaction costs increase. This finding relates to how players in the current article choose volume fractions of transactions in two currencies, depending on their support for each currency which in turn depends on each currency's transaction efficiency, and depending on other factors.

Fernández-Villaverde and Sanches (2019) specify a price stable equilibrium, and some less desirable equilibria, for multiple competing privately issued fiat currencies in a Lagos-Wright environment. Their approach has a linkage to the analysis of two coexisting currencies in the current article.

Almosova (2018) evaluates costly circulation of private currencies, impacted by verification of transactions, mining costs, etc. She finds that sufficiently low costs of private currency circulation (mining costs) are needed to put downward pressure on the inflation for the public currency. Cryptocurrency competition may not cause price stability. These insights relate to the current article where players may choose a fixed-supply currency to avoid the inflation in the variable-supply currency.

Benigno et al. (2019) evaluate a global cryptocurrency and two national currencies. They find that different interest rates may cause the national currency to be abandoned or the zero lower bound may be approached. They argue that ensuring an independent monetary policy, free capital flows, and a fixed exchange rate may become even less possible. As a comparison, the current article evaluates various other conditions that may cause a currency to be abandoned.

Rahman (2018) considers how monetary policy is impacted by fiat and digital currency competition. He argues that a purely private arrangement of digital currencies cannot cause socially efficient allocation, and that optimal monetary policy at the Friedman rule will be socially inefficient. These insights suggest the need to understand the nature of currency competition.

Verdier (2021) analyzes how competition in the deposit and lending markets is impacted by a digital currency. She finds that the digital currency crowds out bank deposits causing increasing bank lending rates. That insight furthermore illustrates how currency competition can cause substantial disruption, which suggests a need to understand the evolutionary dynamics.

Central bank digital currencies and cryptocurrencies. Caginalp and Caginalp (2019) analyze how the wealthy divide their assets between a cryptocurrency and a home currency, similarly to how the current article analyzes players choosing how to transact in two currencies. Additionally they evaluate how a government can confiscate some of the players' assets.

Blakstad and Allen (2018) evaluate various conditions for issuing central bank digital currencies, and risks and possibilities associated with cryptocurrencies. Their analysis relates to the current article where two currencies may be supported differently, and the variable-supply currency may be designed with different characteristics related to facilitating money printing/withdrawal and inflation/deflation.

Masciandaro (2018) analyzes the evolution of different media of payments depending on individual preferences, similarly to this article modeling this evolution. They assess the implications for monetary policy, addressing the zero lower bound constraint for interest rates, and banking policy, e.g. risks of bank disintermediation when the opportunity-cost discrepancies between currencies decrease. That latter focus is partly or indirectly present in the current article in the sense that the abandonment of a variable-supply currency may cause banks to change how they operate.

Benigno (2021) argues that competing currencies may cause central banks to lose control of the nominal interest rate and inflation which depend on structural factors. Cryptocurrencies may set lower bounds on interest rates and inflation. The implication of that insight may be the kind of coexistence of two currencies, or one currency going extinct, as analyzed in the current article.

Asimakopoulos et al. (2019) evaluate substitution between a government currency and a cryptocurrency, depending on preferences, technology and monetary policy shocks, akin to how the current article considers players' substitution between currencies.

The cryptocurrency market. ElBahrawy et al. (2017) analyze the 2013–2017 evolutionary dynamics of market shares of cryptocurrencies. They find several stable statistical properties, e.g. the market share distribution, turnover, and number of active cryptocurrencies. The current article confines attention to the evolutionary dynamics of two currencies.

Caporale et al. (2018) find that cryptocurrencies' past and future values are positively correlated, with changing degree over time. They argue that this constitutes market inefficiency, enabling the generation of abnormal profits. Partly related, the current article shows how players' utilities change over time depending on how they transact in two currencies.

ElBahrawy et al. (2019) evaluate the interplay between online Wikipedia attention and market performance of cryptocurrencies. They find that tightly knit editors impact Wikipedia and that trading based on Wikipedia views mostly performs better than baseline strategies, apart from buying and holding during explosive market expansion. This also illustrates how players' utilities change over time depending on various strategies, and analyzed in this article.

White (2014) evaluates the market shares of Bitcoin and altcoins, similarly to this article evaluating players' volume fractions of transactions in two currencies.

Sapkota and Grobys (2021) identify market inefficiency where privacy coins exhibit market equilibrium unrelated to non-privacy coins. They suggest that the result may be due to criminals preferring non-privacy coins with high liquidity and anonymity. Their approach shows how players consciously choose between currencies with different properties, as in the current article.

Milunovich (2018) determines weak connectedness between six major asset classes and five cryptocurrencies, and mostly strong connectedness within each of these two groups. If such weak connectedness proves to be common for multiple currencies, that suggests the need to understand how players choose between multiple currencies with different characteristics, as in the current article.

Gandal and Halaburda (2016) characterize recent cryptocurrency competition as winner-take-all, and early competition as no winner-take-all. That more recent insight may reflect the finding in this article of players gradually moving towards favoring one or the other currency.

Game theoretic analyses. Imhof and Nowak (2006) consider a stochastic frequency dependent Wright–Fisher process to determine the survival of two strategies. They specify two absorbing states for the Markov process, where homogeneous populations choose either strategy A or strategy B. Players typically abandon a strategy occurring less frequently than 1/3 in an unstable equilibrium. That corresponds partly to this article's finding of players often preferring one or the other currency.

Lewenberg et al. (2015) apply cooperative game theory to determine that Bitcoin mining pools may find it challenging to distribute rewards in a stable way, causing players to switch pools frequently. That, in turn, may cause fluctuations which suggests the importance of applying evolutionary dynamics to assess players preferences over time.

Article organization. Section "The model" presents the model. Section "Analyzing the model" analyzes the model. Section "Discussion and future research" discusses the results. Section "Conclusion" concludes.

The model

Nomenclature. *Parameters*

g Fixed-supply currency

n Variable-supply fiat currency

t_0 Initial time, $t_0 \geq 0$

T Final time, $T \geq t_0$

j Time counting variable, $t_0 \leq j \leq T$

i Player of kind $i, i = 1,2$

s_{it} Player i's support of currency g relative to currency n at time t, $0 \leq s_{it} \leq 1$

μ_i Scaling proportionality parameter in player i's utilities u_{igt} and u_{int}, $\mu_i \geq 0$

m_i Scaling exponent in player i's utilities u_{igt} and u_{int}, $m_i \geq 0$

S_j Supply at discrete time j of the variable-supply fiat currency n, $S_j \in \mathbb{R}$

π_j Inflation at time j, $\pi_j \in \mathbb{R}$

α_i Player i's Cobb Douglas elasticity for money supply S_j, $0 \leq \alpha_i \leq 1$

k_i Player i's process sensitivity for the fraction p_{it} in the replicator equation, $k_i \geq 0$

h Process sensitivity for the fraction q_{1t} in the replicator equation, $h \geq 0$

Independent variables

t Time, $t \geq t_0$

p_{it} Volume fraction of player i's transactions in currency g at time t, $0 \leq p_{it} \leq 1$

q_{it} Fraction q_{it} of players of kind i at time t, $0 \leq q_{it} \leq 1$, $q_{1t} = 1 - q_{2t}$

Dependent variables

p_t Volume fraction of all players' transactions in currency g at time t, $0 \leq p_t \leq 1$

u_{igt} Player i's utility of transacting in the fixed-supply currency g at time t, $u_{igt} \geq 0$

u_{int} Player i's utility of transacting in the variable-supply currency n, $u_{int} \geq 0$

u_{it} Player i's weighted utility of transacting in both currencies, $u_{it} \geq 0$

u_t Society's utility weighing the utilities of all players of both kinds, $u_t \geq 0$

Overview of the model. Section "Simplified player utilities" presents the simplified player utilities where two kinds of players receive a fixed utility depending on their support of a fixed-supply currency to two different extents. They also receive a variable utility of transacting in the variable-supply currency depending on money printing/withdrawal of that currency and inflation/deflation. Section "More realistic player utilities" generalizes so that the two kinds of players' utilities also depend on their support of a given currency, the volume fraction of all players' (of both kinds) transactions in the given currency, and the fraction of players of the same kind as the player being analyzed. Section "Replicator dynamics" introduces three replicator equations specifying each player's volume fraction of transactions in each currency, and which kind of player each player prefers to be.

Simplified player utilities. Consider two kinds of players referred to as kind $i, i = 1, 2$. Assume that player i (i.e. player of kind i) earns a simplified utility u_{igst} of transacting in the fixed-supply currency g proportional to player i's support s_{it}, $0 \leq s_{it} \leq 1$, of currency g relative to currency n at time t, i.e.

$$u_{igst} = 0.5 s_{it} \quad (1)$$

where the scaling 0.5 is chosen to ensure comparison with the generalization in the next section. Assume further that player i's utility u_{inst} of transacting in the variable-supply currency n is proportional to its support $1 - s_{it}$ of currency n. Player i's utility u_{inst} also depends on the variable money supply S_j and inflation/deflation π_j expressed on the Cobb Douglas form with elasticities α_i and $1 - \alpha_i$, respectively, $0 \leq \alpha_i \leq 1$. We assume money supply S_j, $S_j \in \mathbb{R}$, at the discrete times $j = t_0, t_0 + 1, \ldots, T$, where $t_0 \geq 0$ is the initial time and T is the final time. Any time interval of length 1 applies, e.g. year, month, week, day, etc. Thus $S_{j+1} - S_j$ is the changed supply from time j to time $j+1$, $\sum_{j=t_0}^{t-1}\left(S_{j+1} - S_j\right)$ is the changed supply from $j = t_0$ to $j = t - 1$, and $\frac{S_{t_0} + \sum_{j=t_0}^{t-1}(S_{j+1} - S_j)}{S_{t_0}}$ is the supply at time t divided by the supply at time t_0 which expresses player i's purchasing power at time t relative to its purchasing power at time t_0 without inflation. With inflation π_j, $\pi_j \in \mathbb{R}$, at time $j = t_0, \ldots, T$, an asset valued as 1 at time $j = t_0$ is valued as $\frac{1}{\prod_{j=t_0+1}^{t}(1+\pi_j)}$ at time $j = t$, thus degrading the asset value due to accumulative inflation if $\prod_{j=t_0+1}^{t}\left(1 + \pi_j\right) > 1$, and increasing the asset value otherwise. Thus player i's simplified utility of transacting in the variable-supply currency n is

$$u_{inst} = 0.5\left(1 - s_{it}\right)\left(\frac{S_{t_0} + \sum_{j=t_0}^{t-1}\left(S_{j+1} - S_j\right)}{S_{t_0}}\right)^{\alpha_i}\left(\frac{1}{\prod_{j=t_0+1}^{t}\left(1 + \pi_j\right)}\right)^{1-\alpha_i} \quad (2)$$

If $\alpha_i > 0.5$, player i assigns more weight to purchasing power than to inflation/deflation, and conversely if $\alpha_i < 0.5$. Equal weights $\alpha_i = 0.5$ can theoretically be conceptualized as equating the two last Cobb Douglas terms in Eq. (2) with 1 where player i's adjusted purchasing power from adjusted money supply $S_{j+1} - S_j$ is exactly offset by inflation/deflation π_j through time.

More realistic player utilities. A fraction q_{it} of the players are of kind i at time t, where $q_{1t} = 1 - q_{2t}$, $0 \leq q_{1t} \leq 1$. Player i chooses a volume fraction p_{it} of its transactions in currency g, and the remaining volume fraction $1 - p_{it}$ of its transactions in currency n, see Fig. 1 which exemplifies with $p_{1t} > p_{2t}$ and $q_{1t} < q_{2t}$, but generally $0 \leq p_{it} \leq 1$, $0 \leq q_{it} \leq 1$, $i = 1, 2$.

Hence the volume fraction p_t at time t of all players' transactions in currency g is the weighted sum of each player i's volume fraction p_{it} in currency g, weighted by the fraction q_{it} of each kind of player $i, i = 1, 2$, i.e.

$$p_t = p_{1t}q_{1t} + p_{2t}q_{2t} \quad (3)$$

More realistically than the previous section "Simplified player utilities", assume that player i earns a utility u_{igt} of transacting in the fixed-supply currency g proportional to three factors, i.e. its support s_{it} of currency g relative to currency n, the volume fraction p_t of all players' (of both kinds) transactions in currency g, and the fraction q_{it} of players of kind i. We operationalize the latter as $1 + \mu_i q_{it}^{m_i}$, where μ_i, $\mu_i \geq 0$ is a scaling proportionality parameter, and m_i, $m_i \geq 0$, is a scaling exponent. Thus a negligible fraction $q_{it} \approx 0$ causes the proportionality parameter ≈ 1, and a dominant fraction $q_{it} = 1$ causes the proportionality parameter $1 + \mu_i$. Generalizing Eq. (1), player i's utility of transacting in the fixed-supply currency g is

$$u_{igt} = s_{it}\left(p_{1t}q_{1t} + p_{2t}q_{2t}\right)\left(1 + \mu_i q_{it}^{m_i}\right) \quad (4)$$

Analogously, player i's utility of transacting in the variable-supply currency n is proportional to the same three factors, i.e. its support $1 - s_{it}$ of currency n, the volume fraction $1 - p_t$ of all

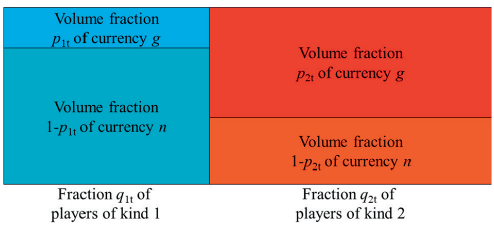

Fig. 1 Volume fractions p_{1t} and p_{2t} of transactions in currencies g and n for two kinds of players of different fractions q_{1t} and q_{2t}. Player i, $i = 1, 2$, chooses a volume fraction p_{it} of its transactions in currency g, and $1 - p_{it}$ in currency n, $0 \leq p_{it} \leq 1$, $0 \leq q_{it} \leq 1$, $q_{1t} + q_{2t} = 1$, $i = 1, 2$.

players' transactions in currency n, and $1+\mu_i q_{it}^{m_i}$. Generalizing Eq. (2), player i's utility of transacting in the variable-supply currency n is

$$u_{int} = (1-s_{it})(1-p_{1t}q_{1t}-p_{2t}q_{2t})(1+\mu_i q_{it}^{m_i}) \times \left(\frac{S_{t_0}+\sum_{j=t_0}^{t-1}(S_{j+1}-S_j)}{S_{t_0}}\right)^{\alpha_i}\left(\frac{1}{\prod_{j=t_0+1}^{t}(1+\pi_j)}\right)^{1-\alpha_i} \quad (5)$$

Equations (4), (5) simplify to Eqs. (1), (2) when $p_{it}=q_{it}=0.5$ and $\mu_i=0$. Player i's utility at time t is the weighted combination of its volume fraction p_{it} of transactions in the fixed-supply currency g, and its remaining volume fraction $1-p_{it}$ in the variable-supply currency n, i.e.

$$u_{it} = p_{it}u_{igt} + (1-p_{it})u_{int} \quad (6)$$

Society's utility, comprising all players of both kinds, is

$$u_t = q_1 u_{1t} + (1-q_1)u_{2t} \quad (7)$$

Replicator dynamics

Player i's volume of transactions in the fixed-supply currency g. To analyze the evolution of the fraction p_{it} of player i's volume of transactions in the fixed-supply currency g, causing $1-p_{it}$ to be in currency n, the replicator equation (Taylor and Jonker, 1978; Weibull, 1997)

$$\frac{\partial p_{it}}{\partial t} = k_i p_{it}(u_{igt}-u_{it}) = k_i p_{it}(1-p_{it})(u_{igt}-u_{int}) \quad (8)$$

is applied, inserting Eq. (6), where $k_i>0$ is the process sensitivity, i.e. how rapidly the fraction p_{it} changes. Intermediate k_i causes a stable process, while high and low p_{it} give quick and slow changes, respectively. The right-hand side of Eq. (8) is proportional to the difference $u_{igt}-u_{it}$ between player i's utility of transacting in the fixed-supply currency g and the weighted combination of both utilities in Eq. (6), and also proportional to the difference $u_{igt}-u_{int}$ between player i's utility of transacting in the fixed-supply currency g and the variable-supply currency n. When u_{igt} exceeds u_{it} or u_{int}, the fraction p_{it} increases, and decreases otherwise. The right-hand side of Eq. (8) is furthermore proportional to $p_{nt}(1-p_{nt})$ which is inverse U shaped with a maximum at $p_{it}=0.5$ and minima at $p_{it}=0$ and $p_{it}=1$. The fractions p_{it} and $1-p_{it}$ change most quickly when equally large, and most slowly when one fraction dominates the other.

The fraction q_{1t} of players of kind 1. If we allow each player of kind 1 to change its preferences so as to be of kind 2, and each player of kind 2 to be of kind 1, we can analyze the analogous evolution of the fraction q_{1t} of players of kind 1, causing $q_{2t}=1-q_{1t}$ to be of kind 2, i.e.

$$\frac{\partial q_{1t}}{\partial t} = hq_{1t}(u_{1t}-u_t) = hq_{1t}(1-q_{1t})(u_{1t}-u_{2t}) \quad (9)$$

where Eq. (7) is inserted and the process sensitivity $h>0$ is interpreted analogously to $k_i>0$ in Eq. (8).

Analyzing the model

The US 1659–2021. Figure 2a, b plots the US M2 money supply S_j (Federal Reserve, 2022) and the US inflation π_i (CPI Inflation Calculator, 2022) from time $t_0=1959$ to time $T=2021$. Figure 2c uses Eqs. (4), (5) and the empirics in Fig. 2a, b to plot player i's utilities u_{igt} and u_{int} of transacting in both currencies, assuming support $s_{it}=0.5$, equal volume fractions $p_{it}=0.5$ of transactions in both currencies, equal fractions $q_{it}=0.5$ of both kinds of players, scaling proportionality parameter $\mu_i=0$, and Cobb Douglas elasticities $\alpha_i=0.6, 0.5, 0.35, 0.2$. Player i's utility is constant at $u_{igt}=0.25$ since currency g has no changes in supply and no inflation. High and intermediate weights $\alpha_i=0.6$ and $\alpha_i=0.5$ for changes in money supply S_j causes player i's utility u_{int} to increase. Low weight $\alpha_i=0.35$ causes u_{int} to oscillate slightly above and below $u_{int}=0.25$. Very low weight $\alpha_i=0.2$ causes u_{int} to decrease overall. Figure 2c uses Eqs. (6) and (7) to plot player i's weighted utility u_{iAt} of transacting in both currencies and society's utility u_{At} weighing the utilities of all players of both kinds. These two utilities $u_{iAt}=u_{At}$ are equal since $p_{it}=q_{it}=0.5$. Since $u_{igt}=0.25$, the weighted utilities $u_{iAt}=u_{At}$ increase less for $\alpha_i=0.6$ and $\alpha_i=0.5$ and decrease less for $\alpha_i=0.2$.

Replicator dynamics with simplified utilities u_{igst} and u_{inst} in Eqs. (1) and (2). Figure 3 applies the simplified utilities u_{igst} and u_{inst} in Eqs. (1), (2) and the replicator equation in Eq. (8) to plot player i's fraction p_{it} 1959–2021 with the same assumptions as in Fig. 2, i.e. $q_{it}=0.5$, $\mu_i=0$, and $0.01\le s_{it}\le 0.99$. Player i's process sensitivity and initial condition are $k_i=p_{it_0}=0.5$. Figure 3a assumes the high weight $\alpha_i=0.6$ for money supply S_j. With low support $s_{it}\le 0.5$ for the fixed-supply currency g relative to the variable-supply currency n, the fraction p_{it} of transactions in currency g decreases towards zero. With higher support $s_{it}=0.6$, the fraction increases to a maximum $p_{it}=0.59$ in 1972, and thereafter decreases towards $\lim_{t\to T}p_{it}\approx 0$. That eventual decrease occurs because of the high weight $\alpha_i=0.6$ assigned to money supply S_j, which for the US 1959–2021 has meant preferable money printing, which is impossible for the fixed-supply currency g. With higher support $s_{it}=0.7$, the fraction increases to a maximum $p_{it}=0.84$ in 1990, and thereafter decreases. With very high support $s_{it}=0.99$, the fraction increases towards $\lim_{t\to T}p_{it}\approx 1$. Hence sufficiently high support s_{it} for currency g can cause player i to prefer it even with high weight assigned to money supply S_j. Figure 3b assumes the low weight $\alpha_i=0.2$ for money supply S_j. High support $s_{it}\ge 0.6$ then causes the fraction p_{it} to quickly increase towards $\lim_{t\to T}p_{it}\approx 1$. Intermediate support $s_{it}=0.5$ causes the fraction p_{it} to decrease marginally to $p_{it}=0.498$ in 1968, and thereafter increase towards $\lim_{t\to T}p_{it}\approx 1$. Support $s_{it}=0.4$ causes p_{it} to decrease to $p_{it}=0.32$ in 1979, and thereafter to increase. Support $s_{it}=0.3$ causes p_{it} to decrease to $p_{it}=0.115$ in 2000, and thereafter to increase marginally to $p_{it}=0.126$ in 2021. Negligible support $s_{it}=0.01$ causes p_{it} to decrease quickly to $\lim_{t\to T}p_{it}\approx 0$.

Figure 3c, d makes the same assumptions as Fig. 3a, b except that the process sensitivity is 10 times higher, i.e. $k_i=5$. That causes p_{it} to approach $\lim_{t\to T}p_{it}\approx 0$ more quickly when $s_{it}\le 0.3$ and approach $\lim_{t\to T}p_{it}\approx 1$ more quickly when $s_{it}\ge 0.99$. In Fig. 3c where $\alpha_i=0.6$, p_{it} when $s_{it}=0.6$ reaches a higher maximum $p_{it}=0.59$ than in Fig. 3a, but in the same year 1972. Also in Fig. 3c, p_{it} when $s_{it}=0.7$ reaches a maximum extremely close to 1 (determined numerically as $p_{it}=0.9999999314$), which is higher than in Fig. 3a, and in the same year 1990, and thereafter decreases towards $\lim_{t\to T}p_{it}\approx 0$. Similarly in Fig. 3d where $\alpha_i=0.2$, p_{it} when $s_{it}=0.5$ reaches a lower minimum $p_{it}=0.476$ than in Fig. 3b, and in the same year 1968, and thereafter increases towards $\lim_{t\to T}p_{it}\approx 1$. Also in Fig. 3d, p_{it} when $s_{it}=0.4$ reaches a minimum extremely close to 0 (determined numerically as $p_{it}=0.000429$), which is lower than in Fig. 3b, and in the same year 1979, and thereafter increases towards $\lim_{t\to T}p_{it}\approx 1$.

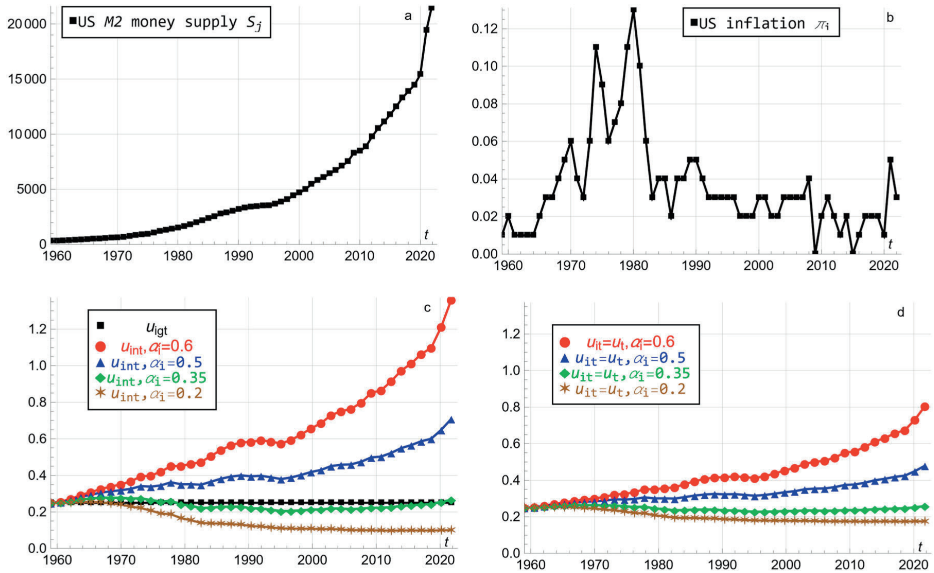

Fig. 2 US M2 money supply, US inflation, player utilities and society's utility. a US M2 money supply S_j 1959–2021 in $billion. **b** US inflation π_i 1959–2021. **c** and **d** Player i's utilities u_{igt}, u_{int}, u_{it}, u_t as functions of time t when $s_{it} = p_{it} = q_{it} = 0.5$, $\mu_i = 0$ and $\alpha_i = 0.6, 0.5, 0.35, 0.2$.

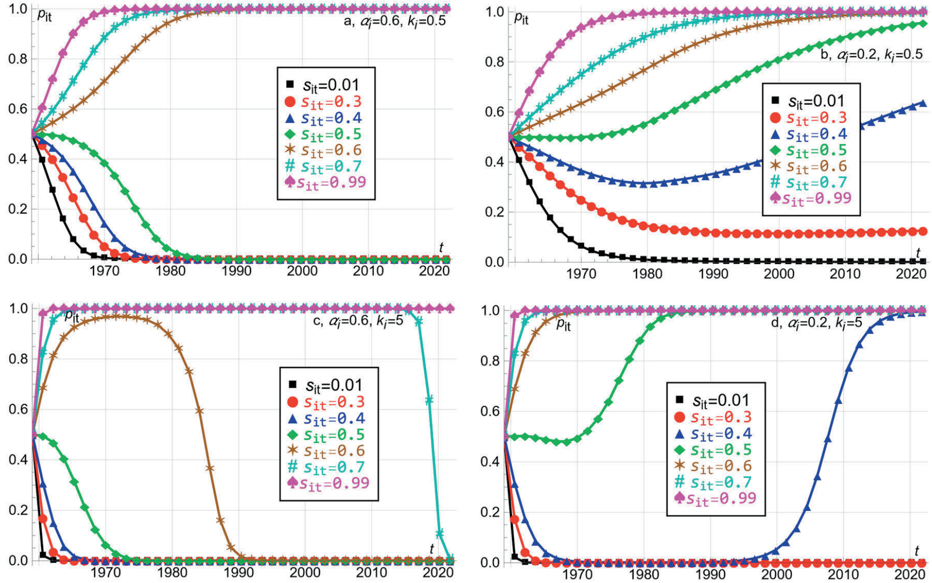

Fig. 3 The volume fraction p_{it} of player i's transactions in currency g at time t 1959–2021 with simplified utilities u_{igst} and u_{inst} in Eqs. (1) and (2) when $p_{it_0} = 0.5$, $\mu_i = 0$, and $0.01 \le s_{it} \le 0.99$. a $\alpha_i = 0.6$, $k = 0.5$, **b** $\alpha_i = 0.2$, $k = 0.5$, **c** $\alpha_i = 0.6$, $k = 5$ and **d** $\alpha_i = 0.2$, $k = 5$.

Replicator dynamics with the utilities u_{igt} and u_{int} in Eqs. (4) and (5). Figure 4 applies the utilities u_{igt} and u_{int} in Eqs. (4), (5), and Eq. (8) to plot p_{it} with the same assumptions as in Fig. 3, i.e. $q_{it} = p_{it_0} = k_i = 0.5$, $\mu_i = 0$, and $0.01 \le s_{it} \le 0.99$. Accounting for p_{it} in the utilities u_{igt} and u_{int} causes p_{it} to approach $\lim_{t\to T} p_{it} \approx 0$ or $\lim_{t\to T} p_{it} \approx 1$ more quickly than in Fig. 3. With high weight $\alpha_i = 0.6$ assigned to money supply S_j, two curves that approach $\lim_{t\to T} p_{it} \approx 0$ or eventually decrease favoring currency n in Fig. 3a, approach $\lim_{t\to T} p_{it} \approx 1$ in Fig. 4a so that player i prefers currency g instead. First, with high support $s_{it} = 0.7$ for currency g, p_{it}>0.5 until 2019 in Fig. 3a which positively impacts player i's utility u_{igt}

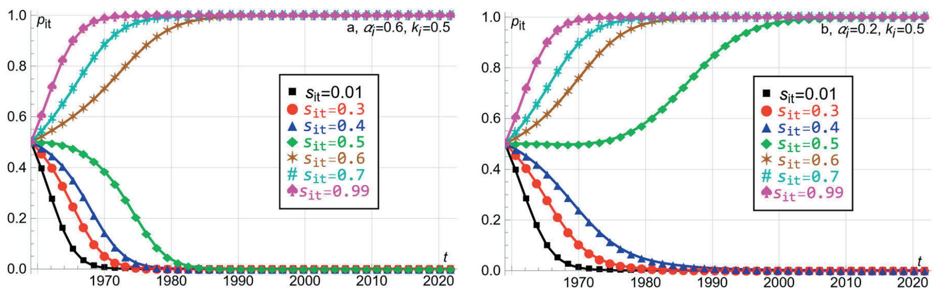

Fig. 4 The volume fraction p_{it} of player i's transactions in currency g at time t 1959–2021 with the utilities u_{igt} and u_{int} in Eqs. (4) and (5) when $q_{it} = p_{it_0} = k_i = 0.5$, $\mu_i = 0$, and $0.01 \le s_{it} \le 0.99$. a $\alpha_i = 0.6$ and **b** $\alpha_i = 0.2$.

causing player i to favor currency g in Fig. 4a. Second, with slightly lower support $s_{it} = 0.6$ for currency g, p_{it}>0.5 until 1985 in Fig. 3a which is sufficient for player i to quickly favor currency g in Fig. 4a, contrary to Fig. 3a. With low weight $\alpha_i = 0.2$ assigned to money supply S_j, only one curve that eventually increases in Fig. 3b, with support $s_{it} = 0.4$, quickly decreases in Fig. 4b. That curve eventually increases in Fig. 3b since player i's utility u_{igt} does not depend on p_{it}. That enables player i to favor currency g since low weight $\alpha_i = 0.2$ assigned to money supply S_j causes player i to prefer to avoid the inflation associated with currency n. The opposite result follow in Fig. 4b since p_{it}<0.5 until 2008, causing p_{it} to quickly decrease towards $\lim_{t\to T} p_{it} \approx 0$ where currency n is preferred.

Replicator dynamics when players support currency g differently with $s_{1t} \neq s_{2t}$. This section assumes that the two kinds of players support currency g differently with $s_{1t} \neq s_{2t}$. Figure 5 applies Eq. (8) to plot the volume fractions p_{1t} and p_{2t} of player i's transactions, $i = 1, 2$, in currency g with the same assumptions as in Fig. 4, i.e. $q_{it} = p_{it_0} = k_i = 0.5$, $\mu_i = 0$, and $0.01 \le s_{it} \le 0.99$. Additionally, $s_{1t} \neq s_{2t}$. With high weight $\alpha_i = 0.6$ assigned to money supply S_j, negligible support $s_{1t} = 0.01$ by player 1 and more support $s_{2t} \le 0.7$ by player 2 cause both volume fractions to eventually approach $\lim_{t\to T} p_{it} \approx 0$ favoring currency n, though p_{2t} initially experiences an inverse U shape. Although the high support $s_{1t} = s_{2t} = 0.7$ comfortably enables both players to eventually transact exclusively in currency g in Fig. 4a, $\lim_{t\to T} p_{2t} \approx 1$, the opposite result follows in Fig. 5a since player 1 supports currency g much less at $s_{1t} = 0.01$. Negligible support $s_{1t} = 0.01$ by player 1 and overwhelming support $s_{2t} = 0.99$ by player 2 cause opposite results for the two players, i.e. $\lim_{t\to T} p_{1t} \approx 0$ for player 1 and $\lim_{t\to T} p_{2t} \approx 1$ for player 2. Support $s_{1t} = 0.3$ by player 1 and more support $s_{2t} = 0.7$ by player 2 cause both volume fractions to eventually approach $\lim_{t\to T} p_{it} \approx 0$ favoring currency n, though p_{2t} initially experiences a higher inverse U shape than when $s_{1t} = 0.01$. Support $s_{1t} = 0.3$ by player 1 and overwhelming support $s_{2t} = 0.99$ by player 2 also cause opposite results for the two players, although player 1's volume fraction p_{1t} approaches $\lim_{t\to T} p_{1t} \approx 0$ more slowly than when $s_{1t} = 0.01$, $\lim_{t\to T} p_{2t} \approx 1$. Support $s_{1t} = 0.4$ by player 1 and more support $s_{2t} = 0.7$ by player 2 cause both volume fractions to eventually approach $\lim_{t\to T} p_{it} \approx 0$ favoring currency n, though p_{2t} initially experiences a higher inverse U shape than when $s_{1t} = 0.3$. Support $s_{1t} = 0.4$ by player 1 and overwhelming support $s_{2t} = 0.99$ by player 2 interestingly cause both volume fractions to eventually approach $\lim_{t\to T} p_{it} \approx 0$ favoring currency g. Although support $s_{1t} = s_{2t} = 0.4$ causes both players to eventually transact exclusively in currency n in Fig. 4a, $\lim_{t\to T} p_{2t} \approx 0$, the opposite result follows in Fig. 5b since player 2 supports currency g much more at $s_{1t} = 0.99$, which enables player 1 to also eventually support currency g. Support $s_{1t} = 0.5$ by player 1 and more support $s_{2t} = 0.6$ by player 2 cause both volume fractions to eventually approach $\lim_{t\to T} p_{it} \approx 0$ favoring currency n. Both fractions approach $\lim_{t\to T} p_{it} \approx 0$ slowly, and p_{2t} initially experiences an inverse U shape. Support $s_{1t} = 0.5$ by player 1 and more support $s_{2t} \ge 0.7$ by player 2 cause both volume fractions to eventually approach $\lim_{t\to T} p_{it} \approx 1$ favoring currency g. This interesting result shows that when $s_{1t} = 0.5$ for player 1, merely increasing player 2's support from $s_{2t} = 0.6$ to $s_{2t} = 0.7$ causes both players to eventually change their preferences from currency n to currency g.

With low weight $\alpha_i = 0.2$ assigned to money supply S_j, both players generally prefer currency g more easily. Negligible support $s_{1t} = 0.01$ by player 1 and more support $s_{2t} = 0.6$ by player 2 cause both volume fractions to eventually approach $\lim_{t\to T} p_{it} \approx 0$ favoring currency n, though p_{2t} in Fig. 5c initially experiences a lower inverse U shape than in Fig. 5a. Negligible support $s_{1t} = 0.01$ by player 1 and more support $s_{2t} \ge 0.7$ by player 2 cause opposite results for the two players, i.e. $\lim_{t\to T} p_{1t} \approx 0$ for player 1 and $\lim_{t\to T} p_{2t} \approx 1$ for player 2, so that player 2 eventually prefers currency g. This result in Fig. 5c differs from Fig. 5a when $s_{2t} = 0.7$ where $s_{2t} = 0.7$ causes both players to eventually prefer currency n. Support $s_{1t} = 0.3$ by player 1 and more support $s_{2t} = 0.5$ by player 2 cause both volume fractions to eventually approach $\lim_{t\to T} p_{it} \approx 0$ favoring currency n. Support $s_{1t} = 0.3$ by player 1 and even more support $s_{2t} = 0.6$ by player 2 cause the fraction p_{1t} for player 1 to decrease towards $\lim_{t\to T} p_{1t} \approx 0$, while the fraction p_{2t} for player 2 increases overall extremely slowly towards $\lim_{t\to T} p_{2t} \approx 0.89$ in 2021, in major support of currency g. Support $s_{1t} = 0.3$ by player 1 and yet more support $s_{2t} = 0.7$ by player 2 cause player 2's fraction p_{2t} to increase quickly towards $\lim_{t\to T} p_{2t} \approx 1$. Player 1's fraction p_{1t} is U shaped towards a minimum, and thereafter increases slowly towards $\lim_{t\to T} p_{1t} \approx 0.30$ in 2021. Although player 1 supports currency g modestly at $s_{1t} = 0.3$, player 2's higher support $s_{2t} = 0.7$ causes player 1 to choose currency g to some modest extent. Support $s_{1t} = 0.3$ by player 1 and overwhelming support $s_{2t} = 0.99$ by player 2 cause player 2's fraction p_{2t} to increase quickly towards $\lim_{t\to T} p_{2t} \approx 1$. Player 1's fraction p_{1t} is first U shaped towards a minimum that is

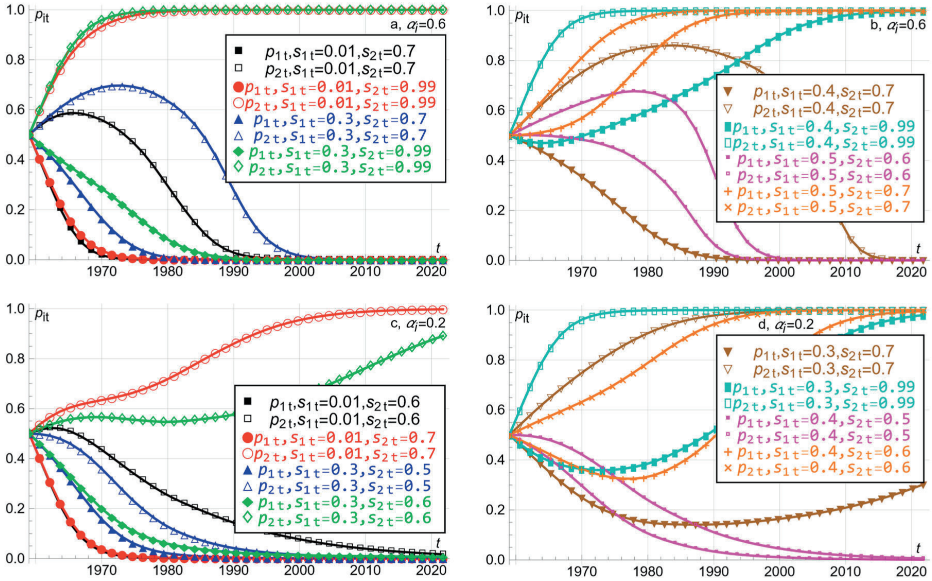

Fig. 5 The volume fractions p_{1t} and p_{2t} of the two kinds of players' transactions in currency g at time t 1959–2021 with different support $s_{1t} \neq s_{2t}$ when $q_{it} = p_{it_0} = k_i = 0.5$, $\mu_i = 0$, and $0.01 \le s_{it} \le 0.99$. a and **b** $\alpha_i = 0.6$. **c** and **d** $\alpha_i = 0.2$.

higher than when $s_{2t} = 0.7$, and thereafter increases logistically towards $\lim_{t \to T} p_{1t} \approx 0.98$. Despite low support $s_{1t} = 0.3$, player 1 eventually supports currency g substantially. Support $s_{1t} = 0.4$ by player 1 and more support $s_{2t} = 0.5$ by player 2 cause both volume fractions to slowly and eventually approach $\lim_{t \to T} p_{it} \approx 0$ favoring currency n. Support $s_{1t} = 0.4$ by player 1 and more support $s_{2t} \ge 0.6$ by player 2 cause player 2's fraction p_{2t} to increase towards $\lim_{t \to T} p_{2t} \approx 1$, while player 1's fraction p_{1t} is U shaped towards a minimum (when $s_{2t} = 0.6$) and thereafter increases towards $\lim_{t \to T} p_{1t} \approx 1$.

Replicator dynamics when the fraction q_{it} of players of kind i changes. This section assumes that the fraction q_{it} of players of kind i changes through time. Figure 6 applies Eqs. (8), (9) to plot the volume fractions p_{1t} and p_{2t} of player i's transactions, $i = 1, 2$, in currency g and the fraction q_{1t} of players of kind 1 with the same assumptions as in Fig. 5 except that q_{1t} varies instead of $q_{it} = 0.5$, i.e. $p_{it_0} = k_i = 0.5$, $\mu_i = 0$, $0.01 \le s_{it} \le 0.99$, $s_{1t} \neq s_{2t}$. Additionally, we assume the process sensitivity $h = 0.5$ for the fraction q_{1t} and initial condition $q_{1t_0} = 0.5$

With high weight $\alpha_i = 0.6$ assigned to money supply S_j, the first three combinations of curves in Fig. 5 with support (s_{1t}, s_{2t}) equal to $(0.01, 0.7)$, $(0.01, 0.99)$, $(0.3, 0.7)$ eventually implying $\lim_{t \to T} p_{1t} \approx 0$, cause the fraction q_{1t} of players of kind 1 to increase towards 1. According to Eq. (9), the players prefer to be of kind 1 when $u_{1t} \ge u_{2t}$, i.e. when $p_{1t}u_{1gt} + (1 - p_{1t})u_{1nt} \ge p_{2t}u_{2gt} + (1 - p_{2t})u_{2nt}$ according to Eq. (6), which approaches $u_{1nt} \ge u_{2nt}$ when $\lim_{t \to T} p_{1t} \approx 0$. The three support combinations $(0.01, 0.7)$, $(0.01, 0.99)$, $(0.3, 0.7)$ satisfy $s_{1t} \le s_{2t} \Leftrightarrow 1 - s_{1t} \ge 1 - s_{2t}$ which is inserted into Eq. (5) to give $u_{1nt} \ge u_{2nt}$ when $\lim_{t \to T} p_{1t} \approx 0$. Non-mathematically, players prefer to be of kind 1 since they prefer currency n which gives higher utility $u_{1nt} \ge u_{2nt}$ when $s_{1t} \le s_{2t}$. That is, the players converge towards transacting in currency n compatibly with kind 1 supporting currency n much more than currency g. With support $(s_{1t}, s_{2t}) = (0.3, 0.99)$, player 2's volume fraction p_{2t} of transactions in currency g approaches $\lim_{t \to T} p_{2t} \approx 1$ in Fig. 5, and in Fig. 6 $\lim_{t \to T} p_{it} \approx 1$, which causes the opposite result where players prefer to be of kind 2. That is, $u_{1t} \le u_{2t}$ implies $p_{1t}u_{1gt} + (1 - p_{1t})u_{1nt} \le p_{2t}u_{2gt} + (1 - p_{2t})u_{2nt}$ approaches $u_{1gt} \le u_{2gt}$ when $\lim_{t \to T} p_{it} \approx 1$. Support $(s_{1t}, s_{2t}) = (0.3, 0.99)$ means that $s_{1t} \le s_{2t}$ which is inserted into Eq. (4) to give $u_{1gt} \le u_{2gt}$ when $\lim_{t \to T} p_{it} \approx 1$. Non-mathematically, players prefer to be of kind 2 since they prefer currency g which gives higher utility $u_{2gt} \ge u_{1gt}$ when $s_{2t} \ge s_{1t}$. That is, the players converge towards transacting in currency g compatibly with kind 2 supporting currency g much more than currency n.

With this insight the interpretations of the subsequent panels in Fig. 6 is straightforward. That is, $\lim_{t \to T} p_{it} \approx 0$ so that players eventually prefer to transact in currency n implies that players prefer to be of kind 1 which gives higher utility $u_{1nt} \ge u_{2nt}$ when $s_{1t} \le s_{2t}$. In contrast, $\lim_{t \to T} p_{it} \approx 1$ so that players eventually prefer to transact in currency g implies that players prefer to be of kind 2 which gives higher utility $u_{2gt} \ge u_{1gt}$ when $s_{2t} \ge s_{1t}$.

Replicator dynamics with positive scaling proportionality parameter μ_i. This section assumes that the scaling proportionality parameter μ_i in player i's utilities u_{igt} and u_{int} is positive. When μ_i increases, player i's utilities u_{igt} and u_{int} in Eqs. (4) and (5) of transacting in both currencies g and n increase equally much. The increase is proportional to the fraction q_{it} of players of kind i at time t raised to the parameter m_i. If both μ_1 and μ_2 increase equally much, both u_{igt} and u_{int} increase which in the replicator Eq. (8) can be interpreted as increasing the process sensitivity k_i, which means quicker changes which are otherwise qualitatively similar to Fig. 6. Figure 7 makes the same

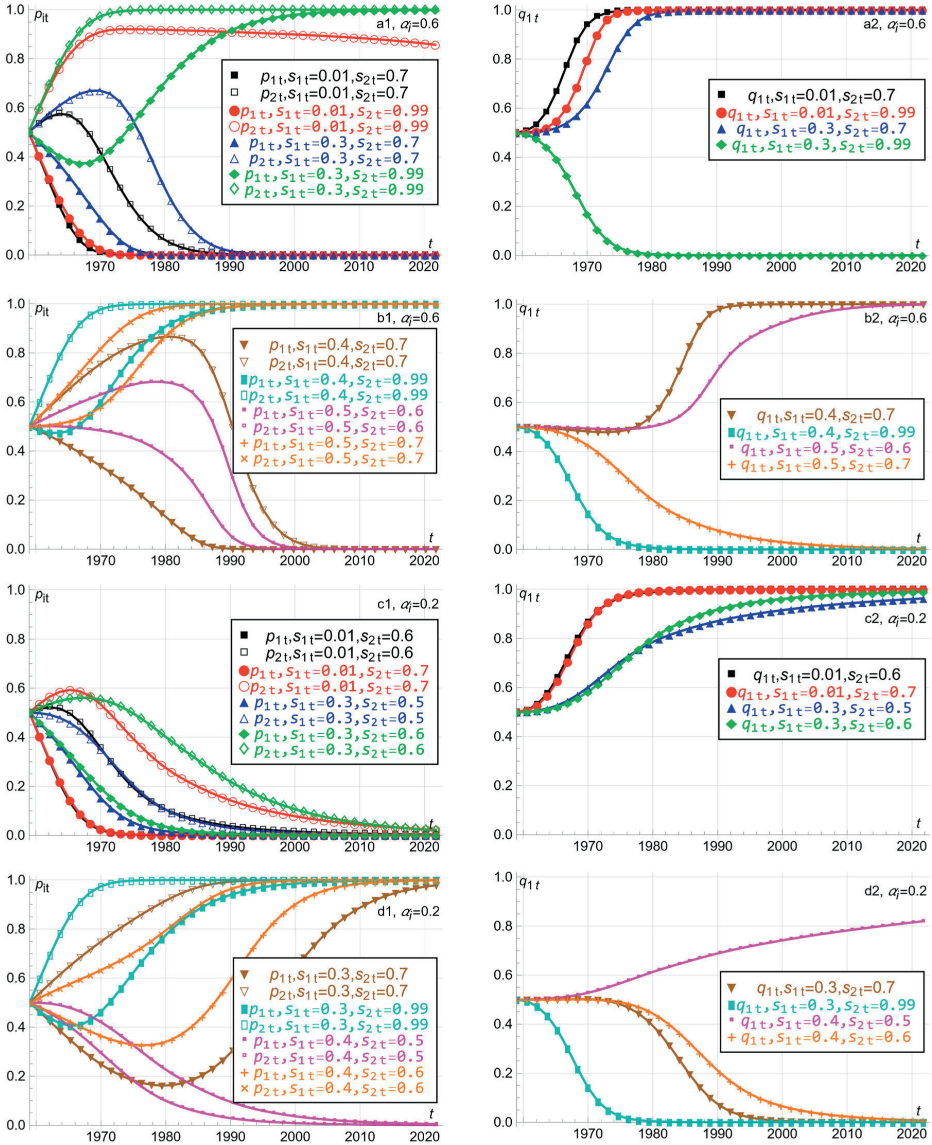

Fig. 6 The fractions p_{1t}, p_{2t}, q_{1t} at time t 1959–2021 with different support $s_{1t} \neq s_{2t}$ when $q_{it_0} = p_{it_0} = k_i = h = 0.5$, $\mu_i = 0$, and $0.01 \le s_{it} \le 0.99$. a1, **a**2, **b**1, **b**2 $\alpha_i = 0.6$. **c**1, **c**2, **d**1, **d**2 $\alpha_i = 0.2$.

assumptions as in Fig. 6 except that $\mu_2 = 1$ and $\mu_1 = 0$, i.e. $q_{1t_0} = p_{it_0} = k_i = h = 0.5$, $\mu_i = 0$, $0.01 \le s_{it} \le 0.99$, $s_{1t} \neq s_{2t}$. The higher $\mu_2 = 1 > \mu_1 = 0$ means that players to a higher extent than in Fig. 6 tend to prefer to be of kind 2 which gives higher utilities u_{2gt} and u_{2nt}. Hence Fig. 7 shows three, four, two, four curves (summing to 13 curves out of 16 possible curves) for the fraction q_{1t} of players of kind 1 at time t approaching $\lim_{t \to T} q_{1t} \approx 0$, as compared with one, two, zero, three curves (summing to only six curves), respectively, approaching $\lim_{t \to T} q_{1t} \approx 0$ in Fig. 6. In Fig. 7a1 the low support $s_{1t} = 0.01$ of player 1 for currency g causes both players to eventually not transact in currency g when $s_{2t} = 0.7$, as explained for Fig. 6, which implies that players prefer to be of kind 1 since they prefer currency n which gives higher utility $u_{1nt} \ge u_{2nt}$ when $s_{1t} \le s_{2t}$. The corresponding curve q_{1t} in Fig. 7a2 gives $\lim_{t \to T} q_{1t} \approx 1$, while the other three curves with higher support $s_{1t} + s_{2t}$ give $\lim_{t \to T} q_{1t} \approx 0$ so that the players prefer to be of kind 2. Fig. 7b1, b2 with higher support $s_{1t} + s_{2t}$ shows a clearer trend where $\lim_{t \to T} p_{it} \approx 0$ and $\lim_{t \to T} q_{1t} \approx 0$ so that players

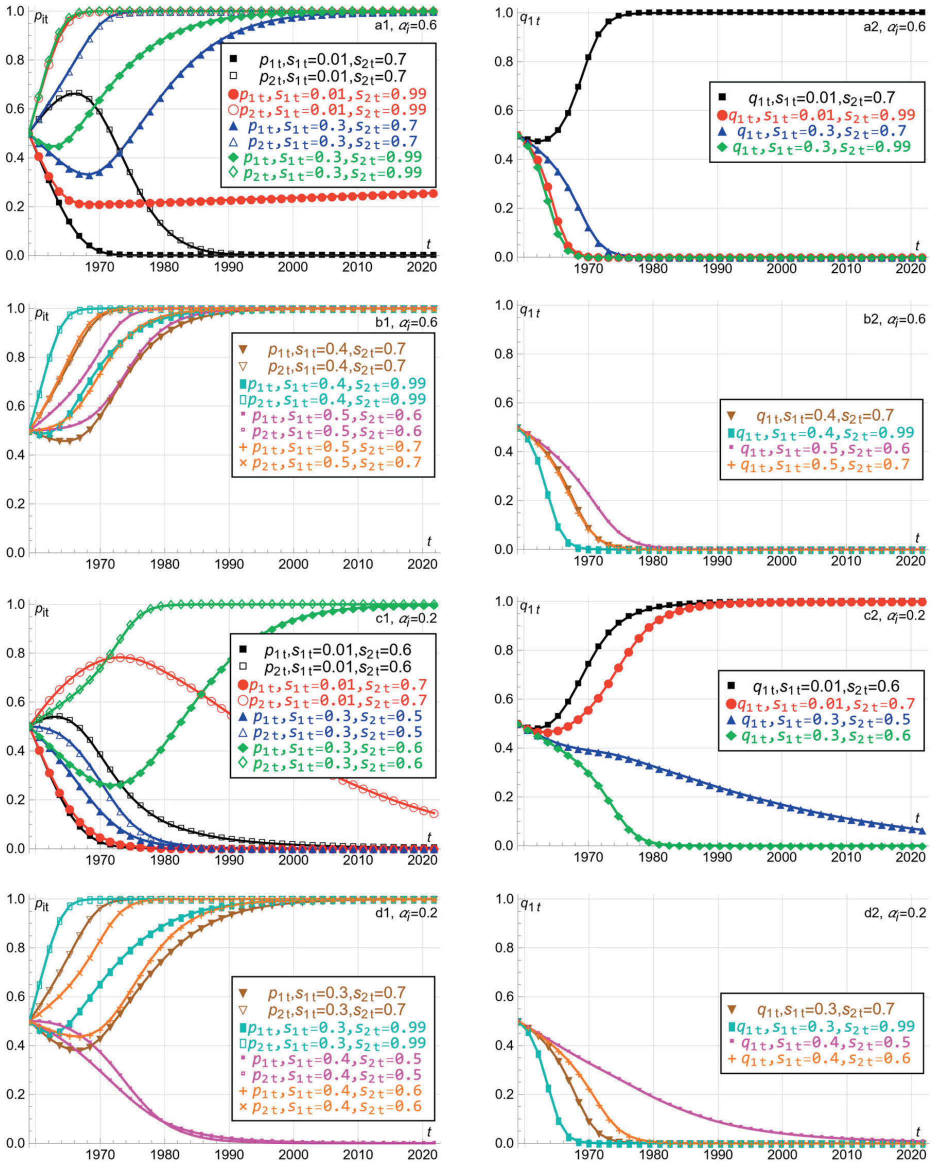

Fig. 7 The fractions p_{1t}, p_{2t}, q_{1t} at time t 1959–2021 with different support $s_{1t} \neq s_{2t}$ when $q_{it_0} = p_{it_0} = k_i = h = 0.5$, $\mu_2 = 1$, $\mu_1 = 0$, and $0.01 \le s_{it} \le 0.99$. **a**1, **a**2, **b**1, **b**2 $\alpha_i = 0.6$. **c**1, **c**2, **d**1, **d**2 $\alpha_i = 0.2$.

prefer to be of kind 2. Figure 7c2 shows two curves, with support (s_{1t}, s_{2t}) equal to $(0.3, 0.5)$, $(0.3, 0.6)$, eventually approaching $\lim_{t\to T} q_{1t} \approx 0$ so that players prefer to be of kind 2, in contrast to Fig. 6c2 which has no such curves. Figure 7d2 shows how all the four curves eventually approach $\lim_{t\to T} q_{1t} \approx 0$ so that players prefer to be of kind 2. Figure 7d2 also shows how it is possible for both players to eventually prefer no transactions in currency g, $\lim_{t\to T} p_{it} \approx 1$, while at the same time the fraction q_{1t} of players of kind 1 slowly decreases.

Discussion and future research

New currencies, especially these in digital format, may induce more currency competition. The competition may become especially fierce between fixed-supply and variable-supply currencies. Fixed-supply currencies rigidly avoids inflation/deflation which would otherwise be induced by altering the money supply. Variable-supply currencies allow more flexibility by allowing money printing during critical events (e.g. wars and recession), but requires fiscal discipline thereafter to avoid inflation.

To understand the competition, a player is assumed to earn a utility depending on its support of and volume of transactions in a given currency, and the fraction of players of the same kind as itself. A player may be any individual or collective unit. Essential in the article is how a player values money printing/withdrawal on the one hand versus inflation/deflation on the other hand. A time delay usually exists from the former to the latter. Batini (2006), Batini and Nelson (2001) and Friedman and Schwartz (1982) suggest that it takes over one year from money printing until inflation. Hence temptation may exist to increase money supply in the short run and postpone worrying about the subsequent inflation. The 1959–2021 US money supply and inflation data suggest that money printing and inflation indeed occur.

With high weight assigned to money supply relative to inflation, this article finds that players are more inclined to prefer the variable-supply currency. They thereby benefit from the temporarily increased purchasing power enabled by the increased money supply. Such players may not have excessively large time horizons, since then they might value the future negative consequences of inflation. This assumes that the player itself indeed can access the increased money supply. In contrast, low weight assigned to money supply relative to inflation induces players to be more inclined to prefer the fixed-supply currency, to avoid the negative impact of inflation.

When two kinds of players support two currencies differently, the players' fractions of transactions in the two currencies may exhibit substantial variation, e.g. be inverse U shaped or U shaped before converging towards preferring one or the other currency. This relates to earlier studies of how players choose between multiple currencies, see e.g. Schilling and Uhlig (2019), Fernández-Villaverde and Sanches (2019), Almosova (2018), Benigno et al. (2019). For example, assume high weight assigned to money supply, and that one player supports the fixed-supply currency much less than the other player. The first player may quickly abandon the fixed-supply currency which fails to offer additional money supply. The second player may initially support the fixed-supply currency increasingly, but may thereafter be influenced by the first player and also abandon the fixed-supply currency, thus potentially being negatively impacted by inflation. In contrast, assume low weight assigned to money supply, and that one player supports the fixed-supply currency much more than the other player. The first player may prefer the fixed-supply currency which provides a hedge against inflation. The second player may initially support the variable-supply currency increasingly, but may thereafter be influenced by the first player and also prefer the fixed-supply currency, thus potentially not benefitting from the increased money supply. The two currencies may obtain different market shares, as also analyzed ElBahrawy et al. (2017) and Imhof and Nowak (2006). These results indicate how countries or societies through various evolutionary dynamics may transform themselves into using one or another currency, or a combination of several currencies, potentially for different purposes. This in turn may impact a country's financial markets, monetary policy, and interaction with other countries.

We next allow players to choose which kind of player they can be. That can be realistic when a player prefers to transact in currencies that many other players transact in, thus being less influenced by how the player individually supports each currency independently of the other players. The analysis shows that players may choose to be of a kind supporting a given currency if that support is much higher than the other kind's support of the same currency. The first kind of player may thus become more common, while the second kind player becomes less common.

We finally enable a player's utility of transacting in a given currency to be proportional to the fraction of players of the same kind as the given player. Thus players not only choose what kind of player they want to be, but they may receive higher utility for being of one kind rather than of another kind, regardless of the players' support for each currency and their volume fractions of transactions in each currency. When the proportional impact of being a certain kind of player increases equally for both kinds of players, the players' fractions of transactions in each currency change more quickly, as if the process sensitivity in the replicator equation increases. When the proportional impact increases more for one kind of player, players increasingly prefer to be of that kind.

Future research, which implicitly indicates limitations of the current article, may extend the analysis to more features than money supply and inflation. More than two currencies and more than two kinds of players may be analyzed. Each kind of player's utility may depend on further features related to each currency's backing, convenience, confidentiality, transaction efficiency, financial stability, and security. Players may be assumed to apply different currencies for different purposes. Different kinds of players gaining different access to increased money supply, or suffering differently from money contraction, may be analyzed. Alternative player risk attitudes and time preferences may be evaluated. Empirics from other world regions may be incorporated. Additional players may be analyzed, e.g. players in different countries accessing different currencies, private versus public players, governmental agencies imposing regulation and taxation, and currency competition between countries.

Conclusion

This article builds a model of two kinds of players who can choose between two currencies, i.e. a fixed-supply currency (e.g. Bitcoin) and a variable-supply currency (e.g. a fiat currency or a central bank digital currency). A player may be any individual or collective unit. A variable-supply currency enables money printing or money withdrawal, and may be associated with inflation or deflation. Comparing fixed-supply and variable-supply currencies has become relevant due to new currencies emerging which incorporate supply, ownership, decentralization, regulation, confirmation of transactions, geographical extension, etc. differently.

A player's utility of transacting in a given currency is proportional to three factors, i.e. the player's support of that currency, the volume fraction of all players' (of both kinds) transactions in that currency, and the fraction of players of the same kind as the given player. A currency's support depends on its financial stability, transaction efficiency, backing, convenience, confidentiality, and security. Additionally, a player's utility of transacting in the variable-supply currency is proportional to a Cobb Douglas utility of two factors. The first factor is the initial money supply plus the accumulative money printing (positive) and money withdrawal (negative) in the numerator, divided by the initial money supply in the denominator. The second factor is the inverse of the accumulative inflation (positive) and deflation (negative when measured as a percentage). If the output elasticity for the first ratio is high, money printing/withdrawal is highly valued relative to inflation/deflation, and conversely if the output elasticity for the second ratio is high.

The players' utility of transacting in the variable-supply currency is illustrated for various output elasticities for 1959–2021. The exponentially increasing US M2 money supply and the positive inflation cause this utility to increase over time with high output elasticity, and decrease with low output elasticity. Such changing utilities over time constitute policy tools for how to adjust money supply/withdrawal and inflation/deflation.

Three replicator equations are developed based on the players' utilities. Two of these model each kind of player's volume fractions of transactions in each currency over time. The third models the evolution of the fraction of each kind of player over time, i.e. how players choose to be of one or the other kind.

High weight assigned to money supply relative to inflation causes players to more likely prefer the variable-supply currency, to gain from the increased money supply, and conversely prefer the fixed-supply currency given low weight assigned to money supply. When the two kinds of players support the two currencies differently, the players' fractions of transactions in the two currencies may be inverse U shaped or U shaped before converging towards preferring one or the other currency. When players can choose which kind of player to be, players may choose to be of a kind supporting a given currency if that support is especially high. When a player's utility of transacting in a given currency is proportional to the fraction of players of the same kind as the given player, and the proportional impact is higher for one kind of player, players tend to prefer to be of that kind.

Data availability

The article contains no associated data. All data generated or analyzed during this study are included in this published article.

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Additional information

Correspondence and requests for materials should be addressed to Kjell Hausken.

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Article

Competition between Variable–Supply and Fixed–Supply Currencies

Guizhou Wang and Kjell Hausken *

Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway
* Correspondence: kjell.hausken@uis.no

Abstract: For one variable–supply currency in isolation, one player's Cobb–Douglas utility depends on the current supply divided by the initial supply, multiplied by the inverse of the accumulative inflation/deflation. With equal weight assigned to both factors, money printing outweighs inflation, and money withdrawal outweighs deflation. The study design is to analyze how competition between one variable–supply and one fixed–supply currency impacts the player's choice of currency. Applying the 1959–2021 US M2 money supply data and the 1635–2021 US inflation data, the player's utility increases over time when assigning high weight to money printing/withdrawal and increases less or decreases overall when assigning high weight to inflation/deflation. With different player support for the two currencies, depending on each currency's backing, convenience, confidentiality, transaction efficiency, financial stability, and security, replicator dynamics is used to determine the player's volume fraction of transactions in each currency. Low, high, increasing, and decreasing support of a currency are analyzed. Each fraction may increase, decrease, be inverse U–shaped, U–shaped, and approach low or high levels over time. For example, high weight assigned to money printing may cause the player to eventually prefer the variable–supply currency unless the player supports the fixed–supply currency highly and increasingly.

Keywords: digital currencies; currency competition; money supply; inflation; replicator dynamics; cryptocurrencies; central bank digital currencies

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1. Introduction

1.1. Background

The emergence of new digital currencies raises questions about how these will compete depending on their characteristics. Historically, currencies have been associated with nations, such as the USD, CNY, EUR, etc. Nakamoto (2008) demonstrated successfully how a decentralized currency (Bitcoin) can be successfully built on a blockchain by applying proof of work technology with no centralized authority. Thereafter 20,178 cryptocurrencies have emerged (with a market cap of USD 915 billion) with great variation in the degree of decentralization, consensus mechanisms (e.g., proof of stake), supply, burning of coins, etc.[1] The introduction of such currencies, combined with central banks expanding their digital currencies, changes the nature of currency competition. Currencies can have all kinds of characteristics related to supply, ownership, decentralization, regulation, confirmation of transactions, geographical extension, etc.

The Federal Reserve Bank of Boston has conducted payment surveys since 2008. According to the latest 2020 Survey of Consumer Payment Choice (Foster et al. 2021), in 2020, consumers in the US, on average, made 68 payments per month. The top three payment methods are debit cards (23 payments) and credit cards (18 payments), followed by cash (14 payments). These three payment methods account for 80% of all payments by numbers (Greene and Stavins 2021). In 2020, cash accounts for 19% of all payments, a drop of 7% from 2019. This illustrates how payment methods can evolve within fiat currencies.

1.2. Contribution

This article analyzes currency competition focusing explicitly on supply and inflation. One variable–supply currency is considered where money can be printed and withdrawn and be subject to inflation or deflation. Both these two concerns have historically been important. Money printing enabled by a variable–supply currency offers additional options not available for a fixed–supply currency. One example is Roosewelt's 1933–1939 New Deal to recover the economy. Another example is war funding, e.g., World War I and World War II. The additional options may cause disadvantages. For example, USD 1 in 2022 buys 1.22% of what it would buy in 1695, which is a poor store of value for this time period. Variable–supply currencies have historically not implemented mechanics to ensure that one unit of a currency generates the same purchasing power on average over certain periods of time. Theoretically, such mechanics would enable financing a New Deal or a war with money printing if corresponding money withdrawals were implemented thereafter. A variable–supply currency with such mechanics would be a better store of value.

As a benchmark competitor, a fixed–supply currency is considered where money printing/withdrawal and inflation/deflation are impossible. Such a currency may be a good store of value and may potentially compete with a variable–supply currency which may lose its purchasing power over time. Historically, a fixed–supply currency has been close to impossible. The closest has been gold, which scores higher than Bitcoin on established history, and scores lower than Bitcoin on portability, divisibility, censorship resistance, verifiability, and scarcity (Ikkurty 2019). Both gold and Bitcoin score high on durability and fungibility (BYBIT Learn 2021). Gold, which is a currency under the current system (Mitchell 2021), has historically approximated fixed supply, with 1.5% additional gold mined in 2020.[2] Bitcoin has a fixed supply of 21 million coins. As of January 2022, 18.9 million Bitcoin have been mined, i.e., 90% (Hayes 2022). The remaining 2.1 million Bitcoin will be mined until approximately 2140.

First, a variable–supply currency is analyzed in isolation. A player's Cobb–Douglas utility is a product of two ratios. The first ratio is the initial supply plus accumulative money printing/withdrawal in the numerator, divided by the initial supply. The second ratio equals the inverse of accumulative inflation/deflation. With equal weight to both ratios, a utility of 1 constitutes a benchmark that is exceeded by assigning more weight to money printing, which can be useful to recover or boost the economy. The utility is less than 1 when assigning more weight to inflation, which is useful when seeking to cool down the economy. The US M2 money supply since 1959 and US inflation since 1635 are used to show how a player's utility increases or decreases over time depending on the player's preferences.

Thereafter competition between a variable–supply currency and a fixed–supply currency is analyzed. The two currencies may have different support depending on their backing, convenience, confidentiality, transaction efficiency, financial stability, and security. Replicator dynamics is used to analyze how the player's fractions of transacting in each currency evolve over time depending on the weights assigned to money printing and inflation and whether the support for each currency is constant, increases or decreases over time. Such insight is useful for policy makers and others seeking to determine how to adjust money printing, inflation, and support for various currencies.

2. Literature

The limited literature on this topic is divided into five groups, i.e., currency competition, competition between fiat currencies and cryptocurrencies, CBDCs (central bank digital currencies) and cryptocurrencies, the cryptocurrency market, and game theoretic analyses and decision models.

2.1. Currency Competition

Dowd and Greenaway (1993) develop a framework to analyze currency competition focused on network effects and switching costs. They find that network effects and switching costs seem to make it optimal for an agent to adopt only one currency. The agent is often reluctant to abandon the existing currency even if it is manifestly inferior to a new currency. They argue that parallel currencies are relatively uncommon. Camera et al. (2004) explore the competition between one safe foreign fiat currency, such as the US dollar, and one risky home fiat currency in a decentralized trading environment. They find that traders normally prefer safe foreign currency unless the trade frictions are high. A risky home currency in a poorly functioning economy is prone to dollarization. Dollarization can be reduced by adopting policies aimed at reducing currency risk and enhancing the trading environment. Gawthorpe (2017) adopts the money in utility function approach to explore the competition between a fiat currency and alternative currencies. They show that competition may cause a lower inflation rate compared with only one fiat currency. Wang and Hausken (2021) investigate the competition between a national currency and a global currency among three types of players, i.e., conventionalists, pioneers, and criminals. They consider six utility features of a currency, i.e., backing, convenience, confidentiality, transaction efficiency, financial stability, and security. They also apply replicator dynamics to analyze the evolution of the fractions of the three kinds of players and how they choose among the two currencies.

This article contributes to this literature by considering the competition between a variable–supply currency and a fixed–supply currency. The article focuses mainly on two variable–supply currency features, i.e., money printing/withdrawal and inflation/deflation. Other features, such as backing, convenience, safety, privacy, etc., are also implicitly embedded in the model.

2.2. Competition between Fiat Currencies and Cryptocurrencies

Wang and Hausken (2022b) analyze the evolution of fixed–supply and variable–supply currencies. The latter enable money printing/withdrawal and inflation/deflation. They find that a player's utility of transacting in each currency is proportional to how the player supports that currency, the volume fraction of all the players' transactions in that currency, and the fraction of players of the same kind as the given player. The current article contributes three advances over Wang and Hausken (2022b). First, if inflation empirics are unavailable, we estimate inflation from money printing by assuming a time lag. Second, if money printing empirics are unavailable, we estimate money printing from inflation by assuming a time lag in the opposite time direction. Third, this article purifies the analysis of how one kind of player supports one currency relative to the other currency, while Wang and Hausken (2022b) consider how two kinds of players support one currency relative to the other currency differently. The analysis of one kind of player enables focusing explicitly on how one typical or average player reacts to money printing/withdrawal and inflation/deflation depending on supporting the two currencies equivalently or differently.

Schilling and Uhlig (2019) analyze agents choosing between a fiat currency and a cryptocurrency. For example, fiat currencies are currently useful for most purchases, while cryptocurrencies may enable tax evasion, anonymity, and censorship resistance. Value–added tax and transaction fees to miners also play a role. They find that substitution decreases with asymmetry in exchange fees and transaction costs. Their analysis corresponds to the different support for the two currencies analyzed in this article, which depends on the currencies' transaction efficiencies.

Fernández-Villaverde and Sanches (2019) consider competition between privately issued fiat currencies. They determine a price stable equilibrium for multiple currencies in a Lagos–Wright environment, corresponding to two coexisting currencies in the current article and various less desirable equilibria. Almosova (2018) supplements their model by assuming costly circulation of private currencies due to mining costs, verification of transactions, etc. Although cryptocurrency competition will not cause price stability, with less costly private currency circulation, competition will cause downward pressure on the inflation of the public currency. Rahman (2018) investigates how fiat and digital currency competition impact monetary policy. He finds that a socially efficient allocation cannot follow from a purely private arrangement of digital currencies.

Lagos and Wright (2005) propose a framework for policy analysis based on the frictions that are essential for money. They allow the agents to interact periodically in both decentralized and centralized markets. Their model estimates that the welfare cost of inflation equals 3–5% of consumption. The framework can be used to analyze how the different regimes, such as one currency versus two currencies, cause different outcomes. Benigno et al. (2022) analyze two national currencies and a global cryptocurrency. They find that deviating from interest rate equality may imply approaching the zero lower bound or abandoning the national currency. They conclude that simultaneously ensuring a fixed exchange rate, free capital flows, and an independent monetary policy becomes even less possible. Verdier (2021) analyzes how a digital currency impacts competition in the deposit and lending markets. She finds increasing bank lending rates as a consequence of the digital currency crowding out bank deposits.

Hong et al. (2018) investigate the potential crowding out effect in a regime consisting of a fiat currency and a digital currency. The crowding–out effect occurs only under extreme conditions, i.e., high costs for one currency and low costs for the other currency. Obu and Ukpere (2022) investigate the impact of cryptocurrencies on the effectiveness of the fiscal policy. They find that government purchases decrease with households' adoption of cryptocurrencies. Sissoko (2021) explores the conceptual world where currencies are convertible into the numeraire consumption goods at a fixed rate. Then nobody wants to hold money over time. He points out that it is possible to establish a banking system in such an environment. The ability to increase the money supply according to societal needs is essential for the banking system's efficiency.

This article contributes to this literature by considering how a variable–supply fiat currency competes with a fixed–supply currency such as Bitcoin. Changing supply and inflation/deflation for the variable–supply fiat currency is explored, together with how the player chooses between a variable–supply currency and a fixed–supply currency over time. The replicator equation is applied to show the dynamic evolution of the volume fractions of the two currencies.

2.3. CBDCs and Cryptocurrencies

Caginalp and Caginalp (2019) analyze asset allocation between a home currency and a cryptocurrency when the government confiscates some of the players' assets. Blakstad and Allen (2018) evaluate which possibilities and risks cryptocurrencies offer for central banks and individuals and the challenges of issuing CBDCs. Masciandaro (2018) assess how different media of payments may evolve depending on individual preferences, akin to how the two currencies in the current article may evolve over time. Belke and Beretta (2020) suggest that central banks need to embrace the technology underlying cryptocurrencies. They suggest that central banks issuing cryptocurrencies may be subject to the disadvantages of cryptocurrencies and few benefits. Benigno (2021) reasons that currency competition causes the nominal interest rate and inflation to be determined by the time discount factor, the exit rate, and the fixed cost of entry, which can challenge the function of central banking. Asimakopoulos et al. (2019) find a substitution effect between the real balances of government currency and cryptocurrencies as a consequence of preferences, technology, and monetary policy shocks. This article relates to this literature by assessing CBDCs and cryptocurrencies from the supply perspective. A CBDC is usually a variable–supply currency. A cryptocurrency such as Bitcoin is a fixed–supply currency. The article presents a model that shows the competition and evolution of a variable–supply currency and a fixed–supply currency.

2.4. The Cryptocurrency Market

ElBahrawy et al. (2017) evaluate the fluctuating evolution of market shares of 1469 cryptocurrencies between April 2013 and May 2017. Caporale et al. (2018) determine a positive correlation between cryptocurrencies' past and future values. ElBahrawy et al. (2019) assess the linkage between online attention towards digital currencies on Wikipedia and market dynamics for digital currencies. White (2014) assesses the different market shares of Bitcoin and altcoins, akin to the current article assessing the volume fractions of transactions for two currencies. Sapkota and Grobys (2021) find no relation between the submarket equilibria of privacy coins and non–privacy coins for the top ten cryptocurrencies in 2016–2018. Milunovich (2018) estimates weak connectedness between five popular cryptocurrencies as one group and six major asset classes as a second group, and strong connectedness within each group, with a few exceptions. Gandal and Halaburda (2016) determine no winner–take–all effects in early cryptocurrency competition and strong network effects and winner–take–all dynamics more recently. The best well–known cryptocurrency is Bitcoin. It has a limited supply of 21 million coins. This article relates to this literature by assessing the competition between a variable–supply currency, such as fiat money, and a fixed–supply cryptocurrency, such as Bitcoin. The market share of the two currencies is captured by the volume fractions of the two currencies. The model shows the dynamic evolution of the market shares of the two currencies.

2.5. Game Theoretic Analyses and Decision Models

Imhof and Nowak (2006) analyze a stochastic frequency–dependent Wright–Fisher process to specify which of two strategies survive. They find that the Markov process has two absorbing states corresponding to homogeneous populations choosing either strategy A or strategy B. Lewenberg et al. (2015) find that it is difficult or impossible to distribute rewards in a stable way for a pooled Bitcoin mining and rewards cooperative game and that players continuously prefer to switch pools. Wang and Hausken (2022a) present a two–period decision model between a central bank and a household. They analyze the household's asset portfolio choice among production, consumption, CBDC, and non–CBDC, such as Bitcoin. This article related to this literature by considering a player's choice between a variable–supply currency and a fixed–supply currency and how this choice is made over time.

2.6. Literature Summary and Additions to the Literature Gap

The literature commonly analyzes the competition between currencies and focuses on different currencies' features, i.e., network effects and switching costs (Dowd and Greenaway 1993), safety, risk, and trade frictions (Camera et al. 2004), switching costs, inflation and network externalities (Gawthorpe 2017), six utility features of a national currency and a global currency (Wang and Hausken 2021), etc. This is one of the first articles that focuses on two essential features of currencies, which are supply and inflation/deflation. Thus, this article adds to this literature gap by exploring currency competition from the supply and inflation/deflation perspective.

Recent literature explores the competition between fiat currencies and cryptocurrencies, e.g., the substitution effects under asymmetry in transaction costs (Asimakopoulos et al. 2019; Schilling and Uhlig 2019), the coexistence and equilibrium of multiple currencies (Fernández-Villaverde and Sanches 2019), the impact on monetary policy and fiscal policy (Benigno et al. 2022; Obu and Ukpere 2022; Rahman 2018), the impact on the deposit and lending market (Verdier 2021), and the crowding out effects under a multiple currencies regime (Bian et al. 2021; Hong et al. 2018). In addition, the literature commonly investigates the relationship between CBDCs and cryptocurrencies (Belke and Beretta 2020; Benigno 2021; Blakstad and Allen 2018). The existing literature barely explores the player's choice between two currencies with respect to the supply and inflation/deflation features. This article adds to this literature gap by demonstrating the evolution of a player's choice between a variable–supply currency and a fixed–supply currency over time. The analysis mainly focuses on the supply and inflation/deflation features and incorporates how the player supports one currency relative to the other currency.

The literature furthermore evaluates the cryptocurrency market, e.g., the market shares of Bitcoin and altcoins (White 2014), the evolution of cryptocurrencies' market shares (ElBahrawy et al. 2017), and the equilibria of the cryptocurrency market (Sapkota and Grobys 2021; Yi et al. 2022). This article fills this literature gap by investigating how the market share of a fixed–supply cryptocurrency such as Bitcoin evolves over time in competition with a variable–supply currency. The market share is represented by the currency's transaction volume. Game theoretic models and decision models are widely used in academic research (Hausken and Welburn 2022; Imhof and Nowak 2006; Prat and Walter 2021; Wang and Hausken 2022a). This article adds to this literature by demonstrating a player's choice of a variable–supply currency versus a fixed–supply currency and the dynamic evolution of the volume fractions of the two currencies over time.

3. The Model

The article models one player receiving different Cobb–Douglas utilities depending on its choice of either a variable–supply fiat currency or a fixed–supply currency. The player mainly considers the two features of a currency, i.e., printing/withdrawal and inflation/deflation. Additional factors, i.e., transaction efficiency, banking, anonymity, security, confidentiality, finality, and stability, are comprised of one parameter, which expresses the player's support of a variable–supply currency relative to a fixed–supply currency.

The six dependent or outcome variables are the player's Cobb–Douglas utility of holding a fixed–supply currency, the player's Cobb–Douglas utility of holding a fixed–supply currency when the variable–supply currency is subject to money printing, the player's Cobb–Douglas utility of holding a fixed–supply currency when the variable–supply currency is subject to inflation, the player's Cobb–Douglas utility of holding a fixed–supply currency when a variable–supply currency is available, the player's Cobb–Douglas utility of holding a variable–supply currency, and the player's Cobb–Douglas utility of holding both a variable–supply currency and a fixed–supply currency in a certain weighted combination. The dynamic competition between a fixed–supply currency and a variable–supply currency is presented by the evolution of the volume fraction of the player's transactions in the variable–supply fiat currency using the replicator equation. The model demonstrates how a variable–supply currency competes with a fixed–supply currency over time.

3.1. One Variable–Supply Fiat Currency n

Consider a fiat currency, which may be a national currency with variable supply s_i at the discrete times $i = t_0, t_0+1, t_0+2, \ldots, T$, where $t_0 \geq 0$ and any time interval of length 1 applies, e.g., year, month, week, day, etc., and T is the final time. Hence $s_{i+1} - s_i$ is the amount printed (if positive) or withdrawn (if negative) from time i to time $i+1$. Summing up, $\sum_{i=t_0}^{t-1}(s_{i+1} - s_i)$ is the amount printed or withdrawn from time $i = t_0$ to time $i = t-1$. Hence $\frac{s_{t_0}+\sum_{i=t_0}^{t-1}(s_{i+1}-s_i)}{s_{t_0}}$ is the money supply at time t divided by the money supply at time t_0, which can be considered as a player's purchasing power at time t relative to the purchasing power at time t_0 without inflation.

Assume inflation π_i at time $i = t_0, t_0+1, \ldots, T$. Hence an asset valued at 1 at time $i = t_0$ is valued as $\frac{1}{1+\pi_{t_0+1}}$ at time $i = t_0+1$, $\frac{1}{(1+\pi_{t_0+1})(1+\pi_{t_0+2})}$ at time $i = t_0+2, \ldots$, and $\frac{1}{\prod_{i=t_0+1}^{t}(1+\pi_i)}$ at time $i = t$, which is the degraded asset value due to accumulative inflation from time t_0 to time t. The terms $\frac{s_{t_0}+\sum_{i=t_0}^{t-1}(s_{i+1}-s_i)}{s_{t_0}}$ and $\frac{1}{\prod_{i=t_0+1}^{t}(1+\pi_i)}$ are not stationary. Instead, they are affected by the currency supply s_t and the inflation π_t. Thus, both terms evolve over time.

Multiplying $\frac{s_{t_0}+\sum_{i=t_0}^{t-1}(s_{i+1}-s_i)}{s_{t_0}}$ raised to the Cobb–Douglas elasticity α, $0 \leq \alpha \leq 1$, with the degraded asset value $\frac{1}{\prod_{i=t_0+1}^{t}(1+\pi_i)}$ raised to the Cobb–Douglas elasticity $1-\alpha$ gives the player's Cobb–Douglas utility

$$u_{nt} = \left(\frac{s_{t_0}+\sum_{i=t_0}^{t-1}(s_{i+1}-s_i)}{s_{t_0}}\right)^{\alpha}\left(\frac{1}{\prod_{i=t_0+1}^{t}(1+\pi_i)}\right)^{1-\alpha} \tag{1}$$

at time t for holding a fiat currency n subject to variable money supply s_t and inflation π_t at time t, $t \geq t_0$. If $\alpha > 0.5$, the player assigns more weight to advantageous purchasing power than to disadvantageous inflation, and conversely if $\alpha < 0.5$. The player weighs the two considerations against each other. Equal weights $\alpha = 0.5$ is an especially interesting benchmark since the constant utility $u_{nt} = 1$ can be envisioned where the player's increased purchasing power from money printing $s_{i+1} - s_i$ is exactly offset by inflation π_i through time, or money withdrawal $s_{i+1} - s_i$ is exactly offset by deflation π_i through time. If inflation π_t is strictly positive in the long run (i.e., $\pi_\infty > 0$), then the utility u_{nt} converges to zero, i.e., $\lim_{t\to\infty} u_{nt} = 0$. This property holds only when inflation π_t is sufficiently high through time t, i.e., when the impact of inflation π_t is greater than the impact of the currency supply s_t. In the long run, the evolution of the utility u_{nt} depends on the currency supply s_t and the inflation π_t.

Overall, (1) expresses the player's Cobb–Douglas utility from the currency supply s_t and inflation π_t. This conception captures reality to some extent. For the player, a higher inflation π_t means currency devaluation. Thus, the player's utility u_{nt} decreases with inflation π_t. This article adopts the money–in–the–utility approach as in (1). It is one of the fundamental approaches in academic research, especially in economics and finance. The money–in–the–utility approach has a long history and is an important tool in economic research. The idea is that the utility function measures the player's preferences on a basket of goods and services. As an early pioneer, Ramsey (1928) assumes that the representative agent makes decisions by maximizing its utility. Sidrauski (1967) similarly conceptualizes a money–in–the–utility function. More recent examples are Block and Heineke (1975); Chen and Guo (2014); Ganelli and Tervala (2010); Mian et al. (2021); Obstfeld (1981); Wachter and Yogo (2010).

If inflation empirics are unavailable, and money printing empirics prior to time t_0 are unavailable or ignored, inflation can be estimated from money printing. Assume that money printing at time i gives inflation at time $i+\tau$, $\tau \geq 0$. Hence, when $t - t_0 > \tau$, we invert the ratio for the player's purchasing power at time t relative to the purchasing power at time t_0 without inflation, and account for the time delay of τ by summing from $i = t_0 + \tau$ to $i = t-1$, instead of summing from $i = t_0$ to $i = t-1$. Hence, no inflation occurs from time t_0 to time $t_0 + \tau$. Equation (1) is thus replaced by

$$u_{nMt} = \begin{cases} \left(\frac{s_{t_0}+\sum_{i=t_0}^{t-1}(s_{i+1}-s_i)}{s_{t_0}}\right)^{\alpha} & if\ t-t_0 \leq \tau \\ \left(\frac{s_{t_0}+\sum_{i=t_0}^{t-1}(s_{i+1}-s_i)}{s_{t_0}}\right)^{\alpha}\left(\frac{s_{t_0}}{s_{t_0}+\sum_{i=t_0+\tau}^{t-1}(s_{i+1}-s_i)}\right)^{1-\alpha} & if\ t-t_0 > \tau \end{cases} \tag{2}$$

where, evidently, the inflation term vanishes when $t - t_0 \leq \tau$.

If money printing empirics are unavailable, and inflation empirics prior to time t_0 are unavailable or ignored, money printing can be estimated from inflation. Assume that inflation at time $i+\tau$ is due to money printing at time i. For the inflation term, we sum from $i = t_0 + 1 + \tau$ to $i = t$ instead of summing from $i = t_0 + 1$ to $i = t$. Hence, no inflation occurs from time t_0 to time $t_0 + \tau$. Equation (1) is thus replaced by

$$u_{nIt} = \begin{cases} \left(\prod_{i=t_0+1}^{t} (1+\pi_i) \right)^{\alpha} \; if \; t - t_0 \le \tau \\ \left(\prod_{i=t_0+1}^{t} (1+\pi_i) \right)^{\alpha} \left(\frac{1}{\prod_{i=t_0+1+\tau}^{t}(1+\pi_i)} \right)^{1-\alpha} \; if \; t - t_0 > \tau \end{cases} \quad (3)$$

3.2. One Variable–Supply Fiat Currency n Competing with One Fixed–Supply Currency g

Assume that a variable–supply fiat currency n competes with a fixed–supply currency g, which may be a global currency, e.g., Bitcoin, which eventually (in ca. year 2140) has a fixed supply of 21 million coins. A player comparing which of two currencies to use will account for additional factors beyond money printing and inflation. We comprise these factors into one parameter h_t, $0 \le h_t \le 1$, at time t, which expresses the player's support of the fixed–supply currency g relative to the variable–supply currency n at time t. The player supports currency g more than currency n when $0.5 < h_t \le 1$, supports currency n more than currency g when $0 \le h_t < 0.5$, supports exclusively currency g when $h_t = 1$, supports the currencies equally much when $h_t = 0.5$, and supports exclusively currency n when $h_t = 0$.[3] Multiplying $1 - h_t$ with (1) gives the player's utility

$$u_{ngt} = \left(\frac{s_{t_0} + \sum_{i=t_0}^{t-1}(s_{i+1} - s_i)}{s_{t_0}} \right)^{\alpha} \left(\frac{1}{\prod_{i=t_0+1}^{t}(1+\pi_i)} \right)^{1-\alpha} (1 - h_t) \quad (4)$$

for transacting with the fiat currency n.

Conversely, since currency g is not subject to money printing and inflation, the two first terms in (4) disappear. Hence, the player's utility for transacting with the fixed–supply currency g is

$$u_{gnt} = h_t \quad (5)$$

Assume that the player at time t chooses a volume fraction p_{nt} of its transactions to be in the variable–supply fiat currency n, and the remaining volume fraction $1 - p_{nt}$ to be in the fixed–supply currency g. The player's utility at time t is thus the weighted combination

$$u_t = p_{nt} u_{ngt} + (1 - p_{nt}) u_{gnt} \quad (6)$$

One interesting aspect of the money–in–the–utility approach arises when multiple currencies may potentially coexist simultaneously. This article incorporates two currencies, i.e., a variable–supply currency n and a fixed–supply currency g, assigned different weights or probabilities p_{nt} and $1 - p_{nt}$. Thus, (6) captures the player's weighted utility u_t, accounting for two currencies.

3.3. Replicator Dynamics

To determine the evolution of the fraction p_{nt} of the player's transactions in the variable–supply fiat currency n, we apply the replicator equation (Taylor and Jonker 1978; Weibull 1997, p. 69)

$$\frac{\partial p_{nt}}{\partial t} = k p_{nt}(u_{ngt} - u_t) = k p_{nt}(1 - p_{nt})(u_{ngt} - u_{gnt}) \quad (7)$$

where (6) has been inserted. In (7), $k > 0$ is the sensitivity or rapidity of change of the process. When k is intermediate, the process is stable. The process changes rapidly when k is high, and slowly when k is low. The right–hand side of (7) is proportional to the difference $u_{ngt} - u_t$ between the player's utility of using the variable–supply fiat currency n and the weighted combination of both utilities in (6), and also proportional to the difference $u_{ngt} - u_{gnt}$ between the player's utility of using the variable–supply fiat currency n and the utility of using the fixed–supply currency g. Hence, when the former exceeds the latter, the fraction p_{nt} increases and conversely decreases when the former is lower than the latter. The right–hand side of (7) is also proportional to the product $p_{nt}(1 - p_{nt})$ of both fractions, which is inverse U–shaped with a maximum at $p_{nt} = 0.5$ and minima when $p_{nt} = 0$ and $p_{nt} = 1$. Hence, the fractions p_{nt} and $1 - p_{nt}$ change most rapidly when they are equally large, which means that the player chooses equal volume fractions $p_{nt} = 1 - p_{nt} = 0.5$ for the two currencies. The evolution of the fraction p_{nt} of the player's volume of transactions in currency n at time t depends on the Cobb–Douglas elasticity α and the currency support parameter h_t. In the long run, only one currency survives. Specifically, the process always evolves toward one or the other currency, eventually surviving exclusionarily, which may take some time, dependent on the initial conditions, the sensitivity parameter k, and the model parameters.

4. Analyzing the Model

4.1. The US 1635–2021

Figure 1a shows the US M2 money supply s_i at time i, i.e., 1959–2021, interpreted as M2, which includes currency, and certain deposit and money market accounts, increasing from USD 289.8 billion in January 1959 referred to as time t_0 to USD 21,425.9 billion in November 2021 referred to as time T (Federal Reserve 2022). Figure 1b shows the US inflation π_i at time i, i.e., 1959–2021, with a maximum 13% in 1980 and a minimum of 0% in 2009 and 2015 (CPI Inflation Calculator 2022). Figure 1c,d, with different time scales, insert the empirics in Figure 1a,b into (1) and plot the player's utility u_{nt} for the five Cobb–Douglas elasticities $\alpha = 0.6,\ 0.5,\ 0.4,\ 0.3,\ 0.2$. More weight $\alpha = 0.6$ to money printing than inflation causes u_{nt} to increase overall. The intermediate elasticity $\alpha = 0.5$, discussed after (1), is especially interesting. Equal weights assigned to money printing and inflation causes the player's utility u_{nt} to increase overall from 1959 to 2021. When $\alpha = 0.4$, i.e., less weight is assigned to advantageous money printing than to disadvantageous inflation, the player's utility remains above utility $u_{nt} = 1$ throughout, reaching minima of $u_{nt} = 1.01$ in 1981 and 1996. When $\alpha = 0.3$, i.e., even less weight assigned to money printing than to inflation, the player's utility is initially inverse U–shaped and crosses below $u_{nt} = 1$ in 1974, remaining below $u_{nt} = 1$ thereafter. When $\alpha = 0.2$, the player's utility is $u_{nt} = 1.00$ in 1959 and 1960 (rising briefly to $u_{nt} = 1.01$ halfway through 1959). Thereafter u_{nt} is inverse U–shaped, reaches $u_{nt} = 1$ in 1967, increases briefly to $u_{nt} = 1.02$ through 1967, and finally crosses below $u_{nt} = 1$ in 1968, where it remains thereafter.

Figure 1e assumes the time lag $\tau = 2$ years from money printing to inflation and insert the money printing empirics in Figure 1a into (2) and plot the player's utility u_{nt} for the five Cobb–Douglas elasticities $\alpha = 0.6, 0.5, 0.4, 0.3, 0.2$, thus not applying the inflation empirics. Batini (2006), Batini and Nelson (2001) and Friedman and Schwartz (1982) find that it takes more than one year from money printing until inflation. Figure 1e gives overall lower player utility than Figure 1c, possibly because inflation estimated from money printing may cause more estimated inflation than the empirical inflation in Figure 1b. The benchmark elasticity $\alpha = 0.5$, i.e., equal weights assigned to money printing and inflation, causes the player's utility u_{nt} to increase marginally to $u_{nt} = 1.05$ in 1961 due to the time lag $\tau = 2$ years from money printing to inflation, with subsequent asymptotic decrease towards $\lim_{t \to T} u_{nt} = 1.00065$ at time T. That illustrates a short–term temptation to print money even with equal weights assigned to money printing and inflation.

Figure 1f assumes the time lag $\tau = 2$ years from money printing to inflation and inserts the inflation empirics in Figure 1b into (3) and plots the player's utility u_{nt} for the five Cobb–Douglas elasticities $\alpha = 0.6,\ 0.5,\ 0.4,\ 0.3,\ 0.2$, thus not applying the money printing empirics. Figure 1f also gives overall lower player utility than Figure 1c, possibly because money printing estimated from inflation may cause less estimated money printing than the empirical money printing in Figure 1a. The benchmark elasticity $\alpha = 0.5$, i.e., equal weights assigned to money printing and inflation, causes the player's utility u_{nt} to increase marginally from $u_{nt} = 1$ in 1959 to $u_{nt} = 1.00995$ in 1960, $u_{nt} = 1.01499$ in 1961, where it remains thereafter.

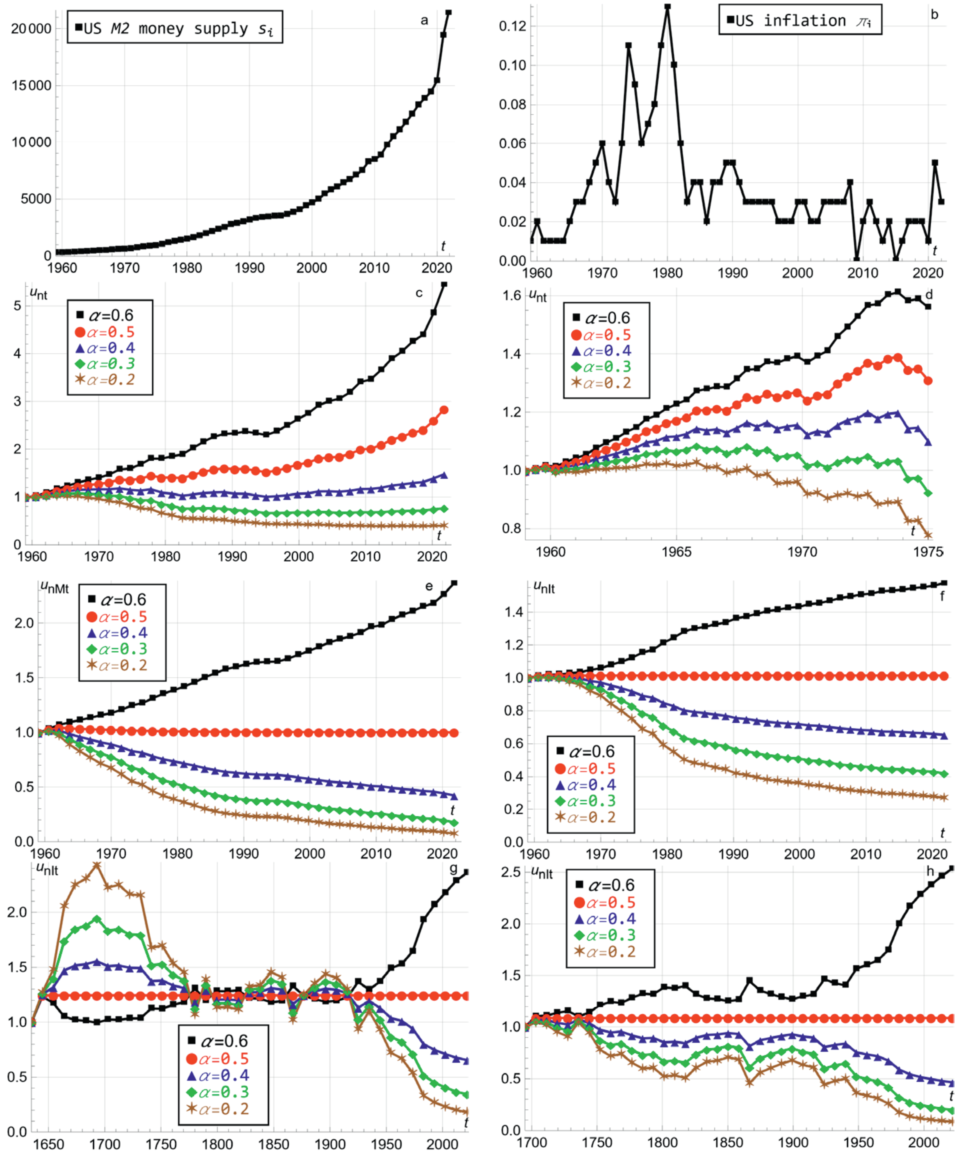

Figure 1. Panel (**a**): US M2 money supply s_i 1959–2021 in USD billion. Panel (**b**): US inflation π_i 1959–2021. Panels (**c**–**h**): The player's utility u_{nt}, u_{nMt}, u_{nIt} as a function of time t for the Cobb–Douglas elasticities $\alpha = 0.6, 0.5, 0.4, 0.3, 0.2$. Panel (**c**): Equation (1) 1959–2021. Panel (**d**): Equation (1) 1959–1975. Panel (**e**): Equation (2) 1959–2021 based on money printing empirics. Panel (**f**): Equation (3) 1959–2021 based on inflation empirics. Panel (**g**): Equation (3) 1635–2021 based on inflation empirics. Panel (**h**): Equation (3) 1695–2021 based on inflation empirics.

Figure 1g replicates Figure 1f for 1635–2021. High weight $\alpha = 0.2$ assigned to inflation causes the player's utility u_{nt} to be inverse U–shaped and remain above $u_{nt} = 1$ until 1864. That occurs because of the substantial deflation, especially in 1635–1650 (CPI Inflation Calculator 2022). Hence, in contrast, high weight $\alpha = 0.6$ assigned to money printing causes the player's utility u_{nt} to be U–shaped and with minima $u_{nt} = 0.99$ in 1693 and 1695. After 1917, this gets reversed due to less deflation and more consistent inflation. Overall, USD 1 in 2022 buys 2.98% of what it would buy in 1635 (CPI Inflation Calculator 2022).

Figure 1h replicates Figure 1g,f for 1695–2021. The year 1695 is chosen since USD 1 in 2022 buys 1.22% of what it would buy in 1695, which is the lowest percentage for 1635–2021 (CPI Inflation Calculator 2022). Eliminating the 1635–1695 deflation causes Figure 1h to be more reminiscent of Figure 1c–f.

4.2. Analysis Applying Replicator Dynamics

Money printing and inflation generally proceed such that the evolution of the fraction p_t of the player's volume of transactions in the variable–supply fiat currency n has no analytical solution.[4] Hence we illustrate the replicator equation in (7) with simulations. Figure 2 applies the same empirics and makes the same assumptions as in Figure 1c, with sensitivity $k = 0.5$, initial condition $p_{nt_0} = 0.5$, and seven different parameters h_t for the player's support of currency g relative to currency n at time t.

Figure 2a assumes the Cobb–Douglas elasticity $\alpha = 0.6$, which causes the rapidly increasing player's utility u_{nt} in Figure 1c due to the high weight $\alpha = 0.6$ assigned to money printing. With negligible support $h_t = 0.01$ for the fixed–supply currency g, the fraction p_{nt} of the player's volume of transactions in currency n at time t increases rapidly and asymptotically towards $\lim_{t\to 2021} p_{nt} \approx 1$ determined numerically. With increasing support $h_t = 0.3$, $h_t = 0.4$, $h_t = 0.5$ for currency g, the fraction p_{nt} increases more slowly towards $\lim_{t\to T} p_{nt} \approx 1$. When $h_t = 0.6$, which means more support for currency g than for the variable–supply currency n at time t, the fraction p_{nt} first decreases towards a minimum $p_{nt} = 0.33$ in 1972 since the player's utility u_{nt} in Figure 1c is still too low, and thereafter increases towards $\lim_{t\to 2021} p_{nt} \approx 1$ as the player's utility u_{nt} in Figure 1c increases. When $h_t = 0.7$, the same, but more pronounced logic applies. The difference is that p_{nt} fails to approach $\lim_{t\to T} p_{nt} \approx 1$ approximatively by 2021, but can be expected to do so beyond 2021. Finally, with overwhelming support $h_t = 0.99$ for currency g, the high player's utility u_{nt} in Figure 1c is too low when multiplied with $1 - h_t$ in (4). Hence the fraction p_{nt} of the player's volume of transactions in currency n at time t decreases rapidly and asymptotically towards $\lim_{t\to 2021} p_{nt} \approx 0$ determined numerically.

Figure 2b assumes the lower Cobb–Douglas elasticity $\alpha = 0.2$, which initially causes an inverse U–shaped, and thereafter overall decreasing, player utility u_{nt} in Figure 1c due to the low weight $\alpha = 0.2$ assigned to money printing. With low support $h_t = 0.01$ and $h_t = 0.3$ for currency g, the fraction p_{nt} increases asymptotically towards $\lim_{t\to T} p_{nt} \approx 1$, but more slowly than in Figure 2a. With higher support $h_t = 0.4$ for the fixed–supply currency g, the fraction p_{nt} first increases towards a maximum $p_{nt} = 0.82$ in 1980 since the player's utility u_{nt} in Figure 1c is still too high and thereafter decreases, causing the majority of the volume of transactions in currency g, not quite reaching $\lim_{t\to T} p_{nt} \approx 0$ by 2021, but can be expected to do so beyond 2021. With equal support $h_t = 0.5$ for both currencies, the fraction p_{nt} first increases marginally towards a maximum $p_{nt} = 0.505$ in 1967, and thereafter decreases towards $\lim_{t\to T} p_{nt} \approx 0$ with all transactions in currency g. With higher support $h_t = 0.5$ for both currencies, $h_t = 0.6$, $h_t = 0.7$, $h_t = 0.99$ for currency g, the fraction p_{nt} decreases more quickly towards $\lim_{t\to T} p_{nt} \approx 0$.

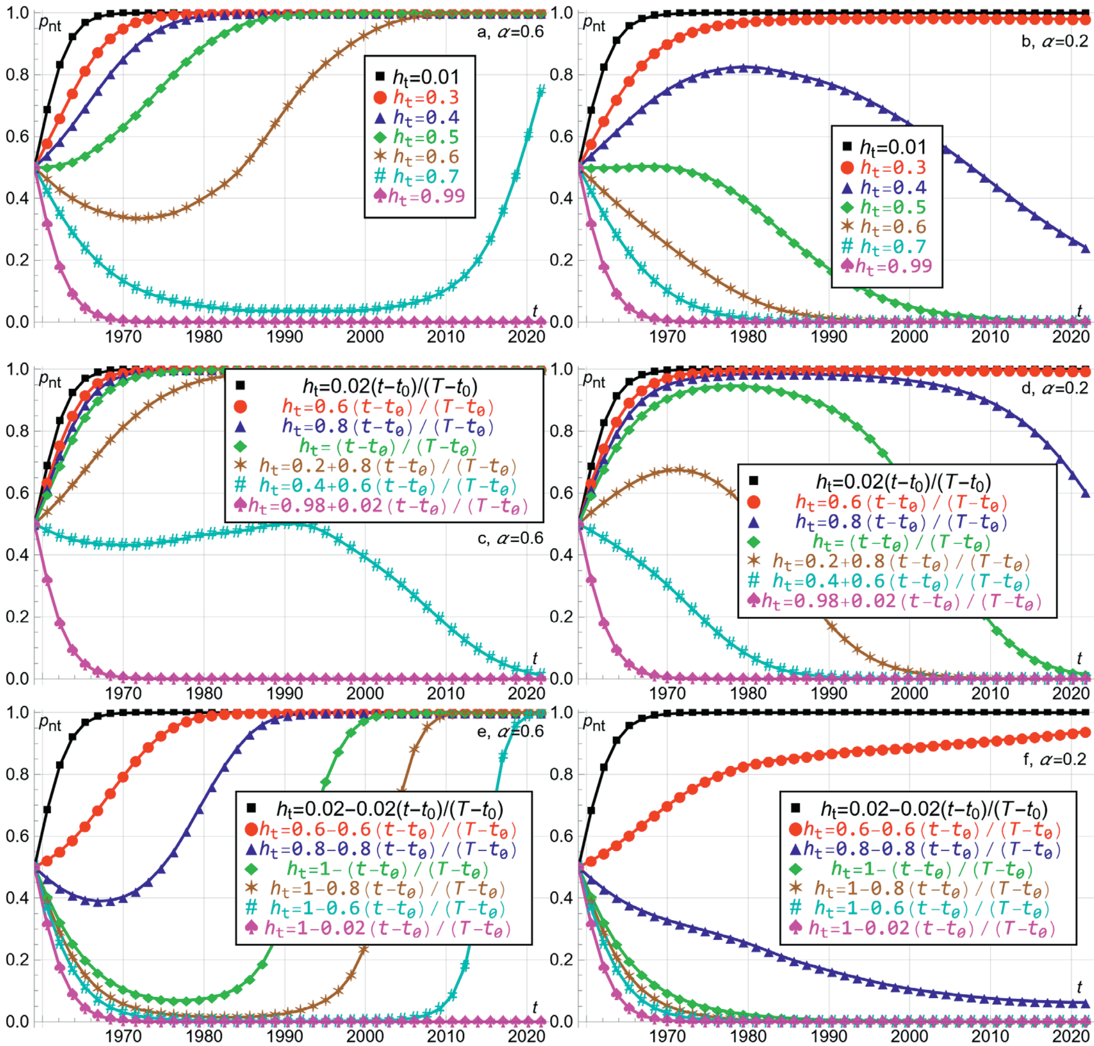

Figure 2. The fraction p_{nt} of the player's volume of transactions in currency n at time t 1959–2021 when $k = p_{nt_0} = 0.5$, applying the empirics in Figure 1c. Panels (**a**,**c**,**e**): $\alpha = 0.6$. Panels (**b**,**d**,**f**): $\alpha = 0.2$. Panels (**a**,**b**): Seven constant support parameters between $h_t = 0.01$ and $h_t = 0.99$. Panels (**c**,**d**): Seven linearly increasing support parameters h_t. Panels (**e**,**f**): Seven linearly decreasing support parameters h_t.

Figure 2c,d assume linearly increasing support h_t for currency g, adjusted to equal the support h_t in Figure 2a,b at the midway point $t = t_0 + (T - t_0)/2 \approx 1990$, constrained to be not less than $h_t = 0$ at the initial time $t = t_0$, and constrained to be maximally $h_t = 1$ at the final time $t = T$. Low initial support h_t for currency g means high initial support $1 - h_t$ for the variable–supply currency c. Hence the low initial support h_t for the five first linear equations in Figure 2c for $\alpha = 0.6$ causes a more rapid increase in the fraction p_{nt} towards $\lim_{t \to T} p_{nt} \approx 1$ than in Figure 2a. When $h_t = 0.4 + 0.6(t - t_0)/(T - t_0)$, which gives $h_t = 0.7$

at the midway point $t \approx 1990$, the remarkable situation arises where the fraction p_{nt} is initially U–shaped towards the maximum $p_{nt} = 0.5007$ in 1990, and thereafter decreases reaching $\lim_{t \to T} p_{nt} \approx 0.01$ in 2021. This result is the opposite of the result in Figure 2a and arises since the linearly increasing support h_t exceeds $h_t = 0.7$ after 1990, which means more support for the fixed–supply currency g. Hence, although currency c before 1990 enjoys more support in Figure 2c than in Figure 2a, after 1990, the reverse is the case. For the final curve, the results are similar except that the fraction p_{nt} initially decreases more slowly towards $\lim_{t \to T} p_{nt} \approx 0$ than in Figure 2a.

Figure 2d, with the lower Cobb–Douglas elasticity $\alpha = 0.2$, causes more slow asymptotic increase in the fraction p_{nt} towards $\lim_{t \to T} p_{nt} \approx 1$ for the first two linear equations compared with $\alpha = 0.6$ in Figure 2c. Already for the third linear equation with support $h_t = 0.8(t - t_0)/(T - t_0)$, which gives $h_t = 04$ at the midway point $t \approx 1990$, asymptotic increase towards $\lim_{t \to T} p_{nt} \approx 1$ cannot be sustained because of the low weight $\alpha = 0.2$ assigned to money printing. After a maximum $p_{nt} = 0.985$ in 1982, the fraction p_{nt} decreases towards $p_{nt} = 0.60$ in 2021. For the fourth linear equation h_t the maximum $p_{nt} = 0.95$ is reached in 1978, with a subsequent decrease towards $\lim_{t \to T} p_{nt} \approx 0.01$ in 2021. For the fifth linear equation h_t the maximum $p_{nt} = 0.67$ is reached in 1971, with subsequent decrease towards $\lim_{t \to T} p_{nt} \approx 0$ in 2021. For the two final linear equations for h_t the fraction p_{nt} of the player's volume of transactions in currency n at time t decreases relatively rapidly towards $\lim_{t \to T} p_{nt} \approx 0$.

Figure 2e,f assume linearly decreasing support h_t for currency g, adjusted to equal the support h_t in Figure 2a,b at the midway point $t = t_0 + (T - t_0)/2 \approx 1990$, constrained to be maximally $h_t = 1$ at the initial time $t = t_0$, and constrained to be not less than $h_t = 0$ at $t = T$. High initial support h_t for currency g means low initial support $1 - h_t$ for the variable–supply currency c. Hence the high initial support h_t for the two first linear equations in Figure 2e for $\alpha = 0.6$ causes more slow increase in the fraction p_{nt} towards $\lim_{t \to T} p_{nt} \approx 1$ than in Figure 2a. For the linear equations number 3, 4, 5, 6 the fraction p_{nt} reaches minima $p_{nt} = 0.39, 0.068, 0.014, 0.0014$ in 1967, 1977, 1983, 1990, respectively, before increasing towards $\lim_{t \to T} p_{nt} \approx 1$ and exhaustive support of the fixed–supply currency c. This result arises because the support h_t of the variable–supply currency g is too low and decreasing after 1990. For the final curve, the results are similar except that the fraction p_{nt} initially decreases more rapidly towards $\lim_{t \to T} p_{nt} \approx 0$ than in Figure 2a.

Figure 2f, with the lower Cobb–Douglas elasticity $\alpha = 0.2$, causes a slower asymptotic increase in the fraction p_{nt} towards $\lim_{t \to T} p_{nt} \approx 1$ for the first linear equation compared with $\alpha = 0.6$ in Figure 2e. For the second linear equation the increase is slower. The fraction p_{nt} of the player's volume of transactions in currency n at time t only reaches $\lim_{t \to T} p_{nt} \approx 0.94$ in 2021. Already for the third linear equation an increasing fraction p_{nt} cannot be sustained. Instead, the fraction p_{nt} decreases towards $\lim_{t \to T} p_{nt} \approx 0.06$ in 2021. For the remaining linear equations, the fraction p_{nt} decreases rapidly towards $\lim_{t \to T} p_{nt} \approx 0$ in 2021.

5. Summarizing the Results

The article first analyzes the variable–supply currency in isolation. A ratio is established with the initial supply in the denominator and the initial supply plus accumulative money printing (positive) and money withdrawal (negative) in the numerator. A second ratio is established with 1 in the numerator and accumulative inflation (positive) and deflation (negative when measured as a percentage) in the denominator. A Cobb–Douglas utility is established for a player with one output elasticity for each of the two ratios, which are multiplied with each other. The player may be a consumer, firm, organization, or any individual or collective actor conceptualizing a utility for money supply subject to money printing/withdrawal and inflation/deflation. If the output elasticity for the first ratio is high, money printing/withdrawal is assigned a high weight relative to inflation/deflation, and conversely, if the output elasticity for the second ratio is high. When the two output elasticities are equal, and money printing is outweighed by inflation, or money withdrawal is outweighed by deflation, the product of the two ratios equals 1. When inflation empirics are unavailable, a second utility is developed where inflation is calculated from money printing accounting for a time delay. When money printing empirics are unavailable, a third utility is developed where money printing is calculated from inflation accounting for a time delay.

The article shows how the US M2 money supply has increased exponentially since 1959 and how the US inflation has changed since 1635. These empirical data are used to plot the player's utility since 1959 for five different output elasticities. With high output elasticity for money printing, the player's utility has increased overall exponentially since 1959. With lower output elasticity for money printing, the player's utility increases less and eventually decreases overall when money printing is assigned a low weight, which means that inflation is assigned a high weight. Curves such as these provide policy tools for how to weigh the challenging and partly opposing concerns of money printing and inflation against each other. Similar curves are plotted assuming that inflation and money printing empirics, respectively, are unavailable.

The inflation data since 1635 are used to plot the player's utility for the five output elasticities. The strong deflationary periods 1635–1695 imply high utility for assigning high weight to inflation/deflation and thus low weight to money printing (estimated from inflation). Applying the inflation data since 1695 causes the player's utility to be qualitatively similar to the player's utility since 1959. The reason is that USD 1 in 2022 buys 1.22% of what it would buy in 1695, which is the lowest percentage since 1635.

The article next analyzes one variable–supply fiat currency competing with one fixed–supply currency. The latter is assumed to have a certain support that expresses the utility of transacting in it. That support ranges from 0 to 1 and may change over time. A currency's support depends on its backing, convenience, confidentiality, transaction efficiency, financial stability, and security. The Cobb–Douglas utility of the variable–supply fiat currency is multiplied by 1 minus the support of the fixed–supply currency. A player's utility of transacting in both currencies is a weighted sum of the two utilities, weighted by the volume fraction of transactions in each currency. With this conceptualization, the replicator dynamics can be used to determine how the fraction of a player's volume of transactions in each currency evolves over time. The player continuously changes the fraction to maximize its utility.

We first assume a high weight assigned to money printing. With low support for the fixed–supply currency, the fraction of a player's volume of transactions in the variable–supply currency quickly approaches 1. With higher support of the fixed–supply currency, the fraction may temporarily decrease but will eventually increase, except for very high support for the fixed–supply currency.

We thereafter assume a low weight assigned to money printing. Then very low support for the fixed–supply currency still causes the fraction of a player's volume of transactions in the variable–supply currency to approach 1. With higher support of the fixed–supply currency, the fraction may temporarily increase but will eventually decrease, especially for very high support for the fixed–supply currency, in which case the decrease is rapid.

We next consider linearly increasing support for the fixed–supply currency over time. With high weight assigned to money printing and low but linearly increasing support for the fixed–supply currency, the fraction of a player's volume of transactions in the variable–supply currency approaches 1 quickly. With higher and linearly increasing support for the fixed–supply currency, the fraction may increase temporarily and eventually decrease. Conducting the same analysis with a low weight assigned to money printing may cause the fraction to increase temporarily and thereafter decrease.

We finally analyze linearly decreasing support for the fixed–supply currency over time. With high weight assigned to money printing and low or intermediate, and linearly decreasing support for the fixed–supply currency, the fraction of a player's volume of transactions in the variable–supply currency may decrease temporarily and thereafter increase towards 1. Conducting the same analysis with a low weight assigned to money printing may cause the fraction to increase for low and decreasing support for the fixed–supply currency and to decrease with slightly higher and decreasing support for the fixed–supply currency.

6. Discussion, Policy Implications, Limitations, and Future Research

Research on cryptocurrencies has increased in recent years. Examples of foci are how cryptocurrencies, such as, e.g., Bitcoin compete with fiat currencies such as CBDCs, and the impact of cryptocurrencies on monetary policy, fiscal policy, welfare, and disintermediation of commercial banks. In this context, this article's analysis builds intuition on some aspects of the currency competition between a variable–supply currency and a fixed–supply currency.

First, the article provides insight for policymakers by focusing on two features of competing currencies, i.e., supply and inflation/deflation. A player's support of one currency relative to the other currency is analyzed. A poorly supported currency is prone to decreasing prevalence in the long run. The findings provide useful insights for central banks and governments seeking to adjust the money supply, inflation rate, and the currency's support in the presence of multiple currencies.

Second, the replicator equation presents the evolution of the volume fractions of the two competing currencies. The Cobb–Douglas elasticity for money printing, the Cobb–Douglas elasticity for inflation, and the player's support for one currency relative to the other currency determine the player's volume fraction of transactions in each currency evolutionarily. Therefore, in addition to the money supply and inflation/deflation, policy makers may account for the support of a currency when setting monetary policy.

Third, considering the importance of support for a currency by many different actors beyond the one player modeled in this article, central banks may analyze the sources of support for various currencies, e.g., backing, convenience, confidentiality, transaction fees, transaction efficiency, financial stability, security, purchasing power risk, privacy, etc. The central bank may thereafter choose measures to improve the support of its own fiat currency, in daily use, for borrowing and saving, for cross–border payments, etc.

Fourth, financial investors, individuals, and cryptocurrency developers may find it beneficial to understand the backing of the various currencies when making decisions.

Fifth, the findings provide insights for policy analysis based on money printing/withdrawal, inflation/deflation, and currency support, which determine the volume fractions of transactions in the various currencies. The different degrees of money printing/withdrawal, inflation/deflation, and currency support cause various outcomes.

Sixth, in this digitalized era, central banks around the world are embracing CBDCs.[5] At the time of writing, 105 countries, representing over 95 percent of global GDP, explore CBDCs. Eleven countries have already launched CBDCs. CBDCs may face various challenges, perhaps especially from various cryptocurrencies such as Bitcoin. Central banks may enhance CBDCs' competitiveness by implementing policies aimed at improving the backing of CBDCs, reducing transaction frictions, limiting inflation, and improving the payment environment.

Seventh, the results indicate how a player may transform into using one variable–supply currency and one fixed–supply currency or a combination of two currencies through evolutionary dynamics. This, in turn, may affect the financial markets, monetary policy, fiscal policy, taxes, cross–border payments, etc. Therefore, central banks may pay more attention to the independence and effectiveness of the monetary policy and fiscal policy when facing currency competition. The evolution and adoption of a non–fiat cryptocurrency might potentially undermine the effectiveness of the current monetary policy. This article intends to shed light on how this evolution may play out.

Overall, the article provides policy implications on how to weigh the challenges deriving from money printing, inflation/deflation, and the relative support of variable–supply and fixed–supply currencies.

The two currencies case is the simplest case for multiple currencies. This article seeks to capture the essentials of the phenomenon by focusing on the simple case of competition between two currencies, assuming that one currency has a variable supply while the other currency has a fixed supply. Analyzing only two currencies is also a limitation since today's world has more than two currencies. The evolution and potential stationary coexistence of multiple currencies may be explored in future research. To address further limitations, future research may analyze currency competition accounting for characteristics other than supply and inflation and alternatives to the money–in–the–utility function. Different time preferences and risk attitudes may be assessed. Empirics from countries other than the US may be considered. Different kinds of players with different preferences may be incorporated. Governmental regulation and taxation may be included. Other approaches for incorporating multiple currencies may be assessed, e.g., substitution, individual preferences, switching costs, and fractions of prevalence for various currencies.

7. Conclusions

This article analyzes variable–supply and fixed–supply currencies and competition between digital currencies. This involves money printing, money withdrawal, inflation, and deflation. Competition between currencies may become more common as digital currencies emerge with different characteristics pertaining to supply, ownership, decentralization, regulation, confirmation of transactions, geographical extension, etc. This article analyzes competition between two currencies focusing explicitly on supply and inflation/deflation. One currency has variable supply, which has been historically the most common. Variable supply means that money can be printed or withdrawn from circulation. Money withdrawal is sometimes referred to as burning money. The other currency has a fixed supply, which means that money can neither be printed nor withdrawn from circulation.

A Cobb–Douglas utility is developed for a player accounting for money printing/ withdrawal and inflation/deflation. The article shows how the player weighs these concerns against each other, first for one variable–supply currency in isolation and thereafter in competition with a fixed–supply currency. Empirics are the US M2 money supply 1959–2021 and the US inflation data 1635–2021.

The player's utility is generalized to account for a weighted combination of a variable–supply fiat currency and a fixed–supply currency, accounting for each currency's support which depends on its backing, convenience, confidentiality, transaction efficiency, financial stability, and security. Replicator dynamics illustrate how the player's volume of transactions in each currency evolves over time.

With high weight assigned to money printing, the player eventually prefers the variable–supply currency, which takes longer with moderately higher support of the fixed–supply currency. With low weight assigned to money printing, the same result follows with low support of the fixed–supply currency. However, with higher support for the fixed–supply currency, the player eventually prefers the fixed–supply currency.

With high weight assigned to money printing and low but linearly increasing support for the fixed–supply currency, the player eventually prefers the variable–supply currency. With higher and linearly increasing support for the fixed–supply currency, the player eventually prefers the fixed–supply currency.

With high weight assigned to money printing and low or intermediate, and linearly decreasing support for the fixed–supply currency, the player may temporarily prefer the fixed–supply currency but will eventually prefer the variable–supply currency.

Finally, low weight is assigned to money printing. Then low and decreasing support for the fixed–supply currency may cause the player to eventually prefer the variable–supply currency, while slightly higher and decreasing support for the fixed–supply currency may cause the player to eventually prefer the fixed–supply currency.

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Nomenclature

Parameters

n	Variable–supply fiat currency
g	Fixed–supply currency
t_0	Initial time, $t_0 \geq 0$
T	Final time, $T \geq t_0$
i	Time counting variable, $t_0 \leq i \leq T$
τ	Time lag from money printing to inflation, $\tau \geq 0$
s_i	Supply at discrete time i of the variable–supply fiat currency n, $s_i \in \mathbb{R}$
π_i	Inflation at time i, $\pi_i \in \mathbb{R}$
α	Cobb–Douglas elasticity expressing weight assigned to money printing, $0 \leq \alpha \leq 1$
h_t	The player's support of currency g relative to currency n at time t, $0 \leq h_t \leq 1$
k	Parameter for the sensitivity or rapidity of change of the replicator equation, $k > 0$

Independent variables

t	Time, $t \geq t_0$
p_{nt}	Volume fraction of the player's transactions in currency n at time t, $0 \leq p_t \leq 1$

Dependent variables

u_{nt}	Player's Cobb–Douglas utility of holding currency n at time t, $u_{nt} \geq 0$
u_{nMt}	Player's utility of holding currency n at time t based on money printing, $u_{nMt} \geq 0$
u_{nIt}	Player's utility of holding currency n at time t based on inflation, $u_{nIt} \geq 0$
u_{ngt}	Player's utility of holding currency n at time t when currency g is available, $u_{ngt} \geq 0$
u_{gnt}	Player's utility of holding currency g at time t when currency n is available, $u_{gnt} \geq 0$
u_t	Player's utility of holding currencies n and g at time t, $u_t \geq 0$

Notes

1. https://coinmarketcap.com/, retrieved 11 July 2022.
2. In total, 197,576 metric tons have been mined (gold.org 2022), and 3030 metric tons were produced in 2020 (Basov 2022).
3. We may operationalize h_t as comprising six factors, i.e., backing (of currency n relative to currency g) by actors, systems, or characteristics that users respect and trust; convenience, e.g., few and easily understood operations when purchasing goods and services; confidentiality, striking balances between privacy, availability, accessibility, and discrimination; transaction efficiency, i.e., low cost, fast speed, affordability, and finality in terms of how many confirmations are needed for transactional approval; financial stability, which usually depends on conditions in the given country; and security, see, e.g., Allen et al. (2020) and Kiff et al. (2020) for the security of blockchain–based currencies.
4. For the special case that $k(u_{ngt} - u_{gnt}) = Kt^m$ where K and m are parameters, which depend on time t in a special manner and depend on time t when $m = 0$, the solution of (7) is $p_t = \frac{1}{1+\left(\frac{1}{p_{t_0}}-1\right)e^{-\frac{K}{1+m}(t^{1+m}-t_0^{1+m})}}$, where $\frac{1}{p_{t_0}} - 1 > 0$ when $0 \leq p_{t_0} < 1$, $\lim_{t\to\infty} e^{-\frac{K}{1+m}(t^{1+m}-t_0^{1+m})} = 0$ causing $\lim_{t\to\infty} p_t = 1$ when $\frac{K}{1+m} > 0$, $\lim_{t\to\infty} e^{-\frac{K}{1+m}(t^{1+m}-t_0^{1+m})} = \infty$ causing $\lim_{t\to\infty} p_t = 0$ when $\frac{K}{1+m} < 0$, and $\lim_{t\to\infty} p_t = p_{t_0}$ when $\frac{K}{1+m} = 0$. Hence, either one currency excludes the other currency, or the fraction p_t equals the initial fraction p_{t_0} at time t_0.
5. https://www.atlanticcouncil.org/cbdctracker/, retrieved 12 October 2022.

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Hard money and fiat money in an inflationary world

Guizhou Wang [1], Kjell Hausken [*,2]

Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway

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ABSTRACT

The purpose is to determine whether a borrower prefers to borrow hard and fiat money from a bank to buy other assets from a seller, whether the seller wants to sell, how the nontraders are impacted, and whether the bank prefers to lend money and print or withdraw fiat money. The method is to compare the agents' and bank's Cobb Douglas utilities over two periods. The conclusions are that the bank prefers to print fiat money to a certain extent. Fiat money printing benefits the borrower/buyer which prefers inflation, benefits the bank if not excessive, and hurts the seller and nontraders. The seller and nontraders prefer a hard money economy or a fiat economy where the bank withdraws money to ensure deflation. More nontraders decrease inflation since the bank's money printing gets spread across more agents. The article provides further results illustrated by varying 64 parameters relative to a benchmark.

1. Introduction

This article introduces hard money and fiat money in a two-period economy. The big general idea in the article is to model three kinds of agents and a bank intended to capture a major part of what occurs in today's economies. The three kinds of agents are an agent which is a borrower and buyer, an agent which is a seller, and nontrading agents. The borrower borrows hard money and fiat money from the bank and buys other assets from the seller. The seller and nontrading agents hold hard money, fiat money and other assets. An agent's Cobb Douglas utility depends on its asset portfolio, that is, on whether the agent holds hard money, fiat money, other assets, loans in hard money, or loans in fiat money. The bank, which also has a Cobb Douglas utility, can lend hard money and fiat money to the borrower, earning interest, and can print and withdraw fiat money which may cause inflation or deflation which impacts the agents. In the model, the bank is a unified actor that represents a central bank and one or several commercial banks.

The article's research question and purpose are to determine how the three kinds of agents and bank are impacted in their Cobb Douglas utilities over two time periods when operating as specified, i.e. borrowing, selling, holding money and assets, printing and withdrawing fiat money, etc. For example, does the borrower prefer to borrow hard money or fiat money excessively or to a limited extent to acquire other assets? Does the seller want to sell other assets? How are the nontrading agents impacted by holding money and assets? How is the bank impacted by lending hard money and fiat money? Does the bank want to print or withdraw fiat money? The economic approach in the article, with three kinds of agents and one bank, is designed with the intention of being especially well

* Corresponding author.
E-mail address: kjell.hausken@uis.no (K. Hausken).
[1] ORCID: orcid.org/0000-0001-5297-8105
[2] ORCID: orcid.org/0000-0001-7319-3876

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equipped to match and answer these questions.

For hard money printing, withdrawal, inflation, and deflation are assumed to be infeasible. Gold does not have a fixed supply, i.e. the global gold supply increased by approximately 2% per year on average since 2013 (World Gold Council, 2023). The current annual growth rate of the Bitcoin supply is approximately 1.64% (Money Printer, 2023), which gradually decreases to zero over time until the maximum supply of 21 million Bitcoins is reached in ca 2140. Although the supply of gold and Bitcoin are currently not fixed, they are considered as two approximate examples of hard money. The opposite is assumed for fiat money. The US dollar is not hard money, but fiat money, per this article's definition.

A model is formulated to analyze the coexistence of hard and fiat money. Agent 1 borrows hard and fiat money from the bank and buys other assets from a seller. Agent 2 sells some of its other assets to agent 1 and does not borrow from the bank. Agent i, as a nontrading agent, $i = 3,\ldots,n$, does not borrow, lend, buy, or sell. Its asset portfolio remains unchanged over the two periods. The article introduces a theoretical model for studying the competition between hard and fiat money, analyzing the effects on agents and one bank of printing and withdrawing fiat money. It examines the implications on inflation and deflation of borrowing, lending, buying and selling.

Inflation, i.e. the rate at which the average price of goods or services increases over time, generally depends on the money supply (Gorton, 2023) and various other factors such as production, the logistics of making goods or services available, and consumer preferences. In this article's model, the adjustment of the money supply gets linked to other assets through the borrower/buyer buying other assets from the seller at a certain value, and through the nontraders holding other assets with a certain valuation. Hence in the agents' utilities, the price or value of these other assets changes depending on the adjustment of the money supply which causes inflation or deflation.

The impact of printing and withdrawing fiat money for the bank and the agents is examined. Comparisons are drawn between borrowing hard and fiat money. The article demonstrates how the utilities of the bank, of the agent which is a borrower and a buyer, of the agent which is a seller, and of the nontrading agents change as the values of the parameters for hard and fiat money vary. The resultant insights may enable central banks and individuals to develop a superior understanding of borrowing, buying, lending, selling, inflation, deflation, money printing, and withdrawal in a fiat economy and in a hard money economy.

The article analyzes inflation and deflation resulting from printing and withdrawing fiat money, abstracting away demand and supply shocks which require more extensive modeling. The nontrading agents are shown to be vulnerable in a fiat economy with money printing. A borrower and buyer benefit from borrowing fiat money. A seller benefits when the bank withdraws fiat money. The bank benefits from printing fiat money to a certain extent. The article explores hard and fiat money in a two-period economy. Inflation caused by fiat money printing or deflation caused by fiat money withdrawal are part of the article's research topic. The article analyzes the effects on one unified bank and multiple agents of printing and withdrawing fiat money. It compares the impacts of borrowing hard money versus fiat money. The article illustrates how the utilities of the bank, borrowing agents, buying agents, selling agents, and nontrading agents change with varying parameter values for hard and fiat money. It highlights the vulnerability of nontrading agents in an economy that employs fiat money printing. Advantages are discussed of borrowing fiat money for borrowers and buyers, while sellers benefit when the bank withdraws fiat money. The article posits that a bank within certain limits can derive benefits from printing fiat money.

The article more generally demonstrates how the bank, the borrower and buyer, the seller, and the nontrading agents get impacted by changes in parameter values. Features of hard money are illustrated, i.e. limited supply and outside the bank's control. Borrowing hard and fiat money is shown to have different impacts. The existence of hard money decreases the impact of inflation caused by printing fiat money. The article presents a benchmark where the bank prefers to lend hard money and fiat money to agent 1, agent 1 prefers to borrow hard and fiat money from the bank to buy other assets from agent 2, and agent 2 prefers to sell some of its assets to agent 1. The article illustrates how changing each of 64 parameter values relative to the benchmark impacts the preferences of the bank and the three kinds of agents. The article offers insights into questions such as the impacts of borrowing, buying, lending, selling, inflation, deflation, money printing, and withdrawal in both hard and fiat money economies on the bank's and agents' utilities. This article supplements the almost nonexistent analyses of the interaction between hard and fiat money. Overall, the article sheds light on the coexistence of hard and fiat money, providing valuable insights into their dynamics.

Section 2 presents the background. Section 3 reviews the literature. Section 4 presents the model. Section 5 analyzes the model. Section 6 illustrates the model. Section 7 provides an interpretation. Section 8 covers policy implications. Section 9 discusses the results. Section 10 covers limitations and avenues for future research. Section 11 concludes.

2. Background

Bitcoin (Nakamoto, 2008) is a decentralized digital currency which operates on a peer-to-peer network. It is not backed by any physical asset, government, or central authority. Historically, hard money approximated by gold, has been adopted widely. Bitcoin, without centralized parties or intermediaries, has different potential than physical gold related to censorship resistance, verifiability, portability, divisibility, convenience, and scarcity (Ikkurty, 2019). Central banks explore CBDCs (central bank digital currencies) to build efficient fiat payment systems and compete with cryptocurrencies. Hard money has real value and commands broad acceptance as a medium of exchange. Hard money is scarce, decentralized, has fixed supply, is difficult to counterfeit or manipulate, and cannot be printed. One further example is representative money (Nicholson, 1888; Steiner, 1941) that is backed by and redeemable for gold. Bitcoin's status as a hard currency is being debated. Although both gold and Bitcoin resemble hard money, current empirics illustrate differences. Long et al. (2021) apply the nonlinear autoregressive distributed lag model to show that gold can, while Bitcoin cannot, hedge against uncertainties to varying degrees. Wen et al. (2022) show during the Covid-19 pandemic that gold is, while Bitcoin is not,

Table 1
How printing and withdrawing fiat money impacts fiat money holders and borrowers. Downward arrow ↓ means negative impact. Upward arrow ↑ means positive impact.

	Fiat money holder	Fiat money borrower
Printing fiat money	↓	↑
Withdrawing fiat money	↑	↓

a safe haven for oil and stock markets. Some of this may be related to gold's market capitalization at ca $13 trillion[3] compared against Bitcoin's $0.6 trillion.[4] Boissay et al. (2022) describe the blockchain scalability and how high fees may fragment the crypto landscape, implying that, at least for now, cryptocurrencies cannot be a substitutive form of fiat money. This may change as the Lightning network and other innovations emerge.

Hard money has been used as a medium of exchange and as a store of value throughout history, partly because it is valuable and scarce. Precious metals, such as gold and silver, were adopted as money by historical civilizations around the world. Coins were frequently produced from metals, e.g. gold, silver, copper, which simplified transactions and promoted trading between various communities. Hard money is still recognized as a store of value in modern society, particularly during times of economic crises, because hard money is thought to retain value better than fiat money, which is susceptible to inflation. Hard money encompasses a form of currency or monetary system that relies on a commodity, i.e. a fixed asset with intrinsic value or decentralized consensus. Two approximate examples are Bitcoin and gold. Hard money is characterized by a fixed or limited supply, which sets it apart from fiat money which lacks physical asset backing and derives its value solely from trust in the government. Hard money provides a perceived sense of stability and limits the potential for inflation or devaluation, as the supply is constrained. In contrast, fiat money relies on trust in the government and central banks. Its value typically decreases over time due to fiat money printing or inflation.

The supply of Bitcoin is fixed at ca. 21 million. Bitcoin can be viewed as a form of hard money and is legal tender in two countries, i.e. El Salvador and the Central African Republic. More countries may adopt Bitcoin as legal tender in the future. Iwamura et al. (2019) discuss the potential competition between Bitcoin and central bank-issued fiat money. Ammous (2018) suggests a Bitcoin standard for nations.

Central banks are responsible for the issuance and governance of fiat money. Central banks can vary the supply of fiat money supply by printing it, for example by buying bonds and securities from the open market, or by withdrawing it, e.g. selling bonds and securities to the open market and thus destroying or burning the earned fiat money. However, central banks cannot print hard money since the supply of hard money is fixed. Hard money has the advantage of being a more reliable store of value than fiat money, which is not backed by tangible goods. Hard money is less vulnerable to inflation than fiat money due to its limited supply. Hard money thus provides stability for individuals and companies.

If fiat money is printed excessively, inflation and a decrease in the purchasing power of the currency may follow, which can disproportionately affect those who hold it, particularly those on fixed incomes or with savings in cash. Borrowers may benefit from expansions of the money supply if it causes lower interest rates and easier access to credit, but this can also contribute to inflation and a devaluation of the currency. The Cantillon effect (Murphy, 1986) is that the distribution of newly created money reaches different kinds of agents at different points in time which can affect the relative prices of goods and services disproportionately, also impacted by production and consumption patterns, market competition, and government policies. The withdrawal of fiat money from the economy may make it more expensive for borrowers to service their loans, but it can also lead to deflation and a decrease in economic activity, which can harm both borrowers and savers.

Printing fiat money may not necessarily entail confiscation or violation of property rights if done responsibly to maintain the stability and value of the currency. Obtaining such stability can be challenging and depends on supply, demand and other factors. With certain assumptions, printing fiat money does entail confiscation and violation of the property rights of those who hold it. Then borrowers benefit from expansions of the money supply, and withdrawing fiat money benefits savers and makes it more expensive for borrowers to service their pre-existing loans. Table 1 illustrates the negative ↓ and positive ↑ impact of printing and withdrawing fiat money on fiat money holders and borrowers. Since money printing dilutes the monetary value, fiat holders and borrowers are negatively and positively impacted, respectively. Money withdrawal has the opposite impact.

3. Literature

The limited literature on this topic covers five topics, namely hard money, competition between fiat currencies, competition between cryptocurrencies and fiat currencies, cryptocurrencies and CBDCs, inflation and currencies, and gametheoretic analyses.

3.1. Hard money

Fisher (1920) warns that "irredeemable paper money has almost invariably proved a curse to the country employing it." The world will experience unstoppable inflation unless the leading nations implement commodity or hard money standards. Cooper et al. (1982)

[3] https://8marketcap.com/metals/, retrieved August 7, 2023.
[4] https://coinmarketcap.com/, retrieved August 7, 2023.

point out that the primary motivation for reviving the gold standard is to eliminate inflation and to maintain a stable noninflationary environment. They propose a commodity standard that goes beyond gold. In their view, such a standard would stabilize general price levels. Friedman and Schwartz (1986) summarize the main pillars of monetary reform, namely the government monopoly on money creation, free banking, and the determination of units of account. They point out that Austrian economists support hard money and oppose discretionary money management. Ammous (2018) points out that individuals will gradually migrate from national money to hard money, which preserves value more effectively. Examples include seashells, glass beads, iron, copper, and other primitive forms of money, which were eventually replaced by gold and silver. Ammous and D'Andrea (2022) investigate the link between time preferences, money, and hard money. They point out that fiat money is expected to lose value over time due to inflation, which increases uncertainty, thus disincentivizing saving. However, forms of hard money such as Bitcoin are expected to maintain their value and purchasing power over time. Therefore, hard money reduces uncertainty and encourages savings. A hard money standard can lead to higher levels of social development. This article contributes to this literature by exploring the different impact of loans in hard and fiat money on various agents. Bibi (2023) explores the nature of money, focusing on cryptocurrencies such as Bitcoin and their potential impact on monetary systems. The author argues that state acceptance and citizens' adoption are crucial for Bitcoin to become money. The article highlights the potential influence of factors on the success and sustainability of Bitcoin, e.g. institutional pressures, convenience, environmental concerns, and the emergence of CBDCs. This article focuses mainly on the incentive of the bank to offer hard and fiat money loans and on the agents' incentives to apply for loans of the two kinds. The bank cannot print hard money, but it can create fiat money.

3.2. Competition between fiat currencies

Fernández-Villaverde and Sanches (2019) develop a model of competition between privately issued fiat currencies. They introduce entrepreneurs who can issue private currencies in a Lagos-Wright environment. They found that competing private currencies can coexist, but their coexistence does not necessarily result in efficiency or stability. Dowd and Greenaway (1993) analyze currency competition. They discover that network effects and switching costs cause agents to favor the use of a single currency. Mafi (2003) investigates the relationship between currency competition and inflation. She finds that countries in which citizens are legally allowed to hold foreign currencies tend to have lower average inflation rates. This result suggests that currency competition could lead to lower inflation. Eichengreen (2005) adopt a historical approach to competition between reserve currencies. He points out that competition for reserve-currency status is not a winner-takes-all game. Instead, it is likely that multiple currencies will continue to hold that status in the future. He predicts that the dollar and the euro will likely remain the dominant reserve currencies for the foreseeable future. Martin and Schreft (2006) challenge the traditional view that currencies cannot coexist. They demonstrate the existence of equilibria in which outside money is issued competitively. The findings show that it is unclear whether competing currency issuers can produce allocations superior to those that result from a monopolist issuer. Gawthorpe (2017) also shows that currency competition can lead to lower inflation rates than the exclusive use of a single fiat currency. Wang and Hausken (2021a) investigate competition between a national currency and a global currency across three different groups of agents, namely conventionalists, pioneers, and criminals. Currency features such as backing, convenience, confidentiality, transaction efficiency, financial stability, and security are represented in the model. The authors show how the three kinds of agents choose between the two currencies. Ron and Valeonti (2023) show during the US Civil War how more democratic governing institutions in the North impacted the legitimacy of tax policies and enabled more effective backing of the currency to cause moderate inflation, as opposed to the South which experienced hyperinflation. This article contributes to this literature by investigating competition between hard and fiat money. The bank can print and withdraw fiat money, causing increased inflation. The supply of hard money is fixed. The article shows how printing and withdrawing fiat money affects borrowers, non-borrowers, buyers, and sellers in an economy.

3.3. Competition between cryptocurrencies and fiat currencies

Almosova (2018) considers the calculation costs that private currencies entail, such as the expenses associated with mining and transaction verification. She finds that currency competition does not lead to price stability. However, the circulation of less costly private currencies exerts downward pressure on inflation. Schilling and Uhlig (2019) examine competition between a fiat currency that is used for daily payments and a cryptocurrency that can be used to avoid taxes, to maintain anonymity, and to resist repression. The results show that the substitution effect between fiat currencies and cryptocurrencies declines as asymmetries in trading costs and exchange fees become more pronounced. Senner and Sornette (2019) think that forms of fixed-supply money such as Bitcoin are negatively affected by their speculative and deflationary designs. The supply of stablecoins such as Tether can be varied. However, neither Bitcoin nor stablecoins are backed by governments or central banks. The authors contend that existing cryptocurrencies cannot replace fiat money. Jumde and Cho (2020) explore whether cryptocurrencies could eventually overtake fiat money. They employ the analytic hierarchy process method. Nine factors, namely accessibility, constant utility, value-common assets, stability, convertibility, divisibility, liquidity, volatility, and possibility of speculation, are introduced to analyze the performance of cryptocurrencies and fiat money. The findings show that fiat money is preferred to cryptocurrencies. Levulytė and Šapkauskienė (2021) explore the connections between cryptocurrencies and fiat money from the perspective of the three functions of money, i.e. medium of exchange, a unit of account, and store of value. They point out that cryptocurrencies such as Bitcoin and Ethereum are useful for cross-border transactions.

The results also show that fluctuations in cryptocurrency prices are affected by fluctuations in the prices of fiat currencies. Sissoko (2021) discusses the hypothetical scenario in which agents can buy goods at fixed rates by using various currencies. He emphasizes that a financial system can be established accordingly. The effectiveness of the banking system depends on its capacity to increase the money supply in response to societal needs. Wang and Hausken (2022a) explore how competition between a variable-supply currency, such as fiat money, and a fixed-supply currency, such as Bitcoin, impacts agents' choices of currency. They rely on a money-in-utility approach. They consider money printing and withdrawal, and an agent's support of money, i.e. backing, convenience, transaction efficiency, financial stability, confidentiality, and security. They analyze the dynamic volume fractions of transactions in two currencies over time. Yu (2023) adopts a search theoretic model to explore the conditions under which fiat money and cryptocurrencies coexist. For cryptocurrencies to exist, the inflation rate in a stationary monetary equilibrium must be zero. The growth rate of the money stock determines the inflation rate for fiat currencies. The findings show that cryptocurrencies can coexist with fiat money. In addition, under the zero-inflation equilibrium, bans on cryptocurrency may decrease social welfare due to the inflation tax. Helmi et al. (2023) apply a time-varying vector autoregressive model to examine the impact of CBDC news on financial and cryptocurrency markets. They find that CBDC uncertainty and volatility index shocks contribute significantly to cryptocurrency uncertainty and Bitcoin return shocks. This article considers competition between variable-supply fiat money and forms of fixed-supply hard money, such as Bitcoin. Agents gain utility by holding hard money, fiat money, other assets, and by borrowing. The article studies how the bank can lend hard or fiat money to the agents and the impact of that lending on the bank and the agents.

3.4. Cryptocurrencies and central bank digital currencies (CBDCs)

Belke and Beretta (2020) recommend that central banks accept the technology that powers cryptocurrency, and develop a well-regulated two-tier system by engaging in innovation in the domain of payment infrastructures. Nabilou (2020) points out that cryptocurrencies such as Bitcoin may pose risks to the monopoly of central banks over the issuance of money, to price stability, to the smooth operation of payment systems, to the execution of monetary policy, and to the stability of financial institutions. Accordingly, central banks explore CBDCs. He notes that the European Central Bank must overcome several legal challenges before introducing CBDCs at the Eurozone level. Laboure et al. (2021) summarize the evolution of cryptocurrencies and CBDCs. They predict that cryptocurrencies and fiat money will coexist in the near future. They also note that numerous concerns, including ones that have to do with energy efficiency, transaction speed, identity problems, and regulation, must be addressed before cryptocurrencies can be accepted widely. Scharnowski (2022) explores market reactions to speeches on CBDCs from the perspective of cryptocurrency investors. He finds that cryptocurrency prices tend to react more strongly to positive speeches, while negative CBDC sentiment has a slight amplifying effect. The findings indicate that investors do not view CBDCs as a threat to cryptocurrencies. Benigno et al. (2022) examine competition between national currencies such as CBDCs and global cryptocurrencies such as Bitcoin in a two-country economy with complete markets. They conclude that national nominal interest rates must be equal in the two countries at the time when a global cryptocurrency is adopted. Deviations from interest rate equality indicate that there is a risk of the national currency being abandoned. They call this feature of the model "crypto-enforced monetary policy synchronization." Adrian and Mancini-Griffoli (2021) consider benefits and risks of digital money compared with traditional money, and assess digital money backed with central bank reserves as a private-public partnership. Ayadi et al. (2023) employ a Cross-Quantilogram model. They reveal a negative association between the CBDC uncertainty index and the returns of cryptocurrencies and stablecoins. This article adds to the literature by evaluating competition between forms of hard money approximated by Bitcoin, which are supported by a proof-of-work consensus mechanism, and forms of fiat money exemplified by CBDCs, paper money, and coins. CBDCs are one form of fiat money that central banks issue, support and supervise. Hard currencies, conversely, typically have a fixed supply because they are backed by assets such as commodities and gold or by consensus algorithms. This article contains a model that illustrates the effects of hard and fiat money lending and borrowing on the economy. It also discusses the conditions under which a bank is prepared to lend and those under which agents are willing to borrow from the bank to buy assets from other agents.

3.5. Inflation and currencies

Sakurai and Kurosaki (2023) find that major cryptocurrencies become slightly better inflation hedges after the reopening after the Covid-19 pandemic, regardless of whether they have a maximum supply cap. Xin and Jiang (2023) develop a dynamic stochastic general equilibrium model to show that CBDCs can stabilize the economic fluctuations caused by a negative interest rate policy implemented by interest rate adjustment to reach various economic objectives such as monetary stimulation, stable exchange rates, and desired inflation levels. Feres (2021) analyzes how the US Federal Reserve handles crises associated with fiat, debt and inflation. He recommends a transition to a monetary system backed by a finite commodity. Messay (2023) develops an idealized model as a thought experiment to show that an international fiat currency issued by one or several core countries is a main factor impacting national economic development, and that seigniorage accrued to developed countries by consuming more than they produce is at the expense of the developing countries in the Global South. This article analyzes how inflation relates to the coexistence of hard money and fiat money.

3.6. Gametheoretic analyses

Welburn and Hausken (2017) explore economic crises from a gametheoretic perspective. They introduce six kinds of agents, i.e. countries, central banks, intergovernmental financial organizations, banks, firms, and households. These agents can adopt various strategies, such as setting interest rates, lending, borrowing, and consuming. The authors use the model to illustrate the European debt crisis. Hart (2020) models the positive input consumption and the negative input pollution with a constant elasticity-of-substitution function. Since pollution has a negative impact, the corresponding exponent, which is the elasticity of the Cobb Douglas utility, is negative. This article uses a similar approach—loans are raised to a negative exponent in the borrower's Cobb Douglas utility because the borrower must pay interest to the bank. Mou et al. (2021) develop two gametheoretic models of CBDC adoption in different countries. The findings indicate that each country should issue a CBDC, regardless of the choices of other countries. The leading country needs to issue a CBDC to maintain its status. Other countries must also introduce a CBDC in order to avoid losing ground in the digital realm. Wang and Hausken (2022b) establish a game between a central bank and a household choosing between a CBDC, a non-CBDC such as Bitcoin, and consumption. The central bank determines the CBDC interest rate, which can be negative. The household chooses its portfolio while accounting for backing, transaction efficiencies, and costs. They demonstrate how the bank and the household choose their strategies. The outcome is determined analytically and illustrated numerically. This article relates to this literature by considering the interactions between a bank and the agents. A bank may choose to lend or not to lend hard or fiat money to an agent. An agent may choose to borrow or not to borrow hard or fiat money from the bank. The other agents may choose to sell or retain their assets or do nothing. The article shows the impact of these strategies.

4. The model

This section develops the model for n agents in Section 4.1, the one bank in Section 4.2, and the inflation rates in Section 4.3. The model is chosen to be minimally complex while simultaneously capturing reality. The model features one bank as a unitary actor, along with n agents consisting of one borrower and buyer, one seller, and $n-2$ nontraders over the two periods. The article establishes the Cobb Douglas utility function (objective function) for both the bank and the agents, following a step-by-step process as outlined in this section. The article employs a money-in-utility approach, where utility is derived from the possession of money or assets. This approach is commonly utilized in economic and financial research. The underlying conception is that the utility function captures an individual's preferences regarding a range of goods and services. Various studies have applied the money-in-utility approach, e.g. Ramsey (1928) and Sidrauski (1967). Recent examples include the research by Chen and Guo (2014), Mian et al. (2021), and Ferrari Minesso et al. (2022). The homogeneity of asset classes is determined by their Cobb Douglas utility elasticities. Appendix A shows the nomenclature.

4.1. The n agents

Subsection 4.1.1 considers period 1 for the three kinds of agents. Subsection 4.1.2 considers period 2 for the three kinds of agents. Agent i, $i=1,\ldots,n$, has a Cobb Douglas utility U_{it} with multiple inputs in period t, $t=1,2$. Agent i can be a household, or any agent, e. g. firm, institution, organization. In period t, agent i holds maximum three kinds of assets with value j_{it}, $j_{it}\geq 0, j=q,m,o$. The article employs a Cobb Douglas utility function and includes assets within the utility function. Other examples applying this approach are Ferrari Minesso et al. (2022); Syarifuddin and Bakhtiar (2022); Wachter and Yogo (2010). The agents assess their utilities across two periods and opt for trading in period 2 if the utility in that period surpasses the utility in period 1. That is a realistic description of an economy to some extent. Therefore, an intertemporal optimization approach is not employed in the article. These assets are hard money q_{it} and fiat money m_{it} deposited in the open market (e.g. in the stock, bond or decentralized finance markets), and other assets o_{it}. Examples of other assets o_{it} are anti-inflationary investments, non-fungible tokens, bonds, stocks, other financial assets, real estate, physical assets, and illegal assets. Holding asset j_{it} earns interest rate I_{jt}, $I_{jt}\in\mathbb{R}, j=q,m,o$, from the open market, as determined by the open market. Each Cobb Douglas input is raised to the Cobb Douglas elasticity α_{ijt}, $\alpha_{ijt}\geq 0$, $j=q,m,o$, which accounts for asset j's liquidity, backing, convenience, confidentiality, transaction efficiency, financial stability, and security.

4.1.1. Period 1

4.1.1.1. Agent 1. Assume, without loss of generality in choice of agent, that agent 1 in period 1 borrows L_{1q1} in hard money and L_{1m1} in fiat money from the bank and buys an asset valued as $L_{1q1}+L_{1m1}$. Agent 1's borrowing interest rate is r_{j1}, $r_{j1}\in\mathbb{R}, j=q,m$. Multiplying agent 1's loan L_{1j1} with $1+r_{j1}$ to account for the interest rate r_{j1}, and inverting since a loan L_{1j1} with interest rate r_{j1} is costly for agent 1 causing negative impact on agent 1's utility U_{11} (just as pollution is costly in Hart's, 2020 model, see Section 3.6), gives the input $\left(\frac{1}{(1+r_{j1})L_{1j1}}\right)^{\alpha_{1jL1}}=\left((1+r_{j1})L_{1j1}\right)^{-\alpha_{1jL1}}$, $j=q,m$, assuming the Cobb Douglas elasticity $\alpha_{ijL1}\geq 0$. Agent 1 uses its entire borrowing $L_{1q1}+L_{1m1}$ to buy other assets. For simplicity, assume that the borrower buys other asset o_{11} in period 1. Adding agent 1's loan $L_{1q1}+L_{1m1}$ to its other assets o_{11} gives $o_{11}+L_{1q1}+L_{1m1}$ which is multiplied with $1+I_{o1}$ to account for the interest rate I_{o1}, and raised to the Cobb Douglas elasticity α_{1o1} which gives the input $\left((1+I_{o1})(o_{11}+L_{1q1}+L_{1m1})\right)^{\alpha_{1o1}}$. Agent 1 holds neither hard money q nor fiat money m in period 1, i.e. $q_{11} = m_{11} = 0$. Requiring constant returns to scale gives $\alpha_{1o1}+\alpha_{1qL1}+\alpha_{1mL1} = 1$. Applying the $Max(1,\bullet)$ function for agent 1's loan $L_{1j1}, j = q,m$, agent 1's period 1 utility is

$$U_{11} = \left((1+I_{o1})(o_{11}+L_{1q1}+L_{1m1})\right)^{\alpha_{1o1}}$$
$$\left(Max(1,(1+r_{q1})L_{1q1})\right)^{-\alpha_{1qL1}}\left(Max(1,(1+r_{m1})L_{1m1})\right)^{-\alpha_{1mL1}} \quad (1)$$

4.1.1.2. Agents $2,\ldots,n$. Assume that agent $i, i = 2,\ldots,n$, in period 1 does not borrow, i.e. $L_{ij1} = 0, j = q,m$, does not sell its other assets o_{i1}, and holds assets with value $j_{i1}, j = q,m,o$. Multiplying agent i's asset j_{i1} with $1+I_{j1}$ to account for the interest rate I_{j1}, and raising to the Cobb Douglas elasticity α_{ij1} gives the input $\left((1+I_{j1})j_{i1}\right)^{\alpha_{ij1}}$. Requiring constant returns to scale gives $\alpha_{iq1} + \alpha_{im1} + \alpha_{io1} = 1$. Hence agent i's period 1 utility is

$$U_{i1} = \left((1+I_{q1})q_{i1}\right)^{\alpha_{iq1}}\left((1+I_{m1})m_{i1}\right)^{\alpha_{im1}}\left((1+I_{o1})o_{i1}\right)^{\alpha_{io1}}, i = 2,\ldots,n \quad (2)$$

4.1.2. Period 2

4.1.2.1. Agent 1. In period 2 agent 1 borrows L_{1q2} in hard money and L_{1m2} in fiat money from the bank and buys an asset valued as $L_{1q2}+L_{1m2}$ from agent 2, without loss of generality. The assets are traded based on their value, regardless of whether they are traded in hard money or fiat money. Agent 1 retains its loans L_{1q1} and L_{1m1} from period 1 to period 2. Adding $L_{1q2}+L_{1m2}$ to agent 1's other assets, adding L_{1q2} and L_{1m2} to agent 1's loans, and applying the $Max(1,\bullet)$ function for agent 1's loans $L_{1j1} + L_{1j2}, j = q,m$, agent 1's period 2 utility is

$$U_{12} = \left((1+I_{o2})(o_{11}+L_{1q1}+L_{1m1}+L_{1q2}+L_{1m2})\right)^{\alpha_{1o2}}$$
$$\left(Max\left(1,(1+r_{q2})\left(L_{1q1}+L_{1q2}\right)\right)\right)^{-\alpha_{1qL2}}$$
$$\frac{\left(Max(1,(1+r_{m2})(L_{1m1}+L_{1m2}))\right)^{-\alpha_{1mL2}}}{(1+\pi_2)^{-\alpha_{1mL2}}} \quad (3)$$

Division with $(1+\pi_2)^{-\alpha_{1mL2}}$ for agent 1's fiat money loan $L_{1m1}+L_{1m2}$ is to account for the inflation rate π_2, $\pi_2 \in \mathbb{R}$. The inflation rate is positive if $\pi_2 > 0$, nonexistent if $\pi_2 = 0$, and negative, i.e. deflation if $\pi_2 < 0$. The negative signs on the Cobb Douglas elasticities α_{1jL2} correspond to the negative signs on α_{1jL1} in (1), due to inverting the base in the function since the loans L_{1q1} and L_{1m1} are costly. Requiring constant returns to scale gives $\alpha_{1o2}+\alpha_{1qL2}+\alpha_{1mL2} = 1$.

4.1.2.2. Agent 2. In period 2 agent 2 sells other assets valued at $L_{1q2}+L_{1m2}$ to agent 1, retaining $o_{21} - L_{1q2} - L_{1m2}$. Multiplying with $1+I_{o2}$ to account for the interest rate I_{o2}, and raising to the Cobb Douglas elasticity α_{2o2} gives the input $\left((1+I_{o2})(o_{21}-L_{1q2}-L_{1m2})\right)^{\alpha_{2o2}}$. Agent 2's sale causes its hard money holding to increase from q_{21} to $q_{21}+L_{1q2}$ which is multiplied with $1+I_{q2}$ to account for the interest rate I_{q2}, and raised to the Cobb Douglas elasticity α_{2q2} which gives the input $\left((1+I_{q2})(q_{21}+L_{1q2})\right)^{\alpha_{2q2}}$. Agent 2's sale causes its fiat money holding to increase from m_{21} to $m_{21}+L_{1m2}$ which is multiplied with $1+I_{m2}$ to account for the interest rate I_{m2}, raised to the Cobb Douglas elasticity α_{2m2}, and divided with $(1+\pi_2)^{\alpha_{2m2}}$ to account for the inflation rate π_2, which gives the input $\frac{\left((1+I_{m2})(m_{21}+L_{1m2})\right)^{\alpha_{2m2}}}{(1+\pi_2)^{\alpha_{2m2}}}$. Agent 2 neither buys nor borrows. Requiring constant returns to scale gives $\alpha_{2q2} + \alpha_{2m2} + \alpha_{2o2} = 1$. Multiplying the three inputs, agent 2's period 2 utility is

$$U_{22} = \left((1+I_{q2})(q_{21}+L_{1q2})\right)^{\alpha_{2q2}}\frac{\left((1+I_{m2})(m_{21}+L_{1m2})\right)^{\alpha_{2m2}}}{(1+\pi_2)^{\alpha_{2m2}}}$$
$$\left((1+I_{o2})(o_{21}-L_{1q2}-L_{1m2})\right)^{\alpha_{2o2}} \quad (4)$$

4.1.2.3. Agents $3,\ldots,n$. Assume that agent $i, i = 3,\ldots,n$, in period 2 neither borrows nor buys nor sells. That is, agent i does nothing, but is subject to the inflation rate π_2. Hence agent i's fiat money holding input is $\frac{\left((1+I_{m2})m_{i2}\right)^{\alpha_{im2}}}{(1+\pi_2)^{\alpha_{im2}}}$, and agent i's period 2 utility is

$$U_{i2} = \left((1+I_{q2})q_{i2}\right)^{\alpha_{iq2}}\frac{\left((1+I_{m2})m_{i2}\right)^{\alpha_{im2}}}{(1+\pi_2)^{\alpha_{im2}}}\left((1+I_{o2})o_{i2}\right)^{\alpha_{io2}}, i = 3,\ldots,n \quad (5)$$

Requiring constant returns to scale gives $\alpha_{iq2} + \alpha_{im2} + \alpha_{io2} = 1$.

4.2. The bank

Three examples of articles assuming that the bank and central banks are one unitary actor are Chen et al. (2017); Gertler and Kiyotaki (2015); Wang and Hausken (2021b). The article employs a similar approach and assumes that the bank and central bank are one unitary actor, which holds an amount of asset j_t, and can lend L_{1jt} to agent 1, $j = q, m, t = 1, 2$. The bank holds no other assets. Therefore, the central bank's role is embedded by the bank actor. Banks have multifarious revenue streams. The bank earns an interest rate $I_{jt}, I_{jt} \in \mathbb{R}$, from the open market, analogously to the n agents. We exclude deposits by the n agents from the bank's utility since the n agents deposit their assets in the open market. Since the bank and the n agents earn the same interest rate I_{jt} in the open market, we may interpret the n agents as depositing their assets in the bank, which further deposits in the open market. The bank's utility U_t in period $t, t = 1, 2$, has two multiplicative inputs pertaining to holding asset $j_t, j = q, m$, and two multiplicative inputs pertaining to earning interest from lending L_{1jt} to agent $1, j = q, m$. Four other examples of articles assuming that the bank has a Cobb Douglas utility function are Goodfriend and McCallum (2007); Mullineaux (1978); Tsai (2013); Wang and Hausken (2022b).

4.2.1. Period 1

In period 1 the bank holds q_1 in hard money and m_1 in fiat money. The bank provides loans L_{1q1} in hard money and L_{1m1} in fiat money to agent 1. After providing the loans, the bank holds $j_1 - L_{1j1}$ in asset $j, j = q, m$, which is multiplied with $1 + I_{j1}$ to account for the interest rate I_{j1}, and raised to the Cobb Douglas elasticity $\beta_{j1}, \beta_{j1} \geq 0$, which gives the input $\left((1 + I_{j1})(j_1 - L_{1j1})\right)^{\beta_{j1}}$. The bank does not print fiat money in period 1. Lending L_{1j1} to agent 1 gives an interest rate r_{j1}. Assume that when the bank lends L_{1j1} to agent 1, the bank retains the utility of the amount it lends out. Hence L_{1j1} is multiplied with $1 + r_{j1}$ instead of $r_{j1}, j = q, m$, and raised to the Cobb Douglas elasticity $\beta_{jL1}, \beta_{jL1} \geq 0, j = q, m$. Requiring constant returns to scale gives $\beta_{q1} + \beta_{m1} + \beta_{qL1} + \beta_{mL1} = 1$. Multiplying the four inputs, and applying the $Max(1, \bullet)$ function for the loans L_{1q1} and L_{1m1}, the bank's period 1 utility is

$$U_1 = \left((1 + I_{q1})(q_1 - L_{1q1})\right)^{\beta_{q1}} \left((1 + I_{m1})(m_1 - L_{1m1})\right)^{\beta_{m1}}$$

$$\left(Max\left(1, (1 + r_{q1})L_{1q1}\right)\right)^{\beta_{qL1}} \left(Max\left(1, (1 + r_{m1})L_{1m1}\right)\right)^{\beta_{mL1}} \tag{6}$$

4.2.2. Period 2

In period 2 the bank provides loans L_{1q2} in hard money and L_{1m2} in fiat money to agent 1. The bank continues in period 2 to hold the loans L_{1q1} and L_{1m1} that agent 1 incurred in period 1. The fiat money loan L_{1m2} is provided by money printing. After lending hard money L_{1q2} to agent 1, the bank holds $q_1 - L_{1q1} - L_{1q2}$ in hard money, which is multiplied with $1 + I_{q2}$ to account for the interest rate I_{q2}, and raised to the Cobb Douglas elasticity $\beta_{q2}, \beta_{q2} \geq 0$, which gives the input $\left((1 + I_{q2})(q_1 - L_{1q1} - L_{1q2})\right)^{\beta_{q2}}$. After printing and lending fiat money L_{1m2} to agent 1, printing an amount $P_{m2}, P_{m2} \geq 0$, of fiat money, and withdrawing an amount $W_{m2}, W_{m2} \geq 0$, of fiat money, the bank holds $m_1 - L_{1m1} + P_{m2} - W_{m2}$ in fiat money, which is multiplied with $1 + I_{m2}$ to account for the interest rate I_{m2}, raised to the Cobb Douglas elasticity $\beta_{m2}, \beta_{m2} \geq 0$, and divided with $(1 + \pi_2)^{\beta_{m2}}$ to account for the inflation rate π_2, which gives the input $\frac{((1+I_{m2})(m_1-L_{1m1}+P_{m2}-W_{m2}))^{\beta_{m2}}}{(1+\pi_2)^{\beta_{m2}}}$. The fiat money loan L_{1m2} to agent 1 is not subtracted in the previous expression since the bank prints the fiat money. Lending hard money L_{1q2} to agent 1 gives an interest rate r_{q2}. Adding agent 1's retained loan L_{1q1} from period 1, $L_{1q1} + L_{1q2}$ is multiplied with $1 + r_{q2}$ and raised to the Cobb Douglas elasticity $\beta_{qL2}, \beta_{qL2} \geq 0$, which gives the input $\left((1 + r_{q2})(L_{1q1} + L_{1q2})\right)^{\beta_{qL2}}$. Lending fiat money L_{1m2} to agent 1 gives an interest rate r_{m2}. Adding agent 1's retained loan L_{1m1} from period 1, $L_{1m1} + L_{1m2}$ is multiplied with $1 + r_{m2}$ and raised to the Cobb Douglas elasticity $\beta_{mL2}, \beta_{mL2} \geq 0$, and divided with $(1 + \pi_2)^{\beta_{m2}}$ to account for the inflation rate π_2, which gives the input $\frac{((1+r_{m2})(L_{1m1}+L_{1m2}))^{\beta_{mL2}}}{(1+\pi_2)^{\beta_{mL2}}}$. $\beta_{q2} + \beta_{m2} + \beta_{qL2} + \beta_{mL2} = 1$. Multiplying the four inputs, the bank's period 2 utility is

$$U_2 = \left((1 + I_{q2})(q_1 - L_{1q1} - L_{1q2})\right)^{\beta_{q2}}$$

$$\frac{\left((1 + I_{m2})(m_1 - L_{1m1} + P_{m2} - W_{m2})\right)^{\beta_{m2}}}{(1 + \pi_2)^{\beta_{m2}}}$$

$$\left((1 + r_{q2})(L_{1q1} + L_{1q2})\right)^{\beta_{qL2}} \frac{\left((1 + r_{m2})(L_{1m1} + L_{1m2})\right)^{\beta_{mL2}}}{(1 + \pi_2)^{\beta_{mL2}}} \tag{7}$$

4.3. The inflation rates π_1 and π_2

The bank cannot print hard money q. Hence no inflation exists for hard money q. To create a reference standard with zero inflation rate $\pi_1 = 0$ in period 1, assume that the bank does not print fiat money m in period 1. Instead, the bank uses its fiat money holding for lending L_{1m1} to agent 1 in period 1. The inflation rate π_2 in period 2 equals a ratio. The numerator is the net increase from period 1 to period 2 in the amounts of hard money q and fiat money m. Since the amount of hard money q does not increase, which is the nature of hard money, the net increase from period 1 to period 2 is $L_{1m2} + P_{m2} - W_{m2}$, where L_{1m2} is what the bank prints to lend to agent 1, P_{m2} is what the bank prints to increase the fiat money circulating amount, and W_{m2} is what the bank withdraws to decrease the fiat money circulating amount. The denominator in the ratio is the amount $\sum_{i=1}^{n} q_{i1} + q_1$ of circulating hard money q in period 1 plus the amount $\sum_{i=1}^{n} m_{i1} + m_1$ of circulating fiat money m in period 1. Thus, the amount of hard money also impacts the inflation rate π_2 in period 2. Hence the inflation rate in period 2 is

$$\pi_2 = \frac{L_{1m2} + P_{m2} - W_{m2}}{\sum_{i=1}^{n} q_{i1} + q_1 + \sum_{i=1}^{n} m_{i1} + m_1} \tag{8}$$

5. Analyzing the model

The model conceptualizes one unitary bank and three kinds of agents, whose behaviors are driven by their respective utilities. The article assumes that the bank and agents act in a manner that maximizes their utilities and compares their utilities over the two periods. Factors that drive the bank's and agents' behavior include holdings of hard money, fiat money, and other assets, and borrowing in hard and fiat money. The bank's behavior is impacted by its holdings of hard money and fiat money, borrowing interest rates in hard money and fiat money, its fiat money printing and fiat money withdrawal.

5.1. Comparing periods 1 and 2

See Appendix B.

Property 1. Agents 1 and 2 prefer to trade if (13) and (14) are satisfied. Agent $i, i = 3,\ldots,n$, prefers the trade between agents 1 and 2 if (15) is satisfied. The bank prefers to lend to agent 1 if (16) is satisfied.

Proof: Eq. (13) implies that agent 1's utility U_{12} in period 2 is higher than its utility U_{11} in period 1, i.e. $U_{11} < U_{12}$. Thus, agent 1 prefers to buy other assets valued as $L_{1q2} + L_{1m2}$ from agent 2 in period 2. Analogously, (14) implies that agent 2's utility U_{22} in period 2 is higher than its utility U_{21} in period 1, i.e. $U_{21} < U_{12}$. Thus, agent 2 prefers to sell other assets valued as $L_{1q2} + L_{1m2}$ to agent 1 in period 2. It follows from (15) that agent $i, i = 3,\ldots,n$, prefers the trade between agents 1 and 2 since its utility U_{i2} in period 2 is higher than its utility U_{i1} in period 1, i.e. $U_{i1} < U_{i2}$. Agent $i, i = 3,\ldots,n$, is unaffected if $U_{i1} = U_{i2}$. Eq. (16) implies that the bank's utility U_2 in period 2 is higher than its utility U_1 in period 1, i.e. $U_1 < U_2$. Thus, the bank prefers to lend $L_{1q2} + L_{1m2}$ to agent 1 in period 2.

Property 1 states that agent 1 prefers to borrow $L_{1q2} + L_{1m2}$ from the bank and buy other assets from agent 2 when $U_{11} < U_{12}$. Agent 2 prefers to sell other assets $L_{1q2} + L_{1m2}$ to agent 1 when $U_{21} < U_{22}$. The bank prefers to lend $L_{1q2} + L_{1m2}$ to agent 1 when $U_1 < U_2$. Agent $i, i = 3,\ldots,n$, prefers the trade between agents 1 and 2 when $U_{i1} < U_{i2}$. Hence agent i is unaffected by the trade between agents 1 and 2 when $U_{i1} = U_{i2}$.

6. Illustrating the model

To illustrate the solution in Section 5, this section alters the model's 64 parameter values relative to the following plausible benchmark parameter values intended to capture the specificities of the context and how the three kinds of agents and bank operate within this context. The benchmark parameter values, and the ranges for the parameter values in the analysis, are chosen carefully with the following objectives in mind: 1. The analysis should capture the most interesting phenomena involved for the three kinds of agents and the bank. 2. The borrower should or should not prefer to borrow hard money and fiat money from the bank in order to buy other assets from the seller. 3. The seller should or should not prefer to sell other assets to the buyer. 4. The nontraders should or should not prefer the trade between the buyer and seller, and should or should not prefer the bank to lend to the borrower and print or withdraw fiat money, though without being able to impact the borrower, seller and bank. 5. The bank should or should not prefer to lend hard money and fiat money to the borrower, and should or should not prefer to print or withdraw fiat money. The analysis is intended to generate valuable insights shown below and believed not to be easily captured by alternative analyses.

Assume that agent 1 has no hard money and no fiat money in the two periods, i.e. $q_{11} = q_{12} = m_{11} = m_{12} = \$0$. Agent 1 also has no other assets before borrowing and buying other assets, i.e. $o_{11} = \$0$. This choice is made to test whether agent 1 may be willing to incur loans from the bank to acquire other assets. For agent 1 assume the loans $L_{1q1} = L_{1m1} = \$10$ in period 1, to enable buying other assets $L_{1q1} + L_{1m1} = \$20$ from agent 2, and the loans $L_{1q2} = L_{1m2} = \$15$ in period 2 to enable buying other assets $L_{1q2} + L_{1m2} = \$30$ from agent 2. Thus, agent 1 holds other assets $o_{11} + L_{1q1} + L_{1m1} = \$20$ after borrowing and buying in period 1, which equals the amount $o_{12} = \$20$ of other assets agent 1 holds before borrowing and buying other assets in period 2. Agent 1 holds other assets $o_{12} + L_{1q2} + L_{1m2} = o_{11} + L_{1q1} + L_{1m1} + L_{1q2} + L_{1m2} = \$50$ after borrowing and buying in period 2.

Assume that agent 2 in period 1 has hard money $q_{21} = \$100$ and fiat money $m_{21} = \$100$, after receiving payments $L_{1q1} = L_{1m1} = \$10$ from agent 1. Agent 2 in period 2 has hard money $q_{21} + L_{1q2} = \$100 + \$15 = \$115$ and fiat money $m_{21} + L_{1m2} = \$100 + \$15 = \$115$ after receiving payments $L_{1q2} = L_{1m2} = \$15$ from agent 1. Agent 2 in period 1 has other assets $o_{21} = \$400$ after selling $L_{1q1} + L_{1m1} = \$20$

to agent 1, chosen to be high to ensure that agent 2 may be willing to sell some of its other assets to agent 1. Agent 2 in period 2 has other assets $o_{21} - L_{1q2} - L_{1m2} = \$400 - \$15 - \$15 = \$370$ after selling other assets $L_{1q2} + L_{1m2} = \$30$ to agent 1.

For agent $i, i = 3,\ldots,n$, assume the benchmark $n = 3$ so that only one agent exists aside from agents 1 and 2, and $q_{i1} = q_{i2} = m_{i1} = m_{i2} = \$100, o_{i1} = o_{i2} = \$400$ so that agent 3 largely resembles agent 2. The differences are that agent 3 does not sell other assets, which agent 2 does, and does not buy other assets, as agent 1 does. Agent 3 is introduced to analyze how an agent can be impacted without buying and selling.

Assume for the benchmark that the bank's period 1 holding of hard money q_1 is the sum of the n agents' holding of hard money, i.e. $q_1 = \sum_{i=1}^{n} q_{i1} = (n-1)q_{i1} = \$200, i = 2,\ldots,n$. Analogously, the bank's period 1 holding of fiat money m_1 is the sum of the n agents' holding of fiat money, i.e. $m_1 = \sum_{i=1}^{n} m_{i1} = (n-1)m_{i1} = \$200, i = 2,\ldots,n$. Since the bank in period 1 lends $L_{1q1} = \$10$ in hard money and $L_{1m1} = \$10$ in fiat money to agent 1, the bank's hard money holding in period 1 is $q_1 - L_{1q1} = \$200 - \$10 = \$190$. Since the bank in period 1 does not print fiat money, and uses its fiat money holding for lending, the bank's fiat money holding in period 1 is $m_1 - L_{1m1} = \$200 - \$10 = \$190$. Analogously, Since the bank in period 2 lends $L_{1q2} = \$15$ in hard money and $L_{1m2} = \$15$ in fiat money to agent 1, the bank's hard money holding in period 2 is $q_1 - L_{1q1} - L_{1q2} = \$200 - \$10 - \$15 = \$175$. Assume the benchmark where the bank in period 2 prints $L_{1m2} = \$15$ in fiat money to furnish the loan to agent 1. The bank does not otherwise print money, i.e. $P_{m2} = \$0$, and does not withdraw money, i.e. $W_{m2} = \$0$. Hence the bank's fiat money holding in period 2 is $m_1 - L_{1m1} + P_{m2} - W_{m2} = \$200 - \$10 - \$0 - \$0 = \$190$.

Agent $i, i = 1,\ldots,n$, has the same Cobb Douglas elasticity for holding other assets in both periods, i.e. $\alpha_{io1} = \alpha_{io2} = 1/2$. Agent 1's Cobb Douglas elasticities of loans in hard money and fiat money are $\alpha_{1qL1} = \alpha_{1qL2} = \alpha_{1mL1} = \alpha_{1mL2} = 1/4$. Agent $i, i = 2,\ldots,n$ has the same Cobb Douglas elasticities for holding hard money and fiat money in both periods, i.e. $\alpha_{iq1} = \alpha_{iq2} = \alpha_{im1} = \alpha_{im2} = 1/4$. The bank has the same Cobb Douglas elasticities for holding hard money, fiat money, hard money lending, and fiat money lending, in both periods, i.e. $\beta_{j1} = \beta_{j2} = \beta_{jL1} = \beta_{jL2} = 1/4, j = q, m$. The inflation rate benchmark is $\pi_2 = 1.875\%$ based on (8), which is close to the common inflation rate target 2% in many fiat economies. The interest rates $I_{jt}, j = q, m, o, t = 1, 2$, for three kinds of assets determined by the open market in the two periods are equivalent, i.e. $I_{q1} = I_{q2} = I_{m1} = I_{m2} = I_{o1} = I_{o2} = 2\%$. The borrowing interest rates $r_{jt}, j = q, m, t = 1, 2$, for hard money q and fiat money m, determined by the bank in the two periods are also equivalent, i.e. $r_{q1} = r_{q2} = r_{m1} = r_{m2} = 5\%$. With these benchmark parameter values the benchmark solution is $U_{11} = 1.39$, $U_{12} = 1.40$, $U_{21} = 204.00$, $U_{22} = 209.43$, $U_{i1} = 204.00$, $U_{i2} = 203.06$, $U_1 = 45.11$, $U_2 = 69.23$, $\pi_2 = 1.875\%$. In the benchmark agents 1 and 2 and the bank prefer period 2 rather than period 1, while agent i prefers period 1 rather than period 2.

Figure 1 illustrates the agents' and the bank's utilities in response to variations in the 64 parameter values, relative to the plausible benchmark parameter values. The x-axis in each panel represents the labeled parameter, displaying the corresponding parameter values. The y-axis represents the utilities of both the agents and the bank. In Figure 1 each of the 64 parameter values is altered from its benchmark marked with vertical dashed lines in each panel, while the other 63 parameter values are kept at their benchmarks. Multiplication of π_2 with 10^4 and 10^2, and multiplication of U_{11} and U_{12} with 200, 20 and 10 are for scaling purposes. The 17 most interesting panels are interpreted in this section. The remaining 47 panels are interpreted in Appendix C.

In Figure 1a, as the number n of agents increases, which is intuitively beneficial for the bank, the bank's utilities U_1 and U_2 increase concavely toward infinity. Agent 1's utility U_{12} decreases slightly since the inflation rate π_2 decreases slightly, which hurts agent 1 because of agent 1's fiat money loans L_{1m1} and L_{1m2}. In contrast, agents 2 and i's utilities U_{22} and U_{i2} increase slightly because the inflation rate π_2 decreases slightly, which benefits agents 2 and i because of their fiat money holdings m_{22} and m_{i2}. The utilities U_{11}, U_{21}, U_{i1} remain constant since neither the number n of agents nor the inflation rate π_2 play a role in period 1. The inflation rate π_2 decreases convexly and asymptotically toward zero due to division with n in (8). The inflation impact of the bank's fiat money printing L_{1m2} to provide agent 1's loan L_{1m2} is spread across more agents. As the number n of agents increases, each agent and the bank experience a lower inflation rate π_2 according to (8).

In Figure 1d, as agent 1's borrowing L_{1q2} in hard money in period 2 increases, the bank's utility U_2 is inverse U shaped. That is, the bank prefers to lend an optimal amount L_{1q2} of hard money to agent 1. The maximum of U_2 is 85.12 when $L_{1q2} = \$90$. The bank's utility U_2 decreases concavely toward zero after the maximum. The bank prefers to lend hard money to agent 1 when $\$0 \le L_{1q2} < \$185.98$. When $\$185.98 < L_{1q2} \le \$190.00$, the bank's utility U_2 is less than U_1. That follows from the nature of the bank's inverse U shaped Cobb Douglas utility U_2, which is low when the bank lends excessively or minimally. Agent 2's utility U_{22} is also inverse U shaped. Agent 2 prefers to sell an optimal amount $L_{1q2} + L_{1m2}$ of its other assets $o_{21} - L_{1q1} - L_{1m1}$ to agent 1 in period 2. That follows from the nature of agent 2's inverse U shaped Cobb Douglas utility U_{22}, which is low when agent 2 sells its other assets excessively or minimally. The maximum of U_{22} is 213.18 when $L_{1q2} = \$61.67$. Agent 2's utility U_{22} decreases concavely after the maximum. Agent 2 wants to sell its other assets $L_{1q2} + L_{1m2}$ to agent 1 when $\$0 \le L_{1m2} < \$143.96$. Hence, agent 2 prefers not to sell too much other assets to agent 1 in period 2. Agent 1's utility U_{12} increases since agent 1 benefits from buying other assets $L_{1q2} + L_{1m2}$ using its borrowing L_{1q2} in hard money. The utilities U_{i1}, U_1, and U_{11} are constant since agent 1's borrowing L_{1q2} in hard money plays no role in period 1. Agent i's utility U_{i2} is constant since L_{1q2} has no impact on agent i in period 2. The inflation rate π_2 remains constant since L_{1q2} plays no role in (8).

In Figure 1f, as agent 1's borrowing L_{1m2} in fiat money in period 2 increases, the inflation rate π_2 increases because L_{1m2} is added to the numerator in (8). Thus, the bank's utility U_2 increases concavely since it prints fiat money for lending in period 2. This implies that the benefit of printing L_{1m2} fiat money for lending overrides the negative impact of holding $m_1 - L_{1m1} + P_{m2} - W_{m2}$ of fiat money from the increasing inflation rate π_2. The bank always prefers to lend fiat money to agent 1 in period 2 since $U_2 > U_1$. Agent 1's utility U_{12} increases since the inflation rate π_2 increases, which benefits agent 1 because of agent 1's fiat money loans L_{1m1} and L_{1m2}. Agent 2's utility U_{22} is inverse U shaped since it prefers to sell an optimal amount L_{1m2} of its other assets o_{22} to agent 1. The maximum of U_{22} is 210.83 when $L_{1m2} = \$42.36$. Agent 2's utility U_{22} decreases concavely toward zero after the maximum. Agent 2 wants to sell other assets $L_{1q2} + L_{1m2}$ to agent 1 when $\$0 \le L_{1m2} < \$110.96$. Agent 2 prefers not to sell too much other assets to agent 1 in period 2. Agent i's utility U_{i2} decreases from $U_{i2} = 204.00$ when $L_{1m2} = \$0$, to $U_{i2} = 203.06$ when $L_{1m2} = \$15$, and thereafter decreases further, because the inflation rate π_2 increases, which hurts agent i because of its fiat money holdings m_{i2}. Thus, agent i suffers from agent 1's borrowing in fiat money L_{1m2} in period 2 without doing anything. The utilities U_{11}, U_{21}, U_{i1}, and U_1 remain constant since neither agent 1's borrowing L_{1m2} in fiat money nor the inflation rate π_2 play a role in period 1.

In Figure 1s, as the bank's fiat money printing P_{m2} in period 2 increases, the inflation rate π_2 increases since P_{m2} is added to the numerator in (8). Interestingly, the bank's utility U_2 is inverse U shaped. It first increases toward a maximum $U_2 = 75.28$ when $P_{m2} = \$435$ and then decreases convexly and asymptotically toward zero. This implies that, before the maximum, the benefit of printing $L_{1m2} + P_{m2}$ fiat money overrides the negative impact of holding $m_1 - L_{1m1} + P_{m2} - W_{m2}$ of fiat money due to the increasing inflation rate π_2. After the maximum, the negative impact of holding $m_1 - L_{1m1} + P_{m2} - W_{m2}$ of fiat money due to the increasing inflation rate π_2 overrides the benefit of printing $L_{1m2} + P_{m2}$ fiat money. Hence the bank prefers to print an optimal amount of fiat money. When $\$0 \le P_{m2} < \$17929.02$, $U_2 > U_1$. The bank prefers not to print more fiat money than \$17929.02 since $U_2 < U_1$ when $P_{m2} > \$17929.02$ in period 2. Agent 1's utility U_{12} increases concavely since the inflation rate π_2 increases, which benefits agent 1 because of its fiat money loans L_{1m1} and L_{1m2}. Agents 2 and i's utilities U_{22} and U_{i2} decrease convexly toward zero. Agents 2 and i are hurt by the increasing inflation rate π_2 due to their holdings m_{22} and m_{i2} of fiat money. Agents 2 and i suffer from the bank's fiat money printing $L_{1m2} + P_{m2}$ in period 2, without agent i doing anything. The utilities U_{11}, U_{21}, U_{i1}, and U_1 remain constant since neither the bank's fiat money printing $L_{1m2} + P_{m2}$ nor the inflation rate π_2 play a role in period 1.

In Figure 1t, conversely, as the bank's fiat money withdrawing W_{m2} in period 2 increases, the inflation rate π_2 decreases and becomes negative when $W_{m2} > \$15$, caused by W_{m2} being subtracted from the numerator in (8). Interestingly, the bank's utility U_2 decreases concavely toward zero. It implies that the negative impact of withdrawing money overrides the benefits of holding $m_1 - L_{1m1} + P_{m2} - W_{m2}$ of fiat money due to the decreasing inflation rate π_2. The bank prefers not to withdraw more fiat money than $W_{m2} = \$168.44$ since $U_2 < U_1$ when $P_{m2} > \$168.44$ in period 2. Agent 1's utility U_{12} decreases sightly since the inflation rate π_2 decreases, which hurts agent 1 because of its fiat money loans L_{1m1} and L_{1m2}. Agents 2 and i's utilities U_{22} and U_{i2} increase. Thus, agents 2 and i benefit in period 2 from the decreasing inflation rate π_2 due to their holdings m_{22} and m_{i2} of fiat money. That is, agents 2 and i benefit from the bank's fiat money withdrawal W_{m2} in period 2, without agent i doing anything. More specifically, agent i's period 2 utility increases from $U_{i2} = 203.06$ when $W_{m2} = \$0$ to $U_{i2} = U_{i1} = 204.00$ when $W_{m2} = \$15$, which exactly matches the bank's money printing $L_{1m2} = \$15$. Thereafter agent i's period 2 utility increases to $U_{i2} = 216.99$ when $W_{m2} = \$190$. The utilities U_{11}, U_{21}, U_{i1}, and U_1 remain constant since neither the bank's fiat money printing $L_{1m2} + P_{m2}$ nor the inflation rate π_2 play a role in period 1.

In Figure 1w and Figure 1z, as the interest rate I_{j1} for holding money $j, j = q, m$, in period 1 increases, which is intuitively beneficial to the bank and agent $i, i = 2, \ldots n$, the three utilities U_1, U_{21}, and U_{i1} increase concavely toward infinity. The bank only wants to lend money $L_{1q2} + L_{1m2}$ to agent 1 in period 2 when $0 \le I_{j1} < 4.66$ in period 2. If the interest rate is too high, $I_{j1} > 4.66$, the bank prefers to hold money j rather than lending it out. The inflation rate π_2 is constant since I_{j1} plays no role in (8). Agent 1's utilities U_{11} and U_{12} are constant since agent 1 holds no money j in the two periods. The utilities U_2, U_{22}, and U_{i2} remain constant since I_{j1} plays no role in period 2.

In Figure 1af and Figure 1ai, as the borrowing interest rate r_{j1} for money $j, j = q, m$ in period 1 increases, which is intuitively beneficial to the bank, the bank's utility U_1 increases concavely toward infinity. The bank only wants to lend money L_{1j2} to agent 1 in period 2 when r_{j1} is sufficiently low, i.e. $0 \le r_{j1} < 4.82$. Agent 1's utility U_{11} decreases convexly toward zero since a higher borrowing interest rate r_{j1} for money j is costly. The inflation rate π_2 is constant since r_{j1} plays no role in (8). The utilities U_{21} and U_{i1} are constant since agents 2 and i do not borrow money j in period 1. The utilities U_2, U_{12}, U_{22}, and U_{i2} remain constant since r_{j1} plays no role in period 2.

In Figure 1am, as agent 1's Cobb Douglas elasticity α_{1o2} for holding $o_{11} + L_{1q1} + L_{1m1} + L_{1q2} + L_{1m2}$ of the other assets in period 2 increases, which is intuitively beneficial to agent 1, the utility U_{12} increases concavely. Agent 1 wants to borrow $L_{1q2} + L_{1m2}$ from the bank in period 2 when α_{1o2} is not too low, i.e. $0.5 < \alpha_{1o2} \le 1$. The inflation rate π_2 is constant since α_{1o2} plays no role in (8). Agent 1's utility U_{11} is constant since α_{1o2} plays no role in period 1. The utilities U_1, U_2, U_{21}, U_{22}, U_{i1}, and U_{i2} remain constant since α_{1o2} has no impact on the bank, agents 2 and i.

In Figure 1as and Figure 1av, as agent 2's Cobb Douglas elasticity α_{2j2} for holding money $j_{21} + L_{1j2}, j = q, m$ in period 2 increases, its utility U_{22} in period 2 decreases convexly because holding other assets $o_{21} - L_{1q2} - L_{1m2}$ becomes less beneficial for agent 1 with decreasing Cobb Douglas elasticity $\alpha_{2o2} = 1 - \alpha_{2q2} - \alpha_{2m2}$. Hence, in contrast to Figure 1as and Figure 1av, agent 2 wants to sell its other assets valued as $L_{1q2} + L_{1m2}$ when α_{2j2} is sufficiently low, i.e. $0 \le \alpha_{2j2} \le 0.27, j = q, m$. The eight variables U_{11}, U_{12}, U_{21}, U_{i1}, U_{i2}, U_1, U_2, π_2 remain constant.

In Figure 1ay and Figure 1bb, as agent i's Cobb Douglas elasticity α_{ij2} for holding money $j_{i2}, j = q, m$ in period 2 increases, its utility U_{i2} in period 2 decreases convexly because holding other assets o_{i2} becomes less beneficial for agent 1 with decreasing Cobb Douglas elasticity $\alpha_{io2} = 1 - \alpha_{iq2} - \alpha_{im2}$. Agent i prefers the trade between agents 1 and 2 when α_{ij1} is sufficiently high, i.e. $0.25 \le \alpha_{iq2} \le 1, j = q, m$. The eight variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i2}, U_1, U_2, π_2 remain constant.

Table 2
Comparing this article's approach and results with those in the literature.

Literature	Comparing this article's approach and results with those in the literature
Adrian and Mancini-Griffoli (2021)	They assess the benefits and risks of digital money compared with traditional money.
Almosova (2018)	She proposes that private currencies exert downward pressure on inflation.
Ammous (2018)	He argues that hard money will eventually replace fiat money while this article assesses coexistence.
Ammous and D'Andrea (2022)	They suggest that hard money maintains value over time and that a hard money standard fosters higher levels of social development.
Ayadi et al. (2023)	They show that the CBDC uncertainty index impacts the return on cryptocurrencies negatively.
Belke and Beretta (2020)	They argue that central banks should embrace the technology of hard money.
Benchimol and Fourçans (2012)	They separate central banks and commercial banks as different players.
Benigno et al. (2022)	They propose a crypto-enforced monetary policy synchronization when hard money and fiat money coexist.
Boissay et al. (2022)	They suggest that hard money currently cannot substitute fiat money, while this article allows coexistence.
Chen et al. (2017)	They assume that the commercial banks and central banks are one unitary actor.
Chen and Guo (2014)	They adopt a money-in-utility approach, as this article also does, where utility is obtained from holding assets.
Cooper et al. (1982)	They propose that a hard money standard aims to reduce inflation, consistently with this article.
Dowd and Greenaway (1993)	They argue that network effects and switching costs are driving forces for players to use one currency.
Eichengreen (2005)	He suggests that multiple reserve currencies will continue to coexist.
Feres (2021)	He proposes a hard money based monetary system.
Fernández-Villaverde and Sanches (2019)	They point out that competing private currencies can coexist.
Ferrari Minesso et al. (2022)	They adopt a money-in-utility approach, as this article also does, where utility is obtained from holding assets.
Fisher (1920)	He suggests a hard money standard to control the unstoppable inflation associated with a fiat money standard, which is a finding compatible with this article.
Friedman and Schwartz (1986)	They support hard money standards and oppose the government monopoly on fiat money creation.
Gawthorpe (2017)	He suggests that currency competition causes lower inflation rates.
Gertler and Kiyotaki (2015)	They assume that the commercial banks and central banks are one unitary actor.
Goodfriend and McCallum (2007)	They assume that banks have Cobb Douglas utility functions, which this article also assumes.
Gorton (2023)	He argues that inflation generally depends on the fiat money supply, consistently with this article.
Hart (2020)	He proposes a negative exponent for the elasticity of the Cobb Douglas utility for pollution as a negative impact factor, which this article also does for the borrower's hard money and fiat money loans.
Helmi et al. (2023)	They find that CBDC uncertainty and volatility index shocks significantly impact the volatility of hard money approximated by Bitcoin.
Engelhardt (1996)	He considers the resource constraints for the players and banks.
Iacoviello (2005)	He assesses the resource constraints for the players and banks.
Ikkurty (2019)	He argues that hard money approximated by Bitcoin has features such as censorship resistance, verifiability, portability, divisibility, convenience, and scarcity.
Iwamura et al. (2019)	They believe that hard money is unlikely to replace fiat money such CBDC, which to some extent differs from this article which illustrates coexistence.
Jumde and Cho (2020)	They suggest that hard money will eventually overtake fiat money.
Kadiyala (1972)	He suggests that a Cobb Douglas utility is appropriate for even distributions of multiple assets.
Laboure et al. (2021)	They claim that cryptocurrencies and fiat money will coexist.
Levulytė and Šapkauskienė (2021)	They highlight that hard money is advantageous for international transactions.
Long et al. (2021)	He contends that hard money approximated by gold can, while hard money approximated by Bitcoin cannot hedge against uncertainties to varying degrees.
Mafi (2003)	She argues that currency competition causes lower inflation.
Messay (2023)	She suggests that an international currency issued by one or several major countries is the driving factor that impacts national economic development at the expense of the Global South.
Martin and Schreft (2006)	They demonstrate the existence of competing currencies.
Mian et al. (2021)	They adopt a money-in-utility approach, as this article also does, where utility is obtained from holding assets.
Mou et al. (2021)	They argue that central banks need to issue their fiat money as CBDCs.
Mullineaux (1978)	He assumes that banks have Cobb Douglas utility functions.
Murphy (1986)	He proposes that the Cantillon effect, i.e. the uneven distribution of wealth and purchasing power that occurs as a result of changes in the fiat money supply, benefits those who receive the new money first at the expense of others.
Nabilou (2020)	He argues that hard money approximated by Bitcoin poses risks to fiat money.
Nakamoto (2008)	He/she/they propose a hard money currency.
Nicholson (1888)	He studies examples of hard money approximated by representative money, which is backed by and redeemable for gold.
Ramsey (1928)	They adopt a money-in-utility approach, which this article also does, where utility is obtained from holding assets.
Ron and Valeonti (2023)	They point out that democratic governing institutions tend to have moderate inflation with fiat money.
Sakurai and Kurosaki (2023)	They find that major cryptocurrencies become slightly more effective safeguards against inflation after the Covid-19 pandemic.
Scharnowski (2022)	He suggests that investors do not view fiat money CBDCs as a threat to cryptocurrencies.
Schilling and Uhlig (2019)	They reveal that as trading cost and exchange fee disparities increase, the substitution effect between fiat money and hard money diminishes.
Schuster and Sigmund (1983)	They propose a replicator dynamics model.
Senner and Sornette (2019)	They argue that hard money cannot replace fiat money.
Sidrauski (1967)	He uses a money-in-utility approach, where utility is obtained from holding assets.
Sissoko (2021)	He suggests that a financial system can be established based on competing currencies, which is compatible with this article.
Steiner (1941)	He studies examples of hard money approximated by representative money, which is backed by and redeemable for gold.
Syarifuddin and Bakhtiar (2022)	They employ a Cobb Douglas utility function for holding assets.
Tsai (2013)	He assumes that banks have Cobb Douglas utility functions, which this article also assumes.

(*continued on next page*)

Table 2 (*continued*)

Literature	Comparing this article's approach and results with those in the literature
Wachter and Yogo (2010)	They employ a Cobb Douglas utility function for holding assets.
Wang and Hausken (2021a)	They show how conventionalists, pioneers, and criminals choose between two currencies.
Wang and Hausken (2021b)	They assume that the commercial banks and central banks are one unitary actor.
Wang and Hausken (2022a)	They explore competition between hard money and fiat money, focusing on money printing and withdrawal, accounting for how an agent supports the two kinds of money.
Wang and Hausken (2022b)	They assume that banks have Cobb Douglas utility functions.
Welburn and Hausken (2017)	They analyze financial crises assuming fiat money.
Wen et al. (2022)	They argue that hard money approximated by gold serves as a safe haven for oil and stock markets, while hard money approximated by Bitcoin does not provide the same level of safety.
Xin and Jiang (2023)	They argue that fiat money such as CBDC can stabilize economic fluctuations arising from a negative interest rate policy.
Yu (2023)	He suggests that fiat money and cryptocurrencies can coexist, which is compatible with this article.

In Figure 1be and Figure 1bh, as the bank's Cobb Douglas elasticity β_{j2} for holding money $j_1 - L_{1j1}, j = q, m$ in period 2 increases, its utility U_2 in period 2 increases convexly because holding hard money $q_1 - L_{1q1} - L_{1q2}$ and fiat money $m_1 - L_{1m1} + P_{m2} - W_{m2}$ becomes more beneficial for the bank with increasing Cobb Douglas elasticity β_{j2}, which overrides the negative impact of decreasing Cobb Douglas elasticity $\beta_{mL2} = 1 - \beta_{q2} - \beta_{m2} - \beta_{qL2}$ for fiat money loans. The bank always wants to give money loans $L_{1q2} + L_{1m2}$ to agent 1 in period 2 since $U_1 < U_2$ for Figure 1be and Figure 1bh. The eight variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, U_{i2}, U_1, π_2 remain constant.

In Figure 1bk, as the bank's Cobb Douglas elasticity β_{qL2} for hard money lending L_{1q2} in period 2 increases, its utility U_2 increases slightly. The reasons are as follows. According to (7), the borrowing interest rates $r_{q2} = r_{m2}$ for hard money and fiat money are the same in period 2, but the bank's loans $L_{1m1} + L_{1m2}$ in fiat money are impacted by the positive inflation rate $\pi_2 = 1.875\%$ in period 2. Thus, the increase in the bank's utility U_2 from holding the hard money loan $L_{1q1} + L_{1q2}$ is higher than the decrease from holding the fiat money loan $L_{1m1} + L_{1m2}$ in period 2 due to the decreasing Cobb Douglas elasticity $\beta_{mL2} = 1 - \beta_{q2} - \beta_{m2} - \beta_{qL2}$. The bank always wants to lend fiat money $L_{1q2} + L_{1m2}$ to agent 1 in period 2 since $U_1 < U_2$. The eight variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, U_{i2}, U_1, π_2 remain constant.

In Figure 1bl, as the bank's Cobb Douglas elasticity $\beta_{qL1} = \beta_{qL2}$ for hard money lending L_{1q1} and $L_{1q1} + L_{1q2}$ in the two periods increases, its utility U_2 increases slightly. The net impact of increasing $\beta_{qL1} = \beta_{qL2}$ is different for the bank in periods 1 and 2. In period 1, according to (6), the bank's decreasing utility from hard money lending L_{1q1} is offset by the bank's increasing utility U_1 from fiat money lending L_{1m1} due to the decreasing Cobb Douglas elasticity $\beta_{mL1} = 1 - \beta_{q1} - \beta_{m1} - \beta_{qL1}$ for fiat money lending L_{1m1}. Thus, the bank's utility U_1 remains constant. In contrast, in period 2 according to (7), the borrowing interest rates $r_{q2} = r_{m2}$ for hard money and fiat money are equivalent, but the bank's fiat money loans $L_{1m1} + L_{1m2}$ are impacted by the positive inflation rate $\pi_2 = 1.875\%$. Thus, the bank's increasing utility U_2 from holding the hard money loan $L_{1q1} + L_{1q2}$ is higher than the decrease from holding the fiat money loan $L_{1m1} + L_{1m2}$ due to the decreasing Cobb Douglas elasticity $\beta_{mL2} = 1 - \beta_{q2} - \beta_{m2} - \beta_{qL2}$. Thus, the bank's utility U_2 increases slightly. The bank always wants to lend fiat money $L_{1q2} + L_{1m2}$ to agent 1 in period 2 since $U_1 < U_2$. The eight variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, U_{i2}, U_1, π_2 remain constant.

Table 2 compares this article's approach and results with those in the literature.

7. Interpreting the model

The authors have identified 24 insights in the previous section.

1. More agents benefit the bank and cause less inflation since the bank's money printing to provide agent 1's loans gets spread across more agents. That causes lower utility for agent 1 which borrows and buys and prefers high inflation, higher utility for agent 2 which sells and prefers low inflation, and higher utility for the nontrading agent i, $i = 3, \ldots, n$ which prefers low inflation.
2. As agent 1's borrowing of hard money increases in period 1, agent 1 benefits from buying other assets. The bank's utility is inverse U shaped. The bank prefers to lend to a certain degree to benefit from agent 1's interest rate payment, but prefers not to lend excessively which depletes its hard money holding.
3. As agent 1's borrowing of hard money increases in period 2, it benefits from buying other assets. The selling agent 2's period 2 utility is inverse U shaped, because it prefers to sell some of its other assets, which are abundant, without, however, depleting its stock. The utility is inverse U shaped as in the previous point.
4. As agent 1's borrowing of fiat money in period 1 increases, it benefits from buying other assets. Analogously to the case of hard money, the bank's utility is inverse U shaped. The bank prefers to lend to a certain degree to benefit from agent 1's interest payments, but prefers not to lend excessively which depletes its holdings of fiat money.
5. As agent 1's borrowing of fiat money increases in period 2, it benefits from buying other assets. The bank's utility increases concavely because it prints fiat money for lending and because it benefits from agent 1's interest payments. The utility of agent 2, a seller, in period 2 takes the shape of an inverted U, as described in point 3. The bank prints fiat money for lending which hurts the nontrading agent i.

6. As the bank prints more fiat money in period 2, its utility is inverse U shaped. The bank prefers to print to a certain degree to benefit from holdings, but prefers not to print excessively which causes extremely high inflation. Agent 1 benefits from buying other assets and prefers high inflation. However, the selling agent 2's utility decreases because it prefers low inflation. Analogously, the nontrading agent *i*'s utility decreases because it suffers from high inflation.
7. Conversely, as the bank's withdrawal of fiat money increases in period 2, the inflation rate decreases, and the bank's utility decreases concavely. Interestingly, the bank prefers to withdraw fiat money to a certain degree to benefit from the decrease in inflation due to its fiat money holding. However, it also strives to avoid excessive withdrawal, which may cause extremely low inflation. Agent 1 suffers a detriment because it buys other assets and thus prefers high inflation. The utilities of the selling agent 2 and the nontrading agent *i* increase because they prefer low inflation.
8. As the interest rate for hard and fiat money increases in period 1, agents 2, *i*, and the bank benefit from holding money. The bank prefers not to lend money to agent 1 in period 2 if the interest rate for holding money in period 1 is excessively high. That is so because the bank benefits from holding money in period 1. Thus, the bank is uninterested in lending in period 2.
9. As the borrowing interest rate for hard or fiat money increases in period 1, the bank's utility increases concavely. The bank prefers not to lend money to agent 1 in period 2 when the borrowing interest rate in period 1 is too high because it benefits from lending in period 1. Thus the bank is uninterested in lending in period 2. Agent 1 intuitively suffers from a high borrowing interest rate.
10. As agent 1's Cobb Douglas elasticity of holding other assets increases in period 1, it benefits from buying other assets. Agent 1 wants to borrow money from the bank in period 2 when its Cobb Douglas elasticity of holding other assets is low, because it benefits from buying other assets in period 1. Hence agent 1 prefers not to buy other assets in period 2.
11. As agent 1's Cobb Douglas elasticity of holding other assets increases in period 2, it benefits from buying other assets. Agent 1 wants to borrow money from the bank in period 2 when its Cobb Douglas elasticity of holding other assets is not low.
12. As agent 2's Cobb Douglas elasticity for holding hard or fiat money in period 1 increases, its utility decreases due to the corresponding decrease in the Cobb Douglas elasticity of holding other assets. Agent 2 wants to sell its other assets when its Cobb Douglas elasticity of holding moneyis sufficiently high.
13. As agent 2's Cobb Douglas elasticity of holding hard or fiat money increases in period 2, its utility decreases due to the decrease in the Cobb Douglas elasticity of holding other assets. In contrast to the previous point, agent 2 wants to sell its other assets when its Cobb Douglas elasticity of holding money is sufficiently low.
14. As agent 2's Cobb Douglas elasticity of holding hard money increases over the two periods, agent 2's utilities decrease convexly, as described in the previous two points. Agent 2 wants to sell other assets when its Cobb Douglas elasticity of holding hard money over the two periods is sufficiently high.
15. As agent 2's Cobb Douglas elasticity of holding fiat money over the two periods increases, agent 2's utilities in the two periods decrease convexly, as in points 11 and 12. Analogously to the previous point, agent 2 wants to sell other assets when its Cobb Douglas elasticity of holding fiat money is sufficiently high.
16. As agent *i*'s Cobb Douglas elasticity of holding hard or fiat money increases in period 1, its utility decreases due to the decrease in the Cobb Douglas elasticity of holding other assets. Interestingly, agent *i* prefers the trade between agents 1 and 2 when its Cobb Douglas elasticity of holding money is sufficiently high.
17. As agent *i*'s Cobb Douglas elasticity of holding hard or fiat money in period 2 increases, its utility decreases convexly, as in the previous point. Agent *i* prefers the trade between agents 1 and 2 when its Cobb Douglas elasticity of holding money is sufficiently high.
18. As the bank's Cobb Douglas elasticity of holding hard or fiat money in period 1 increases, its utility increases convexly. That follows since the bank benefits more from holding money than from lending it due to the decrease in the Cobb Douglas elasticity of money loans. The bank prefers to lend to agent 1 in period 2 when its Cobb Douglas elasticity of holding money is sufficiently low.
19. As the bank's Cobb Douglas elasticity of holding money in period 2 increases, its utility in period 2 increases convexly, as in the previous point. The bank always wants to lend money to agent 1 in period 2.
20. As the bank's Cobb Douglas elasticity of holding hard money over the two periods increases, its utilities increase convexly, as in point 17. The bank wants to provide money loans to agent 1 in period 2 when its Cobb Douglas elasticity of holding hard money is sufficiently low.
21. As the bank's Cobb Douglas elasticity of holding fiat money increases over the two periods, its utilities increase convexly, as in the previous point. The bank wants to give money loans to agent 1 in period 2 when its Cobb Douglas elasticity of holding fiat money is sufficiently low.
22. As the bank's Cobb Douglas elasticity of lending money in period 1 increases, its utility remains constant. Thus, the bank's benefit from the increase in the Cobb Douglas elasticity of holding hard money is offset by the decrease in the Cobb Douglas elasticity of lending hard money in period 1. The bank always wants to lend fiat money to agent 1 in period 2.
23. As the bank's Cobb Douglas elasticity of lending hard money in period 2 increases, its utility increases slightly. That follows since the bank then benefits more from lending hard money than from lending fiat money.
24. As the bank's Cobb Douglas elasticity of lending hard money increases over the two periods, its utility in period 1 remains constant as in point 21, and its utility in period 2 increases slightly, as in point 22.

8. Policy implications

Money plays an essential role in an economy by serving as a medium of exchange, as a unit of account, and as a store of value. Modern society cannot operate without money. This article investigates an economy with both hard and fiat money. The findings offer insights to traders, such as borrowers and sellers, nontraders, policymakers, central banks, and others.

The model incorporates certain aspects of monetary policy, e.g. fiat money printing, borrowing interest rates, and deposit interest rates. First, the article contains insights that may be useful to central banks adjusting the money supply, monetary policy, and the inflation rate. Central banks are commonly responsible for issuing and managing fiat money. Central banks fully control fiat money, but do not control the supply of hard money.

Second, the fixed supply of hard money means that inflation and deflation cannot be manipulated by varying its supply. Thus, the effectiveness of monetary policy in the context of hard money is limited. It is beneficial for central banks to account for the existence of hard money when they design monetary policies.

Third, the results have potential implications for understanding the impact of borrowing hard and fiat money. Borrowers benefit from borrowing both hard and fiat money. Notably, borrowing hard money has no impact on nontrading agents. Fiat money borrowing harms nontrading agents due to inflation following money printing.

Fourth, printing fiat money might boost the economy and increase the amount of fiat money that is available for lending, buying, and other financial activities. The analysis shows that central banks benefit from printing fiat money. However, its utility decreases when printing too much fiat money. Therefore, it is reasonable for central banks to limit the supply of fiat money to a certain degree.

Fifth, the inflation that the printing of fiat money causes is spread across all nontrading agents. The impact of inflation diminishes as the number of nontrading agents increases. Thus, in an economy with many agents, central banks can print more fiat money without causing excessive inflation.

Sixth, central banks benefit from withdrawing fiat money to a limited degree since it decreases causes decreasing inflation. Reducing the amount of fiat money in circulation curbs inflation. However, withdrawal discourages borrowing, buying, selling, and other financial activities. As a whole, withdrawing fiat money is not conducive to economic growth. Prudent implementation is recommended when implementing deflationary monetary policies such as withdrawing fiat money.

Seventh, nontrading agents suffer as a result of fiat money printing, and benefit from fiat money withdrawal. Therefore, as inflation increases, it becomes more sensible for nontrading agents to consider becoming borrowers and buyers of other assets.

Eighth, the findings provide insights to the spread effect of money printing, withdrawal, borrowing, lending, buying, selling, inflation, and deflation, which account for most of the financial activities that unfold in an economy.

Nineth, researchers, individuals, firms, financial analysts, investors, business owners, and others may find the findings informative as they attempt to understand hard money, fiat money, borrowing, buying, and selling.

9. Discussion

The bank's withdrawal of fiat money in period 2 is the only scenario in which the nontrading agent $i, i = 3, \ldots, n$ prefers period 2 over period 1. This shows how vulnerable agent i is or can be in a fiat economy. More generally, the model shows how agent i is negatively affected by changes in parameter values. The negative impact decreases with the number of nontrading agents. In contrast, agent i is unaffected in a hard money economy. That agent 1 borrows hard money does not influence agent i's utility. In a hard money economy, financial activities, e.g. borrowing, lending, buying, and selling, only affect agents as a result of trading. Inflation has no influence on them.

Agent 2, as a seller, also benefits from the bank's withdrawal of fiat money. Analogous to agent i, agent 2 suffers from fiat money printing. In contrast, agent 1, being a borrower and a buyer, prefers the bank to print fiat money and not to withdraw it. The bank favors printing over withdrawal. Specifically, since the bank prints fiat money to lend to agent 1 in period 2, its utility is higher in period 2 than in period 1 except if it prints or withdraws fiat money excessively.

For simplicity, while retaining the key ingredients, the article assumes only one agent which borrows and buys, i.e. agent 1, only one seller, i.e. agent 2, and arbitrarily many nontrading agents, i.e. agent i, $i = 3, \ldots, n$. The notional agent 1 can represent an aggregate of many borrowers and buyers. The seller can be an aggregate of many sellers.

In a fiat economy, the impact of the inflation that printing fiat money causes is split across all agents. Specifically, agent 1 benefits and agent 2 suffers. Agent i, which does not borrow, lend, buy, or sell, also experiences the undesirable impacts of the printing of fiat money. Its asset holdings depreciate as inflation increases. Beyond agent 1, the bank, as an issuer and controller, also benefits from printing fiat money. That benefit stems from the inflation costs that are borne by sellers and nontrading agents. However, the benefit of printing fiat money is limited. The bank cannot increase its utility by printing fiat money continuously, which may cause hyperinflation and harm both the bank and the economy.

In a hard money economy, the bank cannot print hard money to lend to agent 1. Lending and borrowing thus have no impact on inflation, and the utilities of the nontrading agents remain unchanged. Hence the bank cannot transfer costs through inflation like in a fiat economy. The impact of fluctuations in the fiat money supply, which results in inflation or deflation, is diminished by the existence of hard money.

The bank benefits from lending both hard and fiat money because it receives interest payments from agent 1. However, the utility curve of the bank takes the shape of an inverted U, which indicates that the bank prefers to lend to agent 1, up to a certain point. The excessive lending of hard money causes the holdings of the bank to decrease significantly. The excessive lending of fiat money causes hyperinflation. Both affect the utility of the bank adversely.

Agent 1 holds no money or assets before borrowing from the bank and buying assets from agent 2. Therefore, agent 1 is a poor agent compared with agents 2 and *i*. Agent 1 benefits from buying other assets using its borrowing from the bank. That follows both for hard and fiat money. Agent 1, as a borrower, prefers high inflation, which results in lower interest payments. However, only borrowing fiat money can cause inflation to increase if the bank prints fiat money. Agent 1 prefers fiat money to hard money. Fiat money is favored by borrowers and buyers, but it harms sellers and nontrading agents. Agent 2 possesses abundant other assets and benefits from selling some of its other assets in exchange for hard or fiat money. However, agent 2's willingness to sell its other assets is limited. Therefore, the agent 2's utility takes the shape of the letter U.

When the bank and agents 2 and *i* suddenly become rich in period 1, i.e. their holdings of hard or fiat money increase in period 1, intuitively, their utilities increase. Therefore, a money airdrop in period 1 is beneficial to the economy. In addition, the inflation rate decreases in period 2 if such an airdrop has occurred in period 1. The foregoing indicates that an increase in holdings of hard and fiat money in period 1 diminishes the inflation in period 2. The impact of a money airdrop in period 1 is analogous to that of an increase in the number of agents in an economy. An other-asset airdrop in period 1 also benefits the economy, but it has no impact on inflation in period 2.

When the bank benefits excessively in period 1, which may occur as a result of an increase in the deposit interest rate, in the borrowing interest rate, or in the Cobb Douglas elasticities of holding or lending hard or fiat money, the bank loses interest in lending to agent 1 in period 2. That follows because the bank benefits significantly in period 1. Analogously, when agent 1 benefits excessively in period 1, for instance due to a dramatic increase in its Cobb Douglas elasticity of holding other assets, it loses interest in borrowing and in buying other assets in period 2.

10. Limitations and future research

One limitation of this article pertains to the nature of a Cobb Douglas utility. Limited amounts of one kind of assets combined with abundant amounts of another kind of assets causes low utility. Kadiyala (1972) suggests that a Cobb Douglas utility is more suitable for even distributions of multiple assets. In the present article, the issue is mitigated by introducing a Max function. Future studies may identify and formulate alternative utility functions to account for other phenomena. The proposed model provides some mathematical development followed by Property 1. Mathematical development, e.g. in the sense of equilibrium determination, is not analyzed in the article. Future research may adopt a gametheoretic approach and examine the equilibrium between the bank and the agents. Future research may explore extensions to the model concerning hard money, e.g. where the hard money supply increases, but the growth rate decreases over time, or the burning of hard money causing a decreased available amount of money. Future research may incorporate real-world data as a supplementary source to verify the model's findings. Another potential limitation is that the article does not examine the agents' and bank's resource constraints (Engelhardt, 1996; Iacoviello, 2005). Future studies may introduce wages, limits on borrowing and selling, maximum lending amounts, capital adequacy requirements, and other regulatory prescriptions. Future research may reduce the number of nontraders and assume that each buyer, seller, and nontrader are represented by a [0, 1]-continuum, formulating a representative agent's problem for each type. In addition, future research may combine models and incorporate more structure on preferences and constraints of the agents' problem, e.g. a Lagos-Wright monetary model, a money-in-utility function mode, and a cash-in-advance-constraint model (Benigno et al., 2022). Another limitation is that inflation is solely attributed to changes in the fiat money supply. Future research may enhance the modeling of inflation by incorporating other relevant factors, e.g. the money velocity, quantity of produced goods, and transaction efficiency. It would be valuable to explore the influence of agents' expectations, such as how a seller's willingness to sell debt in fiat money may be driven by its expectations regarding central banks' fiat money printing.

While hard money is less susceptible to inflation due to its limited supply, the lack of flexibility in adjusting the money supply can cause economic instability and crises. In a fixed supply hard money economy, demand and supply shocks can cause price fluctuations, creating economic instability. This suggests that fiat money economies may continue to exist, since they allow for greater flexibility in managing the money supply to support economic growth and stability. The model accounts for this by modeling how the agents and bank weigh hard money against fiat money in their Cobb Douglas utility functions. Future research can analyze how demand and supply shocks impact inflation, and how governmental agencies and central banks can regulate. Future research may explore the issue of pricing in trading assets and analyze how the prices of assets are determined.

Future research may also introduce multiple borrowers, buyers, and sellers with different preferences and beliefs, which may enable more robust analyses, and generalize this article's aggregation of agents into the specific agent kinds assumed in this article. The bank may be split into a central bank and commercial banks. Several banks and governments may be introduced. Risk averse agents and banks may be modeled, see e.g. Benchimol and Fourçans (2012). This article divides agents into three kinds, i.e. borrower and buyer, seller, and nontrader. In the real world, an agent may choose to borrow, to buy, and to sell. Restricting the analysis to hard money, fiat money, and other assets is a limitation because other assets have different characteristics, e.g. stocks, bonds, and financial derivatives. There are also different kinds of hard money, approximated by e.g. Bitcoin and gold, and different kinds of fiat money, e.g. paper money, coins, CBDCs. Future research may analyze portfolios and competition between multiple kinds of assets. Future research may expand the model to cover more than two time periods. Techniques such as replicator dynamics (Schuster and Sigmund, 1983) may be applied to capture dynamic evolutionary patterns and determine the potential of the stationary coexistence of hard money and fiat money. A more sophisticated analysis of the competition between hard and fiat money would account for factors other than supply and inflation, e.g. transaction efficiency, convenience, security, and monetary policy. Empirical analyses can be employed to support the theoretical and simulation results.

11. Conclusion

A two-period economy is analyzed with one borrower/buyer (which can be an aggregate of many borrowers/buyers), one seller (which can be an aggregate of many sellers), and arbitrarily many nontraders. The article focuses on the actions of one unitary bank and multiple agents, comparing their utilities over the two periods. They choose their actions, e.g. borrow, buy, sell, lend, to maximize their utilities. Period 1 is a benchmark where the bank neither prints nor withdraws fiat money, causing ceteris paribus neither inflation nor deflation. In period 2 the bank prints fiat money to lend to the borrower/buyer, which causes inflation, and it can additionally print and withdraw fiat money. That impacts the fiat money supply causing inflation or deflation. The adjustment of the money supply gets linked to other assets through the borrower/buyer buying other assets from the seller at a certain value, and through the nontraders holding other assets with a certain valuation, which causes inflation or deflation and impacts the agents' utilities. The bank cannot print or withdraw hard money. Periods 1 and 2 are compared to analyze the impact on the agents and the bank. Instead of determining equilibria gametheoretically through maximizing behavior, the article assesses and compares the agents' and the bank's utilities in the two periods. If an agent's or the bank's utility in period 2 exceeds the agent's or the bank's utility in period 1, the agent or the bank prefers trading based on the higher utility in period 2.

Fiat money printing benefits the borrower/buyer which prefers inflation, benefits the bank if not excessive, and hurts the seller and nontraders. Sellers and nontraders bear the costs of inflation. The seller and the nontraders prefer fiat money withdrawal which causes deflation. Fiat money borrowing causes inflation because the bank prints to lend. The nontraders are vulnerable in a fiat economy with money printing, but unaffected in a hard money economy. More nontraders decrease inflation since the bank's money printing gets distributed across more agents. That benefits the seller, nontraders and the bank, and hurts the borrower/buyer. A hard or fiat money airdrop in period 1 decreases the inflation in period 2. The bank prefers not to lend to the borrower/buyer in period 2 if it benefits excessively in period 1. The borrower/buyer prefers not to borrow and buy other assets in period 2 if it benefits excessively in period 1.

In a fiat economy, inflation and deflation impact all agents. In a hard money economy the bank cannot transfer the costs of inflation to the agents. In a hard money economy with borrowing and lending, ceteris paribus, neither inflation nor deflation occur. Hence the nontraders holding hard money and other assets are not impacted. The borrower/buyer, the seller, and the bank are impacted in a hard money economy by their portfolio changes between hard money, other assets, loans, and the associated interest rates.

The borrower/buyer benefits from buying other assets using its hard and fiat money borrowing from the bank if two conditions are met. First, the borrower/buyer must value other assets more than the interest payment of the loan. Second, the borrower/buyer must ensure that the fiat money loan is sufficiently high compared with the hard money loan so that the borrower/buyer benefits sufficiently from the inflation caused by the bank's money printing to provide the loan.

The seller benefits from selling some of its other assets for hard and fiat money if two conditions are met. First, the seller must value hard and fiat money more than the other assets that it sells. Second, the seller must ensure that it receives sufficiently little fiat money relative to hard money for the other assets that it sells so that it does not suffer excessively from the inflation caused by the bank's fiat money printing to provide the loan to the borrower/buyer of the other assets.

As lending increases, the borrower/buyer's, the seller's and the bank's utilities take the shape of an inverted U. Excessive lending of hard or fiat money does not benefit the bank which prefers a balanced portfolio between money holdings and lending which earns interest payment from the borrower/buyer. The borrower/buyer prefers a balanced portfolio between other assets earning interest and loans incurring interest payments. The seller prefers a balanced portfolio between money holdings and other assets. The seller and nontraders prefer not to be hurt by inflation. Thus they prefer a hard money economy or a fiat economy where the bank withdraws money to ensure deflation. The article provides further results illustrated by varying 64 parameters relative to a benchmark. Supplementing the general understanding of debtors desiring inflation to reduce the value of their debt and creditors being averse to inflation, the article provides a more nuanced analysis and sheds light on specific aspects of this relationship. By examining the dynamics and interplay between debtors, creditors, and banks, the article contributes to the existing literature by providing empirical evidence and a deeper understanding of how inflation expectations impact their decision-making processes. The findings provide insights into the complex motivations and strategic considerations of these actors, which have implications for policymaking and risk management in the financial sector.

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Data Availability

No data was used for the research described in the article.

Appendix A. Nomenclature

General parameters

n	Number of agents, $n \geq 1$.
t	Time period, $t = 1, 2$.
q	Hard money, $q \geq 0$.
m	Fiat money, $m \geq 0$.
o	Other assets, $o \geq 0$.
I_{jt}	Interest rate for asset j determined by the open market in period t, $j = q, m, o$, $I_{jt} \in \mathbb{R}$.

Parameters for agent 1

α_{1jLt}	Agent 1's Cobb Douglas elasticity for borrowing L_{jt} of asset j in period t, $j = q, m$, $\alpha_{1jLt} \geq 0$.

Parameters for agent i, i=1,…,n

α_{ijt}	Agent i's Cobb Douglas elasticity for holding j_{it} of asset j in period t, $j = q, m, o$, $\alpha_{ij} \geq 0$.

Parameters for the bank

β_{jt}	The bank's Cobb Douglas elasticity for holding j_t of asset j in period t, $j = q, m$, $t = 1, 2, \beta_{jt} \geq 0$.
β_{jLt}	The bank's Cobb Douglas elasticity for lending L_{1jt} to the agent 1 in period t, $j = q, m$, $\beta_{jLt} \geq 0$.

Agent 1's parameter or free choice variable

L_{1jt}	Agent 1's borrowing of hard or fiat money j in period t, $j = q, m$, $L_{1jt} \geq 0$.

Agent i's parameter or free choice variable i, i=1,…,n

j_{it}	Agent i's holding of three kinds of assets in period t, $j = q, m, o$, $j_{it} \geq 0$.

The bank's parameters or free choice variables

j_t	The bank's holding of two kinds of assets j in period t, $j = q, m$, $j_t \geq 0$.
r_{jt}	The n agents' borrowing interest rate for hard money and fiat money j in period t, $j = q, m$, $r_{jt} \in \mathbb{R}$.
P_{m2}	The bank's printing of fiat money m in period 2, $P_{m2} \geq 0$.
W_{m2}	The bank's destruction of fiat money m in period 2, $W_{m2} \geq 0$.

Dependent variable

π_t	Inflation rate in period t, $\pi_t \geq 0$.

Agent i's dependent variable i, i=1,…,n

U_{it}	Agent i's Cobb Douglas utility in period t, $U_{it} \geq 0$.

The bank's dependent variable

U_t	The bank's Cobb Douglas utility in period t, $U_t \geq 0$.

Appendix B. Comparing periods 1 and 2

Dividing (3) by (1), agent 1 prefers to borrow $L_{1q2} + L_{1m2}$ if

$$\frac{\left((1+I_{o2})(o_{11}+L_{1q1}+L_{1m1}+L_{1m2}+L_{1q2})\right)^{\alpha_{1o2}}}{\left((1+I_{o1})(o_{11}+L_{1q1}+L_{1m1})\right)^{\alpha_{1o1}}}$$

$$\frac{\left((1+r_{q2})(L_{1q1}+L_{1q2})\right)^{-\alpha_{1qL2}}\left((1+r_{m2})(L_{1m1}+L_{1m2})\right)^{-\alpha_{1mL2}}}{\left((1+r_{q1})L_{1q1}\right)^{-\alpha_{1qL1}}\left((1+r_{m1})L_{1m1}\right)^{-\alpha_{1mL1}}(1+\pi_2)^{-\alpha_{1mL2}}} > 1 \tag{9}$$

Dividing (4) by (2), agent 2 prefers to sell an amount $L_{1q2} + L_{1m2} = o_{21} - o_{22}$ of its other assets if

$$\frac{\left((1+I_{q2})(q_{21}+L_{1q2})\right)^{\alpha_{2q2}}\left((1+I_{m2})(m_{21}+L_{1m2})\right)^{\alpha_{2m2}}}{\left((1+I_{q1})q_{21}\right)^{\alpha_{2q1}}\left((1+I_{m1})m_{21}\right)^{\alpha_{2m1}}(1+\pi_2)^{\alpha_{2m2}}}$$

$$\frac{\left((1+I_{o2})(o_{21}-L_{1q2}-L_{1m2})\right)^{\alpha_{2o2}}}{\left((1+I_{o1})o_{21}\right)^{\alpha_{2o1}}} > 1 \tag{10}$$

Dividing (5) by (2), agents $3, \ldots, n$ prefer the trade between agents 1 and 2 if

$$\frac{\left((1+I_{q2})(q_{i1}+L_{iq2})\right)^{\alpha_{iq2}}\left((1+I_{m2})m_{i2}\right)^{\alpha_{im2}}\left((1+I_{o2})o_{i2}\right)^{\alpha_{io2}}}{\left((1+I_{q1})q_{i1}\right)^{\alpha_{iq1}}\left((1+I_{m1})m_{i1}\right)^{\alpha_{im1}}(1+\pi_2)^{\alpha_{im2}}\left((1+I_{o1})o_{i1}\right)^{\alpha_{io1}}} > 1 \tag{11}$$

Dividing (7) by (6), the bank prefers to lend $L_{1q2}+L_{1m2}$ to agent 1 if

$$\frac{\left(\left(1+I_{q2}\right)\left(q_1-L_{1q1}-L_{1q2}\right)\right)^{\beta_{q2}}\left(\left(1+I_{m2}\right)\left(m_1-L_{1m1}+P_{m2}-W_{m2}\right)\right)^{\beta_{m2}}}{\left(\left(1+I_{q1}\right)\left(q_1-L_{1q1}\right)\right)^{\beta_{q1}}\left(\left(1+I_{m1}\right)\left(m_1-L_{1m1}\right)\right)^{\beta_{m1}}\left(1+\pi_2\right)^{\beta_{m2}}}$$

$$\frac{\left(\left(1+r_{q2}\right)\left(L_{1q1}+L_{1q2}\right)\right)^{\beta_{qL2}}\left(\left(1+r_{m2}\right)\left(L_{1m1}+L_{1m2}\right)\right)^{\beta_{mL2}}}{\left(\left(1+r_{q1}\right)L_{1q1}\right)^{\beta_{qL1}}\left(\left(1+r_{m1}\right)L_{1m1}\right)^{\beta_{mL1}}\left(1+\pi_2\right)^{\beta_{mL2}}}>1 \tag{12}$$

Since agent1 has the same three inputs in periods 1 and 2, we set $\alpha_{1j1}=\alpha_{1j2}$, $j=o,qL,mL$. Thus, (9) is simplified as

$$\left(\frac{\left(1+I_{o2}\right)\left(o_{11}+L_{1q1}+L_{1m1}+L_{1m2}+L_{1q2}\right)}{\left(1+I_{o1}\right)\left(o_{11}+L_{1q1}+L_{1m1}\right)}\right)^{\alpha_{1o2}}$$

$$\left(\frac{\left(1+r_{q2}\right)\left(L_{1q1}+L_{1q2}\right)}{\left(1+r_{q1}\right)L_{1q1}}\right)^{-\alpha_{1qL2}}\left(\frac{\left(1+r_{m2}\right)\left(L_{1m1}+L_{1m2}\right)}{\left(1+r_{m1}\right)L_{1m1}\left(1+\pi_2\right)}\right)^{-\alpha_{1mL2}}>1 \tag{13}$$

Since agent 2 has the same three inputs in periods 1 and 2, we set $\alpha_{2j1}=\alpha_{2j2}$, $j=q,m,o$. Thus, (10) is simplified as

$$\left(\frac{\left(1+I_{q2}\right)\left(q_{21}+L_{1q2}\right)}{\left(1+I_{q1}\right)q_{21}}\right)^{\alpha_{2q2}}\left(\frac{\left(1+I_{m2}\right)\left(m_{21}+L_{1m2}\right)}{\left(1+I_{m1}\right)m_{21}\left(1+\pi_2\right)}\right)^{\alpha_{2m2}}$$

$$\left(\frac{\left(1+I_{o2}\right)\left(o_{21}-L_{1q2}-L_{1m2}\right)}{\left(1+I_{o1}\right)o_{21}}\right)^{\alpha_{2o2}}>1 \tag{14}$$

Since agent i, $i=3,\dots,n$ have the same three inputs in periods 1 and 2, we set $\alpha_{ij1}=\alpha_{ij2}$, $j=q,m,o$. Thus, (11) is simplified as

$$\left(\frac{\left(1+I_{q2}\right)\left(q_{i1}+L_{iq2}\right)}{\left(1+I_{q1}\right)q_{i1}}\right)^{\alpha_{iq2}}\left(\frac{1+I_{m2}}{\left(1+I_{m1}\right)\left(1+\pi_2\right)}\right)^{\alpha_{im2}}\left(\frac{1+I_{o2}}{1+I_{o1}}\right)^{\alpha_{io2}}>1 \tag{15}$$

Since the bank has the same four inputs in periods 1 and 2, we set $\beta_{j1}=\beta_{j2}$, $j=q,m,qL,mL$. Thus, (12) is simplified as

$$\left(\frac{\left(1+I_{q2}\right)\left(q_1-L_{1q1}-L_{1q2}\right)}{\left(1+I_{q1}\right)\left(q_1-L_{1q1}\right)}\right)^{\beta_{q2}}\left(\frac{\left(1+I_{m2}\right)\left(m_1-L_{1m1}+P_{m2}-W_{m2}\right)}{\left(1+I_{m1}\right)\left(m_1-L_{1m1}\right)\left(1+\pi_2\right)}\right)^{\beta_{m2}}$$

$$\left(\frac{\left(1+r_{q2}\right)\left(\left(L_{1q1}+L_{1q2}\right)\right)}{\left(1+r_{q1}\right)L_{1q1}}\right)^{\beta_{qL2}}\left(\frac{\left(1+r_{m2}\right)\left(L_{1m1}+L_{1m2}\right)}{\left(1+r_{m1}\right)L_{1m1}\left(1+\pi_2\right)}\right)^{\beta_{mL2}}>1 \tag{16}$$

Appendix C. Interpretation of 41 of the panels in Figure 1

In Figure 1b, as agent 1's holding o_{11} of other assets in period 1 increases, its utilities U_{11} and U_{12} increase concavely toward infinity. Agent 1 prefers not to borrow $L_{1q2}+L_{1m2}$ of money from the bank since $U_{12}>U_{11}$. The utilities U_{21}, U_{22}, U_{i1}, U_{i2}, U_1, and U_2 remain constant since agent 1's holding o_{11} of other assets has no impact on agents 2 and i, and the bank. The inflation rate π_2 is constant since o_{11} plays no role in (8).

In Figure 1c, as agent 1's borrowing L_{1q1} in hard money in period 1 increases, the bank's utilities U_1 and U_2 are inverse U shaped. The bank prefers to lend an optimal amount L_{1q1} of hard money to agent 1 in period 1. The maximum of U_2 is 85.12 when $L_{1q1}=\$85$. The maximum of U_1 is 68.33 when $L_{1q1}=\$100$. The bank's utilities U_1 and U_2 decrease concavely toward zero after the maximum. The bank prefers to lend hard money L_{1q2} to agent 1 when $\$0\le L_{1q1}<\$175.71$. The bank prefers not to lend too much hard money L_{1q1} to agent 1 in period 1, since then it has a limited amount of hard money q_1-L_{1q1} available for lending in period 2. The nature of the bank's Cobb Douglas utility is such that if it lends excessively in both periods, its utility U_2 is low. Agent 1's utilities U_{11} and U_{12} increase with L_{1q1} since agent 1 benefits from buying other assets using its borrowing L_{1q1}. Agents 2 and i's utilities U_{21}, U_{22}, U_{i1}, and U_{i2} are constant since agent 1's borrowing L_{1q1} in hard money has no impact on agents 2 and i. The inflation rate π_2 is constant since L_{1q1} plays no role in (8).

In Figure 1e, as agent 1's borrowing L_{1m1} in fiat money in period 1 increases, the bank's utilities U_1 and U_2 are inverse U shaped. That is, the bank prefers to lend an optimal amount of fiat money to agent 1 in period 1. The maximum of U_2 is 86.46 when $L_{1q1}=\$92.50$. The maximum of U_1 is 68.33 when $L_{1q1}=\$100$. The bank's utilities U_1 and U_2 decrease concavely after their maxima. Agent 1's utilities U_{11} and U_{12} increase with L_{1m1} since agent 1 benefits from borrowing L_{1m1} in fiat money. Agent 1 prefers not to borrow $L_{1q2}+L_{1m2}$ from the bank since $U_{12}>U_{11}$. Agent 2 and i's utilities U_{21}, U_{22}, U_{i1}, and U_{i2} are constant since agent 1's borrowing L_{1m1} in fiat money has no impact on agent 2 and agent i. The inflation rate π_2 is constant since L_{1m1} plays no role in (8).

In Figure 1g, Figure 1h, and Figure 1i, as agent 2's assets holdings j_{21}, $j = q, m, o$, in period 1 increases, its utilities U_{21} and U_{22} increase toward infinity. The inflation rate π_2 decreases convexly and asymptotically toward zero due to division with j_{21} in (8). Agent 1's utility U_{12} decreases slightly since the inflation rate π_2 decreases slightly, which hurts agent 1 because of agent 1's fiat money loans L_{1m1} and L_{1m2}. In contrast, agent i's utility U_{i2} increases slightly because the inflation rate π_2 decreases slightly, which benefits agent i because of its fiat money holding m_{i2}. The bank's utility U_2 increases slightly since the inflation rate π_2 decreases slightly, which benefits the bank because the benefit of the bank's fiat money holding $m_1 - L_{1m1} + P_{m2} - W_{m2}$ of from the decreasing inflation rate π_2 overrides the negative impact of fiat money lending L_{1m2} from the decreasing inflation rate π_2. The utilities U_1, U_{11}, U_{21}, and U_{i1} remain constant since the inflation rate π_2 plays no role in period 1.

In Figure 1j and Figure 1m, as agent i's money holding j_{i1}, $j = q, m$, in period 1 increases, its utility U_{i1} increases concavely toward infinity. The inflation rate π_2 decreases convexly and asymptotically toward zero due to division with j_{i1} in (8). Agent i's utility U_{i2} increases slightly since agent i benefits from the decreasing inflation rate π_2. Agent 1's utility U_{12} decreases slightly since the inflation rate π_2 decreases slightly, which hurts agent 1 because of its fiat money loans L_{1m1} and L_{1m2}. In contrast, agent 2's utility U_{22} increases slightly because the inflation rate π_2 decreases slightly, which benefits agent 2 because of its fiat money holdings m_{22}. The bank's utility U_2 increases slightly since the inflation rate π_2 decreases slightly, which benefits the bank because the benefits of the bank's fiat money holding of $m_1 - L_{1m1} + P_{m2} - W_{m2}$ from the decreasing inflation rate π_2 override the negative impact of fiat money lending L_{1m2} from the decreasing inflation rate π_2. The utilities U_1, U_{11}, and U_{21} remain constant since the inflation rate π_2 plays no role in period 1.

In Figure 1k and Figure 1n, as agent i's period 2 money holding j_{i2}, $j = q, m$, increases above the benchmark $j_{i2} = \$100$, its period 2 utility U_{i2} increases concavely from $U_{i2} = 203.06$, reaching $U_{i2} = U_{i1} = 204$ when $j_{i2} = \$101.88$, and proceeds concavely toward infinity. The inflation rate π_2 is constant since j_{i2} plays no role in (8). The utilities U_1, U_2, U_{11}, U_{21}, and U_{i1} remain constant since agent i's money holding j_{i2} of plays no role in period 1. The utility U_2 remains constant since the inflation rate π_2 is constant in period 2.

In Figure 1l and Figure 1o, analogously to Figure 1j and Figure 1m, as agent i's money holding $j_{i1} = j_{i2}$, $j = q, m$, in the two periods increases, its utilities U_{21} and U_{22} increase concavely toward infinity. The inflation rate π_2 decreases convexly and asymptotically toward zero due to division with j_{i1} in (8). Agent 1's utility U_{12} decreases slightly since the inflation rate π_2 decreases slightly, which hurts agent 1 because of its fiat money loans L_{1m1} and L_{1m2}. In contrast, agent 2's utility U_{22} increases slightly because the inflation rate π_2 decreases slightly, which benefits agent 2 because of its fiat money holdings m_{22}. The bank's utility U_2 increases slightly since the inflation rate π_2 decreases slightly, overriding the negative impact of fiat money lending L_{1m2}, which benefits the bank because of its fiat money holding $m_1 - L_{1m1} + P_{m2} - W_{m2}$. The utilities U_1, U_{11}, U_{21}, and U_{i1} remain constant since the inflation rate π_2 plays no role in period 1.

In Figure 1p, as agent i's holding o_{i1} of other assets in period 1 increases, its utility U_{i1} increases concavely toward infinity. The inflation rate π_2 is constant since o_{i1} plays no role in (8). The utilities U_1, U_{11}, and U_{21} remain constant since agent i's holding o_{i1} of other assets has no impact on the bank and agents 2 and i. The utility U_{i2} is constant since agent i's holding o_{i2} of other assets is constant in period 2. The utility U_2 remains constant since the inflation rate π_2 is constant in period 2.

In Figure 1q, analogously, as agent i's holding o_{i2} of other assets in period 2 increases, its utility U_{i2} increases concavely toward infinity. The inflation rate π_2 is constant since o_{i2} plays no role in (8). The utilities U_1, U_{11}, U_{21}, and U_{i1} remain constant since agent i's holding o_{i2} of other assets plays no role in period 1. The utility U_2 remains constant since the inflation rate π_2 is constant in period 2.

In Figure 1r, analogously, as agent i's holding $o_{i1} = o_{i2}$ of other assets in the two periods increase, its utilities U_{i1} and U_{i2} increase concavely toward infinity. The inflation rate π_2 is constant since o_{i1} and o_{i2} play no role in (8). The utilities U_1, U_{11}, and U_{21} remain constant since agent i's holding $o_{i1} = o_{i2}$ of other assets plays no role in period 1. The bank's utility U_2 remains constant since the inflation rate π_2 is constant in period 2.

In Figure 1u and Figure 1v, as the bank's money holding j_1, $j = q, m$ in period 1 increases, its utilities U_1 and U_2 increase concavely to infinity. The period 2 inflation rate π_2 decreases convexly and asymptotically toward zero due to division with j_1 in (8). Agent 1's utility U_{12} decreases slightly since the inflation rate π_2 decreases slightly, which hurts agent 1 because of its fiat money loans L_{1m1} and L_{1m2}. In contrast, the utilities U_{22} and U_{i2} increase slightly because the inflation rate π_2 decreases slightly, which benefits agents 2 and i because of their fiat money holdings m_{22} and m_{i2}. More specifically, agent i's utility U_{i2} approaches U_{i1} asymptotically from below as j_1 approaches infinity, i.e. $\lim_{q_1 \to \infty} U_{i2} = U_{i1} = 204.00$. The utilities U_{11}, U_{21}, and U_{i1} remain constant since neither q_1 nor the inflation rate π_2 impact agents 1, 2 and i in period 1.

In Figure 1x and Figure 1aa, as the interest rate I_{j2} for holding money $j, j = q, m$, in period 2 increases, which is intuitively beneficial to the bank and agent $i, i = 2, \ldots n$, the three utilities U_2, U_{22}, and U_{i2} increase concavely toward infinity. The inflation rate π_2 is constant since I_{j2} plays no role in (8). Agent 1's utilities U_{11} and U_{12} are constant since agent 1 holds no money j in the two periods. The utilities U_1, U_{21}, and U_{i1} remain constant since I_{j2} plays no role in period 1.

In Figure 1y and Figure 1ab, as the interest rate $I_{j1} = I_{j2}$ for holding money $j, j = q, m$, in the two periods increases, which is intuitively beneficial to the bank and agent $i, i = 2, \ldots n$, the six utilities U_1 U_2, U_{21}, U_{22}, U_{i1}, and U_{i2} increase concavely toward infinity, equivalently to the three concave increases in Figure 1w and the three concave increases in Figure 1x. The inflation rate π_2 is constant since I_{j1} and I_{j2} play no role in (8). Agent 1's utilities U_{11} and U_{12} are constant since agent 1 holds no money j in the two periods.

In Figure 1ac, as the interest rate I_{o1} for holding other assets o in period 1 increases, which is intuitively beneficial to all the agents, the utilities U_{21}, U_{21}, and U_{i1} increase concavely toward infinity. The inflation rate π_2 is constant since I_{o1} plays no role in (8). The bank's utilities U_1 and U_2 are constant since the bank holds no other assets o in the two periods. The utilities U_2, U_{22}, and U_{i2} remain constant since I_{o1} plays no role in period 2.

In Figure 1ad, as the interest rate I_{o2} for holding other assets o in period 2 increases, which is intuitively beneficial to all the agents, the utilities U_{12}, U_{22}, and U_{i2} increase concavely toward infinity. The inflation rate π_2 is constant since I_{o2} plays no role in (8). The bank's utilities U_1 and U_2 are constant since the bank holds no other assets o in the two periods. The utilities U_1, U_{11}, U_{21}, and U_{i1} remain constant since I_{o2} plays no role in period 1.

In Figure 1ae, as the interest rate $I_{o1} = I_{o2}$ for holding other assets o in the two periods increases, which is intuitively beneficial to all the agents, the utilities U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, and U_{i2} increase concavely toward infinity. The inflation rate π_2 is constant since I_{o1} and I_{o2} play no role in (8). The bank's utilities U_1 and U_2 are constant since it holds no other assets in the two periods.

In Figure 1ag and Figure 1aj, as the borrowing interest rate r_{j2} for money $j, j = q, m$ in period 2 increases, which is intuitively beneficial to the bank, the bank's utility U_2 increases concavely toward infinity. Agent 1's utility U_{12} decreases convexly toward zero since a higher borrowing interest rate r_{j2} for money j is costly. The inflation rate π_2 is constant since r_{j2} plays no role in (8). The utilities U_{22} and U_{i2} are constant since agents 2 and i do not borrow money j in period 2. The utilities U_1, U_{11}, U_{21}, and U_{i1} remain constant since r_{j2} plays no role in period 2.

In Figure 1ah and Figure 1ak, as the borrowing interest rate $r_{j1} = r_{j2}$ for money $j, j = q, m$ in the two periods increases, which is intuitively beneficial to the bank, the bank's utilities U_1 and U_2 increase concavely toward infinity. These two concave increases are equivalent to the concave increases in Figure 1af, Figure 1ai, Figure 1ag and Figure 1aj. Agent 1's utilities U_{11} and U_{12} decrease convexly toward zero since higher borrowing interest rate $r_{j1} = r_{j2}$ for money j is costly. These two convex decreases are equivalent to the convex decreases in Figure 1af, Figure 1ai, Figure 1ag and Figure 1aj. The inflation rate π_2 is constant since r_{j1} and r_{j2} play no role in (8). The utilities U_{21}, U_{22}, U_{i1} and U_{i2} are constant since agents 2 and i do not borrow money j in the two periods. Thus, r_{j1} and r_{j2} play no role for agents 2 and i.

In Figure 1al, as agent 1's Cobb Douglas elasticity α_{1o1} for holding $o_{11} + L_{1q1} + L_{1m1}$ of other assets in period 1 increases, which is intuitively beneficial to agent 1, its utility U_{11} increases concavely. Agent 1 wants to borrow $L_{1q2} + L_{1m2}$ from the bank in period 2 when α_{1o1} is sufficiently low, i.e. $0 \le \alpha_{1o1} < 0.50$. The inflation rate π_2 is constant since α_{1o1} plays no role in (8). Agent 1's utility U_{12} is constant since α_{1o1} plays no role in period 2. The utilities U_1, U_2, U_{21}, U_{22}, U_{i1}, and U_{i2} remain constant since α_{1o1} has no impact on the bank, agents 2 and i.

In Figure 1an, as agent 1's Cobb Douglas elasticity $\alpha_{1o1} = \alpha_{1o2}$ for holding $o_{11} + L_{1q1} + L_{1m1}$ and $o_{11} + L_{1q1} + L_{1m1} + L_{1q2} + L_{1m2}$ of other assets in the two periods increases, its utilities U_{11} and U_{12} increase concavely, and equivalently to the concave increases in Figure 1al and Figure 1am. Agent 1 wants to borrow $L_{1q2} + L_{1m2}$ from the bank in period 2 when $\alpha_{1o1} = \alpha_{1o2}$ is not too low, i.e. $0.5 \le \alpha_{1o1} = \alpha_{1o2} < 1$. The inflation rate π_2 is constant since α_{1o1} and α_{1o2} play no role in (8). The utilities U_1, U_2, U_{21}, U_{22}, U_{i1}, and U_{i2} remain constant since α_{1o1} and α_{1o2} have no impact on the bank, agents 2 and i.

In Figure 1ao, as agent 1's Cobb Douglas elasticity α_{1qL1} for borrowing L_{1q1} in hard money in period 1 increases, its utility U_{11} in period 1 is constant since the bank does not print money to lend L_{1m1} to agent 1. Hence decreasing Cobb Douglas elasticity $\alpha_{1mL1} = 1 - \alpha_{1o1} - \alpha_{1qL1}$ due to increasing Cobb Douglas elasticity α_{1qL1} has no impact since $L_{1q1} = L_{1m1} = \$10$. The nine variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, U_{i2}, U_1, U_2, π_2 remain constant.

In Figure 1ap, as agent 1's Cobb Douglas elasticity α_{1qL2} for borrowing L_{1q2} in hard money in period 2 increases, its utility U_{12} in period 2 decreases slightly because the positive inflation rate $\pi_2 = 1.875\%$ becomes less beneficial for agent 1 when lower Cobb Douglas elasticity $\alpha_{1mL2} = 1 - \alpha_{1o2} - \alpha_{1qL2}$ is assigned to borrowing $L_{1m1} + L_{1m2}$ in fiat money. The eight variables U_{11}, U_{21}, U_{22}, U_{i1}, U_{i2}, U_1, U_2, π_2 remain constant.

In Figure 1aq, as agent 1's Cobb Douglas elasticity $\alpha_{1qL1} = \alpha_{1qL2}$ for borrowing L_{1q1} and L_{1q2} in hard money in the two periods increases, the results are as in Figure 1ap where only α_{1qL2} changes while α_{1qL1} is constant. The reason follows from Figure 1ao where the changing Cobb Douglas elasticity α_{1qL1} does not impact the nine variables.

In Figure 1ar and Figure 1au, as agent 2's Cobb Douglas elasticity α_{2j1} for holding money $j_{21}, j = q, m$ in period 1 increases, which means decreasing Cobb Douglas elasticity $\alpha_{2o1} = 1 - \alpha_{2q1} - \alpha_{2m1}$ for holding other assets o_{21}, agent 2's utility U_{21} in period 1 decreases because holding other assets o_{21} becomes less beneficial. Hence agent 2 prefers period 1 when $\alpha_{2j1} < 0.23$ and prefers period 2 when $0.23 \le \alpha_{2j1} \le 1$. That is, agent 2 wants to sell its other assets valued as $L_{1q2} + L_{1m2}$ when α_{2j1} is sufficiently high, i.e. $0.23 \le \alpha_{2j1} \le 1, j = q, m$. The eight variables U_{11}, U_{12}, U_{22}, U_{i1}, U_{i2}, U_1, U_2, π_2 remain constant.

In Figure 1at, as agent 2's Cobb Douglas elasticity $\alpha_{2q1} = \alpha_{2q2}$ for holding hard money q_{21} and $q_{21} + L_{1q2}$ in the two periods increases, its utilities U_{21} and U_{22} decrease convexly, and equivalently to the convex decreases in Figure 1ar and Figure 1as. Agent 2 wants to sell other assets valued as $L_{1q2} + L_{1m2}$ when $\alpha_{2q1} = \alpha_{2q2}$ is sufficiently high, i.e. $0.13 \le \alpha_{2q1} = \alpha_{2q2} \le 1$. The seven variables U_{11}, U_{12}, U_{i1}, U_{i2}, U_1, U_2, π_2 remain constant.

In Figure 1aw, as agent 2's Cobb Douglas elasticity $\alpha_{2m1} = \alpha_{2m2}$ for holding fiat money m_{21} and $m_{21} + L_{1m2}$ in the two periods increases, its utilities U_{21} and U_{22} decrease convexly, and equivalently to the convex decreases in Figure 1au and Figure 1av. Agent 2 wants to sell other assets valued as $L_{1q2} + L_{1m2}$ when $\alpha_{2m1} = \alpha_{2m2}$ is sufficiently high, i.e. $0.12 \le \alpha_{2m1} = \alpha_{2m2} \le 1$. The seven variables U_{11}, U_{12}, U_{i1}, U_{i2}, U_1, U_2, π_2 remain constant.

In Figure 1ax and Figure 1ba, as agent i's Cobb Douglas elasticity α_{ij1} for holding money $j_{i1}, j = q, m$ in period 1 increases, its utility U_{i1} in period 1 decreases convexly because holding other assets o_{i1} becomes less beneficial for agent i with decreasing Cobb Douglas elasticity $\alpha_{io1} = 1 - \alpha_{iq1} - \alpha_{im1}$. Agent i prefers the trade between agents 1 and 2 when α_{ij1} is sufficiently high, i.e. $0.25 \le \alpha_{iq1} \le 1, j = q, m$. The eight variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i2}, U_1, U_2, π_2 remain constant.

In Figure 1az and Figure 1bc, as agent i's Cobb Douglas elasticity $\alpha_{ij1} = \alpha_{ij2}$ for holding money $j_{it}, j = q, m, t = 1, 2$ in the two periods increases, its utilities U_{i1} and U_{i2} decrease convexly, and equivalently to the convex decreases in Figure 1ax and Figure 1ay, Figure 1ba and Figure 1bb. Agent i does not prefer the trade between agents 1 and 2 since $U_{i2} < U_{i1}$ holds for Figure 1az and Figure 1bc. The seven variables U_{11}, U_{12}, U_{21}, U_{22}, U_1, U_2, π_2 remain constant.

In Figure 1bd and Figure 1bg, as the bank's Cobb Douglas elasticity β_{j1} for holding money $j_1 - L_{1j1}, j = q, m$ in period 1 increases, its utility U_1 in period 1 increases convexly because holding money $j_1 - L_{1j1}$ becomes more beneficial for the bank with increasing Cobb Douglas elasticity β_{j1}, which overrides the negative impact of the decreasing Cobb Douglas elasticity $\beta_{mL1} = 1 - \beta_{q1} - \beta_{m1} - \beta_{qL1}$ for fiat money loans. The bank wants to give money loans $L_{1q2} + L_{1m2}$ to agent 1 in period 2 when β_{j1} is sufficiently low, i.e. $0 \le \beta_{j1} \le 0.4$, $j = q, m$. The eight variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, U_{i2}, U_2, π_2 remain constant.

In Figure 1bf, as the bank's Cobb Douglas elasticity $\beta_{q1} = \beta_{q2}$ for holding hard money $q_1 - L_{1q1}$ and $q_1 - L_{1q1} - L_{1q2}$ in the two periods increases, its utilities U_1 and U_2 in the two periods increase convexly because holding money $q_1 - L_{1q1}$ and $q_1 - L_{1q1} - L_{1q2}$ becomes more beneficial for the bank with increasing Cobb Douglas elasticity $\beta_{q1} = \beta_{q2}$, which overrides the negative impact of the decreasing Cobb Douglas elasticity $\beta_{mL1} = 1 - \beta_{q1} - \beta_{m1} - \beta_{qL1} = \beta_{mL2} = 1 - \beta_{q2} - \beta_{m2} - \beta_{qL2}$ for fiat money loans. The bank wants to give money loans $L_{1q2} + L_{1m2}$ to agent 1 in period 2 when $\beta_{q1} = \beta_{q2}$ is sufficiently low, i.e. $0 \le \beta_{q1} = \beta_{q2} \le 0.69$. The seven variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, U_{i2}, π_2 remain constant.

In Figure 1bi, as the bank's Cobb Douglas elasticity $\beta_{m1} = \beta_{m2}$ for holding fiat money $m_1 - L_{1m1}$ and $m_1 - L_{1m1} + P_{m2} - W_{m2}$ in the two periods increases, its utilities U_1 and U_2 in the two periods increase convexly because holding money $m_1 - L_{1m1}$ and $m_1 - L_{1m1} + P_{m2} - W_{m2}$ become more beneficial for the bank with increasing Cobb Douglas elasticity $\beta_{m1} = \beta_{m2}$, which overrides the negative impact of decreasing Cobb Douglas elasticity $\beta_{mL1} = 1 - \beta_{q1} - \beta_{m1} - \beta_{qL1} = \beta_{mL2} = 1 - \beta_{q2} - \beta_{m2} - \beta_{qL2}$ for fiat money loans. The bank wants to give money loans $L_{1q2} + L_{1m2}$ to agent 1 in period 2 when $\beta_{m1} = \beta_{m2}$ is sufficiently low, i.e. $0 \le \beta_{m1} = \beta_{m2} \le 0.72$. The seven variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, U_{i2}, π_2 remain constant.

In Figure 1bj, as the bank's Cobb Douglas elasticity β_{qL1} for hard money lending L_{1q1} in period 1 increases, its utility U_1 remains constant because the benefit of increasing Cobb Douglas elasticity β_{qL1} is offset by the negative impact of decreasing Cobb Douglas elasticity $\beta_{mL1} = 1 - \beta_{q1} - \beta_{m1} - \beta_{qL1}$. The bank benefits from lending money $L_{1q2} + L_{1m2}$ to agent 1 in period 2 because $r_{j2} > I_{j2}$, $j = q, m$. The bank always wants to lend fiat money $L_{1q2} + L_{1m2}$ to agent 1 in period 2 since $U_1 < U_2$. The nine variables U_{11}, U_{12}, U_{21}, U_{22}, U_{i1}, U_{i2}, U_1, U_2, π_2 remain constant.

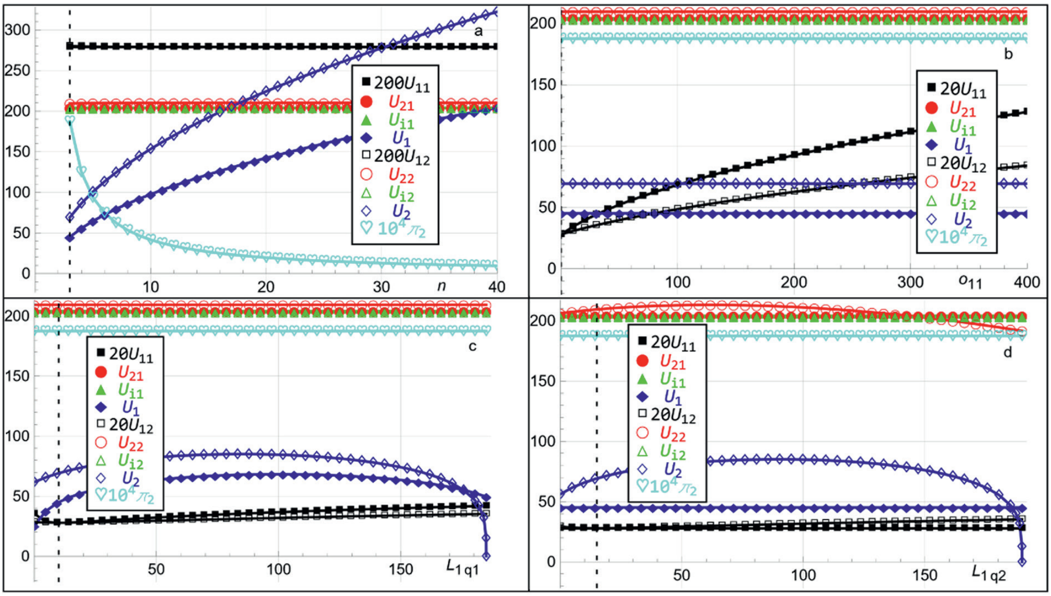


Fig. 1. Agent 1's utilities U_{11} and U_{12}, agent 2's utilities U_{21} and U_{22}, agent i's utilities U_{i1} and U_{i2}, the bank's utilities U_1 and U_2, and the inflation rate π_2, respectively, relative to the benchmark parameter values $q_{11} = q_{12} = m_{11} = m_{12} = o_{11} = \$0$, $L_{1q1} = L_{1m1} = \$10$, $L_{1q2} = L_{1m2} = \$15$, $q_{21} = m_{21} = \$100$, $n = 3$, $q_{i1} = q_{i2} = m_{i1} = m_{i2} = \$100$, $o_{i1} = o_{i2} = \$400$, $q_1 = m_1 = \$200$, $P_{m2} = W_{m2} = \$0$, $\alpha_{io1} = \alpha_{io2} = 1/2$, $i = 1, \ldots, n$, $\alpha_{1qL1} = \alpha_{1qL2} = \alpha_{1mL1} = \alpha_{1mL2} = 1/4$, $\alpha_{iq1} = \alpha_{iq2} = \alpha_{im1} = \alpha_{im2} = 1/4$, $\beta_{q1} = \beta_{q2} = \beta_{qL1} = \beta_{qL2} = 1/4$, $\beta_{m1} = \beta_{m2} = \beta_{mL1} = \beta_{mL2} = 1/4$, $\pi_2 = 1.875\%$, $I_{q1} = I_{q2} = I_{m1} = I_{m2} = I_{o1} = I_{o2} = 2\%$, $r_{q1} = r_{q2} = r_{m1} = r_{m2} = 5\%$.

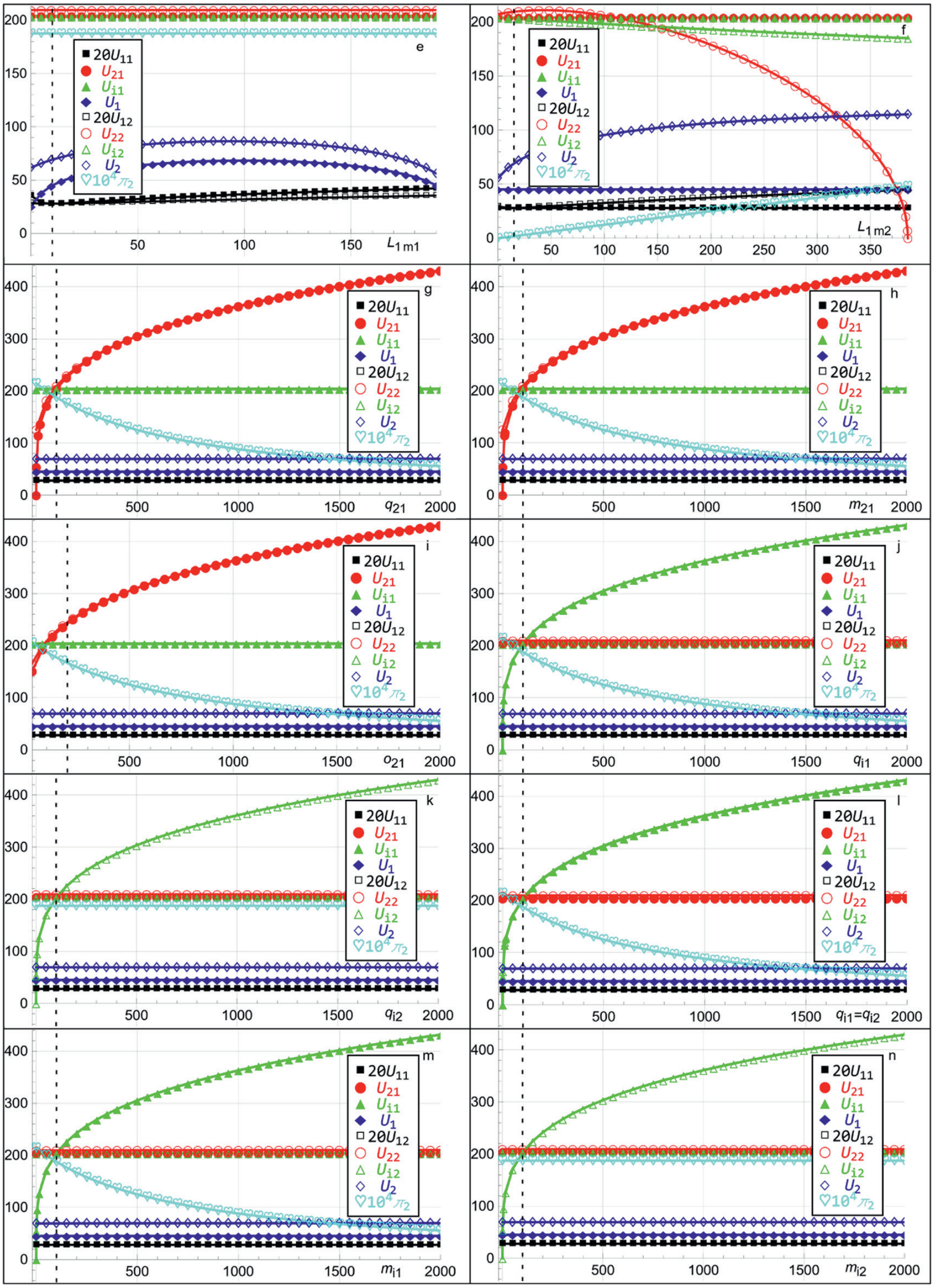
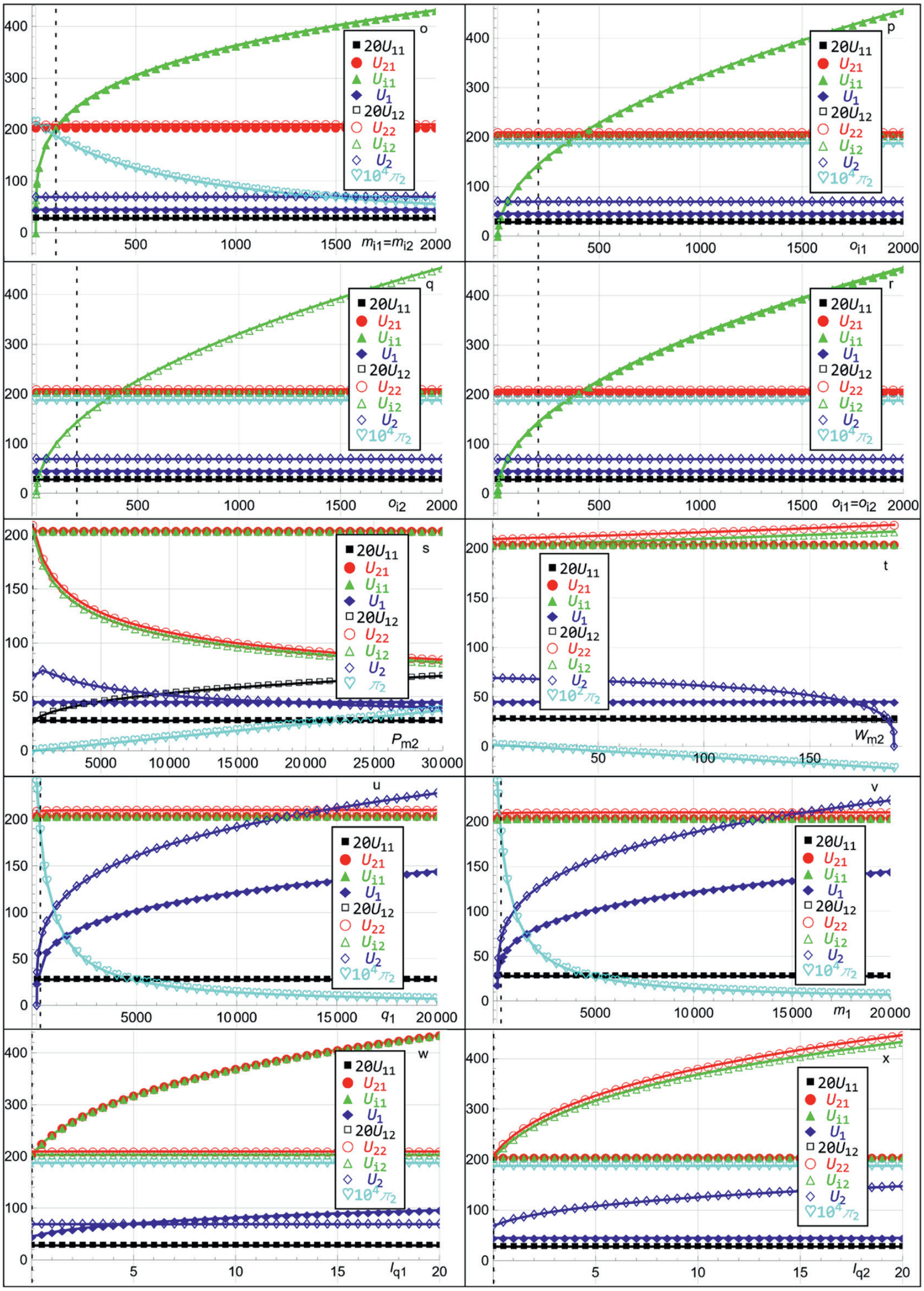


Fig. 1. (*continued*).



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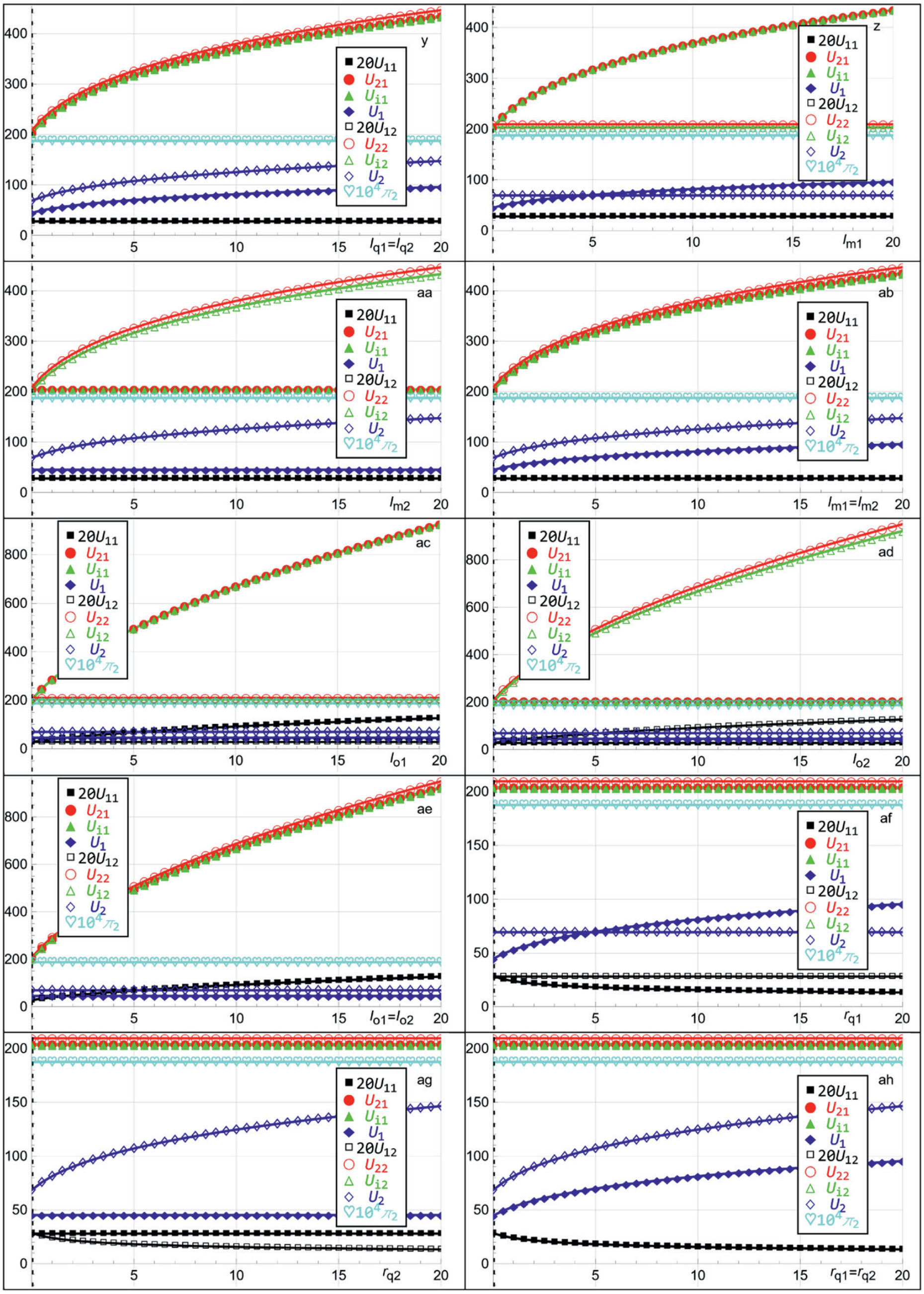

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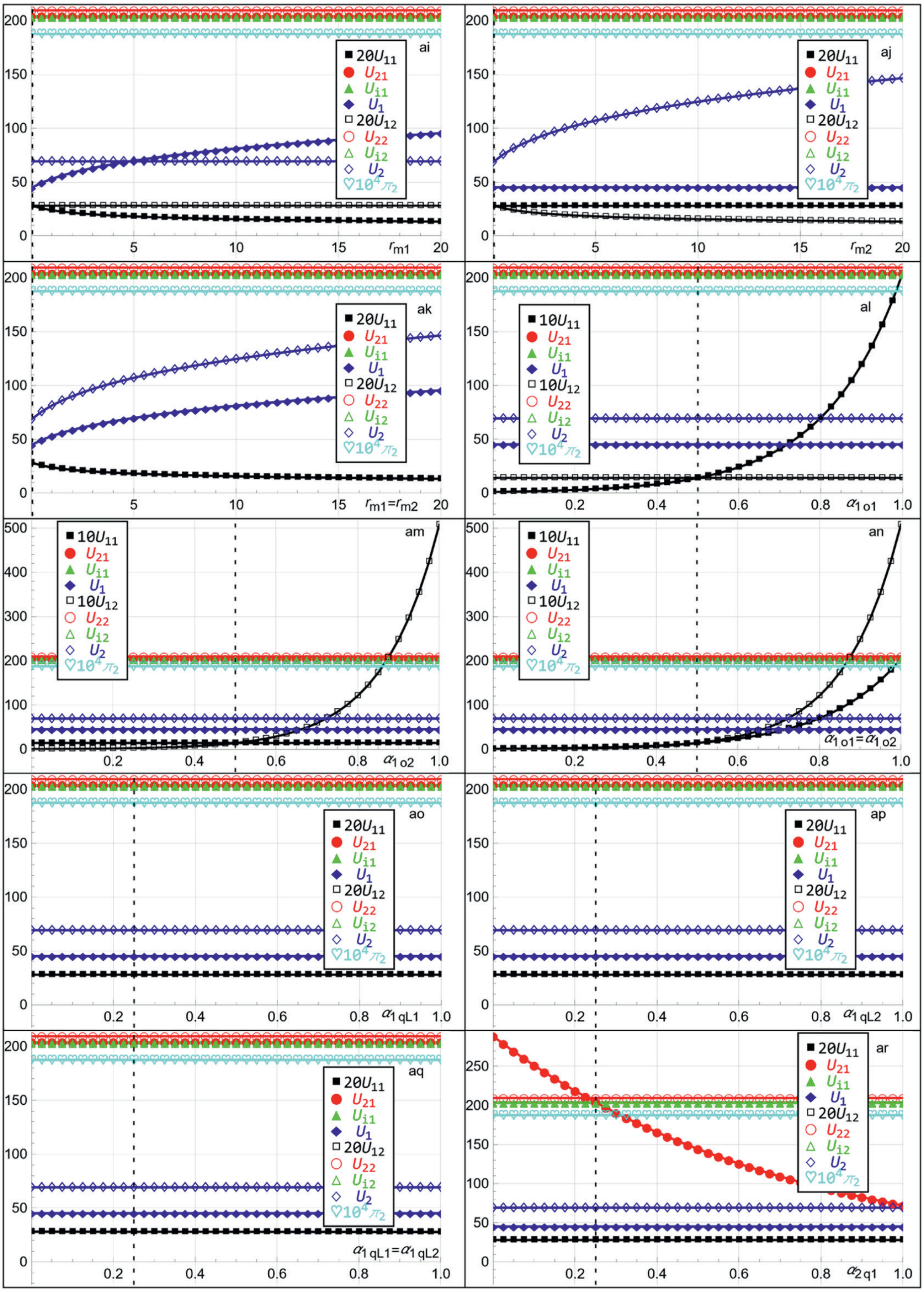

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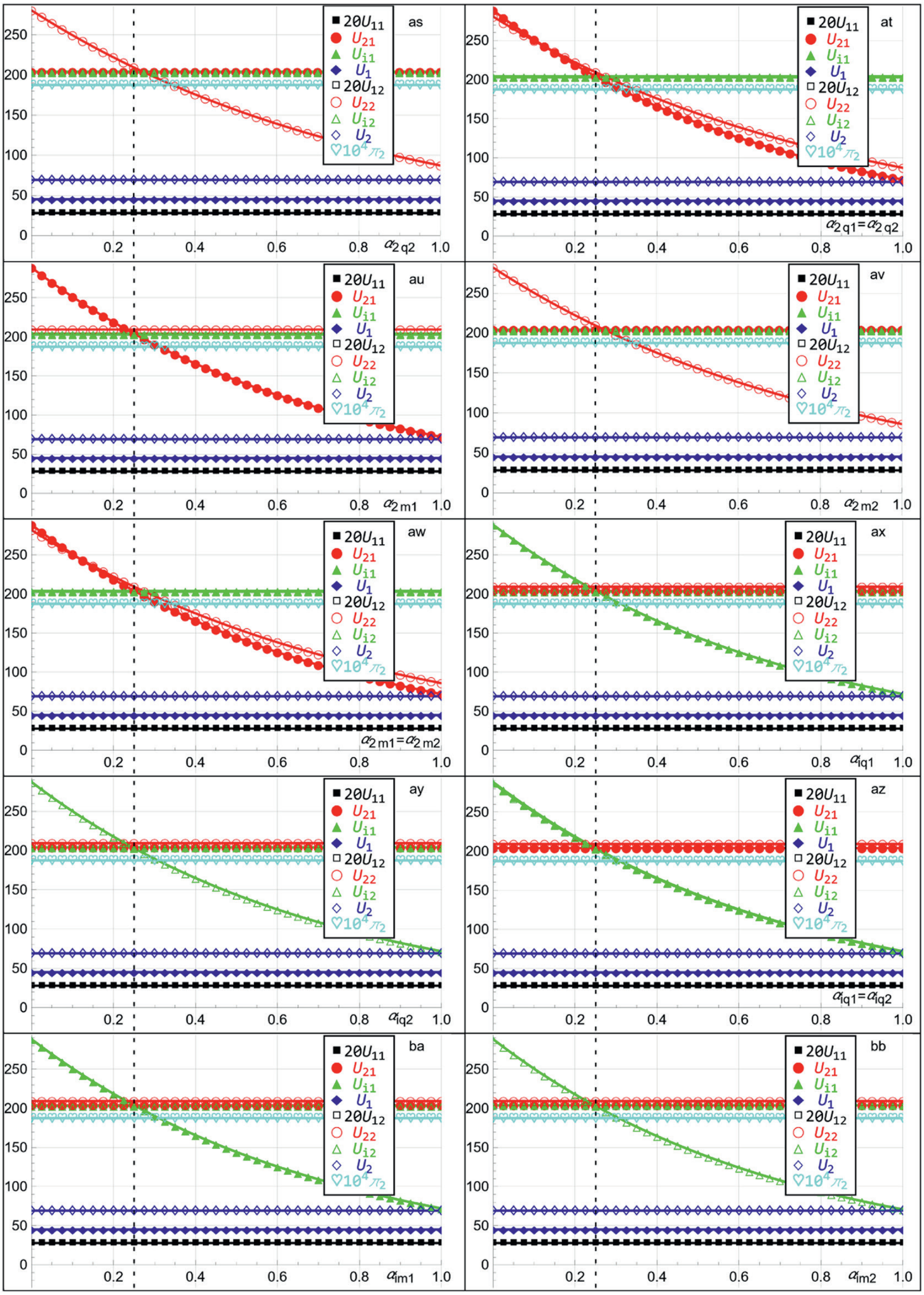

Fig. 1. (*continued*).

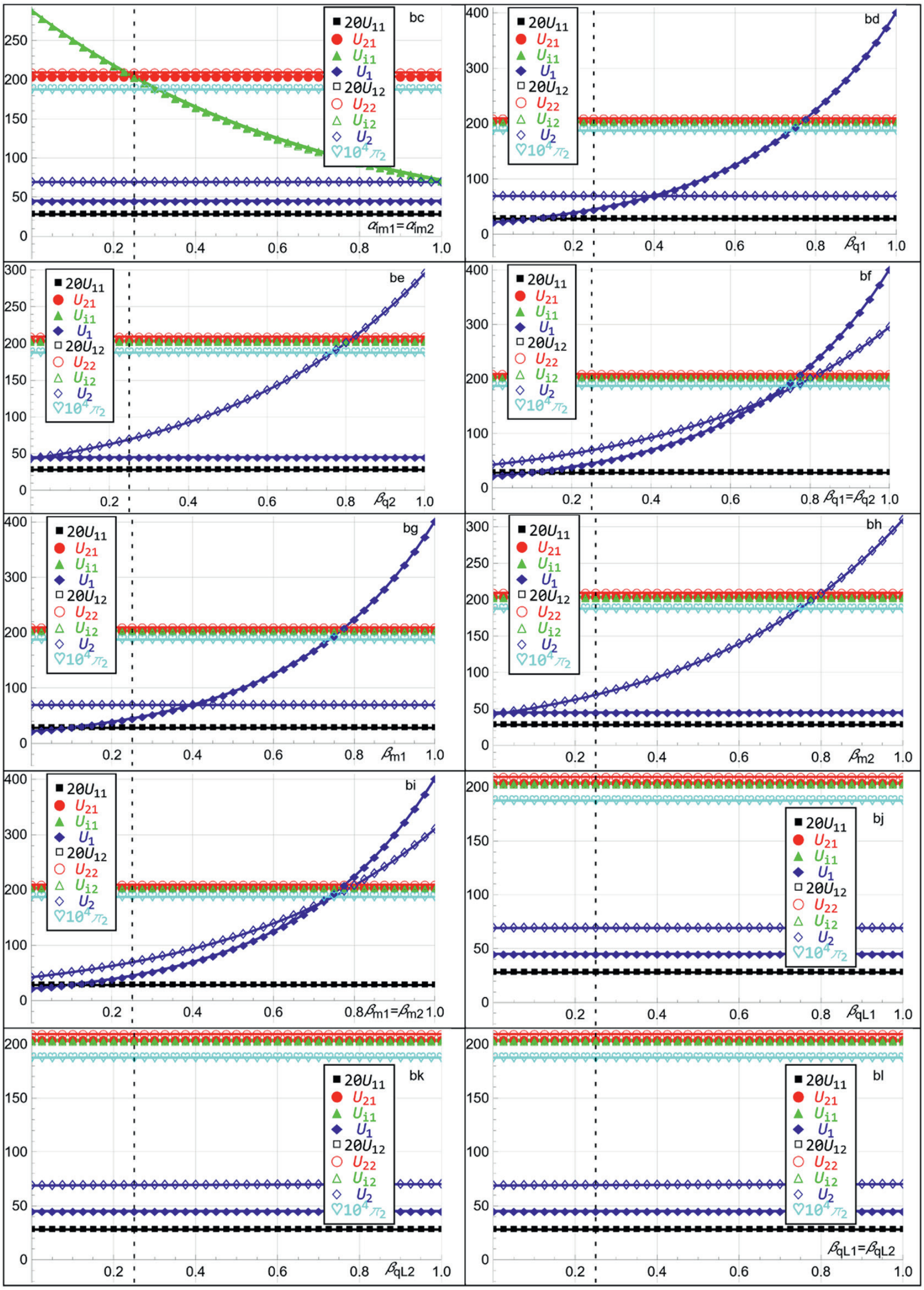

Fig. 1. (*continued*).

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A Bitcoin price prediction model assuming oscillatory growth and lengthening cycles

Guizhou Wang[1] and Kjell Hausken[1]*

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*Corresponding author: Kjell Hausken, Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway.
E-mail: kjell.hausken@uis.no

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David McMillan, University of Stirling, Stirling, United Kingdom

Additional information is available at the end of the article

Abstract: This article's motivation is to understand the volatile Bitcoin price increase. The objective is to develop price estimation methods. The methodology is to present five differential equation models estimated against the 23 July 2010–21 June 2021 Bitcoin data. The findings are that Gompertz growth fits the damped oscillations and lengthening cycles well, and tracks the early data better with the weighted least squares method. Gompertz growth combined with charged capacitor growth tracks the early data even better. Logistic growth is too slow to track the early data. Logistic growth combined with charged capacitor growth to some extent tracks the early data. Pure charged capacitor growth is unrealistic. The dates for the future bull market maxima depend to a low degree on the growth model carrying capacity approached asymptotically, assumed to match gold at $10 trillion, and to be 50 times higher. The implications for traders are to focus on the large standard deviations. Investors should understand the growth potential compared with other asset classes. Regulators should ensure financial stability by focusing on the fluctuations. Central banks should adjust the money supply while acknowledging. Bitcoin competition. Collective units should understand Bitcoin growth models to determine whether to accept Bitcoin transactions.

ABOUT THE AUTHORS

Guizhou Wang is a PhD student at the University of Stavanger, Norway, since 2020-06. His working PhD thesis title is "Game Theoretic Modeling of Economic Systems Involving Digital Currencies." He has published 10 articles in peer reviewed journals. His research fields are digital currencies, game theory, risk analysis, cryptocurrencies, central bank digital currencies, econometrics. He holds a MSc degree in financial economics from the University of Chinese Academy of Sciences (Beijing, China), 2016-09 – 2019-06, focusing on mathematical finance, econometrics, venture capital, and cryptocurrency. He holds a BSc degree in finance from the Jinan University (Guangzhou, China), 2010-09 – 2014-06, focusing on finance, derivatives, and mathematical modeling. Email: guizhou.wang@uis.no.

Kjell Hausken is a professor of economics and societal safety at the University of Stavanger, Norway, since 1999. His research fields are terrorism, societal safety, economics, economic risk management, economics and safety, political economy, information security, public choice, conflict, game theory, reliability, war, crime, risk analysis, disaster prevention, stochastic theory, dynamics, petroleum economics, resilience management. He holds a PhD from the University of Chicago (1990-1994), was a postdoc at the Max Planck Institute for the Studies of Societies (Cologne) 1995-1998, and a visiting scholar at Yale School of Management 1989-1990. He holds a Doctorate Program Degree (HAE) ("Philosophical, Behavioral, and Gametheoretic Negotiation Theory") in Administration from the Norwegian School of Economics and Business Administration (NHH), a MSc degree in electrical engineering, cybernetics, from the Norwegian Institute of Technology (NTNU), focusing on mathematics and statistics, and a minor in Public Law from the University of Oslo. He has published 270 articles in peer reviewed journals, one book, edited two books, is/was on the Editorial Board for Theory and Decision (May 20, 2007 –), Reliability Engineering & System Safety (January 17, 2012 –), and Defence and Peace Economics (December 4, 2007 – December 31, 2015), has refereed 400 submissions for 85 journals, and advises and has advised seven PhD students. Email: kjell.hausken@uis.no.

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1. Introduction

1.1. Background

Since the genesis block was mined on 3 January 2009 at 18:15:05 UTC, the Bitcoin price has increased above 100% per year subject to fluctuations. Understanding the nature of the growth and fluctuations is of paramount importance. A variety of opinions emerge on how the Bitcoin price evolves into the future. Skeptics believe the Bitcoin price is in a bubble and will collapse. Others see Bitcoin, accompanied with layer 2 solutions for scaling (e.g., the Lightning Network) and layer 3 solutions for interoperability, as the future dominant means of payment, measure of value, medium of exchange, basis of credit, standard of postponed payment, store of value, and possibly unit of account. Other cryptocurrencies may contribute. The decentralized nature of Bitcoin, where anyone can run a node which stores the entire blockchain, emerges as a competitor to traditional media of exchange and stores of value which require an intermediary. Thus for example El Salvador on 7 September 2021 and the Central African Republic on 27 April 2022 accepted Bitcoin as legal tender. Cryptocurrencies and their underlying ledger technologies currently impact how most central banks develop digital currencies. These developments can be expected to reshape the financial system.

1.2. Contribution

This article's motivation, objectives, research hypotheses, and research questions are as follows. First, the Bitcoin price has increased apparently unpredictably since 3 January 2009, which suggests a need both to understand the evolution so far and to predict the future evolution. Second, applying Bitcoin's price data since 23 July 2010, methods are developed to estimate and understand this price evolution as accurately as possible. The Bitcoin empirics are such that the methods involve growth models, while accounting for oscillations and lengthening cycles. Third, two different Bitcoin carrying capacities are considered, assumed to express reasonable outer limits for what can be expected over the next decades. Fourth, five differential equation models are compared against each other to determine which is best, applying the least squares method and the weighted least squares method. Fifth, the methods are used to predict the future Bitcoin price and future bull marked maxima.

More specifically, first, a generalized logistic growth model is presented, which depicts the Bitcoin price's growth with four characteristics: logistic growth, damped oscillation, retracement in bear markets, and lengthening cycles. Second, a generalized charged capacitor growth is introduced with damped oscillation, retracement in bear markets, and lengthening cycles. Third, the article introduces what the authors believe are two hitherto unknown theoretical combinations of growth models, i.e., logistic growth combined with charged capacitor growth, and Gompertz growth combined with charged capacitor growth. This gives five models which are solved analytically and analyzed numerically.

The least squares method is applied to estimate the models' parameters. Supplementation is made with the weighted least squares method since the Bitcoin price variance increases over time, exhibiting heteroscedasticity. Based on the three bull market local maxima and three bear market local minima during the period 23 July 2010–21 June 2021, the scaling of the inverse of the cycle length of the sine oscillations, and the scaling of the inverse of the degree of lengthening of each subsequent cycle, are estimated. The amplitude of the oscillations, and the start time adjustment parameter for the sine oscillations, are estimated to predict the future bull market local maxima and bear market local minima.

The article outperforms other alternative approaches and adds to our knowledge in various ways. First, the dynamic nature of the Bitcoin price is such that differential time equations are especially well suited. Such differential equations do exist in the literature, but are perhaps in the minority. Second, the Bitcoin price is not only characterized by dynamics, but by dynamic growth. Hence this article focuses explicitly on growth models. Third, the Bitcoin price is characterized by dynamic oscillatory growth with lengthening cycles, which is explicitly incorporated into the analysis.

1.3. Literature

The existing literature predicts the Bitcoin price applying various methods, occasionally using differential equations, and more generally accounting for statistics, econometrics, machine learning, neural network, deep learning, etc. The literature is divided into five groups, i.e., 1. Differential equations, 2. Bitcoin price dynamics, 3. Gompertz growth and Metcalfe's Law, 4. Machine learning, and 5. Neural network, deep learning and memory models. This article correlates most with the first two groups, while introducing growth with damped oscillatory and lengthening cycles. The last three groups are included for broader positioning.

1.3.1. Differential equations

Relatively few studies apply differential equations to predict the Bitcoin price. K. S. Chen and Huang (2020) adopt a stochastic differential equation to capture the evolution of the Bitcoin price 2015–2018. Their differential equation considers the instantaneous expected return, the instantaneous volatility, and jumps focusing explicitly on the crash after the 17 December 2017 and Brownian motion. They focus on the jump risk distribution of the Bitcoin price and Bitcoin options pricing and hedging. Such a focus on jumps is implicitly present in the current article which determines moves back and forth between bull market maxima and bear market minima. Jalali and Heidari (2020) adopt grey system theory and propose a first order differential equation to predict the Bitcoin price. The approach requires an appropriate time frame. They focus explicitly on five-day predictions. That differs from the current article which predicts over any future time horizon. Wang and Wang (2020) introduce a partial differential equation model to predict the Bitcoin price January 1–31 December 2017. They incorporate the daily Bitcoin transaction volumes and google trends index, and the spatial heterogeneity of chainlet clusters, which proceeds beyond this article's focus. This article differs from these other articles applying differential equations by focusing explicitly on the Bitcoin price growth patterns. That is, the differential equations consider the Bitcoin price, two different Bitcoin carrying capacities, damped oscillations, lengthening cycles, and bull market maxima and bear market minima for five different growth models.

1.3.2. Bitcoin price dynamics

The following articles pertain to Bitcoin price dynamics, but with a different focus and applying other models than in the current article, thus implicitly illustrating a gap in the literature. Statistics and econometrics are widely used methods to forecast the Bitcoin price. Begusic et al. (2018) demonstrate slowly decaying tails in the distributions of Bitcoin returns, and a power law with $2 < \alpha < 2.5$, which means heavier tails than for stocks with alpha around 3. Such slowly decaying tails seem consistent with damped oscillations, and heavy tails seem consistent with the substantial fluctuations between maxima and minima, found in the current article.

Caporale et al. (2019) apply statistical methods for 2013–2018. They find that the frequency of price overreactions is informative about Bitcoin price movements and the Bitcoin price exhibits no seasonality. Their approach constitutes an alternative way of assessing the drive towards bull market maxima and bear market minima.

Roy et al. (2018) apply 2013–2017 data and present an autoregressive integrated moving average model which predicts the Bitcoin price volatility with 90% accuracy, thus also capturing fluctuations between maxima and minima.

Indera et al. (2017) apply 2012–2017 data and develop a multi-layer perceptron-based non-linear autoregressive model to predict the Bitcoin price with good accuracy. They generate moving averages, account for input and output lags, and apply regression analysis, validation and fitting tests. They focus less on the timing and magnitudes of the maxima and minima than the current article.

Cretarola and Figa-Talamanca (2021) apply a continuous time stochastic model to determine how bubbles in the Bitcoin price in 2012–2013 and in 2017 are linked to the correlation between the market attention to Bitcoin and the Bitcoin return being above a threshold, known as market exuberance. Such bubbles are yet another way of assessing bull market maxima. Jana et al. (2021) apply 2013–2019 data to forecast the Bitcoin price through a differential evolution-based regression framework, shown to be superior to six advanced predictive modeling algorithms. Instead of differential equations, they apply polynomial regression on time series.

Further studies consider market attention, market sentiment, active addresses, etc. for Bitcoin price prediction, which is a broader focus than in the current article. Sabalionis et al. (2021) found that the amount of active addresses impacts the Bitcoin and Ethereum prices more than other factors such as google search interest and number of tweets. Haffar and Le Fur (2021) applied a structural vector error correction model to determine that the Bitcoin price in the short run is influenced positively by Asian emerging countries and negatively by North America. In the long run, the influence is negative from all countries in Asia and the Pacific, and positive from Europe. This article adopts a wider range of Bitcoin price data than the above articles, applying growth models to explain and predict the Bitcoin price.

1.3.3. Gompertz growth and Metcalfe's Law

The quick initial increase in Gompertz growth (commonly used for e.g. tumor growth; see Yorke et al. (1993)) is found to be descriptive in the current article. Two other articles have also identified Gompertz growth as descriptive. Peterson (2018) applies the Gompertz curve to capture the inflationary impacts of the creation of new Bitcoin, shown to follow Metcalfe's Law. Patel et al. (2020) found that the price of cryptocurrencies follows a Gompertz growth function, which links the traditional time-value-of-money concepts to Metcalfe's law, and that the growth rate of users impacts the Bitcoin price. This article extends this focus to other growth models, i.e., logistic growth, charged capacitor growth, and combinations of growth models, accounting for damped oscillation and lengthening cycles.

1.3.4. Machine learning

Several studies apply machine learning methods to explain and predict the Bitcoin price. Chevallier et al. (2021) applied six machine learning algorithms to parameterize and disentangle the non-stationary behavior of the Bitcoin price data, as an alternative to classical parameter models. They suggest that machine learning does not teach how to trade due to the substantial Bitcoin price variability, and that long term holding may be preferable. Such a suggestion seems compatible with the current article which determines overall 2010–2021 growth, interrupted by substantial downturns towards bear market minima. Dutta et al. (2020) present a framework of machine learning forecasting methods to predict the Bitcoin's price. They compare various approaches, arguing that the gated recurring unit model with a recurrent dropout performs best. Z. Chen et al. (2020) predict the Bitcoin price with various frequencies data via machine learning techniques. They also incorporate high-dimension features like property and network, trading and market, attention, etc. They show that statistical methods perform better than machine learning algorithms, reaching accuracy of 66% and 65.3%, respectively.

Some articles combine machine learning and econometrics. Mudassir et al. (2020) developed high-performance machine learning-based classification and regression models to predict the Bitcoin price. The models have accuracy of 65% and 64% for next-day forecast and seventh–ninetieth-day forecast, respectively. Gupta and Nain (2021) use time series involving moving averages, autoregressive integrated moving averages, and multiple machine learning approaches including Support Vector Machine, Long Short Term Memory and Gated Recurrent Unit. They compare these models to determine their accuracy. The machine learning approach is challenging since it requires appropriate data input. Long term forecasting is challenging. Instead of machine learning, this article applies differential equations and least squares methods which more directly cause price explanation and prediction.

1.3.5. Neural network, deep learning and memory models

Recent articles adopt neural networks and deep learning to predict the Bitcoin price, which enables the analysis of instructed data including documents, images, and texts. Ji et al. (2019) explored the performance of a deep neural network model, a Long Short-Term Memory model, and a Convolutional Neural Network model, to predict the Bitcoin price. They show that the deep neural network model predicts price increases and decreases nicely, and that classification models are more effective than regression models. Patel et al. (2020) present a Long Short Term Memory and Gated Recurrent Unit based hybrid cryptocurrency prediction model to predict the price of Litecoin and Monero. They found that the model accurately predicts the prices.

Hua (2020) compares the accuracy of predicting the Bitcoin price via Long Short Term Memory model and an Autoregressive Integrated Moving Average model. He finds that the former performs better, but requires more time to train the neural network. Cocco et al. (2021) compared several approaches to predict the Bitcoin price. They show that two-stage frameworks usually outperform one-stage frameworks, except for one-stage Bayesian Neural Network. Jaquart et al. (2021) proposed a stochastic neural network model based on random walk to predict the price of cryptocurrencies. The approach induces a layer-wise randomness into the neural networks to capture market volatility. Using multi-layer perceptron and Long Short Term memory models, they found that the proposed models perform well compared with deterministic models.

Chkili (2021) applies a long memory model and a Markov switching model to determine the Bitcoin price volatility, which relates to the focus in the current article of assessing fluctuations between maxima and minima. A common challenge faced by the deep learning approach is finding the optimal network hyperparameters. That contrasts with the current article which applies least squares methods to estimate the parameters.

1.4 Article organization

Section 2 presents the materials and methods. Section 3 analyzes the model and presents the results. Section 4 discusses the results. Section 5 concludes.

2. Materials and methods

This section identifies and develops the differential equations believed to capture the Bitcoin price evolution most accurately.

2.1. Nomenclature

Parameters

- k Growth rate, $k \in \mathbb{R}$
- K Carrying capacity, $K \geq 0$
- ν Parameter for generalized logistic growth impacting near which asymptote maximum growth occurs, $\nu \geq 0$
- λ Adjustment parameter for combined generalized logistic and charged capacitor growth, $\lambda \in \mathbb{R}$
- α Oscillation amplitude, expressing strength of bull and bear markets, $\alpha \in \mathbb{R}$
- ω Scaling of the inverse of the cycle length of the sine oscillations, $\omega \in \mathbb{R}$
- γ Scaling of the inverse of the degree of lengthening of each subsequent cycle, $\gamma \in \mathbb{R}$
- δ Start time adjustment parameter relative to time $t = t_0$ for the oscillation of the Sin function, $\gamma \in \mathbb{R}$
- t_0 Initial time t

T Final time t
p_0 Initial price p at time $t = t_0$

Independent variable
t Time

Dependent variable
p Price

2.2. Generalized logistic growth

This section generalizes Richards' (1959) model for growth modeling to

$$\begin{aligned} &\frac{\partial p}{\partial t} = (k + \alpha Sin(\omega(t - t_0)^\gamma + \delta))p\left(1 - \left(\frac{p}{K}\right)^v\right) \\ &\Rightarrow p = \frac{K}{\left(1 + \left(\frac{K}{p_0} - 1\right)^\nu e^{-(k+Q)\nu(t-t_0)}\right)^{1/\nu}}, \lim_{t\to\infty} p = K, \\ Q \equiv \frac{i\alpha}{2\gamma} &\left(e^{i\delta} \int_1^\infty \frac{e^{i\omega(t-t_0)^\gamma q}}{q^{\frac{\gamma-1}{\gamma}}} dq - e^{-i\delta} \int_1^\infty \frac{e^{-i\omega(t-t_0)^\gamma q}}{q^{\frac{\gamma-1}{\gamma}}} dq\right), i \equiv \sqrt{-1} \end{aligned} \tag{1}$$

where ∂ means partial differentiation, t means time, t_0 is the start time, $k \in \mathbb{R}$ is the growth rate which expresses how quickly the price p changes, and $K \geq 0$ is the carrying capacity, defined as the maximum sustainable price p. Equation (1) expresses that the price p changes logistically from p_0, $p_0 \geq 0$, at the initial time $t = t_0$ towards $p = K$ as time t approaches infinity. The parameter ν, $\nu \geq 0$, impacts near which asymptote maximum growth occurs.

Whereas Richards (1959) assumes a constant growth rate, (1) supplements the growth rate k with a Sin function and four additional parameters. The Sin function oscillates between +1 and −1 to reflect bull markets with increased growth rate when the Sin function is positive, and bear markets with decreased growth rate when the Sin function is negative.

The parameter $\alpha \in \mathbb{R}$ expresses the strength of the bull and bear markets, and thus the size of the positive and negative amplitudes in the oscillations. Equation (1) simplifies to Richards' (1959) model when $\alpha = 0$ which eliminates the sine oscillations causing $Q = 0$.

The parameter $\omega \in \mathbb{R}$ scales the inverse of the cycle length of the sine oscillations. Higher ω gives shorter cycle length, since ω is multiplied with time t, and higher ω means that each cycle with length 2π gets completed more quickly.

The parameter $\gamma \in \mathbb{R}$ scales the inverse of the degree of lengthening of each subsequent cycle as time t progresses. Lower γ gives more lengthening of each subsequent cycle as time t progresses. Mathematically, if we consider $\gamma = 1$ as a common benchmark giving linear oscillatory progression through time t, decreasing gamma below 1 causes $\omega(t - t_0)^\gamma < \omega(t - t_0)$, and hence more time t is needed for each subsequent cycle with length 2π to be completed. In contrast, increasing gamma above 1 causes $\omega(t - t_0)^\gamma > \omega(t - t_0)$, and hence less time t is needed for each subsequent cycle with length 2π to be completed. Equation (1) also simplifies to Richards' (1959) model at the limit when $\lim_{\gamma\infty} Q = 0$, since each subsequent cycle gets completed immediately and thus the sine oscillations have no impact.

The parameter $\delta \in \mathbb{R}$ adjusts the start time at time $t = 0$ for the oscillation of the Sin function. For example, if $\delta = 0$, $Sin(\omega t^\gamma + \delta) = 0$ when $t = 0$, which gives zero amplitude and thus no impact of the Sin function at time $t = 0$.

2.2.1. Conventional logistic growth
Inserting $\nu = 1$ into (1) gives conventional logistic growth (Lotka, 1924; Verhulst, 1845) with oscillation, retracement in bear markets, and lengthening cycles, where both the initial value asymptote $t = t_0$ and the future value asymptote $t \to \infty$ are approached symmetrically.

2.2.2 Gompertz growth
Inserting the limit $\nu - >0^+$ into (1) gives conventional Gompertz (1825, p. 518) logistic growth with oscillation, retracement in bear markets, and lengthening cycles, i.e.,

$$\frac{\partial p}{\partial t} = (k + \alpha Sin(\omega(t - t_0)^\gamma + \delta))pLn\left(\frac{K}{p}\right) \Rightarrow p = K\left(\frac{p_0}{K}\right)^{e^{-(k+R)(t-t_0)}},$$
$$R \equiv \frac{i\alpha}{2\gamma(t-t_0)\omega^{\frac{1}{\gamma}}}\left(\begin{array}{c} e^{\frac{i(\pi+2\gamma\delta)}{2\gamma}}\left(\int\limits_{-i(t-t_0)^\gamma\omega}^{\infty} q^{\frac{1}{\gamma}-1}e^{-q}dq - 1\right) \\ -e^{-\frac{i(\pi+2\gamma\delta)}{2\gamma}}\left(\int\limits_{i(t-t_0)^\gamma\omega}^{\infty} q^{\frac{1}{\gamma}-1}e^{-q}dq - 1\right)\end{array}\right) \quad (2)$$

where Ln is the natural logarithm, and the initial value asymptote $t = t_0$ is approached more quickly than the future value asymptote $t \to \infty$. Equation (2) simplifies to Richards' (1959) model when $R = 0$.

2.3. Generalized charged capacitor growth
This section assumes generalized charged capacitor growth with damped oscillation, retracement in bear markets, and lengthening cycles, with growth rate $(k + \alpha Sin(\omega(t - t_0)^\gamma + \delta))$, i.e.,

$$\begin{array}{c}\frac{\partial p}{\partial t} = (k + \alpha Sin(\omega(t - t_0)^\gamma + \delta))\left(1 - \left(\frac{p}{K}\right)^\nu\right) \\ \Rightarrow p = InverseFunction\left[-{}_2F_1\left(1, \frac{1}{v}, 1 + \frac{1}{v}, \left(\frac{\#}{K}\right)^\nu\right)\#\&\right] \\ \left[-(k+Q)(t - t_0) - p_{0\,2}F_1\left(1, \frac{1}{v}, 1 + \frac{1}{v}, \left(\frac{p_0}{K}\right)^\nu\right)\right], \lim\limits_{t\to\infty} p = K\end{array} \quad (3)$$

where Q is defined in (1), which simplifies to

$$\begin{array}{c}\frac{\partial p}{\partial t} = (k + \alpha Sin(\omega(t - t_0)^\gamma + \delta))\left(1 - \frac{p}{K}\right) \\ \Rightarrow p = K - (K - p_0)e^{-(k+Q)(t-t_0)/K}, \lim\limits_{t\to\infty} p = K\end{array} \quad (4)$$

when $\nu = 1$, and simplifies to $\frac{\partial p}{\partial t} = 0 \Rightarrow p = p_0$, i.e., no growth, when $\nu = 0$. The function ${}_2F_1(a, b, c, z)$ is hypergeometric with power series assuming $|z|<1$. The Mathematica (www.wolfram.com) notation in (3) is as follows: $InverseFunction[f]$ is the inverse of the function f, defined so that $InverseFunction[f][y]$ gives the value of x for which $f[x] = y$. The symbol $\#$ is the first argument supplied to a pure function, so that $f[\#]\&[x]$ evaluates to $f[x]$. The symbol & expresses the end of the argument. Charged capacitor growth expresses a quick initial price increase, due to the high value of the right-hand side of the differential Equation (3) when p_0 is low.

2.4. Combined generalized logistic and charged capacitor growth
Combining (3) and (1) gives

$$\begin{array}{c}\frac{\partial p}{\partial t} = (k + \alpha Sin(\omega(t - t_0)^\gamma + \delta))p^\lambda\left(1 - \left(\frac{p}{K}\right)^\nu\right) \\ \Rightarrow p = InverseFunction\left[\frac{-\#^{1-\lambda}}{1-\lambda}{}_2F_1\left(1, \frac{1-\lambda}{v}, 1 + \frac{1-\lambda}{v}, \left(\frac{\#}{K}\right)^\nu\right)\&\right] \\ \left[-(k+Q)(t - t_0) - \frac{p_0^{1-\lambda}}{1-\lambda}{}_2F_1\left(1, \frac{1-\lambda}{v}, 1 + \frac{1-\lambda}{v}, \left(\frac{p_0}{K}\right)^\nu\right)\right], \lim\limits_{t\to\infty} p = K\end{array} \quad (5)$$

where the price p is raised to an exponent λ, $\lambda \in \mathbb{R}$, and ${}_2F_1(a,b,c,z)$ is defined in (3). The adjustment parameter λ can be thought of as weighing generalized logistic growth (conventional logistic growth and Gompertz growth) and charged capacitor growth against each other. Equation (5) simplifies to generalized logistic growth in (1) when $\lambda = 1$, simplifies to Gompertz growth in (2) when $\lambda = 1$ at the limit $v \to 0^+$, and simplifies to generalized charged capacitor growth in (3) when $\lambda = 0$. Hence $0<\lambda<1$ enables growth intermediate between quick generalized charged capacitor growth when $\lambda = 0$, and slower generalized logistic growth when $\lambda = 1$. Compared with generalized logistic growth in (4) when $\lambda>1$, initial growth when the price p is low is damped since p^λ in (5) is comparatively low, and eventual growth is amplified more when the price p is high since p^λ in (5) is comparatively high.

2.4.1 Combined logistic and charged capacitor growth

Inserting $\nu = 1$ into (5) gives combined conventional logistic growth and generalized charged capacitor growth expressed as

$$\begin{aligned}&\frac{\partial p}{\partial t} = (k + \alpha Sin(\omega(t-t_0)^\gamma + \delta))p^\lambda\left(1-\frac{p}{K}\right)\\ &\Rightarrow p = InverseFunction\left[\frac{\#^{1-\lambda}}{K(1-\lambda)}{}_2F_1\left(1, 1-\lambda, 2-\lambda, \frac{\#}{K}\right)\&\right]\\ &\left[\frac{(k+Q)(t-t_0)}{K} + K^{-\lambda}\int_0^{p_0/K}\frac{q^{-\lambda}}{1-q}dq\right], \lim_{t\to\infty} p = K\end{aligned} \tag{6}$$

2.4.2 Combined Gompertz and charged capacitor growth

Inserting the limit $v \to 0^+$ into (5) gives combined Gompertz growth and generalized charged capacitor growth expressed as

$$\begin{aligned}&\frac{\partial p}{\partial t} = (k + \alpha Sin(\omega(t-t_0)^\gamma + \delta))p^\lambda Ln\left(\frac{K}{p}\right)\\ &\Rightarrow p = InverseFunction\left[e^{(1-\lambda)Ln(K)}\int_{(1-\lambda)Ln\left(\frac{K}{\#}\right)}^{\infty}\frac{e^{-q}}{q}dq\&\right]\\ &\left[(k+Q)(t-t_0) + K^{1-\lambda}\int_{(1-\lambda)Ln\left(\frac{K}{p_0}\right)}^{\infty}\frac{e^{-q}}{q}dq\right], \lim_{t\to\infty} p = K\end{aligned} \tag{7}$$

3. Results

The results are presented over seven subsections. Sections 3.1 and 3.2 assume no oscillation and estimate and predict the Bitcoin price with the various methods developed in the previous section. Sections 3.3 and 3.4 generate results needed to account for oscillation. These results pertain to Bitcoin price maxima and minima, cycle length and cycle lengthening. Sections 3.5 and 3.6 allow for oscillation and predict the Bitcoin price with the various methods. Section 3.7 estimates future bull market maxima.

More specifically, section 3.1 estimates the Bitcoin price assuming no oscillation amplitude α = 0 and the Bitcoin carrying capacity K = $476,190 which corresponds to Bitcoin eventually approaching the market capitalization of gold estimated at $10 trillion. Section 3.2 repeats the exercise for the 50 times higher Bitcoin carrying capacity K = $23,809,524 which corresponds to Bitcoin eventually approaching a market capitalization of $500 trillion. Section 3.3 determines the three bull market local maxima and the three bear market local minima which have been established at the writing of this article. Section 3.4 estimates the scaling ω of the inverse of the cycle length of the sine oscillations and the scaling γ of the inverse of the degree of lengthening of each subsequent cycle. Section 3.5 allows for oscillation amplitude α ≥ 0 as determined by the previous two subsections, and estimates the Bitcoin price assuming the carrying capacity K = $476,190. Section 3.6 repeats the exercise for the 50 times higher Bitcoin carrying capacity K = $23,809,524.

Section 3.7 estimates bull market local maxima 4,5,6,7,8 assuming the carrying capacities K = \$476,190 and K = \$23,809,524.

3.1. Bitcoin carrying capacity k=\$476,190 and no oscillation amplitude α=0

The Bitcoin carrying capacity K is estimated as the maximum sustainable market capitalization divided by the circulating supply. The Bitcoin circulating supply is capped at 21 million coins, expected to be mined by ca year 2140. Estimating Bitcoin's maximum sustainable market capitalization is extremely uncertain. This section assumes that Bitcoin approaches a maximum sustainable market capitalization of \$10 trillion, which is similar to the market capitalization of gold.[1] The comparison with gold is made since it would constitute a major milestone if Bitcoin were to reach it. Dividing \$10 trillion with 21 million coins gives the Bitcoin carrying capacity $K = \$476,190$. Using daily midnight 11:59.99 pm UTC closing time Bitcoin data,[2] the initial Bitcoin price at the initial time 23 July 2010 is $p_0 = \$0.04951$. Figure 1 shows the empirical price p_E for the period 23 July 2010–21 June 2021, which increases overall, with intermittent decreases.[3]

The subsequent seven curves in each panel in Figure 1 assume $\alpha = 0$, $K = \$476,190$ and $p_0 = \$0.04951$ and estimate the growth rate k, and the two parameters λ and β, for the models in section 2.2. These seven curves increase strictly, in contrast to the empirical price p_E, due to the nature of growth models. In sections 3.5 and 3.6 oscillatory growth is modeled.

3.1.1. Least squares method

Applying the least squares method, Figure 1(a) shows the historical estimates. Figure 1(b) predicts until 1 January 2100. Using data over ca 11 years to predict ca 79 more years into the future, i.e., a ratio 79/11 ≈ 7.2, entails some uncertainty for the more distant future.

The curve p_L estimates the growth rate $k = 1.28$ by assuming logistic growth in (1) for $\nu = 1$, approaching $K = \$476,190$ more quickly than the other six curves. The curve p_L appears almost linear on a logarithmic plot with base 10.

The curve p_G estimates the growth rate $k = 0.16$ by assuming Gompertz growth in (2). The curve p_G is concave to reflect that the future value asymptote $t \to \infty$ is approached more gradually than

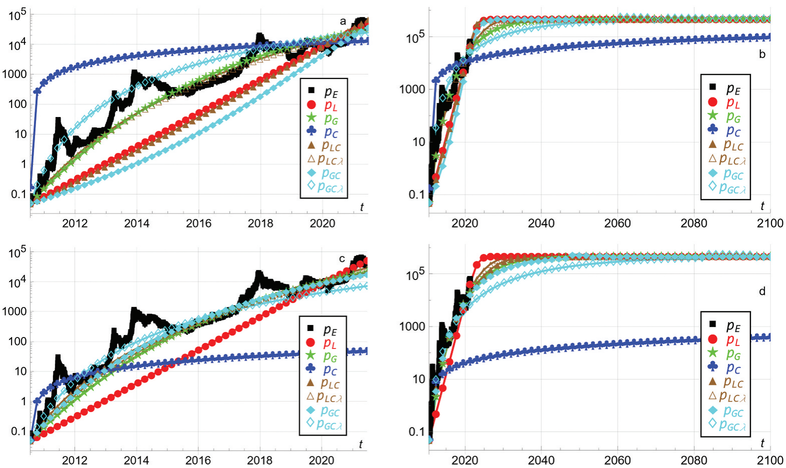

Figure 1. Assuming no oscillation amplitude $\alpha = 0$, the empirical price p_E, logistic growth p_L, Gompertz growth p_G, charged capacitor growth p_C, combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$, and combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$, for 23 July 2010–21 June 2021 (panels a and c) and until 1 January 2100 (panels b and d), $K = \$476,190$. Panels a and b apply the least squares method. Panels c and d apply the weighted least squares method.

the initial value asymptote $t = t_0$. That is, initial growth is quick, and $K = \$476,190$ is approached more slowly.

The curve p_C estimates the growth rate $k = 1178$ by assuming charged capacitor growth in (4). The curve p_C is extremely concave. It initially increases more quickly than the other six curves, and eventually approaches $K = \$476,190$ more slowly than the other six curves.

The curve p_{LC} estimates the growth rate $k = 1.21$ and adjustment parameter $\lambda = 1.02$ by assuming combined logistic and charged capacitor growth in (6). Since $\lambda > 1$, initial growth for the curve p_{LC} is slower than for the curve p_L for logistic growth, see section 2.4.

The curve $p_{LC\lambda}$ is intermediate between the curve p_L for logistic growth and p_C for charged capacitor growth. This is obtained by assuming $\lambda = 0.88$ and using the least squares method to optimize the growth rate k which gives $k = 2.18$. The curve $p_{LC\lambda}$ is similar to Gompertz growth p_G.

The curve p_{GC} estimates the growth rate $k = 0.08$ and adjustment parameter $\lambda = 1.17$ by assuming combined Gompertz and charged capacitor growth in (7). Since $\lambda > 1$, initial growth for the curve p_{GC} is slower than for the curve p_G for Gompertz growth, see section 2.4.

The curve $p_{GC\lambda}$ is intermediate between the curve p_G for Gompertz growth and p_C for charged capacitor growth. This is obtained by assuming $\lambda = 0.88$ and using the least squares method to optimize the growth rate k which gives $k = 0.32$. The curve $p_{GC\lambda}$ initially increases more quickly than all the other curves except the curve p_C for charged capacitor growth.

3.1.2. Weighted least squares method

Applying the weighted least squares method, Figure 1(c) shows the historical estimates. Figure 1(d) predicts until 1 January 2100. The Bitcoin data exhibits heteroscedasticity so that the variance increases over time. That is, the Bitcoin price was \$0.04951 on 23 July, 2010, with a few cents variation over the subsequent months until \$1 was exceeded on 17 February 2011. In contrast, the Bitcoin price was \$32,950 on 21 June 2021, with several thousand US\$ variation over the preceding months until \$1 was exceeded 17 February 2011. Hence the least squares method is more influenced by recent data than early data. This section assigns more weight to the earlier data by dividing each squared difference (between the model prediction and the data) at each time t with the 20-week moving variance in the data, i.e., the variance over 140 days from time t to time $t + 139$. The variance calculation is constrained by the final time T so that at time $T - 1$ the variance over only the two final days at $T - 1$ and T is determined.

The two curves p_L for logistic growth and p_G for Gompertz growth have the same and slightly lower growth rates $k = 1.28$ and $k = 0.15$ as Figure 1(a) in section 3.1.1.

The curve p_C for charged capacitor growth has the much lower growth rate $k = 4.32$. That is because the early data is weighed more heavily, and more recent data is discounted. Hence the model prediction is worse for the more recent data, and the curve p_C needs more time to approach the carrying capacity $K = \$476,190$.

The curve p_{LC} for combined logistic and charged capacitor growth estimates the higher growth rate $k = 2.33$ and lower adjustment parameter $\lambda = 0.85$, compared with Figure 1(a). Weighing the early data more heavily causes more rapid initial growth.

The curve $p_{LC\lambda}$ for combined logistic and charged capacitor growth when $\lambda = 0.88$ has the lower growth rate $k = 2.10$ compared with Figure 1(a), since it becomes less important to adjust to the recent data.

The curve p_{GC} for combined Gompertz and charged capacitor growth estimates the higher growth rate $k = 0.17$ and lower adjustment parameter $\lambda = 0.97$, compared with Figure 1a. Weighing the early data more heavily causes more rapid initial growth.

The curve $p_{GC\lambda}$ for combined Gompertz and charged capacitor growth when $\lambda = 0.88$ has the lower growth rate $k = 0.22$ compared with Figure 1(a), since it becomes less important to adjust to the recent data.

3.2. Bitcoin carrying capacity K=$23,809,524 and no oscillation amplitude α=0

As an alternative, assume that Bitcoin in the future eradicates all or most other digital currencies, overtakes gold, bonds, and most other assets except physical real estate and various other physical assets. That may suggest a maximum sustainable market capitalization of $500 trillion, which may account for future inflation of the US$. Dividing $500 trillion with 21 million coins gives the Bitcoin carrying capacity $K = \$23,809,524$. Figure 2 replicates Figure 1 for $K = \$23,809,524$.

The subsequent seven curves in each panel in Figure 1 assume $\alpha = 0$, $K = \$23,809,524$ and $p_0 = \$0.04951$ and estimate the growth rate k, and the two parameters λ and β, for the models in section 2.2, using the least squares method. Figure 1(a) shows the historical estimates. Figure 1(b) predicts until 1 January 2100.

3.2.1. Least squares method

Applying the least squares method, Figure 2(a) shows the historical estimates. Figure 2(b) predicts until 1 January 2100. Figure 2 gives similar parameter estimates to those in Figure 1.

The curve p_L estimates slightly lower growth rate $k = 1.27$ compared with Figure 1 for logistic growth in (1) for $\nu = 1$.

The curve p_G estimates lower growth rate $k = 0.10$ compared with Figure 1 for Gompertz growth in (2).

The curve p_C estimates slightly lower growth rate $k = 1171$ compared with Figure 1 for charged capacitor growth in (4).

The curve p_{LC} estimates slightly higher growth rate $k = 1.23$ and slightly lower adjustment parameter $\lambda = 1.01$, compared with Figure 1 for combined logistic and charged capacitor growth in (6).

The curve $p_{LC\lambda}$ estimates slightly lower growth rate $k = 2.17$ when assuming the same adjustment parameter $\lambda = 0.88$, compared with Figure 1 for combined logistic and charged capacitor growth in (6).

The curve p_{GC} estimates higher growth rate $k = 0.14$ and lower adjustment parameter $\lambda = 0.94$ compared with Figure 1 for combined Gompertz and charged capacitor growth in (7).

The curve $p_{GC\lambda}$ estimates lower growth rate $k = 0.20$ compared with Figure 1 when assuming adjustment parameter $\lambda = 0.88$, for combined Gompertz and charged capacitor growth in (7).

3.2.2. Weighted least squares method

Applying the same weighted least squares method as in section 3.1.2, Figure 2(c) shows the historical estimates. Figure 2(d) predicts until 1 January 2100.

The two curves p_L for logistic growth and p_G for Gompertz growth have the same and slightly higher growth rates $k = 1.28$ and $k = 0.11$ compared with Figure 2(a) in section 3.2.1.

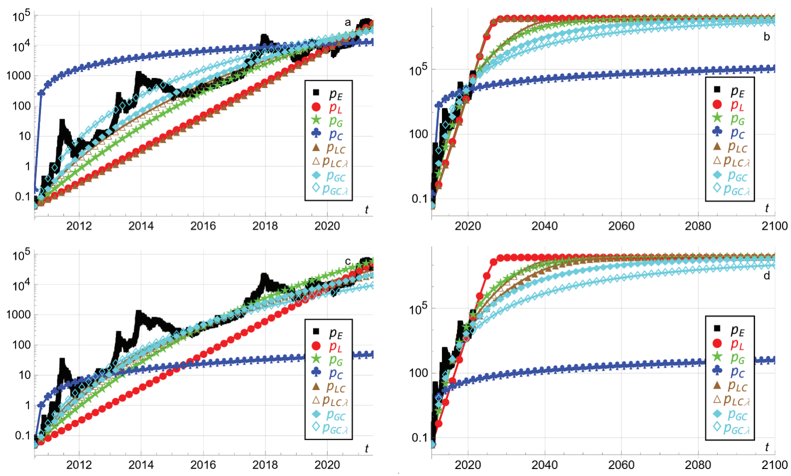

Figure 2. Assuming no oscillation amplitude $\alpha = 0$, the empirical price p_E, logistic growth p_L, Gompertz growth p_G, charged capacitor growth p_C, combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$, and combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$, for 23 July 2010–21 June 2021 (panels a and c) and until 1 January 2100 (panels b and d), $K = \$23,809,524$. Panels a and b apply the least squares method. Panels c and d apply the weighted least squares method.

The curve p_C for charged capacitor growth has the much lower growth rate $k = 4.32$. That is because the early data is weighed more heavily, and more recent data is discounted. Hence the model prediction is worse for the more recent data, and the curve p_C needs more time to approach the carrying capacity $K = \$23,809,524$.

The curve p_{LC} for combined logistic and charged capacitor growth estimates the higher growth rate $k = 2.33$ and lower adjustment parameter $\lambda = 0.85$, compared with Figure 2(a). Weighing the early data more heavily causes more rapid initial growth.

The curve $p_{LC\lambda}$ for combined logistic and charged capacitor growth when $\lambda = 0.88$ has the lower growth rate $k = 2.10$ compared with Figure 2(a), since it becomes less important to adjust to the recent data.

The curve p_{GC} for combined Gompertz and charged capacitor growth estimates slightly lower growth rate $k = 0.13$ and the same adjustment parameter $\lambda = 0.94$, compared with Figure 2(a).

The curve $p_{GC\lambda}$ for combined Gompertz and charged capacitor growth when $\lambda = 0.88$ has the lower growth rate $k = 0.15$ compared with Figure 2(a), since it becomes less important to adjust to the recent data.

3.3. Determining the three bull market local maxima and the three bear market local minima

The three bull market local maxima since 23 July 2010, are as follows:

$29.6 on 14 June, 2011, expressed as $t_{1max} = 2011.449315$, i.e., 327 days after the start date 23 July 2010 which is day 1.

$1131.992853 on 29 November, 2013 expressed as $t_{2max} = 2013.909589$, i.e., 1226 days after the start date 23 July 2010 which is day 1.

$19,378.35059 on 16 December, 2017 expressed as $t_{3max} = 2017.956164$, i.e., 2704 days after the start date 23 July 2010 which is day 1.

This gives 1226–327 = 899 days, i.e., 2.46027 years, from bull market local maximum 1 to bull market local maximum 2, and 2704–1226 = 1478 days, i.e., 4.04657 years, from bull market local maximum 2 to bull market local maximum 3.

The three bear market local minima since 23 July 2010 are as follows:

$2.2 on 20 November, 2011, expressed as $t_{1min} = 2011.884932$, i.e., 486 days after the start date 23 July 2010 which is day 1.

$178.712075 on 14 January, 2015 expressed as $t_{2min} = 2015.035616$, i.e., 1637 days after the start date 23 July 2010 which is day 1.

$3226.92952 on 14 December, 2018 expressed as $t_{3min} = 2018.950685$, i.e., 3067 days after the start date 23 July 2010 which is day 1.

This gives 1637–486 = 1151 days, i.e., 3.15068 years, from bear market local minimum 1 to bear market local minimum 2, and 3067–1637 = 1430 days, i.e., 3.915069 years, from bear market local minimum 2 to bear market local minimum 3.

The modeling assumes oscillations in the sense that a maximum is followed by a minimum, then a new maximum, etc.

3.4. Estimating the scaling ω of the inverse of the cycle length of the sine oscillations and the scaling γ of the inverse of the degree of lengthening of each subsequent cycle

This section estimates the scaling ω of the inverse of the cycle length of the sine oscillations, and the scaling γ of the inverse of the degree of lengthening of each subsequent cycle. The oscillatory growth rate with damped oscillation, retracement in bear markets, and lengthening cycles in all the equations in section 2 contain the sine of $\omega(t-t_0)^\gamma+\delta$. One cycle has time length 2π. Hence the two equations

$$(\omega(t_{2max}-t_0)^\gamma+\delta)-(\omega(t_{1max}-t_0)^\gamma+\delta)=2\pi,(\omega(t_{3max}-t_0)^\gamma+\delta)-(\omega(t_{2max}-t_0)^\gamma+\delta)=2\pi \tag{8}$$

express the time length from bull market local maximum 1 to bull market local maximum 2, and the time length from bull market local maximum 2 to bull market local maximum 3, respectively. Solving (8) by using t_{1max}, t_{2max}, t_{3max} from section 3.3 gives $\omega_{max}=7.05885$ and $\gamma_{max}=0.499872$.

Analogously, the two equations

$$(\omega(t_{2min}-t_0)^\gamma+\delta)-(\omega(t_{1min}-t_0)^\gamma+\delta)=2\pi,(\omega(t_{3min}-t_0)^\gamma+\delta)-(\omega(t_{2min}-t_0)^\gamma+\delta)=2\pi \tag{9}$$

express the time length from bear market local minimum 1 to bear market local minimum 2, and the time length from bear market local minimum 2 to bear market local minimum 3, respectively. Solving (9) by using t_{1min}, t_{2min}, t_{3min} from section 3.3 gives $\omega_{min}=3.45348$ and $\gamma_{min}=0.744082$. The average of ω_{max} and ω_{min} is $\omega=5.25616$. The average of γ_{max} and γ_{min} is $\gamma=0.621977$.

3.5. Bitcoin carrying capacity K=$476,190 and oscillation amplitude α>0

This section assumes positive oscillation amplitude $\alpha\geq 0$ and assumes the same k, γ, and β as when $\alpha=0$. Since $\omega=5.25616$ and $\gamma=0.621977$ are estimated in the previous section, we only have to estimate α and δ. Figure 3 shows the empirical price p_E for the period 23 July 2010–21 June 2021.

3.5.1. Least squares method

Applying the least squares method, Figure 3(a) shows the historical estimates. Figure 3(b) predicts until 1 January 2040.

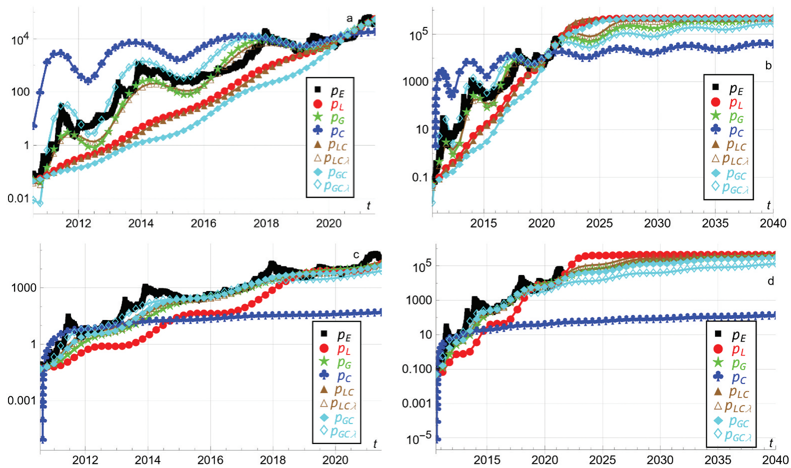

Figure 3. Assuming oscillation amplitude $\alpha\geq 0$, the empirical price p_E, logistic growth p_L, Gompertz growth p_G, charged capacitor growth p_C, combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$, and combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$, for 23 July 2010–21 June 2021 (panels a and c) and until 1 January 2040 (panels b and d), $K=\$476,190$. Panels a and b apply the least squares method. Panels c and d apply the weighted least squares method.

The curve p_L estimates the oscillation amplitude $\alpha = 0.48$ and start time adjustment parameter $\delta = 3.90$ for logistic growth in (1) for $\nu = 1$ and $k = 1.28$. The oscillation amplitude α is moderate relative to the almost linear curve in Figure 1.

The curve p_G estimates the oscillation amplitude $\alpha = 0.37$ and start time adjustment parameter $\delta = 4.29$ for Gompertz growth in (2) when $k = 0.16$. The oscillation amplitude α is higher than for the curve p_L, and conforms with the bull and bear markets.

The curve p_C estimates the oscillation amplitude $\alpha = 7209$ and start time adjustment parameter $\delta = 5.03$ for charged capacitor growth in (4) when $k = 1178$. The curve p_C starts with extreme concavity, thereafter oscillates according to the bull and bear markets, and eventually approaches $K = \$476,190$ slowly.

The curve p_{LC} estimates the oscillation amplitude $\alpha = 0.39$ and start time adjustment parameter $\delta = 3.05$ for combined logistic and charged capacitor growth in (6) when $k = 1.21$ and $\lambda = 1.02$. The curve p_{LC} oscillates similarly to Gompertz growth p_G.

The curve $p_{LC\lambda}$ estimates the oscillation amplitude $\alpha = 3.93$ and start time adjustment parameter $\delta = 4.12$ for combined logistic and charged capacitor growth in (6) when $k = 2.18$ and $\lambda = 0.88$. The curve $p_{LC\lambda}$ oscillates similarly to Gompertz growth p_G.

The curve p_{GC} estimates the oscillation amplitude $\alpha = 0.04$ and start time adjustment parameter $\delta = 5.05$ for combined logistic and charged capacitor growth in (6) when $k = 0.08$ and $\lambda = 1.17$. The curve p_{GC} initially grows slower than all the other curves.

The curve $p_{GC\lambda}$ estimates the oscillation amplitude $\alpha = 0.91$ and start time adjustment parameter $\delta = 4.55$ for combined Gompertz and charged capacitor growth in (7) when $k = 0.32$ and $\lambda = 0.88$. The curve is intermediate between Gompertz growth p_G and combined logistic and charged capacitor growth p_{LC} on the one hand, and charged capacitor growth p_C on the other hand. The curve p_{GC} conforms with the bull and bear markets.

3.5.2. Weighted least squares method

Applying the weighted least squares method, Figure 3(c) shows the historical estimates. Figure 3(d) predicts until 1 January 2040.

The curve p_L estimates the oscillation amplitude $\alpha = 1.42$ and start time adjustment parameter $\delta = 2.31$ for logistic growth in (1) for $\nu = 1$ and $k = 1.28$. The oscillation amplitude α is higher and the start time adjustment parameter δ is lower compared with Figure 3(a).

The curve p_G estimates the oscillation amplitude $\alpha = 0.08$ and start time adjustment parameter $\delta = 2.48$ for Gompertz growth in (2) when $k = 0.15$. Both the oscillation amplitude α and the start time adjustment parameter δ are lower compared with Figure 3(a).

The curve p_C estimates the oscillation amplitude $\alpha = 3.13$ and start time adjustment parameter $\delta = 4.28$ for charged capacitor growth in (4) when $k = 4.32$. The oscillation amplitude α is substantially lower, impacted by the much lower growth rate $k = 4.32$, and the start time adjustment parameter δ is lower, compared with Figure 3(a). The curve p_C eventually approaches $K = \$476,190$ slowly.

The curve p_{LC} estimates the oscillation amplitude $\alpha = 1.88$ and start time adjustment parameter $\delta = 3.24$ for combined logistic and charged capacitor growth in (6) when $k = 2.33$ and $\lambda = 0.85$. Both the oscillation amplitude α and the start time adjustment parameter δ are higher compared with Figure 3(a). The curve p_{LC} oscillates similarly to Gompertz growth p_G.

The curve $p_{LC\lambda}$ estimates the oscillation amplitude $\alpha = 1.65$ and start time adjustment parameter $\delta = 3.02$ for combined logistic and charged capacitor growth in (6) when $k = 2.10$ and $\lambda = 0.88$. Both the oscillation amplitude α and the start time adjustment parameter δ are lower compared with Figure 3(a). The curve $p_{LC\lambda}$ also oscillates similarly to Gompertz growth p_G.

The curve p_{GC} estimates the oscillation amplitude $\alpha = 0.15$ and start time adjustment parameter $\delta = 3.26$ for combined logistic and charged capacitor growth in (6) when $k = 0.17$ and $\lambda = 0.97$. The oscillation amplitude α is higher and the start time adjustment parameter δ is lower compared with Figure 3(a). The curve p_{GC} also oscillates similarly to Gompertz growth p_G.

The curve $p_{GC\lambda}$ estimates the oscillation amplitude $\alpha = 0.20$ and start time adjustment parameter $\delta = 3.93$ for combined Gompertz and charged capacitor growth in (7) when $k = 0.22$ and $\lambda = 0.88$. Both the oscillation amplitude α and the start time adjustment parameter δ are lower compared with Figure 3(a). The curve $p_{GC\lambda}$ initially oscillates around higher values than the other curves except charged capacitor growth p_C.

3.6. Bitcoin carrying capacity K=$23,809,524 and oscillation amplitude α>0

This section replicates the previous section with the higher Bitcoin carrying capacity $K = \$23,809,524$. Figure 4 replicates Figure 3 for $K = \$23,809,524$.

3.6.1. Least squares method

Applying the least squares method, Figure 4(a) shows the historical estimates. Figure 4(b) predicts until 1 January 2040. Figure (4) gives similar parameter estimates to those in Figure 3.

The curve p_L estimates lower oscillation amplitude $\alpha = 0.34$ and slightly higher start time adjustment parameter $\delta = 4.00$ for logistic growth compared with Figure 3, assuming $\nu = 1$ and $k = 1.27$.

The curve p_G estimates lower oscillation amplitude $\alpha = 0.17$ and slightly higher start time adjustment parameter $\delta = 4.34$ for Gompertz growth compared with Figure 3, assuming $k = 0.10$.

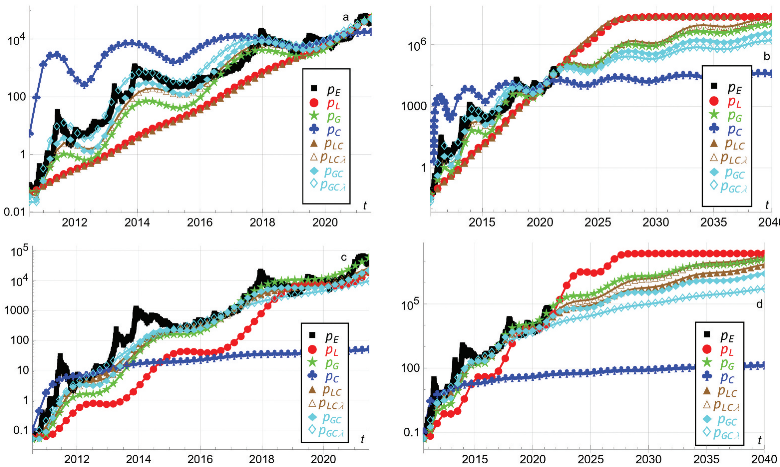

Figure 4. Assuming oscillations $\alpha \geq 0$, the empirical price p_E, logistic growth p_L, Gompertz growth p_G, charged capacitor growth p_C, combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$, and combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$, for 23 July 2010–21 June 2021 (panels a and c) and until 1 January 2040 (panels b and d), $K = \$23,809,524$. Panels a and b apply the least squares method. Panels c and d apply the weighted least squares method.

The curve p_C estimates slightly lower oscillation amplitude $\alpha = 7200$ and slightly lower start time adjustment parameter $\delta = 5.00$ for charged capacitor growth compared with Figure 3, assuming $k = 1171$.

The curve p_{LC} estimates lower oscillation amplitude $\alpha = 0.22$ and higher start time adjustment parameter $\delta = 3.69$ for combined logistic and charged capacitor growth compared with Figure 3, assuming $k = 1.23$ and $\lambda = 1.01$.

The curve $p_{LC\lambda}$ estimates lower oscillation amplitude $\alpha = 3.58$ and lower start time adjustment parameter $\delta = 4.08$ for combined logistic and charged capacitor growth compared with Figure 3, assuming $k = 2.17$ and $\lambda = 0.88$.

The curve p_{GC} estimates substantially higher oscillation amplitude $\alpha = 0.29$ and lower start time adjustment parameter $\delta = 4.22$ for combined logistic and charged capacitor growth compared with Figure 3, assuming $k = 0.14$ and $\lambda = 0.94$.

The curve $p_{GC\lambda}$ estimates the same oscillation amplitude $\alpha = 0.91$ and the same start time adjustment parameter $\delta = 4.55$ for combined Gompertz and charged capacitor growth compared with Figure 3, assuming $k = 0.2$ and $\lambda = 0.88$.

3.6.2. Weighted least squares method

Applying the weighted least squares method, Figure 4(c) shows the historical estimates. Figure 4(d) predicts until 1 January 2040.

The curve p_L estimates substantially higher oscillation amplitude $\alpha = 1.53$ and lower start time adjustment parameter $\delta = 2.38$ compared with Figure 4(a) for logistic growth, assuming $\nu = 1$ and $k = 1.27$ as in Figure 2.

The curve p_G estimates lower oscillation amplitude $\alpha = 0.11$ and lower start time adjustment parameter $\delta = 3.40$ compared with Figure 4(a) for Gompertz growth, assuming $k = 0.11$ as in Figure 2.

The curve p_C estimates substantially lower oscillation amplitude $\alpha = 3.13$ and lower start time adjustment parameter $\delta = 4.28$ compared with Figure 4(a) for charged capacitor growth, assuming $k = 4.32$ as in Figure 2. The difference between Figure 4(c) and Figure 4(a) is similar to the difference between Figure 2(c) and Figure 2(a).

The curve p_{LC} estimates substantially higher oscillation amplitude $\alpha = 1.66$ and lower start time adjustment parameter $\delta = 3.23$ compared with Figure 4(a) for combined logistic and charged capacitor growth, assuming $k = 2.33$ and $\lambda = 0.85$ as in Figure (2).

The curve $p_{LC\lambda}$ estimates lower oscillation amplitude $\alpha = 1.65$ and lower start time adjustment parameter $\delta = 3.00$ compared with Figure 4(a) for combined logistic and charged capacitor growth, assuming $k = 2.10$ and $\lambda = 0.88$ as in Figure 2.

The curve p_{GC} estimates lower oscillation amplitude $\alpha = 0.15$ and start time adjustment parameter $\delta = 3.5$ compared with Figure 4(a) for combined logistic and charged capacitor growth, assuming $k = 0.13$ and $\lambda = 0.94$ as in Figure 2.

The curve $p_{GC\lambda}$ estimates the much lower oscillation amplitude $\alpha = 0.20$ and lower start time adjustment parameter $\delta = 3.93$ compared with Figure 4(a) for combined Gompertz and charged capacitor growth, assuming $k = 0.15$ and $\lambda = 0.88$ as in Figure 2.

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Table 1. Dates and magnitudes of bull market local maxima 4,5,6,7,8 when $K = \$476,190$ and $K = \$23,809,524$, predicted until year 2050

Bull market local maximum	4	5	6	7	8
Logistic growth p_L	Asymptotic. 07/06/24; 431,785. Asymptotic. 05/05/24; 3,360,314.	Asymptotic. 01/28/30; 476,153. Asymptotic. 11/18/29; 23,694,323.	Asymptotic. Asymptotic. Asymptotic. 02/06/36; 23,809,485.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.
Gompertz growth p_G	05/22/22; 75,506. Asymptotic. 06/29/22; 107,064. Asymptotic.	08/25/27; 221,873. Asymptotic. 10/08/27; 1,032,330. Asymptotic.	08/11/33; 358,456. Asymptotic. 09/29/33; 4,343,140. Asymptotic.	03/21/40; 432,860. Asymptotic. 05/15/40; 10,021,700. Asymptotic.	06/11/47; 462,526. Asymptotic. 08/09/47; 15,732,787. Asymptotic.
Charged capacitor growth p_C	08/08/21; 18,032. Asymptotic. 08/15/21; 18,317. Asymptotic.	09/29/26; 24,619. Asymptotic. 10/07/26; 25,173. Asymptotic.	08/05/32; 31,829. Asymptotic. 08/14/32; 32,789. Asymptotic.	02/07/39; 39,587. Asymptotic. 02/17/39; 41,119. Asymptotic.	03/25/46; 47,827. Asymptotic. 04/04/46; 50,127. Asymptotic.
Combined logistic and charged capacitor growth p_{LC}	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.
Combined logistic and charged capacitor growth $p_{LC\lambda}$	08/19/22; 113,012. Asymptotic. 09/16/22; 133,710. Asymptotic.	12/05/27; 391,125. Asymptotic. 01/07/28; 1,497,123. Asymptotic.	12/03/33; 470,084. Asymptotic. 01/08/34; 8,753,549. Asymptotic.	07/25/40; 475,908. Asymptotic. 09/03/40; 19,460,471. Asymptotic.	10/26/47; 476,180. Asymptotic. 12/08/47; 23,181,174. Asymptotic.
Combined Gompertz and charged capacitor growth p_{GC}	Asymptotic. Asymptotic. 06/26/22; 94,049. 06/27/23; 65,062.	Asymptotic. Asymptotic. 10/05/27; 512,728. 11/27/28; 367,353.	Asymptotic. Asymptotic. 09/25/33; 1,714,019. 01/06/35; 1,286,048.	Asymptotic. Asymptotic. 05/10/40; 4,022,084. 10/06/41; 3,169,232.	Asymptotic. Asymptotic. 08/04/47; 7,301,533. 02/12/49; 6,030,620.
Combined Gompertz and charged capacitor growth $p_{GC\lambda}$	02/12/22; 46,652. Asymptotic. 04/28/22; 68,116. Asymptotic.	05/03/27; 110,018. Asymptotic. 07/29/27; 264,925. Asymptotic.	04/05/33; 191,648. Asymptotic. 07/11/33; 736,127. Asymptotic.	11/02/39; 273,762. Asymptotic. 02/16/40; 1,611,829. Asymptotic.	01/10/47; 342,930. Asymptotic. 05/06/47; 2,960,784. Asymptotic.

(Continued)

Table 1. (*Continued*)

Bull market local maximum	4	5	6	7	8
Date and local maximum expected value ± standard deviation	12/12/22 ± 314. 11/29/22 ± 397. 05/18/22 ± 94. 07/06/24 ± 0. 12/21/22 ± 288. 07/02/22 ± 58. 11/30/23 ± 221. 166,739 ± 178,774. 78,390 ± 33,274. 431,785 ± 0. 638,053 ± 1,333,874. 100,735 ± 27,301. 1,712,688 ± 2,330,095.	04/15/28 ± 359. 03/30/28 ± 454. 08/21/27 ± 108. 01/28/30 ± 0. 04/25/28 ± 329. 10/12/27 ± 66. 05/24/29 ± 252. 299,792 ± 164,854. 241,005 ± 141,527. 476,153 ± 0. 4,561,464 ± 9,384,608. 826,777 ± 549,521. 12,030,838 ± 16,494,659.	02/07/34 ± 327. 08/06/33 ± 121. 08/06/33 ± 121. Asymptotic. 05/10/34 ± 367. 10/03/33 ± 74. 07/23/35 ± 280. 340,063 ± 140,126. 340,063 ± 140,126. Asymptotic. 6,773,728 ± 8,856,776. 3,886,709 ± 3,584,281. 12,547,767 ± 15,926,475.	06/27/40 ± 211. 03/16/40 ± 133. 03/16/40 ± 133. Asymptotic. 08/28/40 ± 237. 05/19/40 ± 82. 10/06/41 ± 0. 394,177 ± 106,480. 394,177 ± 106,480. Asymptotic. 7,657,063 ± 7,328,208. 8,779,021 ± 7,950,571. 3,169,232 ± 0.	09/25/47 ± 228. 06/06/47 ± 145. 06/06/47 ± 145. Asymptotic. 12/01/47 ± 257. 08/14/47 ± 89. 02/12/49 ± 0. 427,212 ± 73,309. 427,212 ± 73,309. Asymptotic. 11,041,380 ± 8,273,060. 12,294,070 ± 8,988,687. 6,030,620 ± 0.

Each cell has four entries, except in the bottom row. Each entry ends with period ".". The dates are in the format month/day/year for the least squares method and the weighted least squares method in entries 1 and 2 when $K = \$476,190$. Entries 3 and 4 are for the least squares method and the weighted least squares method when $K = \$23,809,524$. After each date the price follows in \$. Asymptotic means that no local maximum exists, but that the Bitcoin price p approaches K asymptotically. In the bottom row each cell has 13 entries. Each entry ends with period ".". This gives seven entries with dates and six entries with local maxima expressed as expected value ± standard deviation in days and \$. Entry 1 gives the date for both methods and both $K = \$476,190$ and $K = \$23,809,524$. Entries 2 and 8 apply for both methods when $K = \$476,190$. Entries 3 and 9 apply for the least squares method when $K = \$476,190$. Entries 4 and 10 apply for the weighted least squares method when $K = \$476,190$. Entries 5 and 11 apply for both methods when $K = \$23,809,524$. Entries 6 and 12 apply for the least squares method when $K = \$23,809,524$. Entries 7 and 13 apply for the weighted least squares method when $K = \$23,809,524$.

Table 2. Dates and magnitudes of bear market local minima 4,5,6,7,8 when $K = \$476,190$ and $K = \$23,809,524$, predicted until year 2050

Bear market local minima	4	5	6	7	8
Logistic growth p_L	Asymptotic. 04/09/25; 428,795. Asymptotic. 05/01/25; 2,898,043.	Asymptotic. 12/07/30; 476,150. Asymptotic. 01/01/31; 23,669,932.	Asymptotic. Asymptotic. Asymptotic. 05/05/37; 23,809,475.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.
Gompertz growth p_G	03/05/24; 45,736. Asymptotic. 12/28/23; 73,350. Asymptotic.	09/10/29; 174,196. Asymptotic. 06/24/29; 802,142. Asymptotic.	11/22/35; 323,674. Asymptotic. 08/27/35; 3,725,750. Asymptotic.	09/22/42; 416,631. Asymptotic. 06/19/42; 9,196,504. Asymptotic.	02/27/50; 456,629. Asymptotic. 11/16/49; 15,043,891. Asymptotic.
Charged capacitor growth p_C	10/24/23; 9,710. Asymptotic. 11/01/23; 9,743. Asymptotic.	04/12/29; 15,198. Asymptotic. 04/21/29; 15,342. Asymptotic.	06/07/35; 21,449. Asymptotic. 06/16/35; 21,799. Asymptotic.	03/21/42; 28,365. Asymptotic. 04/01/42; 29,046. Asymptotic.	08/11/49; 35,866. Asymptotic. 08/22/49; 37,031. Asymptotic.
Combined logistic and charged capacitor growth p_{LC}	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.	Asymptotic. Asymptotic. Asymptotic. Asymptotic.
Combined logistic and charged capacitor growth $p_{LC\lambda}$	03/17/24; 78,327. Asymptotic. 03/09/24; 95,424. Asymptotic.	09/24/29; 355,689. Asymptotic. 09/14/29; 1,139,873. Asymptotic.	12/07/35; 466,389. Asymptotic. 11/27/35; 7,376,641. Asymptotic.	10/08/42; 475,712. Asymptotic. 09/27/42; 18,476,294. Asymptotic.	03/16/50; 476,173. Asymptotic. 03/04/50; 22,992,469. Asymptotic.
Combined Gompertz and charged capacitor growth p_{GC}	Asymptotic. Asymptotic. 03/11/24; 57,377. 05/03/24; 62,860.	Asymptotic. Asymptotic. 09/17/29; 360,351. 11/16/29; 358,295.	Asymptotic. Asymptotic. 11/29/35; 1,334,243. 02/04/36; 1,263,066.	Asymptotic. Asymptotic. 09/30/42; 3,371,727. 12/12/42; 3,128,527.	Asymptotic. Asymptotic. 03/07/50; 6,458,819. 05/25/50; 5,975,591.
Combined Gompertz and charged capacitor growth $p_{GC\lambda}$	01/17/24; 27,008. Asymptotic. 01/14/24; 45,585. Asymptotic.	07/18/29; 77,314. Asymptotic. 07/13/29; 196,739. Asymptotic.	09/22/35; 152,541. Asymptotic. 09/18/35; 587,238. Asymptotic.	07/17/42; 236,548. Asymptotic. 07/12/42; 1,354,016. Asymptotic.	12/16/49; 312,871. Asymptotic. 12/11/49; 2,585,103. Asymptotic.

(*Continued*)

Bear market local minima	4	5	6	7	8
Date and local minima expected value ± standard deviation	05/17/24 ± 182. 06/04/24 ± 208. 02/22/24 ± 32. 04/09/25 ± 0. 05/06/24 ± 182. 02/06/24 ± 38. 10/31/24 ± 257. 144,966 ± 190,403. 50,357 ± 25,970. 428,795 ± 0. 538,773 ± 1,155,924. 67,934 ± 21,571. 1,480,452 ± 2,004,777.	12/02/29 ± 206. 12/22/29 ± 235. 08/27/29 ± 36. 12/07/30 ± 0. 11/18/29 ± 207. 08/09/29 ± 43. 06/09/30 ± 291. 270,837 ± 179,019. 202,400 ± 141,314. 476,150 ± 0. 4,421,222 ± 9,436,336. 624,776 ± 428,143. 12,014,113 ± 16,483,817.	01/07/36 ± 188. 11/06/35 ± 40. 11/06/35 ± 40. Asymptotic. 02/07/36 ± 229. 10/18/35 ± 48. 09/19/36 ± 322. 314,201 ± 157,138. 314,201 ± 157,138. Asymptotic. 6,349,402 ± 8,910,871. 3,255,968 ± 3,055,912. 12,536,271 ± 15,942,719.	09/06/42 ± 58. 09/05/42 ± 44. 09/05/42 ± 44. Asymptotic. 09/07/42 ± 70. 08/14/42 ± 53. 12/12/42 ± 0. 376,297 ± 124,579. 376,297 ± 124,579. Asymptotic. 7,105,414 ± 7,008,998. 8,099,635 ± 7,675,386. 3,128,527 ± 0.	02/09/50 ± 63. 02/08/50 ± 48. 02/08/50 ± 48. Asymptotic. 02/10/50 ± 76. 01/16/50 ± 57. 05/25/50 ± 0. 415,224 ± 89,177. 415,224 ± 89,177. Asymptotic. 10,611,175 ± 8,308,037. 11,770,071 ± 9,114,698. 5,975,591 ± 0.

The structure is the same as for Table 1, but applies for local minima instead of local maxima.

3.7. Estimating bull market local maxima 4,5,6,7,8 when K=$476,190 and K=$23,809,524

Table 1 predicts the dates and magnitudes of the five future Bitcoin bull market local price maxima, assuming $K = \$476,190$ and $K = \$23,809,524$, and assuming logistic growth p_L, Gompertz growth p_G, charged capacitor growth p_C, combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$, and combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$.

Table 2 analogously predicts the dates and magnitudes of the five future Bitcoin bear market local price minima, assuming $K = \$476,190$ and $K = \$23,809,524$, and assuming logistic growth p_L, Gompertz growth p_G, charged capacitor growth p_C, combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$, and combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$.

3.7.1. Bitcoin carrying capacity K=$476,190

Assuming the Bitcoin carrying capacity $K = \$476,190$, for logistic growth p_L no local maxima exist with the least squares method, which means that the Bitcoin price p increases monotonically and asymptotically towards $K = \$476,190$ with no local maxima. The absence of local maxima is due to logistic growth p_L approaching $K = \$476,190$ more quickly than the other six curves (see Figure 1), and also due to logistic growth p_L exhibiting limited oscillation, thus not tracking the empirics very well. Two local maxima exist with the weighted least squares method, after which the Bitcoin price p increases monotonically towards $K = \$476,190$. The presence of two local maxima is intermediate between no local maxima and five local maxima, reflecting more oscillation due to more weight being assigned to the early data, and better tracking of the early empirics. Since logistic growth p_L approaches $K = \$476,190$ more quickly than the other six curves, both local maxima are above $431,785, at 7/6/24 and 1/28/30.

For Gompertz growth p_G, five local maxima exist with the least squares method, starting with $75,506 at 5/22/22, and ending with $462,526 at 6/11/47. These lower local maxima, compared with logistic growth p_L, arise since Gompertz growth p_G approaches the Bitcoin carrying capacity $K = \$476,190$ more slowly. No local maxima exist with the weighted least squares method, reflecting less oscillation than with the least squares method.

For charged capacitor growth p_C five local maxima exist with the least squares method. Since charged capacitor growth p_C approaches the Bitcoin carrying capacity $K = \$476,190$ slowly (see Figure 1), the local maxima are low, ranging from \$18,032 at 8/8/21 to \$47,827 at 3/25/46. No local maxima exist with the weighted least squares method, reflecting the much lower growth rate $k = 4.32$, and inability to track the empirical oscillations.

For combined logistic and charged capacitor growth p_{LC}, no local maxima exist with the least squares method and the weighted least squares method. This result is influenced by no local maxima existing for logistic growth p_L with the least squares method, and no local maxima existing for charged capacitor growth p_C with the weighted least squares method.

For combined logistic and charged capacitor growth $p_{LC\lambda}$, five local maxima exist with the least squares method, starting with \$113,012 at 8/19/22, and ending with \$476,180 at 10/26/47. No local maxima exist with the weighted least squares method, reflecting less oscillation.

For combined Gompertz and charged capacitor growth p_{GC}, no local maxima exist with the least squares method and the weighted least squares method, due to less oscillation.

For combined Gompertz and charged capacitor growth $p_{GC\lambda}$, five local maxima exist with the least squares method, starting with \$46,652 at 2/12/22, and ending with \$342,930 at 1/10/47. No local maxima exist with the weighted least squares method, reflecting less oscillation.

3.7.2. Bitcoin carrying capacity K=\$23,809,524

Assuming the Bitcoin carrying capacity $K = \$23,809,524$, the results are remarkably similar to when $K = \$476,190$. The local maxima mostly occur at the similar time t, but are naturally higher. For logistic growth p_L, no local maxima exist with the least squares method, while three local maxima exist with the weighted least squares method. These range from \$3,360,314 at 5/5/24 to \$23,809,485 at 2/6/36.

For Gompertz growth p_G the five local maxima with the least squares method range from \$107,064 at 6/29/22 to \$15,732,787 at 8/9/47, i.e., lower than for logistic growth p_L. No local maxima exist with the weighted least squares method.

For charged capacitor growth p_C the five local maxima with the least squares method only slightly exceed the local maxima when $K = \$476,190$, and almost at the same time, ranging from \$18,317 at 8/15/21 to \$50,127 at 4/4/46. No local maxima exist with the weighted least squares method.

For combined logistic and charged capacitor growth p_{LC}, no local maxima exist with the least squares method and the weighted least squares method.

For combined logistic and charged capacitor growth $p_{LC\lambda}$, five local maxima exist with the least squares method, starting with \$133,710 at 9/16/22, and ending with \$23,181,174 at 12/8/47. No local maxima exist with the weighted least squares method.

For combined Gompertz and charged capacitor growth p_{GC}, local maxima exist with both the least squares method and the weighted least squares method, contrary to when $K = \$476,190$. With the least squares method the five local maxima range from \$94,049 at 6/26/22 to \$7,301,533 at 8/4/47. With the weighted least squares method the five local maxima range from the lower \$65,062 at 6/26/22 to the lower \$6,030,620 at 2/12/49.

For combined Gompertz and charged capacitor growth $p_{GC\lambda}$, five local maxima exist with the least squares method, starting with \$68,116 at 4/28/22, and ending with \$2,960,784 at 5/6/47. No local maxima exist with the weighted least squares method.

4. Discussion

Ten results in the previous section are noteworthy. First, based on the current market capitalization of gold at approximately $10 trillion, the Bitcoin carrying capacity is estimated as $K = \$476,190$. Using the least squares method, without modeling oscillations, logistic growth p_L appears nearly linear with growth rate $k = 1.28$ on a logarithmic plot with base 10. It initially increases more slowly, and eventually approaches $K = \$476,190$ more quickly. Gompertz growth p_G with growth rate $k = 0.16$ is fast at the beginning and approaches $K = \$476,190$ slowly. Charged capacitor growth p_C initially grows much faster than the other six curves, with growth rate $k = 1178$, and approaches $K = \$476,190$ much slower than the other six curves. Combined logistic and charged capacitor growth p_{LC} is slower than logistic growth p_L but faster than combined Gompertz and charged capacitor growth p_{GC}. The curve p_{LC} estimates the growth rate $k = 1.21$ and adjustment parameter $\lambda = 1.02$. Assuming $\lambda = 0.88,$ the curve $p_{LC\lambda}$ with growth rate $k = 2.18$ is intermediate between p_L for logistic growth and p_C for charged capacitor growth. Combined Gompertz and charged capacitor growth p_{GC} displays initial slow growth rate $k = 0.08$, but eventually approaches $K = \$476,190$ quickly. Assuming $\lambda = 0.88,$ the curve $p_{GC\lambda}$ with growth rate $k = 0.32$ is intermediate between p_G for Gompertz growth and p_C for charged capacitor growth, and grows as the second fastest among the six curves. Summing up impressionistically, the curves $p_{GC\lambda}$, $p_{LC\lambda}$ and p_G in Figure 1(a) fit the data relatively well.

Second, applying the weighted least squares method when $K = \$476,190$, early data (with low price fluctuations measured in US$) is weighed more heavily than late data (with high price fluctuations measured in US$), which eliminates or ameliorates the impact of heteroscedasticity since more equal weight is assigned over the period 23 July 2010–21 June 2021, causing some similar and some different results. Logistic growth p_L estimates the same growth rate $k = 1.28$. Gompertz growth p_G has the higher growth rate $k = 0.15$. Charged capacitor growth p_C has the much lower growth rate $k = 4.32$, and approaches $K = \$476,190$ more slowly than with the least squares method. Combined logistic and charged capacitor growth p_{LC} has higher growth rate $k = 2.33$ and lower adjustment parameter $\lambda = 0.85$. Assuming $\lambda = 0.88$, the curve $p_{LC\lambda}$ estimates the lower growth rate $k = 2.10$. Combined Gompertz and charged capacitor growth p_{GC} has higher growth rate $k = 0.17$ and lower adjustment parameter $\lambda = 0.97$. Assuming $\lambda = 0.88,$ the curve $p_{GC\lambda}$ has lower growth rate $k = 0.22$. Summing up, the curves p_G, p_{GC}, $p_{GC\lambda}$, p_{LC} and $p_{LC\lambda}$ in Figure 1(c) are relatively similar with seemingly good fit to the data.

Third, based on a Bitcoin market capitalization of $500 trillion, the Bitcoin carrying capacity is estimated as $K = \$23,809,524$, i.e., 50 times higher than $K = \$476,190$. The results and especially the dates of the local maxima are similar, but the local maxima are higher. Using the least squares method, without modeling oscillations, logistic growth p_L is slightly lower at $k = 1.27$. Gompertz growth p_G is lower at $k = 0.10$. Charged capacitor growth p_C is slightly lower at $k = 1171$. Combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$ are similar at $k = 1.23$ and $k = 2.17$. Combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$ (assuming $\lambda = 0.88$) are higher at $k = 0.14$ and lower at $k = 0.20$, respectively. Summing up, the curves p_G, p_{GC}, $p_{LC\lambda}$, $p_{GC\lambda}$ in Figure 2(a) seem to fit the data well.

Fourth, applying the weighted least squares method when $K = \$23,809,524$, logistic growth p_L and Gompertz growth p_G are similar at $k = 1.27$ and $k = 0.11$. Charged capacitor growth p_C is much lower than with the least squares method, at $k = 4.32$ (due to weighing early data more heavily). Combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$ are higher at $k = 2.33$ and lower at $k = 2.10$, respectively. Combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$ are slightly lower at $k = 0.13$ and lower at $k = 0.15$, respectively. Summing up, the curves p_G, p_{GC}, p_{LC}, $p_{LC\lambda}$, $p_{GC\lambda}$ in Figure 2(c) seem to fit the data well.

Fifth, the three bull market local maxima during the period 23 July 2010–21 June 2021 are used to estimate the scaling of the inverse of the cycle length of the sine oscillations as $\omega_{max} = 7.05885$, and the inverse of the degree of lengthening of each subsequent cycle as $\gamma_{max} = 0.499872$. The three bear market local minima during the period 23 July 2010–21 June 2021 are analogously used to estimate $\omega_{min} = 3.45348$ and $\gamma_{min} = 0.744082$. Taking the average, $\omega = 5.25616$ as the scaling of the inverse of the cycle length of the sine oscillations, and $\gamma = 0.621977$ as the inverse of the degree of lengthening of each subsequent cycle, are used in the remainder of the article.

Sixth, applying the same growth rate k and adjustment parameter λ as estimated without assuming oscillations (i.e., when $\alpha = 0$), and applying $\omega = 5.25616$ and $\gamma = 0.621977$, the oscillation amplitude α and the start time adjustment parameter δ are estimated to model oscillatory growth for the models. With Bitcoin carrying capacity $K = \$476,190$ and applying the least squares method, logistic growth p_L oscillates minimally at the amplitude $\alpha = 0.48$. Gompertz growth p_G oscillates more at $\alpha = 0.37$. Charged capacitor growth p_C oscillates moderately at $\alpha = 7209$. Combined logistic and charged capacitor growth p_{LC} oscillates similarly to logistic growth p_L at $\alpha = 0.39$. The curve $p_{LC\lambda}$ oscillates similarly to Gompertz growth p_G at $\alpha = 3.93$. Combined Gompertz and charged capacitor growth p_{GC} oscillates minimally at $\alpha = 0.04$. The curve $p_{GC\lambda}$ oscillates at $\alpha = 0.91$. Summing up, the curves p_G, p_{GC}, $p_{LC\lambda}$, $p_{GC\lambda}$ in Figure 3(a) seemingly oscillate nicely according to the data.

Seventh, applying the weighted least squares method when $K = \$476,190$, logistic growth p_L oscillates more at $\alpha = 1.42$. Gompertz growth p_G oscillates less at $\alpha = 0.08$. Charged capacitor growth p_C oscillates minimally at $\alpha = 3.13$. Combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$ oscillate similarly to Gompertz growth p_G at $\alpha = 1.88$ and $\alpha = 1.65$. Combined Gompertz and charged capacitor growth p_{GC} oscillates at $\alpha = 0.15$. The curve $p_{GC\lambda}$ oscillates at $\alpha = 0.20$. Summing up, the curves p_G, p_{GC}, p_{LC}, $p_{LC\lambda}$, $p_{GC\lambda}$ in Figure 3(c) seem to oscillate according to the data.

Eighth, with Bitcoin carrying capacity $K = \$23,809,524$ and applying the least squares method, logistic growth p_L and combined logistic and charged capacitor growth p_{LC} oscillate similarly and minimally at $\alpha = 0.34$ and $\alpha = 0.22$. Gompertz growth p_G oscillates at $\alpha = 0.17$. Charged capacitor growth p_C oscillates moderately at $\alpha = 7200$. The curve $p_{LC\lambda}$ oscillates similarly to Gompertz growth p_G at $\alpha = 3.58$. Combined Gompertz and charged capacitor p_{GC} oscillates at $\alpha = 0.29$. The curve $p_{GC\lambda}$ oscillates at $\alpha = 0.91$. Summing up, the curves p_G, p_{GC}, $p_{LC\lambda}$, $p_{GC\lambda}$ in Figure 4(a) seemingly oscillate according to the data.

Ninth, applying the weighted least squares method when $K = \$23,809,524$, logistic growth p_L again oscillates more at $\alpha = 1.53$. Gompertz growth p_G and combined logistic and charged capacitor growth p_{LC} and $p_{LC\lambda}$ oscillate similarly at $\alpha = 0.11$, $\alpha = 1.66$, and $\alpha = 1.65$. Charged capacitor growth p_C oscillates at $\alpha = 3.13$. Combined Gompertz and charged capacitor growth p_{GC} and $p_{GC\lambda}$ oscillate at $\alpha = 0.15$ and $\alpha = 0.20$. Summing up, the curves p_G, p_{GC}, p_{LC}, $p_{LC\lambda}$, $p_{GC\lambda}$ in Figure 4(c) seem to oscillate according to the data.

Tenth, applying the two Bitcoin carrying capacities $K = \$476,190$ and $K = \$23,809,524$, the bull market local maxima 4,5,6,7,8 and bear market local minima 4,5,6,7,8 are estimated until 2050. These dates depend to a low degree on the growth model carrying capacity K. The magnitudes of the local maxima and local minima of course depend on K, assumed to vary broadly to assess the implications.

If the Bitcoin price evolves until the year 2100 as predicted in this article, that has substantial implications for today's financial system. First, Bitcoin may become a more dominant investment class competing with today's classes, i.e., stocks, bonds, real estate, money market instruments, non-inflationary instruments (minerals, art, etc.), etc. Second, if Bitcoin layer 2 solutions become common, as in El Salvador, such solutions may spread to more countries, and likely first to the world's countries with the weakest currencies or countries without their own currency. The insights in this article may be useful for all humans, i.e., consumers choosing between Bitcoin layer 2 solutions and alternative payment rails, investors, politicians and regulators choosing how to regulate Bitcoin, regulators and developers of asset classes competing with Bitcoin, financial institutions competing with Bitcoin or developing Bitcoin-based instruments, and central banks developing digital currencies.

5. Conclusion

The motivation for this article is the apparently unpredictable Bitcoin price evolution since 3 January 2009, and the need for methods to understand the evolution so far and predict the future evolution. The methods are differential equation growth models incorporating oscillation and lengthening cycles. The analysis is interesting for traders (with time horizons from microseconds to months or years) exchanging Bitcoin with other cryptocurrencies, fiat currencies and asset classes, and savers and investors choosing Bitcoin as a mid term or long term store of value. The article is also relevant for regulators, central banks administering and developing competing currencies with specifically designed characteristics, banks offering competing financial products, collective units assessing whether to offer Bitcoin transactions, and countries assessing whether to accept/reject Bitcoin mining and trading, and whether to accept/reject Bitcoin as legal tender. Regulators want to understand Bitcoin to determine where and how Bitcoin trading and investing can occur, which Bitcoin-related financial products can be developed, how Bitcoin can interact and operate within the existing financial system, and which risk factors are involved. The study is unique in that a minority of other studies account for the dynamics of the Bitcoin price evolution with differential time equations. Further uniqueness consists in incorporating oscillation and lengthening cycles into growth models.

One of the main contributions of this article is to explain the Bitcoin price and predict its future evolution. The past evolution has been embedded within a structure of growth subject to damped oscillations and lengthening cycles. Future bull market maxima and bear market minima are predicted. Earlier studies mostly apply other methods to predict the Bitcoin price, or compare the accuracy of different prediction models, see e.g., Jana et al. (2021); Roy et al. (2018). This article develops differential equations which is uncommon in the literature. Differential time equations enable a different kind of dynamic understanding and explanation, which furthermore enable prediction. The differential equations assume Bitcoin price growth towards two different carrying capacities, subject to damped oscillations and lengthening cycles. Existing studies, e.g., K. S. Chen and Huang (2020); Wang and Wang (2020), capture some aspects of differential equations such as volatility, Bitcoin trading volume, market sentiment, etc. This article additionally incorporates oscillations which express the strength of past and future bull and bear markets, overall approaching one of two different Bitcoin carrying capacities. The authors believe that past studies unsatisfactorily, or at least differently, predict the Bitcoin price in future bull and bear markets. Acknowledging that the Bitcoin price, according to the best models developed in this article, is more influenced by recent data than early data, this article also adopts the weighted least squares method to estimate the parameters. Other studies incorporate the volatility in the models, see e.g., K. S. Chen and Huang (2020); Jaquart et al. (2021). This article furthermore uses a wider time range of the past Bitcoin prices to identify the optimum model parameters, i.e., 23 July 2010–21 June 2021, than has been common elsewhere in the literature, benefiting from more time having elapsed since Bitcoin's emergence. Earlier studies mostly apply shorter data time ranges, see e.g., Caporale et al. (2019); Cocco et al. (2021); Cretarola and Figa-Talamanca (2021); Gupta and Nain (2021).

The parameters in the differential equation growth models are estimated with the least squares method against the 23 July 2010–21 June 2021 empirical data. The weighted least squares method is applied to account for heteroscedasticity. Logistic growth, Gompertz growth, charged capacitor growth, and two hitherto unknown combinations of these are merged with oscillation and damped lengthening cycles for increased realism.

For each of the five models the growth rate is estimated. Logistic growth is initially slow and eventually quick towards the asymptote. Gompertz growth is initially quick and thereafter slow.

Charged capacitor growth is initially too quick and thereafter too slow. As theoretically novel contributions, logistic and Gompertz growth combined with charged capacitor growth exhibit intermediate growth rates, depending on an additional adjustment parameter which weighs the combination. This parameter is determined optimally (using the least squares method and the weighted least squares method) and by assumption, yielding seven growth curves in addition to the empirical curve.

The three bull market local maxima and the three bear market local minima in the available empirics are used to estimate the scaling of the inverse of the cycle length of the sine oscillations, and the scaling of the inverse of the degree of lengthening of each subsequent cycle. Two additional parameters are estimated, i.e., the oscillation amplitude, which expresses the strength of the bull and bear markets, and the start time adjustment parameter for the sine oscillations.

Gompertz growth tracks the growth and oscillations in the empirical data quite well, and tracks the early data better with the weighted least squares method which weighs the early data more heavily. Gompertz growth combined with charged capacitor growth tracks the early data even better since initial growth is quicker. Logistic growth is too slow to track the early empirical data, even when applying the weighted least squares method. Logistic growth combined with charged capacitor growth to some extent tracks the early data. Pure charged capacitor growth is judged to be least realistic.

Six of the curves (abandoning pure charged capacitor growth) are used to estimate the expected value $\pm$ the standard deviation of the dates of the future bull market local maxima and bear market local minima. These dates depend to a low degree on the growth model carrying capacities, approached asymptotically. The magnitudes of the bull market local maxima depend indeed on the two carrying capacities. When the carrying capacity is $476,190 to reflect the current market capitalization $10 trillion of gold, the future bull market local maxima and bear market local minima are lower than when the carrying capacity is $23,809,524 to reflect a $500 trillion market capitalization. The large standard deviations in the estimates are common for new assets in their early stages, and reflect the different predictions of the various models.

Modeling the Bitcoin price as oscillatory growth does not mean that the Bitcoin price can be expected to eventually stabilize towards a horizontal asymptote in the long run. The authors expect growth models to describe the Bitcoin price over the next few bull market local maxima towards various hypothetical carrying capacities. As cryptocurrency markets mature, at some point growth models will become less descriptive. Then alternative models may come into play. Examples of other kinds of evolution are the price fluctuations of more mature asset classes such as gold, stocks, bonds and real estate over the last centuries. Competition with other asset classes and means of exchange, and governmental policies, may increasingly impact the future Bitcoin price.

The implications of the study for all market participants are to be especially cognizant of Gompertz growth combined with charged capacitor growth of the Bitcoin price, and to realize that no growth is unlimited forever. Short term traders should focus on the large standard deviations which may indicate where to impose stop loss orders. Long term investors can focus less on the standard deviations and more on the Bitcoin price Gompertz growth compared with the potential growth of competing asset classes. Regulators focus on the stability and legality of the financial system which suggests a focus on the standard deviations and the fluctuations between the bull market maxima and bear market minima. Central banks focus on financial stability, which relates to inflation, unemployment, interest rates, and exchange rates. They should adjust the money supply of a fiat currency or a specifically designed central bank digital currency while acknowledging potential competition from a fixed supply and highly volatile cryptocurrency. Banks should adjust their competing financial products to account for the volatility and potential growth of the Bitcoin price. Collective units such as firms, institutions, governmental units (e.g., tax authorities), and countries need to account for the standard deviations and fluctuations of the Bitcoin price in order to determine whether to accept or reject Bitcoin transactions. For example, El Salvador addresses this by pricing goods and services in US$ while accepting Bitcoin transactions.

Future research may extend the analysis to other cryptocurrencies (e.g., Ethereum, Cardano, Polkadot, Chainlink) or other phenomena exhibiting growth. Other aspects to include are Bitcoin's hash rate, mining difficulty, network value to transactions, active addresses and new addresses, on chain transaction volume, Bitcoin's electricity consumption, renewable energy consumption, institutional investors, and other assets such bonds and stocks. The five models in this article may be generalized to include more parameters, and may be merged with other models, e.g., the stock-to-flow model, machine learning, neural networks, deep learning, and econometrics. The models may incorporate regulatory intervention, the policies and attitudes of various countries, and competition with other currencies and asset classes. Further extensions can be made to extreme value theory and stochastic analysis with probability distributions.

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Author details
Guizhou Wang[1]
E-mail: guizhou.wang@uis.no
ORCID ID: http://orcid.org/0000-0001-5297-8105
Kjell Hausken[1]
E-mail: kjell.hausken@uis.no
ORCID ID: http://orcid.org/0000-0001-7319-3876
[1] Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway.

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Notes
1. https://8marketcap.com/metals. Retrieved February 20, 2022.
2. https://messari.io/asset/bitcoin/historical. Retrieved February 20, 2022.
3. The Mathematica 13 software package (www.wolfram.com) was used. The codes used for the simulations are available from the authors upon request.

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The article contains no associated data. All data generated or analyzed during this study is included in this published article.

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Article

Governmental Taxation of Households Choosing between a National Currency and a Cryptocurrency

Guizhou Wang and Kjell Hausken *

Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway; pobewang@outlook.com
* Correspondence: kjell.hausken@uis.no; Tel.: +47-51-831632; Fax: +47-51-831550

Abstract: A game between a representative household and a government was analyzed. The household chose which fractions of two currencies to hold, e.g., a national currency such as a Central Bank Digital Currency (CBDC) and a global currency such as Bitcoin or Facebook's Diem, and chose the tax evasion probability for each currency. The government chose, for each currency, the probability of detecting and prosecuting tax evasion, the tax rate, and the penalty factor imposed on the household when tax evasion was successfully detected and prosecuted. The household's fraction of the national currency, the government's monitoring probability of the national currency, and the penalty factor imposed on the global currency, increased in the household's Cobb Douglas output elasticity for the national currency. The household's probabilities of tax evasion on both currencies increased in the government's Cobb Douglas output elasticity for the national currency. The government's taxation on both currencies decreased in the output elasticity for the national currency. High output elasticity for the national currency eventually induced the government to tax that currency more than the global currency. The household's probability of tax evasion on the global currency increased in the government's output elasticity for that currency. The household was less (more) likely to tax evade on the national (global) currency if the government valued taxation and penalty on the national (global) currency. The results are illustrated numerically where each of the eight parameter values was varied relative to a benchmark.

Keywords: digital currency; cryptocurrency; CBDC; Bitcoin; game theory; taxation; household; government

JEL Classification: C72; H26

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1. Introduction

1.1. Background

Digital currencies are receiving increasing attention as central banks launch Central Bank Digital Currencies (CBDCs) (https://cbdctracker.org/, retrieved 7 April 2021), companies develop currencies (e.g., Facebook's Diem), and individuals, institutions, and others (e.g., Tesla, Grayscale, MicroStrategy, Square) buy Bitcoin and other cryptocurrencies. As of 7 April 2021, 9162 cryptocurrencies contributed to a market cap of $1.9 trillion (https://coinmarketcap.com/, retrieved 7 April 2021).

Cryptocurrencies work via the distributed ledger technology or blockchain. Blockchain is a decentralized technology spread across many nodes that manage and record transactions. The transactions are stored in multiple nodes that are permanent, verifiable, and unchangeable. Cryptocurrencies have no physical form, are typically not issued by a central authority, and are controlled through networks with varying degrees of decentralization. The first cryptocurrency was Bitcoin that emerged through the genesis block 3 January 2009 at 18:15:05 UTC.

Advantages of cryptocurrency included typical avoidance of inflation (e.g., through a fixed limited supply for Bitcoin or burning coins for the Binance coin), self-governance, disintermediation (no central party), security, privacy, cost-effective transaction modes (especially for cross borders payments), instant or quick, and 24/7/365 accessibility, etc. Disadvantages of cryptocurrencies include possible use for illegal transactions (e.g., by applying privacy coins such as Monero), challenges of market fluctuations, no security or remedy in case of loss, limited scalability for some cryptocurrencies, etc.

Cryptocurrencies, and especially privacy coins like Monero, Verge, Zcash, etc., might enable tax evasion, which challenges regulators. Households might correctly or incorrectly assess and compare governments' abilities to monitor storage and transactions and enforce regulations for cryptocurrencies and government-issued currencies. Marian [1] suggests that cryptocurrencies could replace tax havens as the weapon-of-choice for tax-evaders.

These developments induce households to determine what fractions of each currency to hold, how to evade tax on each currency, and induce governments to determine how to tax, monitor tax evasion, and punish tax evasion, on each currency.

1.2. Contribution

This article models a game between a representative household and a government. The household chooses three strategies, i.e., the fractions to hold and the probabilities of tax evasion for two currencies. The government chooses six strategies, i.e., tax rates, tax monitoring, and punishments for tax evasion, for two currencies. The national currency offers the most common usage within a nation, e.g., purchasing and selling goods and services, paying taxes, and saving for retirement. The global currency generally offers opportunities beyond the national borders, e.g., user autonomy, discretion, peer-to-peer focus, and tax evasion.

The players' choices cause the household to assess four fractions for each currency; i.e., legally permitted for the household to keep, successful tax evasion, unsuccessful tax evasion, and the tax fraction paid voluntarily. The household has a Cobb Douglas expected utility with one output elasticity for each currency. The government has a Cobb Douglas expected utility with four output elasticities, i.e., one output elasticity for each currency reflecting its identification with the household, and one output elasticity for each currency reflecting its preference for taxation and penalties on unsuccessful tax evasion.

This article proposes a new way to formulate the government's utility. The government represents its households. Hence, the government is to some extent assumed to identify with each household, and benefits when the household benefits. The government also benefits from the household paying taxes, and benefits from the household paying a penalty when the government successfully monitors, and thus detects and prosecutes tax evasion.

The article analytically determines how eight parameters, intended to capture the phenomenon, impact the players' nine strategies and two expected utilities. Sensitivity analysis shows the variation in the government's monitoring probabilities, tax rates, penalty factors, and expected utility, and the household's fractions of the two currencies and the probability of tax evasion for each currency, as each parameter value varies relative to a benchmark. The results are discussed in terms of economic intuition and policy implications. The article contributes to all four areas of the literature reviewed in the next section.

1.3. Literature

The literature is divided into four areas, i.e., CBDC and cryptocurrencies, currency competition, game theory analyses, and governmental taxation.

1.3.1. CBDC and Cryptocurrencies

This article relates to this literature by considering one national currency that can be interpreted to be a CBDC and one global currency that can be interpreted to be a cryptocurrency.

Blakstad and Allen [2] summarized the possibilities and risks offered by cryptocurrencies for central banks and individuals.

Brunnermeier and Niepelt [3] developed a generic framework of money, liquidity, seignories rents, and financial frictions. They provided sufficient conditions for the equivalence of monetary systems. They proposed that the introduction of CBDC could reduce run risk on banks, rather than increasing it.

Asimakopoulos et al. [4] developed a Dynamic Stochastic General Equilibrium (DSGE) model to assess the economic consequences of cryptocurrencies. Applying Bayesian techniques using US and crypto markets monthly data for the period 2013:M6-2019:M3, they found a strong substitution effect between the real balances of government currency and the real balances of cryptocurrency.

Sapkota and Grobys [5] divided cryptocurrencies into privacy and non-privacy coins. They explored whether asset market equilibria exist in the cryptocurrency markets. By analyzing ten cryptocurrencies with the highest market capitalization in each submarket in the 2016–2018 period, they found that privacy coins and non-privacy coins expressed two distinct unrelated market equilibria.

Allen et al. [6] enumerated the fundamental technical design challenges facing CBDC designers, with a particular focus on performance, privacy, and security. They summarized the main potential benefits of CBDC, namely, efficiency, a broader tax base, flexible monetary policy, payment backstop, and financial inclusion.

1.3.2. Currency Competition

This article relates to this literature by considering competition between one national currency and one global currency, in the sense that each household chooses optimally how much to hold of each.

Gandal and Halaburda [7] evaluated the impact of network effects on competition in the cryptocurrency market. They found no winner-take-all effects in the early period since November 2013 (when data collection started) until April 2014, but strong network effects and winner-take-all dynamics from April 2014 until February 2016.

Benigno [8] stated that multiple currencies could compromise the primary function of a central bank. Additionally, they found that with many competing currencies issued by profit-maximizing actors, both the nominal interest rate and the inflation could not be manipulated, but were instead determined by structural factors, such as the intertemporal discount factor, the exit rate, and the fixed entry cost.

Fernández-Villaverde and Sanches [9] considered competition between privately issued fiat currencies. They found that an equilibrium existed in which price stability was consistent with competing private monies, and also, that a continuum of equilibrium trajectories existed with the property, such that the value of private currencies monotonically converged to zero.

Benigno et al. [10] evaluated a two-country economy with complete markets, two national currencies, and a global cryptocurrency. They suggest that deviating from interest rate equality might imply approaching the zero lower bound or the abandonment of the national currency, referred to as Crypto-Enforced Monetary Policy Synchronization (CEMPS). Hence, the impossibility of jointly ensuring a fixed exchange rate, free capital flows, and an independent monetary policy (the classic Impossible Trinity) becomes even less reconcilable.

1.3.3. Game Theory Analyses

This article relates to this literature by considering a game between a government and a representative household.

Wang [11] set up a game theory model to analyze the implications of tax evasion for the optimal design of CBDC. He discussed several scenarios where CBDC had different anonymity compared to cash. For example, if CBDC offered less anonymity than cash, introducing CBDC would decrease tax evasion. If CBDC provided a high level of anonymity but low interest rate, then it would decrease the agents' output. However, if CBDC offered low anonymity and a high interest rate, it would increase the output and aggregate the welfare.

Zhang et al. [12] assessed the tax preferences of enterprise income for comprehensive utilization of resources. They theoretically explored the game tax preference policy for energy conservation and emission reduction. They found that increasing camouflage cost and expected cost of risk could effectively prevent the generation of enterprise frauds.

Caginalp and Caginalp [13] determined the game theory equilibria for cryptocurrencies. The players divided their assets between the home currency and the cryptocurrency. The government decided the probability of seizing a fraction of the players' assets. The conditions for existence and uniqueness of Nash equilibria were established.

Wang and Hausken [14] analyzed competition between a national currency and a global currency, both of which had specific characteristics in an economy. The replicator equation was used to illustrate how conventionalists (which prefer to be in the majority) tend to compete against the pioneers and criminals (which prefer to be in the minority), under various conditions.

Welburn and Hausken [15,16] theoretically analyzed the economic crises game, assuming six kinds of players, i.e., countries, central banks, banks, firms, households, and financial inter-governmental organizations. Players have strategies such as setting interest rates, lending, borrowing, producing, consuming, investing, importing, exporting, defaulting, and penalizing default.

1.3.4. Taxation

This article related to this literature by considering how a government taxes, monitors, and punishes tax evasion, and how a representative household might evade tax on two currencies.

Reviews

Alm [17] reviewed how to measure, explain, and control tax evasion. The examples were to analyze shadow economies, experimental methods, survey evidence, assess currency demand, and trace evasion in transactions financed by currencies.

Andreoni et al. [18] theoretically and empirically reviewed the literature on tax compliance. They pointed out that the theoretical models only served as rough guides for empirical research. They recommended more work on exploring the psychological, moral, and social impacts on tax compliance activities, more attention to the dynamic and complex institutional framework of tax compliance, and more empirical research outside the USA jurisdiction.

Governmental Taxation

Brito et al. [19] analyzed the optimal income tax problem when consumers work for many periods. The results indicated that when the government commits to future tax schedules, intertemporal nonstationary tax schedules could relax the self-selection constraints and lead to Pareto improvements.

Lai and Liao [20] investigated the optimal capital income taxation in heterogeneous agent economies, featuring endogenous government spending. They pointed out that the long-run optimal capital tax rate should not be zero when the competitive equilibrium risk-free interest rate differed from the subjective time discount rate. The results could be extended to a wide range of model economies.

Liu [21] explored how government preferences affected the choices of capital tax rates in the presence of tax competition. The article suggests that countries emphasizing economic development tend to choose lower corporate income tax rates than countries emphasizing regional equality.

Raurich [22] developed an endogenous growth model with an endogenous labor supply. He pointed out that the dynamic equilibrium might exhibit local indeterminacy when labor income is heavily taxed.

Economides et al. [23] presented a general equilibrium model of endogenous growth with productive and non-productive public goods and services. They solved for Ramsey second-best optimal policy. The findings differed from the benchmark case of the social planner's first-best allocation and depended crucially on whether public goods and services were subject to congestion.

Chen and Guo [24] explored the theoretical interrelations between progressive income taxation and macroeconomic (in)stability. The results showed that progressive taxation operated like an automatic destabilizer that generated equilibrium indeterminacy and belief-driven fluctuations in the economy, which differed from traditional Keynesian-type stabilization policies.

Bacchetta and Perazzi [25] discussed a monetary reform in Switzerland. Based on a simple infinite-horizon open-economy model, they pointed out that a tradeoff existed between a reduction in distortionary labor taxes and an increase in the opportunity cost of holding money.

Tax Evasion and Punishment

Becker [26] and Hausken and Moxnes [27] recommended optimal public and private policies to combat illegal behavior. They showed that optimal enforcement depended on the cost of catching and convicting offenders, the nature of punishments, and the responses of offenders to changes in enforcement. Similarly, this article showed how households responded to punishments for tax evasion.

Allingham and Sandmo [28] explored static and dynamic aspects of the taxpayer's decisions on tax evasion. In the static model, they found that the penalty rate and the probability of detection were substitutes for each other. In the dynamic analysis, they showed that consistent rational individuals always declared more taxes than myopic short-sighted tax-evading individuals. Extending Allingham and Sandmo's [28] work, Yitzhaki [29] showed that if a penalty was imposed on the evaded tax, no contradiction existed between an income and a substitution effect. Furthermore, if the taxpayer had absolute risk aversion, which decreased with income, increased taxation causes decreased tax evasion. This article supported the finding, when varying how the government identified with the household's output elasticity for the national currency (see Section 4) and when varying the government's elasticity for the national currency, when valuing taxation and penalty on unsuccessful tax evasion (see Section 4), and otherwise supported the opposite result or that one variable did not vary when the other variable varied.

Myles and Naylor [30] set out a model of tax evasion that captured a benefit of conforming with non-evaders and of adhering to the social custom of non-evasion. They showed that both equilibria with no evasion and with taxpayers choosing to evade could exist. Similarly, this article showed how households might respond differently to the government's taxation, monitoring, and punishment.

Slemrod and Yitzhaki [31] presented theoretical models that integrate tax avoidance and evasion into the overall decision problem faced by taxpayers. They also developed a taxonomy of efficiency costs and introduced a general theory of optimal tax systems. They found that when the tax structure changed, individuals might change their consumption basket.

Experimental Work on Tax Evasion

Torgler [32] summarized experimental findings on tax morale and tax compliance, focusing on personal income tax morale, and social and institutional factors. He argued for the infeasibility of testing the predictions of the level of tax compliance models. In addition, social and institutional factors were important factors on tax compliance.

Kleven et al. [33] presented a tax enforcement field experiment in Denmark. They found that tax evasion was near zero for income subject to third-party reporting, and was much higher for self-reported income. In addition, marginal tax rates impacted tax evasion positively for self-reported income, but the effect was small compared to legal avoidance and behavioral responses. Additionally, prior audits and threat-of-audit letters significantly impacted self-reported income, but did not impact third-party reported income.

Empirical Work on Tax Evasion

Ariyo and William [34] estimated that for 1975–2010, 42.54–79.32% of the Nigerian underground economy and tax evasion constituted 2.09–6.75% of the Gross Domestic Product.

Bittencourt et al. [35] found for 150 cases that less (more) financial development and a more (less) inflation caused a bigger (smaller) shadow economy with related tax evasion, during 1980–2009.

Hanlon et al. [36] assessed "round tripping" tax evasion where funds in offshore tax havens were invested in U.S. securities markets. They found that the incentives to evade U.S. taxation and expected costs of evasion detection affected the amount of foreign portfolio investment in U.S. debt and equity markets.

Tax Morale and Alternatives to Expected Utility Theory

Luttmer and Singhal [37] pointed out that apart from tax tools like tax rate, detection probality, and penalties imposed if evasion was detected, tax morale including nonpecuniary motivations were important factors in tax compliance decisions. Drawing on evidence from experiments, they demonstrated that tax morale operated through many underlying mechanisms.

Dhami and al-Nowaihi [38] contended that the expected utility theory failed to explain tax evasion activities. They found that the cumulative prospect theory provided a much more satisfactory explanation of tax evasion.

1.4. Article Organization

Section 2 presents the model. Section 3 analyzes the model. Section 4 illustrates the solution. Section 5 discusses the results and provides economic intuition and policy implications. Section 6 concludes.

2. The Model

2.1. Two Currencies n and g

Appendix A shows the nomenclature. Consider an economy with two available currencies. The first currency n is national and offers the most common usage, and especially legal usage, within the economy. Examples of usage were for making various purchases or paying taxes. The government has complete control and dominance over the national currency n, e.g., by adjusting tax rates and inflation. We can think of the currency n as a CBDC. The second currency g is global and outside the control of the government. It offers more limited usage, e.g., cannot be used for all kinds of purchases, but offers other opportunities, e.g., user autonomy, discretion, peer-to-peer focus, no banking fees, tax evasion, black market payments, criminal activities, and a potentially high return. We might think of currency g as a cryptocurrency such as Bitcoin, Zcash, or Facebook's Diem.

The household pays taxes for holding the two currencies, and can choose tax evasion with a probability for each currency. If tax evasion is detected and prosecuted by the government, the household has to pay a penalty. Owing to the features of the two currencies, the probabilities of tax evasion, tax rates, probabilities of detecting tax evasion, and penalty factors if tax evasion is detected, generally differ. Figure 1 illustrates the two currencies n and g.

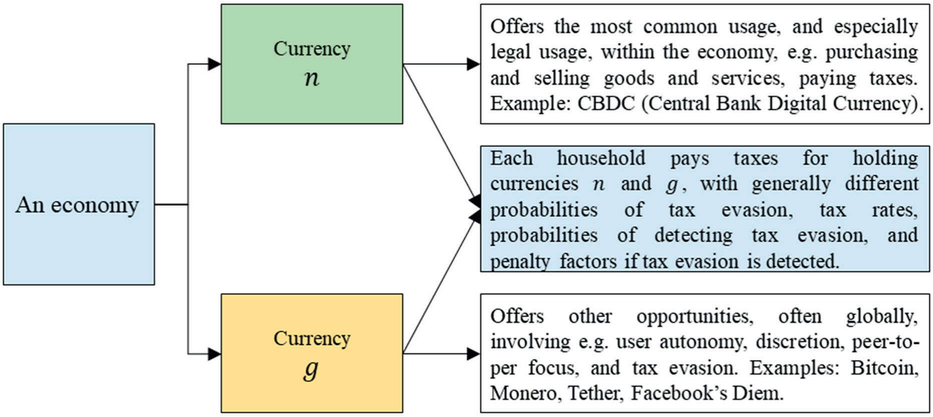

Figure 1. An economy with two currencies n and g.

2.2. Two Kinds of Players: Households and One Government

Consider an economy with a representative household and a government. The household chooses the fraction to hold currency n, causing the remaining fraction to be held in currency g, and chooses the tax evasion probability for each currency. The government is the second player. It completely controls the national currency n, but has no control of the global currency g. However, the government can set the tax rates, the probabilities of detecting tax evasion, and the penalty factors if tax evasion is detected, for both currencies. We consider a non-cooperative one-period game. The households and government choose their strategies simultaneously and independently. The players are interlinked as in Figure 2.

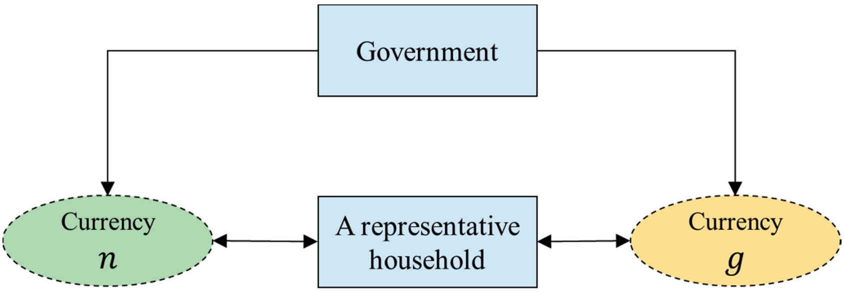

Figure 2. The government and a representative household involved in a national currency n and a global currency g.

2.3. The Players' Strategic Choices

The representative household simultaneously chooses three strategies to maximize its expected utility U. It chooses its fraction x, $0 \leq x \leq 1$ of currency n, causing the remaining fraction $1 - x$ to be held in currency g. Additionally, it chooses the tax evasion probability p_j, $0 \leq p_j \leq 1$, for currency j, $j = n, g$.

The government chooses six strategies simultaneously to maximize its expected utility u. It chooses the probability m_j, $0 \leq m_j \leq 1$ of detecting and prosecuting tax evasion on currency j. Additionally, it chooses the tax rate $\tau_j, \tau_j \geq 0$ for currency j. Finally, it chooses the penalty factor $P_j, P_j \geq 0$, imposed on each household when tax evasion is successfully detected and prosecuted on currency j, $j = n, g$. Table 1 shows the players' strategies descriptions and strategy sets.

Table 1. Player descriptions and strategy sets.

Player	Strategies Description	Strategy Set
A representative household	Chooses its fraction $x,\ 0 < x \leq 1$, of currency n, causing the remaining fraction $1 - x$, to be held in currency g. Chooses the tax evasion probability p_n for currency n and tax evasion probability p_g for currency g.	$\{x, p_n,\ p_g\}$
Government	Chooses the probability m_j,$0 \leq m_j \leq 1$, of detecting and prosecuting tax evasion on currency j. Chooses the tax rate $\tau_j, \tau_j \geq 0$, for currency j. Chooses the penalty factor $P_j, P_j \geq 0$, imposed on each household when tax evasion is successfully detected and prosecuted on currency $j,\ j = n, g$.	$\left\{ \begin{array}{c} m_n,\ m_g, \tau_n, \\ \tau_g, P_n, P_g \end{array} \right\}$

2.4. *The Household's Strategies and Expected Utility*

Assume that a representative household evades taxes on currency j with probability p_j, $0 \leq p_j \leq 1$, $j = n, g$, which is detected and prosecuted by the government with probability m_j, $0 \leq m_j \leq 1$. With a tax rate τ_j, $0 \leq \tau_j \leq 1$, for currency j, the household's expected tax payment fraction on currency j is $\left(1 - p_j\right)\tau_j$, paid voluntarily. With zero government detection $m_j = 0$, the household's expected income fraction from tax evasion on currency j is $p_j\tau_j$. With 100% government detection and prosecution $m_j = 1$, the household's expected income fraction from tax evasion on currency j is 0. Generally, the household's expected income fraction from tax evasion on currency j is $\left(1 - m_j\right)p_j\tau_j$, i.e., successful tax evasion. Hence, the household's expected expense fraction without penalty from unsuccessful tax evasion on currency j is $m_jp_j\tau_j$. We assume that the government penalizes unsuccessful tax evasion by adjusting $m_jp_j\tau_j$ in two ways. First, $m_jp_j\tau_j$ is multiplied with a penalty factor P_j, $P_j \geq 0$, chosen by the government as a free choice variable. Second, $P_jm_jp_j\tau_j$ is assumed to depend on the representative household's tax evasion probability p_j in a more flexible manner by replacing p_j with $p_j^{\lambda_j}$, where p_j is a parameter, which gives $m_j\tau_jP_jp_j^{\lambda_j}$ as the household's expense from unsuccessful tax evasion. We require $\lambda_j \geq 0$ since the household's expected expense for tax evasion should increase as the household's tax evasion probability increases, $\partial\left(m_j\tau_jP_jp_j^{\lambda_j}\right)/p_j \geq 0$. Tax evasion should not be beneficial. We might interpret $P_jp_j^{\lambda_j-1}$ as the government's penalty, which is multiplied with the household's expected expense fraction $m_jp_j\tau_j$ from unsuccessful tax evasion on currency j, to give $m_j\tau_jP_jp_j^{\lambda_j}$. Hence, the household keeps a fraction

$$f_j = 1 - \left(1 - p_j\right)\tau_j - m_j\tau_jP_jp_j^{\lambda_j} \tag{1}$$

of currency j, which is multiplied with the fraction x of currency n, and multiplied with the fraction $1 - x$ of currency g, to determine how much of the two currencies n and g the household owns. The fraction f_j is positive when $P_j \leq \frac{1-\left(1-p_j\right)\tau_j}{m_j\tau_jp_j^{\lambda_j}}$, and is otherwise negative.

We apply the Cobb Douglas expected utility for both players, since it is widely used within economics and since it explicitly captures tradeoff players strike between multiple conflicting or partly conflicting objectives. For the household that includes which currencies to hold and with which probabilities to tax evade, assume that the household has a Cobb Douglas expected utility with output elasticity α, $0 \leq \alpha \leq 1$, associated with currency n, and $1 - \alpha$ associated with currency g, i.e.,

$$U = \begin{cases} \left(\left(1 - (1 - p_n)\tau_n - m_n \tau_n P_n p_n^{\lambda_n} \right) x \right)^{\alpha} \\ \times \left(\left(1 - (1 - p_g)\tau_g - m_g \tau_g P_g p_g^{\lambda_g} \right) (1 - x) \right)^{1-\alpha} \\ \quad if\ P_j \le \frac{1-(1-p_j)\tau_j}{m_j \tau_j p_j^{\lambda_j+1}}, j = n, g \\ 0\ otherwise \end{cases} \tag{2}$$

where $U = 0$ means that the penalty factor P_j is so high that the household goes into debt. This is illustrated in Figure 3. The household's three free choice variables are its fraction x of currency n, which causes the remaining fraction $1 - x$ to be held in currency g, and its tax evasion probability p_j for currency j, $j = n, g$.

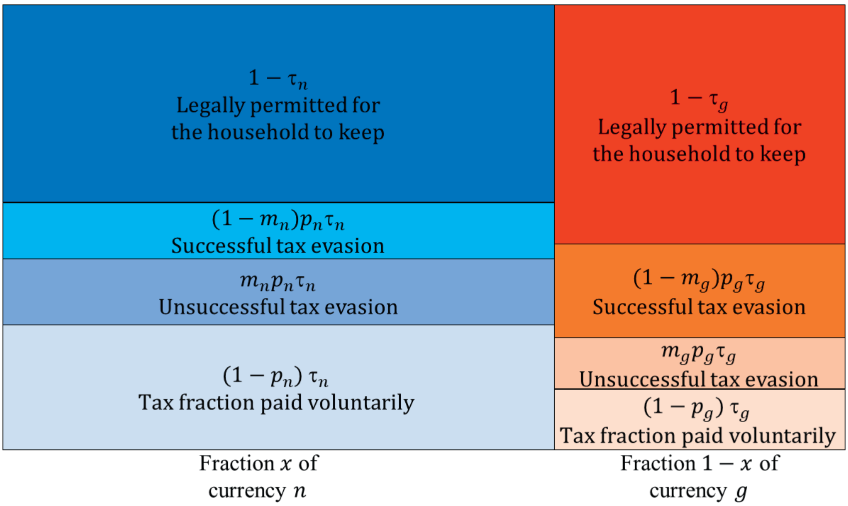

Figure 3. Fractions x and $1 - x$ of the household's currencies n and g, each divided into four subgroup fractions, i.e., $1 - \tau_j$ as legally permitted for the household to keep, $\left(1 - m_j\right) p_j \tau_j$ as successful tax evasion, $m_j p_j \tau_j$ as unsuccessful tax evasion, and $\left(1 - p_j\right) \tau_j$ as the tax fraction paid voluntarily, $j = n, g$.

The output elasticities α and $1 - \alpha$ for the two currencies n and g account in a deep sense for the benefits and costs of holding, acquiring, and transacting with the two currencies. Cryptocurrencies are freely available. Once acquired, no costs exist of holding them, and interest might be earned. If we think of currency g as Bitcoin, these benefits and costs changed since the genesis block in 2009. The early Bitcoin adopters operated in a segmented market, possessing competence beyond the majority of households. Over the last years, the market has broadened, become less segmented, is more easily accessible through multiple entry points, and is more user-friendly. Users learned to use crypto wallets, which are of five types—mobile, desktop, paper, hardware, online, and mobile wallets. Users operate on platforms and exchanges such as ImToken, Metamask, TrustWallet, TokenPlus, Binance, OKEx, Huobi, Coinbase, etc. Users download apps such as Abra from the internet on their cellphone, and create their own cryptocurrency addresses, where they buy, sell, exchange, and earn interest on cryptocurrencies. Buying cryptocurrencies has become similar to buying stocks and is almost costless. Cryptocurrencies are gradually incorporated into the conventional financial system, exemplified with Paypal, which currently offers Bitcoin, Bitcoin Cash, Ethereum, and Litecoin. To the extent the representative household perceives holding a global currency g such as Bitcoin as less straightforward than holding a government-issued national currency n, the household assigns lower output elasticity $1-\alpha$ to the global currency g, and thus higher output elasticity α to the national currency n.

2.5. The Government's Strategies and Expected Utility

The challenge in modeling the government is that it cannot identify 100% with each household individually, because of the collective action dilemma, including the objective of maximizing the expected utility or welfare of all households. The government also cannot minimize the expected utility of each household since then it will not be reelected. Hence, we assume that the government to some extent identifies with and represents each household, and benefits when the household benefits. A straightforward way of accomplishing that objective is to incorporate the household's expected utility U in Equation (2) into the government's expected utility u. That implicitly means that the government to some extent, as determined by the parameters and the players' strategic choices, internalizes all advantages of the household, including the advantage of evading taxes for the household. Since internalizing that advantage cannot be taken too far, we assume that the government also benefits from the household paying taxes, and benefits from the household paying a penalty when the government successfully monitors, and thus detects and prosecutes tax evasion. The government finally has a cost expenditure of choosing the monitoring probability m_j, $j=n,g$. These multiple conflicting or partly conflicting objectives of the government are obtained by assuming a more extensive Cobb Douglas expected utility for the government, expressed per household as

$$U=\begin{cases}\left(\left(1-(1-p_n)\tau_n-m_n\tau_nP_np_n^{\lambda_n}\right)x\right)^{\beta_n} \\ \times\left(\left(1-(1-p_g)\tau_g-m_g\tau_gP_gp_g^{\lambda_g}\right)(1-x)\right)^{\beta_g} \\ \times\left(\left((1-p_n)\tau_n+m_n\tau_nP_np_n^{\lambda_n}\right)x-a_nm_n\right)^{\gamma_n} \\ \times\left(\left((1-p_g)\tau_g+m_g\tau_gP_gp_g^{\lambda_g}\right)(1-x)-a_gm_g\right)^{1-\beta_n-\beta_g-\gamma_n} \\ \quad if\ P_j\leq\frac{1-(1-p_j)\tau_j}{m_j\tau_jp_j^{\lambda_j}}, j=n,g \\ 0\ otherwise\end{cases} \tag{3}$$

which has four multiplicative terms. The first two terms in Equation (3) are equivalent to the two terms in Equation (2), except that α and $1-\alpha$ are replaced with β_n and β_g, respectively, $0\leq\beta_n,\beta_g\leq1$. That replacement means that although the government identifies with the household, the government is enabled to prioritize differently and have other output elasticities for the two currencies n and g than the household. For the special case when the government has the same ratio $\alpha/(1-\alpha)=\beta_n/\beta_g$ between the two currencies n and g as the household, we get

$$\frac{\alpha}{1-\alpha}=\frac{\beta_n}{\beta_g}\Leftrightarrow\alpha=\frac{\beta_n}{\beta_n+\beta_g} \tag{4}$$

which we do not require the government to adhere to. The third and fourth terms in Equation (3), for currencies n and g, respectively, express that the government maximizes the sum of two terms and a subtracted third term raised to the output elasticities γ_n and $1-\beta_n-\beta_g-\gamma_n$, respectively, $0\leq\gamma_n\leq1, 0\leq1-\beta_n-\beta_g-\gamma_n\leq1$, for currencies n and g. Term 1 is the household's tax fraction paid voluntarily, multiplied with the currency fraction, i.e., $(1-p_n)\tau_nx$ and $\left(1-p_g\right)\tau_g(1-x)$, for currencies n and g, respectively. Term 2 is the household's unsuccessful tax evasion multiplied with the penalty and currency fraction, i.e., $m_n\tau_nP_np_n^{\lambda_n}x$ and $m_g\tau_gP_gp_g^{\lambda_g}(1-x)$, for currencies n and g, respectively. Term 3 is the household's unit cost a_j, $a_j\geq0$, of choosing the monitoring probability m_j, multiplied with m_j, $j=n,g$. Since m_j is a probability, the unit cost a_j has to be scaled so that $0\leq m_j\leq1$.

The government's six free choice variables are its probability m_j of detecting and prosecuting tax evasion on currency j, the tax rate τ_j on currency j, and the penalty factor P_j imposed on each household when tax evasion is successfully detected and prosecuted on currency j, $j = n, g$. The government and each household choose their free choice variables simultaneously and independently. Analyzing such a stationary situation reflects reality in the sense that governments in general, and households over time, adapt their preferences and strategies to each other, making it difficult to state that one player chooses a strategy over some other player.

3. Analyzing the Model

3.1. Analyzing the Household

Appendix B shows that the household chooses to hold the fraction

$$x = \begin{cases} \alpha \; if \; P_j \leq \frac{1-(1-p_j)\tau_j}{m_j \tau_j p_j^{\lambda_j}}, j = n, g \\ undetermined \; otherwise \end{cases} \tag{5}$$

of currency n, and thus the remaining fraction $1 - x$ of currency g, and chooses the probability

$$p_j = \begin{cases} \frac{1}{(m_j P_j \lambda_j)^{1/(\lambda_j - 1)}} \; if \; P_j \leq \frac{1-(1-p_j)\tau_j}{m_j \tau_j p_j^{\lambda_j}} \; and \; 0 \leq p_j \leq 1, j = n, g \\ undetermined \; or \; 1 \; otherwise \end{cases} \tag{6}$$

of tax evasion on currency j, $j = n, g$.

3.2. Analyzing the Government

Appendix C shows that the government chooses the free choice variables

$$\begin{aligned} & m_n = \frac{x}{a_n}, m_g = \frac{1-x}{a_g}, \tau_n = \frac{\gamma_n}{(1-p_n)(\beta_n+\gamma_n)}, \tau_g = \frac{1-\beta_n-\beta_g-\gamma_n}{(1-p_g)(1-\beta_n-\gamma_n)}, \\ & P_n = \frac{a_n(1-p_n)\beta_n}{p_n^{\lambda_n} x \gamma_n}, P_g = \frac{a_g(1-p_g)\beta_g}{p_g^{\lambda_g}(1-x)(1-\beta_n-\beta_g-\gamma_n)}, \\ & 0 \leq m_n \leq 1 \Leftrightarrow a_n \geq x \, , \, 0 \leq m_g \leq 1 \Leftrightarrow a_g \geq 1-x \, , \, 0 \leq \tau_n \leq 1 \Leftrightarrow 0 \leq p_n \leq \frac{\beta_n}{\beta_n+\gamma_n}, \\ & 0 \leq \tau_g \leq 1 \Leftrightarrow 0 \leq p_g \leq \frac{\beta_g}{1-\beta_n-\gamma_n} \end{aligned} \tag{7}$$

3.3. Analyzing the Household and Government Together

Property 1. *The household's and the government's strategies are*

$$\begin{aligned} & x = \alpha, p_n = \frac{\lambda_n \beta_n}{\lambda_n \beta_n + \gamma_n}, p_g = \frac{\lambda_g \beta_g}{1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g}, \\ & m_n = \frac{\alpha}{a_n}, m_g = \frac{1-\alpha}{a_g}, \tau_n = \frac{\lambda_n \beta_n + \gamma_n}{\beta_n + \gamma_n}, \tau_g = \frac{1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g}{1-\beta_n-\gamma_n}, \\ & P_n = \frac{a_n}{\lambda_n \alpha}\left(\frac{\lambda_n \beta_n}{\lambda_n \beta_n + \gamma_n}\right)^{1-\lambda_n}, P_g = \frac{a_g}{\lambda_g(1-\alpha)}\left(\frac{\lambda_g \beta_g}{1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g}\right)^{1-\lambda_g}, \\ & U = u = 0, a_n \geq \alpha, \; a_g \geq 1-\alpha, \; 0 \leq \lambda_j \leq 1, \; j = n, g \end{aligned} \tag{8}$$

Proof. Appendix D. □

Property 2. (1): $\frac{\partial x}{\partial \alpha} \geq 0$, $\frac{\partial(1-x)}{\partial \alpha} \leq 0$, $\frac{\partial m_n}{\partial \alpha} \geq 0$, $\frac{\partial m_g}{\partial \alpha} \leq 0$, $\frac{\partial P_n}{\partial \alpha} \leq 0$, $\frac{\partial^2 P_n}{\partial \alpha^2} \geq 0$, $\frac{\partial P_g}{\partial \alpha} \geq 0$, $\frac{\partial^2 P_g}{\partial \alpha^2} \geq 0$, $\frac{\partial p_n}{\partial \alpha} = \frac{\partial p_g}{\partial \alpha} = \frac{\partial \tau_n}{\partial \alpha} = \frac{\partial \tau_g}{\partial \alpha} = 0$. (2): $\frac{\partial p_n}{\partial \lambda_n} \geq 0$, $\frac{\partial^2 p_n}{\partial \lambda_n^2} \geq 0$, $\frac{\partial \tau_n}{\partial \lambda_n} \geq 0$, $\frac{\partial x}{\partial \lambda_n} = \frac{\partial(1-x)}{\partial \lambda_n} = \frac{\partial p_g}{\partial \lambda_n} = \frac{\partial m_n}{\partial \lambda_n} = \frac{\partial m_g}{\partial \lambda_n} = \frac{\partial \tau_g}{\partial \lambda_n} = \frac{\partial P_g}{\partial \lambda_n} = 0$. (3): $\frac{\partial p_g}{\partial \lambda_g} \geq 0$, $\frac{\partial^2 p_g}{\partial \beta_g^2} \geq 0$, $\frac{\partial \tau_g}{\partial \lambda_g} \geq 0$, $\frac{\partial x}{\partial \lambda_g} = \frac{\partial(1-x)}{\partial \lambda_g} = \frac{\partial p_n}{\partial \lambda_g} = \frac{\partial m_n}{\partial \lambda_g} = \frac{\partial m_g}{\partial \lambda_g} = \frac{\partial \tau_n}{\partial \lambda_g} = \frac{\partial P_n}{\partial \lambda_g} = 0$. (4): $\frac{\partial p_n}{\partial \beta_n} \geq 0$, $\frac{\partial^2 p_n}{\partial \beta_n^2} \leq 0$, $\frac{\partial p_g}{\partial \beta_n} \geq 0$, $\frac{\partial^2 p_g}{\partial \beta_n^2} \geq 0$, $\frac{\partial \tau_n}{\partial \beta_n} \leq 0$, $\frac{\partial^2 \tau_n}{\partial \beta_n^2} \geq 0$, $\frac{\partial \tau_g}{\partial \beta_n} \leq 0$, $\frac{\partial^2 \tau_g}{\partial \beta_n^2} \leq 0$, $\frac{\partial P_n}{\partial \beta_n} \geq 0$, $\frac{\partial^2 P_n}{\partial \beta_n^2} \leq 0$, $\frac{\partial P_g}{\partial \beta_n} \geq 0$, $\frac{\partial x}{\partial \beta_n} = \frac{\partial(1-x)}{\partial \beta_n} = \frac{\partial m_n}{\partial \beta_n} = \frac{\partial m_g}{\partial \beta_n} = 0$. (5): $\frac{\partial p_g}{\partial \beta_g} \geq 0$, $\frac{\partial^2 p_g}{\partial \beta_g^2} \geq$

$0, \frac{\partial \tau_g}{\partial \beta_g} \leq 0, \frac{\partial P_g}{\partial \beta_g} \geq 0, \frac{\partial x}{\partial \beta_g} = \frac{\partial(1-x)}{\partial \beta_g} = \frac{\partial p_n}{\partial \beta_g} = \frac{\partial m_n}{\partial \beta_g} = \frac{\partial m_g}{\partial \beta_g} = \frac{\partial \tau_n}{\partial \beta_g} = \frac{\partial P_n}{\partial \beta_g} = 0$. (6): $\frac{\partial p_n}{\partial \gamma_n} \leq 0, \frac{\partial^2 p_n}{\partial \gamma_n^2} \geq 0, \frac{\partial p_g}{\partial \gamma_n} \geq 0, \frac{\partial^2 p_g}{\partial \gamma_n^2} \geq 0, \frac{\partial \tau_n}{\partial \gamma_n} \geq 0, \frac{\partial^2 \tau_n}{\partial \gamma_n^2} \leq 0, \frac{\partial \tau_g}{\partial \gamma_n} \leq 0, \frac{\partial^2 \tau_g}{\partial \gamma_n^2} \leq 0, \frac{\partial P_n}{\partial \gamma_n} \leq 0, \frac{\partial^2 P_n}{\partial \gamma_n^2} \geq 0, \frac{\partial P_g}{\partial \gamma_n} \geq 0, \frac{\partial^2 P_g}{\partial \gamma_n^2} \geq 0, \frac{\partial x}{\partial \gamma_n} = \frac{\partial(1-x)}{\partial \gamma_n} = \frac{\partial m_n}{\partial \gamma_n} = \frac{\partial m_g}{\partial \gamma_n} = 0$. (7): $\frac{\partial m_n}{\partial a_n} \leq 0, \frac{\partial^2 m_n}{\partial a_n^2} \geq 0, \frac{\partial P_n}{\partial a_n} \geq 0, \frac{\partial x}{\partial a_n} = \frac{\partial(1-x)}{\partial a_n} = \frac{\partial p_n}{\partial a_n} = \frac{\partial p_g}{\partial a_n} = \frac{\partial m_g}{\partial a_n} = \frac{\partial \tau_n}{\partial a_n} = \frac{\partial \tau_g}{\partial a_n} = \frac{\partial P_g}{\partial a_n} = 0$. (8): $\frac{\partial m_g}{\partial a_g} \leq 0, \frac{\partial^2 m_g}{\partial a_g^2} \geq 0, \frac{\partial P_g}{\partial a_g} \geq 0, \frac{\partial x}{\partial a_g} = \frac{\partial(1-x)}{\partial a_g} = \frac{\partial p_n}{\partial a_g} = \frac{\partial p_g}{\partial a_g} = \frac{\partial m_n}{\partial a_g} = \frac{\partial \tau_n}{\partial a_g} = \frac{\partial \tau_g}{\partial a_g} = \frac{\partial P_n}{\partial a_g} = 0$.

Proof. Follows from Equations (A12)–(A19) in Appendix E. □

Property 2 states that, first, the household's fraction x of currency n, the government's monitoring probability m_n of currency n, and the government's penalty factor P_g imposed on each household's holding of currency g, increase linearly, linearly, and convexly in the household's output elasticity α for currency n. Conversely, the household's fraction $1-x$ of currency g, the government's monitoring probability m_g of currency g, and the government's penalty factor P_n imposed on each household's holding of currency n, decrease linearly, linearly, and convexly in α. The remaining variables are independent of α.

Second and third, the household's probability p_j of tax evasion on currency j and the government's taxation τ_j on currency j increase concavely and linearly, respectively, in the exponential tax evasion parameter λ_j. The remaining variables except P_j are independent of λ_j, $j = n, g$.

Fourth, the household's probabilities p_n and p_g of tax evasion on currencies n and g increase linearly and convexly in the government's output elasticity β_n for currency n. The government's taxation τ_n and τ_g on currencies n and g decrease concavely and convexly in β_n. This decrease follows since increasing β_n causes the government to identify more strongly with the household in Equation (3), and the household prefers low taxation. That the decrease is concave versus convex follows since high output elasticity β_n for currency n eventually induces the government to tax currency n more than currency g. Furthermore, higher β_n means lower output elasticity $1-\beta_n-\beta_g-\gamma_n$ for the fourth term in Equation (3), which expresses lower government weight assigned to income from taxation and penalty on tax evasion associated with currency g. The government's penalty factors P_n and P_g imposed on each household's holding of currencies n and g increase concavely and convexly in β_n. The remaining variables are independent of β_n.

Fifth, the household's probability p_g of tax evasion on currency g increases convexly in the government's output elasticity β_g for the same currency g, as currency g becomes more valuable for the household. The government's taxation τ_g on currency g decreases linearly in β_g, as the government identifies more strongly with the household and thus prefers to impose fewer costs on the household. The government's penalty factor P_g imposed on each household's holding of currency g increases convexly in β_g, as the government seeks to curtail the household's probability p_g of tax evasion on currency g. The remaining variables are independent of β_g.

Sixth, the household's probabilities p_n and p_g of tax evasion on currencies n and g decreases concavely and increases convexly in the government's output elasticity γ_n for currency n when valuing taxation τ_n and valuing penalty P_n on unsuccessful tax evasion on currency n. Thus, the household is less (more) likely to evade tax on currency n (g) if the government values taxation τ_n (τ_g) and penalty P_n (P_g). The government's taxation τ_n and τ_g on currencies n and g increases concavely and decreases convexly in γ_n. The increase follows since increasing γ_n causes the government to identify less strongly with the household's preference for low taxation τ_n on currency n, and instead to value taxation τ_n and penalty P_n. The decrease follows, conversely, since the government's higher valuation of taxation τ_n and penalty P_n on currency n implies a lower valuation of taxation τ_g and penalty P_g on currency g. The government's penalty factors P_n and P_g imposed on each household's holding of currencies n and g which decreases concavely and increases convexly in γ_n. The remaining variables are independent of γ_n.

Seventh and eighth, the government's monitoring probability m_j of currency j decreases concavely in the unit cost a_j of choosing m_j, while the government's penalty factor P_j imposed on each household's holding of currency j increases linearly. The remaining variables are independent of a_j. The remaining variables are independent of a_j, $j = n, g$.

4. Illustrating the Solution

To illustrate the solution in Property 1 in Section 3.3, this section alters the eight parameter values α,λ_n, λ_g,β_n, γ_n,β_g,a_n, a_g relative to the benchmark parameter values $\alpha = 4/5$, $\lambda_n = \lambda_g = 1/5$, $\beta_n = \gamma_n = 2/5$, $\beta_g = 1/10$, $a_n = a_g = 1$.

First, $\alpha = 4/5$ reflects that the national currency n might be more common than the global currency g, in this illustration, four times more common. Second and third, $\lambda_n = \lambda_g = 1/5$ express that the household's expense $m_j \tau_j P_j p_j^{\lambda_j}$ from unsuccessful tax evasion increases concavely in the representative household's tax evasion probability p_j. Fourth, fifth, and sixth, $\beta_n = \gamma_n = 2/5 = \alpha/2$ and $\beta_g = (1-\alpha)/2 = 1/10$ preserve the same ratio $\alpha/(1-\alpha) = \beta_n/\beta_g = \gamma_n/(1-\beta_n-\beta_g-\gamma_n) = 4$ for how the household and government assign output elasticities to the national currency n versus the global currency g. That is, both the household and the government assign a four times higher output elasticity to currency n than to currency g in their Cobb Douglas expected utilities U and u, and the government does so for both first terms in Equation (3) pertaining to its identification with the household, and for the last two terms in Equation (3) pertaining to how the government benefits from taxation income and income from the household's penalty payment from unsuccessful tax evasion. Seventh and eighth, the government's unit effort costs $a_n = a_g = 1$ of choosing the monitoring probability m_j are the simplest possible benchmarks that satisfy $a_n \geq \alpha$ and $a_g \geq 1-\alpha$. In Figure 4, each of the eight parameter values is altered from its benchmark, while the other seven parameter values are kept at their benchmarks. Division of P_j with 20 is for scaling purposes.

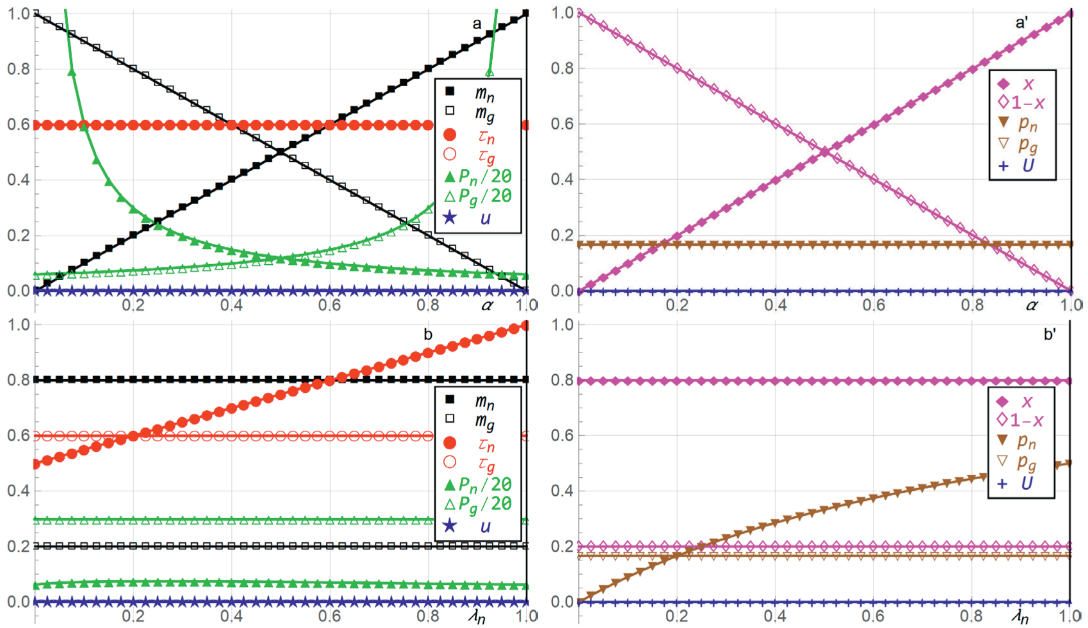

Figure 4. *Cont.*

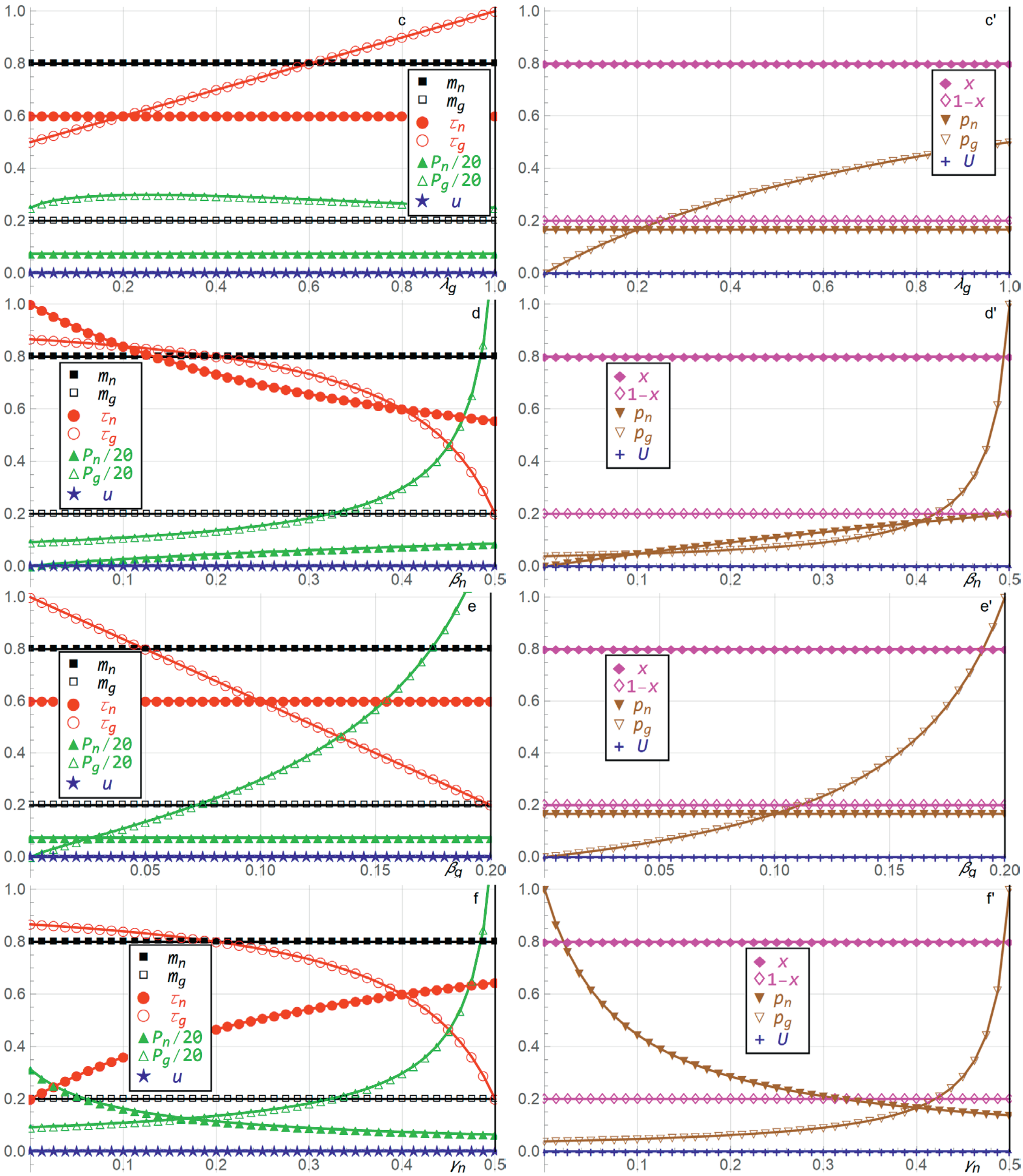

Figure 4. *Cont.*

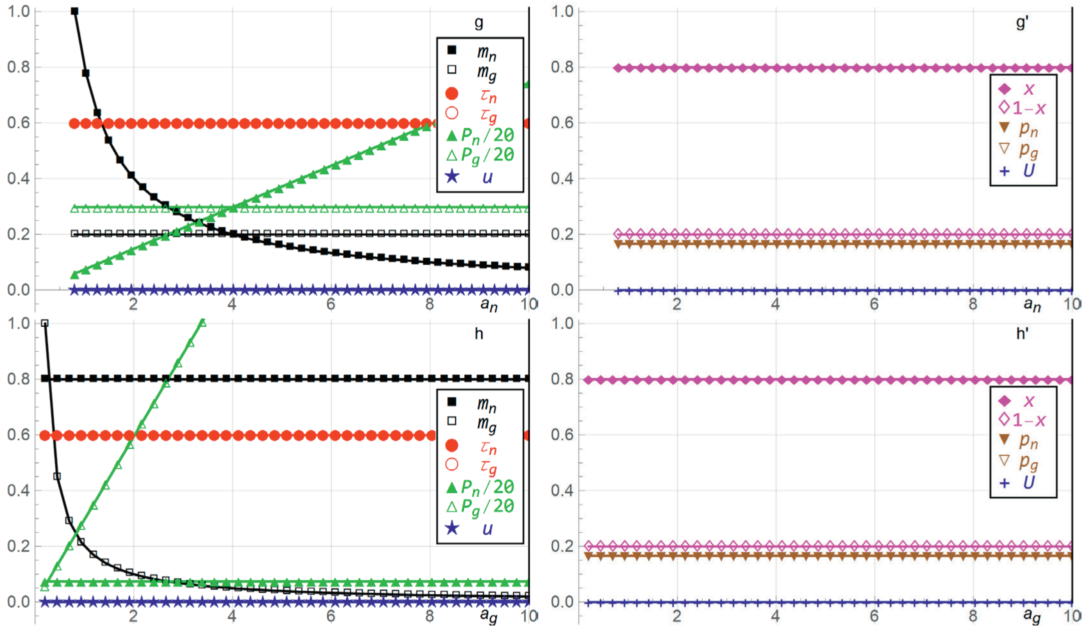

Figure 4. The government's monitoring probability m_j, taxation τ_j, penalty factor P_j, and expected utility u, and the household's fractions x and $1-x$ of currencies n and g, probability p_j of tax evasion on currency j, and expected utility U, as functions of the eight parameter values α,λ_n, λ_g,β_n, γ_n,β_g,a_n, a_g relative to the benchmark parameter values $\alpha = 4/5$, $\lambda_n = \lambda_g = 1/5$, $\beta_n = \gamma_n = 2/5$, $\beta_g = 1/10$, $a_n = a_g = 1$. The eight double panels, for the eight parameters α,λ_n, λ_g,β_n, γ_n,β_g,a_n, a_g, are referred to as (**a**,**a′**); (**b**,**b′**); (**c**,**c′**); (**d**,**d′**); (**e**,**e′**); (**f**,**f′**); (**g**,**g′**); and (**h**,**h′**). Division of P_j with 20 is for scaling purposes, $j = n, g$.

In Figure 4a,a′, as the household's output elasticity α for currency n increases, the household's fraction x of currency n increases linearly, the government's monitoring probability m_n of currency n increases linearly, and the government's penalty factor P_g imposed on each household's holding of currency g increases convexly; while the household's fraction $1-x$ of currency g decreases linearly, the government's monitoring probability m_g of currency g decreases linearly, and the government's penalty factor P_n imposed on each household's holding of currency n decreases convexly; and the remaining variables are constant.

In Figure 4b,b′,c,c′, as the exponential tax evasion parameter λ_j increases, the household's probability p_j of tax evasion on currency j increases concavely, and the government's taxation τ_j on currency j increases linearly; the government's penalty factor P_j imposed on each household's holding of currency j is relatively constant, and the remaining variables are constant, $j = n, g$.

In Figure 4d,d′, as the government's output elasticity β_n for currency n increases, the household's probabilities p_n and p_g of tax evasion on currencies n and g increase linearly and convexly, and the government's taxation τ_n and τ_g on currencies n and g decrease concavely and convexly. That causes taxation τ_g to be quite low when β_n is high, since the government then taxes currency n more than currency g. Furthermore, increasing β_n causes the government's penalty factors P_n and P_g imposed on each household's holding of currencies n and g to increase concavely and convexly, and the remaining variables are constant.

In Figure 4e,e′, as the government's output elasticity β_g for currency g increases, the household's probability p_g of tax evasion on currency g increases convexly, the government's taxation τ_g on currency g decreases linearly, and the government's penalty factor P_g imposed on each household's holding of currency g increases convexly. The remaining variables are constant.

In Figure 4f,f′, as the government's output elasticity γ_n for currency n when valuing taxation τ_n and valuing penalty P_n on unsuccessful tax evasion on currency n increases, the household's probabilities p_n and p_g of tax evasion on currencies n and g decreases concavely and increases convexly. Furthermore, as γ_n increases, the government's taxation τ_n and τ_g on currencies n and g increases concavely and decreases convexly, the government's penalty factors P_n and P_g imposed on each household's holding of currencies n and g decreases concavely and increases convexly, and the remaining variables are constant.

In Figure 4g,g′,4h,h′, as the government's unit cost a_j of choosing the monitoring probability m_j of currency j increases, the government's monitoring probability m_j of currency j decreases concavely, the government's penalty factor P_j imposed on each household's holding of currency j increases linearly, and the remaining variables are constant, $j = n, g$.

5. Discussion, Economic Intuition, and Policy Implications

Eight results in the previous section are particularly noteworthy. First, the household's fraction x of the national currency n, the government's monitoring probability m_n of the national currency n, and the penalty factor P_g imposed on holding the global currency g, increase linearly, linearly, and convexly in the household's output elasticity α for the national currency n. It is assumed that as one currency becomes more important, valuable, and useful for the household, it holds more of it, which causes the government to monitor it more thoroughly. More extensive monitoring of one currency is accompanied with a lower penalty factor for that currency, and a higher penalty factor for the other currency. This inverse correlation between monitoring m_j and the penalty factor P_j, shown in mboxfigfig:games-1159039-f004a,a′, causes the household to choose a constant probability p_j of tax evasion on currency j. The policy implication is that governments should be cognizant of this inverse correlation between monitoring m_j and the penalty factor P_j, which can be implemented in laws and procedures. For example, increased monitoring m_j without decreasing the penalty factor P_j as shown in Figure 4a,a′ cannot be expected to cause the household to choose a constant probability p_j of tax evasion on currency j, but can instead cause the household to choose a lower probability p_j of tax evasion on currency j since the penalty factor P_j is too high.

Second and third, the household's probability p_j of tax evasion and the government's taxation τ_j increase concavely and linearly, respectively, in the exponential tax evasion parameter λ_j for each currency j. The mathematical reason can be seen from Equation (2) where higher λ_j causes lower $p_j^{\lambda_j}$, since $0 \leq p_j \leq 1$, which dilutes the impact of monitoring m_j and the penalty factor P_j through the term $m_j \tau_j P_j p_j^{\lambda_j}$, causing higher probability p_j of tax evasion. The government's natural response is to tax more, which is expressed with higher τ_j. The intuition is that if the government's structure of monitoring and penalties becomes more lenient, expressed with higher λ_j, the household will evade tax more, and will face higher taxation. The policy implication is that governments should holistically recognize the relationship between monitoring, penalties, the amount of taxation, and how households evade tax under these conditions.

Fourth, the household's probabilities p_n and p_g of tax evasion on both currencies n and g increase in the government's output elasticity β_n for the national currency n. Furthermore, the government's taxation τ_j on both currencies decrease, and the penalty factor P_j increase, in β_n. Additionally, a high β_n eventually induces the government to tax that currency n more than the global currency g. Since higher β_n means that the government identifies more with the household, and thus becomes more altruistic, it is assumed that the household exploits the government's altruism through more tax evasion. Additionally, the household enjoys less taxation, although the government eventually taxes currency n, which it values, more than currency g, and eventually suffers higher penalties. The policy implication is that governments should realize that identifying too much with households, by becoming more altruistic, and lowering taxes, with a possible objective of appeasing citizens and ensuring reelection, might cause the households to exploit the situation by evading tax even more.

Fifth, and similarly fourth, the household's probability p_g of tax evasion on currency g increases in the government's output elasticity β_g. The government's taxation τ_g on currency g decreases in β_g, as the government identifies more strongly with the household. The government's penalty factor P_g imposed on each household's holding of currency g increases in β_g. The intuition is again that the household exploits the government's altruism through more tax evasion, enjoys less taxation, although eventually there is more taxation on the currency that the government values most, and eventually suffers higher penalties. The policy implication is again that governments should recognize the relationship between being altruistic, being exploited through different probabilities of tax evasion on the two currencies, and imposing adequate taxes and penalties.

Sixth, the household's probabilities p_n and p_g of tax evasion on currencies n and g decreases and increases in the government's output elasticity γ_n for currency n, which values taxation τ_n and penalty P_n on unsuccessful tax evasion on currency n. Furthermore, the household is less likely to evade tax on the national currency n if the government values taxation τ_n and penalty P_n, expressed with γ_n, on the national currency n. The results are opposite for currency g, as shown in Sections 3 and 4. The intuition is that a higher γ_n, which implies valuing taxation and penalties for tax evasion, causes the government to be less altruistic towards the household regarding the national currency n, which causes more taxation with a lower associated penalty factor, and less tax evasion. Intuitively, higher γ_n has the opposite impact for the global currency g. The policy implication is that governments should assess how they value taxation and penalties for tax evasion, which impacts how households evade tax differently on national and global currencies.

Seventh and eighth, the government's monitoring probability m_j of each currency j decreases in the unit cost a_j of monitoring, counteracted by the penalty factor P_j imposed on each household's holding of each currency increase. This causes the tax rates τ_n and τ_g and the household's probabilities p_n and p_g of tax evasion to be constant. The intuition is that the government compensates for a low (high) monitoring probability m_j, as regulated by the unit cost a_j of monitoring, by choosing a high (low) penalty factor P_j. The model thus predicts, for example, that if the government is less able to monitor transactions and enforce regulations in cryptocurrencies, expressed by a high unit costs of monitoring, then it should impose higher penalties on each household's holding of cryptocurrencies when taxes are evaded. Whether that happens in practice is an interesting empirical question that should be analyzed in future research. For example, if the government's unit cost a_g of monitoring in Figure 4g,g′ is extremely high causing the monitoring probability m_g to be extremely low, then a variety of consequences are possible. For example, the government might not be able to impose and enforce payment of sufficiently high penalties as predicted by the model, due to laws, regulations, and customs placing upper bounds on penalties, or households being unable to pay excessive penalties, for example. Alternatively, households might in practice not follow the expected utility theory when facing an extremely low monitoring probability m_g of being detected and prosecuted for tax evasion, and might choose to ignore the probability of being monitored. The policy implication is that governments should be cognizant of the relationship between how they choose monitoring efforts and penalties for tax evasion, and how this relationship impacts their own taxation and the households' tax evasion.

6. Conclusions

This article presents a game between a government and a representative household holding two currencies, which can generally be any two assets, subject to taxation. The two currencies are a national currency, e.g., a CBDC and a global currency, e.g., Bitcoin, Zcash, or Facebook's Diem, which might have limited usage within a nation. The global currency might offer other opportunities, e.g., tax evasion, user autonomy, discretion, peer-to-peer focus, no banking fees, payment on the black market, criminal activities, and potential return.

The household makes three strategic choices to maximize its Cobb Douglas expected utility with two output elasticities associated with the two currencies. Due to the different opportunities, usage, values, etc. provided by the two currencies, the household chooses to hold one fraction in the national currency, and the remaining fraction in the global currency. Additionally, the household chooses the tax evasion probability on each currency.

The government makes six strategic choices, i.e., the probability of detecting and prosecuting tax evasion on each currency, the tax rate on each currency, and the penalty factor imposed on each household when tax evasion is successfully detected and prosecuted for each currency. The government has a Cobb Douglas expected utility with four output elasticities, minus costs of choosing the monitoring probabilities. Two output elasticities are associated with the two currencies as the government identifies with the household. The two remaining output elasticities are due to the government benefitting from taxes and penalties. The government incurs a cost of choosing the monitoring probability.

The article analytically determines the players' nine strategic choices and expected utilities. Many results are in line with logic. Some results illustrate aspects that the governments and households should be cognizant of. The household prefers low taxation. The government identifies partly with each household, since it is either elected by the households or needs support from the households, but also needs income from taxation and might receive penalty payments for detecting tax evasion. The players' strategic choices are closely related to their output elasticities for the two currencies, and to the government's output elasticities that value taxation and penalties for tax evasion.

The household's fraction of the national currency, the government's monitoring probability of the national currency, and the penalty factor imposed on the global currency, increase the household's output elasticity for the national currency. The household's probability of tax evasion and the government's taxation increase in the exponential tax evasion parameter for each currency. The household's probabilities of tax evasion on both currencies increase in the government's output elasticity for the national currency. The government's taxation on both currencies decrease in the output elasticity for the national currency.

High output elasticity for the national currency eventually induces the government to tax that currency more than the global currency. The household's probability of tax evasion on the global currency increases in the government's output elasticity for that currency. The household is less (more) likely to tax evade on the national (global) currency if the government values taxation and penalty on the national (global) currency. The government's monitoring probability of each currency decreases in the unit cost of monitoring. The government's penalty factor imposed on each household's holding of each currency increases in the unit cost of monitoring. The results are illustrated numerically where each of eight parameter values are varied relative to a benchmark.

Future research should compile and assess empirical support for how households and governments choose strategies for national and global currencies, and assess common output elasticities in Cobb Douglas expected utilities for currencies. Such empirical support should be assessed against the fractions that a representative household chooses for each currency, and the probabilities the households choose for tax evasion on currencies. The government's probability of detecting and prosecuting tax evasion, the tax rate, and the penalty factor imposed on each household when tax evasion is successfully detected and prosecuted, should be empirically assessed for each currency.

Future research might also model more than two currencies, and additional players such as firms, multiple governments in multiple countries, central banks, banks, and international financial institutions. Various alternatives to the players' expected utilities might be evaluated, i.e., backing, convenience, confidentiality, transaction efficiency, financial stability, and security, as perceived by each player. More complexity and multiple time periods might also be incorporated.

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Appendix A. Nomenclature

Appendix A.1. Parameters

j Currency of kind j, $j = n, g$.
n National currency.
g Global currency.
α The household's output elasticity for currency n, $0 \leq \alpha \leq 1$.
$1 - \alpha$ The household's output elasticity for currency g, $0 \leq \alpha \leq 1$.
λ_j Exponential tax evasion parameter, $0 \leq \lambda_j \leq \infty$, $j = n, g$.
β_n The government's output elasticity for currency n when identifying with the household, $0 \leq \beta_n \leq 1$.
β_g The government's output elasticity for currency g when identifying with the household, $0 \leq \alpha \leq 1$.
γ_n The government's output elasticity for currency n when valuing taxation τ_n and valuing penalty P_n on unsuccessful tax evasion on currency n, $0 \leq \gamma_n \leq 1$.
$1 - \beta_n - \beta_g - \gamma_n$ The government's output elasticity for currency g when valuing taxation and valuing penalties on unsuccessful tax evasion, of currency g, $0 \leq 1 - \beta_n - \beta_g - \gamma_n \leq 1$.
a_j Unit cost of choosing the monitoring probability n_j, $a_j \geq 0$, $j = n, g$.

Appendix A.2. Household's Free Choice Variables

p_j Household's probability of tax evasion on currency j, $j = n, g$, $0 \leq p_j \leq 1$.
x Household's fraction of currency n, $0 \leq x \leq 1$.

Appendix A.3. Government's Free Choice Variables

m_j Government's probability of monitoring and thus detecting and prosecuting tax evasion on currency j, $0 \leq m_j \leq 1$, $j = n, g$.
τ_j Household's tax rate on currency j, $0 \leq \tau_j \leq 1$, $j = n, g$.
P_j Government's penalty factor imposed on each household's holding of currency j when tax evasion is successfully detected and prosecuted, $j = n, g$.

Appendix A.4. Dependent Variables

U Household's expected utility.
u Government's expected utility per household.
$1 - x$ Household's fraction of currency g, $0 \leq x \leq 1$.

Appendix B. Determining the Household's Free Choice Variables

Differentiating the household's expected utility U in Equation (2) with respect to its free choice variable x gives

$$\frac{\partial U}{\partial x} = \begin{cases} \left(1-(1-p_n)\tau_n - m_n\tau_n P_n p_n^{\lambda_n}\right)^{\alpha}\left(1-\left(1-p_g\right)\tau_g - m_g\tau_g P_g p_g^{\lambda_g}\right)^{1-\alpha} \\ \times \frac{(\alpha-x)x^{\alpha-1}}{(1-x)^{\alpha}} \; if \; P_j \le \frac{1-\left(1-p_j\right)\tau_j}{m_j\tau_j p_j^{\lambda_j}}, j=n,g \\ 0 \; otherwise \end{cases} \tag{A1}$$

which is equated with zero and solved to yield Equation (5). The second order conditions, inserting $x=\alpha$, are satisfied as negative, i.e.,

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=\alpha} = \begin{cases} -\left(1-(1-p_n)\tau_n - m_n\tau_n P_n p_n^{\lambda_n}\right)^{\alpha}\left(1-\left(1-p_g\right)\tau_g - m_g\tau_g P_g p_g^{\lambda_g}\right)^{1-\alpha} \\ \times \frac{\alpha^{\alpha-1}}{(1-\alpha)^{\alpha}} \; if \; P_j \le \frac{1-\left(1-p_j\right)\tau_j}{m_j\tau_j p_j^{\lambda_j}}, j=n,g \\ 0 \; otherwise \end{cases} \tag{A2}$$

Differentiating the household's expected utility U in Equation (2) with respect to its free choice variables $p_j,\ j=n,g$, gives

$$\frac{\partial U}{\partial p_n} = \begin{cases} \alpha\tau_n\left(1-m_n P_n\lambda_n p_n^{\lambda_n-1}\right)\left(1-(1-p_n)\tau_n - m_n\tau_n P_n p_n^{\lambda_n}\right)^{\alpha-1} \\ \times\left(1-\left(1-p_g\right)\tau_g - m_g\tau_g P_g p_g^{\lambda_g}\right)^{1-\alpha} x^{\alpha}(1-x)^{1-\alpha} \\ if \; P_j \le \frac{1-\left(1-p_j\right)\tau_j}{m_j\tau_j p_j^{\lambda_j}}, j=n,g \\ 0 \; otherwise \end{cases} \tag{A3}$$

and

$$\frac{\partial U}{\partial p_g} = \begin{cases} (1-\alpha)\tau_g\left(1-m_g P_g\lambda_g p_g^{\lambda_g-1}\right)\left(1-(1-p_n)\tau_n - m_n\tau_n P_n p_n^{\lambda_n}\right)^{\alpha} \\ \times\left(1-\left(1-p_g\right)\tau_g - m_g\tau_g P_g p_g^{\lambda_g}\right)^{-\alpha} x^{\alpha}(1-x)^{1-\alpha} \\ if \; P_j \le \frac{1-\left(1-p_j\right)\tau_j}{m_j\tau_j p_j^{\lambda_j}}, j=n,g \\ 0 \; otherwise \end{cases} \tag{A4}$$

which are equated with zero and solved to yield Equation (6). The second order condition for p_n is satisfied as negative, i.e.,

$$\frac{\partial^2 U}{\partial p_n^2} = \begin{cases} -\left(1-(1-p_n)\tau_n - m_n\tau_n P_n p_n^{\lambda_n}\right)^{\alpha-2}\left(1-\left(1-p_g\right)\tau_g - m_g\tau_g P_g p_g^{\lambda_g}\right)^{1-\alpha} \\ \times\alpha\tau_n(1-x)^{1-\alpha}x^{\alpha}\left(m_n P_n p_n^{\lambda_n-2}(\lambda_n-1)\lambda_n\left(1-(1-p_n)\tau_n - m_n\tau_n P_n p_n^{\lambda_n}\right)\right. \\ \left.+(1-\alpha)\tau_n\left(1-m_n P_n\lambda_n p_n^{\lambda_n-1}\right)^2\right) \; if \; P_n \le \frac{1-(1-p_n)\tau_n}{m_n\tau_n p_n^{\lambda_n}} \\ 0 \; otherwise \end{cases} \tag{A5}$$

The second order condition for p_g is analogous.

Appendix C. Determining the Government's Free Choice Variables

Differentiating the government's expected utility u in Equation (3) with respect to its six free choice variables m_j, τ_j, P_j, $j = n, g$, gives

$$N \equiv \left(1 - p_n + m_n P_n p_n^{\lambda_n}\right)\tau_n x, \; G \equiv \left(1 - p_g - m_g P_g p_g^{\lambda_g}\right)\tau_g(1-x),$$

$$\frac{\partial u}{\partial m_n} = \begin{cases} -x(1-x-G)^{\beta_g}\left(G - a_g m_g\right)^{1-\beta_n-\beta_g-\gamma_n}(x-N)^{\beta_n-1}(N-a_n m_n)^{\gamma_n-1} \\ \times\begin{pmatrix} m_n p_n^{2\lambda_n} P_n^2 x(\beta_n+\gamma_n)\tau_n^2 + a_n\gamma_n(1+(-1+p_n)\tau_n) \\ -p_n^{\lambda_n} P_n \tau_n(a_n m_n(\beta_n+\gamma_n) + x(\gamma_n+(-1+p_n)(\beta_n+\gamma_n)\tau_n)) \end{pmatrix} \\ \qquad if\ P_j \le \frac{1-(1-p_j)\tau_j}{m_j \tau_j p_j^{\lambda_j}}, j = n, g \\ 0\ otherwise \end{cases} \tag{A6}$$

and

$$\frac{\partial u}{\partial m_g} = \begin{cases} (-1+x)(1-x-G)^{\beta_g-1}\left(G-a_g m_g\right)^{-\beta_n-\beta_g-\gamma_n}(x-N)^{\beta_n}(N-a_n m_n)^{\gamma_n} \\ \times(a_g m_g p_g^{\lambda_g} P_g(-1+\beta_n+\gamma_n)\tau_g - a_g\left(-1+\beta_g+\beta_n+\gamma_n\right) \\ \times\left(1+\left(-1+p_g\right)\tau_g\right) + p_g^{\lambda_g} P_g(-1+x)\tau_g(-\beta_g \\ +(-1+\beta_n+\gamma_n)(-1+\tau_g+p_g(-1+m_g p_g^{\lambda_g-1} P_g)\tau_g))) \\ \qquad if\ P_j \le \frac{1-(1-p_j)\tau_j}{m_j \tau_j p_j^{\lambda_j}}, j = n, g \\ 0\ otherwise \end{cases} \tag{A7}$$

and

$$\frac{\partial u}{\partial \tau_n} = \begin{cases} -(1+p_n\left(-1+m_n p_n^{\lambda_n-1} P_n\right)x(1-x-G)^{\beta_g}(x-N)^{\beta_n-1} \\ \times\left(G-a_g m_g\right)^{1-\beta_n-\beta_g-\gamma_n}(N-a_n m_n)^{\gamma_n-1}(-a_n m_n \beta_n - x\gamma_n \\ +\left(1+p_n\left(-1+m_n p_n^{\lambda_n-1} P_n\right)\right)x(\beta_n+\gamma_n)\tau_n)) \\ \qquad if\ P_j \le \frac{1-(1-p_j)\tau_j}{m_j \tau_j p_j^{\lambda_j}}, j = n, g \\ 0\ otherwise \end{cases} \tag{A8}$$

and

$$\frac{\partial u}{\partial \tau_g} = \begin{cases} \left(1+p_g\left(-1+m_g p_g^{\lambda_g-1} P_g\right)\right)(-1+x)(1-x-G)^{\beta_g-1}\left(G-a_g m_g\right)^{-\beta_n-\beta_g-\gamma_n} \\ \times(\beta_g - a_g m_g \beta_g - x\beta_g - (-1+\beta_n+\gamma_n)(-1+\tau_g+p_g(-1+ \\ m_g p_g^{\lambda_g-1} P_g)\tau_g) + x(-1+\beta_n+\gamma_n)(-1+\tau_g \\ +p_g\left(-1+m_g p_g^{\lambda_g-1} P_g\right)\tau_g))(x-N)^{\beta_n}(N-a_n m_n)^{\gamma_n} \\ \qquad if\ P_j \le \frac{1-(1-p_j)\tau_j}{m_j \tau_j p_j^{\lambda_j}}, j = n, g \\ 0\ otherwise \end{cases} \tag{A9}$$

and

$$\frac{\partial u}{\partial P_n} = \begin{cases} -(1-x-G)^{\beta_g}\left(G-a_g m_g\right)^{1-\beta_n-\beta_g-\gamma_n}(x-N)^{\beta_n-1}(N-a_n m_n)^{\gamma_n-1} \\ \times m_n p_n^{\lambda_n} x \tau_n\left(-a_n m_n \beta_n - x\gamma_n + \left(1+p_n\left(-1+m_n p_n^{\lambda_n-1} P_n\right)\right)x(\beta_n+\gamma_n)\tau_n\right) \\ \qquad if\ P_j \le \frac{1-(1-p_j)\tau_j}{m_j \tau_j p_j^{\lambda_j}}, j = n, g \\ 0\ otherwise \end{cases} \tag{A10}$$

and

$$\frac{\partial u}{\partial P_g} = \begin{cases} m_g p_g^{\lambda_g}(-1+x)\tau_g(1-x-G)^{\beta_g-1}(G-a_g m_g)^{-\beta_n-\beta_g-\gamma_n}(\beta_g - a_g m_g \beta_g \\ -x\beta_g - (-1+\beta_n+\gamma_n)\left(-1+\tau_g+p_g\left(-1+m_g p_g^{\lambda_g-1}P_g\right)\tau_g\right) + x(-1+\beta_n+ \\ \gamma_n)(-1+\tau_g+p_g(-1+m_g p_g^{\lambda_g-1}P_g)\tau_g))(x-N)^{\beta_n}(N-a_n m_n)^{\gamma_n} \\ \quad if\ P_j \le \frac{1-(1-p_j)\tau_j}{m_j\tau_j p_j^{\lambda_j}}, j=n,g \\ 0\ otherwise \end{cases} \quad \text{(A11)}$$

Equating the first order conditions in Equations (A6)–(A11) with zero and solving gives Equation (7), which are valid when the inequalities are satisfied. The if-test $P_j \le \frac{1-(1-p_j)\tau_j}{m_j\tau_j p_j^{\lambda_j}}, j=n,g$, is omitted in Equation (7) since it is always satisfied. It can be shown that the second order conditions are satisfied as negative.

Appendix D. Proof of Property 1

Equations (5)–(7) constitute nine equations with the nine unknown variables x, p_j, m_j, τ_j, P_j, $j=n,g$, which are solved to yield Equation (8). Just as the if-test $P_j \le \frac{1-(1-p_j)\tau_j}{m_j\tau_j p_j^{\lambda_j}}, j=n,g$, is omitted in Equation (7) since it is always satisfied, it is also omitted for x, p_n and p_g in Equation (8) since it is always satisfied. The inequalities $a_n \ge \alpha$ and $a_g \ge 1-\alpha$ follow from Equation (7) when $x=\alpha$. The inequality $0 \le \lambda_j \le 1$, $j=n,g$, follows since $\lambda_j > 1$ would cause taxation $\tau_n > 1$ in Equation (8), which is not meaningful.

Appendix E. First Order and Second Order Derivatives for Property 2

Differentiating Equation (8) when $a_n \ge \alpha,\ a_g \ge 1-\alpha,\ 0 \le \lambda_j \le 1,\ j=n,g$, gives

$$\begin{aligned} &\tfrac{\partial x}{\partial \alpha} = 1,\ \tfrac{\partial (1-x)}{\partial \alpha} = -1,\ \tfrac{\partial p_n}{\partial \alpha} = \tfrac{\partial p_g}{\partial \alpha} = \tfrac{\partial \tau_n}{\partial \alpha} = \tfrac{\partial \tau_g}{\partial \alpha} = 0,\ \tfrac{\partial m_n}{\partial \alpha} = \tfrac{1}{a_n},\ \tfrac{\partial m_g}{\partial \alpha} = \tfrac{-1}{a_g}, \\ &\tfrac{\partial P_n}{\partial \alpha} = \tfrac{-a_n}{\lambda_n \alpha^2}\left(\tfrac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)^{1-\lambda_n},\ \tfrac{\partial^2 P_n}{\partial \alpha^2} = \tfrac{2a_n}{\lambda_n\alpha^3}\left(\tfrac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)^{1-\lambda_n}, \\ &\tfrac{\partial P_g}{\partial \alpha} = \tfrac{a_g}{\lambda_g(1-\alpha)^2}\left(\tfrac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g}\right)^{1-\lambda_g}, \\ &\tfrac{\partial^2 P_g}{\partial \alpha^2} = \tfrac{2a_g}{\lambda_g(1-\alpha)^3}\left(\tfrac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g}\right)^{1-\lambda_g} \end{aligned} \quad \text{(A12)}$$

$$\begin{aligned} &\tfrac{\partial x}{\partial \lambda_n} = \tfrac{\partial (1-x)}{\partial \lambda_n} = \tfrac{\partial p_g}{\partial \lambda_n} = \tfrac{\partial m_n}{\partial \lambda_n} = \tfrac{\partial m_g}{\partial \lambda_n} = \tfrac{\partial \tau_g}{\partial \lambda_n} = \tfrac{\partial P_g}{\partial \lambda_n} = 0, \\ &\tfrac{\partial p_n}{\partial \lambda_n} = \tfrac{\beta_n\gamma_n}{(\lambda_n\beta_n+\gamma_n)^2},\ \tfrac{\partial^2 p_n}{\partial \lambda_n^2} = \tfrac{-2\beta_n^2\gamma_n}{(\lambda_n\beta_n+\gamma_n)^3},\ \tfrac{\partial \tau_n}{\partial \lambda_n} = \tfrac{\beta_n}{\beta_n+\gamma_n}, \\ &\tfrac{\partial P_n}{\partial \lambda_n} = \tfrac{-a_n\beta_n}{\alpha(\lambda_n\beta_n+\gamma_n)^2}\left(\tfrac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)^{-\lambda_n}\left(\beta_n+\gamma_n+(\lambda_n\beta_n+\gamma_n)Ln\left(\tfrac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)\right) \end{aligned} \quad \text{(A13)}$$

$$\begin{aligned} &\tfrac{\partial x}{\partial \lambda_g} = \tfrac{\partial (1-x)}{\partial \lambda_g} = \tfrac{\partial p_n}{\partial \lambda_g} = \tfrac{\partial m_n}{\partial \lambda_g} = \tfrac{\partial m_g}{\partial \lambda_g} = \tfrac{\partial \tau_n}{\partial \lambda_g} = \tfrac{\partial P_n}{\partial \lambda_g} = 0, \\ &\tfrac{\partial p_g}{\partial \lambda_g} = \tfrac{\beta_g(1-\beta_n-\beta_g-\gamma_n)}{(1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g)^2},\ \tfrac{\partial^2 p_g}{\partial \beta_g^2} = \tfrac{2\beta_g^2(1-\beta_n-\beta_g-\gamma_n)}{(1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g)^3}, \\ &\tfrac{\partial \tau_g}{\partial \lambda_g} = \tfrac{\beta_g}{1-\beta_n-\gamma_n}, \\ &\tfrac{\partial P_g}{\partial \lambda_g} = \tfrac{a_g\beta_g}{(1-\alpha)(1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g)^2}\left(\tfrac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g}\right)^{-\lambda_g} \\ &\times\left(1-\beta_n-\gamma_n+(1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g)Ln\left(\tfrac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-(1-\lambda_g)\beta_g}\right)\right) \end{aligned} \quad \text{(A14)}$$

$$\begin{aligned}
&\frac{\partial x}{\partial \beta_n}=\frac{\partial(1-x)}{\partial \beta_n}=\frac{\partial m_n}{\partial \beta_n}=\frac{\partial m_g}{\partial \beta_n}=0, \frac{\partial p_n}{\partial \beta_n}=\frac{\lambda_n\gamma_n}{(\lambda_n\beta_n+\gamma_n)^2}, \frac{\partial^2 p_n}{\partial \beta_n^2}=\frac{-2\lambda_n^2\gamma_n}{(\lambda_n\beta_n+\gamma_n)^3},\\
&\frac{\partial p_g}{\partial \beta_n}=\frac{\lambda_g\beta_g}{\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^2}, \frac{\partial^2 p_g}{\partial \beta_n^2}=\frac{2\lambda_g\beta_g}{\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^3},\\
&\frac{\partial \tau_n}{\partial \beta_n}=\frac{-(1-\lambda_n)\gamma_n}{(\beta_n+\gamma_n)^2}, \frac{\partial^2 \tau_n}{\partial \beta_n^2}=\frac{2\gamma_n(1-\lambda_n)}{(\beta_n+\gamma_n)^3}, \frac{\partial \tau_g}{\partial \beta_n}=\frac{-\left(1-\lambda_g\right)\beta_g}{(1-\beta_n-\gamma_n)^2},\\
&\frac{\partial^2 \tau_g}{\partial \beta_n^2}=\frac{-2\beta_g\left(1-\lambda_g\right)}{(1-\beta_n-\gamma_n)^3}, \frac{\partial P_n}{\partial \beta_n}=\frac{a_n(1-\lambda_n)\gamma_n}{\alpha(\lambda_n\beta_n+\gamma_n)^2}\left(\frac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)^{-\lambda_n},\\
&\frac{\partial^2 P_n}{\partial \beta_n^2}=\frac{-a_n\gamma_n(2\beta_n+\gamma_n)(1-\lambda_n)\lambda_n}{\alpha\beta_n(\lambda_n\beta_n+\gamma_n)^3}\left(\frac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)^{-\lambda_n},\\
&\frac{\partial P_g}{\partial \beta_n}=\frac{a_g\left(1-\lambda_g\right)\beta_g}{(1-\alpha)\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^2}\left(\frac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g}\right)^{-\lambda_g},\\
&\frac{\partial^2 P_g}{\partial \beta_n^2}=\frac{a_g\beta_g\left(2-\lambda_g\right)\left(1-\lambda_g\right)}{(1-\alpha)\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^3}\left(\frac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g}\right)^{-\lambda_g}
\end{aligned} \tag{A15}$$

$$\begin{aligned}
&\frac{\partial x}{\partial \beta_g}=\frac{\partial(1-x)}{\partial \beta_g}=\frac{\partial p_n}{\partial \beta_g}=\frac{\partial m_n}{\partial \beta_g}=\frac{\partial m_g}{\partial \beta_g}=\frac{\partial \tau_n}{\partial \beta_g}=\frac{\partial P_n}{\partial \beta_g}=0,\\
&\frac{\partial p_g}{\partial \beta_g}=\frac{\lambda_g(1-\beta_n-\gamma_n)}{\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^2}, \frac{\partial^2 p_g}{\partial \beta_g^2}=\frac{2(1-\beta_n-\gamma_n)\left(1-\lambda_g\right)\lambda_g}{\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^3},\\
&\frac{\partial \tau_g}{\partial \beta_g}=\frac{-\left(1-\lambda_g\right)}{1-\beta_n-\gamma_n},\\
&\frac{\partial P_g}{\partial \beta_g}=\frac{a_g(1-\beta_n-\gamma_n)\left(1-\lambda_g\right)}{(1-\alpha)\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^2}\left(\frac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g}\right)^{-\lambda_g}
\end{aligned} \tag{A16}$$

$$\begin{aligned}
&\frac{\partial x}{\partial \gamma_n}=\frac{\partial(1-x)}{\partial \gamma_n}=\frac{\partial m_n}{\partial \gamma_n}=\frac{\partial m_g}{\partial \gamma_n}=0, \frac{\partial p_n}{\partial \gamma_n}=\frac{-\lambda_n\beta_n}{(\lambda_n\beta_n+\gamma_n)^2}, \frac{\partial^2 p_n}{\partial \gamma_n^2}=\frac{2\lambda_n\beta_n}{(\lambda_n\beta_n+\gamma_n)^3},\\
&\frac{\partial p_g}{\partial \gamma_n}=\frac{\lambda_g\beta_g}{\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^2}, \frac{\partial^2 p_g}{\partial \gamma_n^2}=\frac{2\lambda_g\beta_g}{\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^3},\\
&\frac{\partial \tau_n}{\partial \gamma_n}=\frac{(1-\lambda_n)\beta_n}{(\beta_n+\gamma_n)^2}, \frac{\partial^2 \tau_n}{\partial \gamma_n^2}=\frac{-2\beta_n(1-\lambda_n)}{(\beta_n+\gamma_n)^3}, \frac{\partial \tau_g}{\partial \gamma_n}=\frac{-\left(1-\lambda_g\right)\beta_g}{(1-\beta_n-\gamma_n)^2},\\
&\frac{\partial^2 \tau_g}{\partial \gamma_n^2}=\frac{-2\beta_g\left(1-\lambda_g\right)}{(1-\beta_n-\gamma_n)^2}, \frac{\partial P_n}{\partial \gamma_n}=\frac{-a_n(1-\lambda_n)\beta_n}{\alpha(\lambda_n\beta_n+\gamma_n)^2}\left(\frac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)^{-\lambda_n},\\
&\frac{\partial^2 P_n}{\partial \gamma_n^2}=\frac{a_n\beta_n(2-\lambda_n)(1-\lambda_n)}{\alpha(\lambda_n\beta_n+\gamma_n)^3}\left(\frac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)^{-\lambda_n},\\
&\frac{\partial P_g}{\partial \gamma_n}=\frac{a_g\left(1-\lambda_g\right)\beta_g}{(1-\alpha)\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^2}\left(\frac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g}\right)^{-\lambda_g},\\
&\frac{\partial^2 P_g}{\partial \gamma_n^2}=\frac{a_g\beta_g\left(2-\lambda_g\right)\left(1-\lambda_g\right)}{(1-\alpha)\left(1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g\right)^2}\left(\frac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g}\right)^{-\lambda_g}
\end{aligned} \tag{A17}$$

$$\begin{aligned}
&\frac{\partial x}{\partial a_n}=\frac{\partial(1-x)}{\partial a_n}=\frac{\partial p_n}{\partial a_n}=\frac{\partial p_g}{\partial a_n}=\frac{\partial m_g}{\partial a_n}=\frac{\partial \tau_n}{\partial a_n}=\frac{\partial \tau_g}{\partial a_n}=\frac{\partial P_g}{\partial a_n}=0,\\
&\frac{\partial m_n}{\partial a_n}=\frac{-\alpha}{a_n^2}, \frac{\partial^2 m_n}{a_n^2}=\frac{2\alpha}{a_n^3}, \frac{\partial P_n}{\partial a_n}=\frac{1}{\alpha\lambda_n}\left(\frac{\lambda_n\beta_n}{\lambda_n\beta_n+\gamma_n}\right)^{1-\lambda_n}
\end{aligned} \tag{A18}$$

$$\begin{aligned}
&\frac{\partial x}{\partial a_g}=\frac{\partial(1-x)}{\partial a_g}=\frac{\partial p_n}{\partial a_g}=\frac{\partial p_g}{\partial a_g}=\frac{\partial m_n}{\partial a_g}=\frac{\partial \tau_n}{\partial a_g}=\frac{\partial \tau_g}{\partial a_g}=\frac{\partial P_n}{\partial a_g}=0,\\
&\frac{\partial m_g}{\partial a_g}=\frac{-(1-\alpha)}{a_g^2}, \frac{\partial^2 m_g}{a_g^2}=\frac{2(1-\alpha)}{a_g^2},\\
&\frac{\partial P_g}{\partial a_g}=\frac{1}{(1-\alpha)\lambda_g}\left(\frac{\lambda_g\beta_g}{1-\beta_n-\gamma_n-\left(1-\lambda_g\right)\beta_g}\right)^{1-\lambda_g}
\end{aligned} \tag{A19}$$

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A game between central banks and households involving central bank digital currencies, other digital currencies and negative interest rates

Guizhou Wang & Kjell Hausken

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A game between central banks and households involving central bank digital currencies, other digital currencies and negative interest rates

Guizhou Wang[1] and Kjell Hausken[1*]

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*Corresponding author: Kjell Hausken, Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway
E-mail: kjell.hausken@uis.no

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David McMillan, University of Stirling, Stirling, United Kingdom

Additional information is available at the end of the article

Abstract: Central Bank Digital Currencies (CBDCs) enable negative interest rates. A game is analyzed between a central bank (accounting for the government's interest) and a representative household choosing to consume, hold CBDC, or hold non-CBDC. The central bank chooses negative interest rate when it realizes that the household is willing to pay the central bank for holding CBDC. The household pays the negative interest rate because of its Cobb Douglas preferences whereby it values holding CBDC while simultaneously holding the competitive non-CBDC with a given interest rate, consuming with various output elasticities, and accounting for transaction efficiencies and costs. More explicitly, intuition and how the players benefit are provided for the following results: The central bank chooses more negative interest rate when the household's output elasticity for consumption increases, the household's output elasticity for holding CBDC decreases, the CBDC and non-CBDC transaction efficiencies increase, the household's transaction efficiency for consumption decreases, the household's scaling of the transaction cost increases, the scaling parameter for the central bank's profit per household decreases, the household's monetary energy decreases, and the non-CBDC interest rate decreases. The results are determined analytically and illustrated numerically where each of nine parameter values is varied relative to a benchmark.

Subjects: Public Finance; Corporate Finance; Banking

Keywords: central bank; central bank digital currency; digital currency; negative interest rates; cryptocurrency; game theory; household; government

JEL Classification Numbers: C72; H26

1. Introduction

1.1. Background

The digitization of currency revolutionizes mankind's use of currencies. Increasingly many central banks research Central Bank Digital Currencies (CBDCs), or have progressed to proof of concept or pilots, or have launched CBDCs (https://cbdctracker.org/). Commonly stated reasons are to promote financial inclusion and simplify the implementation of monetary and fiscal policy. CBDC developments are enabled and incentivized by new technological opportunities, potentially or partly as a countermovement, competitor or alternative to cryptocurrencies controlled by algorithms or actors (https://coinmarketcap.com). One early and essential cryptocurrency is Nakamoto's (2008) "proof of work" blockchain based electronic cash system labeled Bitcoin.[1] Whereas CBDCs are digital currencies developed by central banks (which are centralized authorities), cryptocurrencies are digital currencies where transactions are recorded and verified through cryptography by a decentralized system. Less common cash usage incentivizes central banks to popularize more acceptable and easily applicable electronic currencies. Some central banks and their associated governments may prefer CBDCs designed to record and possibly control households' transactions. In recent years credit and debit cards, wire transfers and various other forms of payments have gradually replaced cash. CBDCs may continue such replacements of cash. A survey by the Bank for International Settlements shows that currently, central banks representing a fifth of the world's population are likely to issue a general purpose CBDC in the next three years (Boar & Wehrli, 2021). Households in countries adopting CBDCs as legal tender, and prohibiting all alternatives as legal tender, are forced to adopt their country's CBDC (unless they can function through commodity exchange). Countries can more commonly be expected to accept alternatives to CBDCs so that households can choose among alternatives. 10 July 2022, 20,172 cryptocurrencies contribute to a market cap of $931 billion.[2] The crypto fields of decentralized finance (DeFi) and non-fungible Tokens (NFT) develop rapidly.

Digital currencies give rise to new possibilities, including differences across currencies regarding transaction efficiencies, convenience, universal accessibility, confidentiality, financial stability, monetary policy, security, privacy, etc. Meanwhile, it also brings various challenges such as new infrastructures, new household behaviors, potentially more efficient and flexible monetary policies, and new functions or disintermediation for banks. Specifically, CBDCs enable central banks to implement negative interests.

Traditionally, the zero-lower bound on interest rate has been a challenge for central banks with paper money. The reasons are multifarious, i.e. the store of value of money requires a non-negative return, potentially adverse implications for bank profitability, and a potentially weak monetary transmission mechanism as the interest rate decreases towards zero.[3] Under various accommodative policy regimes, various regions and countries such as the Euro area, Denmark, Sweden, Japan, and Switzerland have implemented negative interest rates.

1.2. Contribution

This article develops a game model between a representative household and a central bank which includes the government's interest. This approach grounded in game theory, which has earned 18 Nobel prizes from 1970 to 2017, constitutes the theoretical underpinning of the study. The household converts its resources or monetary energy strategically into consumption, holding of CBDC issued by the central bank with a given interest rate, and holding of non-CBDC which earns an interest rate and can be any asset not issued by and not controlled by a central bank. Each household's allocation highlights the potential relation between CBDC and non-CBDC, and furthermore the relation to consumption. Each household's allocation impacts the central bank's monetary policy, which in turn may impact how non-CBDCs evolve. The central bank chooses the CBDC interest rate, which can be negative or positive. The household has a Cobb Douglas utility with three elasticities, accounting for its strategic choices. The central bank identifies partly with each household, but additionally pays interest to each household when it is positive.

The emergence of digital currencies (CBDCs or non-CBDC) makes it easier to implement negative interest rates, which incentivize consumption rather than saving. CBDC holders subject to negative interest rates are easily subtracted what they owe on the ledger, whereas holders of physical cash not recorded on a ledger must actively provide through some channel cash they possess as interest payment. When a household experiences a negative interest rate, it pays a storage charge instead of earning positive interest. A central bank may have multiple reasons for choosing negative interest rates, e.g., to avoid recession, stimulate economic activity, and avoid deflation. An actor controlling a non-CBDC may choose negative interest rates for similar reasons, and to compete with CBDCs.

This article's research question and objective are to determine how a household earns utility and allocates monetary energy between consumption, holding CBDC and holding non-CBDC depending on the interest rate of CBDC chosen by the central bank and the non-CBDC interest rate (both of which may be positive or negative), and depending on various preferences, transaction efficiencies and other factors.

The household's utility accounts for consumption, CBDC and non-CBDC having different transaction efficiencies. A transaction efficiency function is presented which increases with holding CBDC and non-CBDC, and decreases with consumption. The model illustrates how different transaction efficiencies and interest rates of CBDC and non-CBDC impact the players' strategic choices.

The impact of nine parameters is analyzed analytically and numerically. These are the household's monetary energy; output elasticities for consumption and CBDC (which implicitly determines the elasticity for non-CBDC); and transaction efficiencies for CBDC, non-CBDC and consumption; the scaling of the household's transaction cost; the scaling of the central bank's profit, and the non-CBDC interest rate. These parameters are interesting to study since they impact the players' strategies, utility and profit. Each parameter has an independent impact on the model, which is essential since it enables identifying which specific ingredients of the model has which specific impact. Numerical analysis illustrates variation of each parameter value relative to a benchmark. The article contributes to all the four areas of the literature reviewed in the next section.

1.3. Literature

The literature is divided into four groups, i.e., CBDC design and economy; game theoretic analyses; negative interest rates; and CBDC, monetary policy and policy implications. These four groups are interconnected and relevant as follows. Since the central bank is one of the two players in the article, the first group is about CBDC design and the economy, which provides a foundation for the central bank as a player and crucially impacts how the central bank operates. The second group, naturally, is game theoretic analysis, to illustrate the linkage to the current article which applies game theory as a tool. The third group is about negative interest rates, which some central banks have already started to explore. CBDCs contain the unique feature of being technologically able to implement negative interest rates, which may potentially become important in the future. The fourth group is CBDC, monetary policy and policy implications, which extends from the other three groups into the real economy through policy implications.

1.3.1. CBDC design and economy

Kiff et al. (2020) explore the issuance considerations of retail CBDC which the general public has access to it. They review CBDC research, and summarize the operating models, design considerations and risk management of issuing CBDC. Similarly, Allen et al. (2020) show that CBDC brings a range of new possibilities, but also causes many challenges. They investigate the technical challenges facing CBDC designers, focusing on performance, privacy, and security. They summarize the main potential benefits of CBDC, i.e. efficiency, a broader tax base, flexible monetary policy, payment backstop, and financial inclusion. Ozili (2022) reviews the literature, points out that the motivation of a CBDC is to improve the monetary policy, enhance digital payment efficiency, and increase financial inclusion. He points out limitations of CBDC design, and challenges in meeting multiple competing goals. He finds that a CBDC has cash-like attributes and is a liability of the issuing central bank. Carapella and Flemming (2020) also review the literature, and assess how CBDCs impact commercial banks, monetary policy and financial stability. Oh and Zhang (2020) analyze a CBDC in a two-sector monetary model with a formal and an informal economy. They show that tax reduction and a positive CBDC interest rate are useful to enhance CBDC adoption and improve its effectiveness. This article contributes to this literature by considering how a representative household chooses strategies impacted by the CBDC interest rate, impacted by the non-CBDC interest rate, consumption and various transaction efficiencies.

1.3.2. Game theoretic analyses

This article contributes to this literature by considering a game between the central bank choosing the CBDC interest rate and a representative household choosing consumption, holding CBDC and holding non-CBDC. Wijsman (2021) analyzes households which can earn positive or negative interest rate at one bank, can switch to another bank subject to switching costs, or can invest alternatively. His approach relates to the current article where households also have two possibilities for saving (CBDC and non-CBDC), and have an alternative which is consumption instead of investment. His switching costs have some linkage to transaction costs in the current article. Wijsman (2021) finds that banks may decrease their interest rates if switching costs are higher and alternative household investments are less attractive. He also finds that high switching costs prevent banks from attracting savers from competitors, and less attractive alternatives for households may cause expensive wars of attrition between banks.

Wang and Hausken (2021) consider a game between a representative household choosing to hold a national currency and a global currency, and a government choosing how to tax the two currencies, and how to detect, prosecute and impose penalties for tax evasion. Jia (2020) develops an overlapping generations model to explore the macroeconomic impact of negative interest rates on CBDC. He finds that a negative CBDC interest rate induces agents to save less and consume more, which in turn leads to a decrease in capital investment and output. This article presents related results for how a household saves CBDC or non-CBDC with negative CBDC interest rates. George et al. (2020) evaluate the macroeconomic implications of a CBDC with an adjustable interest rate. They extend the analysis to an open-economy context with foreign capital flows. The study shows that a CBDC with an adjustable interest rate is welfare-improving, and that a quantity rule delivers the best welfare outcome for society.

Welburn and Hausken (2015, 2017) adopt game theory to explore economic crises. They analyze six kinds of players, i.e., countries, central banks, banks, firms, households, and financial intergovernmental organizations. Players have various strategies such as setting interest rates, lending, borrowing, producing, consuming, investing, defaulting, etc. This article considers only two players, i.e. a representative household and the central bank, with specific strategies and utilities for each.

1.3.3. Negative interest rates

This article contributes to this literature by considering how a central bank may choose a negative interest rate impacting, and being impacted by a representative household's consumption, holding of CBDC and non-CBDC, and transaction efficiencies. Davoodalhosseini et al. (2020) argue that an interest-bearing CBDC could be a versatile instrument, which may enhance monetary policy theatrically, i.e., break below the effective lower bound of interest rates, enable non-linear transfer, reduce incentives to adopt alternative means of payments, etc. But in practice the expected benefits might be small. Partly related, the current article shows how an interest-bearing CBDC can operate in conjunction with an interest-bearing non-CBDC for a household which also consumes.

Rognlie (2016) explores monetary policy with negative interest rates. He finds that gains from negative interest rates depend inversely on the level and elasticity of currency demand, that negative interest rates stabilize aggregate demand, but inefficiently subsidize the paper currency.

Altavilla et al. (2019) apply confidential data from the euro area to show that well-performed banks can pass negative rates on to their corporate depositors without experiencing decreased funding. Additionally, a negative interest rate policy can provide further stimulus to the economy via firms' asset rebalancing. The findings challenge the view that conventional monetary policy becomes ineffective when policy rates reach the zero-lower bound.

Assenmacher and Krogstrup (2018) think that cash prevents central banks from cutting interest rates much below zero. They analyze the practical feasibility of adopting electronic money, which could remove the lower bound constraint on monetary policy. The result is feasible electronic money fully restoring the monetary policy space with negative interest rates.

Grasselli and Lipton (2019) point out that CBDC can overcome the lower bound for interest rates imposed by physical cash. They construct a stock-flow macroeconomic model to investigate the theoretical effectiveness of negative interest rates. They find that negative interest rates can be an effective tool for macroeconomic stabilization.

David-Pur et al. (2020) provide experimental evidence on how zero and negative interest rates impact investments. They show that a zero-interest rate is more efficient than a negative interest rate in terms of the impact on people's willingness to borrow money and take risks. But there is no impact of the difference between a positive and a negative interest rate on the change in the allocation of risky assets in investment portfolios.

1.3.4. CBDC, monetary policy and policy implications

This article contributes to this literature by allowing positive and negative CBDC interest rates. Bordo and Levin (2017) analyze how digital cash enhances the effectiveness of monetary policy. They argue that a CBDC may potentially facilitate many aspects of monetary policy, thus potentially improving the stability of the financial system. Asimakopoulos et al. (2019) set up a dynamic stochastic general equilibrium model to evaluate the economic consequences of cryptocurrencies. Using US and crypto markets monthly data for the period 2013:M6-2019:M3, a substitution effect is found between the real balances of government currency and cryptocurrency.

Beniak (2019) explores hypothetical challenges of CBDC implementation for monetary policy, and the impact on the broader economy. Based on an overview of the literature, he concludes that CBDC impacts central bank interest rates, monetary policy implementation and the transmission mechanism. The scale of these effects depends on the design and demand for this new form of money.

Kim and Kwon (2019) apply a monetary general equilibrium model to explore the implications of CBDC on financial stability. The study shows that deposits in CBDC accounts decrease the supply of private credit by commercial banks, which has a negative effect on financial stability via increasing the likelihood of a bank panic. However, once the central bank can lend all the deposits in CBDC account to commercial banks, an increase in the quantity of CBDC can enhance financial stability.

Bindseil (2020) reviews the CBDC advantages, i.e. efficient payments, anti-illegal activities, strengthened monetary policy (negative interest rates are possible), higher seigniories income, etc. Possible risks are structural disintermediation of banks, systemic runs on banks, centralization of the credit allocation process within the central bank, etc. They propose a two-tier remuneration of CBDC as a solution.

Bindseil and Fabio (2020) point out that a two-tier remuneration system for the CBDC would be an efficient solution to issues like bank disintermediation, negative interest rate policy, financial stability, etc. A tiered remuneration of CBDC would achieve four key objectives, namely, offering attractive CBDC as a means of payment to households, offering CBDC in a quantitatively unconstrained manner to any holder (not just citizens), controlling the risks of structural or cyclical bank disintermediation, and enabling negative interest rates.

1.4. Article organization

Section 2 presents the model. Section 3 analyzes the model. Section 4 illustrates the solution. Section 5 discusses the results and provides economic intuition and policy implications. Section 6 presents shortcomings and future research. Section 7 concludes.

2. The model

A non-cooperative static simultaneous-move one-period game is played between a representative household and a unitary player comprising the interests and capabilities of a central bank and a government, referred to as the central bank, for simplicity. The household and central bank choose their strategies simultaneously and independently. The static analysis is assumed to represent a stationary situation through time where the players adapt optimally to each other in a manner that can be expressed at one point in time. The stationary situation implicitly accounts for the nature of interest rates where resources usually have to be held for a certain amount of time in order for interest to be earned. Mathematically this amount of time can be made arbitrarily small. Hence, in a stationary situation, interest can be assumed earned at the same point in time in which the players choose their strategies and earn their utilities. Appendix A shows the nomenclature.

2.1. The household's strategic choices and utility

The representative household has available monetary energy r, which also can be interpreted as resources, converted at unit cost 1 into consumption c, CBDC (Central Bank Digital Currency) m, and some non-CBDC q, i.e.

$$r = c + m + q \tag{1}$$

where c, m, q are scaled equivalently on some appropriate scale, which may be any scale, e.g., of monetary nature. Hence, since c, m, q are scaled equivalently in (1), we assume no coefficients before c, m, q, which means that the coefficients equal 1. Equation (1) means that the household accepts and adopts both CBDC m and non-CBDC q. The household demands optimal amounts of CBDC m and non-CBDC q, and weighs these demands against its consumption c to maximize its utility U developed below.

A CBDC m is in this model interpreted as any currency issued by the central bank with an interest rate I_m, $I_m \in \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, which includes e.g., the Chinese e-CNY. A non-CBDC q is interpreted as any asset earning an interest rate I_q, $I_q \in \mathbb{R}$, and which is not issued by and not controlled by a central bank. We may think of the non-CBDC q as a cryptocurrency such as Bitcoin. Both interest rates I_m and I_q can be positive or negative. That means that the non-CBDC q can earn a higher or lower interest rate than the CBDC m, as illustrated e.g., in Figure 1 panel i. The broad definitions of CBDC m and non-CBDC q in (1) work fine for the purpose of this article, where the household allocates its monetary energy r into the three destinations consumption c, CBDC m with interest rate I_m, and some non-CBDC q with interest rate I_q.

We develop the household's Cobb Douglas utility in four steps. First, the household has a Cobb Douglas utility with three output elasticities $\alpha, \beta, 1-\alpha-\beta$, $0 \le \alpha \le 1$, $0 \le \beta \le 1$, $0 \le 1-\alpha-\beta \le 1$, for consumption c, CBDC m, and some non-CBDC q, i.e.

$$U_1 = c^{\alpha} m^{\beta} q^{1-\alpha-\beta} \tag{2}$$

which expresses constant returns to scale, since the three exponents sum to 1. Second, the household earns interest I_m, $I_m \in \mathbb{R}$, on CBDC m, and earns interest I_q, $I_q \in \mathbb{R}$, on the non-CBDC q. Interest rates are usually positive, but can for digital currencies, and especially for CBDC m, be negative. Earning interest rates I_m and I_q on CBDC m and non-CBDC q means multiplying m and q with $1 + I_m$ and $1 + I_q$, respectively. Incorporating these multiplications into (2) gives

$$U_2 = c^{\alpha} (m(1 + I_m))^{\beta} \left(q(1 + I_q)\right)^{1-\alpha-\beta} \tag{3}$$

Third, a simultaneous-move game is analyzed which can be interpreted as a stationary situation where time plays no role. Equation (1) is interpreted so that the household converts its resources r into consumption c, CBDC m, and non-CBDC q. This conversion involves transaction costs which impacts the household's utility. In order to transact between consumption c, CBDC m, and non-CBDC q, the household seeks to obtain high transaction efficiency, which means that it has to pay

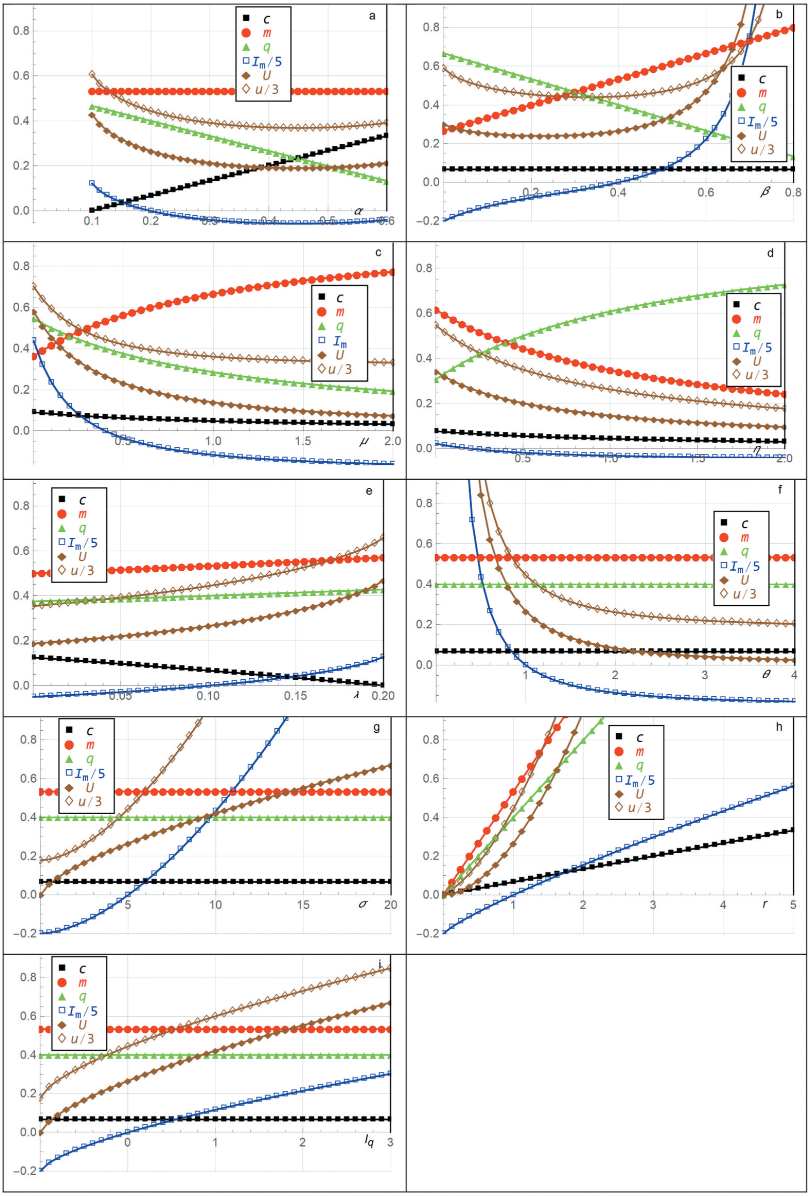

Figure 1. The household's consumption c, holding of CBDC m, holding of non-CBDC q, and utility U, and the central bank's interest rate I_m and profit u, as functions of the nine parameter values $\alpha, \lambda_n, \lambda_g, \beta_n, \gamma_n, \theta_g, a_n, a_g$ relative to the benchmark parameter values $\alpha = \eta = 1/5, \beta = \mu = 2/5, \lambda = 1/10, \theta = r = 1, \sigma = 5, I_q = 0$. Division of I_m with 5 and u with 3 is for scaling purposes.

transaction costs. Transactions are never free. Costs are always involved when transacting. With other conditions unchanged, a high transaction efficiency means a lower transaction cost. We define the household's transaction efficiency E to increase with holding CBDC m and holding non-CBDC q, and decrease with consumption c, i.e.

$$E = \frac{m^{\mu} q^{\eta}}{\theta c^{\lambda}} \tag{4}$$

where μ, $\mu \geq 0$, is the household's transaction efficiency for CBDC m; η, $\eta \geq 0$, is the household's transaction efficiency for non-CBDC q. The parameter λ is the household's transaction efficiency for consumption c, and $1/\theta, \theta \geq 0$, scales the degree or level of the household's transaction efficiency. We require $0 \leq \lambda \leq \alpha$ so that the household benefits positively from consumption, expressed as c^{α} in (3), despite the transaction cost $1/c^{\lambda}$ in (4). We also assume $\eta \geq \lambda$, so that the household's transaction efficiency η for non-CBDC q is higher than or equal to the household's transaction efficiency λ for consumption c.

The transaction efficiency E in (4) satisfies $\frac{\partial E}{\partial c} \leq 0, \frac{\partial E}{\partial m} \geq 0, \frac{\partial E}{\partial q} \geq 0, \frac{\partial^2 E}{\partial c^2} \geq 0, \frac{\partial^2 E}{\partial m^2} \leq 0 \text{ when } \mu \leq 1, \frac{\partial^2 E}{\partial q^2} \leq 0 \text{ when } \eta \leq 1, \frac{\partial^2 E}{\partial c \partial m} \leq 0, \frac{\partial^2 E}{\partial c \partial q} \leq 0$, see Appendix B. Thus E decreases convexly in consumption c, and increases in the CBDC m and the non-CBDC q. For related accounts of the transaction efficiency E, usually conceptualized as the transaction cost $1/E$, see, Feenstra (1986), Bougheas (1994), and Saygılı (2012).

The literature usually considers the inverse $1/E$ of (4) interpreted as the transaction cost, where θ scales the transaction cost. Higher transaction efficiency for CBDC m than for non-CBDC q, to sustain negative interest rates $I_m<0$ on CBDC m, requires $\mu>\eta$, which we generally do not require since we in principle can envision even more negative interest rates $I_q<I_m<0$ for non-CBDC q. Multiplying (4) with (3) gives the household's utility

$$U_3 = c^{\alpha} (m(1 + I_m))^{\beta} \left(q\left(1 + I_q\right)\right)^{1-\alpha-\beta} \frac{m^{\mu} q^{\eta}}{\theta c^{\lambda}} \tag{5}$$

Fourth, the household's resource constraint in (1) expresses that the household has two free choice variables, i.e. consumption c and CBDC m, where non-CBDC $q = r - c - m$ follows from solving (1) with respect to q. Inserting $q = r - c - m$ into (5) gives

$$U = c^{\alpha} (m(1 + I_m))^{\beta} \left((r - c - m)\left(1 + I_q\right)\right)^{1-\alpha-\beta} \frac{m^{\mu} (r - c - m)^{\eta}}{\theta c^{\lambda}} \tag{6}$$

which has two strategic choice variables c and m, and which is the household's utility U which we analyze in the remainder of the article.

2.2. The central bank's strategic choice and profit

We consider the central bank and the government as one unitary player, referred to as the central bank for simplicity, with the ability to choose the CBDC interest rate I_m, $I_m \in \mathbb{R}$. Common objectives for central banks usually include financial stability including price stability, and controlling inflation, unemployment, interest rates, or exchange rates. To obtain these objectives central banks are often assumed to choose discretionary policies. Some literature, e.g., Taylor (1993), assumes that central banks follow certain rules, without evidence of specific rules actually being applied. One may hypothesize that central banks follow certain norms, e.g., as philosophically expounded by Kant (1785) for which evidence is also not apparent. Given the common presence of maximizing behavior for players tasked with reaching objectives, this article assumes that also the central bank maximizes to reach its stated objectives. Although the literature agrees that central banks have objectives, the literature does not agree on what central banks actually maximize to reach these objectives. One might assume that central banks minimize deviations from specified targets related to financial stability including price stability, inflation, unemployment, interest rates, or exchange rates. One problem with that approach is that it is not directly linked to what each household may perceive as its objectives. Each household may not agree with the specified targets, may not agree with which of the many objectives the central bank seeks to reach, or may consider the central bank's objectives as too abstract. As outlined in the previous section 2.1, each household may find it easier to focus more concretely on its resource allocation into consumption, holding CBDC and holding non-CBDC, instead of somehow conceptualizing the general price level or some of the other central bank's objectives. To formalize how the central bank maximizes to reach objectives, this article assumes that the central bank identifies partly with each household, with the utility in (6). That assumption is made by reasoning that the central bank's wide-ranging objectives listed above are compatible with creating an environment within which each household can flourish in the sense of maximizing its utility. That the central bank's profit per household function is linear in the household's utility is assumed to be a suitable first approximation. Future research may explore whether various kinds of nonlinear relationships may be appropriate. Additionally, the central bank pays interest mI_m to each household, which is subtracted from (6) to yield the central bank's profit per household

$$u = \sigma c^{\alpha}(m(1+I_m))^{\beta}\left((r-c-m)(1+I_q)\right)^{1-\alpha-\beta}\frac{m^{\mu}(r-c-m)^{\eta}}{\theta c^{\lambda}} - mI_m \tag{7}$$

where the parameter σ, $\sigma>0$, is multiplied with the first term for scaling purposes. That is, the subtracted term mI_m is measured along some monetary scale, and σ enables the first term to be measured along the same monetary scale, and hence we refer to u as profit. Equation (7) expresses that the central bank identifies partly with each household, weighted with the parameter σ, and subtracting the interest mI_m paid to each household.

2.3. Methodology

The article applies non-cooperative game theory (Fujiwara-Greve, 2015; Von Neumann & Morgenstern, 1944) assuming two players, i.e. a representative household and a central bank. Each player is fully rational and has complete information about the game and all parameter values. The players choose their strategies simultaneously and independently to maximize their utilities. For the household the utility is a Cobb Douglas utility multiplied with a transaction efficiency E. For the central bank the utility is a profit function defined as a benefit minus a cost mI_m. Both players' utilities depend on the two players' three strategic choice variables c, m and I_m. The game is a so-called variable sum game which means that the sum of the players' utilities depend on their strategies. The game's solution amounts to determining a Nash equilibrium Nash (1951) from which no player prefers to deviate unilaterally when choosing its strategy.

3. Analyzing the model

3.1. Analyzing the household

Lemma 1. The household's consumption c, holding of CBDC m, and holding of non-CBDC q are

$$c = \frac{r(\alpha-\lambda)}{1+\eta-\lambda+\mu}, m = \frac{r(\beta+\mu)}{1+\eta-\lambda+\mu}, q = \frac{r(1-\alpha-\beta+\eta)}{1+\eta-\lambda+\mu} \tag{8}$$

with characteristics shown in and discussed after the Proposition.

Proof. Appendix C.

3.2. Analyzing the central bank

Lemma 2. The central bank's CBDC interest rate I_m for the household's holding of CBDC m is

$$I_m = \left(\frac{\theta(\beta+\mu)^{1-\beta-\mu}}{\sigma\beta(\alpha-\lambda)^{\alpha-\lambda}}\left(\frac{r}{1+\eta-\lambda+\mu}\right)^{\lambda-\mu-\eta}(1+\eta-\alpha-\beta)^{\alpha+\beta-\eta-1}\left(1+I_q\right)^{\alpha+\beta-1}\right)^{\frac{1}{\beta-1}} - 1 \tag{9}$$

with characteristics shown in and discussed after the Proposition.

Proof. Appendix D.

3.3. Analyzing the household and the central bank

Lemma 3. The household's utility U and the central bank's profit per household u are

$$U = \left(1+I_q\right)^{1-\frac{\alpha}{1-\beta}} r^{\frac{1-\beta+\eta-\lambda+\mu}{1-\beta}} \beta^{\frac{\beta}{1-\beta}} (1-\alpha-\beta+\eta)^{1-\frac{\alpha-\eta}{1-\beta}} \theta^{\frac{-1}{1-\beta}} (\alpha-\lambda)^{\frac{\alpha-\lambda}{1-\beta}} (\beta+\mu)^{\frac{\mu}{1-\beta}} (1+\eta-\lambda+\mu)^{\frac{-(1-\beta+\eta-\lambda+\mu)}{1-\beta}} \sigma^{\frac{\beta}{1-\beta}} \tag{10}$$

$$\begin{aligned} u = {} & \left(1+I_q\right)^{1-\frac{\alpha}{1-\beta}} r^{\frac{1-\beta+\eta-\lambda+\mu}{1-\beta}} \beta^{\frac{\beta}{1-\beta}} (1-\alpha-\beta+\eta)^{1-\frac{\alpha-\eta}{1-\beta}} \theta^{\frac{-1}{1-\beta}} (\alpha-\lambda)^{\frac{\alpha-\lambda}{1-\beta}} (\beta+\mu)^{\frac{\mu}{1-\beta}} \\ & (1+\eta-\lambda+\mu)^{\frac{-(1-\beta+\eta-\lambda+\mu)}{1-\beta}} \sigma^{\frac{1}{1-\beta}} + \frac{1}{1+\eta-\lambda+\mu} r(\beta+\mu \\ & -\left(1+I_q\right)^{1-\frac{\alpha}{1-\beta}} r^{\frac{\eta-\lambda+\mu}{1-\beta}} \beta^{\frac{1}{1-\beta}} (1-\alpha-\beta+\eta)^{\frac{1-\alpha-\beta+\eta}{1-\beta}} \theta^{\frac{-1}{1-\beta}} (\alpha-\lambda)^{\frac{\alpha-\lambda}{1-\beta}} (\beta+\mu)^{\frac{\mu}{1-\beta}} (1+\eta-\lambda+\mu)^{\frac{\lambda-\mu-\eta}{1-\beta}} \sigma^{\frac{1}{1-\beta}}) \end{aligned} \tag{11}$$

with characteristics shown in and discussed after the Proposition.

Proof. Follows from inserting (8) and (9) into (6) and (7).

Proposition

$$\frac{\partial c}{\partial \alpha} \geq 0, \frac{\partial c}{\partial \beta} = 0, \frac{\partial c}{\partial \mu} = \frac{\partial c}{\partial \eta} \leq 0, \frac{\partial^2 c}{\partial \mu^2} = \frac{\partial^2 c}{\partial \eta^2} \geq 0, \frac{\partial c}{\partial \lambda} \leq 0, \frac{\partial^2 c}{\partial \lambda^2} \leq 0, \frac{\partial c}{\partial \theta} = \frac{\partial c}{\partial \sigma} = \frac{\partial c}{\partial I_q} = 0,$$
$$\frac{\partial c}{\partial r} \geq 0, \frac{\partial m}{\partial \alpha} = 0, \frac{\partial m}{\partial \beta} \geq 0, \frac{\partial m}{\partial \mu} \geq 0, \frac{\partial^2 m}{\partial \mu^2} \leq 0, \frac{\partial m}{\partial \eta} = -\frac{\partial m}{\partial \lambda} \leq 0, \frac{\partial^2 m}{\partial \eta^2} = \frac{\partial^2 m}{\partial \lambda^2} \geq 0, \frac{\partial m}{\partial \theta} = \frac{\partial m}{\partial \sigma} =$$
$$\frac{\partial m}{\partial I_q} = 0, \frac{\partial m}{\partial r} \geq 0, \frac{\partial q}{\partial \alpha} \leq 0, \frac{\partial q}{\partial \beta} \leq 0, \frac{\partial q}{\partial \mu} = -\frac{\partial q}{\partial \lambda} \leq 0, \frac{\partial^2 q}{\partial \mu^2} = \frac{\partial^2 q}{\partial \lambda^2} \geq 0, \frac{\partial q}{\partial \eta} \geq 0, \frac{\partial^2 q}{\partial \eta^2} \leq 0, \frac{\partial q}{\partial \theta} = \frac{\partial q}{\partial \sigma} =$$
$$\frac{\partial q}{\partial I_q} = 0, \frac{\partial q}{\partial r} \geq 0, \frac{\partial I_m}{\partial \alpha} \propto -Ln\left(1+I_q\right) - Ln(1+\eta-\alpha-\beta) + Ln(\alpha-\lambda), \frac{\partial I_m}{\partial \theta} \leq 0, \frac{\partial^2 I_m}{\partial \theta^2} \geq 0,$$
$$\frac{\partial I_m}{\partial \sigma} \geq 0, \frac{\partial^2 I_m}{\partial \sigma^2} \geq 0, \frac{\partial I_m}{\partial r} \geq 0 \text{ when } \eta-\lambda+\mu \geq 0, \frac{\partial^2 I_m}{\partial r^2} \geq 0 \text{ when } (\eta-\lambda+\mu)(-1+\beta+\eta-\lambda+\mu)$$
$$\geq 0, \frac{\partial I_m}{\partial I_q} \geq 0, \frac{\partial^2 I_m}{\partial I_q^2} \leq 0, \frac{\partial U}{\partial \alpha} \propto -Ln\left(1+I_q\right) - Ln(1+\eta-\alpha-\beta) + Ln(\alpha-\lambda),$$
$$\frac{\partial U}{\partial \theta} \leq 0, \frac{\partial^2 U}{\partial \theta^2} \geq 0, \frac{\partial U}{\partial \sigma} \geq 0, \frac{\partial^2 U}{\partial \sigma^2} \leq 0 \text{ when } 2\beta \leq 1, \frac{\partial U}{\partial r} \geq 0 \text{ when } 1-\beta+\eta-\lambda+\mu \geq 0,$$
$$\frac{\partial^2 U}{\partial r^2} \text{ when } \eta-\lambda+\mu \geq 0, \frac{\partial U}{\partial I_q} \geq 0, \frac{\partial^2 U}{\partial I_q^2} \leq 0$$

Proof. Follows from (22), (23), (24), (25) in Appendix E, where $\alpha \geq \lambda$ implies $1+\eta-\lambda+\mu \geq 0$ since $1 \geq \alpha$.

The Proposition states, first, that the household's consumption c, holding of CBDC m, and holding of non-CBDC q, increases, is independent, and decreases in its output elasticity α for consumption c. That is, as consumption becomes more important, the household consumes more and holds less non-CBDC q. When $-Ln\left(1+I_q\right) - Ln(1+\eta-\alpha-\beta) + Ln(\alpha-\lambda) \leq 0$, which is satisfied when α is not too high, increasing α causes the central bank to decrease its interest rate I_m, which is consistent with higher consumption c, and causes lower household's utility U, consistently with the lower interest rate I_m.

Second, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, is independent, increases, and decreases in its output elasticity β for holding CBDC m. That is, as holding CBDC m becomes more important, the household holds more CBDC m, and holds less non-CBDC q.

Third, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, decreases convexly, increases concavely, and decreases convexly, in its transaction efficiency μ for CBDC m. That is, as CBDC m transactions become more efficient, the household holds more CBDC m, consumes less, and holds less non-CBDC q.

Fourth, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, decreases convexly, decreases convexly, and increases concavely, in its transaction efficiency η for non-CBDC q. That is, as non-CBDC transactions become more efficient, the household holds more non-CBDC q, consumes less, and holds less CBDC m.

Fifth, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, decreases concavely, increases convexly, and increases convexly, in its transaction efficiency λ for consumption c. That is, as consumption c transactions become more efficient, which in (6) implies less weight to consumption c due to the term $c^{\alpha-\lambda}$, the household consumes less, and holds more CBDC m and more non-CBDC q.

Sixth, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, are independent of the household's scaling θ of the transaction cost. The central bank's interest rate I_m and the household's utility U decrease convexly in θ. That is, higher transaction cost θ is costly for the household. That cost is to some extent experienced by the central bank in (7) which compensates by choosing lower interest rate I_m which makes the second cost term $-mI_m$ lower in absolute value, and positive if the interest rate I_m is negative.

Seventh, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, are independent of the scaling parameter σ for the central bank's profit. The central bank's interest rate I_m increases convexly in σ, which according to (7) enables the central bank to profit substantially. The household's utility U increases concavely in σ when $2\beta \leq 1$, and otherwise increases convexly, as the household benefits from the higher interest rate I_m.

Eighth, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, increase linearly in the household's monetary energy, or resources, r. When $\eta - \lambda + \mu \geq 0$, the central bank's interest rate I_m increases in r, as the central bank identifies partly with the household's utility in (6), and pays higher interest rate I_m on the household's increased holding of CBDC m. When $1 - \beta + \eta - \lambda + \mu \geq 0$, the household's utility U increases in r, as the household benefits from the higher interest rate I_m on its increased holding of CBDC m.

Ninth, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, are independent of the non-CBDC's interest rate I_q. The central bank's interest rate I_m and the household's utility U increase concavely in I_q. That is, the household benefits from the higher interest rate I_q on its holding of non-CBDC q, which induces the central bank competitively to increase its interest rate I_m to prevent the household from changing its holding from CBDC m to non-CBDC q.

Table 1 summarizes the main results in the Proposition with an upward arrow $\uparrow$, sideways arrow $\rightarrow$, or downward arrow $\downarrow$, respectively, depending on whether the first order derivative

Table 1. Upward arrow ↑, sideways arrow →, or downward arrow ↓, respectively, depending on whether the first order derivative (listed first) and second order derivative (listed second) in the proposition are positive, zero or negative. The derivatives are for the variable in the row with respect to the parameter in the column.

Variable	α	β	μ	η	λ	θ	σ	r	l_q
c	↑→	→	↓↑	↓↑	↓↓	→	→	↑→	→
m	→	↑→	↑↓	↓↑	↑↑	→	→	↑→	→
q	↓→	↓→	↓↑	↑↓	↑↑	→	→	↑→	→
l_m						↓↑	↑↑		↑↓
U						↓↑	↑		↑↓

(listed first) and second order derivative (listed second) are positive, zero or negative. The derivatives are for the variable in the row with respect to the parameter in the column. Empty cells means that the signs of the derivatives contain if-conditions as expressed in the Proposition and Appendix E. Only one sideways arrow $\rightarrow$ is listed if the first order derivative and all higher order derivatives equal zero. Only the upward arrow $\uparrow$ is listed for $\frac{\partial U}{\partial \sigma} \geq 0$ since $\frac{\partial^2 U}{\partial \sigma^2} \leq 0$ *when* $2\beta \leq 1$.

4. Illustrating the solution

To illustrate the solution in section 3, this section alters the nine parameter values $\alpha, \beta, \mu, \eta, \lambda, \theta, \sigma, r, I_q$ relative to the benchmark parameter values $\alpha = \eta = 1/5, \beta = \mu = 2/5, \lambda = 1/10, \theta = r = 1, \sigma = 5, I_q = 0$. First, $\alpha = 1/5$ expresses relatively low weight or elasticity for consumption *c*. Second, $\beta = 1 - \alpha - \beta = 2/5$ reflects equal and higher weight or elasticity for CBDC *m* and non-CBDC *q*. Third, $\eta = 1/5$ reflects intermediate transaction efficiency for non-CBDC *q*. Fourth, $\mu = 2/5$ reflects twice as high transaction efficiency for CBDC *m*. Fifth, $\lambda = 1/10$ reflects low transaction efficiency for consumption *c*. Sixth, $I_q = 0$ expresses zero interest rate for non-CBDC *q*, as a plausible benchmark relative to which the CBDC interest rate I_m may be higher or lower. Seventh, $\sigma = 5$ is chosen so that the CBDC interest rate $I_m = 0$ at the benchmark. Eighth, $\theta = r = 1$ are chosen due to simplicity and since the value 1 seems plausible when no other value may appear more plausible. With these benchmark parameter values the benchmark solution is $c = 1/15 \approx 0.067, m = 8/15 \approx 0.53, q = 2/5 = 0.4, I_m/5 = 0.00, U = 0.27, u/3 = 0.44$. In Figure 1 each of the nine parameter values is altered from its benchmark, while the other eight parameter values are kept at their benchmarks. Division of I_m with 5 and *u* with 3 is for scaling purposes.

In Figure 1a, as the household's output elasticity α for consumption *c* increases, its consumption *c*, holding of CBDC *m*, and holding of non-CBDC *q*, increases, is independent, and decreases. When α is high, the household values consumption *c* more and non-CBDC *q* less. Except when α is very high, as α increases, the central bank's interest rate I_m decreases and becomes negative when $\alpha > 1/5$. Furthermore, the household's utility *U* decreases since it earns less interest on its holding of CBDC *m*, and the central bank's profit per household *u* decreases since it identifies partly with the household as expressed in (7) compared with (6).

In Figure 1b, as the household's output elasticity β for CBDC *m* increases, its consumption *c*, holding of CBDC *m*, and holding of non-CBDC *q*, is independent, increases, and decreases. When β is high, the household values CBDC *m* more and non-CBDC *q* less. Valuing CBDC *m* more is consistent with higher CBDC interest rate I_m, which eventually causes higher household's utility *U* and higher central bank's profit per household *u*.

In Figure 1c as the household's output transaction efficiency μ for CBDC *m* increases, its consumption *c*, holding of CBDC *m*, and holding of non-CBDC *q*, decreases convexly, increases concavely, and decreases convexly. More efficient CBDC *m* transactions cause the household to hold more CBDC *m*, consume less, and hold less non-CBDC *q*. That the household holds more CBDC *m* is costly for the central bank, as expressed with $-mI_m$ in (7), which is negative when $I_m \geq 0$. Hence, as μ increases, the central bank decreases its interest rate I_m which eventually becomes negative. That's costly for the household which receives decreasing utility *U*, and costly for the central bank which identifies partly with the household and receives decreasing profit *u*.

In Figure 1d, as the household's transaction efficiency η for non-CBDC *q* increases, its consumption *c*, holding of CBDC *m*, and holding of non-CBDC *q*, decreases convexly, decreases convexly, and increases concavely. More efficient non-CBDC *q* transactions cause the household to hold less CBDC *m*, consume less, and hold more non-CBDC *q*. With the specified parameter values, that causes the central bank to decrease its interest rate I_m marginally, causing the household's utility *U* and the central bank's profit per household *u* to decrease.

In Figure 1e, as the household's transaction efficiency λ for consumption *c* increases, its consumption *c*, holding of CBDC *m*, and holding of non-CBDC *q*, decreases concavely, increases convexly, and increases convexly. More efficient consumption c transactions enable the household to hold more CBDC m and more non-CBDC q, and consume less. The central bank responds by increasing its CBDC interest rate I_m, which causes higher household's utility U and higher central bank's profit per household u.

In Figure 1f, as the household's scaling θ of the transaction cost increases, its consumption c, holding of CBDC m, and holding of non-CBDC q, do not change. The higher cost θ has to be born by someone, so the central bank decreases its interest rate I_m which becomes negative, and the household's utility U and the central bank's profit per household u decrease.

In Figure 1g, as the scaling parameter σ for the central bank's profit per household increases, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, do not change. In contrast to higher θ which is a cost, higher σ is a benefit, and thus the central bank increases its interest rate I_m, and the household's utility U and the central bank's profit per household u increase.

In Figure 1h, as the household's monetary energy, or resources, r, increases, its consumption c, holding of CBDC m, and holding of non-CBDC q, increase. That's beneficial for both players causing the central bank's interest rate I_m and profit u, and the household's utility U, to increase.

In Figure 1i, as the non-CBDC interest rate I_q increases, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, do not change. Higher I_q causes the central bank to increase its CBDC interest rate I_m, which causes the central bank's interest rate I_m and profit u, and the household's utility U, to increase.

5. Discussion, economic intuition and policy implications

Nine results in the previous section are noteworthy. First, as the household's output elasticity α for consumption increases, it consumes more, holds the same amount of CBDC m, and holds less non-CBDC q. Except when α is high, the central bank's interest rate I_m and the players' utility U and profit u decrease. The intuition is that higher household consumption causes the household to decrease holding something. It chooses to hold less non-CBDC q. The central bank's decreased benefit from the positive term in (7) induces it to strike a different tradeoff or balance between benefit and cost expressed with the negative term in (7), causing decreased and negative CBDC interest rate I_m. As α increases from the low value $\alpha = 1/10$, we get the conventional relationship where the household responds to decreasing CBDC interest rate I_m by consuming more. Interestingly, as α increases above $\alpha = 1/5$ and the central bank's interest rate I_m becomes negative, the household pays the central bank for holding its CBDC m. That is possible according to the Cobb Douglas logic in (6) since the household values holding CBDC m, despite having to pay for it, in combination with the other ingredients of (6). Naturally, a limit exists for how much the household is willing to pay the central bank. Hence the central bank's negative interest rate I_m levels out and starts increasing from a minimum $I_m = -0.057$ when $\alpha = 0.45$. The increasing I_m in principle curtails the household's consumption c, which nevertheless continues to increase since α as the household's output elasticity α for consumption c constitutes a stronger force and has higher impact. The policy implication is that the household and central bank should be conscious about how they impact each other. Negative CBDC interest rate I_m can indeed be associated with increased consumption c. The central bank needs to assess the household's Cobb Douglas preferences broadly within the economy, to determine how negative the CBDC interest rate I_m can be allowed to be.

Second, and as a contrast, as the household's output elasticity β for holding CBDC m increases, it holds more CBDC m and less non-CBDC q, the CBDC interest rate I_m increases, and the players' utility U and profit u eventually increase. The intuition is that the household chooses to hold CBDC m or non-CBDC q depending on what it considers most valuable. Furthermore, if holding CBDC m is sufficiently valuable for the household, the central bank increases its interest rate I_m from negative to positive.

The household benefits in terms of the interest payment. The central bank benefits due to identifying partly with the household, which offsets its cost of the interest payment to the household. The policy implication is to be conscious of how a household assesses the value of holding CBDC m relative to holding non-CBDC q, which impacts the household's strategies choices and the CBDC interest rate I_m.

Third, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, decreases, increases, and decreases, in its transaction efficiency μ for CBDC m. The CBDC interest rate I_m decreases and becomes negative, and the players' utility U and profit u decrease. The intuition is that more efficient CBDC m transactions encourage the household to consume less, hold more CBDC m, and hold less non-CBDC q. That the household holds more CBDC m is costly for the central bank unless it decreases its interest rate I_m to become negative so that it receives interest payment from the household for holding CBDC m. The household receives decreasing utility U due to paying increased interest rate to the central bank. The central bank receives decreased profit u due to identifying partly with the household. The policy implication is to realize the implications of increased CBDC transaction efficiency μ, eventually causing negative CBDC interest rate I_m because of the household's increased holding of CBDC m.

Fourth, the household's consumption c, holding of CBDC m, and holding of non-CBDC q, decreases, decreases, and increases, in its transaction efficiency η for non-CBDC q. The CBDC interest rate I_m decreases and becomes negative, and the players' utility U and profit u decrease. The intuition is that more efficient non-CBDC q transactions encourage the household to consume less, hold less CBDC m, and hold more non-CBDC q. That's costly for the central bank which has to pay more in interest to the household. These results are qualitatively in the same direction as for the third result when the transaction efficiency μ for CBDC m increases, except that the household's holding of CBDC m and holding of non-CBDC q, intuitively, move in the opposite direction. The explanation is that both the transaction efficiencies μ and η appear in the numerator in (4), which both have the opposite impact compared with the impact of the household's transaction efficiency λ for consumption c, which appears in the denominator in (4). The policy implication is to realize the implications of increased non-CBDC transaction efficiency η, eventually causing negative CBDC interest rate I_m because of the household's increased holding of non-CBDC q.

Fifth, as the household's transaction efficiency λ for consumption c increases, its consumption c, holding of CBDC m, and holding of non-CBDC q, decreases, increases, and increases. The central bank increases its interest rate I_m, and the players' utility U and profit u increase. The intuition is that more efficient consumption c transactions enable the household to consume less, and hold more CBDC m and more non-CBDC q. The central bank appreciates this decreased consumption and responds by increasing its CBDC interest rate I_m, which is the opposite of results 3 and 4 where the CBDC interest rate I_m decreases. The policy implication is to realize that increasing the household's transaction efficiency λ for consumption c eventually causes positive CBDC interest rate I_m, contrary to results 3 and 4 where increasing transaction efficiencies μ and η for CBDC m and non-CBDC q eventually cause negative CBDC interest rate I_m.

Sixth, the CBDC interest rate I_m and the players' utility U and profit u decrease in the household's scaling θ of the transaction cost. The intuition is that a higher transaction cost θ is expensive for the household, which is partly experienced by the central bank, and compensated by choosing lower and negative CBDC interest rate I_m. The policy implication is to be conscious about the scaling θ of the household's transaction cost, which dysfunctionally can cause negative CBDC interest rate I_m and low players' utility U and profit u.

Seventh, and in contrast to the sixth result, the CBDC interest rate I_m and the players' utility U and profit u increase in the scaling parameter σ for the central bank's profit per household u. The intuition is that higher σ benefits the central bank, enabling it to pay higher and eventually positive CBDC interest rate I_m to the household, incurred as a cost mI_m in (7), which in turn benefits the central bank which identifies partly with the household. The policy implication is to realize which factors constitute a benefit for the central bank, which is weighed against the central bank's potential cost of paying interest to the household for holding CBDC m.

Eighth, as the household's monetary energy, or resources, r, increases, the players' three free choice variables c, m, I_m, and the three dependent variables q, U, u, increase. The intuition is that a more resourceful household can consume more and hold more CBDC m and more non-CBDC q, which benefits the household and the central bank which identifies partly with the household. This in turn enables the central bank to pay more interest to the household for holding CBDC m. The policy implication is to assess how each household can be made more resourceful, which causes all the variables to increase.

Ninth, the CBDC interest rate I_m and the players' utility U and profit u increase in the non-CBDC interest rate I_q. The intuition is that the central bank faces the competition from the higher non-CBDC interest rate I_q by increasing its own CBDC interest rate I_m. The household benefits from holding non-CBDC q due to the higher non-CBDC interest rate I_q, which causes the central bank to benefit due to identifying partly with the household. This in turn enables the central bank to pay higher CBDC interest rate I_m to ensure that the household keeps holding CBDC m. Hence a reinforcing virtuous circle (the opposite of a vicious circle) arises which benefits everyone. The policy implication is to realize the positive relationship between the CBDC interest rate I_q and the non-CBDC interest rate I_q.

6. Shortcomings and future research

Future research, which implicitly specifies shortcomings of the current research, should consider several CBDCs and non-CBDCs, including other assets such as bonds, stocks, etc. Additional players can be introduced, such as distinguishing between the central banks and governments, modeling commercial banks, firms, financial institutions, accounting for different kinds of households, etc. Alternative functional forms may be explored. Non-functional forms may also be explored, which may enable more generality, but fewer analytical solutions. Empirical evidence should be compiled for how households choose consumption, holding of CBDC and non-CBDC, with positive and negative CBDC interest rates. Households with different characteristics can be incorporated. The players may be assigned different risk attitudes. The players' Cobb Douglas utilities may account for additional factors beyond transaction efficiency, such as privacy, convenience, security, taxes. The players' strategy sets may be extended. For example, each potentially different household may be allowed to choose production and leisure in addition to consumption. The analysis may be generalized to account for more than one time period, and allow players to move in various sequences or simultaneously in repeated games. Digital currencies are a relatively new innovation with markets that may be subject to rapid price swings, fluctuations and uncertainty. The sensitivity analysis in the current article accounts for substantial variation in nine parameter values, which may change with arbitrary rapidity in the sense that the time dimension is not present in the current model. A dynamic analysis accounting for the time dimension may capture the implications over time of price swings, fluctuations, uncertainty, etc. from multiple angles.

7. Conclusion

This article presents a game model between a representative household and a central bank assumed to incorporate the interests of a government. The household has resources converted into consumption, holding of CBDC (Central Bank Digital Currency) controlled by a central bank, and holding of non-CBDC which can be any asset not issued by and not controlled by a central bank. The central bank determines its interest rate. The non-CBDC also has an interest rate. Both these two interest rates can be positive or negative. A Cobb Douglas utility with three elasticities for the household is developed, which represents consumption, holding of CBDC, and holding of non-CBDC. This conceptualization is assumed to be realistic for how households operate in the real world, i.e. choosing to consume while also choosing to hold two currencies with different interest rates and transaction efficiencies. The central bank identifies partly with each household, and pays interest to each household, which is subtracted to yield the central bank profit per household.

The article determines the household's consumption and holding of CBDC and the central bank's interest rate analytically, from which the dependent variables follow. Various interesting results follow. First, as the household's output elasticity for consumption increases, it consumes more and holds less non-CBDC, while the CBDC interest rate decreases and becomes negative. The central bank eventually imposes negative CBDC interest rate on the household since it identifies partly with the household which substitutes from holding non-CBDC and into consumption.

Second, as the household's output elasticity for holding CBDC increases, it holds more CBDC and less non-CBDC. Hence in contrast, the central bank eventually imposes positive CBDC interest rate on the household since it identifies partly with the household which substitutes from holding non-CBDC and into holding CBDC.

Third and fourth, the household's consumption, holding of CBDC, and holding of non-CBDC, decreases, increases (decreases), and decreases (increases), in its transaction efficiency for CBDC (non-CBDC). Increasing both the CBDC and non-CBDC transaction efficiencies eventually induces the central bank to choose negative interest rate, since it otherwise either must pay the household too much in interest or must identify with the household's decreased utility from consuming less and holding less CBDC.

Fifth, as the household's transaction efficiency for consumption increases, it consumes less, and holds more CBDC and more non-CBDC. In contrast to the third and fourth results, that encourages the central bank to increase its interest rate which becomes positive. The central bank pays more interest to the household, but identifies with the household and benefits from the household's benefit.

Sixth, the CBDC interest rate and the players' utility and profit decrease in the household's transaction cost, which is detrimental for both players, causing the central bank to burden the household with negative interest rates.

Seventh, and in contrast to the sixth result, the CBDC interest rate and the players' utility and profit increase in the scaling parameter for the central bank's profit, which benefits both players.

Eighth, as the household's monetary energy, or resources, increases, the household consumes more and holds more CBDC and non-CBDC, and the central bank increases its interest rate.

Ninth, the CBDC interest rate and the players' utility and profit increase in the non-CBDC interest rate. A higher non-CBDC interest rate induces the central bank competitively to increase the CBDC interest rate, to prevent the household from changing its holding from CBDC to non-CBDC.

The results are illustrated numerically, varying nine parameter values relative to a benchmark.

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Author details
Guizhou Wang[1]
Kjell Hausken[1]
E-mail: kjell.hausken@uis.no
ORCID ID: http://orcid.org/0000-0001-7319-3876
[1] Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway.

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Both authors contributed to all parts of the article.

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Notes

1. The Bitcoin White Paper was published by Satoshi Nakamoto on metzdowd.com's Cryptography Mailing List on October 31, 2008. It was subsequently published in Decentralized Business Review; https://www.debr.io/article/21260.
2. https://coinmarketcap.com/, retrieved 10 July 2022.
3. Other drawbacks of paper currencies are that they are less easily tracked, need to be replaced, can be lost and counterfeited, and can be cumbersome to transport.

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Appendix A Nomenclature

Parameters	
r	Household's monetary energy, or resources, $r \geq 0$
α	Household's output elasticity for consumption c, $\lambda \leq \alpha \leq 1$
β	Household's output elasticity for CBDC m, $0 \leq \beta \leq 1$
$1-\alpha-\beta$	Household's output elasticity for non-CBDC q, $0 \leq 1-\alpha-\beta \leq 1$
I_q	Interest rate, $I_q \in \mathbb{R}$
μ	Household's transaction efficiency for CBDC m, $\mu \geq 0$
η	Household's transaction efficiency for non-CBDC q, $\eta \geq \lambda$
λ	Household's transaction efficiency for consumption c, $0 \leq \lambda \leq \alpha$
θ	Scaling or degree or level of the household's transaction cost, $\theta \geq 0$
σ	Scaling parameter for the central bank's profit, $\sigma > 0$
Household's free choice variables	
c	Household's consumption, $0 \leq c \leq r$
m	Household's holding of CBDC, $0 \leq m \leq r$
Central bank's free choice variable	
I_m	CBDC interest rate for the household's holding of CBDC m, $I_m \in \mathbb{R}$
Dependent variables	
U	Household's utility
u	Central bank's profit per household
$q=r-m-c$	Household's holding of non-CBDC, $0 \leq q = r-m-c \leq r$
E	Household's transaction efficiency

Appendix B The derivatives for the transaction efficiency *E*

Differentiating the transaction efficiency E in (4) with respect to c, m and q gives

$$\frac{\partial E}{\partial c} = -\frac{c^{-1-\lambda}m^{\mu}q^{\eta}\lambda}{\theta} \leq 0, \frac{\partial E}{\partial m} = \frac{c^{-\lambda}m^{-1+\mu}q^{\eta}\mu}{\theta} \geq 0, \frac{\partial E}{\partial q} = \frac{c^{-\lambda}m^{\mu}q^{-1+\eta}\eta}{\theta} \geq 0 \tag{12}$$

The second derivatives of the transaction efficiency E in (4) with respect to c, m and q gives

$$\begin{aligned}
&\frac{\partial^2 E}{\partial c^2} = \frac{c^{-2-\lambda}m^{\mu}q^{\eta}\lambda(1+\lambda)}{\theta} \geq 0, \\
&\frac{\partial^2 E}{\partial m^2} = \frac{c^{-\lambda}m^{-2+\mu}q^{\eta}(-1+\mu)\mu}{\theta} \leq 0 \text{ when } \mu \leq 1, \\
&\frac{\partial^2 E}{\partial q^2} = \frac{c^{-\lambda}m^{\mu}q^{-2+\eta}(-1+\eta)\eta}{\theta} \leq 0 \text{ when } \mu \leq 1, \\
&\frac{\partial^2 E}{\partial c \partial m} = -\frac{c^{-1-\lambda}m^{-1+\mu}q^{\eta}\lambda\mu}{\theta} \leq 0, \\
&\frac{\partial^2 E}{\partial c \partial m} = -\frac{c^{-1-\lambda}m^{-1+\mu}q^{\eta}\lambda\mu}{\theta} \leq 0
\end{aligned} \tag{13}$$

Appendix C Proof of Lemma 1

Differentiating the household's utility U in (6) with respect to its free choice variables c and m gives

$$\frac{\partial U}{\partial c}=\frac{1}{\theta}c^{\alpha-\lambda-1}(1+I_m)^{\beta}\left(1+I_q\right)^{1-\alpha-\beta}m^{\beta+\mu}(r-c-m)^{\eta-\alpha-\beta}\times((r-m)(\alpha-\lambda)-c(1-\beta+\eta-\lambda)) \tag{14}$$

$$\frac{\partial U}{\partial m}=-\frac{1}{\theta}c^{\alpha-\lambda}(1+I_m)^{\beta}\left(1+I_q\right)^{1-\alpha-\beta}m^{-1+\beta+\mu}(r-c-m)^{-\alpha-\beta+\eta}((c-r)(\beta+\mu)+m(1-\alpha+\eta+\mu)) \tag{15}$$

which are equated with zero and solved to yield c and m in (10). The dependent variable q follows from solving (1) with respect to q and inserting c and m. The second order conditions, inserting (15) and (10), are

$$\begin{aligned}\frac{\partial^2 U}{\partial c^2}&=-\frac{(\beta+\mu)(1-\beta+\eta-\lambda)}{\theta(\alpha-\lambda)}(1+I_m)^{\beta}\left(1+I_q\right)^{1-\alpha-\beta}q^{-\alpha-\beta+\eta}c^{\alpha-\lambda}m^{-1+\beta+\mu}\le 0,\\ \frac{\partial^2 U}{\partial m^2}&=-\frac{(1-\alpha+\eta+\mu)}{\theta}(1+I_m)^{\beta}\left(1+I_q\right)^{1-\alpha-\beta}q^{-\alpha-\beta+\eta}c^{\alpha-\lambda}m^{-1+\beta+\mu}\le 0\end{aligned} \tag{16}$$

The term $1-\beta+\eta-\lambda$ in (16) equals $\eta-\lambda$ when β has its maximum $\beta=1$. Hence $\frac{\partial^2 U}{\partial c^2}\le 0$ when $\eta\ge\lambda$. Since the household has two decision variables c and m, we determine the Hessian matrix

$$H=\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}=\begin{bmatrix}\frac{\partial^2 U}{\partial m^2} & \frac{\partial^2 U}{\partial mc}\\ \frac{\partial^2 U}{\partial cm} & \frac{\partial^2 U}{\partial c^2}\end{bmatrix} \tag{17}$$

$$=\frac{-(1+I_m)^{\beta}\left(1+I_q\right)^{1-\alpha-\beta}}{\theta q^{\alpha+\beta-\eta}c^{-\alpha+\lambda}m^{1-\beta-\mu}}\begin{bmatrix}1-\alpha+\eta+\mu & \beta+\mu\\ \beta+\mu & \frac{(\beta+\mu)(1-\beta+\eta-\lambda)}{(\alpha-\lambda)}\end{bmatrix}$$

To show that H in (17) is negative semi-definite, it is sufficient to show that (1) $|H_{11}|\le 0$ and (2) $\begin{vmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{vmatrix}\ge 0$ hold. Condition 1 obviously holds since $|H_{11}|=H_{11}=-\frac{(1-\alpha+\eta+\mu)}{\theta}(1+I_m)^{\beta}\left(1+I_q\right)^{1-\alpha-\beta}q^{-\alpha-\beta+\eta}c^{\alpha-\lambda}m^{-1+\beta+\mu}\le 0$. Condition 2 also holds, $\begin{vmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{vmatrix}=\frac{r(\alpha-\lambda)(\beta+\mu)}{\theta^2}(1+I_m)^{2\beta}\left(1+I_q\right)^{-2(-1+\alpha+\beta)}q^{1-2(\alpha+\beta-\eta)}c^{2(-1+\alpha-\lambda)}m^{2(-1+\beta+\mu)}\ge 0$, since $\alpha\ge\lambda$.

Appendix D Proof of Lemma 2

Differentiating the central bank's profit per household u in (7) with respect to its free choice variable I_m gives

$$\frac{\partial u}{\partial I_m}=\sigma c^{\alpha}m^{\beta}\left((r-c-m)\left(1+I_q\right)\right)^{1-\alpha-\beta}\frac{m^{\mu}(r-c-m)^{\eta}}{\theta c^{\lambda}}\beta(1+I_m)^{\beta-1}-m \tag{18}$$

which is equated with zero and solved to yield

$$I_m=\left(\frac{\theta}{\sigma\beta}c^{\lambda-\alpha}m^{1-\beta-\mu}(r-c-m)^{\alpha+\beta-\eta-1}\left(1+I_q\right)^{\alpha+\beta-1}\right)^{\frac{1}{\beta-1}}-1 \tag{19}$$

The second order conditions, inserting (15), are satisfied as negative, i.e.

$$\frac{\partial^2 u}{\partial I_m^2}=\frac{(\beta-1)\beta\sigma}{\theta}c^{\alpha-\lambda}(1+I_m)^{-2+\beta}\left(1+I_q\right)^{1-\alpha-\beta}m^{\beta+\mu}(-c-m+r)^{1-\alpha-\beta+\eta}\leq 0 \tag{20}$$

Inserting (8) into (19) gives (9).

Appendix E Proof of the Proposition

Differentiating (8) for c, m, q, differentiating (9) for I_m, and differentiating (6) (inserting (8) and (9)) for U, give

$$\frac{\partial c}{\partial \alpha}=\frac{r}{1+\eta-\lambda+\mu},\frac{\partial c}{\partial \beta}=0,\frac{\partial c}{\partial \mu}=\frac{\partial c}{\partial \eta}=\frac{-r(\alpha-\lambda)}{(1+\eta-\lambda+\mu)^2},\frac{\partial^2 c}{\partial \mu^2}=\frac{\partial^2 c}{\partial \eta^2}=\frac{2r(\alpha-\lambda)}{(1+\eta-\lambda+\mu)^3},$$
$$\frac{\partial c}{\partial \lambda}=\frac{-r(1-\alpha+\eta+\mu)}{(1+\eta-\lambda+\mu)^2},\frac{\partial^2 c}{\partial \lambda^2}=\frac{-2r(1-\alpha+\eta+\mu)}{(1+\eta-\lambda+\mu)^3},\frac{\partial c}{\partial \theta}=\frac{\partial c}{\partial \sigma}=\frac{\partial c}{\partial I_q}=0,\frac{\partial c}{\partial r}=\frac{\alpha-\lambda}{1+\eta-\lambda+\mu} \tag{21}$$

$$\frac{\partial m}{\partial \alpha}=0,\frac{\partial m}{\partial \beta}=\frac{r}{1+\eta-\lambda+\mu},\frac{\partial m}{\partial \mu}=\frac{r(1+\eta-\lambda-\beta)}{(1+\eta-\lambda+\mu)^2},\frac{\partial^2 m}{\partial \mu^2}=\frac{-2r(1+\eta-\lambda-\beta)}{(1+\eta-\lambda+\mu)^3},$$
$$\frac{\partial m}{\partial \eta}=-\frac{\partial m}{\partial \lambda}=\frac{-r(\beta+\mu)}{(1+\eta-\lambda+\mu)^2},\frac{\partial^2 m}{\partial \eta^2}=\frac{\partial^2 m}{\partial \lambda^2}=\frac{2r(\beta+\mu)}{(1+\eta-\lambda+\mu)^3},\frac{\partial m}{\partial \theta}=\frac{\partial m}{\partial \sigma}=\frac{\partial m}{\partial I_q}=0,$$
$$\frac{\partial m}{\partial r}=\frac{\beta+\mu}{1+\eta-\lambda+\mu} \tag{22}$$

$$\frac{\partial q}{\partial \alpha}=\frac{-r}{1+\eta-\lambda+\mu},\frac{\partial q}{\partial \beta}=\frac{-r}{1+\eta-\lambda+\mu},\frac{\partial q}{\partial \mu}=-\frac{\partial q}{\partial \lambda}=\frac{-r(1+\eta-\alpha-\beta)}{(1+\eta-\lambda+\mu)^2},$$
$$\frac{\partial^2 q}{\partial \mu^2}=\frac{\partial^2 q}{\partial \lambda^2}=\frac{2r(1+\eta-\alpha-\beta)}{(1+\eta-\lambda+\mu)^3},\frac{\partial q}{\partial \eta}=\frac{r(\alpha+\beta-\lambda+\mu)}{(1+\eta-\lambda+\mu)^2},\frac{\partial^2 q}{\partial \eta^2}=\frac{-2r(\alpha+\beta-\lambda+\mu)}{(1+\eta-\lambda+\mu)^3},$$
$$\frac{\partial q}{\partial \theta}=\frac{\partial q}{\partial \sigma}=\frac{\partial q}{\partial I_q}=0,\frac{\partial q}{\partial r}=\frac{1-\alpha-\beta+\eta}{1+\eta-\lambda+\mu} \tag{23}$$

$$\frac{\partial I_m}{\partial \alpha}\propto -Ln\left(1+I_q\right)-Ln(1+\eta-\alpha-\beta)+Ln(\alpha-\lambda),\frac{\partial I_m}{\partial \theta}\propto -1+\beta,\frac{\partial^2 I_m}{\partial \theta^2}\propto\frac{2-\beta}{(1-\beta)^2},$$
$$\frac{\partial I_m}{\partial \sigma}\propto\frac{1}{1-\beta},\frac{\partial^2 I_m}{\partial \sigma^2}\propto\frac{\beta}{(1-\beta)^2},\frac{\partial I_m}{\partial r}\propto\frac{\eta-\lambda+\mu}{1-\beta},\frac{\partial^2 I_m}{\partial r^2}\propto\frac{(\eta-\lambda+\mu)(-1+\beta+\eta-\lambda+\mu)}{(1-\beta)^2},$$
$$\frac{\partial I_m}{\partial I_q}\propto\frac{1-\alpha-\beta}{1-\beta},\frac{\partial^2 I_m}{\partial I_q^2}\propto\frac{-1+\alpha+\beta}{(1-\beta)^2} \tag{24}$$

$$\frac{\partial U}{\partial \alpha} \propto -Ln(1+I_q) - Ln(1+\eta-\alpha-\beta) + Ln(\alpha-\lambda), \frac{\partial U}{\partial \theta} \propto -1+\beta, \frac{\partial^2 U}{\partial \theta^2} \propto \frac{2-\beta}{(1-\beta)^2},$$
$$\frac{\partial U}{\partial \sigma} \propto \frac{1}{1-\beta}, \frac{\partial^2 U}{\partial \sigma^2} \propto \frac{-1+2\beta}{(1-\beta)^2}, \frac{\partial U}{\partial r} \propto \frac{1-\beta+\eta-\lambda+\mu}{1-\beta}, \frac{\partial^2 U}{\partial r^2} \propto \frac{(\eta-\lambda+\mu)(1-\beta+\eta-\lambda+\mu)}{(1-\beta)^2},$$
$$\frac{\partial U}{\partial I_q} \propto \frac{1-\alpha-\beta}{1-\beta}, \frac{\partial^2 U}{\partial I_q^2} \propto \frac{-1+\alpha+\beta}{(1-\beta)^2} \quad (25)$$

where $\propto$ means proportional to.

9. Wang, G. and Hausken, K. (2022), "A Two-Period Decision Model for Central Bank Digital Currencies and Households," International Journal of Finance & Banking Studies 11, 2, 49-68

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A Two-Period Decision Model for Central Bank Digital Currencies and Households

Guizhou Wang [(a)], *Kjell Hausken* [(a)*]

[(a)] *Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway*

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ABSTRACT

Central bank digital currencies (CBDCs) give rise to many possibilities including those of negative interest rates. A two-period decision model is presented between one central bank and one representative household. The central bank applies the Taylor (1993) rule to choose its interest rate. The household allocates its resources strategically to production, consumption, CBDC holding, and non-CBDC holding. The results are determined analytically and illustrated numerically by varying 19 parameter values. Interesting novelties of the article are that the central bank may choose negative CBDC interest rates when the household holds far more CBDC than non-CBDC, for low inflation rates, low real interest rates, low household's potential production, low weight assigned to inflation in the Taylor (1993) rule, high target inflation rate, and high household's production parameter. That usually causes the household to decrease its CBDC holding and increase its non-CBDC holding, production and consumption. The central bank may increase its CBDC interest rate to compete with an increasing non-CBDC interest rate if the household's transaction efficiencies for CBDC and non-CBDC increase, or the household's transaction efficiency for consumption decreases. Shocks to production, inflation and interest rates are analyzed.

Introduction

Background

Technological developments in cryptography and blockchain have made digital currencies worldwide accessible. Central banks increasingly explore and develop CBDCs (central bank digital currencies). The Bank for International Settlements predicts that central banks for 20% of 7.9 billion people can be expected to issue CBDCs within three years (Boar & Wehrli, 2021). New cryptocurrencies emerge every day. December 30, 2021, 16,211 cryptocurrencies contribute to a market cap of $1.8 trillion.[1]. G. Wang, Zhang, Yu, and Ning (2021) provide a holistic picture of cryptocurrencies and blockchain research. Bhimani, Hausken, and Arif (2022) assess cryptocurrency adoption.

Digital currencies provide new possibilities that include higher transaction efficiencies, universal accessibility, confidentiality and privacy, flexible monetary policy, etc. Theoretically, low or negative interest rates can stimulate production and consumption. Some countries currently choose negative interest rates. For example, Blanke and Krogstrup (2016) cite the negative interest rates -0.75% for Switzerland, -0.5% for Denmark, and -0.1% for Japan. CBDSc make a negative interest policy more widely feasible, which can impact the economy substantially. That suggests a need for thorough analysis.

[1] https://coinmarketcap.com/, retrieved April 28, 2022.

* *Corresponding author. ORCID ID: orcid.org/0000-0001-7319-3876*

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Contribution

This article is the first in a series of two articles. This first article builds the decision model involving the central bank applying the Taylor (1993) rule and a representative household choosing strategically and compares with a benchmark solution assumed to be common in practice. The second article, G. Wang and Hausken (2022), compares with the empirics for the US, China and Russia.

The objective and research question intended to fill the current research gap are to explore the relationship between positive and negative CBDC interest rates and a household's production, consumption, CBDC holding and non-CBDC holding. A CBDC in this article can be interpreted as money supply M2 issued by the central bank. A two-period decision model is developed involving a central bank and a representative household. The central bank applies the Taylor (1993) rule to determine its positive or negative CBDC interest rate. A Cobb Douglas utility with four elasticities for the household accounts for the household allocating its resources strategically to production, consumption, CBDC holding, and non-CBDC holding.

A central bank fully controls its monetary policy and applies a variety of policy instruments, sometimes referred to as discretionary policy. Although no central bank officially uses the Taylor (1993) rule, the rule is frequently used as indicative of what a central bank does or may do. Even for central banks occasionally or more permanently applying a fixed exchange rate strategy, the rule may be indicative if economic conditions are comparable to other countries. The rule was proposed by Taylor (1993) in 1992 to stabilize economic policy by determining an interest rate based on inflation and production.

The four elasticities are adjusted by the CBDC and non-CBDC interest rates, and the household's transaction efficiency which increases with the household's CBDC and non-CBDC holdings and decreases with consumption. Solutions are provided analytically and numerically relative to a benchmark for how 19 parameters impact the central bank's application of the Taylor (1993) rule and the household's strategies. The impacts are analyzed of shocks to production, inflation, and the CBDC, non-CBDC, and real interest rates.

Article organization

Section 2 provides a literature review. Section 3 develops the methodology and the model. Section 4 examines the model. Section 5 shows and exemplifies the solution. Section 6 analyzes shocks to production, inflation, the CBDC interest rate, the non-CBDC interest rate, and the real interest rate. Section 7 discusses the results with economic interpretation. Section 8 concludes.

Literature Review

The literature has four categories. First, CBDCs enable central banks to implement negative interest rates which may become an important policy. Second, the central bank, one of the two actors in the article, provide and design the CBDC, and assess its impact. The third part presents decision theoretic analyses. The fourth part is about CBDCs and policy implications.

Negative interest rates

The article explores how a negative CBDC interest rate is connected to a household's allocations into production, consumption, and holding CBDC and non-CBDC. Grasselli and Lipton (2019) think that CBDCs enable the central bank to overcome any lower interest rate bound. They build a stock-flow macroeconomic model to explore the theoretical effectiveness of negative interest rates. They show that negative interest rates have lower impact on consumption than on investment. In contrast, we show that negative interests greatly and positively impacts both production and consumption.

Czudaj (2020) evaluates the effectiveness of negative interest rates based on expectations data from surveys for 44 economies 2002-2017. He finds reduced expectations for 10-year government bond yields and 3-month money market interest rates, and positive impact on GDP growth and preventing deflation, consistently with the current article.

Jia (2020) presents a model to investigate the macroeconomic impact of negative interest rates on CBDC. He shows that negative interest rates compel agents to save less and consume more, which in turn leads to declining capital investment and output. In the current article agents also save less CBDC and consume more. In contrast, the current article finds that agents save more non-CBDC and produce more.

M. Davoodalhosseini, Rivadeneyra, and Zhu (2020) suggest that an interest-bearing CBDC is a versatile instrument for a central bank. Theoretically, it may boost monetary policy. For instance, it can impose a negative interest rate, carry out non-linear transfers, decrease incentives to use alternative means of payments like cash, etc. That partly relates to the current article's finding that a household's CBDC holding is typically the opposite of its non-CBDC holding.

Assenmacher and Krogstrup (2021) think that digital money removes how monetary policy is constrained by a lower bound. They investigate how a central bank may construct and run a negative interest rate system. They show that without the lower bound constraint, the central bank can stabilize the economy by applying conventional policies. With low to intermediate real interest rates, the central bank can make deflationary spirals and the length of business cycle downturns less likely. That partly relates to the current article's finding of how the CBDC interest greatly impacts each household.

Meaning, Dyson, Barker, and Claytona (2021) point out that central banks may pay positive, zero or negative interest rates, and may impose different rates on different CBDC holders. This flexibility could be an important monetary policy instrument, to stabilize inflation and output, and regulate demand. That issue can be explored in a future extension of the current article by assuming households with different characteristics.

Mooij (2021) explores the legal framework in the Eurozone, including whether CBDCs could be classified as legal tender, and used as a monetary policy instrument. He concludes that the ECB mandate legally permits using CBDCs. He suggests that a CBDC enabling interest can decrease the negative lower bound to near zero, and that negative CBDC interest rates may cause capital flows into cryptocurrencies. That is consistent with the current article's finding that a household's CBDC holding is usually the opposite of its non-CBDC holding.

Some research has focused on pre-commitment rules, dynamic consistency and optimal policy related to negative interest rate. For example, Borio and Zabai (2018) find that various unconventional policies including negative interest rates in varying degrees influence financial conditions. They suggest that the policies are exceptional, for use in specific circumstances, and that the cost-benefit balance is likely to deteriorate over time. They criticize prevailing analyses of helicopter money and explore the risks associated with negative nominal interest rates. Ferrero and Neri (2017) assess reasons for historically low interest rates, including structural factors and cyclical and financial phenomena. They frame their assessment around a so-called natural interest rate and a transmission mechanism for money. They attempt to specify possible policy changes that may follow.

Decision theoretic analyses

The article considers a decision model involving the central bank choosing the CBDC interest rate and a representative household choosing production, consumption, holding CBDC and holding non-CBDC. G. Wang and Hausken (2021) build a model involving a representative household selecting a cryptocurrency or a national currency, analogously to the current article where a household chooses whether to hold CBDC or non-CBDC, and selects the probability of tax evasion for each currency. The government decides how to tax the two currencies, and how to detect and impose penalties for tax evasion. Welburn and Hausken (2015, 2017) investigate economic crises. Six kinds of players are included. These are countries, central banks, firms, banks, households, and financial inter-governmental organizations. Each player has multiple strategies, i.e. choosing interest rate, borrowing, lending, producing, consuming, investing, defaulting, etc.

CBDC design and the economy

The article considers the features of CBDC, i.e., higher transaction efficiencies for consumption, compared to non-CBDCs (including Bitcoin, bonds, stocks, etc.) and flexible monetary policies that include negative interest rates. Kiff et al. (2020) review the literature on central bank experiments, present main considerations on retail CBDCs, and provide a structured framework for CBDC issuance. Allen et al. (2020) argues that CBDCs may achieve a broad range of new capabilities, e.g., frictionless payments, new financial instruments, direct disbursements, broader tax bases, financial inclusion, the overcoming of technological vulnerabilities, etc. But CBDCs also lead to various challenges related to privacy, security, disintermediation of the banking system, etc. They summarize the basic technical design choices of CBDCs, especially as they relate to privacy, security, and performance. Auer and Böhme (2020) focus on retail CBDCs. They depict the CBDC pyramid that maps consumer needs into the CBDC design choice of central banks. They argue that the retail CBDC design needs to make tradeoffs between being secure, accessible, convenient and the safeguarding of privacy. Additional assessments addressed in the current article are how a household compares CBDC against non-CBDC, production and consumption.

Agur, Ari, and Dell'Ariccia (2021) present an optimal CBDC design, where each agent holds cash, CBDCs and bank deposits. The agent chooses based on its preference for anonymity and security. They find that the optimal CBDC design entails a tradeoff between bank intermediation and the maintenance of various payment instruments. H. Wang and Gao (2021) investigate various types of CBDCs and their implications on regulation and global financial networks. They suggest that the optimal CBDC networks will be decentralized, and cause monetary policy diffusion without regulatory convergence. Lee, Yan, and Wang (2021) explore how a CBDC structure can keep a balance between benefits and risks. Advantages of CBDC include inclusiveness, cost-saving, managed anonymity, lower cross-border payments, transaction efficiency, security, and more. The risks of CBDCs include bank disintermediation, blockchain-based technology vulnerabilities, and the regulation of shadow and derivate markets of CBDCs. They conclude that CBDCs will become the primary tools of the digital economy.

Urbinati et al. (2021) present the status quo of CBDC-related work worldwide. They illustrate a potential digital euro solution that will combine an account-based platform and distributed ledger technology. Based on the experiments, they find that this combination may provide a sound solution for regulations and retail demand. Choi, Henry, Lehar, Reardon, and Safavi-Naini (2021) introduce a hypothetical retail CBDC design for the Bank of Canada. They think that the design is sound and feasible because it is scalable, resilient, privacy-centric, and universally accessible. Boar and Wehrli (2021) survey worldwide CBDC developments. They find that central banks for 20% of 7.9 billion people can be expected to issue CBDCs within three years.

CBDCs and policy implications

This article relates to this literature by exploring positive and negative CBDC interest rates, and transaction efficiencies. Böser and Gersbach (2020) examine the impact of an interest-bearing CBDC on bank activities and monetary policy. They point out that setting appropriate collateral requirements will boost aggregate productivity. However, if households hold massive amounts of CBDCs, policy with restrictive collateral requirements is risky for banks related to liquidity. That may induce the central bank to abandon these policies. This illustrates the dilemmas faced by central banks when issuing CBDCs. S. M. Davoodalhosseini (2021) explores the optimal monetary policy when an agent chooses between cash and a CBDC. He finds that only a CBDC may be used if its cost is limited, since more efficient allocations can be achieved.

Beniak (2019) discusses potential challenges of CBDC implementation for monetary policy. He points out that CBDCs will impact the interest rates of the central bank, implementation of policy, and the mechanism for transmission. These impacts depend on the design of, and the demand for, CBDC. Bindseil (2020) summarizes the advantages of CBDCs, which include efficient payments, anti-illegal activities, flexible monetary policy with a negative interest rate, etc. The potential risks of CBDCs are bank disintermediation, systemic runs on banks, possible centralization within the central bank, etc. He introduces a two-tier remuneration of CBDC as a solution. Bindseil and Fabio (2020) think that a two-tier CBDC provides a sound solution to issues like bank disintermediation, negative interest rate policy, financial stability, etc. The CBDC with tiered remuneration has four key objectives, including being an attractive means of payment, being universally accessible, depressing the risks of structural bank disintermediation, and providing negative interest rates.

Methodology: The model

This section specifies how the central bank determines the interest rate through the Taylor (1993) rule. The household's resource constraint for production, consumption, CBDC holding and non-CBDC holding, is specified. The household's utility is built up gradually over four steps. Appendix A shows the nomenclature.

The central bank's Taylor (1993) rule application

The central bank applies in period 1 the Taylor (1993) rule to determine the interest rate

$$I_m = \max\left\{\pi + I_r + a_\pi(\pi - \pi^*) + a_p Log\left(\frac{p^h}{\bar{p}^h}\right), z\right\} \tag{1}$$

where I_r, $I_r \in \mathbb{R}$, is the equilibrium real interest rate, where $\mathbb{R}$ is the set of all real numbers; π, $\pi \in \mathbb{R}$, is the inflation rate (which can be positive or negative); π^*, $\pi^* \in \mathbb{R}$, is the desired inflation rate; p^h, $p \geq 0$, is the representative household's production; h is a production parameter; $\bar{p}^h$, $\bar{p} \geq 0$, is the household's potential production (which can be sustained over the long term); Log is the logarithm with base ten; a_π, $a_\pi \geq 0$, is the weight assigned to inflation; a_p, $a_p \geq 0$, is the weight assigned to production; and z, $z \leq 0$, is the negative lower bound on the interest rate I_m.

The household's strategic choices and utility

The representative household has resources r which comprise labor capacity and convertible assets. The resources r are in period 2 converted at unit cost a into production p, and converted at unit cost 1 into consumption c, CBDC (Central Bank Digital Currency) m, and non-CBDC q, i.e.

$$r = ap + c + m + q \tag{2}$$

where c, m, q are equivalently scaled on a suitable scale, e.g. US$. As CBDCs are not widely available at the time of writing this article, we may interpret CBDC as money supply M2 that the central bank issues, made available to the household. The household's production p follows from applying its labor capacity which may generate a salary or useful products. The non-CBDC q can be a cryptocurrency such as Bitcoin, a CBDC from another central bank, or any asset. The CBDC m and non-CBDC q are money demands which in (2) have the same interpretation as resource allocation into any asset. The household's production p causes productive output p^h, where $h = 1$ means linear production, $h > 1$ means convex production, $0 < h < 1$ means concave production, and $h = 0$ means no production.

The household's Cobb Douglas utility is advanced in four steps. First, the household's Cobb Douglas utility has four output elasticities. The first is $\alpha - MI_m - QI_q$ for production, $0 \leq \alpha - MI_m - QI_q \leq 1$, where α is the basic elasticity from which the CBDC interest rate I_m and the non-CBDC interest rate I_q, with weights M and Q, are subtracted. The reasoning is that when the interest rates I_m and I_q increase, production decreases as is commonly observed, and is thus assigned lower elasticity or weight.

The second elasticity is $\beta - MI_m - QI_q$ for consumption, $0 \leq \beta - MI_m - QI_q \leq 1$, where, analogously, β is the basic elasticity from which the CBDC interest rate I_m and the non-CBDC interest rate I_q, with weights M and Q, are subtracted. The reasoning is that when the interest rates I_m and I_q increase, consumption decreases as is commonly observed, and is thus assigned lower elasticity or weight.

The third elasticity is $\gamma + 2MI_m$ for saving CBDC m, $0 \leq \gamma + 2MI_m \leq 1$, where γ is the basic elasticity to which the CBDC interest rate I_m, with weight $2M$, is added. The reasoning is that when the interest rate I_m increases, the household assigns higher elasticity or weight to saving CBDC m. The weight $2M$ is chosen to ensure that the four elasticities sum to 1.

The fourth elasticity is $1 - \alpha - \beta - \gamma + 2QI_q$ for saving non-CBDC q, $0 \leq 1 - \alpha - \beta - \gamma + 2QI_q \leq 1$, where $1 - \alpha - \beta - \gamma$ is the basic elasticity to which the non-CBDC interest rate I_q, with weight $2Q$, is added. The reasoning is that when the interest rate I_q increases, the household assigns higher elasticity or weight to saving non-CBDC q. The weight $2Q$ is chosen to ensure that the four elasticities sum to 1. The household's utility is thus

$$U_1 = p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} m^{\gamma + 2MI_m} q^{1-\alpha-\beta-\gamma+2QI_q} \quad (3)$$

which manifests constant returns to scale. Since the Cobb Douglas elasticities sum to 1, increasing one elasticity means that at least one other elasticity must decrease. For example, if α and β increase, assigning higher weight to production p and consumption c, The four exponents sum to 1. Second, the household earns interest I_m on CBDC m, and earns interest I_q, $I_q \in \mathbb{R}$, on the non-CBDC q. Interest rates are, at least historically, mostly positive. For digital currencies, including CBDC m and non-CBDC q, interest rates can be negative. Hence, we multiply m and q with $1 + I_m$ and $1 + I_q$, respectively, to denote how interest rates are earned. Absorbing these multiplications into (3) gives

$$U_2 = p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} \left(m(1 + I_m)\right)^{\gamma + 2MI_m} \left(q(1 + I_q)\right)^{1-\alpha-\beta-\gamma+2QI_q} \quad (4)$$

Third, to transact between consumption c, CBDC m, and non-CBDC q, the household has to pay transaction costs. The household prefers high transaction efficiency, which ceteris paribus expresses lower transaction cost. The household's transaction efficiency E is modeled to increase with holding CBDC m and holding non-CBDC q, and decrease with consumption c, i.e.

$$E = \frac{m^{\mu} q^{\eta}}{\theta c^{\lambda}} \quad (5)$$

where μ, $\mu \geq 0$, is the household's transaction efficiency for CBDC m; η, $\eta \geq 0$, is the household's transaction efficiency for non-CBDC q. The parameter λ is the household's transaction efficiency for consumption c, and $1/\theta$, $\theta \geq 0$, scales the degree or level of the household's transaction efficiency. The requirement $0 \leq \lambda \leq \beta \leq 1$ expresses that the household prefers consumption, shown as c^{β} in (4), although incurring the transaction cost $1/c^{\lambda}$ in (5). The assumption $\eta \geq \lambda$ ensures that the household's transaction efficiency η for non-CBDC q is higher than or equal to the household's transaction efficiency λ for consumption c.

In (5) the transaction efficiency E satisfies $\frac{\partial E}{\partial c} \leq 0, \frac{\partial E}{\partial m} \geq 0, \frac{\partial E}{\partial q} \geq 0, \frac{\partial^2 E}{\partial c^2} \geq 0, \frac{\partial^2 E}{\partial m^2} \leq 0 \; when \; \mu \leq 1, \frac{\partial^2 E}{\partial q^2} \leq 0 \; when \; \eta \leq 1, \frac{\partial^2 E}{\partial c \partial m} \leq 0, \frac{\partial^2 E}{\partial c \partial q} \leq 0$, see Appendix B. Hence E decreases convexly in consumption c, and increases in the CBDC m and the non-CBDC q. For other accounts of the transaction efficiency E, often expressed as the transaction cost $1/E$, see Feenstra (1986), Bougheas (1994), and Saygılı (2012).

The inverse $1/E$ of E in (5), interpreted as the transaction cost, is commonly analyzed in the literature, where θ scales the transaction cost. Higher transaction efficiency for CBDC m than for non-CBDC q, to enable negative interest rates $I_m < 0$ on CBDC m, requires $\mu > \eta$. This article does not impose that requirement since we may have even more negative interest rates $I_q < I_m < 0$ for non-CBDC q. Multiplying (5) with (4) gives the household's utility

$$U_3 = p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} \left(m(1 + I_m)\right)^{\gamma + 2MI_m} \left(q(1 + I_q)\right)^{1-\alpha-\beta-\gamma+2QI_q} \frac{m^{\mu} q^{\eta}}{\theta c^{\lambda}} \quad (6)$$

Fourth, the household's resource constraint in (2) shows that the household has three free choice variables, i.e. production p, consumption c and CBDC m, where non-CBDC $q = r - ap - c - m$ follows from solving (2) with respect to q. Inserting $q = r - ap - c - m$ into (6) gives the household's utility

$$\begin{aligned} U &= p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} \left(m(1 + I_m)\right)^{\gamma + 2MI_m} \\ &\times \left((r - ap - c - m)(1 + I_q)\right)^{1-\alpha-\beta-\gamma+2QI_q} \frac{m^{\mu}(r - ap - c - m)^{\eta}}{\theta c^{\lambda}} \end{aligned} \quad (7)$$

which has three strategic choice variables p, c and m, and which is the household's utility U which we now proceed to analyze.

We analyze a two-period decision model. In period 1 the central bank applies the Taylor (1993) rule to determine its interest rate I_m. In period 2 the household makes three strategic choices, i.e., production p, consumption c, and its holding m of CBDC. Applying (1) gives the household's holding q of non-CBDC.

Analyzing the model

This section determines the household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U. An implicit solution is presented for the CBDC interest rate I_m. The signs of the first and second order derivatives of the variables are determined. Further analysis is provided when the CBDC interest rate I_m is a parameter.

Analyzing the household

Assumption 1.

$$\{p \geq 0, c \geq 0, m \geq 0, q \geq 0, U \geq 0\} \Leftrightarrow \left\{\begin{matrix} 0 \leq \alpha - MI_m - QI_q \leq 1, \\ \beta - \lambda - MI_m - QI_q \geq 0, \\ \gamma + 2MI_m + \mu \geq 0, \\ 1 - \alpha - \beta - \gamma + \eta + 2QI_q \geq 0, \\ 1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu \geq 0 \end{matrix}\right\} \tag{8}$$

Property 1. When Assumption 1 holds, the household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U, are

$$\begin{aligned}
p &= \frac{rh\left(\alpha - MI_m - QI_q\right)}{a\left(1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu\right)}, \\
c &= \frac{r\left(\beta - \lambda - MI_m - QI_q\right)}{1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu}, \\
m &= \frac{r(\gamma + 2MI_m + \mu)}{1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu}, \\
q &= \frac{r\left(1 - \alpha - \beta - \gamma + \eta + 2QI_q\right)}{1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu}, \\
U &= \frac{\left(1 - \alpha - \beta - \gamma + \eta + 2QI_q\right)}{\theta\left(\beta - \lambda - MI_m - QI_q\right)}(1 + I_m)^{\gamma + 2MI_m}\left(1 + I_q\right)^{1-\alpha-\beta-\gamma+2QI_q} \\
&\times \left(\frac{-rh\left(\alpha - MI_m - QI_q\right)}{a\left((1-h)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1\right)}\right)^{h\left(\alpha - MI_m - QI_q\right)} \\
&\times \left(\frac{-r\left(1 - \alpha - \beta - \gamma + \eta + 2QI_q\right)}{(1-h)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{-\alpha-\beta-\gamma+\eta+2QI_q} \\
&\times \left(\frac{-r\left(\beta - \lambda - MI_m - QI_q\right)}{(1-h)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{1+\beta-\lambda-MI_m-QI_q} \\
&\times \left(\frac{-r(\gamma + 2MI_m + \mu)}{(1-h)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{\gamma+2MI_m+\mu}
\end{aligned} \tag{9}$$

Proof. Appendix C. ■

Analyzing the central bank

Property 2. When Assumption 1 holds, the central bank's CBDC interest rate I_m for the household's CBDC holding m is

$$I_m = \max\left\{\pi + I_r + a_\pi(\pi - \pi^*) + a_p h Log\left(\frac{rh\left(\alpha - MI_m - QI_q\right)}{a\left(1 - (1-h)\left(\alpha - MI_m - QI_q\right) + \eta - \lambda + \mu\right)\bar{p}}\right), z\right\} \tag{10}$$

Proof. Follows from inserting p in (9) into (1). ■

Since the CBDC interest rate I_m appears on the left-hand side and twice inside the logarithm Log with base ten in (10), I_m has no analytical solution and is determined numerically.

Analyzing the household and the central bank

The CBDC interest rate I_m in (10) depends on $r,a,\alpha,M,Q,I_q,\mu,\eta,\lambda,I_r,\pi,\pi^*,h,\bar{p},a_\pi,a_p,z$, and hence does not depend on β,γ,θ. Assume that $\bar{p}=\frac{kr}{a}$, which means that the household's potential production $\bar{p}$ is a fraction k, $0\leq k\leq 1$, where k is a parameter, of the maximum possible production $p=\frac{r}{a}$ obtained when $c=m=q=0$ in (2). Then I_m in (10) also does not depend on r and a. Hence Property 3 determines the derivatives of p,c,m,q,U with respect to β,γ,θ,r,a when $\bar{p}=kr/a$.

Property 3. When Assumption 1 holds, $\frac{\partial p}{\partial \beta}=\frac{\partial p}{\partial \gamma}=\frac{\partial p}{\partial \theta}=0,\frac{\partial^2 p}{\partial \beta^2}=\frac{\partial^2 p}{\partial \gamma^2}=\frac{\partial^2 p}{\partial \theta^2}=0,\frac{\partial p}{\partial r}\geq 0,\frac{\partial^2 p}{\partial r^2}=0,\frac{\partial p}{\partial a}\leq 0,\frac{\partial^2 p}{\partial a^2}\geq 0,\frac{\partial c}{\partial \beta}\geq 0,\frac{\partial^2 c}{\partial \beta^2}=0,\frac{\partial c}{\partial \gamma}=\frac{\partial c}{\partial \gamma^2}=0,\frac{\partial c}{\partial \theta}=\frac{\partial c}{\partial \theta^2}=0,\frac{\partial c}{\partial r}\geq 0,\frac{\partial^2 c}{\partial r^2}=0,\frac{\partial c}{\partial a}=\frac{\partial^2 c}{\partial a^2}=0,\frac{\partial m}{\partial \beta}=\frac{\partial^2 m}{\partial \beta^2}=0,\frac{\partial m}{\partial \gamma}\geq 0,\frac{\partial^2 m}{\partial \gamma^2}=0,\frac{\partial m}{\partial \theta}=\frac{\partial^2 m}{\partial \theta^2}=0,\frac{\partial m}{\partial r}\geq 0,\frac{\partial^2 m}{\partial r^2}=0,\frac{\partial m}{\partial a}=\frac{\partial^2 m}{\partial a^2}=0,\frac{\partial q}{\partial \beta}\leq 0,\frac{\partial q}{\partial \gamma}\leq 0,\frac{\partial^2 q}{\partial \beta^2}=\frac{\partial^2 q}{\partial \gamma^2}=0,\frac{\partial q}{\partial \theta}=\frac{\partial^2 q}{\partial \theta^2}=0,\frac{\partial q}{\partial r}\geq 0,\frac{\partial^2 q}{\partial r^2}=0,\frac{\partial q}{\partial a}=\frac{\partial^2 q}{\partial a^2}=0,\frac{\partial U}{\partial \beta}\lesseqgtr 0,\frac{\partial^2 U}{\partial \beta^2}>0,\frac{\partial U}{\partial \gamma}\lesseqgtr 0,\frac{\partial^2 U}{\partial \gamma^2}>0,\frac{\partial U}{\partial \theta}<0,\frac{\partial^2 U}{\partial \theta^2}>0,\frac{\partial U}{\partial r}>0,\frac{\partial^2 U}{\partial r^2}\lesseqgtr 0,\frac{\partial U}{\partial a}<0,\frac{\partial^2 U}{\partial a^2}\lesseqgtr 0.$

Proof. Follows from (19), (20), (21), (22), (23) in Appendix D. ■

Property 3 states, first, that the household's consumption c increases linearly, while its non-CBDC holding q decreases linearly, in its output elasticity β for consumption c. As β increases, the household values consumption c more and values non-CBDC q less. The household's production p and CBDC holding m are independent of β. The household's utility U can increase or decrease in β.

Second, the household's CBDC holding m increases linearly, while its non-CBDC holding q decreases linearly, in its output elasticity γ for holding CBDC m. As γ increases, the household values CBDC m more and values non-CBDC q less. The household's production p and CBDC holding m are independent of γ. Also, here the household's utility U can increase or decrease in γ.

Third, the household's production p, consumption c, CBDC holding m, and non-CBDC holding q, are independent of the household's scaling θ of the transaction cost. That's because θ appears only in the denominator in (7), and hence does not impact the household's strategic choices. However, a high θ is costly and impacts the household's utility U which decreases convexly.

Fourth, the household's production p, consumption c, CBDC holding m, and non-CBDC holding q, increase linearly in the household's resources r. That's because more resources are beneficial for the household. Hence the household's utility U also increases in r.

Fifth, the household's production p decreases convexly, causing the household's utility U also to decrease, in the household's unit production cost a. The household's consumption c, CBDC holding m, and non-CBDC holding q are independent of a.

Analyzing the household when I_m is a parameter

Property 4. Assume that Assumption 1 holds, and that I_m is a parameter.

$\frac{\partial p}{\partial I_m}\leq 0,\frac{\partial^2 q}{\partial I_m^2}\geq 0,\frac{\partial m}{\partial I_m}\lesseqgtr 0,\frac{\partial^2 m}{\partial I_m^2}\lesseqgtr 0.$

If $0\leq h\leq 1,\frac{\partial^2 p}{\partial I_m^2}\geq 0,\frac{\partial c}{\partial I_m}\leq 0,\frac{\partial^2 c}{\partial I_m^2}\geq 0,\frac{\partial q}{\partial I_m}\leq 0.$

If $h\geq 1,\frac{\partial^2 p}{\partial I_m^2}\leq 0,\frac{\partial q}{\partial I_m}\geq 0.$

If $h\gg 1,\frac{\partial c}{\partial I_m}\geq 0,\frac{\partial^2 c}{\partial I_m^2}\geq 0.$

Proof. Follows from (24) in Appendix E. ■

Property 4 states, first, that the household's production p decreases in the CBDC interest rate I_m, since the subtraction of MI_m in the numerator in (9) has higher impact than the role of MI_m in the denominator in (9).[2]

Second, the household's consumption c decreases convexly in I_m if $0\leq h<1$, and decreases linearly in I_m if $h=1$. The household's consumption c can increase in I_m if h is sufficiently above 1 as specified in (24).

Third, the household's non-CBDC holding q decreases convexly in I_m if $0\leq h\leq 1$, due to the competing CBDC m offering more favorable interest rate I_m, and otherwise increases concavely due to the high household's production parameter $h>1$.

Fourth, the household's CBDC holding m usually increases in I_m, $\frac{\partial m}{\partial I_m}\geq 0$, which tends to make holding CBDC m more attractive.[3]

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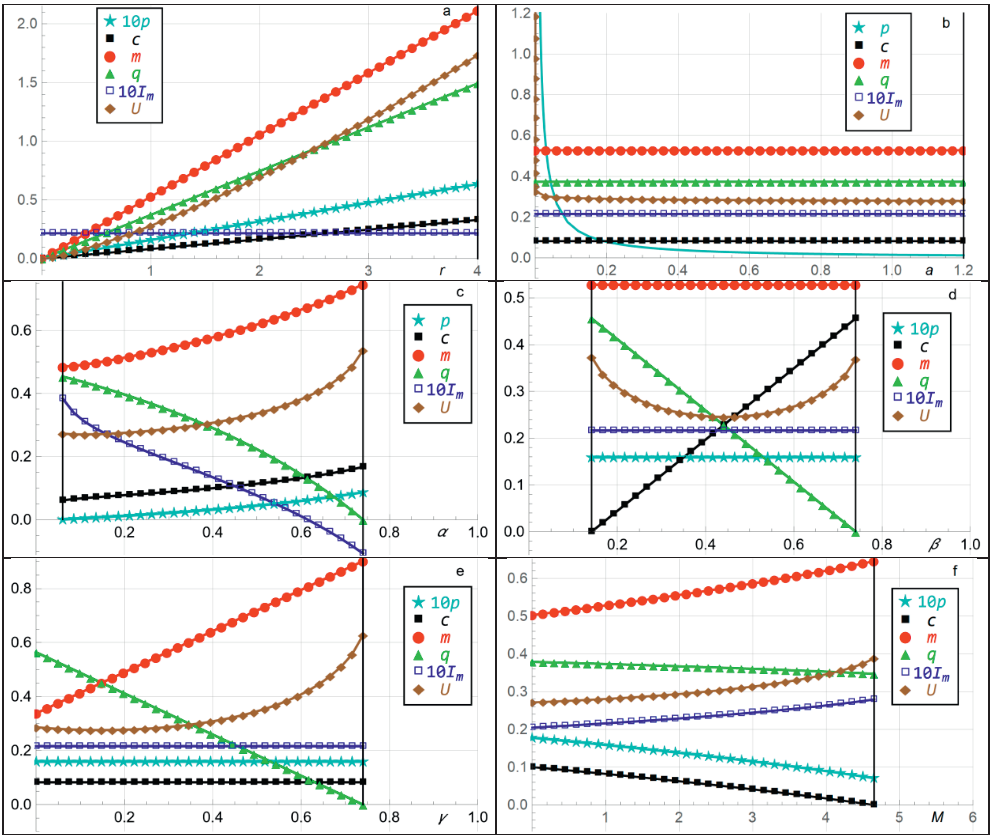
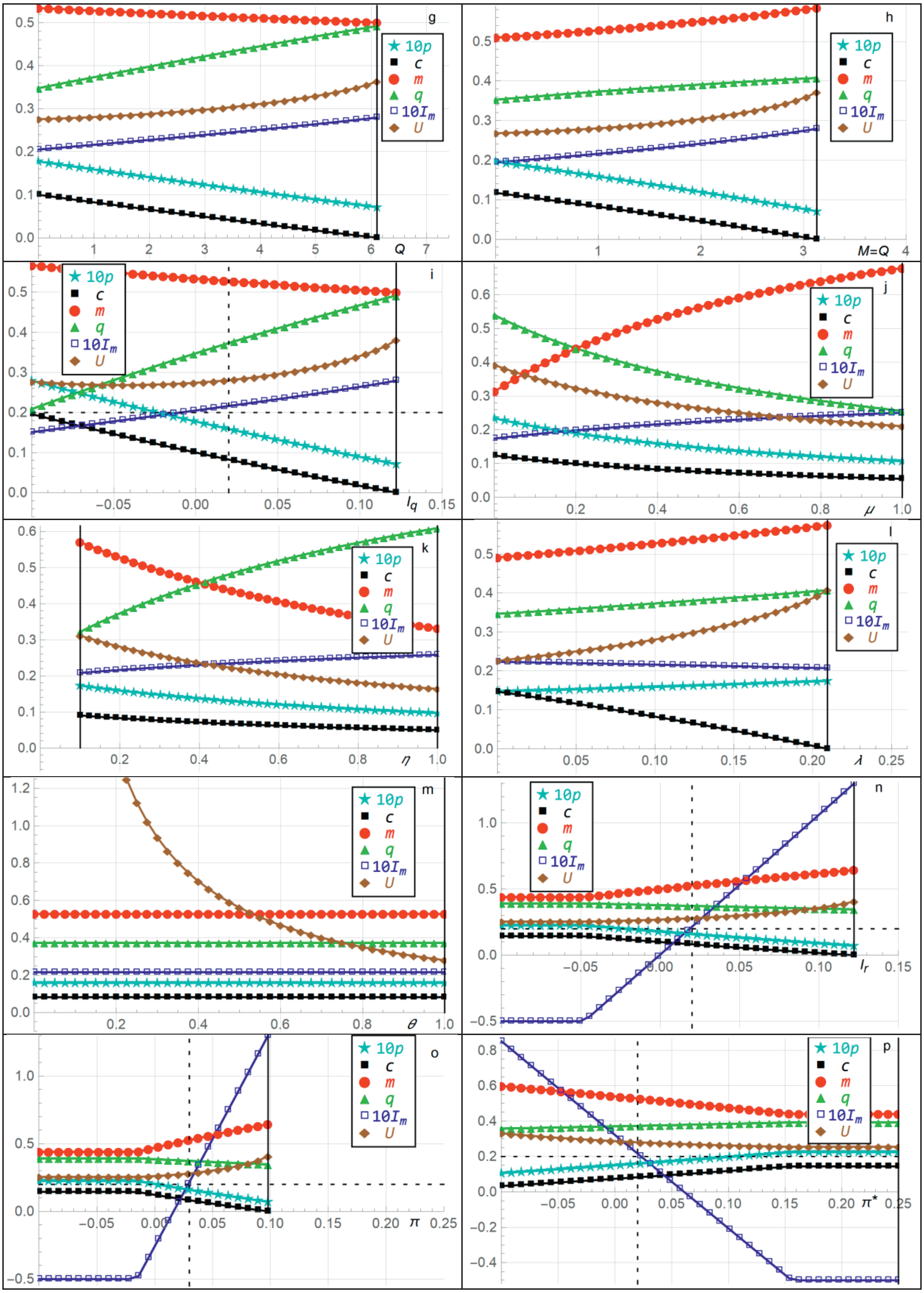
[2] If $h>1$, the production p decreases concavely. If $0\leq h<1$, the production p decreases convexly. If $h=1$, the production p decreases linearly.

[3] However, numerical simulation has shown that extreme parameter values, such as negative non-CBDC interest rate I_q, low h, high α, low μ and high λ, may cause the household's CBDC holding m to decrease in I_m.

Illustrating the solution

This section varies the parameter values $r, a, \alpha, \beta, \gamma, M, Q, M = Q, I_q, \mu, \eta, \lambda, \theta, I_r, \pi, \pi^*, h, \bar{p}, a_\pi, a_p$ relative to a benchmark. The benchmark values are $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = I_r = 2\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 3\%, \pi^* = 2\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. The benchmark is chosen to be realistic in practice. First, $\alpha = \beta = \frac{1}{4}$ expresses equal weight or elasticity for production p, consumption c, respectively. Second, $\gamma = 1 - \alpha - \beta - \gamma = 1/4$ reflects identical weight or elasticity for CBDC m and non-CBDC q. Third, $\eta = \frac{1}{5}$ depicts a middle-level transaction efficiency for non-CBDC. Fourth, $\mu = \frac{2}{5}$ expresses a higher transaction efficiency or lower cost for CBDC m. Fifth, $\lambda = \frac{1}{10}$ expresses low transaction efficiency for consumption c. Sixth, $I_q = 2\%$ reflects an intermediate interest rate for non-CBDC. $I_r = 2\%$ expresses a desired or equilibrium real interest rate. Seventh, $r = a = M = Q = 1$ are chosen for the sake of simplicity, and value one is also plausible. Eighth, $\pi = 3\%$ presents the inflation rate. Nineth, $\pi^* = 2\%$ reflects a desired inflation rate. Tenth, $\bar{p} = \frac{r}{2a} = \frac{1}{2}$ expresses the potential production, which is 50% of what can be produced if the entire resource r is allocated to production. Eleventh, $h = \frac{1}{10}$ reflects a concave production function for the household. Twelfth, $a_\pi = a_p = \frac{1}{2}$ expresses the common equal weight assigned to inflation and production in the Taylor (1993) rule. Thirteenth, $z = -5\%$ is the negative lower bound on the CBDC interest rate I_m. With these benchmark parameter values, the benchmark solution is $I_m = 2.16\%, p = 0.0159, c = 0.0826, m = 0.5282, q = 0.373, U = 0.281$. In Figure 1 each of the 20 parameters values is changed from its benchmark, as illustrated with labels along the horizontal axis, while the other 19 parameter values remain at their benchmarks. The Wolfram Mathematica 13 software package (wolfram.com) has been used. Multiplication of p and I_m with 10 is for scaling purposes.

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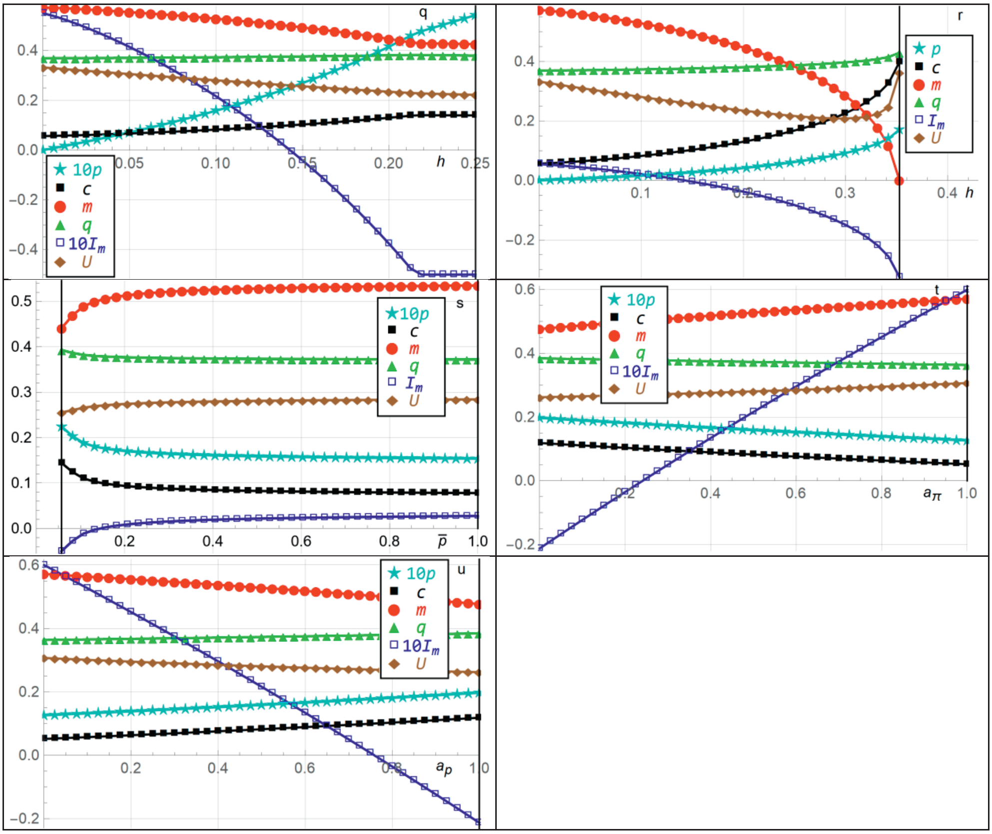

Figure 1: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U, and the CBDC interest rate I_m, as functions of $r, a, \alpha, \beta, \gamma, M, Q, M = Q, I_q, \mu, \eta, \lambda, \theta, I_r, \pi, \pi^*, h, \bar{p}, a_\pi, a_p$ relative to the benchmark parameter values $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = I_r = 2\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 3\%, \pi^* = 2\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Multiplication of p and I_m with 10 is to ensure proper scaling.

In Figure 1a, if the household's resources r increases, which is intuitively beneficial, its production p, consumption c, CBDC holding m, and non-CBDC holding q, increase linearly according to (9). The central bank's CBDC interest rate I_m remains constant, since resources r are abbreviated in the Taylor (1993) rule in (10) since $\bar{p} = \frac{r}{2a}$. The household's utility U increases convexly according to (9). Specifically, production p increases slowly, while CBDC holding m increases rapidly.

In Figure 1b, if the household's unit production cost a increases, its production p and utility U intuitively decrease convexly. The other variables remain constant, and a is abbreviated in (10). The household intuitively benefits from the unit cost a approaching zero, which causes the production p and expected utility U to approach infinity.

In Figure 1c, if the household's output elasticity α for production p increases from $\alpha = 0.06$, its production p, consumption c, CBDC holding m, and the household's utility U, increase convexly, while non-CBDC holding q decreases convexly reaching $q = 0$ when $\alpha > 0.74$, since the output elasticity $1 - \alpha - \beta - \gamma + 2QI_q$ in (7) decreases. When $\alpha < 0.06$, no production p occurs due to subtraction of $MI_m + QI_q$ in (9), where the CBDC interest rate I_m is high to induce the household to save in CBDC m rather than non-CBDC q. The CBDC interest rate I_m decreases and becomes negative when $\alpha > 0.61$, since the household then saves far more in CBDC m than in non-CBDC q, and the central bank can charge the household for saving in CBDC m. When I_m becomes negative, subtraction of MI_m for production p and consumption c in (9) causes addition of $-MI_m$ which is positive. When $\alpha > 0.61$, the household values consumption c more than non-CBDC q.

In Figure 1d, if the household's elasticity β for consumption c increases from $\beta = 0.14$, its consumption c increases linearly. The CBDC interest rate I_m in (10) is independent of β, and hence production p and CBDC holding m in (9) are also independent of β. The household's non-CBDC holding q decreases linearly reaching $q = 0$ when $\beta > 0.74$, since the output elasticity $1 - \alpha - \beta - \gamma + 2QI_q$ in (7) decreases. Interestingly, the household's utility U is U shaped. That's because holding non-CBDC q causes high utility U when β is low, choosing high consumption c causes high utility U when β is high, and U is intermediate when q and c are intermediate at the intermediate $\beta = 0.44$. When $\beta < 0.14$, no consumption c occurs due to subtraction of $MI_m + QI_q - \lambda$ in (9).

In Figure 1e, if the household's output elasticity γ for holding CBDC m increases, its CBDC holding m intuitively increases, and its non-CBDC holding q decreases reaching $q = 0$ when $\gamma > 0.74$, according to (9). When $\gamma > 0.15$, the household values CBDC m more than non-CBDC q. The CBDC interest rate I_m in (10) is independent of γ, and hence production p and consumption c in (9) are also independent of γ. The household's utility U is U shaped, but less symmetric than in Figure 1d. That's because m crosses q at the low value $\gamma = 0.15$, causing high utility U when m is high, while c crosses q at the intermediate $\beta = 0.44$ in Figure 1d.

In Figure 1f, if the household's weight M of the CBDC interest rate I_m in its output elasticities increases, its CBDC holding m increases, while its non-CBDC holding q decreases. Subtracting MI_m in the household's output elasticity for production p and consumption c in (7), as also shown in the numerator for p and c in (9), causes production p and consumption c to decrease. Consumption c eventually decreases to $c = 0$ when $M > 4.65$. Interestingly, the CBDC interest rate I_m increases, which is not intuitively obvious. It illustrates the multiple tradeoffs that the central bank has to make. The household benefits more from holding CBDC m at an increasing CBDC interest rate I_m, than the costs of decreasing q, p, c. Its utility U thus increases.

In Figure 1g, if the household's weight Q of the non-CBDC interest rate I_q in its output elasticities increases, its non-CBDC holding q increases, while its CBDC holding m decreases. That intuitively stands in contrast to Figure 1f. The other four curves are qualitatively similar to Figure 1f, since the household merely shifts its interest from CBDC m to non-CBDC q. That is, subtracting QI_q in the household's output elasticity for production p and consumption c in (7) causes production p and consumption c to decrease. Consumption c eventually decreases to $c = 0$ when $Q > 6.10$. The CBDC interest rate I_m and the household's utility U increase.

In Figure 1h, if the household's equal weights $M = Q$ of the CBDC and non-CBDC interest rates I_m and I_q in its output elasticities increase, its holding of both CBDC m and non-CBDC q increase. That follows since $2MI_m$ and $2QI_q$ are added to the output elasticities in (7). In contrast, MI_m and QI_q are subtracted from the output elasticities for production p and consumption c in (7), causing these to decrease. Consumption c eventually decreases to $c = 0$ when $M = Q > 3.13$, which is a lower value than in Figure 1f and Figure 1g. As in Figure 1f and Figure 1g, and for the same reason, the CBDC interest rate I_m and the household's utility U increase.

In Figure 1i, if the non-CBDC interest rate I_q increases, the household's non-CBDC holding q increases, while its CBDC holding m decreases. The six variables are qualitatively similar to Figure 1g as functions of Q. That's intuitive since QI_q always appear multiplicatively together in (9) and (10), and never separately alone. Hence production p decreases, and consumption c decreases reaching $c = 0$ when $I_q > 12\%$. That is, the household stops consuming when the non-CBDC interest rate I_q is high, and saves non-CBDC q instead. The CBDC interest rate I_m increases to compete with the increasing I_q, and the household's utility U increases convexly.

In Figure 1j, if the household's transaction efficiency μ for CBDC m increases, its CBDC holding m increases concavely, while its production p, consumption c, non-CBDC holding q, and utility U decrease convexly. When $\mu > 0.20$, the household values CBDC m more than non-CBDC q. The central bank increases its CBDC interest rate I_m.

In Figure 1k, if the household's transaction efficiency η for non-CBDC q increases, its non-CBDC holding q increases concavely, while its production p, consumption c, CBDC holding m, and utility U decrease convexly. Hence the CBDC m and non-CBDC q have switched roles compared to Figure 1j, and p, c, U are qualitatively similar. When $\eta > 0.41$, the household values non-CBDC q more than CBDC m. Interestingly, the central bank increases its CBDC interest rate I_m.

In Figure 1l, if the household's transaction efficiency λ for consumption c increases, its consumption c decreases according to (9), eventually reaching $c = 0$ when $\lambda > 0.21$. In contrast, it saves more. Hence both its CBDC holding m and non-CBDC holding q increase. The household's production p increases marginally, and its utility increases convexly. The central bank decreases its CBDC interest rate I_m.

In Figure 1m, if the household's scaling θ of the transaction cost increases, causing the transaction efficiency E in (5) to decrease, only its utility U is affected and decreases convexly. The other variables, i.e., production p, consumption c, CBDC holding m, non-CBDC holding q, and the CBDC interest rate I_m, remain unchanged.

In Figure 1n, if the equilibrium real interest rate I_r increases, the CBDC interest rate I_m increases according to (10). That induces the household to increase its CBDC holding m, decrease its non-CBDC holding q, and decrease its production p and consumption c which decreases to $c = 0$ when $I_r > 12\%$. The household's utility U increases.

In Figure 1o, if the inflation rate π increases, the impact is qualitatively similar to Figure 1n, except that the CBDC interest rate I_m becomes negative when the inflation rate π decreases below $\pi = 1.6\%$. That's because π appears twice on the right hand side of (10), and $\pi - \pi^*$ is negative when π decreases below the desired inflation rate $\pi^* = 2\%$. The central bank thus combats low and decreasing inflation π below the target inflation π^* by choosing negative CBDC interest rate I_m, thus inducing the household to increase its consumption c, production p, and non-CBDC holding q, and decrease its CBDC holding m, which causes convexly decreasing utility U. The household's consumption c decreases to $c = 0$ when $\pi > 9.80\%$.

In Figure 1p, if the desired inflation rate π^* increases, the impact is opposite that of Figure 1o. All the variables move in the opposite direction. That follows from the term $\pi - \pi^*$ in (10) and the minus sign before π^*. The CBDC interest rate I_m becomes negative when the desired inflation rate π^* increases above $\pi^* = 6.1\%$. When π^* increases above $\pi^* = 15.6\%$, the CBDC interest rate I_m decreases to its negative lower bound $z = -5\%$, causing all the six variables to remain constant when $\pi^* > 15.58\%$. As π^* increases, the household's consumption c, production p, and non-CBDC holding q increase, while its CBDC holding m and its utility U decrease.

In Figure 1q, if the household's production parameter h increases, so that it produces more effectively, its production p, consumption c, and non-CBDC holding q increase, while its CBDC holding m and utility U decrease. The CBDC interest rate I_m becomes negative when h increases above $h = 0.143$. When h increases above $h = 0.215$, the CBDC interest rate I_m decreases to its negative lower bound $z = -5\%$, causing all the six variables to remain constant.

Figure 1r replicates Figure 1q with no lower bound $z = -\infty$ on the interest rate I_m. Then the interest rate I_m decreases to $I_m = -32.5\%$ when the household eliminates its CBDC holding m to $m = 0$ when $h > 0.353$. As h increases to $h = 0.353$, the other four variables increase. That is, the household's production p, consumption c, and non-CBDC holding q increase, and the utility U is U shaped with a minimum at $h = 0.296$ and thereafter increases. The situation when $h > 0.353$ models a world with no central bank where, with these parameter values, the household benefits from high utility U.

In Figure 1s, if the household's potential production $\bar{p}$ increases to its maximum $\bar{p} = r/a = 1$, the central bank increases its CBDC interest rate to $I_m = 2.74\%$. Applying (9), that causes the household to increase its CBDC holding m, which increases its utility U, and decrease its production p, consumption c, and non-CBDC holding q. As $\bar{p}$ decreases, the CBDC interest rate I_m becomes negative when $\bar{p} < 0.144$, and decreases to the negative lower bound $z = -5\%$ when $\bar{p} = 0.067$.

In Figure 1t, if the weight assigned to inflation a_π in the Taylor (1993) rule increases, the impact is qualitatively similar to Figure 1o where the inflation rate π increases. That can be seen mathematically from the term $a_\pi(\pi - \pi^*)$ in (10). The CBDC interest rate I_m becomes negative when a_π decreases below $a_\pi = 0.241$. Since inflation then is assigned low weight a_π, and production is assigned higher weight $a_p = 1 - a_\pi$, the household chooses lower CBDC holding m, and chooses higher production p, consumption c, and non-CBDC holding q, which causes lower utility U.

In Figure 1u, if the weight assigned to production a_p in the Taylor (1993) rule increases, the impact is opposite that of Figure 1s, since $a_p = 1 - a_\pi$. Thus the CBDC interest rate I_m becomes negative when a_p increases above $a_p = 0.759$. Furthermore, the household chooses lower CBDC holding m, and chooses higher production p, consumption c, and non-CBDC holding q, which causes lower utility U.

Shocks to production, inflation, interest rates of CBDC and non-CBDC, and real interest rate

Shocks to production p

The household's production is characterized by its unit cost a of production considered in Figure 1b, and its production parameter h considered in Figure 1q and Figure 1r. Figure 1b shows increased household production p and household utility U, and the other variables are constant, as a decreases. Figure 1q and Figure 1r show increased household production p, consumption c, and non-CBDC holding q, and decreased CBDC holding m and CBDC interest rate I_m, and decreased utility U, up to a certain point, as h increases.

Shocks to inflation π and target inflation π^*

Inflation is characterized by the inflation rate π considered in Figure 1o, and the desired or target inflation rate π^* considered in Figure 1p. Figure 1o shows decreased household production p, consumption c and non-CBDC holding q, and increased CBDC holding m, utility U and CBDC interest rate I_m, as π increases. Figure 1p shows all the variables moving in the opposite direction. Hence the household prefers high inflation rate π and low target inflation rate π^*.

Shocks to the CBDC interest rate I_m

The CBDC interest rate I_m is the central bank's free choice variable. Shocks to I_m may occur if the central bank were to depart from the optimal solution analyzed in the previous sections. Considering I_m as a parameter, Figure 2 plots the household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U, as functions of I_m ranging from $I_m = -20\%$ to $I_m = 20\%$.

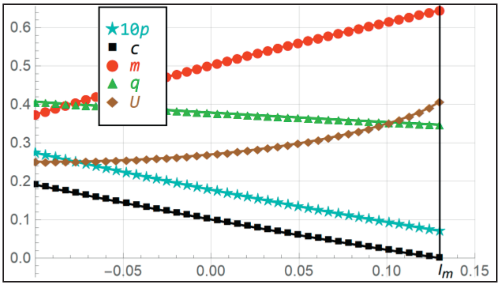

Figure 2: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U as functions of the CBDC interest rate I_m as a parameter relative to the benchmark $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = I_r = 2\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 3\%, \pi^* = h = 2\%, \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Multiplication of p with 10 is for scaling purposes.

In Figure 2, if the CBDC interest rate I_m increases, the five variables change as follows: First, and intuitively, the household's CBDC holding m increases. When $I_m > -0.8\%$, the household holds more CBDC m than non-CBDC q. Second, the household's non-CBDC holding q decreases due to the substitution effect between CBDC m and non-CBDC q. Third and fourth, holding more CBDC m induces the household to decrease its production p and consumption c. The household's consumption c eventually decreases to $c = 0$ when $I_m > 13\%$. Fifth, the household's utility U is U shaped, reaching a minimum of approximately 0.25 when the CBDC interest rate $I_m = -0.94\%$. Hence the household prefers I_m to be low or high. When I_m is low, the household derives moderately high utility U due to high production p and consumption c, and substantial non-CBDC holding q. When I_m is high, the household derives high utility U due to substantial CBDC holding m at a high CBDC interest rate I_m.

Shocks to the non-CBDC interest rate I_q

Shocks to the non-CBDC interest rate I_q is considered in Figure 1i which shows decreased household production p, consumption c, and CBDC holding m, and increased non-CBDC holding q, utility U, and CBDC interest rate I_m, as I_q increases. That benefits the household. The central bank needs to increase its interest rate I_m to compete.

Shocks to the real interest rate I_r

Shocks to the real interest rate I_r is considered in Figure 1n which shows decreased household production p, consumption c, and non-CBDC holding q, and increased CBDC holding m, utility U, and CBDC interest rate I_m, as I_r increases. Again, that benefits the household, and the central bank increases its interest rate I_m to compete.

Discussion and economic interpretation

The following results in the previous sections are noteworthy, related to varying 19 parameter values relative to a benchmark.

The household's production, consumption, CBDC holding and non-CBDC holding increase as its resources increase. The intuition is that it is beneficial to have more resources. Thus, as resources increase, the household's utility increases.

If the household's unit production cost increases, it intuitively decreases its production, which decreases the household's utility. The household's consumption, CBDC holding, and non-CBDC holding remain constant if the unit production cost changes.

If the household's output elasticity for production increases, it increases its production, consumption, and CBDC holding. However, the non-CBDC holding decreases since the output elasticities sum to one and higher elasticity assigned to production means lower elasticity assigned to non-CBDC holding. The CBDC interest rate becomes negative when the household holds far more CBDC than non-CBDC. Since the household saves substantially in CBDC, the central bank sees no reason to encourage further saving.

If the household's elasticity for consumption increases, it increases its consumption, but decreases its non-CBDC holding since higher elasticity assigned to consumption means lower elasticity assigned to non-CBDC holding. The CBDC interest rate, production, and CBDC holding remain constant since the CBDC interest rate is independent of the elasticity for consumption. The household's utility is U shaped in the elasticity for consumption. That is, it prefers either high or low elasticity for consumption.

If the household's output elasticity for holding CBDC increases, it increases its CBDC holding, but decreases its non-CBDC holding. The CBDC interest rate is independent of the output elasticity for holding CBDC. Thus, the household's production and consumption remain constant.

If the household's weight of the CBDC interest rate in its elasticities increases, it increases its CBDC holding and decreases its non-CBDC holding. This is similar to the increase of the output elasticity for holding CBDC. The increase of the CBDC interest rate shows the multiple tradeoffs that the central bank strikes. The household's utility increases from holding more CBDC if the CBDC interest rate increases, which offsets the decrease in production, consumption, and non-CBDC holding.

Intuitively and in contrast to the previous point, if the household's weight of the non-CBDC interest rate in its output elasticities increases, it increases its non-CBDC holding and decreases its CBDC holding.

We consider the case when the weight of the CBDC interest rate in the output elasticities equals the weight of the non-CBDC interest rate. If the household's equal weights increase, the household increases both its CBDC holding and its non-CBDC holding, and the central bank increases its CBDC interest rate. Since the weights are subtracted from production and consumption, production and consumption decrease.

If the non-CBDC interest rate increases, the household increases its non-CBDC holding and decreases its CBDC holding. This is because holding non-CBDC becomes more attractive since the household benefits from gaining higher interest from non-CBDC. The central bank chooses to increase the CBDC interest rate to compete with the non-CBDC. Hence a household's CBDC holding is typically the opposite of its non-CBDC holding, as found by Mooij (2021) and partly found by M. Davoodalhosseini et al. (2020).

If the household's transaction efficiency for CBDC increases, it intuitively increases its CBDC holding, and decreases its non-CBDC holding. The central bank increases its CBDC interest rate to support the household's CBDC holding.

In contrast to the previous point, if the household's transaction efficiency for non-CBDC increases, it increases its non-CBDC holding, and decreases its CBDC holding. Interestingly, also here the central bank increases its CBDC interest rate, to compete with the non-CBDC.

If the household's transaction efficiency for consumption increases, its consumption decreases since transactions become more costly. However, the household increases its CBDC holding and non-CBDC holding, and the central bank decreases its CBDC interest rate.

If the household's scaling of the transaction efficiency increases, the transaction efficiency decreases accordingly. Only the household's utility is impacted, and it decreases.

If the real interest rate increases, the CBDC interest rate increases, in accordance with the Taylor (1993) rule. Thus, if the real interest rate increases, the household holds more CBDC and less non-CBDC. Meanwhile, the household's production and consumption decrease.

Similarly to the previous point, if the inflation rate increases, the CBDC interest rate increases, encouraging more CBDC saving and less non-CBDC saving. The central bank combats low inflation via a negative CBDC interest rate.

In contrast to the previous point, if the target inflation rate increases, the CBDC interest rate decreases. The central bank combats a high target inflation rate through a negative CBDC interest rate. That, in turn, induces agents to save less CBDC and consume more, as also found by Jia (2020).

If the household's production parameter increases, it increases its production, consumption, and non-CBDC holding, but decreases its CBDC holding. This is because the household produces more effectively. The central bank decreases its CBDC interest rate, which also enhances the household's production and consumption.

If the household's potential production increases, the central bank increases its CBDC interest rate. Thus, the household increases its CBDC holding, and decreases its production, consumption, and non-CBDC holding.

If the weight assigned to inflation in the Taylor (1993) rule increases, the impact is similar to increasing the inflation rate. The central bank chooses negative interest rate when inflation is assigned low weight. Since a higher weight assigned to inflation means a lower weight assigned to production, the household's production and consumption decrease. In contrast, Grasselli and Lipton (2019) show that negative interest rates have lower impact on consumption than on investment.

The following further results are noteworthy, related to analyzing the impacts of shocks to production, inflation, the CBDC interest rate, the non-CBDC interest rate, and the real interest rate.

Production shocks are captured by the unit cost of production and production parameter. If the unit cost decreases, the household's production increases, and the other variables remain constant. If the production parameter increases, the household's production, consumption, and non-CBDC holding increase, while the household's CBDC holding and utility, and the CBDC interest rate, decrease.

Inflation shocks are characterized by changes to the inflation rate and the target inflation rate. If the inflation rate increases, the household's production, consumption, and non-CBDC holding decrease, and the CBDC interest rate, CBDC holding and utility increase.

If the CBDC interest rate increases, the household increases its CBDC holding, and decreases its non-CBDC holding. The household decreases its production and consumption. Its utility is U shaped. When the CBDC interest rate is low, the household gains utility from production, consumption, and non-CBDC holding. When the CBDC interest rate is high, the household gains utility from holding CBDC with high CBDC interest return.

The non-CBDC interest shock shows that if the non-CBDC interest rate increases, the household decreases its production, consumption, and CBDC holding, increases its non-CBDC holding, and eventually earns higher utility.

The real interest rate shock shows that if the real interest rate increases, the household decreases its production, consumption, and non-CBDC holding, and increases its CBDC holding, while the central bank increases its CBDC interest rate.

Conclusion

The article explores a two-period decision model between a central bank and a representative household. The central bank applies the Taylor (1993) rule to choose a positive or negative interest rate. The representative household owns resources or energy allocated into production, consumption, CBDC (central bank digital currency) holding, and non-CBDC holding. The non-CBDC holding can be various cryptocurrencies like Bitcoin, Ethereum, etc., or stocks, bonds, real estate, etc. A Cobb Douglas utility with elasticities for the household's allocations is presented, and adjusted by the CBDC interest rate, the non-CBDC interest rate, and the transaction efficiency. In period 1, the central bank chooses its interest rate. In period 2, the household determines its production, consumption, CBDC holding and non-CBDC holding.

The article shows that if the household's output elasticities for production, consumption, CBDC holding, and non-CBDC holding change, the household's strategies change as expected. The central bank chooses negative interest rate when the household holds far more CBDC than non-CBDC, to discourage further saving. Increasing the non-CBDC interest rate, which causes the household to hold more non-CBDC and less CBDC, induces the central bank to increase its CBDC interest rate to compete with the threat from the attractive non-CBDC interest rate. Increasing the household's transaction efficiencies for CBDC and non-CBDC cause the central bank to increase its CBDC interest rate, to support the household's holding of CBDC and compete with the non-CBDC, respectively. However, increasing the household's transaction efficiency for consumption has the opposite impact of decreasing the CBDC interest rate. Decreasing the real interest rate or the inflation rate or the household's potential production or the weight assigned to inflation in the Taylor (1993) rule, or increasing the target inflation rate or the household's production parameter, causes lower and eventually negative CBDC interest rate, which induces the household to hold less CBDC, more non-CBDC, produce and consume more, and earn lower utility.

Positive shocks to production cause lower and eventually negative CBDC interest rate. The household holds less CBDC and earns lower utility, but produces and consumes more and holds more non-CBDC. Positive inflation shocks cause the household to hold more CBDC and earn higher utility due to a higher CBDC interest rate, while the production, consumption, and non-CBDC holding decrease. Positive shocks to the CBDC interest rate cause the household to hold more CBDC and less non-CBDC, and conversely for positive shocks to the non-CBDC interest rate. Both these two shocks cause the household to produce and consume less and eventually earn higher utility. Positive shocks to the real interest rate cause higher CBDC interest rate.

Future research, which implicitly illustrates limitations of the article, should consider the interactions of several CBDCs and non-CBDCs. More players can be introduced, e.g. governments, commercial banks, firms, etc. Various negative interest rate bounds, and corner solutions can be analyzed. In addition, the burning and issuance of CBDCs and non-CBDCs should be analyzed. Expansion should be made to heterogeneous households. Each household's Cobb Douglas utility should be expanded to account for additional factors such as safety, convenience, taxes, etc. The analysis can also be generalized to allow each household and one or multiple central banks to choose their strategies simultaneously or sequentially in one-period or repeated games. More extensive empirical research should be conducted.

Appendix A Nomenclature

Parameters

r	Household's monetary energy, or resources, $r \geq 0$
a	Household's unit cost of production, $a \geq 0$
α	Household's output elasticity for production p, $0 \leq \alpha \leq 1$
β	Household's output elasticity for consumption c, $0 \leq \lambda \leq \beta \leq 1$
γ	Household's output elasticity for CBDC m, $0 \leq \gamma \leq 1$
M	Household's weight of the CBDC interest rate I_m in its output elasticities, $M \geq 0$
Q	Household's weight of the non-CBDC interest rate I_q in its output elasticities, $Q \geq 0$
$1-\alpha-\beta-\gamma+2QI_q$	Household's output elasticity for non-CBDC q, $0 \leq 1-\alpha-\beta-\gamma+2QI_q \leq 1$
I_q	Non-CBDC interest rate, $I_q \in \mathbb{R}$
μ	Household's transaction efficiency for CBDC m, $\mu \geq 0$
η	Household's transaction efficiency for non-CBDC q, $\eta \geq \lambda$
λ	Household's transaction efficiency for consumption c, $0 \leq \lambda \leq \beta \leq 1$
θ	Scaling or degree or level of the household's transaction cost, $\theta \geq 0$.
I_r	The equilibrium real interest rate, $I_r \in \mathbb{R}$
π	The inflation rate, $\pi \in \mathbb{R}$
π^*	The desired or target inflation rate, $\pi^* \in \mathbb{R}$
h	The household's production parameter, $h \geq 0$

$\bar{p}^h$ The household's potential production, $0 \leq \bar{p} \leq r/a$
a_π The weight assigned to inflation in the Taylor (1993) rule, $0 \leq a_\pi \leq 1$
$a_p = 1 - a_\pi$ The weight assigned to production in the Taylor (1993) rule, $0 \leq a_p \leq 1$
z The negative lower bound on the interest rate I_m, $z \leq 0$

Household's free choice variables
p Household's production, $0 \leq p \leq r/a$
c Household's consumption, $0 \leq c \leq r$
m Household's CBDC holding, $0 \leq m \leq r$

Dependent variables
I_m CBDC interest rate for the household's CBDC holding m, $I_m \in \mathbb{R}$
U Household's utility
$q = r - ap - c - m$ Household's non-CBDC holding, $0 \leq q = r - ap - c - m \leq r$
E Household's transaction efficiency

Appendix B The derivatives for the transaction efficiency E

Differentiating the transaction efficiency E in (5) with respect to c, m and q gives

$$\frac{\partial E}{\partial c} = -\frac{c^{-1-\lambda}m^{\mu}q^{\eta}\lambda}{\theta} \leq 0, \frac{\partial E}{\partial m} = \frac{c^{-\lambda}m^{-1+\mu}q^{\eta}\mu}{\theta} \geq 0, \frac{\partial E}{\partial q} = \frac{c^{-\lambda}m^{\mu}q^{-1+\eta}\eta}{\theta} \geq 0 \tag{11}$$

The second derivatives of the transaction efficiency E in (5) with respect to c, m and q gives

$$\begin{aligned} \frac{\partial^2 E}{\partial c^2} &= \frac{c^{-2-\lambda}m^{\mu}q^{\eta}\lambda(1+\lambda)}{\theta} \geq 0, \\ \frac{\partial^2 E}{\partial m^2} &= \frac{c^{-\lambda}m^{-2+\mu}q^{\eta}(-1+\mu)\mu}{\theta} \leq 0, when\ \mu \leq 1, \\ \frac{\partial^2 E}{\partial q^2} &= \frac{c^{-\lambda}m^{\mu}q^{-2+\eta}(-1+\eta)\eta}{\theta} \leq 0, when\ \mu \leq 1, \\ \frac{\partial^2 E}{\partial c \partial m} &= -\frac{c^{-1-\lambda}m^{-1+\mu}q^{\eta}\lambda\mu}{\theta} \leq 0, \\ \frac{\partial^2 E}{\partial c \partial q} &= -\frac{c^{-1-\lambda}m^{\mu}q^{-1+\eta}\eta\lambda}{\theta} \leq 0 \end{aligned} \tag{12}$$

Appendix C Proof of Property 1

Calculating the defivative of the household's utility U in (7) with respect to its free choice variables p, c and m, and equating to zero, gives

$$\begin{aligned} \frac{\partial U}{\partial p} &= \frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}(1+I_q)^{1-\alpha-\beta-\gamma+2QI_q} \\ &\times m^{\gamma+2MI_m+\mu}p^{-1+h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q} \\ &\times \Big(h(r-c-m-ap)(\alpha-MI_m-QI_q) - ap(1-\alpha-\beta-\gamma+\eta+2QI_q)\Big) = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{\partial U}{\partial c} &= \frac{1}{\theta}c^{-1+\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}(1+I_q)^{1-\alpha-\beta-\gamma+2QI_q} \\ &\times m^{\gamma+2MI_m+\mu}p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q} \\ &\times \Big((r-m-ap)(\beta-\lambda-MI_m-QI_q) - c(1-MI_m+QI_q-\alpha-\gamma+\eta-\lambda)\Big) = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{\partial U}{\partial m} &= \frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}(1+I_q)^{1-\alpha-\beta-\gamma+2QI_q} \\ &\times m^{-1+\gamma+2MI_m+\mu}p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q} \\ &\times \big((r-c-ap)(\gamma+2MI_m+\mu) - m(1-\alpha-\beta+\eta+\mu+2MI_m+2IqQ)\big) = 0, \end{aligned} \tag{15}$$

which are solved to yield p, c and m in (9). The dependent variable q follows from solving (2) with respect to q and inserting p, c and m. The second order conditions, inserting (14) to (14), are

$$\begin{aligned} \frac{\partial^2 U}{\partial p^2} &= -\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_m}(1+I_m)^{\gamma+2MI_m}(1+I_q)^{1-\alpha-\beta-\gamma+2QI_q} \\ &\times m^{\gamma+2MI_m+\mu}p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q} \end{aligned} \tag{16}$$

$$\times\left(\frac{2ah(\alpha-MI_m-QI_q)(1-\alpha-\beta-\gamma+\eta+2QI_q)}{p}\right.$$
$$+\frac{h(r-c-m-ap)\left(1-h(\alpha-MI_m-QI_q)\right)(\alpha-MI_m-QI_q)}{p^2}$$
$$\left.+\frac{a^2(\alpha+\beta+\gamma-\eta-2QI_q)(1-\alpha-\beta-\gamma+\eta+2QI_q)}{r-c-m-ap}\right),$$

$$\frac{\partial^2 U}{\partial c^2}=-\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_m}(1+I_m)^{\gamma+2MI_m}(1+I_q)^{1-\alpha-\beta-\gamma+2QI_q}$$
$$\times m^{\gamma+2MI_m+\mu}p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}$$
$$\times\left(\frac{2(1-\alpha-\beta-\gamma+\eta+2QI_q)(\beta-\lambda-MI_m-QI_m)}{c}\right.$$
$$+\frac{(\alpha+\beta+\gamma-\eta-2QI_q)(1-\alpha-\beta-\gamma+\eta+2QI_q)}{r-c-m-ap} \tag{17}$$
$$\left.+\frac{(r-c-m-ap)(\beta-\lambda-MI_m-QI_m)(1-\beta+\lambda+MI_m+QI_q)}{c^2}\right),$$

$$\frac{\partial^2 U}{\partial m^2}=-\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_m}(1+I_m)^{\gamma+2MI_m}(1+I_q)^{1-\alpha-\beta-\gamma+2QI_q}$$
$$\times m^{\gamma+2MI_m+\mu}p^{h(\alpha-MI_m-QI_q)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}$$
$$\times\left(\frac{2(1-\alpha-\beta-\gamma+\eta+2I_qQ)(\gamma+2MI_m+\mu)}{m}\right.$$
$$+\frac{(\alpha+\beta+\gamma-\eta-2QI_q)(1-\alpha-\beta-\gamma+\eta+2QI_q)}{r-c-m-ap} \tag{18}$$
$$\left.+\frac{(r-c-m-ap)(1-\gamma-\mu-2MI_m)(\gamma+2MI_m+\mu)}{m^2}\right)$$

Appendix D Proof of Property 3

Differentiating (9) gives

$$\frac{\partial p}{\partial \beta}=\frac{\partial p}{\partial \gamma}=\frac{\partial p}{\partial \theta}=0,\frac{\partial^2 p}{\partial \beta^2}=\frac{\partial^2 p}{\partial \gamma^2}=\frac{\partial^2 p}{\partial \theta^2}=0,$$
$$\frac{\partial p}{\partial r}=\frac{h(\alpha-MI_m-QI_q)}{a\left(1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu\right)},\frac{\partial^2 p}{\partial r^2}=0,$$
$$\frac{\partial p}{\partial a}=\frac{-hr(\alpha-MI_m-QI_q)}{a^2\left(1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu\right)}, \tag{19}$$
$$\frac{\partial^2 p}{\partial a^2}=\frac{2hr(\alpha-MI_m-QI_q)}{a^3\left(1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu\right)}$$

$$\frac{\partial c}{\partial \beta}=\frac{r}{1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu},\frac{\partial^2 c}{\partial \beta^2}=0,$$
$$\frac{\partial c}{\partial \gamma}=\frac{\partial c}{\partial \gamma^2}=0,\frac{\partial c}{\partial \theta}=\frac{\partial c}{\partial \theta^2}=0, \tag{20}$$
$$\frac{\partial c}{\partial r}=\frac{\beta-\lambda-MI_m-QI_q}{1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu},\frac{\partial^2 c}{\partial r^2}=0,$$
$$\frac{\partial c}{\partial a}=\frac{\partial^2 c}{\partial a^2}=0$$

$$\frac{\partial m}{\partial \beta}=\frac{\partial^2 m}{\partial \beta^2}=0,$$
$$\frac{\partial m}{\partial \gamma}=\frac{r}{1-(1-h)(\alpha-MI_m-QI_q)+\eta-\lambda+\mu},\frac{\partial^2 m}{\partial \gamma^2}=0, \tag{21}$$
$$\frac{\partial m}{\partial \theta}=\frac{\partial^2 m}{\partial \theta^2}=0,$$

$$\frac{\partial m}{\partial r}=\frac{r}{1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu},\frac{\partial^2 m}{\partial r^2}=0,$$
$$\frac{\partial m}{\partial a}=\frac{\partial^2 m}{\partial a^2}=0$$

$$\frac{\partial q}{\partial \beta}=\frac{\partial q}{\partial \gamma}=\frac{-r}{1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu},\frac{\partial^2 q}{\partial \beta^2}=\frac{\partial^2 q}{\partial \gamma^2}=0,$$
$$\frac{\partial q}{\partial \theta}=\frac{\partial^2 q}{\partial \theta^2}=0,$$
$$\frac{\partial q}{\partial r}=\frac{1-\alpha-\beta-\gamma+\eta+2QI_m}{1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu},\frac{\partial^2 q}{\partial r^2}=0, \tag{22}$$
$$\frac{\partial q}{\partial a}=\frac{\partial^2 q}{\partial a^2}=0$$

$$\frac{\partial U}{\partial \beta}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q}Log\left(\frac{c}{\left(1+I_q\right)(r-c-m-ap)}\right),$$
$$\frac{\partial^2 U}{\partial \beta^2}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q}$$
$$\times\left(Log\left(\frac{\left(1+I_q\right)(r-c-m-ap)}{c}\right)\right)^2,$$
$$\frac{\partial U}{\partial \gamma}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q}Log\left(\frac{m(1+I_m)}{\left(1+I_q\right)(r-c-m-ap)}\right),$$
$$\frac{\partial^2 U}{\partial \gamma^2}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}$$
$$\times m^{\gamma+2MI_m+\mu}p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q}$$
$$\times\left(Log\left(\frac{m(1+I_m)}{\left(1+I_q\right)(r-c-m-ap)}\right)\right)^2, \tag{23}$$
$$\frac{\partial U}{\partial \theta}=-\frac{1}{\theta^2}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q},$$
$$\frac{\partial^2 U}{\partial \theta^2}=\frac{2}{\theta^3}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{1-\alpha-\beta-\gamma+\eta+2QI_q},$$
$$\frac{\partial U}{\partial r}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right),$$
$$\frac{\partial^2 U}{\partial r^2}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{-1-\alpha-\beta-\gamma+\eta+2QI_q}\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right)$$
$$\times\left(-\alpha-\beta-\gamma+\eta+2QI_q\right),$$
$$\frac{\partial U}{\partial a}=-\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{1+h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{-\alpha-\beta-\gamma+\eta+2QI_q}\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right),$$
$$\frac{\partial^2 U}{\partial a^2}=\frac{1}{\theta}c^{\beta-\lambda-MI_m-QI_q}(1+I_m)^{\gamma+2MI_m}\left(1+I_q\right)^{1-\alpha-\beta-\gamma+2QI_q}m^{\gamma+2MI_m+\mu}$$
$$\times p^{2+h\left(\alpha-MI_m-QI_q\right)}(r-c-m-ap)^{-1-\alpha-\beta-\gamma+\eta+2QI_q}\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right)$$
$$\times\left(-\alpha-\beta-\gamma+\eta+2QI_q\right)$$

Appendix E Proof of Property 4

Differentiating (9) when I_m is a parameter gives

$$\frac{\partial p}{\partial I_m}=-\frac{rhM(1+\eta-\lambda+\mu)}{a\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^2},$$
$$\frac{\partial^2 p}{\partial I_m^2}=\frac{2rhM^2(1-h)(1+\eta-\lambda+\mu)}{a\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^3},$$
$$\frac{\partial c}{\partial I_m}=-\frac{rM(1+(1-h)(\beta-\alpha-\lambda)+\eta-\lambda+\mu)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^2},$$
$$\frac{\partial^2 c}{\partial I_m^2}=\frac{2rM^2(1-h)(1+(1-h)(\beta-\alpha-\lambda)+\eta-\lambda+\mu)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^3},$$
$$\frac{\partial m}{\partial I_m}=\frac{rM(2-2\left((1-h)\left(\alpha-QI_q\right)-\eta+\lambda\right)-(1-h)\gamma+(1+h)\mu)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^2}, \quad (24)$$
$$\frac{\partial^2 m}{\partial I_m^2}=-\frac{rM^2(1-h)\left(2-2\left((1-h)QI_q+\alpha-\eta+\lambda\right)-\gamma+\mu+h(2\alpha+\gamma+\mu)\right)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^3},$$
$$\frac{\partial q}{\partial I_m}=-\frac{(1-h)Mr(1-\alpha-\beta-\gamma+\eta+2QI_q)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^2},$$
$$\frac{\partial^2 q}{\partial I_m^2}=\frac{2rM^2(1-h)^2\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right)}{\left(1-(1-h)\left(\alpha-MI_m-QI_q\right)+\eta-\lambda+\mu\right)^3}$$

Combining $0\le\alpha-MI_m-QI_q\le 1$ and $\beta-\lambda-MI_m-QI_q\ge 0$ gives $\beta-\alpha-\lambda\ge MI_m+QI_q-\alpha\ge -1$. Hence $1+(1-h)(\beta-\alpha-\lambda)\ge 0$ if $0\le h<1$, causing $\frac{\partial c}{\partial I_m}<0$ and $\frac{\partial^2 c}{\partial I_m^2}>0$.

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Data Availability Statement: The data presented in this study are available on request from the corresponding author. The data are not publicly available due to privacy.

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Comparative Analysis of Households and Digital Currencies for the US, China and Russia

Guizhou Wang [(a)], *Kjell Hausken* [(a)*]

[(a)] *Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway*

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ABSTRACT

In a two-period decision model, a central bank chooses a CBDC (central bank digital currency) interest rate and a representative household allocates resources into production, consumption, CBDC holding, and non-CBDC holding. The model's analytical results and a plausible benchmark are compared with the empirics for the US, China and Russia. Interesting novelties of the article are that the model predicts that the US in 2021/2022 should choose 7.56% *rather than 0.125% CBDC interest to combat its high October 2021 empirical inflation of 6.2%. That would induce households to hold more CBDC, hold less non-CBDC, and produce and consume less. In contrast, the model predicts that China should choose a low* 2.99% *rather than* 3.85% *CBDC interest rate. That would decrease each household's CBDC holding and increase the low inflation. The model predicts that Russia should choose* 6.82% *rather than* 6.75% *CBDC interest rate. Russia's strategy is remarkably consistent with the model's predictions. The model predicts that the central bank should choose negative CBDC interest rate when the inflation and real interest rate are low, and the inflation target is high. The article shows how extremely high inflation, which increases the CBDC interest rate, makes production and consumption nearly impossible, unless the real interest rate is extremely negative.*

Introduction

Central banks investigate CBDCs (central bank digital currencies) (Boar & Wehrli, 2021; Urbinati et al., 2021), and cryptocurrencies continue to be adopted (Bhimani, Hausken, & Arif, 2022; G. Wang, Zhang, Yu, & Ning, 2021). This article is the second in a series of two articles. The first article, G. Wang and Hausken (2022), builds a decision model with a central bank applying the Taylor (1993) rule and a representative household choosing strategically, and compares with a plausible benchmark solution. This second article compares with the empirics for the US, China and Russia.

This article briefly summarizes the model and results of G. Wang and Hausken (2022). Compared with the benchmark solution in G. Wang and Hausken (2022), the article explores the empirical data of the US, China and Russia. The model recommends that the US in 2021/2022 should choose a CBDC interest rate far above its 0.125% empirical interest rate. The CBDC can be interpreted as money suppy M2 issued by the central bank. China should choose a lower CBDC interest rate than its 3.85% empirical interest rate. Russia should choose a CBDC interest rate slightly above its 6.75% empirical interest rate. The article shows how the central bank should choose negative CBDC interest rate when the inflation and real interest rate are low, and the inflation target is high. The article explores the implications of increased inflation rates. Extremely high inflation, which increases the CBDC interest rate, makes production and consumption nearly impossible, unless the real interest rate is extremely negative.

Negative interest rates have already occurred in Switzerland, Denmark, and Japan (Blanke & Krogstrup, 2016), and may become easier to implement with CBDCs which may potentially enable universal accessibility, flexible policy, confidentiality and privacy and higher transaction efficiencies. Whereas Grasselli and Lipton (2019) find that negative interest rates impact consumption less than investment, this article shows high and positive impact of negative interests on both production and consumption. While Jia (2020) finds that negative interest rates induce agents to consume more and save less, this article finds that agents produce more and

* *Corresponding author. ORCID ID: orcid.org/0000-0001-7319-3876*

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save more non-CBDC. Both Mooij (2021) and this article find that negative CBDC interest rates may cause the agents to hold more CBDCs.

Just as this article considers the decisions of central banks and households, G. Wang and Hausken (2021) consider a household choosing between a cryptocurrency or a national currency. Welburn and Hausken (2015, 2017) extend beyond these two players, to countries, firms, banks, and financial inter-governmental organizations.

Regarding CBDC design, see Agur, Ari, and Dell'Ariccia (2021). Kiff et al. (2020), Auer and Böhme (2020) and Choi, Henry, Lehar, Reardon, and Safavi-Naini (2021) evaluate retail CBDCs and structured frameworks for CBDC issuance, and Allen et al. (2020) assess capabilities and challenges for CBDCs. H. Wang and Gao (2021) focus more on the types of CBDCs and how they impact regulation and global financial networks, while Lee, Yan, and Wang (2021) assess benefits and risks of CBDCs.

Böser and Gersbach (2020) assess how an interest-bearing CBDC impact bank activities and policy, and Davoodalhosseini (2021) investigates the suitable policy when choosing between cash and a CBDC. Beniak (2019) evaluates how CBDCs may impact policy. Bindseil (2020) and Bindseil and Fabio (2020) assesses benefits and risks of CBDCs. They recommend a two-tier remuneration which enables payment, universal accessibility, possible avoidance of bank disintermediation, and the possibility of negative interest rates.

Article organization

Section 2 presents the model. Section 3 analyzes the model. Section 4 compares the empirical data of the US, China and Russia. Section 5 assesses the impact of high inflation and hyperinflation. Section 6 discusses the results and concludes.

Methodology: The model

In period 1 the central bank uses the Taylor (1993) rule to determine its interest rate

$$I_m = \max\left\{\pi + I_r + a_\pi(\pi - \pi^*) + a_p Log\left(\frac{p^h}{\bar{p}^h}\right), z\right\} \quad (1)$$

where I_r is the equilibrium real interest rate; π is the inflation rate; π^* is the desired inflation rate; p^h is the representative household's production; h is a production parameter; $\bar{p}^h$, is the household's potential production; Log is the logarithm with base ten; a_π is the weight assigned to inflation; $a_p = 1 - a_\pi$ is the weight assigned to production; and z is the negative lower bound on the interest rate I_m.

In period 2 the representative household chooses its production p, consumption c, and CBDC holding m, causing the non-CBDC holding $q = r - ap - c - m$, where r is the household's resources and a is the household's unit production cost. The household's utility is

$$U = p^{h(\alpha - MI_m - QI_q)} c^{\beta - MI_m - QI_q} \left(m(1 + I_m)\right)^{\gamma + 2MI_m} \times \left((r - ap - c - m)(1 + I_q)\right)^{1-\alpha-\beta-\gamma+2QI_q} \frac{m^\mu (r - ap - c - m)^\eta}{\theta c^\lambda} \quad (2)$$

where α is the household's output elasticity for production p, $0 \le \alpha \le 1$, β is the household's output elasticity for consumption c, $0 \le \lambda \le \beta \le 1$, γ is the household's output elasticity for CBDC m, $0 \le \gamma \le 1$, M is the household's weight of the CBDC interest rate I_m in its output elasticities, Q is the household's weight of the non-CBDC interest rate I_q in its output elasticities, $1 - \alpha - \beta - \gamma + 2QI_q$ is the household's output elasticity for non-CBDC q, $0 \le 1 - \alpha - \beta - \gamma + 2QI_q \le 1$, I_q is the non-CBDC interest rate, μ is the household's transaction efficiency for CBDC m, η is the household's transaction efficiency for non-CBDC q, $\lambda$is the household's transaction efficiency for consumption c, and θ is the scaling or degree or level of the household's transaction cost, $\theta \ge 0$.

Analyzing the model

When $p \ge 0, c \ge 0, m \ge 0, q \ge 0, U \ge 0$, the household's production p, consumption c, CBDC holding m, non-CBDC holding q, and utility U, are

$$\begin{aligned}
p &= \frac{rh(\alpha - MI_m - QI_q)}{a\left(1 - (1-h)(\alpha - MI_m - QI_q) + \eta - \lambda + \mu\right)}, \\
c &= \frac{r(\beta - \lambda - MI_m - QI_q)}{1 - (1-h)(\alpha - MI_m - QI_q) + \eta - \lambda + \mu}, \\
m &= \frac{r(\gamma + 2MI_m + \mu)}{1 - (1-h)(\alpha - MI_m - QI_q) + \eta - \lambda + \mu}, \\
q &= \frac{r(1 - \alpha - \beta - \gamma + \eta + 2QI_q)}{1 - (1-h)(\alpha - MI_m - QI_q) + \eta - \lambda + \mu}, \\
U &= \frac{(1 - \alpha - \beta - \gamma + \eta + 2QI_q)}{\theta(\beta - \lambda - MI_m - QI_q)}(1 + I_m)^{\gamma + 2MI_m}(1 + I_q)^{1-\alpha-\beta-\gamma+2QI_q}
\end{aligned} \quad (3)$$

$$\times\left(\frac{-rh\left(\alpha - MI_m - QI_q\right)}{a\left(\left(1-h\right)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1\right)}\right)^{h\left(\alpha - MI_m - QI_q\right)}$$
$$\times\left(\frac{-r\left(1-\alpha-\beta-\gamma+\eta+2QI_q\right)}{\left(1-h\right)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{-\alpha-\beta-\gamma+\eta+2QI_q}$$
$$\times\left(\frac{-r\left(\beta-\lambda - MI_m - QI_q\right)}{\left(1-h\right)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{1+\beta-\lambda-MI_m-QI_q}$$
$$\times\left(\frac{-r\left(\gamma+2MI_m+\mu\right)}{\left(1-h\right)\left(\alpha - MI_m - QI_q\right) - \eta + \lambda - \mu - 1}\right)^{\gamma+2MI_m+\mu}$$

which are inserted into (1) to give the central bank's CBDC interest rate I_m, i.e.

$$I_m = \max\left\{\pi + I_r + a_\pi(\pi-\pi^*) + a_p h Log\left(\frac{rh\left(\alpha - MI_m - QI_q\right)}{a\left(1-\left(1-h\right)\left(\alpha - MI_m - QI_q\right)+\eta-\lambda+\mu\right)\bar{p}}\right), z\right\} \quad (4)$$

Proof. See G. Wang and Hausken (2022). ∎

Figure 1 is plotted in G. Wang and Hausken (2022).

Figure 1. See G. Wang and Hausken (2022).
Figure 2 is plotted in G. Wang and Hausken (2022).

Figure 2. See G. Wang and Hausken (2022).

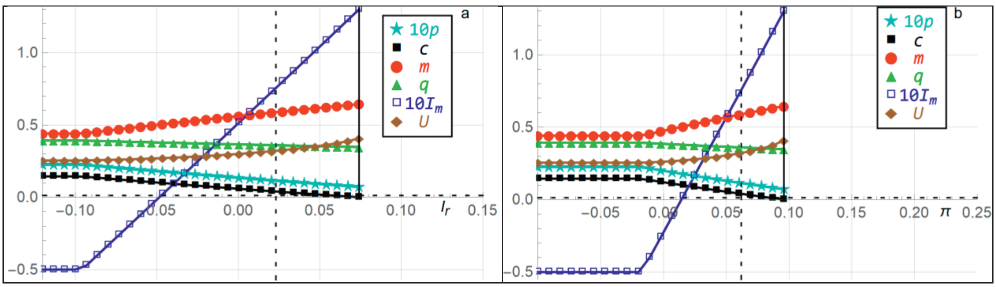
Comparing the US, China and Russia

The US

The Federal Open Market Committee (2021) maintained the target range for the federal funds rate (refers to CBDC interest rate I_m) at $0\% - 0.25\%$ on September 22, 2021. We choose the midpoint of this range, that is $I_m = 0.125\%$. The US real interest rate was $I_r = 2.305\%$ in 2020 (The World Bank, 2021c). The US annual inflation rate was $\pi = 6.2\%$ for the 12 months ending October 31, 2021 (The US Labor Department, 2021). The Federal Open Market Committee (2021) seeks to achieve an average target inflation rate at $\pi^* = 2\%$ in the long-run. Table 1 summarizes these numbers.

Table 1: Empirical CBDC interest rate I_m, model CBDC interest rate I_m, empirical equilibrium real interest rate I_r, empirical inflation rate π, and empirical desired or target inflation rate π^*, for the US, China and Russia.

Parameters	The US	China	Russia
Empirical CBDC interest rate I_m	0.125%	3.85%	6.75%
Model CBDC interest rate I_m	7.56%	2.99%	6.82%
Empirical real interest rate I_r	2.305%	3.6535 %	5.83%
Empirical inflation rate π	6.2%	2.419%	3.382%
Empirical target inflation rate π^*	2%	3%	4%



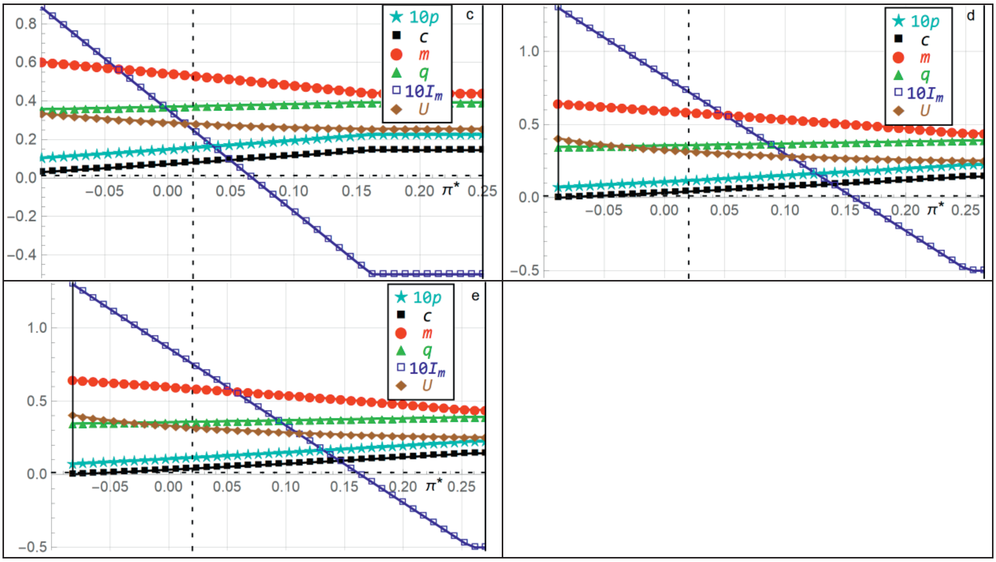


Figure 3: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, utility U, and the CBDC interest rate I_m for the US, as functions of the real interest rate I_r, inflation rate π, and target inflation rate π^*, respectively, relative to the benchmark parameter values $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = 2\%, I_r = 2.305\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 6.2\%, \pi^* = 2\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Multiplication of p and I_m with 10 is for scaling purposes.

Figure 3a plots p, c, m, q, U, I_m as functions of the real interest rate I_r when the inflation rate $\pi = 6.2\%$, which is higher than $\pi = 3\%$ in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > 7.4\%$, which is lower than $I_r > 12.21\%$ in Figure 1n. The higher inflation rate $\pi = 6.2\% > 3\%$ decreases consumption c in Figure 3a. Compared to Figure 1n, in Figure 3a the household chooses lower consumption c, lower production p, and holds less non-CBDC q. The household holds more CBDC m and earns higher utility U. The CBDC interest rate I_m becomes negative when $I_r < -4.85\%$, which is lower than $I_r < 0.00\%$ in Figure 1n. The model thus predicts a higher CBDC interest rate I_m when the inflation rate is $\pi = 6.2\%$ in Figure 3a compared to $\pi = 3\%$ in Figure 1n. That follows from the logic of the Taylor (1993) rule in (4). The central bank combats high inflation rate $\pi = 6.2\%$ by increasing its CBDC interest rate I_m, to make saving in the form of holding CBDC m more attractive than consumption c, which is lower in Figure 3a than in Figure 1n. Mathematically, high inflation $\pi = 6.2\%$ on the right hand side in (4) causes high CBDC interest rate I_m on the left hand side in (4). For example, the CBDC interest rate is $I_m = 7.56\%$ at the benchmark $I_r = 2.305\%$ in Figure 3a, which is higher than $I_m = 2.48\%$ when $I_r = 2.305\%$ in Figure 1n, and much higher than the empirical $I_m = 0.125\%$ in Table 1. That seems remarkable. The model and the Taylor (1993) rule predict that the US CBDC interest rate I_m should be substantially higher, $I_m = 7.56\%$, than the empirical $I_m = 0.125\%$, in order to induce holding more CBDC m, and suppress the high inflation $\pi = 6.2\%$.

Figure 3b plots p, c, m, q, U, I_m as functions of the inflation rate π, when the real interest rate $I_r = 2.305\%$, which is higher than $I_r = 2\%$ in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > 9.60\%$, which is slightly lower than $I_r > 9.80\%$ in Figure 1o. The CBDC interest rate I_m becomes negative when $\pi < 1.43\%$, which is lower than $\pi < 1.63\%$ in Figure 1o. Hence the higher real interest rate $I_r = 2.305\%$ decreases the consumption c and increases the CBDC interest rate I_m. The CBDC interest rate is $I_m = 7.56\%$ at the benchmark $\pi = 6.2\%$, which is higher than $I_m = 7.24\%$ when $\pi = 6.2\%$ in Figure 1o. Both these I_m are substantially higher than I_m =0.125% in Table 1.

Figure 3c plots p, c, m, q, U, I_m as functions of the target inflation rate π^* for the same real interest rate $I_r = 2.305\%$, which is higher than $I_r = 2\%$ in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 6.71\%$, which is higher than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 16.19\%$, which is higher than $\pi^* > 15.58\%$ in Figure 1p. Hence the higher real interest rate $I_r = 2.305\%$ increases the target inflation rate π^* and the CBDC interest rate I_m. The CBDC interest rate is $I_m = 2.48\%$ at the benchmark $\pi^* = 2\%$, which is higher than $I_m = 0.125\%$ in Table 1, and also higher than $I_m = 2.00\%$ in Figure 1p when $\pi^* = 2\%$.

Figure 3d plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the inflation rate is $\pi = 6.2\%$, which is higher than $\pi = 3\%$ in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 15.70\%$, which is much higher than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 25.18\%$, which is much higher than $\pi^* > 15.58\%$ in Figure 1p. Hence the higher inflation rate $\pi = 6.2\%$ greatly increases the target inflation rate π^* and the CBDC interest rate I_m. The CBDC interest rate is $I_m = 7.24\%$ at the benchmark $\pi^* = 2\%$, which is much higher than $I_m = 0.125\%$ in Table 1, and also higher than $I_m = 2.00\%$ in Figure 1p when $\pi^* = 2\%$.

Figure 3e plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate is $I_r = 2.305\%$ and the inflation rate is $\pi = 6.2\%$. All the other parameter values are as the benchmarks in Figure 1. It is the combination of Figure 3c and Figure 3d. The CBDC interest rate I_m becomes negative when $\pi^* > 16.31\%$, which is much higher than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 25.79\%$, which is much higher than $\pi^* > 15.58\%$ in Figure 1p. Hence the higher inflation rate is $\pi = 6.2\%$ and the higher real interest rate $I_r = 2.305\%$ greatly increases the target inflation rate π^* and the CBDC interest rate I_m. The CBDC interest rate is $I_m = 7.56\%$ at the benchmark $\pi^* = 2\%$, which is much higher than $I_m = 0.125\%$ in Table 1, and also higher than $I_m = 2.00\%$ in Figure 1p when $\pi^* = 2\%$.

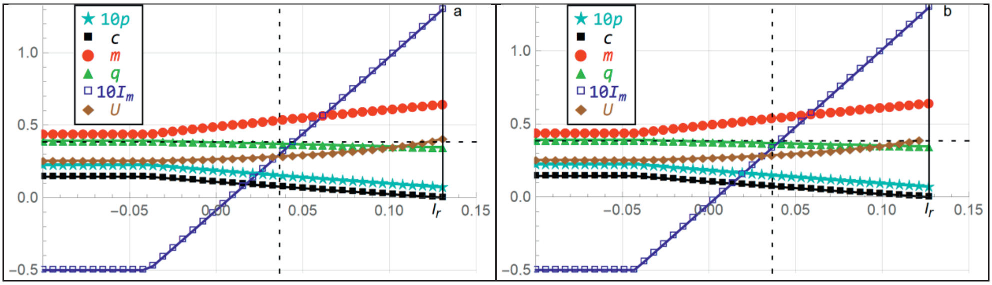
The empirical US inflation rate $\pi = 6.2\%$ is much higher than the empirical CBDC interest rate $I_m = 0.125\%$. Thus, the gap between the predicted CBDC interest rate I_m and the US empirical CBDC interest rate I_m is large, at the real interest rate benchmark I_r and the target inflation benchmark π^*. The model predicts that the US CBDC interest rate I_m should be substantially higher than $I_m = 0.125\%$. The higher real interest rate I_r decreases the consumption c, increases the CBDC interest rate I_m, and increases the target inflation rate π^*. The higher inflation rate increases the target inflation rate π^* and increases the CBDC interest rate I_m.

Table 2: Interpretation of Figure 3 for the US compared to Figure 1.

The US	Changed parameter values from the benchmark in Figure 1	c decreases to zero when	c, p, m, q reach constant values when	I_m becomes negative when	I_m at the benchmark
Figure 3a	$\pi = 6.2\%$	$I_r > 7.4\%$	$I_r < -9.59\%$	$I_r < -4.85\%$	$I_m = 7.56\%$ at $I_r = 2.305\%$
Figure 3b	$I_r = 2.305\%$	$\pi > 9.60\%$	$\pi < -1.73\%$	$\pi < 1.43\%$	$I_m = 7.56\%$ at $\pi = 6.2\%$
Figure 3c	$I_r = 2.305\%$	$\pi^* < -17.80\%$	$\pi^* > 16.19\%$	$\pi^* > 6.71\%$	$I_m = 2.48\%$ at $\pi^* = 2\%$
Figure 3d	$\pi = 6.2\%$	$\pi^* < -8.81\%$	$\pi^* > 25.18\%$	$\pi^* > 15.70\%$	$I_m = 7.24\%$ at $\pi^* = 2\%$
Figure 3e	$I_r = 2.305\%$ $\pi = 6.2\%$	$\pi^* < -8.20\%$	$\pi^* > 25.79\%$	$\pi^* > 16.31\%$	$I_m = 7.56\%$ at $\pi^* = 2\%$
Figure 1n	$I_r = 2\%$	$I_r > 12.21\%$	$I_r < -4.79\%$	$I_r < 0.00\%$	$I_m = 2.48\%$ at $I_r = 2.305\%$
Figure 1o	$\pi = 3\%$	$I_r > 9.80\%$	$I_r < -1.53\%$	$\pi < 1.63\%$	$I_m = 7.24\%$ at $\pi = 6.2\%$
Figure 1p	$\pi^* = 2\%$	$\pi^* < -18.41\%$	$\pi^* > 15.58\%$	$\pi^* > 6.10\%$	$I_m = 2.00\%$ at $\pi^* = 2\%$

China

The People's Bank of China kept its interest rate unchanged since October 2015. The China interest rate has on average been I_m =3.85% over the last year (Gang, 2021). The China real interest rate is $I_r = 3.6535$ % in 2020, the China annual inflation rate is $\pi = 2.419\%$, according to the World Bank (The World Bank, 2021a). The State Council of China (2020) set the inflation target $\pi^* = 3\%$ for the year 2021, just as in 2020.



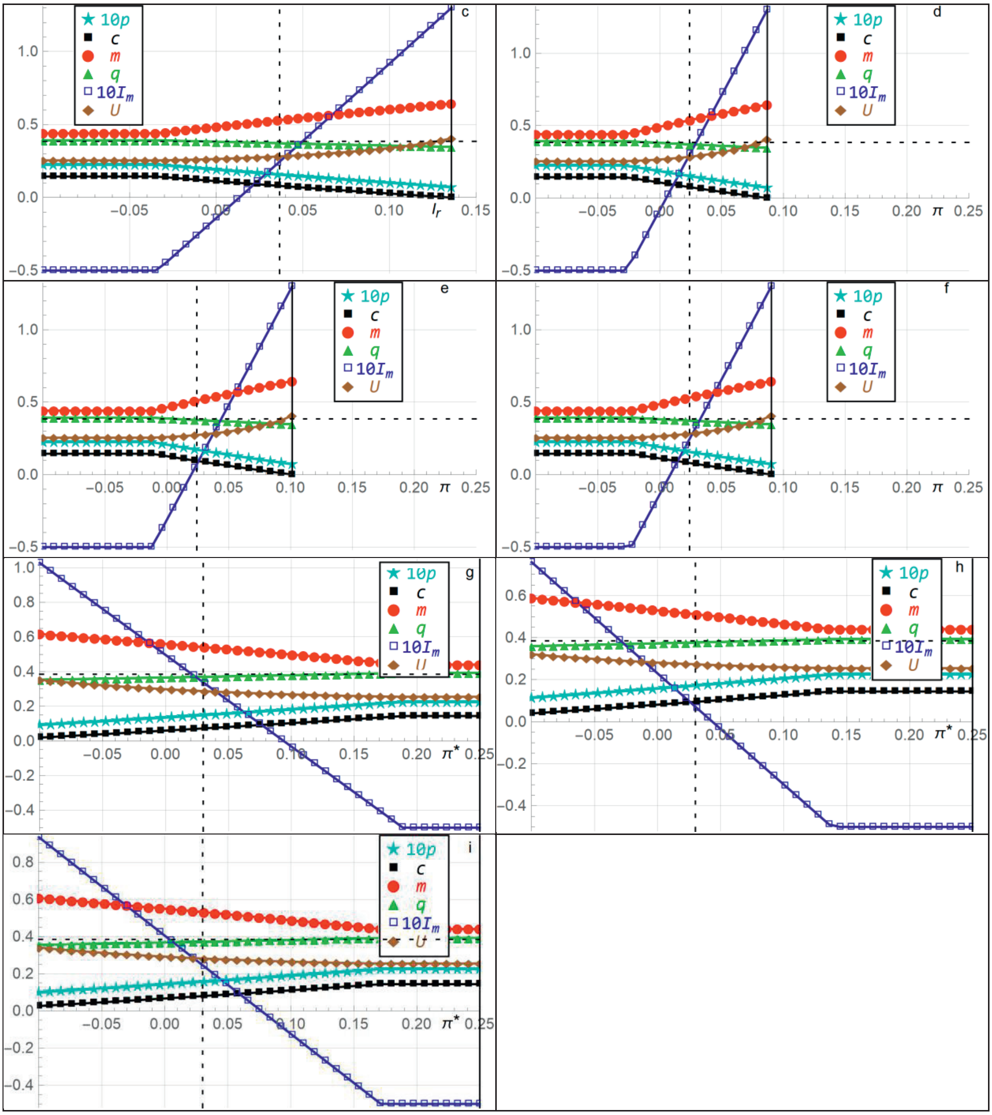


Figure 4: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, utility U, and the CBDC interest rate I_m for China, as functions of the real interest rate I_r, inflation rate π, and target inflation rate π^*, respectively, relative to the benchmark parameter values $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = 2\%, I_r = 3.6535\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 2.419\%, \pi^* = 3\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Multiplication of p and I_m with 10 is for scaling purposes.

Figure 4a plots p, c, m, q, U, I_m as functions of the real interest rate I_r, when the inflation rate $\pi = 2.419\%$, which is lower than $\pi = 3\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > 13.08\%$, which is slightly higher than $I_r > 12.21\%$ in Figure 1n. The lower inflation rate $\pi = 2.419\% < 3\%$ increases slightly the consumption c in Figure 4a compared to Figure 1n, in contrast to the decreased consumption c in Figure 3a for the US. Compared to Figure 1n in Figure 4a, the household chooses higher consumption c, higher production p, and holds more non-CBDC q, in contrast to Figure 3a for the US. The household holds less CBDC m and earns lower utility U, also in contrast to Figure 3 for the US. The CBDC interest rate I_m becomes negative when $I_r < 0.82\%$ which is higher than $I_r < 0.00\%$ in Figure 1n, and much higher than $I_r < -4.85\%$ in Figure 3a for the US. The model thus predicts a lower CBDC interest rate I_m when the inflation rate is $\pi = 2.419\%$ in Figure 4a compared to $\pi = 3\%$ in Figure 1n. That follows from the logic of the Taylor (1993) rule in (4). The central bank responds to low inflation rate $\pi = 2.419\%$ by decreasing its CBDC interest rate I_m, to make saving in the form of holding CBDC m less attractive than consumption c, which is higher in Figure 4a than in Figure 1n. Mathematically, low inflation $\pi = 2.419\%$ on the right hand side in (4) causes low CBDC interest rate I_m on the left hand side in (4). For example, the CBDC interest rate is $I_m = 2.99\%$ at the benchmark $I_r = 3.6535\%$ in Figure 4a, which is lower than $I_m = 3.91\%$ when $I_r = 3.6535\%$ in Figure 1n, and also lower than the empirical $I_m = 3.85\%$ in Table 1. The model predicts partly in accordance with the empirics. The model and the Taylor (1993) rule predict that China's CBDC interest rate I_m should be lower, $I_m = 2.99\%$, than the empirical $I_m = 3.85\%$, in order to induce holding less CBDC m, and increase the low inflation rate $\pi = 2.419\%$ towards its target $\pi^* = 3\%$.

Figure 4b plots p, c, m, q, U, I_m as functions of the real interest rate I_r, when the target inflation rate $\pi^* = 3\%$, which is higher than $\pi^* = 2\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > 12.71\%$, which is higher than $I_r > 12.21\%$ in Figure 1n. The CBDC interest rate I_m becomes negative when $I_r < 0.45\%$, which is higher than $I_r < 0.00\%$ in Figure 1n. Hence the higher target inflation rate $\pi^* = 3\%$ increases the consumption c and correspondingly decreases the CBDC interest rate I_m. Both of these are in contrast to the US in Figure 3b. Accordingly, the CBDC interest rate is $I_m = 3.38\%$ at the benchmark $I_r = 3.6535\%$, which is lower than $I_m = 3.91\%$ when $I_r = 3.6535\%$ in Figure 1n, and lower than $I_m = 3.85\%$ in Table 1.

Figure 4c plots p, c, m, q, U, I_m as functions of the real interest rate I_r,when the inflation rate $\pi = 2.419\%$ and the target inflation rate $\pi^* = 3\%$, thus combining the assumptions for Figure 4a and Figure 4b. All the other parameter values are as the benchmark in Figure 1. The consumption c decreases and the CBDC interest rate I_m increases. More specifically, the household's consumption c decreases to $c = 0$ when $I_r > 13.58\%$, which is higher than $I_r > 12.21\%$ in Figure 1n. The CBDC interest rate I_m becomes negative when $I_r < 1.32\%$, which is higher than $I_r < 0.00\%$ in Figure 1n. The CBDC interest rate is $I_m = 2.46\%$ at the benchmark $I_r = 3.6535\%$, which is lower than $I_m = 3.91\%$ when $I_r = 3.6535\%$ in Figure 1n, and also lower than 3.85% in Table 1.

Figure 4d plots p, c, m, q, U, I_m as functions of the inflation rate π, when the real interest rate $I_r = 3.6535\%$, which is higher than $I_r = 2\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $\pi > 8.70\%$, which is lower than $\pi > 9.80\%$ in Figure 1o. The CBDC interest rate I_m becomes negative when $\pi < 0.53\%$, which is lower than $\pi < 1.63\%$ in Figure 1o. Hence the higher real interest rate $I_r = 3.6535\%$ decreases the consumption c and correspondingly increases the CBDC interest rate I_m. The CBDC interest rate is $I_m = 2.99\%$ at the benchmark $\pi = 2.419\%$, which is higher than $I_m = 1.24\%$ when $\pi = 2.419\%$ in Figure 1o, but lower than $I_m = 3.85\%$ in Table 1. Hence China empirically chooses a higher CBDC interest rate $I_m = 3.85\%$ than $I_m = 2.99\%$ predicted by the model, which is the opposite of what the US does.

Figure 4e plots p, c, m, q, U, I_m, as functions of the inflation rate π, when the target inflation rate $\pi^* = 3\%$, which is higher than $\pi^* = 2\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $\pi > 10.14\%$, which is higher than $\pi > 9.80\%$ in Figure 1o. The CBDC interest rate I_m becomes negative when $\pi < 1.97\%$, which is higher than $\pi < 1.63\%$ in Figure 1o. Hence the higher target inflation rate $\pi^* = 3\%$ increases the consumption c and correspondingly decreases the CBDC interest rate I_m. The CBDC interest rate is $I_m = 0.71\%$ at the benchmark $\pi = 2.419\%$, which is lower than $I_m = 1.24\%$ when $\pi = 2.419\%$ in Figure 1o, and also lower than 3.85% in Table 1. Again, China empirically chooses a higher CBDC interest rate $I_m = 3.85\%$ than $I_m = 0.71\%$ predicted by the model, which is the opposite of what the US does.

Figure 4f plots p, c, m, q, U, I_m, as functions of the inflation rate π, when the real interest rate $I_r = 3.6535\%$ and the target inflation rate $\pi^* = 3\%$, thus combining the assumptions for Figure 4d and Figure 4e. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $\pi > 9.03\%$, which is lower than $\pi > 9.80\%$ in Figure 1o. The CBDC interest rate I_m becomes negative when $\pi < 0.87\%$, which is lower than $\pi < 1.63\%$ in Figure 1o. Hence $I_r = 3.6535\%$ and $\pi^* = 3\%$ increase the consumption c and correspondingly decreases the CBDC interest rate I_m. The results are intermediate between those of Figure 4d and Figure 4e which pull in opposite directions. More specifically, the CBDC interest rate is $I_m = 2.46\%$ at the benchmark $\pi = 2.419\%$, which is higher than $I_m = 1.24\%$ when $\pi = 2.419\%$ in Figure 1o, and lower than 3.85% in Table 1.

Figure 4g plots p, c, m, q, U, I_m, as functions of the target inflation rate π^*, when the real interest rate $I_r = 3.6535\%$, which is higher than $I_r = 2\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 9.40\%$, which is higher than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 18.89\%$, which is higher than $\pi^* > 15.58\%$ in Figure 1p. The CBDC interest rate is $I_m = 3.38\%$ at the benchmark $\pi^* = 3\%$, which is higher than $I_m = 1.63\%$ when $\pi^* = 3\%$ in Figure 1p, but lower than 3.85% in Table 1. Thus, the higher real interest rate $I_r = 3.6535\%$ increases the target inflation rate π^*, but decreases the CBDC interest rate I_m, which is contrary to the US.

Figure 4h plots p, c, m, q, U, I_m, as functions of the target inflation rate π^*, when the inflation rate $\pi = 2.419\%$, which is lower than $\pi = 3\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 4.35\%$, which is lower than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 13.84\%$, which is lower than $\pi^* > 15.58\%$ in Figure 1p. The CBDC interest rate is $I_m = 0.71\%$ at the benchmark $\pi^* = 3\%$, which is lower than $I_m = 1.63\%$ when $\pi^* = 3\%$ in Figure 1p, and much lower than 3.85% in Table 1. The lower inflation rate $\pi = 2.419\%$ decreases the CBDC interest rate I_m, and decreases the target inflation rate π^*.

Figure 4i plots p, c, m, q, U, I_m, as functions of the target inflation rate π^*, when the real interest rate $I_r = 3.6535\%$ and the inflation rate $\pi = 2.419\%$, thus combining the assumptions for Figure 4g and Figure 4h. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 7.66\%$, which is higher than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 17.15\%$, which is higher than $\pi^* > 15.58\%$ in Figure 1p. The CBDC interest rate is $I_m = 2.46\%$ at the benchmark $\pi^* = 3\%$, which is higher than $I_m = 1.63\%$ when $\pi^* = 3\%$ in Figure 1p, but lower than 3.85% in Table 1. Thus, the real interest rate $I_r = 3.6535\%$ combined with the lower inflation rate $\pi = 2.419\%$, increase target inflation rate π^* and decrease the CBDC interest rate I_m.

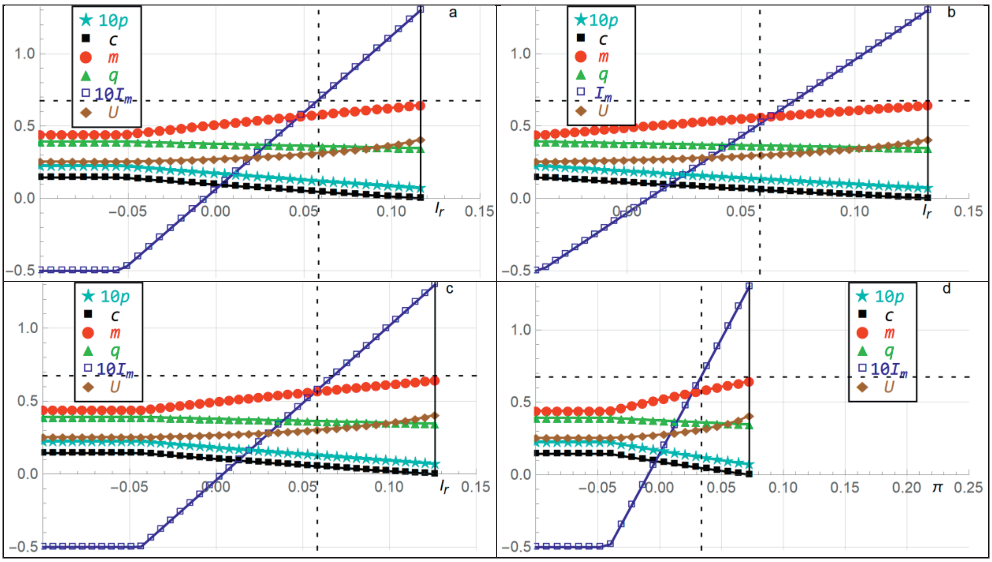
The gap between the empirical inflation rate $\pi = 2.419\%$ and the empirical CBDC interest rate $I_m = 3.6535\%$ is much lower for China than for the US. The model predicts that China's CBDC interest rate I_m should be slightly lower. China empirically chooses a higher CBDC interest rate I_m predicted by the model, which is contrary to the US. The higher real interest rate increases the target inflation rate π^*, but decreases the CBDC interest rate I_m. The higher target inflation rate π^* increases the consumption c and decreases the CBDC interest rate I_m. The lower inflation rate π decreases the CBDC interest rate I_m, and decreases the target inflation rate π^*.

Table 3: Interpretation of Figure 4 for China compared to Figure 1.

China	Changed parameter values from the benchmark in Figure 1	c decreases to zero when	c, p, m, q reach constant values when	I_m becomes negative when	I_m at the benchmark
Figure 4a	$\pi = 2.419\%$	$I_r > 13.08\%$	$I_r < -3.92\%$	$I_r < 0.82\%$	$I_m = 2.99\%$ at $I_r = 3.6535\%$
Figure 4b	$\pi^* = 3\%$	$I_r > 12.71\%$	$I_r < -4.29\%$	$I_r < 0.45\%$	$I_m = 3.38\%$ at $I_r = 3.6535\%$
Figure 4c	$\pi = 2.419\%$ $\pi^* = 3\%$	$I_r > 13.58\%$	$I_r < -3.42\%$	$I_r < 1.32\%$	$I_m = 2.46\%$ at $I_r = 3.6535\%$
Figure 4d	$I_r = 3.6535\%$	$\pi > 8.70\%$	$\pi < -2.63\%$	$\pi < 0.53\%$	$I_m = 2.99\%$ at $\pi = 2.419\%$
Figure 4e	$\pi^* = 3\%$	$\pi > 10.14\%$	$\pi < -1.19\%$	$\pi < 1.97\%$	$I_m = 0.71\%$ at $\pi = 2.419\%$
Figure 4f	$I_r = 3.6535\%$ $\pi^* = 3\%$	$\pi > 9.03\%$	$\pi < -2.3\%$	$\pi < 0.87\%$	$I_m = 2.46\%$ at $\pi = 2.419\%$
Figure 4g	$I_r = 3.6535\%$	$\pi^* < -15.1\%$	$\pi^* > 18.89\%$	$\pi^* > 9.40\%$	$I_m = 3.38\%$ at $\pi^* = 3\%$
Figure 4h	$\pi = 2.419\%$	$\pi^* < -20.15\%$	$\pi^* > 13.84\%$	$\pi^* > 4.35\%$	$I_m = 0.71\%$ at $\pi^* = 3\%$
Figure 4 i	$I_r = 3.6535\%$ $\pi = 2.419\%$	$\pi^* < -16.85\%$	$\pi^* > 17.15\%$	$\pi^* > 7.66\%$	$I_m = 2.46\%$ at $\pi^* = 3\%$
Figure 1n	$I_r = 2\%$	$I_r > 12.21\%$	$I_r < -4.79\%$	$I_r < 0.00\%$	$I_m = 3.91\%$ at $I_r = 3.6535\%$
Figure 1o	$\pi = 3\%$	$I_r > 9.80\%$	$I_r < -1.53\%$	$\pi < 1.63\%$	$I_m = 1.24\%$ at $\pi = 2.419\%$
Figure 1p	$\pi^* = 2\%$	$\pi^* < -18.41\%$	$\pi^* > 15.58\%$	$\pi^* > 6.10\%$	$I_m = 1.63\%$ at $\pi^* = 3\%$

Russia

The Bank of Russia (2021) set its interest rate to $I_m = 6.75\%$ September 10, 2021. Russia's real interest rate is $I_r = 5.83\%$ in 2020 and its annual inflation rate is $\pi = 3.382\%$ (The World Bank, 2021b). The Bank of Russia (2021) set its inflation target rate $\pi^* = 4\%$.



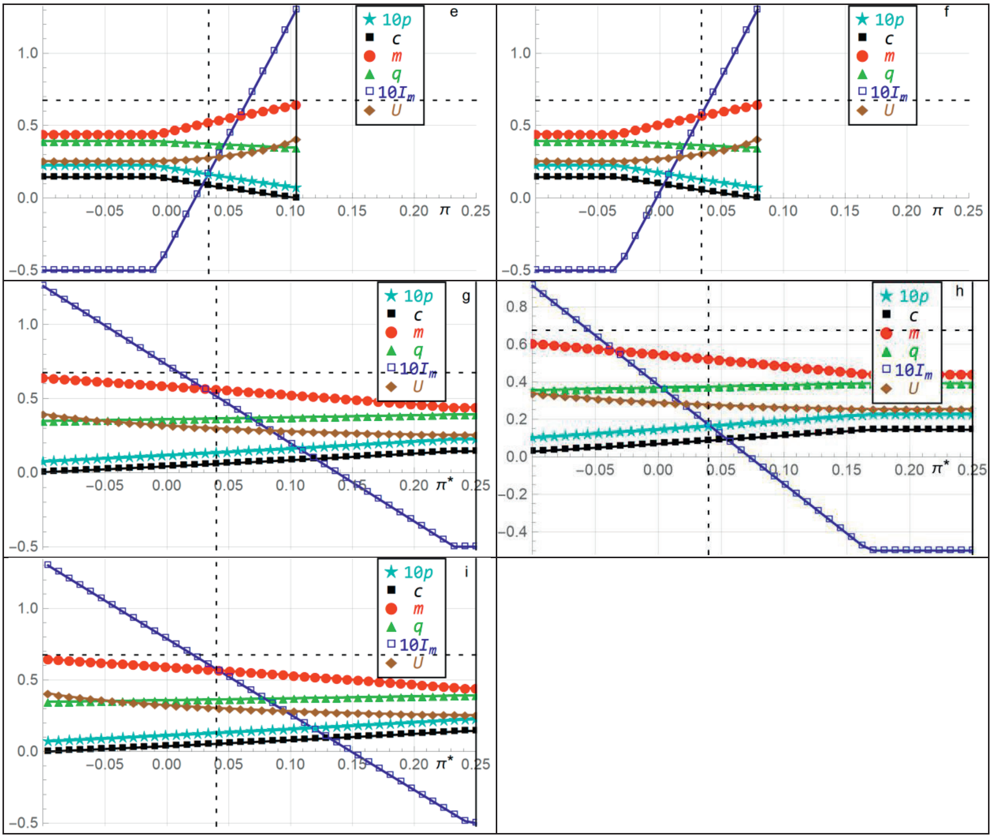


Figure 5: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, utility U, and the CBDC interest rate I_m for Russia, as functions of the real interest rate I_r, inflation rate π, and target inflation rate π^*, respectively, relative to the benchmark parameter values $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = 2\%, I_r = 5.83\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi = 3.382\%, \pi^* = 4\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Multiplication of p and I_m with 10 is for scaling purposes.

Figure 5a plots p, c, m, q, U, I_m as functions of the real interest rate I_r, when the inflation rate $\pi = 3.382\%$, which is higher than $\pi = 3\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > 11.63\%$, which is slightly lower than $I_r > 12.21\%$ in Figure 1n. The higher inflation rate $\pi = 3.382\% > 3\%$ decreases consumption c in Figure 5a. Compared to Figure 1n, in Figure 5a the household chooses lower consumption c, lower production p, and holds less non-CBDC q. The household holds more CBDC m and earns higher utility U. The CBDC interest rate I_m becomes negative when $I_r < -0.62\%$, which is lower than $I_r < 0.00\%$ in Figure 1n. The model thus predicts a higher CBDC interest rate I_m when the inflation rate is $\pi = 3.382\%$ in Figure 5a compared to $\pi = 3\%$ in Figure 1n. Analogously to Figure 3a for the US, that follows from the logic of the Taylor (1993) rule in (4). The central bank combats high inflation rate $\pi = 3.382\%$ by increasing its CBDC interest rate I_m, to make saving in the form of holding CBDC m more attractive than consumption c, which is lower in Figure 5a than in Figure 1n. Mathematically, high inflation $\pi = 3.382\%$ on the right hand side in (4) causes high CBDC interest rate I_m on the left hand side in (4). For example, the CBDC interest rate is $I_m = 6.82\%$ at the benchmark $I_r = 5.83\%$ in Figure 5a, which is higher than $I_m = 6.21\%$ when $I_r = 5.83\%$ in Figure 1n, and slightly higher than the empirical $I_m = 6.75\%$ in Table 1. We interpret this to mean that the model and the Taylor (1993) rule predict appropriately and in accordance with the current empirics for Russia. Interestingly, the model shows that Russia chooses a slightly higher CBDC interest rate I_m to suppress the inflation rate π. But its empirical inflation rate $\pi = 3.382$ is lower than its target inflation rate $\pi^* = 4\%$. The model suggests that Russia should choose a slightly lower CBDC interest rate I_m, which decreases the household's CBDC holding m, and encourages the household to consume and produce more.

Figure 5b plots p, c, m, q, U, I_m, as functions of the real interest rate I_r, when the target inflation rate $\pi^* = 4\%$. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > 13.20\%$, which is higher than $I_r > 12.21\%$ in Figure 1n. The CBDC interest rate I_m becomes negative when $I_r < 0.95\%$, which is higher than $I_r < 0.00\%$ in Figure 1n. Hence the higher target inflation rate $\pi^* = 4\%$ increases the consumption c and correspondingly decreases the CBDC interest rate I_m. The impact of the higher target inflation rate is in contrast to the US in Figure 3b, but the same as for China in Figure

4b. The CBDC interest rate is $I_m = 5.15\%$ at the benchmark $I_r = 5.83\%$, which is lower than $I_m = 6.21\%$ when $I_r = 5.83\%$ in Figure 1n, and also lower than $I_m = 6.75\%$ in Table 1.

Figure 5c plots p, c, m, q, U, I_m, as functions of the real interest rate I_r, when the inflation rate $\pi = 3.382\%$ and the target inflation rate $\pi^* = 4\%$, thus combining the assumptions for Figure 4d and Figure 5e. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > 12.63\%$, which is slighter higher than $I_r > 12.21\%$ in Figure 1n. The CBDC interest rate I_m becomes negative when $I_r < 0.38\%$, which is higher than $I_r < 0.00\%$ in Figure 1n. Thus, the higher inflation rate $\pi = 3.382\%$ combined with the target inflation rate $\pi^* = 4\%$ increase the consumption c slightly, and decrease the CBDC interest rate I_m slightly. The CBDC interest rate is $I_m = 5.76\%$ at the benchmark $I_r = 5.83\%$, which is lower than $I_m = 6.21\%$ when $I_r = 5.83\%$ in Figure 1n, and also lower than 6.75% in Table 1.

Figure 5d plots p, c, m, q, U, I_m, as functions of the inflation rate π, when the real interest rate $I_r = 5.83\%$, which is higher than $I_r = 2\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $\pi > 7.25\%$, which is lower than $\pi > 9.80\%$ in Figure 1o. The CBDC interest rate I_m becomes negative when $\pi < -0.92\%$, which is lower than $\pi < 1.63\%$ in Figure 1o. The CBDC interest rate is $I_m = 6.82\%$ at the benchmark $\pi = 3.382\%$, which is higher than $I_m = 3.46\%$ when $\pi = 3.382\%$ in Figure 1o, and slightly higher than $\pi = 6.75\%$ in Table 1. Thus, the higher real interest rate $I_r = 5.83\%$ decreases the consumption c and increases the CBDC interest rate I_m.

Figure 5e plots p, c, m, q, U, I_m, as functions of the inflation rate π, when the target inflation rate $\pi^* = 4\%$. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $\pi > 10.47\%$, which is higher than $\pi > 9.80\%$ in Figure 1o. The CBDC interest rate I_m becomes negative when $\pi < 2.3\%$, which is higher than $\pi < 1.63\%$ in Figure 1o. The CBDC interest rate is $I_m = 1.71\%$ at the benchmark $\pi = 3.382\%$, which is lower than $I_m = 3.46\%$ when $\pi = 3.382\%$ in Figure 1o, and much lower than $\pi = 6.75\%$ in Table 1. Notably, the higher target inflation rate $\pi^* = 4\%$ decreases CBDC interest rate I_m. Again, the model predicts that Russia should choose a lower CBDC interest rate I_m.

Figure 5f plots p, c, m, q, U, I_m, as functions of the inflation rate π, when the real interest rate $I_r = 5.83\%$ and the target inflation rate $\pi^* = 4\%$. Both parameter values are higher than in Figure 1. Figure 5f thus combines the assumptions for Figure 5d and Figure 5e. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $\pi > 7.92\%$, which is lower than $\pi > 9.80\%$ in Figure 1o. The CBDC interest rate I_m becomes negative when $\pi < -0.25\%$, which is lower than $\pi < 1.63\%$ in Figure 1o. The CBDC interest rate is $I_m = 5.76\%$ at the benchmark $\pi = 3.382\%$, which is higher than $I_m = 3.46\%$ when $\pi = 3.382\%$ in Figure 1o, but lower than $\pi = 6.75\%$ in Table 1. The impact of the higher real interest rate $I_r = 5.83\%$ is greater than the higher target inflation rate $\pi^* = 4\%$. Thus, the household's consumption c decreases compared to Figure 1o.

Figure 5g plots p, c, m, q, U, I_m, as functions of the target inflation rate π^*, when the real interest rate $I_r = 5.83\%$, which is higher than $I_r = 2\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 13.76\%$, which is higher than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 23.24\%$, which is higher than $\pi^* > 15.58\%$ in Figure 1p. The CBDC interest rate is $I_m = 5.15\%$ at the benchmark $\pi^* = 4\%$, which is much higher than $I_m = 1.11\%$ when $\pi^* = 4\%$ in Figure 1p, but lower than 6.75% in Table 1. Hence, the higher real interest rate $I_r = 5.83\%$ increases the target inflation rate π^*, but decreases the CBDC interest rate I_m. The impact of the higher interest rate I_r is the same as for China in Figure 4g for the target inflation rate π^* and the CBDC interest rate I_m, but in contrast to the US for the CBDC interest rate I_m.

Figure 5h plots p, c, m, q, U, I_m, as functions of the target inflation rate π^*, when the inflation rate $\pi = 3.382\%$, which is higher than $\pi = 3\%$ in Figure 1. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 7.24\%$, which is higher than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 16.73\%$, which is higher than $\pi^* > 15.58\%$ in Figure 1p. The CBDC interest rate is $I_m = 1.71\%$ at benchmark $\pi^* = 4\%$, which is much lower than $I_m = 1.11\%$ when $\pi^* = 4\%$ in Figure 1p, and much lower than $\pi^* = 6.5\%$ in Table 1. The higher inflation rate $\pi = 3.382\%$ increases the target inflation rate π^*, but decreases the CBDC interest rate I_m.

Figure 5i plots p, c, m, q, U, I_m, as functions of the target inflation rate π^*, when the real interest rate $I_r = 5.83\%$ and the inflation rate $\pi = 3.382\%$. Both parameter values are higher than in Figure 1. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 14.90\%$, which is much higher than $\pi^* > 6.10\%$ in Figure 1p. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 24.39\%$, which is higher than $\pi^* > 15.58\%$ in Figure 1p. The CBDC interest rate is $I_m = 5.76\%$ at the benchmark $\pi^* = 4\%$, which is much higher than $I_m = 1.11\%$ when $\pi^* = 4\%$ in Figure 1p, but slightly lower than $\pi^* = 6.5\%$ in Table 1. Hence, the higher real interest rate $I_r = 5.83\%$ and the higher inflation rate $\pi = 3.382\%$ CBDC interest rate I_m, and increase the target inflation rate π^*.

The Russia inflation rate $\pi = 3.382\%$ is lower than the CBDC interest rate $I_m = 6.75\%$. The gap between the predicted CBDC interest rate I_m and the Russia empirical CBDC interest rate I_m is intermediate between The US and China. The model predicts that Russia chooses a slightly higher CBDC interest rate I_m to suppress the inflation rate π. Notably, the change of real interest rate I_r has a higher impact on the CBDC interest rate I_m, the change of the inflation rate π has a lower impact on the CBDC interest rate I_m. This holds for the three countries' empirical data. Table 1 shows the empirical data of the four variables I_m, I_r, π, π^* for the US, China and Russia.

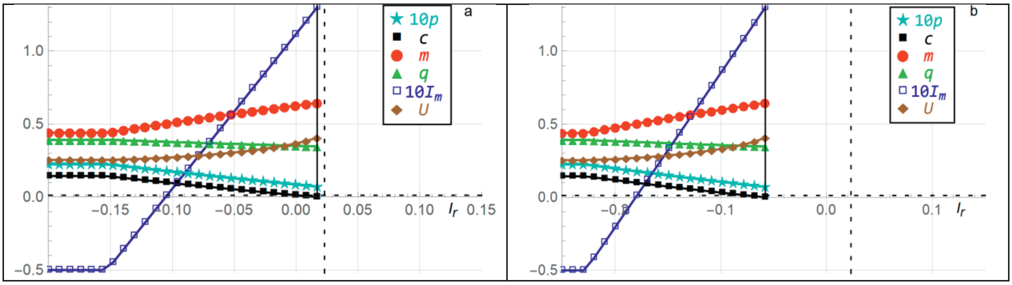
Table 4: Interpretation of Figure 5 for Russia compared to Figure 1.

Russia	Changed parameter values from the benchmark in Figure 1	c decreases to zero when	c, p, m, q reach constant values when	I_m becomes negative when	I_m at the benchmark
Figure 5a	$\pi = 3.382\%$	$I_r > 11.63\%$	$I_r < -5.36\%$	$I_r < -0.62\%$	$I_m = 6.82\%$ at $I_r = 5.83\%$
Figure 5b	$\pi^* = 4\%$	$I_r > 13.20\%$	$I_r < -3.79\%$	$I_r < 0.95\%$	$I_m = 5.15\%$ at $I_r = 5.83\%$
Figure 5c	$\pi = 3.382\%$ $\pi^* = 4\%$	$I_r > 12.63\%$	$I_r < -4.36\%$	$I_r < 0.38\%$	$I_m = 5.76\%$ at $I_r = 5.83\%$
Figure 5d	$I_r = 5.83\%$	$\pi > 7.25\%$	$\pi < -4.08\%$	$\pi < -0.92\%$	$I_m = 6.82\%$ at $\pi = 3.382\%$
Figure 5e	$\pi^* = 4\%$	$\pi > 10.47\%$	$\pi < -0.86\%$	$\pi < 2.3\%$	$I_m = 1.71\%$ at $\pi = 3.382\%$
Figure 5f	$I_r = 5.83\%$ $\pi^* = 4\%$	$\pi > 7.92\%$	$\pi < -3.41\%$	$\pi < -0.25\%$	$I_m = 5.76\%$ at $\pi = 3.382\%$
Figure 5g	$I_r = 5.83\%$	$\pi^* < -10.75\%$	$\pi^* > 23.24\%$	$\pi^* > 13.76\%$	$I_m = 5.15\%$ at $\pi^* = 4\%$
Figure 5h	$\pi = 3.382\%$	$\pi^* < -17.26\%$	$\pi^* > 16.73\%$	$\pi^* > 7.24\%$	$I_m = 1.71\%$ at $\pi^* = 4\%$
Figure 5i	$I_r = 5.83\%$ $\pi = 3.382\%$	$\pi^* < -9.6\%$	$\pi^* > 24.39\%$	$\pi^* > 14.90\%$	$I_m = 5.76\%$ at $\pi^* = 4\%$
Figure 1n	$I_r = 2\%$	$I_r > 12.21\%$	$I_r < -4.79\%$	$I_r < 0.00\%$	$I_m = 6.21\%$ at $I_r = 5.83\%$ i
Figure 1o	$\pi = 3\%$	$I_r > 9.80\%$	$I_r < -1.53\%$	$\pi < 1.63\%$	$I_m = 3.46\%$ at $\pi = 3.382\%$
Figure 1p	$\pi^* = 2\%$	$\pi^* < -18.41\%$	$\pi^* > 15.58\%$	$\pi^* > 6.10\%$	$I_m = 1.11\%$ at $\pi^* = 4\%$

Assessing higher inflation rates π for the US, China and Russia

This section analyzes the implications of hypothetically higher inflation rates $\pi = 10\%$ and $\pi = 15\%$ for the US, China and Russia. The relevance of such an analysis is underscored by Turkey's annual inflation increasing to a three-year high of 21.31% in November 2021.[1] Hanke and Krus (2013) summarize 56 worldwide hyperinflation examples. The highest is $\pi = 2.93 \times 10^{177}\%$ per year ($\pi = 4.19 \times 10^{16}\%$ per month) in Hungary in July 1946. We consider $\pi = 2{,}688{,}670\%$ Venezuela, January 2019 (Descifrado, 2019) for analysis.

The US



[1] https://www.reuters.com/world/middle-east/turkish-inflation-jumps-3-year-high-amid-lira-plunge-2021-12-03/, retrieved April 22, 2022.

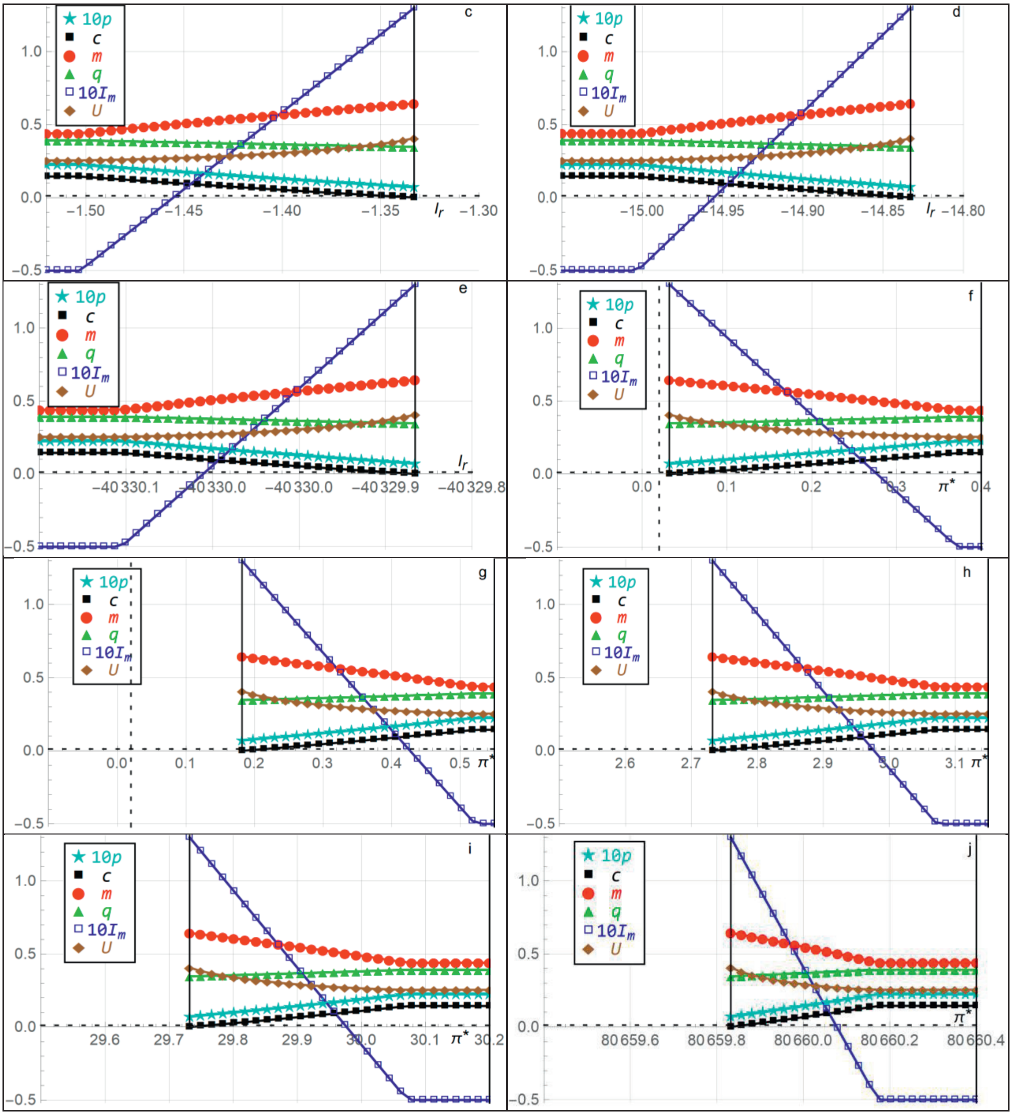


Figure 6: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, utility U, and the CBDC interest rate I_m, as functions of the real interest rate I_r, inflation rate π, and target inflation rate π^*, respectively, relative to the benchmark parameter values $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = 2\%, I_r = 2.305\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi^* = 2\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Panels a and f: $\pi = 10\%$. Panels b and g: $\pi = 15\%$. Panels c and h: $\pi = 100\%$. Panels d and i: $\pi = 1000\%$. Panels e and j: $\pi = 2{,}688{,}670\%$. Multiplication of p and I_m with 10 is for scaling purposes.

Figure 6a plots p, c, m, q, U, I_m as functions of the real interest rate I_r when the inflation rate $\pi = 10\%$, which is higher than $\pi = 3\%$ in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > 1.71\%$, which is lower than $I_r > 12.21\%$ in Figure 1n and lower than $I_r > 7.4\%$ in Figure 3a. The higher inflation rate $\pi = 10\% > 3\%$ decreases consumption c in Figure 6a. The CBDC interest rate I_m becomes negative when $I_r < -10.55\%$, which is lower than $I_r < 0.00\%$ in Figure 1n and lower than $I_r < -4.85\%$ in Figure 3a. Thus, the curves move to the left compared to Figure 1n and Figure 3a. When consumption c decreases to $c = 0$, the CBDC interest rate is $I_m = 13.0\%$. Again, the central bank combats high inflation rate $\pi = 6.2\%$ by increasing its CBDC interest rate I_m, to make saving in the form of holding CBDC m more attractive than consumption c. But it is costly since the CBDC interest rate I_m goes up a lot.

Figure 6b plots p, c, m, q, U, I_m as functions of the real interest rate I_r when the inflation rate $\pi = 15\%$. All the other parameter values are as the benchmarks in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > -5.8\%$, which is lower than

$I_r > 1.71\%$ in Figure 6a. Thus, the higher inflation rate $\pi = 15\%$ decreases consumption c in Figure 6b. The CBDC interest rate I_m becomes negative when $I_r < -18.05\%$, which is lower than $I_r < -10.55\%$ in Figure 6a. Again, the curves move to the left compared to Figure 1n, Figure 3a and Figure 6a.

Figure 6c plots p, c, m, q, U, I_m as functions of the real interest rate I_r when the inflation rate $\pi = 100\%$. All the other parameter values are as the benchmarks in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > -133.3\%$, which is lower than $I_r > -5.8\%$ in Figure 6b. The CBDC interest rate I_m becomes negative when $I_r < -145.55\%$, which is lower than $I_r < -18.05\%$ in Figure 6b. The curves move to the left compared to Figure 1n, Figure 3a, Figure 6a, Figure 6b.

Figure 6d plots p, c, m, q, U, I_m as functions of the real interest rate I_r when the inflation rate $\pi = 1000\%$. All the other parameter values are as the benchmarks in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > -1483.3\%$, which is lower than $I_r > -133.3\%$ in Figure 6c. The CBDC interest rate I_m becomes negative when $I_r < -1495.55\%$.

Figure 6e plots p, c, m, q, U, I_m as functions of the real interest rate I_r when the inflation rate $\pi = 2{,}688{,}670\%$, as in Venezuela, January 2019. All the other parameter values are as the benchmarks in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > -4{,}032{,}988.3\%$. The CBDC interest rate I_m becomes negative when $I_r < -4{,}033{,}000.55\%$. The high Venezuela inflation rate $\pi = 2{,}688{,}670\%$ makes consumption c almost impossible, unless the real interest rate I_r is extremely and unrealistically negative.

Figure 6f plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate is $I_r = 2.305\%$ and the inflation rate is $\pi = 10\%$. Both the real interest rate and the inflation rate are higher than in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 27.71\%$, which is much higher than $\pi^* > 6.10\%$ in Figure 1p and higher than $\pi^* > 16.31\%$ in Figure 3e. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 37.19\%$, which is much higher than $\pi^* > 15.58\%$ in Figure 1p and higher than $\pi^* > 25.79\%$ in Figure 3e. Thus, the curves move to the right compared to Figure 1p and Figure 3e. The higher inflation rate $\pi = 10\%$ and the higher real interest rate $I_r = 2.305\%$ greatly increase the target inflation rate π^* and decrease the CBDC interest rate I_m. The household consumption c decreases to $c = 0$ when $\pi^* < 3.2\%$, where the CBDC interest rate is $I_m = 13.00\%$.

Figure 6g plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate is $I_r = 2.305\%$ and the inflation rate is $\pi = 15\%$. Both the real interest rate and the inflation rate are higher than in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 42.71\%$. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 52.20\%$. The higher inflation rate $\pi = 15\%$ and the higher real interest rate $I_r = 2.305\%$ greatly increases the target inflation rate π^* and increase the CBDC interest rate I_m. The household consumption c decreases to $c = 0$ when $\pi^* < 18.20\%$, where the CBDC interest rate is $I_m = 13.00\%$.

Figure 6h plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate is $I_r = 2.305\%$ and the inflation rate is $\pi = 100\%$. Both the real interest rate and the inflation rate are higher than in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 297.71\%$. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 307.20\%$. The higher inflation rate $\pi = 100\%$ and the higher real interest rate $I_r = 2.305\%$ greatly increases the target inflation rate π^* and increase the CBDC interest rate I_m. The household consumption c decreases to $c = 0$ when $\pi^* < 18.20\%$, where the CBDC interest rate is $I_m = 13.00\%$.

Figure 6i plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate is $I_r = 2.305\%$ and the inflation rate is $\pi = 1000\%$. Both the real interest rate and the inflation rate are higher than in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 2997.8\%$. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 3072.0\%$. The higher inflation rate $\pi = 1000\%$ and the higher real interest rate $I_r = 2.305\%$ greatly increases the target inflation rate π^* and increase the CBDC interest rate I_m. The household consumption c decreases to $c = 0$ when $\pi^* < 2937.20\%$, where the CBDC interest rate is $I_m = 13.00\%$.

Figure 6j plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate is $I_r = 2.305\%$ and the inflation rate is $\pi = 2{,}688{,}670\%$. Both the real interest rate and the inflation rate are higher than in Figure 1. All the other parameter values are as the benchmarks in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 8{,}066{,}007.71\%$. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 8{,}066{,}017.19\%$. The extremely high inflation rate $\pi = 2{,}688{,}670\%$ and the higher real interest rate $I_r = 2.305\%$ greatly increase the target inflation rate π^* and increase the CBDC interest rate I_m.

Table 5: Implication summary of higher inflation rates for the US.

The US	Changed parameter values from the benchmark in Figure 1	c decreases to zero when	I_m becomes negative when	p, c, m, q reach constant values when	I_m at the benchmark	How curves change compared to Figure 3
Figure 6a	$\pi = 10\%$	$I_r > 1.71\%$	$I_r < -10.55\%$	$I_r < 15.29\%$	$I_m = 13.64\%$	Left
Figure 6b	$\pi = 15\%$	$I_r > -5.8\%$	$I_r < -18.05\%$	$I_r < 22.79\%$	$I_m = 22.09\%$	Left
Figure 6c	$\pi = 100\%$	$I_r > -133.3\%$	$I_r < -145.55\%$	$I_r < -150.29\%$	$I_m = 1495.41\%$	Left

Figure 6d	$\pi = 1000\%$	$I_r > -1483.3\%$	$I_r < -1495.55\%$	$I_r < -1500.29\%$	$I_m = 14998.17$	Left
Figure 6e	$\pi =$ 2,688,670%.	I_r $> -4{,}032{,}988.3\%$	I_r $< -4{,}033{,}000.55\%$	I_r $< 4033005.29\%$	I_m $= 40{,}330{,}048.78\%$	Left
Figure 6f	$\pi = 10\%$ $I_r = 2.305\%$	$\pi^* < 3.2\%$	$\pi^* > 27.71\%$	$\pi^* > 37.19\%$	$I_m = 13.64\%$	Right
Figure 6g	$\pi = 15\%$ $I_r = 2.305\%$	$\pi^* < 18.2\%$	$\pi^* > 42.71\%$	$\pi^* > 52.20\%$	$I_m = 22.09\%$	Right
Figure 6h	$\pi = 100\%$ $I_r = 2.305\%$	$\pi^* < 273.2\%$	$\pi^* > 297.71\%$	$\pi^* > 307.20\%$	$I_m = 1495.41\%$	Right
Figure 6i	$\pi = 1000\%$ $I_r = 2.305\%$	$\pi^* < 2973.2\%$	$\pi^* > 2997.8\%$	$\pi^* > 3072.0\%$	$I_m = 14998.2\%$	Right
Figure 6j	π $= 2{,}688{,}670\%$ $I_r = 2.305\%$	π^* $< 8{,}065{,}983.2\%$	π^* $> 8{,}066{,}007.71\%$	π^* $> 8{,}066{,}017.19\%$	I_m $= 40{,}330{,}048.78\%$	Right
Figure 1n	$\pi = 3\%$	$I_r > 12.21\%$	$I_r < 0.00\%$	$I_r < -4.79\%$	$I_m = 3.91\%$	Right
Figure 1p	$I_r = 2\%$	$\pi^* < -18.41\%$	$\pi^* > 6.10\%$	$\pi^* > 15.58\%$	$I_m = 1.63\%$	Left

China

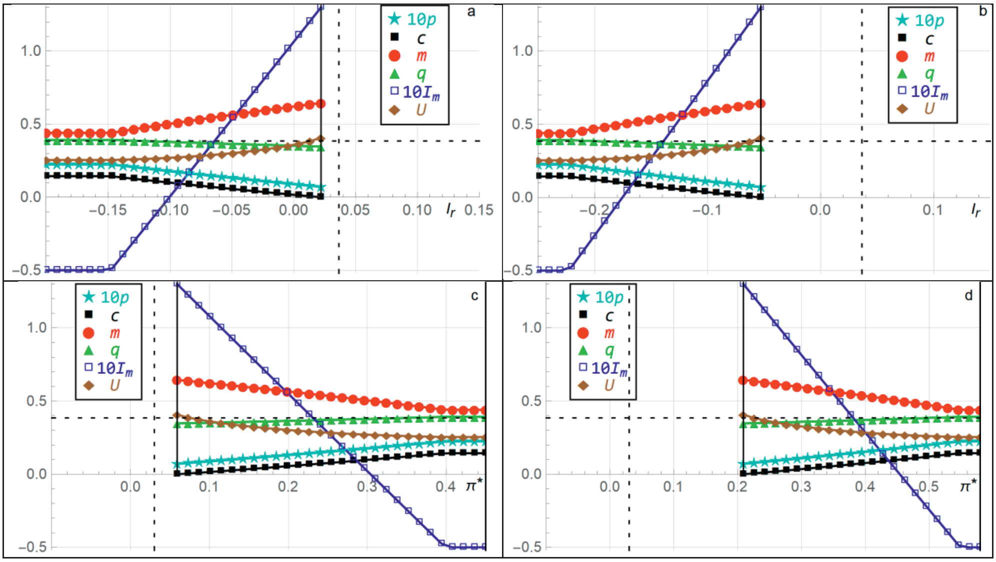


Figure 7: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, utility U, and the CBDC interest rate I_m, as functions of the real interest rate I_r, inflation rate π, and target inflation rate π^*, respectively, relative to the benchmark parameter values $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = 2\%, I_r = 3.6535\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \quad \pi^* = 3\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Panels a and c: $\pi = 10\%$. Panels b and d: $\pi = 15\%$. Multiplication of p and I_m with 10 is for scaling purposes.

Figure 7a plots p, c, m, q, U, I_m as functions of the real interest rate I_r, when the inflation rate $\pi = 10\%$ and the target inflation rate $\pi^* = 3\%$. All the other parameter values are as the benchmark in Figure 1. The consumption c decreases and the CBDC interest rate I_m increases. More specifically, the household's consumption c decreases to $c = 0$ when $I_r > 2.2\%$, which is much lower than $I_r >$ 12.21% in Figure 1n, and also much lower than $I_r > 13.58\%$ in Figure 4c. The CBDC interest rate I_m becomes negative when $I_r <$ −10.05%, which is lower than $I_r < 0.00\%$ in Figure 1n, and lower than $I_r < 1.32\%$ in Figure 4c. Thus, the curves move to the left compared to Figure 1n and Figure 4c. The high inflation rate $\pi = 10\%$ decreases the consumption c and decreases the real interest rate I_r. The central bank increases its interest to combat inflation.

Figure 7b plots p, c, m, q, U, I_m as functions of the real interest rate I_r, when the inflation rate $\pi = 15\%$ and the target inflation rate $\pi^* = 3\%$. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > -5.3\%$, which is much lower than $I_r > 12.21\%$ in Figure 1n, much lower than $I_r > 13.58\%$ in Figure 4c, and lower than $I_r >$ 2.2% in Figure 7a. The CBDC interest rate I_m becomes negative when $I_r < -17.55\%$, which is lower than $I_r < 0.00\%$ in Figure 1n, lower than $I_r < 1.32\%$ in Figure 4c, and lower than $I_r < -10.05\%$ in Figure 7a. Again, the curves move to the right even further compared to Figure 1n, Figure 4c and Figure 7a. The high inflation rate $\pi = 15\%$ decreases the consumption c and decreases the real interest rate I_r.

Figure 7c plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate $I_r = 3.6535\%$ and the inflation rate $\pi = 10\%$. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 30.40\%$, which is much higher than $\pi^* > 6.10\%$ in Figure 1p and higher than $\pi^* > 7.66\%$ in Figure 4i. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 39.89\%$, which is much higher than $\pi^* > 15.58\%$ in Figure 1p and higher than $\pi^* > 17.15\%$ in Figure 4i. Thus, the curves move to the right compared to Figure 1p and Figure 4i. The higher inflation rate $\pi = 10\%$ and the higher real interest rate $I_r = 3.6535\%$ greatly increase the target inflation rate π^*. The household consumption c decreases to $c = 0$ when $\pi^* < 5.9\%$, the CBDC interest rate is $I_m = 13.00\%$ at this point.

Figure 7d plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate $I_r = 3.6535\%$ and the inflation rate $\pi = 15\%$. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 45.4\%$, which is much higher than $\pi^* > 6.10\%$ in Figure 1p, much higher than $\pi^* > 7.66\%$ in Figure 4I, and higher than $\pi^* > 30.40\%$ in Figure 7c. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 45.41\%$, which is higher than $\pi^* > 15.58\%$ in Figure 1p, higher than $\pi^* > 17.15\%$ in Figure 4i, and higher than $\pi^* > 39.89\%$ in Figure 7c. Again, the curves move to the right even further compared to Figure 1p, Figure 4i and Figure 7c. The higher inflation rate $\pi = 15\%$ and the higher real interest rate $I_r = 3.6535\%$ greatly increase the target inflation rate π^*. The household consumption c decreases to $c = 0$ when $\pi^* < 5.9\%$, the CBDC interest rate is $I_m = 13.00\%$ at this point.

Table 6: Implication summary of higher inflation rates for China.

China	Changed parameter values from the benchmark in Figure 1	c decreases to zero when	I_m becomes negative when	p, c, m, q reach constant values when	I_m at the benchmark	How curves change compared to Figure 4
Figure 7a	$\pi = 10\%$ $\pi^* = 3\%$	$I_r > 2.2\%$	$I_r < -10.05\%$	$I_r < -14.79\%$	$I_m = 14.56\%$	Left
Figure 7b	$\pi = 15\%$ $\pi^* = 3\%$.	$I_r > -5.3\%$	$I_r < -17.55\%$	$I_r < -22.29\%$	$I_m = 23.21\%$	Left
Figure 7c	$\pi = 10\%$ $I_r = 3.6535\%$	$\pi^* < 5.9\%$	$\pi^* > 30.40\%$	$\pi^* > 39.89\%$	$I_m = 14.56\%$	Right
Figure 7d	$\pi = 15\%$ $I_r = 3.6535\%$	$\pi^* < 20.9\%$	$\pi^* > 45.4\%$	$\pi^* > 45.41\%$	$I_m = 23.21\%$	Right
Figure 1n	$\pi = 3\%$	$I_r > 12.21\%$	$I_r < 0.00\%$	$I_r < -4.79\%$	$I_m = 3.91\%$	Right
Figure 1p	$I_r = 2\%$	$\pi^* < -18.41\%$	$\pi^* > 6.10\%$	$\pi^* > 15.58\%$	$I_m = 1.63\%$	Left

Russia

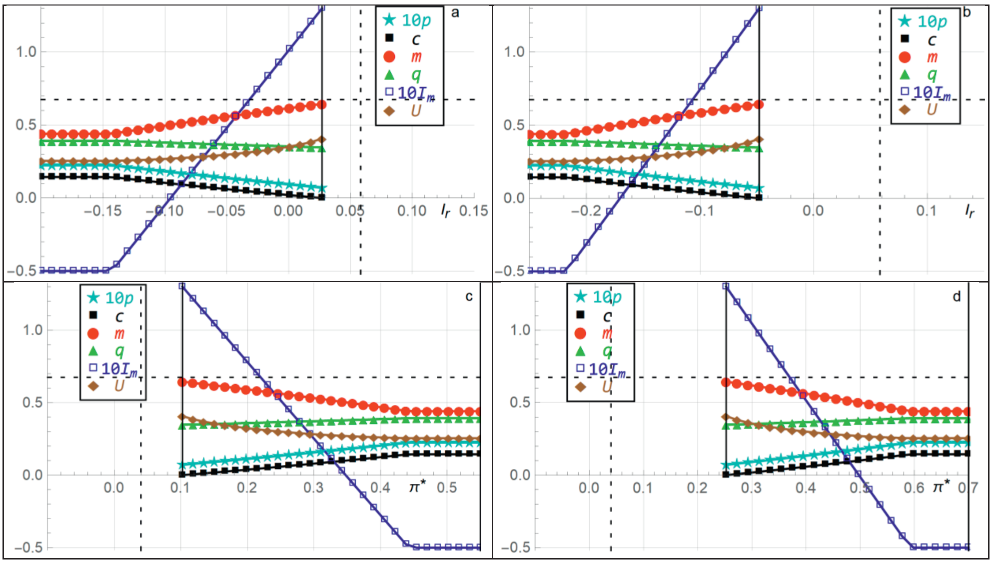


Figure 8: The household's production p, consumption c, CBDC holding m, non-CBDC holding q, utility U, and the CBDC interest rate I_m, as functions of the real interest rate I_r, inflation rate π, and target inflation rate π^*, respectively, relative to the benchmark

parameter values $\alpha = \beta = \gamma = \frac{1}{4}, r = a = M = Q = 1, I_q = 2\%, I_r = 5.83\%, \eta = \frac{1}{5}, \mu = \frac{2}{5}, \lambda = \frac{1}{10}, \pi^* = 4\%, h = \frac{1}{10}, \bar{p} = \frac{1}{2}, a_\pi = a_p = \frac{1}{2}, z = -5\%$. Panels a and c: $\pi = 10\%$. Panels b and d: $\pi = 15\%$. Multiplication of p and I_m with 10 is for scaling purposes.

Figure 8a plots p, c, m, q, U, I_m as functions of the real interest rate I_r, when the inflation rate $\pi = 10\%$ and the target inflation rate $\pi^* = 4\%$. All the other parameter values are as the benchmark in Figure 1. The consumption c decreases and the CBDC interest rate I_m increases. More specifically, the household's consumption c decreases to $c = 0$ when $I_r > 2.7\%$, which is much lower than $I_r > 12.21\%$ in Figure 1n, and lower than $I_r > 12.63\%$ in Figure 5c. The CBDC interest rate I_m becomes negative when $I_r < -9.5\%$, which is lower than $I_r < 0.00\%$ in Figure 1n, and lower than $I_r < 0.38\%$ in Figure 5c. Thus, the curves move to the left compared to Figure 1n and Figure 5c. The high inflation rate $\pi = 10\%$ decreases the consumption c and decreases the real interest rate I_r.

Figure 8b plots p, c, m, q, U, I_m as functions of the real interest rate I_r, when the inflation rate $\pi = 15\%$ and the target inflation rate $\pi^* = 4\%$. All the other parameter values are as the benchmark in Figure 1. The household's consumption c decreases to $c = 0$ when $I_r > -4.8\%$, which is much lower than $I_r > 12.21\%$ in Figure 1n, lower than $I_r > 12.63\%$ in Figure 5c, and lower than $I_r > 2.7\%$ in Figure 8a. The CBDC interest rate I_m becomes negative when $I_r < -17.0\%$, which is lower than $I_r < 0.00\%$ in Figure 1n, lower than $I_r < 0.38\%$ in Figure 5c, and lower than $I_r < -9.5\%$ in Figure 8a. Again, the curves move to the left even further compared to Figure 1n, Figure 5c, and Figure 8a. The higher inflation rate $\pi = 15\%$ further decreases the consumption c and decreases the real interest rate I_r.

Figure 8c plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate $I_r = 5.83\%$ and the inflation rate $\pi = 10\%$. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 34.76\%$, which is much higher than $\pi^* > 6.10\%$ in Figure 1p and higher than $\pi^* > 14.90\%$ in Figure 5i. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 44.24\%$, which is much higher than $\pi^* > 15.58\%$ in Figure 1p and higher than $\pi^* > 24.39\%$ in Figure 5i. The household consumption c decreases to $c = 0$ when $\pi^* < 10.25\%$. Thus, the curves move to the right compared to Figure 1p and Figure 5i. The higher inflation rate $\pi = 10\%$ and the higher real interest rate $I_r = 5.83\%\%$ greatly increase the target inflation rate π^*.

Figure 8d plots p, c, m, q, U, I_m as functions of the target inflation rate π^*, when the real interest rate $I_r = 5.83\%$ and the inflation rate $\pi = 15\%$. All the other parameter values are as the benchmark in Figure 1. The CBDC interest rate I_m becomes negative when $\pi^* > 49.76\%$, which is much higher than $\pi^* > 6.10\%$ in Figure 1p, higher than $\pi^* > 14.90\%$ in Figure 5i, and higher than $\pi^* > 34.76\%$ Figure 8c. The household consumption c, production p, CBDC holding m and non-CBDC holding q reach constant values when $\pi^* > 59.24\%$, which is much higher than $\pi^* > 15.58\%$ in Figure 1p, higher than $\pi^* > 24.39\%$ in Figure 5i, and higher than $\pi^* > 44.24\%$ in Figure 8c. The household consumption c decreases to $c = 0$ when $\pi^* < 25.25\%$. Again, the curves move to the right even further compared to Figure 1p, Figure 5i, and Figure 8c.

Table 7: Implication summary of higher inflation rates for Russia.

Russia	Changed parameter values from the benchmark in Figure 1	c decreases to zero when	I_m becomes negative when	p, c, m, q reach constant values when	I_m at the benchmark	How curves change compared to Figure 5
Figure 8a	$\pi = 10\%$ $\pi^* = 4\%$	$I_r > 2.7\%$	$I_r < -9.5\%$	$I_r < -14.29\%$	$I_m = 16.37\%$	Left
Figure 8b	$\pi = 15\%$ $\pi^* = 4\%$	$I_r > -4.8\%$	$I_r < -17.0\%$	$I_r < -21.79\%$	$I_m = 24.74\%$	Left
Figure 8c	$\pi = 10\%$ $I_r = 5.83\%$	$\pi^* < 10.25\%$	$\pi^* > 34.76\%$	$\pi^* > 44.24\%$	$I_m = 16.37\%$	Right
Figure 8d	$\pi = 15\%$ $I_r = 5.83\%$	$\pi^* < 25.25\%$	$\pi^* > 49.76\%$	$\pi^* > 59.24\%$	$I_m = 24.74\%$	Right
Figure 1n	$\pi = 3\%$	$I_r > 12.21\%$	$I_r < 0.00\%$	$I_r < -4.79\%$	$I_m = 3.91\%$	Right
Figure 1p	$I_r = 2\%$	$\pi^* < -18.41\%$	$\pi^* > 6.10\%$	$\pi^* > 15.58\%$	$I_m = 1.63\%$	Left

Conclusion

The article extends G. Wang and Hausken (2022) in a series of two articles by comparing a decision model with the empirics for the US, China and Russia. In period 1 the central bank chooses positive or negative interest rate. In period 2 the household allocates its resources into production, consumption, CBDC (central bank digital currency) holding, and non-CBDC holding.

Whereas the benchmark in G. Wang and Hausken (2022) assumed the inflation rate 3% and the target inflation rate 2%, the US's October 2021 empirical inflation rate is 6.2%, with a target 2% inflation rate. The model predicts and quantifies how the US should choose a substantially higher CBDC interest rate 7.56% than its empirical interest rate 0.125%, in order to suppress the high inflation rate. That would encourage the household to hold more CBDC, hold less non-CBDC, and produce and consume less. The central bank should choose negative CBDC interest rate when the inflation and real interest rate are low, and the inflation target is high.

China, in contrast, has a low empirical inflation rate 2.419% below its target inflation rate 3%. The model predicts that China should choose the low CBDC interest rate 2.99%, below its empirical interest rate 3.85%. That would decrease the household's CBDC holding and increase the low inflation rate to the target inflation rate. It would also induce the household to hold more non-CBDC, and produce and consume more.

Russia chooses a strategy in between that of the US and China. Russia's inflation rate is 3.382%, which is below its target inflation rate 4%. The model predicts that Russia should choose the CBDC interest rate 6.82%, which is slightly above its empirical interest rate 6.75%. Compared to the benchmark in G. Wang and Hausken (2022), Russia's high CBDC interest rate 6.82% induces the household to hold slightly more CBDC and earn slightly higher utility, and hold slightly less non-CBDC and produce and consume slightly less.

The article also assesses higher inflation rates for the US, Russia, and China. The highest recent inflation rate 2,688,670% occurred in Venezuela in January 2019. As inflation increases, all curves move to the left compared to the benchmark for the real interest rate. That is, extremely high inflation makes production and consumption almost impossible, unless the real interest rate is extremely negative. The extremely high inflation greatly increases the CBDC interest rate. In contrast, all curves move to the right compared to the benchmark for the target inflation rate. That is, an extremely high target inflation rate makes production and consumption almost impossible, unless the target inflation rate is extremely positive.

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Interest Rates, the Taylor Rule, the Quantity Equation, and the Phillips Curve

Guizhou Wang
University of Stavanger, Norway
Email: guizhou.wang@uis.no

Kjell Hausken
Corresponding Author: University of Stavanger, Norway
Email: kjell.hausken@uis.no

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Abstract

This article combines the Taylor rule, the Friedman's Quantity Equation, and the Phillips curve to explore how deviations in the inflation rate, real GDP, money supply, money velocity, and the unemployment rate interact with the interest rate. The motivation is to understand which factors impact the interest rate and how. Applying monthly United States data from 1 January 1959 to 31 March 2022, the contribution and findings show that the deviation in the inflation rate, the deviation in the real GDP, the deviation in the money supply, the money velocity, and the deviation in the unemployment rate are positively correlated with the interest rate. Regression analysis shows that the deviation in the inflation rate and the deviation in the real GDP are statistically positive and interact with the interest rate, consistently with Taylor. The interest rate increases with the money supply and the money velocity. Multicollinearity exists between the deviation in the real GDP and the deviation in the unemployment rate. The interest rate increases with the deviation in the unemployment rate, consistently with the Phillips curve. The deviation in the inflation rate, the deviation in the money supply, the money velocity, and the deviation in the unemployment rate are good interest rate indicators. The combination explains the interest rate more realistically than the Taylor rule.

Keywords: Interest Rate, Taylor Rule, Quantity Equation, Phillips Curve, Money Supply, Money Velocity, Unemployment, Regression Analysis

JEL Classifications: C5, E24, E4, E5

1. Introduction

Central banks have multiple roles, with goals pertaining to economic growth, optimal employment or low unemployment rate, low inflation rate, exchange stability, financial stability, etc. The widely known Taylor (1993) rule is a tool for central banks to determine interest rates. It predicts and suggests the interest rate with four variables, i.e., the inflation rate, the equilibrium real interest rate, the gap in the inflation rate, and the gap in real GDP (gross domestic product). The Taylor

(1993) rule does not include the money supply, commonly accepted to impact the interest rate. The Quantity Equation (Friedman, 1970) connects the money supply, money velocity, price level (inflation rate), and real GDP. For the money supply, first, based on the law of supply and demand (Gale, 1955), the interest rate is the price of the money supply. Thus, the money supply increase causes the interest rate to decrease. Second, central banks tend to increase the interest rate to prevent massive withdrawals when the money supply increases. In addition, the increase in the money supply may cause inflation. If the inflation rate is high, central banks may be forced to increase the interest rate to stabilize the economy. Money velocity is related to the interest rate. As Taylor (1999) points out, velocity depends on the interest rate and real output or income. Money velocity is the average number of times that a unit of currency is circulated within a time period. Under a certain real output level, the increase of money velocity decreases the money supply. According to the Keynesian money demand theory (Keynes *et al.* 1971), when the money supply decreases, the money velocity has to increase to maintain the balance of the monetary market. Thus, the money velocity has an opposite impact on the interest rate compared with the money supply. Phillips (1958) connects the inflation rate and the unemployment rate in the short run, expressed in the so-called Phillips (1958) curve[1]. It suggests a negative relationship between inflation and unemployment rates in the short run. Taylor (1993) suggests that the inflation rate increases the interest rate. Hence, an inverse relationship is assumed between unemployment and interest rates (Prag, 1994). Therefore, it is reasonable to link the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. To our best knowledge, such combinations remain poorly explored. Thus, against this background, this article combines the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. This research identifies five variables involved in these three equations and explores their interaction with the interest rate.

This article chooses the following five independent variables which may statistically impact the interest rate, i.e., the deviation in the inflation rate, the deviation in the real GDP, the deviation in the money supply, the money velocity, and the deviation in the unemployment rate. This article innovatively explores the combinations of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. We employ the monthly data from 1 January 1959 to 31 March 2022 to explore the impact of these five variables on the interest rate in the United States. The research generalizes the Taylor (1993) rule by introducing money supply and money velocity captured in the Quantity Equation (Friedman, 1970) and the unemployment rate presented in the Phillips (1958) curve. Exploring the combinations of these three equations helps better understand the interactions of these five variables with the interest rate.

Although the money supply is not included in the Taylor (1993) rule, it has received substantial attention. The literature compares the Taylor (1993) rule with other rules, including the money supply rule (Minford *et al.* 2003), the Friedman rule (Srinivasan, 2000), and the solvency rule (Brancaccio and Fontana, 2013). Various studies analyze the Taylor (1993) rule and monetary policy (Asso *et al.* 2010; Auray and Fève, 2003; Castro, 2011; Kliesen, 2019)) or apply the Taylor (1993) rule to analyze central bank digital currencies (Wang and Hausken, 2022). The growth form of the Quantity Equation (Friedman, 1970) indicates the relationship between the inflation rate and changes in the money supply, money velocity, and GDP. Kang (1983) points out that the relationship between the money supply and the interest rate is robust since the money supply has a negative short-term liquidity effect on the interest rate and a positive long-term income effect. Qureshi (2021) investigates the role of money in Federal Reserve policy. The findings indicate that money is a relevant indicator for explaining the monetary policy.[2] The well-known Phillips (1958) curve explores the unemployment rate and suggests an inverse relationship

[1] The modern Phillips curves include a short-run Phillips curve and a long-run Phillips curve (Granger and Jeon, 2011). In the short run, it is commonly accepted that inflation and unemployment rates are inversely related. In the long run, that relationship breaks down (Russell and Banerjee, 2008). The economy maintains the natural unemployment rate regardless of the inflation rate. Thus, there is no tradeoff between inflation and interest rates in the long run. This article uses monthly data. Thus, it is reasonable to assume an inverse relationship between the inflation rate and the unemployment rate, as in a short-run Phillips curve.

[2] For monetary policy in a Central Bank Digital Currency System, see Wijngaard and Van Hee (2021).

between inflation rate and unemployment rate in the short run. It omits the interest rate term. Rocheteau and Rodriguez-Lopez (2014) explore the linkage between the money supply, liquidity (the interplay between the supply and demand for money), unemployment, and interest rates. They find that increased public liquidity (assets serving as media of exchange) causes the real interest rate and unemployment to increase.

The article shows a positive correlation between the interest rate on the one hand and the deviation in the inflation rate, the deviation in the real GDP, the deviation in the money supply, the money velocity, and the deviation in the unemployment rate positively on the other hand. Regression analysis shows that the deviation in the inflation rate and the deviation in the real GDP are statistically positive and interact with the interest rate. The interest rate increases with the money supply and the money velocity. Multicollinearity exists between the deviation in the real GDP and the deviation in the unemployment rate, causing the removal of the deviation in the real GDP. The interest rate increases with the deviation in the unemployment rate. The deviation in the inflation rate, the deviation in the money supply, the money velocity, and the deviation in the unemployment rate are goods interest rate indicators.

The remainder of the article is as follows. Section 2 illustrates the conceptual framework of dependent and independent variables and the analytic approaches. Section 3 presents the empirical data. Section 4 investigates the Pearson correlation between six variables, presents exploratory regression analysis, and contains a discussion. Section 5 summarizes the study giving conclusions.

2. Conceptual framework and the analytic procedures

2.1. Choosing the dependent and independent variables

The nomenclature is shown in Table A1 in the Appendix. This article investigates the variables that impact interest rates by incorporating the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. Interest rate is the dependent variable. We use five independent variables according to the incorporated approaches described as follows:

1. The deviation $(\pi - \pi^*)$ in the inflation rate is present in the Taylor (1993) rule as in Equation (1),

$$i = \pi + r^* + a_\pi(\pi - \pi^*) + a_y Log\left(\frac{y}{\bar{y}}\right), \quad (1)$$

where $i \in \mathbb{R}$ is the interest rate, $\pi \in \mathbb{R}$ is the inflation rate, $\pi^* \in \mathbb{R}$ is the target inflation rate, $r^* \in \mathbb{R}$ is the equilibrium real interest rate, whereas $a_\pi = a_y = 0.5$ are constants, $y \geq 0$ is the real GDP, and $\bar{y} \geq 0$ is the potential real GDP that can be sustained over the long term. The latter is a theoretical estimation of GDP when labor and capital are at their maximum sustainable amounts. Log denotes the logarithm with base ten.

2. The deviation $Log(y/\bar{y})$ in the real GDP is in Equation (1).
3. The deviation $Log(m/m_\tau)$ in the money supply, where $m > 0$ is the money supply that is present in the Quantity Equation (Friedman, 1970) as in Equation (2),

$$m * v = \pi * t, \quad (2)$$

where t is the volume of transactions and $m_\tau \geq 0$ is the money supply at some earlier point τ in time $(\tau \geq 0)$.

4. The money velocity v $(v \geq 0)$ is also present in the Quantity Equation in (2).
5. The deviation $\bar{u} - u$ between the natural unemployment rate $\bar{u}$ $(\bar{u} \geq 0)$ and the unemployment rate u $(u \geq 0)$. Both $\bar{u}$ and u are present in the Phillips (1958) curve in Equation (3),

$$gW = gW^T + f(\bar{u} - u), \quad (3)$$

where gW is the growth rate of money wages, gW^T is the growth trend rate of money wages, and $f(\cdot)$ is a function. Analogously to the Taylor (1993) rule in Equation (1), with the deviation $Log(y/\bar{y})$ in the real GDP and the deviation $\pi - \pi^*$ in the inflation rate, this article introduces the deviation $\bar{u} - u$ in the unemployment rate in Equation (3). Phillips (1958) assumes a negative relationship between the inflation rate π and the unemployment rate u, i.e. a positive relationship between the inflation rate π and the deviation $\bar{u} - u$ in the unemployment rate. Taylor (1993) assumes a positive relationship between the interest rate i and the inflation rate π. Combining assumptions of Phillips (1958) and Taylor (1993) implies a positive relationship between the interest rate i and the deviation $\bar{u} - u$ in the unemployment rate. This implication is consistent with Prag's (1994) finding of an inverse relationship between the interest rate i and unemployment rate u.

2.2. The analytic procedures

The article first shows the Pearson correlation coefficients between six variables. After that, the regression analysis is presented with an interest rate i as a dependent variable and the other five variables as independent variables. The regression analysis is updated and refined by removing insignificant independent variables. Consequently, independent variables which pass the significant test are selected. This approach is exploratory. The total amount of possible combinations with five independent variables is given by $\sum_{x=1}^{5}\binom{5}{x} = 31$, where $\binom{5}{x}$ denotes the binomial coefficient. Furthermore, the regression findings are presented and discussed. The analysis seeks to combine the three equations mentioned above in economics to enhance the understanding of the impact of these five variables on the interest rate i.

3. Empirics for the United States

This article collects and adopts monthly United States data from 1 January 1959 to 31 March 2022 from the following resources. The historical interest rate i is derived from the Board of Governors of the Federal Reserve System (US) (2022a). The inflation rate π data is obtained from the U.S. Bureau of Labor Statistics (2022a). The target inflation rate $\pi^* = 1.5\%$ is estimated from a previous study by Shapiro and Wilson (2019) from 1 January 2000 to 30 December 2007. For the remaining period from 1 January 1959 to 31 March 2022, we adopt the common $\pi^* = 2\%$, which Taylor (1993) also uses from 1 January 1984 to 30 September 1992. The real GDP y is estimated by the U.S. Bureau of Economic Analysis (2022). The real potential GDP $\bar{y}$ is derived from the U.S. Congressional Budget Office (2022b). The M2 money supply m is estimated from the Board of Governors of the Federal Reserve System (2022b). Inspired by previous studies (Batini, 2006; Batini and Nelson, 2001; Friedman and Schwartz, 1982), this study uses the money supply m_τ with a two-year lag. This approach suggests more than a one-year time lag from money printing to inflation. The unemployment rate u is evaluated by the U.S. Bureau of Labor Statistics (2022b). The natural unemployment rate $\bar{u}$ is estimated from the U.S. Congressional Budget Office (2022a).[3] This is the same natural rate of unemployment used in the Phillips (1958) curve. The money velocity v is estimated from the Federal Reserve Bank of St. Louis (2022). For the real GDP y, the real potential GDP $\bar{y}$, the natural unemployment rate $\bar{u}$, and the money velocity v, the quadratic interpolation method is adopted to convert the quarterly data to monthly data. Table 1 illustrates the descriptive statistics for the six variables.

According to Table 1, the sample size is $N = 735$. For the interest rate i, the minimum and maximum are 0.05% in April and May 2020 and 19.10% in July 1981, respectively, with an average of 4.85% and a standard deviation of 3.73%.

[3] The natural unemployment rate is the rate of unemployment arising from all sources except fluctuations in aggregate demand. Starting with the July 2021 report: "An Update to the Budget and Economic Outlook: 2021 to 2031", this series was renamed from "Natural Rate of Unemployment (Long-Term)" to "Noncyclical Rate of Unemployment".

Table 1. Descriptive statistics of six variables

Variable	N	Mean	S.D.	Min.	Median	Max.
(1) i	735	0.0485	0.0373	0.0005	0.0476	0.1910
(2) $\pi - \pi^*$	735	0.0182	0.0281	-0.0396	0.0112	0.1259
(3) $Log\left(\frac{y}{\bar{y}}\right)$	735	-0.0042	0.0105	-0.0523	-0.0041	0.0246
(4) $Log\left(\frac{m}{m_\tau}\right)$	735	0.0595	0.0237	0.0076	0.0577	0.1489
(5) v	735	1.7813	0.2215	1.0711	1.7664	2.1928
(6) $\bar{u} - u$	735	-0.0053	0.0169	-0.1019	-0.0016	0.0245

Notes: In Table 1, Column 1 presents the variable name. Column 2 is the sample size. Column 3 shows the mean of the six variables, whereas Column 4 is the standard deviation. Columns 5, 6, and 7 are the variables' minimum, median, and maximum values, respectively.

4. Correlation and regression analysis
4.1. Correlation

Table 2 shows the Pearson correlation between the interest rate i, the deviation $\pi - \pi^*$ in the inflation rate, the deviation $Log(y/\bar{y})$ in the real GDP, the deviation $Log(m/m_\tau)$ in the money supply, the money velocity v, and the deviation $\bar{u} - u$ in the unemployment rate. Notably, the variables are all positively related to the interest rate i. As shown in the second column of Table 2, the interest rate i has the highest correlation coefficient (0.7267) with the deviation $\pi - \pi^*$ in the inflation rate. This high number indicates that the deviation $\pi - \pi^*$ in the inflation rate is explanatory for the interest rate i. Thereafter follows the money velocity v at 0.3686, which suggests that the money velocity v is also essential for the interest rate i. This relationship has hardly been explored in the existing literature. The deviation $\bar{u} - u$ in the unemployment rate is 0.2201. This lower correlation coefficient expresses weak relation with the interest rate i. That sounds plausible since the unemployment rate u is usually not assumed to impact the interest rate i directly.

Table 2. Correlation matrix

Variables	i	$\pi - \pi^*$	$Log\left(\frac{y}{\bar{y}}\right)$	$Log\left(\frac{m}{m_\tau}\right)$	v	$\bar{u} - u$
i	1.0000					
$\pi - \pi^*$	0.7267	1.0000				
$Log\left(\frac{y}{\bar{y}}\right)$	0.1473	0.0552	1.0000			
$Log\left(\frac{m}{m_\tau}\right)$	0.0875	0.2170	-0.0443	1.0000		
v	0.3686	0.1157	0.2097	-0.5221	1.0000	
$\bar{u} - u$	0.2201	0.1189	0.8847	-0.0916	0.1878	1.0000

Note: Table 2 reports the correlation between the dependent variable interest rate i and five independent variables.

An even lower correlation coefficient exists for the deviation $Log(y/\bar{y})$ in the real GDP at 0.1473. This low correlation suggests that the deviation $Log(y/\bar{y})$ in the real GDP has a weak relationship with the interest rate i. Thereafter follows the deviation $Log(m/m_\tau)$ in the money supply with an even lower correlation coefficient at 0.0875. In that regard, Conrad (2021) argues that the interest rate i decreases in the money supply m. This low correlation coefficient may be explained by the net effect of the money supply m. First, the money supply rule implies a positive relationship between the money supply m and the interest rate i, as Ascari and Ropele (2013) suggest. Second, and in contrast, the interest rate i is the price of the money supply m from the supply and demand perspective. Hence when the money supply m increases, the interest rate i decreases (Carr and Smith, 1972). Therefore, the net effect of the money supply m on the interest rate i may be moderate. Noticeably, the deviation $Log(y/\bar{y})$ in the real GDP has a high correlation coefficient with the deviation $\bar{u}-u$ in the unemployment rate at 0.8847.

4.2. Analysis

This analysis investigates the statistical linear relationship between the dependent variable interest rate i and five independent variables as in Equation (4),

$$i = \beta_0 + \beta_1(\pi - \pi^*) + \beta_2 Log\left(\frac{y}{\bar{y}}\right) + \beta_3 Log\left(\frac{m}{m_\tau}\right) + \beta_4 v + \beta_5(\bar{u} - u), \quad (4)$$

where β_0 is the constant intercept term, $\pi - \pi^*$ is the deviation in the inflation rate, $Log(y/\bar{y})$ is the deviation in the real GDP, $Log(m/m_\tau)$ is the deviation in the money supply, v is the money velocity, and $\bar{u}-u$ refers to the deviation in the unemployment rate. The article enriches the regression analysis by removing the most insignificant independent variables. The significance level 1% is applied. Table 3 shows the results. The null hypothesis is the regression coefficient $\beta_i = 0$ for $i = 0,1,\ldots 5$, which implies no significant statistical relationship between the dependent variable and the independent variables.

Table 3. Regression results for the interest rate

	(1)	(2)	(3)	(4)	(5)
$\pi - \pi^*$	0.9561*** (0.0414)	0.8690*** (0.0384)	0.9389*** (0.0421)	0.8362*** (0.0399)	0.8565*** (0.0391)
$Log\left(\frac{y}{\bar{y}}\right)$	0.3830*** (0.103)	0.1534 (0.0973)	-0.1969 (0.1900)	-0.6188*** (0.1870)	
$Log\left(\frac{m}{m_\tau}\right)$		0.2065*** (0.0467)		0.2486*** (0.0504)	0.2170*** (0.0455)
v		0.0594*** (0.0042)		0.0621*** (0.0044)	0.0588*** (0.0040)
$\bar{u} - u$			0.4081*** (0.1070)	0.5389*** (0.1040)	0.1997*** (0.0572)
Intercept	0.0327*** (0.0009)	-0.0847*** (0.0088)	0.0328*** (0.0009)	-0.0919*** (0.0092)	-0.0837*** (0.0081)
N	735	735	735	735	735
Adj. *R*-squared	0.5383	0.6217	0.5450	0.6334	0.6277

Notes: The numbers are the regression coefficients. Standard errors are in parentheses. *, **, and *** denote significance levels at 10%, 5%, and 1%, respectively. The dependent variable is the interest rate i in regressions (1)-(5). Adj. *R*-squared expresses the adjusted *R*-squared, which shows the percentage of variation explained by the independent variables that affect the dependent variable.

Regression (1) represents the result when the interest rate i is the dependent variable, and $\pi - \pi^*$ and $Log(y/\bar{y})$ are two independent variables. This regression resembles the Taylor (1993) rule. Notably, the regression coefficients for $\pi - \pi^*$ and $Log(y/\bar{y})$ are positive and statistically significant at the 1% significance level.

Regression (2) represents the result when the interest rate i is the dependent variable, and $\pi - \pi^*$, $Log(y/\bar{y})$, $Log(m/m_\tau)$ and v are four independent variables. Since the deviation

$Log(m/m_\tau)$ in the money supply and the money velocity v are added to the Taylor (1993) rule, Regression (2) represents the combination of the Taylor (1993) rule and the Quantity Equation (Friedman, 1970). Again, the regression coefficients for the four independent variables are positive, where $\pi - \pi^*$, $Log(m/m_\tau)$ and v are significant at the 1% significance level. $Log(y/\bar{y})$ is nevertheless insignificant.

Regression (3) represents the result when the interest rate i is the dependent variable, and $\pi - \pi^*$, $Log(y/\bar{y})$, and $\bar{u} - u$ are three independent variables. Since the deviation $\bar{u} - u$ in the unemployment rate is added to the Taylor (1993) rule, Regression (3) represents the combination of the Taylor (1993) rule and the Phillips (1958) curve. As in regression (2), $Log(y/\bar{y})$ is insignificant with a p-value above 10%. The coefficient sign in Regression (3) is negative for $Log(y/\bar{y})$, in contrast to positive coefficients in Regressions (1) and (2). The other two independent variables, $\pi - \pi^*$ and $\bar{u} - u$, are positive and significant at the 1% level.

Regression (4) represents the result when the interest rate i is the dependent variable, and $\pi - \pi^*$, $Log(y/\bar{y})$, $Log(m/m_\tau)$, v, and $\bar{u} - u$ are five independent variables. Regression (2) incorporates the deviation $Log(m/m_\tau)$ in the money supply, the money velocity v, and the deviation $\bar{u} - u$ in the unemployment rate. It represents the combination of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. All five independent variables are statistically significant at the 1% significance level. The coefficient for $Log(y/\bar{y})$ is significant and negative at the 1% significance level.

We further test the potential problem of multicollinearity among the variables in Regression (4). The estimation of the VIF (variance inflation factor) for $Log(y/\bar{y})$, $\bar{u} - u$, $Log(m/m_\tau)$, v, and $\pi - \pi^*$ gives 4.91, 4.89, 1.61, 1.58, and 1.19, respectively, with an average of 2.83. The VIFs for Regression (3) with respect to $\bar{u} - u$, $Log(y/\bar{y})$, and $\pi - \pi^*$ are 4.71, 4.65, and 1.03, respectively. The VIF estimates the severity of the multicollinearity problem in a regression analysis with the ordinary least squares estimation method. Generally, a VIF above 10 expresses a high degree of multicollinearity. A more conservative opinion is that a VIF above 2.5 indicates multicollinearity. According to Table 2, a high correlation coefficient of 0.8847 exists between $Log(y/\bar{y})$ and $\bar{u} - u$. The coefficient sign for $Log(y/\bar{y})$ changes from positive to negative from Regression (2), which contains $Log(y/\bar{y})$ but not $\bar{u} - u$, to Regressions (3) and (4), which contain both $Log(y/\bar{y})$ and $\bar{u} - u$. This suggests a multicollinearity issue in Regressions (3) and (4). Therefore, among $Log(y/\bar{y})$ and $\bar{u} - u$, we remove the independent variable with the highest VIF ($Log(y/\bar{y})$) in Regression (4) and run the regression again. The result is Regression (5), where the VIFs for $Log(m/m_\tau)$, $v, \pi - \pi^*$, and $\bar{u} - u$ are 1.54, 1.51, 1.15, and 1.05, respectively, with an average of 1.31. Findings suggest no multicollinearity concern in Regression (5).

4.3. Discussion and limitations

The regression analysis results in Table 3 suggest a positive impact of $\pi - \pi^*$, $Log(y/\bar{y})$, $Log(m/m_\tau)$, v and $\bar{u} - u$ on the interest rate i. In this article, we begin with the regression analysis illustrating the Taylor (1993) rule, then combine the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. The multicollinearity issue is tested and addressed. Finally, Regression (5) presents a statistically significant result. Based on Regression (5), the coefficient for $\pi - \pi^*$ is statistically significant and positive at 0.8565, which indicates that the deviation $\pi - \pi^*$ in the inflation rate is essential for the interest rate i. The coefficient for $Log(y/\bar{y})$ is also found to be significant and positive at 0.3830 in Regression (1). However, $Log(y/\bar{y})$ is removed in Regression (5) due to multicollinearity, in contrast with Regressions (3) and (4). The result supports the Taylor (1993) rule, which confines attention to $\pi - \pi^*$ and $Log(y/\bar{y})$. The combination of the Taylor (1993) rule and the Quantity Equation (Friedman, 1970) explains interest rate i better, since the adjusted *R*-squared increases from 0.5383 in Regression (1) to 0.6217 in Regression (2). The second highest coefficient in Regression (5) is $Log(m/m_\tau)$ at 0.2170. This finding suggests that the deviation $Log(m/m_\tau)$ in the money supply is an important indicator for the interest rate i. The coefficients for $\bar{u} - u$ and v in Regression (5) are positively significant under the 1% level at 0.1997 and 0.05878, respectively. Hence the best combination of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve is reported in Regression (5), which explains the interest rate i in a superior manner with the adjusted R-squared at 0.6277.

During the study period, the Federal Reserve adopts different operating procedures with respect to the federal funds rate, for example, free-reserves targeting, federal-funds-rate targeting, and non-borrowed reserves targeting, which implies different distributions for the federal funds rate. More recently, between late 2008 and late 2018 and again after March 2020, the Federal Reserve paid interest on both required and excess reserves at a rate at the top of its target range for the federal funds rate. The consequence is the virtual elimination of lending in the federal funds market by private banks (Afonso and Jalles, 2013; Bech and Klee, 2011) and a gradual drying up of that market (Dutkowsky and VanHoose, 2017) except for some borrowing of excess reserves from government-sponsored institutions like the Federal Home Loan Banks, the Federal National Mortgage Association, and the Federal Home Loan Mortgage Corporation by private banks that then held the funds on reserve at the Fed at the higher interest rate on reserves. One limitation is that the article mainly applies the central bank interest rate and does not account for what the central bank actually does. This article investigates the interest rate by extending the Taylor (1993) rule. The prediction is a recommendation or a reference for the central bank. This article finds an interest rule that explains the empirical interest rates better than the Taylor (1993) rule.

Another limitation is that the analysis has not explored the underlying mechanisms and the interactions between the five independent variables. Other potential limitations are the linear relationship assumption implicit in regression analysis and whether the independent variables are independent of each other.

5. Discussion

This article combines the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve to explore the variables which may influence the interest rate. Correlation and regression analyses are adopted to show how these variables interact with the interest rate. The article uses empirical data for the United States. The Pearson correlation coefficients suggest that the deviation in the inflation rate, the deviation in the real GDP, the deviation in the money supply, the money velocity, and the deviation in the unemployment rate are positively correlated with the interest rate. The highest Pearson correlation with the interest rate occurs for the deviation in the inflation rate, followed by the money velocity, the deviation in the unemployment rate, the deviation in the real GDP, and the deviation in the money supply. This ranking from high to low of the correlation coefficients between the interest rate and the five independent variables illustrates the focus variables that interact with the interest rate.

Regression analysis specifies that the deviation in the inflation rate and the deviation in the real GDP are statistically positive and interact with the interest rate. This finding is consistent with the Taylor (1993) rule. Second, the interest rate increases with the money supply and the money velocity. This connection is illustrated by combing the Taylor (1993) rule and the Quantity Equation (Friedman, 1970). Third, multicollinearity is present between the deviation in the real GDP and the deviation in the unemployment rate. Thus, the deviation in the real GDP is removed. Fourth, the interest rate also increases with the deviation in the unemployment rate, which is in line with the Phillips (1958) curve. Final regression suggests that the deviation in the inflation rate, the deviation in the money supply, the money velocity, and the deviation in the unemployment rate are good interest rate indicators. The Pearson correlation and regression analysis contribute to understanding how the five independent variables impact the interest rate. The findings are relevant to how central banks choose interest rate policies.

Future research may explore more comprehensibly potential indirect impact paths for how the five independent variables impact each other and the interest rate and include more variables. Some variables may be operationalized differently, e.g., the potential real GDP, the real equilibrium interest rate, and the natural unemployment rate. Variation and uncertainty in the variables may be accounted for, while a systematic comparison of the data for more countries and different periods is another future research direction.

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Appendix

Table A1. Nomenclature

i	Interest rate, $i \in \mathbb{R}$
π	Inflation rate, $\pi \in \mathbb{R}$
π^*	Target inflation rate, $\pi^* \in \mathbb{R}$
r^*	Equilibrium real interest rate, $r^* \in \mathbb{R}$
a_π	Constant
a_y	Constant
Log	The logarithm with base ten
y	Real GDP (Gross Domestic Product), $y \geq 0$
$\bar{y}$	Real potential GDP, $\bar{y} \geq 0$
m	Money supply, $m > 0$
m_τ	Money supply at some earlier point in time, $m_\tau > 0$
v	Money velocity, $v \geq 0$
$t \geq 0$	Volume of transactions
u	Unemployment rate, $u \geq 0$
$\bar{u}$	Natural unemployment rate, $\bar{u} \geq 0$
gW	Growth rate of money wages, $gW \geq 0$
gW^T	Growth trend rate of money wages, $gW^T \geq 0$

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Guizhou Wang [*]**, *Kjell Hausken*** [**]

[*] *Faculty of Science and Technology, University of Stavanger, Norway*

E-mail: guizhou.wang@uis.no

[**] *Faculty of Science and Technology, University of Stavanger, Norway*

E-mail: kjell.hausken@uis.no

Modeling which Factors Impact Interest Rates

Abstract: The Taylor (1993) rule for determining interest rates is generalized to account for three additional variables: The money supply, money velocity, and the unemployment rate. Thus, five parameters, i.e. weights assigned to the deviation in the inflation rate, the deviation in real GDP (Gross Domestic Product), the deviation in money supply, the deviation in the money velocity, and the deviation in unemployment rate, are introduced and estimated. The article explores and tests various combinations of the Taylor rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. The monthly US January 1, 1959 to March 31, 2022 data are adopted to test the optimal parameter values. Estimating the parameters with the least squares method gives better results than the Taylor rule. The optimal parameter values involve a relatively high weight to the deviation in unemployment rate, and moderate weights are assigned to the deviation in the inflation rate, the deviation in real GDP, the deviation in money supply, and the deviation in the money velocity. The corresponding sum of squares decreases by 42.95% when compared with the Taylor rule.

Keywords: Monetary policy, Taylor rule, Quantity Equation, Phillips curve, interest rates, inflation rate, GDP, money supply, money velocity, unemployment rate.

JEL classification: C6, E24, E50, E47, E52, E58.

1. Introduction

1.1. Background

Central banks are traditionally mandated to achieve certain objectives such as economic growth, low unemployment, price stability, stability of financial markets, etc. The Taylor rule (1993) accounts for some objectives. It predicts interest rates based on five variables: the equilibrium real interest rate, inflation rate, target inflation rate, real GDP (Gross Domestic Product), and the potential real GDP which can be sustained over the long term. Central banks often apply monetary policies including setting interest rates to manage the macroeconomy. Taylor's analysis (1993) has substantial impact on how the interest rate is determined. According to the Taylor rule, the interest rate is adjusted in response to the deviation in GDP and the deviation in the inflation rate. Taylor believes that his rule is a good tool to interpret historical monetary policy. This article questions that belief.

The Taylor rule relies on the deviation in real GDP and the deviation in the inflation rate to obtain the recommended central bank interest rate. It does not account for other variables which may be relevant for the conduct of monetary policy in economic and financial systems, such as money supply, money velocity, unemployment rate, financial market conditions, etc. Thus, the Taylor rule fails to reflect the state of the economy in real time. Another challenge is to precisely estimate the real potential GDP. In addition, the Taylor rule is a backward looking approach. This is also a critique of the current article since it ignores that central banks may be forward looking in setting the interest rates.

The Taylor rule is a well-known technique for central banks to set interest rates. The rule recommends that central banks increase the interest rate when the inflation rate is higher than the target inflation rate and the real GDP is higher than the real potential GDP. It gives equal 0.5 weight to the gap in real GDP and the gap in the inflation rate. It faces criticism because too few variables are incorporated. Other known variables such as money supply, money velocity and unemployment rate, captured by the Quantity Equation (Friedman, 1970) and the Phillips (1958) curve, respectively, may additionally impact the interest rates. Specifically, a lower unemployment rate is one essential objective for central banks. Hence, it is interesting to incorporate these variables into the Taylor rule and explore the associated weights. Other unknown factors not considered in this article, such as economic crisis, fiscal deficit, global interest rates, etc. may also impact the interest rates.

How a central bank determines its interest rate is of particular interest in times of economic turmoil, common through history and, for example, during and in the aftermath of the 2020-2021 pandemic crisis when many countries first decreased and thereafter increased the interest rate to suppress high inflation above the target inflation rate. Changes in money supply impact economies substantially. Central banks commonly adjust the money supply through open market operations. That is, a central bank may increase the money supply by buying government bonds, either from commercial banks or other actors, or new bonds created by the government. The money velocity may also impact monetary policy. For example, a decline in the money velocity may offset an increase in the money supply. The Quantity Equation (Friedman, 1970) shows the relationship between the money supply and the money velocity. Two important objectives of central banks are low unemployment rate, and low inflation commonly preferred at 2%. However, the Taylor rule does not include the money supply, the money velocity and the unemployment rate.

1.2. Contribution

The article generalizes the Taylor rule by introducing the money supply and the money velocity as presented in the Quantity Equation (Friedman, 1970), and the unemployment rate as presented in the Phillips (1958) curve. The monthly US January 1, 1959 to March 31, 2022 data is adopted for empirical analysis. The least squares method is applied to estimate the optimal weights.

In his article, Taylor (1993, p. 202) points out that "this policy rule has the same coefficient on the deviation in the real GDP from trend and the inflation rate." Inspired by this, this article tests different weights assigned to the deviation in real GDP, the deviation in the inflation rate, and three additional variables. The research questions are: How can the Taylor (1993) rule be improved to better account for the money supply, the money velocity and the unemployment rate? What are the optimal weights assigned to the deviations in inflation, real GDP, money supply, money velocity, and unemployment rate?

The theoretical contribution of this research is as follows: First, the article expands the Taylor rule by introducing additional variables, i.e. money supply, money velocity and unemployment rate. Second, the article explores various weights assigned to the deviations in inflation, real GDP, money supply, money velocity, and unemployment rate. Third, the article shows that incorporating the money supply, money velocity and the unemployment rate is more accurate than the Taylor rule. The article provides a better framework for central banks to determine interest rates.

1.3. Literature

The Taylor rule has received substantial interest, with theoretical assessments and empirical testing, earning 12,681 citations in Google Scholar. Taylor (1993) assumes the same 0.5 weight to the deviation in real GDP and the deviation in the inflation rate. These parameter values fit the actual path during the 1987-1992 period very well. Judd and Rudebusch (1998) explore the Federal Reserve's response function to economic development. They point out that the Taylor rule framework helps to summarize the key elements of monetary policy. In his following research, Taylor (1999) updates the weights for the deviation in real GDP and the deviation in the inflation rate at 1 and 0.5, respectively. The reason is that the monetary policy rules have changed considerably over the different periods.

The Quantity Equation (Friedman, 1970) presents an analytical framework to explore the relationship between the money supply, money velocity, price level, and the real GDP. Although the money supply is widely assumed to impact interest rates, it is absent in the Taylor rule, perhaps because it is assumed to impact inflation and consequently may impact interest rates indirectly. The money supply plays an important role in monetary policy. The McCallum (1988) rule is an alternative to the Taylor rule. It recommends a target money supply M0 for the central banks. The McCallum rule is closely related to the Quantity Equation (Friedman, 1970), and recommends the central bank to set the target money supply M0 based on five variables: These are the money supply M0 in the previous period, the average quarterly increase of the money velocity of M0, the desired inflation rate, the long-run average quarterly increase of real GDP, and the quarterly increase of nominal GDP. The McCallum rule performs better than the Taylor rule during crisis periods (Benchimol & Fourçans, 2012). Krušković (2022) investigates the role of central banks in maintaining price stability and achieving their inflation targets through various policy instruments, e.g. interest rate changes, foreign exchange interventions, and asset purchases.

The unemployment rate is also absent in the Taylor rule, but Prag (1994) finds a linkage from the unemployment rate to interest rates. Phillips (1958) also omits analyzing interest rates. Instead he analyzes the relationship between the unemployment rate and inflation. Azam, Khan, and Khan (2022) investigate the validity of the Phillips (1958) curve for eight countries in the Middle East and North Africa region. They find a negative but insignificant trade-off between the inflation and unemployment rates in the short run. Gocer and Ongan (2020) examine the relationship between the inflation and interest rates in the United Kingdom using a nonlinear Autoregressive Distributed Lag model. They show that the nominal interest rate reacts more strongly to increases in inflation than to decreases in inflation. Wang and Hausken (2022b, 2022c) combine the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve, applying different tools and generating results different from the current article.

The literature more commonly compares how interest rate rules compare with money supply rules (Ascari & Ropele, 2013; Auray & Fève, 2003; Minford, Perugini, & Srinivasan, 2003), with solvency rules (Brancaccio & Fontana, 2013), and with the Friedman rule (Srinivasan, 2000). The literature also links the money supply to interest rate targets (Schabert, 2005, 2009) or to exchange rates (Tervala, 2012). The literature furthermore links monetary rules to macroeconomics more generally (Clarida, Gali, & Gertler, 2000), or applies the Taylor rule to build decision models for central bank digital currency (Wang & Hausken, 2022d).

Modified monetary rules appear after the Taylor rule. For example, Orphanides (2003) proposes a first difference rule, relating the current interest rate to its historical value and a year ahead forecast. As an alternative, Bullard (2017) and Kliesen (2019) adjust the Taylor rule, and propose an inertial rule. The rule prescribes a response of the interest rate to the economic developments over time.

1.4. Article organization

Section 2 presents the model. Section 3 analyzes the model with data sources, parameter estimation, and illustrations. Section 4 discusses the results. Section 5 presents limitations and future research. Section 6 provides policy implications. Section 7 concludes.

2. The model

Appendix A shows the nomenclature. This article tests and generalizes the well-known Taylor (1993) rule by incorporating the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. Thus, we include three additional terms: money supply $m_t, m_t > 0$, money velocity v_t, $v_t > 0$, and the unemployment rate $u_t, u_t \geq 0$, at time t, $t \geq 0$, i.e.

$$i_t = \pi_t + r_t^* \pm a_\pi(\pi_t - \pi_t^*) \pm a_y Log\left(\frac{y_t}{\bar{y}_t}\right) \pm a_m Log\left(\frac{m_t}{\bar{m}_t}\right) \pm a_v Log\left(\frac{v_t}{\bar{v}_t}\right)$$
$$\pm a_u(\bar{u}_t - u_t), a_\pi + a_y + a_m + a_v + a_u = 1 \quad (1)$$

where i_t, $i_t \in \mathbb{R}$ is the interest rate at time t, $\mathbb{R}$ is the set of all real numbers. The right hand side of (1) contains $\pi_t + r_t^*$, as in the Taylor rule, where π_t, $\pi_t \in \mathbb{R}$, is the inflation rate and r_t^*, $r_t^* \in \mathbb{R}$, is the equilibrium real interest rate. The subsequent five terms in (1) are preceded with ± where + is the plausible default positive impact on the interest rate i_t, and – is the alternative negative impact on i_t analyzed in section 3. These five terms are expressed as follows: The deviation π_t - π_t^* in inflation rate, where π_t^*, $\pi_t^* \in \mathbb{R}$, is the target inflation rate. The deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP, where y_t, $y_t \geq 0$, is the real GDP, and $\bar{y}_t$, $\bar{y}_t \geq 0$, is the potential real GDP that can be sustained over the long term. The deviation $Log\left(\frac{m_t}{\bar{m}_t}\right)$ in money supply, where m_t, $m_t \geq 0$, is the money supply, and $\bar{m}_t$, $\bar{m}_t \geq 0$, is the potential money supply. The deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity, where v_t, $v_t \geq 0$, is the money velocity, and $\bar{v}_t$, $\bar{v}_t \geq 0$, is the potential money velocity. The deviation $\bar{u}_t - u_t$ in the unemployment rate, where $\bar{u}_t$, $\bar{u}_t \geq 0$ is the natural unemployment rate, and u_t, $u_t \geq 0$ is the unemployment rate. The five nonnegative parameters a_π, a_y, a_m, a_v, a_u are the weights assigned to the deviations in inflation π_t, real GDP y_t, money supply m_t, money velocity v_t, and unemployment rate u_t, respectively. *Log* is the logarithm with a base ten. The sum of the five parameters is assumed to be one, corresponding to Taylor (1993) assuming that $a_\pi + a_y = 0.5 + 0.5 = 1$ when considering only the first two of the five terms.

The deviation π_t - π_t^* in the inflation rate and the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP are the two terms originally included in the Taylor (1993) rule. For the new term, the deviation $Log\left(\frac{m_t}{\bar{m}_t}\right)$ in money supply in (1), the new variable money supply m_t, $m_t \geq 0$ is introduced, as present in the Quantity Equation (Friedman, 1970). The potential money supply $\bar{m}_t$ is estimated using the standard HP filter (Hodrick & Prescott, 1997), which is commonly used in economics to estimate potential real GDP (Michałek, 2010). Regarding the impact of the money supply m_t on the interest rate i_t, on the one hand, Ascari and Ropele (2013) suggest that an increase of money supply m_t will cause the interest rate i_t to increase. Thus, when the money supply m_t increases, central banks may increase the interest rate i_t to prevent savers' extensive withdrawals. On the other hand, the interest rate i_t is the price of the money supply m_t from the supply and demand perspective. Accordingly, C. A. Conrad (2021) suggest that the interest rate i_t decreases when the money supply m_t increases. This article explores both suggestions. The plus sign in (1) assumes a positive relationship between the interest rate i_t and the deviation

$Log\left(\frac{m_t}{\bar{m}_t}\right)$ in the money supply, while the minus sign assumes a negative relationship.

For the new term the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity in (1), the new variable money velocity v_t, $v_t \geq 0$, is introduced. This term is also captured by the Quantity Equation (Friedman, 1970). The Keynesian theory of money demand (Keynes, Moggridge, & Johnson, 1971) suggests that the money velocity v_t needs to increase when the money supply m_t decreases, to keep the balance within the monetary market. Mendizabal (2006) suggests the money velocity v_t has a positive impact on the inflation rate π_t. Taylor (1993) suggests that the inflation rate π_t impacts the interest rate i_t positively. Therefore, we assume a positive relationship between the money velocity v_t and the interest rate i_t. Money velocity v_t is defined as the ratio of nominal GDP to the money supply stock (Federal Reserve Bank of St. Louis, 2022). Similarly, we define the potential money velocity $\bar{v}_t$, $\bar{v}_t \geq 0$ as the ratio of nominal potential GDP to the potential money supply. Thus, in (1) the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in the money velocity is presented on the same structure as the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP.

The new variable unemployment rate u_t is introduced for the new term the deviation $\bar{u}_t - u_t$ in the unemployment rate in (1). A low unemployment rate u_t is one of the most important objectives of a central bank. Thus, central banks may take into account the unemployment rate u_t when setting the interest rate i_t. Phillips (1958) originally investigates the relationship between the unemployment rate u_t and wage growth, Thereafter, Samuelson and Solow (1960) connect the employment rate with the inflation rate. The Phillips (1958) curve illustrates an inverse relationship between the unemployment rate u_t and the inflation rate π_t in the short term. Specifically, the Phillips (1958) curve is divided into a short run Phillips (1958) curve and a long run Phillips (1958) curve (Granger & Jeon, 2011). The unemployment rate u_t and the inflation rate π_t are inversely related in the short run. This relationship breaks down in the long run (Russell & Banerjee, 2008). Since Taylor (1993) assumes a positive correlation between the inflation rate π_t and the interest rate i_t, an inverse relationship is assumed between the interest rate i_t and the unemployment rate u_t, as also suggested by Prag (1994). The deviation $\bar{u}_t - u_t$ in the unemployment rate indicates an inverse relationship between the interest rate i_t and the unemployment rate u_t. Finally, for generality, the article also tests the plus versus minus signs for the five terms, i.e. the deviation $\pi_t - \pi_t^*$ in the inflation rate, the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP, the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity, and the deviation $\bar{u}_t - u_t$ in the unemployment rate.

3. Analyzing the model

3.1. Data sources

Monthly US January 1, 1959 to March 31, 2022 data is collected and compiled from the following sources: The real GDP y_t is estimated from the U.S. Bureau of Economic Analysis (2022). The real potential GDP $\bar{y}_t$ is derived from the U.S. Congressional Budget Office (2022b). The quadratic interpolation method is applied to convert quarterly data to monthly data for the real GDP y_t and the real potential GDP $\bar{y}_t$. The M2 money supply m_t is estimated from the Board of Governors of the Federal Reserve System (US) (2022b). The money velocity v_t is estimated from the Federal Reserve Bank of St. Louis (2022). The unemployment rate u_t is derived from the U.S. Bureau of Labor Statistics (2022b). The natural unemployment rate $\bar{u}_t$ is estimated from the U.S. Congressional Budget Office (2022a). The quadratic interpolation method is used to convert quarterly data to monthly data for $\bar{u}_t$. The inflation rate π_t is derived from the U.S. Bureau of Labor Statistics (2022a). The target inflation rate $\pi_t^* = 1.5\%$ is estimated from Shapiro and Wilson (2019) from January 1, 2000 to December 30, 2007. The common $\pi_t^* = 2\%$ is assumed for the remaining January 1, 1959 to March 31, 2022 period, as Taylor (1993) assumes for January 1, 1984 to September 31, 1992. The common equilibrium real interest rate $r_t^* = 2\%$ is assumed throughout January 1, 1959 to March 31, 2022, used also by Taylor (1993) for January 1, 1984 to September 31, 1992, and consistently with Kiley's (2020) estimation and the long run inflation target specified by the Federal Open Market Committee (The Federal Reserve, 2022). The empirical interest rate i_t is derived from the Board of Governors of the Federal Reserve System (US) (2022a).

3.2. Estimating the parameters and illustrating the solution

Table 1 shows the estimations of the five paramter values a_π, a_y, a_m, a_v, a_u with different combinations of parameter values in (1), obtained using Mathematica 13.1 (https://www.wolfram.com).

Table 1. Curve number, estimated parameter values a_π, a_y, a_m, a_v, a_u, parameter specifics, the number N of free choice variables, and the sum S of the squared differences between the empirical interest rate i_t and the theoretical interest rate i_t in (1). A superscript star * after a number means that the corresponding sign in (1) is changed from plus to minus.

Curve	a_π, a_y, a_m, a_v, a_u	Parameter specifics	*N*	*S*
1	0.5, 0.5, 0, 0, 0	Taylor (1993) rule	0	0.830774
2	0.2, 0.2, 0.2, 0.2, 0.2	Equal weight	0	0.582477
3a	0.2, 0.2, 0.2, 0, 0.4	$a_\pi = a_y = a_m = 0.2$, optimizing a_u when $a_v = 0.4 - a_u$	1	0.577883
3b	0.2, 0.2, 0.2, 0.04*, 0.36	$a_\pi = a_y = a_m = 0.2$, optimizing a_u when $a_v = 0.4 - a_u$	1	0.576750
4a	0.2, 0.2, 0.25, 0, 0.35	$a_\pi = a_y = 0.2$, optimizing a_v and a_u when $a_m = 0.6 - a_v - a_u$	2	0.577109
4b	0.2, 0.2, 0, 0.17*, 0.43	$a_\pi = a_y = 0.2$, optimizing a_v and a_u when $a_m = 0.6 - a_v - a_u$	2	0.576230
4c	0.2, 0.2, 0.03*, 0.17*, 0.4	$a_\pi = a_y = 0.2$, optimizing a_v and a_u when $a_m = 0.6 - a_v - a_u$	2	0.576582
5a	0, 0, 0.37, 0.37, 0.26	$a_\pi = a_y$, $a_m = a_v$, optimizing a_y and a_v when $a_u = 1 - 2a_y - 2a_v$	2	0.499951
5b	0.16*, 0.16*, 0.13, 0.13, 0.4	$a_\pi = a_y$, $a_m = a_v$, optimizing a_y and a_v when $a_u = 1 - 2a_y - 2a_v$	2	0.474088
6a	0, 0, 0.47, 0.18, 0.35	$a_\pi = a_y$, optimizing a_m, a_v and a_u when $a_\pi = a_y = (1 - a_m - a_v - a_u) / 2$	3	0.496629
6b	0.165*, 0.165*, 0.11, 0.13, 0.43	$a_\pi = a_y$, optimizing a_m, a_v and a_u when $a_\pi = a_y = (1 - a_m - a_v - a_u) / 2$	3	0.474051
7a	0.2, 0.41, 0.17, 0, 0.22	$a_\pi = 0.2$, optimizing a_m, a_v and a_u when $a_\pi = 0.8 - a_m - a_v - a_u$	3	0.576203
7b	0.2, 0.42, 0, 0.12*, 0.26	$a_\pi = 0.2$, optimizing a_m, a_v and a_u when $a_\pi = 0.8 - a_m - a_v - a_u$	3	0.574744
7c	0.2, 0.61, 0.06*, 0, 0.13	$a_\pi = 0.2$, optimizing a_m, a_v and a_u when $a_\pi = 0.8 - a_m - a_v - a_u$	3	0.578171
7d	0.2, 0.42, 0*, 0.12*, 0.26	$a_\pi = 0.2$, optimizing a_m, a_v and a_u when $a_\pi = 0.8 - a_m - a_v - a_u$	3	0.574744
8a	0, 0.09, 0.44, 0.15, 0.32	optimizing a_y, a_m, a_v and a_u when $a_\pi = 1 - a_y - a_m - a_v - a_u$	4	0.496512
8b	0.16*, 0, 0.32, 0.21, 0.31	optimizing a_y, a_m, a_v and a_u when $a_\pi = 1 - a_y - a_m - a_v - a_u$	4	0.474937
8c	0.17*, 0.13*, 0.15, 0.15, 0.4	optimizing a_y, a_m, a_v and a_u when $a_\pi = 1 - a_y - a_m - a_v - a_u$	4	**0.473981**
9a	0, 0.31, 0.4, 0.29, 0	$a_u = 0$, Taylor (1993) rule and Quantity Equation (Friedman, 1970), optimizing a_y, a_m and a_v when $a_u = 0$, and $a_\pi = 1 - a_y - a_m - a_v$	3	0.502298
9b	0.16*, 0.32, 0.24, 0.28, 0	$a_u = 0$, Taylor (1993) rule and Quantity Equation (Friedman, 1970), optimizing a_y, a_m and a_v when $a_u = 0$, and $a_\pi = 1 - a_y - a_m - a_v$	3	0.481483

10a	0, 0.98, 0, 0, 0.02	$a_m = a_v = 0$, Taylor (1993) rule and Phillips (1958) curve, optimizing a_y, a_u when $a_m = a_v = 0$, $a_\pi = 1 - a_y - a_u$	2	0.512049
10b	0.17*, 0.66, 0, 0, 0.17	$a_m = a_v = 0$, Taylor (1993) rule and Phillips (1958) curve, optimizing a_y, a_u when $a_m = a_v = 0$, $a_\pi = 1 - a_y - a_u$	2	0.488556
11	0, 0, 0.47, 0.18, 0.35	$a_\pi = a_y = 0$, Quantity Equation (Friedman, 1970) and Phillips (1958) curve, optimizing a_v, and a_u when $a_\pi = a_y = 0$, $a_m = 1 - a_v - a_u$	2	0.496629
12a	0, 0, 0.46, 0.1, 0.44	$a_y = 0$, optimizing a_m, a_v, and a_u when $a_y = 0$, $a_\pi = 1 - a_m - a_v - a_u$	3	0.497407
12b	0.16*, 0, 0.32, 0.21, 0.31	$a_y = 0$, optimizing a_m, a_v, and a_u when $a_y = 0$, $a_\pi = 1 - a_m - a_v - a_u$	3	0.474937
13	0, 0, 0.51, 0.49, 0	$a_\pi = a_y = a_u = 0$, Quantity Equation (Friedman, 1970), optimizing a_v when $a_m = 1 - a_v$	1	0.509102
14	0, 0, 0, 0, 1	$a_\pi = a_y = a_m = a_v = 0$, $a_u = 1$, Phillips (1958) curve	0	0.571153
Average	N/A	The average of the above 27 curves	0	0.510363

Curve 1 represents the Taylor (1993) rule assuming $a_\pi = a_y = 0.5$, $a_m = a_v = a_u = 0$. The sum of squares is relatively high at $S = 0.830774$. Curve 2 assumes equal 0.2 weight for the five parameters. The sum of the squared differences is lower at $S = 0.582477$, i.e. a 29.96% decrease compared with the Taylor (1993) rule in curve 1. Hence equal weights for the five parameters explain the interest rate i_t better than the Taylor (1993) rule. Curves 3a and 3b assume one free choice variable, where a_u is optimized assuming $a_v = 0.4 - a_u$. That causes an even lower sum of squared differences $S = 0.577883$, but with the optimal parameter $a_v = 0$. That suggests that the corresponding sign in (1) may be negative. A negative sign before $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in (1) causes the optimal parameters $a_v = 0.04$ and $a_u = 0.36$, and a marginally lower sum of squared differences $S = 0.576750$. Curves 3a and 3b suggest that the weight a_u assigned to unemployment, not present in the Taylor (1993) rule, may potentially be relatively high, which becomes clearer as we proceed. Curves 4a, 4b, and 4c assume two free choice variables, where a_v and a_u are optimized assuming $a_m = 0.6 - a_v - a_u$. That causes a similar sum of squared differences $S = 0.577109$ in curve 4a. Again, the optimal parameter is $a_v = 0$. Hence, curve 4b tests the negative sign for $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in (1). That causes a slightly lower sum of squares $S = 0.576230$ compared with curve 4a, but with the optimal parameter $a_m = 0$. Assuming negative signs for $Log\left(\frac{m_t}{\bar{m}_t}\right)$ and $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in (1) cause the optimal parameters $a_m = 0.03$ and $a_v = 0.17$ in curve 4c, and a similar sum of squared differences $S = 0.576582$. Curves 4a, 4b, and 4c also suggest that the weight a_u may be relatively high. Curves 5a and 5b assume two free choice variables, where a_y and a_v are optimized assuming $a_\pi = a_y$, $a_m = a_v$, and $a_u = 1 - 2a_y - 2a_v$. That causes a lower sum of squared differences $S = 0.499951$ in curve 5a, but interestingly with the two optimal parameters $a_\pi = a_y = 0$. Assuming negative signs before $(\pi_t - \pi_t^*)$ and $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in (1) yield an even lower sum of squared differences $S = 0.474088$ compared with curve 5a. Curves 6a and 6b assume three free choice variables, where a_m, a_v and u are optimized assuming $a_\pi = a_y = (1 - a_m - a_v - a_u)/2$. That causes a similar sum of squared differences S = 0.496629, but also with the optimal parameters $a_\pi = a_y = 0$. Hence, curve 6b assumes negative signs before $(\pi_t - \pi_t^*)$ and $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in (1). That causes a lower sum of squared differences $S = 0.474051$ compared with curve 6a. Curves 5a, 5b, 6a and 6b suggest negative signs before $(\pi_t - \pi_t^*)$ and $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in (1). Curves 7a, 7b, 7c, and 7d also assume three free choice variables, where a_m, a_v and u are optimized assuming $a_\pi = 0.2$. That causes the sum of squared differences S = 0.576203, 0.574744, 0.578171 and 0.574744, respectively, which are higher compared with curves 5a, 5b, 6a and 6b. The higher sum of squares in curves 7a, 7b, 7c and 7d suggests that the weight a_y assigned to $(\pi_t - \pi_t^*)$ should be lower than 0.2.

Curves 8a, 8b and 8c assume four free choice variables, where a_y, a_m, a_v and a_u are optimized assuming $a_\pi = 1 - a_y - a_m - a_v - a_u$. That causes the sum of squared differences $S = 0.496512$ in curve 8a, but with optimal parameter $a_\pi = 0$. A negative sign before $(\pi_t - \pi_t^*)$ in (1) causes the optimal parameters $a_\pi = 0.16$ and $a_y = 0$, and a marginally lower sum of squared differences $S = 0.474937$ compared with curve 8a. Hence, curve 8c assumes negative signs before $(\pi_t - \pi_t^*)$ and $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in (1), which causes the lowest sum of squared differences $S = 0.473981$ so far, and also the lowest overall in Table 1, and thus marked in bold, i.e. a 42.95% decrease compared with the Taylor (1993) rule in curve 1. The corresponding optimal parameter values are $a_\pi = 0.17$, $a_y = 0.13$, $a_m = 0.15$, $a_v = 0.15$, $a_u = 0.4$. This again suggests that the weight a_u assigned to unemployment rate should be relatively high.

Curves 9a and 9b assume three free choice variables and represents the combination of the Taylor (1993) rule and the Quantity Equation (Friedman, 1970), where a_y, a_m, and a_u are optimized assuming $a_u = 0$ and $a_\pi = 1 - a_y - a_m - a_v$. That causes a sum of squared differences $S = 0.502298$ in curve 9a, but with the optimal parameter $a_\pi = 0$. A negative sign before $(\pi_t - \pi_t^*)$ in (1) causes a lower sum of squared differences $S = 0.481483$ compared with curve 9a. Thus, the combination of the Taylor (1993) rule and the Quantity Equation (Friedman, 1970) explains the interest rate i_t better than the Taylor (1993) rule in curve 1.

Curves 10a and 10b assume two free choice variables and represent the combination of the Taylor (1993) rule and the Phillips (1958) curve, where a_y and a_u are optimized assuming $a_m = a_v = 0$ and $a_\pi = 1 - a_y - a_u$. That causes a sum of squared differences $S = 0.512049$ in curve 10a, but again with the optimal parameter $a_\pi = 0$. Thus, curve 10b assumes the negative sign for $(\pi_t - \pi_t^*)$ in (1). That causes a slightly lower sum of squared differences S = 0.488556 compared with curve 10a. The combination of the Taylor (1993) rule and the Phillips (1958) curve also explain the interest rate i_t better than the Taylor (1993) rule in curve 1.

Curve 11 assumes two free choice variables and represents the combination of the Quantity Equation (Friedman, 1970) and the Phillips (1958) curve, where a_v and a_u are optimized assuming $a_\pi = a_y = 0$, and $a_m = 1 - a_v - a_u$. That causes a sum of squared differences $S = 0.496629$, i.e., a 40.22% decrease compared with the Taylor (1993) rule in curve 1. The combination of the Quantity Equation (Friedman, 1970) and the Phillips (1958) curve also explain the interest rate i_t better than the Taylor (1993) rule in curve 1.

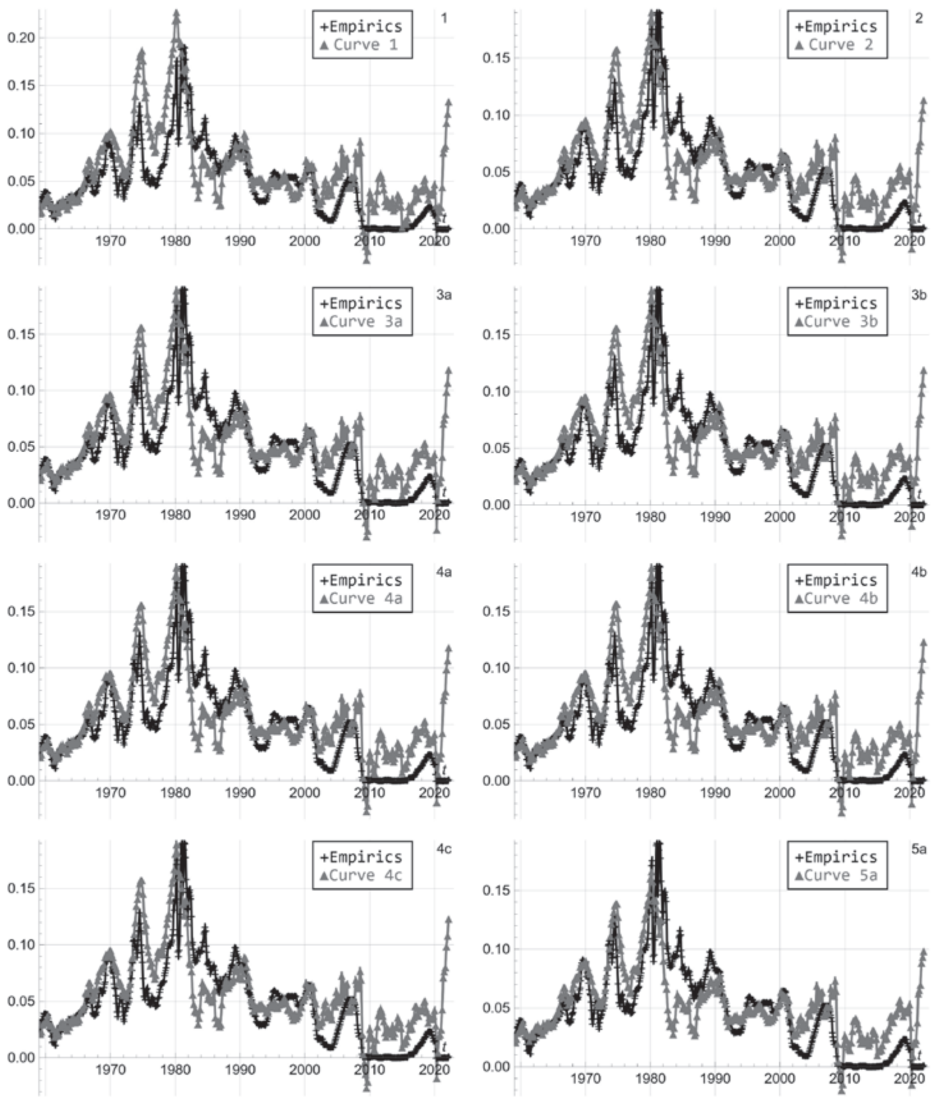
Curves 12a and 12b assume three free choice variables, where a_m, a_v and a_u are optimized assuming $a_y = 0$, and $a_\pi = 1 - a_m - a_v - a_u$. That causes a sum of squared differences $S = 0.497407$ in curve 12a, but again with the optimal parameter $a_\pi = 0$. Assuming a negative sign before $(\pi_t - \pi_t^*)$ in (1) causes for curve 12b the second lowest sum of squared differences $S = 0.474937$ in Table 1. The result happens to be the same as in curve 8b.

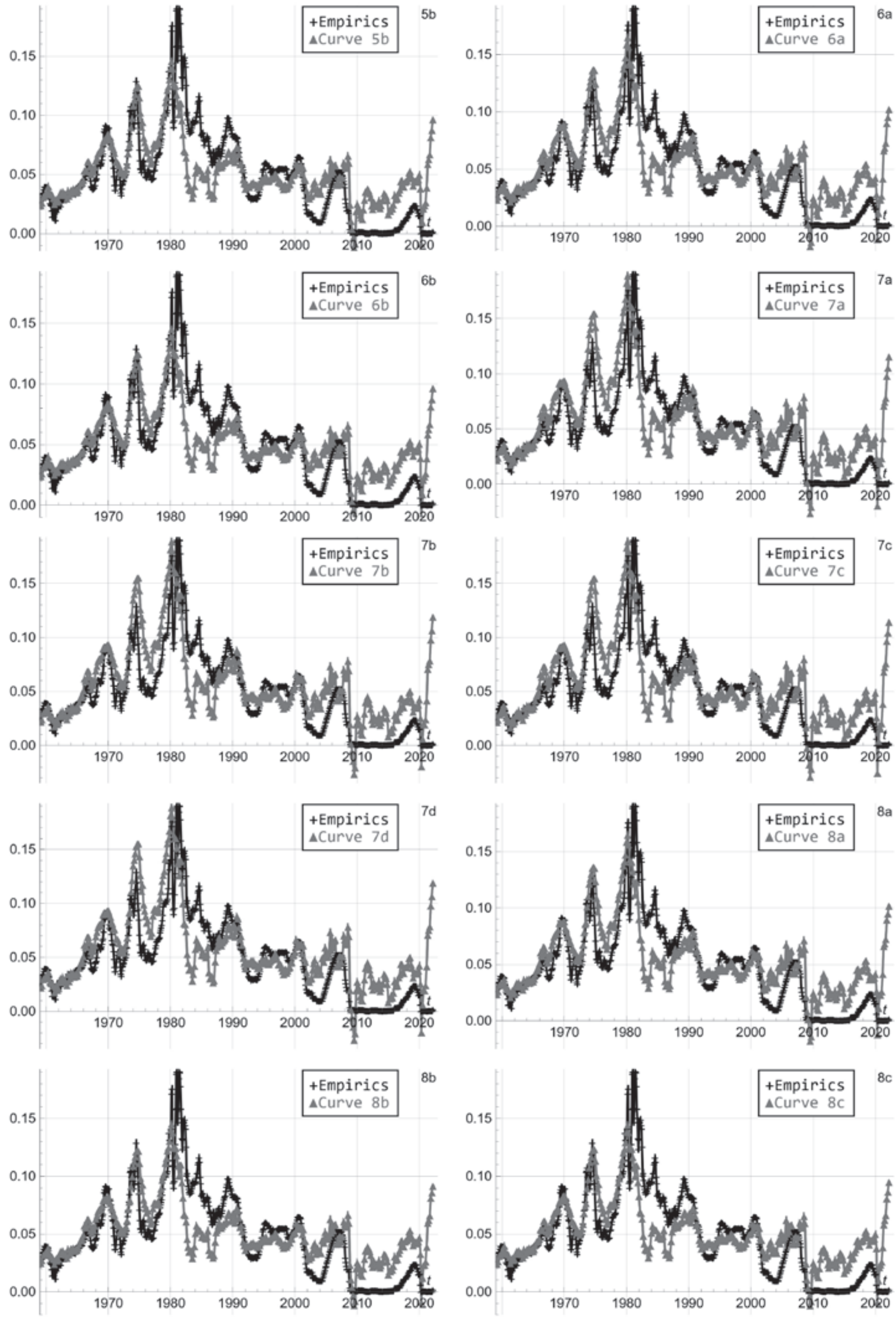
Curve 13 assumes one free choice variable and represents the Quantity Equation (Friedman, 1970), where a_v is optimized assuming $a_\pi = a_y = a_u = 0$, and $a_m = 1 - a_v$. That causes a sum of squared differences $S = 0.509102$. That suggests that the Quantity Equation (Friedman, 1970) explains the interest rate i_t better than the Taylor (1993) rule in curve 1.

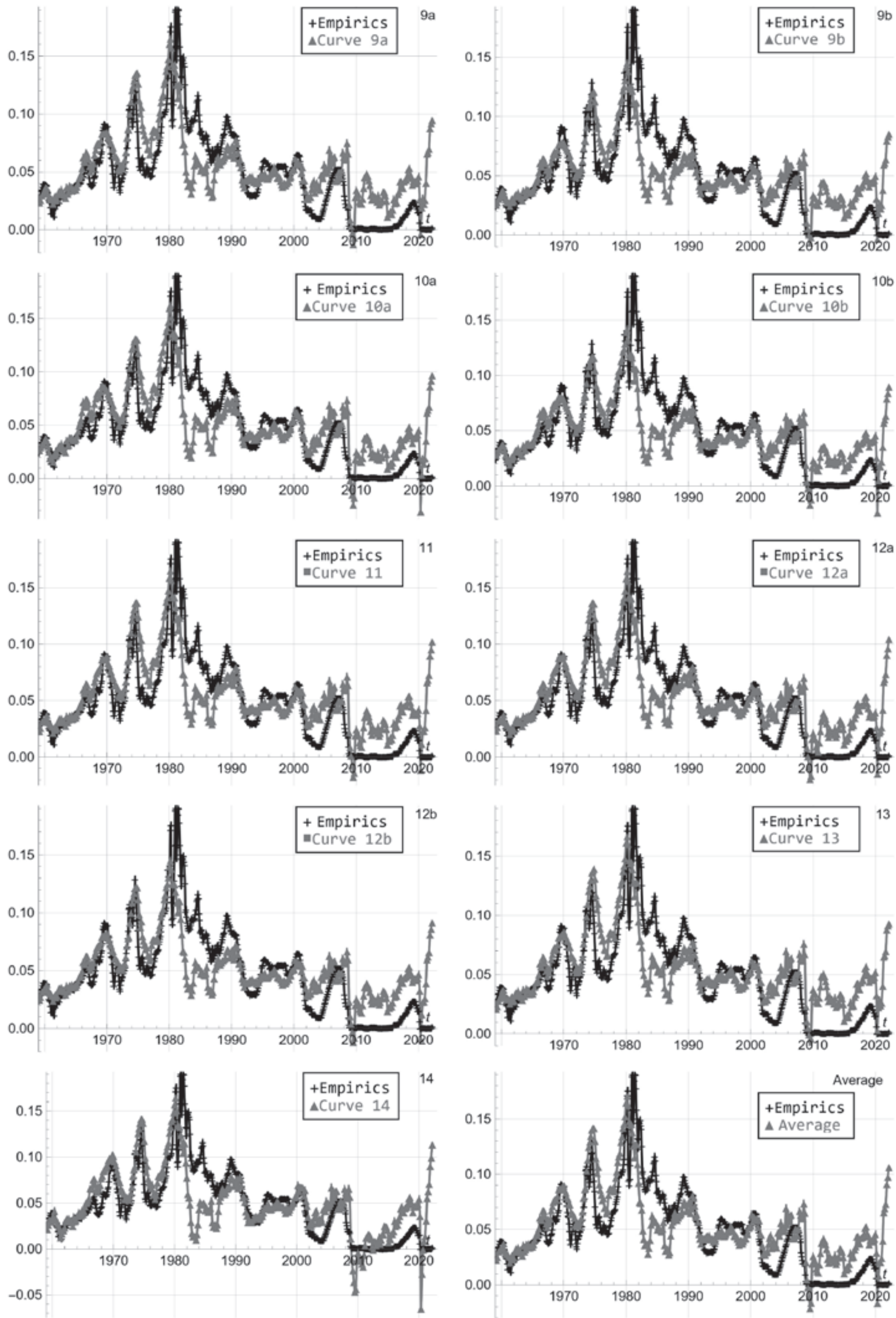
Curve 14 assumes no free choice variables and that only the Phillips (1958) curve is explanatory, i.e. $a_\pi = a_y = a_m = a_v = 0$, $a_u = 1$. The sum of squared differences is $S = 0.571153$.

Curve Average calculates the average of these 27 curves. The corresponding sum of squared differences $S = 0.510363$, i.e. a 38.57% decrease compared with the Taylor (1993) rule in curve 1. Curve Standard deviation calculates the standard deviation on these 27 curves.

Figure 1 plots the empirical interest rate i_t with black "+", together with 27 curves for the interest rate i_t in (1) with red filled triangles according to Table 1. The average and the standard deviation of these 27 curves are shown in the last two panels, which gives 29 panels.




5b
+Empirics
▲Curve 5b
6a
+Empirics
▲Curve 6a
6b
+Empirics
▲Curve 6b
7a
+Empirics
▲Curve 7a
7b
+Empirics
▲Curve 7b
7c
+Empirics
▲Curve 7c
7d
+Empirics
▲Curve 7d
8a
+Empirics
▲Curve 8a
8b
+Empirics
▲Curve 8b
8c
+Empirics
▲Curve 8c
0.00
0.05
0.10
0.15
1970
1980
1990
2000
2010
2020
t


9a
+Empirics
▲Curve 9a
9b
+Empirics
▲Curve 9b
10a
+ Empirics
▲Curve 10a
10b
+ Empirics
▲Curve 10b
11
+Empirics
■Curve 11
12a
+ Empirics
■Curve 12a
12b
+ Empirics
■Curve 12b
13
+Empirics
▲Curve 13
14
+Empirics
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+Empirics
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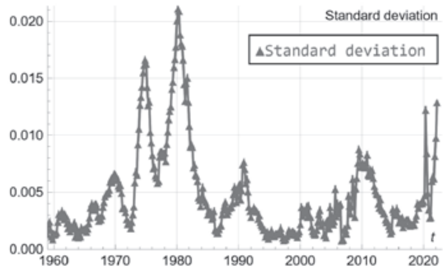

Figure 1. The Monthly US January 1, 1959 to March 31, 2022 empirical interest rate i_t and the interest rate i_t based on (1) with the following parameter values. Curve 1: $a_\pi = a_y = 0.5$, $a_m = a_v = a_u = 0$. Curve 2: $a_\pi = a_y = a_m = a_v = a_u = 0.2$. Curve 3a: $a_\pi = a_y = a_m = 0.2$, $a_v = 0$, $a_u = 0.36$. Curve 3b: $a_\pi = a_y = a_m = 0.2$, $a_v = 0.04$*, $a_u = 0.36$. Curve 4a: $a_\pi = a_y = 0.2$, $a_m = 0.25$, $a_v = 0$, $a_u = 0.35$. Curve 4b: $a_\pi = a_y = 0.2$, $a_m = 0$, $a_v = 0.17$*, $a_u = 0.43$. Curve 4c: $a_\pi = a_y = 0.2$, $a_m = 0.03$, $a_v = 0.17$*, $a_u = 0.4$*. Curve 5a: $a_\pi = a_y = 0$, $a_m = a_v = 0.37$, $a_u = 0.26$. Curve 5b: $a_\pi = a_y = 0.16$*, $a_m = a_v = 0.13$, $a_u = 0.42$. Curve 6a: $a_\pi = a_y = 0$, $a_m = 0.47$, $a_v = 0.18$, $a_u = 0.35$. Curve 6b: $a_\pi = a_y = 0.165$*, $a_m = 0.11$, $a_v = 0.13$, $a_u = 0.43$. Curve 7a: $a_\pi = 0.2$, $a_y = 0.41$, $a_m = 0.17$, $a_v = 0$, $a_u = 0.22$. Curve 7b: $a_\pi =0.2$, $a_y = 42$, $a_m = 0$, $a_v = 0.12$*, $a_u = 0.26$. Curve 7c: $a_\pi = 0.2$, $a_y = 0.61$, $a_m = 0.06$*, $a_v = 0$, $a_u = 0.13$. Curve 7d: $a_\pi = 0.2$, $a_y = 42$, $a_m = 0$*, $a_v = 0.12$*, $a_u = 0.26$. Curve 8a: $a_\pi = 0$, $a_y = 0.09$, $a_m = 0.44$, $a_v = 0.15$, $a_u = 0.32$. Curve 8b: $a_\pi = 0.16$*, $a_y = 0$, $a_m = 0.32$, $a_v = 0.21$, $a_u = 0.31$. Curve 8c: $a_\pi = 0.17$*, $a_y = 0.13$*, $a_m = 0.15$, $a_v = 0.15$, $a_u = 0.4$. Curve 9a: $a_\pi = 0$, $a_y = 0.31$, $a_m = 0.4$, $a_v = 0.29$, $a_u = 0$. Curve 9b: $a_\pi = 0.16$*, $a_y = 0.32$, $a_m = 0.24$, $a_v = 0.28$, $a_u = 0$. Curve 10a: $a_\pi = 0$, $a_y = 0.98$, $a_m = 0$, $a_v = 0$, $a_u = 0.02$. Curve 10b: $a_\pi = 0.17$*, $a_y = 0.66$, $a_m = 0$, $a_v = 0$, $a_u = 0.17$. Curve 11: $a_\pi = 0$, $a_y = 0$, $a_m = 0.47$, $a_v = 0.18$, $a_u = 0.35$. Curve 12a: $a_\pi = 0$, $a_y = 0$, $a_m = 0.46$, $a_v = 0.1$, $a_u = 0.44$. Curve 12b: $a_\pi = 0.16$*, $a_y = 0$, $a_m = 0.32$, $a_v = 0.21$, $a_u = 0.31$. Curve 13: $a_\pi = 0$, $a_y = 0$, $a_m = 0.51$, $a_v = 0.49$, $a_u = 0$. Curve 14: $a_\pi = 0$, $a_y = 0$, $a_m = 0$, $a_v = 0$, $a_u = 1$. Curve Average: The average of these 27 curves. Curve Standard deviation: The standard deviation of these 27 curves A superscript star * after a number means that the corresponding sign in (1) is changed from plus to minus. No superscript star * after a number means that only the plus signs in (1) are used.

These 27 curves are similar in some regards, but they are unique and present different features. Curve 1 presents the Taylor (1993) rule, i.e. assuming $a_\pi = a_y = 0.5$, $a_m = a_v = a_u = 0$. Among the 27 curves, the peak in 1980 for curve 1 is highest compared with the peaks in 1980 for all the 27 curves. Curve 1 predicts negative interest rate i_t from January 2009 to May 2009, and in March 2020. Curve 2 assumes $a_\pi = a_y = a_m = a_v = a_u = 0.2$, which fits the empirical interest rate i_t better than the Taylor (1993) rule. The peak of curve 2 in 1980 is close to the empirical interest rate i_t. The last two curves show the average interest rate i_t of the 27 curves, and the standard deviation of the interest rate i_t, respectively. Overall, the 27 curves show especially high variation for 1980, as the curve Standard deviation shows.

Table 2 shows that the Pearson correlation coefficients are high, ranging from 0.71 to 0.75 between the empirical interest rate i_t and the 27 curves. The correlations are even higher, ranging from 0.92 to 1 among the 27 curves.

Table 2 Pearson correlation coefficients between the empirical interest rate i_t and the 27 curves

Curves	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
(1) Empirical	1.00														
(2) Curve1	0.74	1.00													
(3) Curve2	0.74	1.00	1.00												
(4) Curve3a	0.74	0.99	1.00	1.00											
(5) Curve3b	0.74	0.99	1.00	1.00	1.00										
(6) Curve4a	0.74	0.99	1.00	1.00	1.00	1.00									
(7) Curve4b	0.74	0.99	1.00	1.00	1.00	1.00	1.00								
(8) Curve4c	0.74	0.99	1.00	1.00	1.00	1.00	1.00	1.00							
(9) Curve5a	0.74	0.99	1.00	1.00	1.00	1.00	0.99	0.99	1.00						
(10) Curve5b	0.75	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00					
(11) Curve6a	0.74	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00				
(12) Curve6b	0.75	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00			
(13) Curve7a	0.74	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	1.00		
(14) Curve7b	0.74	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	1.00	1.00	
(15) Curve7c	0.74	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	1.00	0.99	1.00	1.00	1.00
(16) Curve7d	0.74	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	1.00	1.00	1.00
(17) Curve8a	0.74	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
(18) Curve8b	0.74	0.98	1.00	1.00	1.00	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
(19) Curve8c	0.75	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.99
(20) Curve9a	0.74	1.00	1.00	1.00	1.00	1.00	0.99	0.99	1.00	0.99	1.00	0.99	1.00	1.00	1.00
(21) Curve9b	0.74	0.99	1.00	0.99	0.99	0.99	0.99	0.99	1.00	0.99	0.99	0.99	1.00	0.99	1.00
(22) Curve10a	0.73	0.97	0.99	0.99	0.99	0.99	0.98	0.98	0.99	0.98	0.99	0.98	0.99	0.99	0.99
(23) Curve10b	0.73	0.97	0.99	0.99	0.98	0.99	0.98	0.98	0.99	0.99	0.99	0.99	0.99	0.99	0.99
(24) Curve11	0.74	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
(25) Curve12a	0.74	0.98	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99
(26) Curve12b	0.74	0.98	1.00	1.00	1.00	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
(27) Curve13	0.74	1.00	1.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	1.00
(28) Curve14	0.71	0.92	0.95	0.96	0.96	0.96	0.96	0.95	0.97	0.97	0.97	0.97	0.95	0.95	0.96
(29) Curve Average	0.74	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 2 Pearson correlation coefficients between the empirical interest rate i_t and the 27 curves - continued

Curves	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)
(1) Empirical														
(2) Curve1														
(3) Curve2														
(4) Curve3a														
(5) Curve3b														
(6) Curve4a														
(7) Curve4b														
(8) Curve4c														
(9) Curve5a														
(10) Curve5b														
(11) Curve6a														
(12) Curve6b														
(13) Curve7a														
(14) Curve7b														
(15) Curve7c														
(16) Curve7d	1.00													
(17) Curve8a	1.00	1.00												
(18) Curve8b	1.00	1.00	1.00											
(19) Curve8c	0.99	1.00	1.00	1.00										
(20) Curve9a	1.00	1.00	0.99	0.99	1.00									
(21) Curve9b	0.99	0.99	1.00	0.99	1.00	1.00								
(22) Curve10a	0.99	0.99	0.99	0.99	0.99	0.99	1.00							
(23) Curve10b	0.99	0.99	0.99	0.99	0.99	0.99	1.00	1.00						
(24) Curve11	1.00	1.00	1.00	1.00	1.00	0.99	0.99	0.99	1.00					
(25) Curve12a	1.00	1.00	1.00	1.00	0.99	0.99	0.99	0.99	1.00	1.00				
(26) Curve12b	1.00	1.00	1.00	1.00	0.99	1.00	0.99	0.99	1.00	1.00	1.00			
(27) Curve13	0.99	0.99	0.99	0.99	1.00	1.00	0.98	0.98	0.99	0.99	0.99	1.00		
(28) Curve14	0.95	0.97	0.97	0.97	0.94	0.95	0.97	0.98	0.97	0.97	0.97	0.93	1.00	
(29) Curve Average	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.99	1.00	1.00	1.00	0.99	0.96	1.00

4. Discussion

This article expands the Taylor (1993) rule by introducing three additional variables, i.e. the money supply, the money velocity, and the unemployment rate. The article also tests the various weights assigned to the deviations in inflation rate, real GDP, money supply, money velocity, and unemployment rate. Five results in the previous section are noteworthy. First, the Taylor (1993) rule does not explain the US empirical interest rate well. Among the Taylor (1993) rule, the Quantity Equation (Friedman, 1970) and the Phillips (1958) curve, the Quantity Equation (Friedman, 1970) gives the lowest sum of the squared differences between the empirical interest rate and the predicted interest rate, followed by the Phillips (1958) curve and the Taylor (1993) rule, respectively. Second, the combination of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve causes a substantially better result than the Taylor (1993) rule. Thus, incorporating the money supply, the money velocity and the unemployment rate substantially improves the accuracy compared with the Taylor (1993) rule. Third, the weight assigned to the unemployment rate should be relatively high. The Taylor (1993) rule assigns equal 0.5 weight to the deviation in inflation rate and the deviation in real GDP. The findings show that that may not be a good weight combination. Fourth, equal 0.2 weight to the deviations of inflation rate, real GDP, money supply, money velocity and unemployment rate decreases the sum of squared differences compared with the Taylor (1993) rule. Fifth, assuming two combinations, the Taylor (1993) rule and the Quantity Equation (Friedman, 1970) gives best result, followed by the Taylor (1993) rule and the Phillips (1958) curve, and finally, the Quantity Equation (Friedman, 1970) and the Phillips (1958) curve. The results of these three combinations are similar.

The endogeneity problem, i.e. that some independent variables are not independent of the dependent variable, is commonly assessed related to the Taylor (1993) rule. Endogeneity is often problematic in an econometric approach, but can also arise in economics more generally. This article does not apply an econometric approach. The authors believe that endogeneity is a limited or not a problem for this article for the following reasons: The article assumes that the sum of the five weight parameters is one. The authors believe that the three additional variables are not highly correlated. The article does not introduce the money supply and the money velocity into the Taylor (1993) rule directly in (1). The term $Log\left(\frac{m_t}{\bar{m}_t}\right)$ for the money supply is a ratio which eliminates the scaling impact of the money supply. The term $Log\left(\frac{v_t}{\bar{v}_t}\right)$ for the money velocity is a ratio which eliminates the scaling impact of the money velocity. The term $Log\left(\frac{y_t}{\bar{y}_t}\right)$ for the GDP is a ratio which eliminates the scaling impact of the real GDP. Instead, these are logarithms of ratios, and thus not linear combinations of the relevant variables, which eliminates the scaling impact of these variables. A stationary test of endogeneity is common in time series analysis. This article does not use a time series analysis technique. Thus it is not feasible to conduct a stationary test. Instead this article conducts a robustness test by exploring various weights assigned to the five variables.

5. Limitations and future research

Conrad and Eife (2012) point out that the weights discussed in the previous sections are not fixed over time. One limitation of the Taylor (1993) rule, and also of this article assuming additional terms, is thus the assumption of constant weights through time. Future research may explore how these weights change dynamically over time.

Other potential limitations of the Taylor (1993) rule, combined or not combined with the other rules in this article, are the uncertainty of the level of potential real GDP, the long term real equilibrium interest rate, and the natural unemployment rate. One common challenge of the Taylor (1993) rule is to estimate the potential real GDP and thus the real GDP gap, i.e. the difference between the real GDP and the potential real GDP. Orphanides (2001) points out that the real GDP gap can look quite differently today as compared to the view in retrospect in some years. Hence, future research may find a better way to dynamically estimate the real GDP gap. This article assumes that the long term equilibrium real interest rate is 2%, which is commonly accepted, also in the Taylor (1993) rule. Laubach and Williams (2003) suggest that the equilibrium real interest rate is not stable over time. Thus, future research may find a way to better estimate the long term equilibrium real interest rate.

The central bank may adjust interest rates to the desired level gradually, i.e. "interest smoothing" (Judd & Rudebusch, 1998). Future research may incorporate additional lagged variables into the model, and explore non-lagged variables. Another limitation of this article and the Taylor (1993) rule is that these are backward looking approaches. In contrast, Clarida et al. (2000) explore a forward looking interest rate rule and recommend being forward looking in future research.

This article and the Taylor (1993) rule apply an in-sample fit approach. Qin and Enders (2008) compare the properties of the in-sample fit approach and the out-sample fit approach in the Taylor (1993) rule. They suggest that an out-of-sample fit approach may be useful in selecting the alternative interest rate functional forms.

Future research may connect the interest rates in multiple counties and treat the global financial system as a whole. The interaction between interest rates, monetary policy and macroprudential policy may be examined. The data for various countries during different time periods may be explored accounting for specific economic changes. Interest rate rules during times of changes between positive and negative interest rates may be explored (Wang & Hausken, 2022a).

Other factors impacting interest rates may also be explored, e.g. economic crises, fiscal deficits, global interest rates, financial variables such as house prices, stock prices, leverage, oil and commodity prices (Kahn, 2010). Broader economic and financial theories may be incorporated to investigate potential further underlying mechanisms impacting interest rates.

6. Policy implications

Research on interest rates has progressed at a torrid pace in recent years. But central banks still face challenges when choosing monetary policy and determining interest rates, perhaps especially after the 2021-2022 pandemic crisis. The findings in this article provide insights relevant for the policy makers including central banks. First, the Taylor (1993) rule performs poorly in explaining the empirical interest rate. Hence, it is beneficial for the central bank to consider more factors beyond the Taylor rule when determining the interest rate. Second, the article presents a generalized interest rate rule, which combines the Taylor rule, the Quantity Equation (Friedman, 1970) and the Phillips (1958) curve. The model performs better than the Taylor rule. Three additional variables, i.e. the money supply, the money velocity and the unemployment rate help explain the interest rate more convincingly. Therefore, the central bank may consider these additional variables when determining the interest rate. Third, Taylor (1993) assigns equal 0.5 weight to the deviation in the inflation rate and the deviation in the real GDP. However, the article shows that these weights are not optimal. Higher weights assigned to the deviation in the unemployment rate, the deviation in the money supply and the deviation in the money velocity are appropriate. Fourth, interest rates impact households, firms and other actors substantially. For example, a lower interest rate may boost consumption, spending and borrowing. It may also encourage an entrepreneur to borrow funds for expansion, make new investments, and hire more workers. The findings of this article are believed to be helpful for researchers, financial analysts, investors, entrepreneurs, consumers, etc., who may better predict interest rates and make better decisions.

7. Conclusion

This article provides a broad view of monetary policy starting from the Taylor (1993) rule. The article generalizes the Taylor rule to account for the money supply, the money velocity, and the unemployment rate. Thus, the article explores and tests various combinations of the Taylor rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. Five parameters are introduced and estimated; i.e. the weights assigned to the deviation in inflation rate, real GDP, money supply, money velocity, and the unemployment rate. The Taylor rule only has two parameters, i.e. the weights assigned to the deviation in real GDP and the deviation in the inflation rate. Various combinations of parameter values are explored and tested.

The generalized equation is tested using the monthly US January 1, 1959 to March 31, 2022 data. First, the Taylor rule is evaluated against the empirics. Second, equal weight to the five parameters is evaluated. Third, various values for these five parameters are explored and tested, such as equal weight to the deviation in the inflation rate and the deviation in the real GDP, equal weight to the deviation in the money supply and the deviation in the money velocity, and the values that represent various combinations of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve. The findings show that the generalized equation fits the empirical interest rate better and has a lower sum of squares compared with the Taylor rule. Notably, for the optimal values for the five parameters, the weights assigned to the deviation in inflation rate and the deviation in real GDP decrease compared with the Taylor rule. Meanwhile, the weight assigned to the deviation in unemployment rate is relatively high compared with the weights assigned to the deviation in inflation rate and the deviation in real GDP. The weights assigned to the deviation in money supply and the deviation in money velocity are moderate compared with the weights assigned to the deviation in inflation rate and the deviation in real GDP.

Appendix A: Nomenclature

Parameters

a_π Weight assigned to deviation in inflation, $0 \leq a_\pi \leq 1$

a_y Weight assigned to deviation in real GDP, $0 \leq a_y \leq 1$

a_m Weight assigned to deviation in money supply, $0 \leq a_m \leq 1$

a_v Weight assigned to deviation in money velocity, $0 \leq a_m \leq 1$

a_u Weight assigned to deviation in the unemployment rate, $0 \leq a_m \leq 1$

Variables

i_t Interest rate at time t, $i_t \in \mathbb{R}$

π_t Inflation rate, $\pi_t \in \mathbb{R}$

π_t^* Target inflation rate, $\pi_t^* \in \mathbb{R}$

r_t^* Equilibrium real interest rate, $r_t^* \in \mathbb{R}$

y_t Real GDP (Gross Domestic Product), $y_t \geq 0$

$\bar{y}_t$ Real potential GDP, $\bar{y}_t \geq 0$

m_t Money supply at time t, $m_t > 0$

u_t Unemployment rate, $u_t \geq 0$

$\bar{u}_t$ Natural rate of unemployment, $\bar{u}_t \geq 0$

t Time, $t > 0$

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A Generalized Interest Rates Model with Scaling

Guizhou Wang, Kjell Hausken*

Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway. *Email: kjell.hausken@uis.no

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ABSTRACT

The article introduces scaling and generalizes the Taylor (1993) interest rate rule from four terms to seven terms. The three additional terms are the deviation in money supply, the deviation in money velocity, and the deviation in unemployment rate. The four original terms are the inflation rate, the equilibrium real interest rate, the deviation in inflation rate, and the deviation in real GDP (Gross Domestic Product). The weights for the seven terms are estimated via the monthly January 1, 1959-March 31, 2022 US data. All the seven combinations of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve with scaling give substantially better results than both the Taylor (1993; 1999) rules without scaling. The Phillips (1958) curve is best when choosing only one rule with scaling. Combining the Taylor (1993) rule and the Phillips (1958) curve is best when choosing between two rules with scaling.

Keywords: Monetary Policy, Taylor Rules, Phillips Curve, Interest Rate, Inflation Rate, Money Supply, Money Velocity, Unemployment Rate
JEL Classifications: C6, E24, E50, E47, E52, E58

1. INTRODUCTION

1.1. Background

Interest rates have been a hot topic in academic research for a long time. Central banks apply discretion and various rules to adjust interest rates to ensure economic stability and monetary liquidity. The best known policy rule is the Taylor (1993) rule. It recommends that central banks adjust interest rates in response to four terms, i.e. the inflation rate, long term equilibrium real interest rate, deviation in inflation rate, and the deviation in real GDP (Gross Domestic Product). The Taylor (1993) rule has received substantial attention in academic research. Various interest rate rules have emerged after the Taylor (1993) rule, e.g. the Taylor (1999) rule, balanced-approach rule, inertial Taylor rule, effective lower bound-adjusted rule, first-difference rule, etc. (Erceg et al., 2012). The Taylor (1993) rule has four terms, i.e. the inflation rate, the equilibrium real interest rate, the deviation in inflation rate, and the deviation in real GDP. Taylor (1993) assigns equal 0.5 weight to both the deviation in real GDP and the deviation in inflation rate. Subsequently, in his Taylor (1999) rule, he increases the weight for the deviation in real GDP to one. Perhaps surprisingly, both Taylor (1993; 1999) rules assign default weight one to the inflation rate and the equilibrium real interest rate.

1.2. Contribution

Building upon this background, it seems interesting to explore additional phenomena beyond Taylor's (1993; 1999) four terms, and assess how the terms should be scaled relative to each other. The article investigates and generalizes the Taylor (1993) rule from four terms to seven terms on the right hand side to determine the interest rate on the left hand side. The three additional terms are two terms from the Quantity Equation (Friedman, 1970), i.e. the money supply and money velocity, and one term from the Phillips (1958) curve, i.e. the unemployment rate. The article estimates weights for the seven terms, which amounts to scaling them relative to each other. To our best knowledge, this article is the first to explore the scaling issue for Taylor (1993; 1999) rules or generalizations of such rules. The article adopts monthly US January 1, 1959-March 31, 2022 US data in the empirical analysis. The article uses the least squares method to estimate the optimal weights for the seven terms.

1.3. Literature

The Taylor (1993) rule suggests an equal 0.5 weight for the deviation in inflation rate and the deviation in real GDP. The Taylor (1999) rule keeps the 0.5 weight for the deviation in inflation rate, but increases the weight assigned to the deviation in real GDP to one. Several monetary rules are based on the Taylor (1993) rule, e.g. the effective lower bound-adjusted rule (Reifschneider and Williams, 2000). It suggests that the interest rate cannot be lower than the so-called effective lower bound. The first difference rule (Orphanides, 2003) connects the current interest rate to its previous value. The inertial rule (Bullard, 2017; Kliesen, 2019) lowers the interest rate's volatility over time, and points out that the policymaker adjusts the interest rate gradually. Taylor and Williams (2010) provide a comprehensive review of interest rate policy rules.

The Quantity Equation (Friedman, 1970) connects the money supply, money velocity, price level (or inflation rate), and the real GDP. Money supply is widely assumed to impact interest rates. For example, Friedman (1961) suggests that the money supply has a negative effect on the interest rate. Money velocity also relates to the interest rate. Taylor (1999, p. 322) says that "we know that velocity depends on the interest rate and on real output or income." Keynes et al. (1971) suggest an inverse relationship between the money velocity and the money supply. In addition, money velocity may also impact the interest rate via the inflation rate (Mendizabal, 2006). But both money supply and money velocity are absent in the Taylor (1993, 1999) rules. Prag (1994) suggests an inverse relationship between the interest rate and the unemployment rate. The unemployment rate is also absent in the Taylor (1993, 1999) rules.

The literature compares the interest rate rules with other policy rules, e.g. money supply rules (Ascari and Ropele, 2013; Auray and Fève, 2003; Schabert, 2005; Srinivasan, 2000), McCallum rule Razzak (2003), Friedman rule (Srinivasan, 2000), etc. The literature also links monetary policy to macroeconomics (Clarida et al., 2000; Schabert, 2009; Wijngaard and Van Hee, 2021; Woodford, 2001), to the Phillips (1958) curve (Wang and Hausken, 2022a), adopts the Taylor (1993) rule to design decision models (Wang and Hausken, 2022b), and builds dynamic stochastic general equilibrium models (Ferrari Minesso et al., 2022; Oh and Zhang, 2020).

1.4. Article Organization

Section 2 presents the model. Section 3 analyzes the model with data sources, parameter estimation, and illustrations. Section 4 concludes.

2. THE MODEL

Appendix A shows the nomenclature. This article generalizes the Taylor (1993) rule. First, it introduces three additional terms, i.e. money supply m_t, m_t>0, and money velocity v_t, v_t>0, as presented in the Quantity Equation (Friedman, 1970), and unemployment rate u_t, $u_t \geq 0$ as presented in the Phillips (1958) curve. Second, it incorporates scaling for the seven terms, thus making the weights assigned to the seven terms comparable. Thus the interest rate i_t at time t is given by

$$i_t = a_{pi}s_{pi}\pi_t + a_r s_r r_t^* + a_\pi s_\pi\left(\pi_t - \pi_t^*\right) + a_y s_y Log\left(\frac{y_t}{\bar{y}_t}\right)$$

$$+a_m s_m Log\left(\frac{m_t}{\bar{m}_t}\right) + a_v s_v Log\left(\frac{v_t}{\bar{v}_t}\right) + a_u s_u\left(\bar{u}_t - u_t\right),$$

$$s_{pi} \equiv \frac{1}{\left(P_{pi}\sum_{h=1}^{P_{pi}}\pi_h + N_{pi}\left|\sum_{k=1}^{N_{pi}}\pi_k\right|\right)/\left(P_{pi}+N_{pi}\right)},$$

$$s_r \equiv \frac{1}{\left(P_r\sum_{h=1}^{P_r}r_h^* + N_r\left|\sum_{k=1}^{N_r}r_k^*\right|\right)/\left(P_r+N_r\right)},$$

$$s_\pi \equiv \frac{1}{\left(P_\pi\sum_{h=1}^{P_\pi}\left(\pi_h - \pi_h^*\right) + N_\pi\left|\sum_{k=1}^{N_\pi}\left(\pi_k - \pi_k^*\right)\right|\right)/\left(P_\pi+N_\pi\right)},$$

$$s_y \equiv \frac{1}{\left(P_y\sum_{h=1}^{P_y}Log\left(\frac{y_h}{\bar{y}_h}\right) + N_y\left|\sum_{k=1}^{N_y}Log\left(\frac{y_k}{\bar{y}_k}\right)\right|\right)/\left(P_y+N_y\right)}, \tag{1}$$

$$s_m \equiv \frac{1}{\left(P_m\sum_{h=1}^{P_m}Log\left(\frac{m_h}{\bar{m}_h}\right) + N_m\left|\sum_{k=1}^{N_m}Log\left(\frac{m_k}{\bar{m}_k}\right)\right|\right)/\left(P_m+N_m\right)},$$

$$s_v \equiv \frac{1}{\left(P_v\sum_{h=1}^{P_v}Log\left(\frac{v_h}{\bar{v}_h}\right) + N_v\left|\sum_{k=1}^{N_v}Log\left(\frac{v_k}{\bar{v}_k}\right)\right|\right)/\left(P_v+N_v\right)},$$

$$s_u \equiv \frac{1}{\left(P_u\sum_{h=1}^{P_u}\left(\bar{u}_h - u_h\right) + N_u\left|\sum_{k=1}^{N_u}\left(\bar{u}_k - u_k\right)\right|\right)/\left(P_u+N_u\right)}$$

where $i_t \in R$, R is the set of all real numbers, $t \geq 0$, r_t^* is the equilibrium real interest rate, y_t is the real GDP, $y_t \geq 0$, $\bar{y}_t$ is the potential real GDP that can be sustained in the long run, $\bar{y}_t \geq 0$. The right hand side of (1) contains the four original terms in the Taylor (1993) rule, i.e π_t, r_t^*, $\pi_t - \pi_t^*$ and $Log\left(\frac{y_t}{\bar{y}_t}\right)$, where $\pi_t - \pi_t^*$ is the deviation in inflation rate, $Log\left(\frac{y_t}{\bar{y}_t}\right)$ is the deviation in real GDP. The three new terms in (1) are the deviation $Log\left(\frac{m_t}{\bar{m}_t}\right)$ in money supply, the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity, and the deviation $\bar{u}_t - u_t$ in unemployment rate, where, Log is the logarithm with a base of 10, m_t is the money supply, $m_t \geq 0$, $\bar{m}_t$ is the potential money supply, $\bar{m}_t \geq 0$, v_t is the money velocity $v_t \geq 0$, $\bar{v}_t$ is the potential money velocity, $\bar{v}_t \geq 0$, $\bar{u}_t$ is the natural unemployment rate, $\bar{u}_t \geq 0$, and u_t is the unemployment rate, $u_t \geq 0$.

In (1), s_j, j=pi,r,π,y,m,v,u are the scaling parameters for the seven terms. These are the inflation rate π_t, the equilibrium real interest rate r_t^*, the deviation $\pi_t - \pi_t^*$ in inflation rate, the deviation

$Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP, the deviation $Log\left(\frac{m_t}{\bar{m}_t}\right)$ in money supply, the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity, and the deviation $\bar{u}_t - u_t$ in unemployment rate, respectively, where P_j, $j=pi,r,\pi,y,m,v,u$ specifies the number of nonnegative numbers in the data for term j, and N_j, $j=pi,r,\pi,y,m,v,u$ specifies the number of negative numbers in the data for term j. Hence N_j is multiplied by the absolute value of the sum of the negative data points for term j in (1). The sum $P_j+N_j=759$ specifies the number of data points for the period January 1, 1959-March 31, 2022. We introduce P_j and N_j to ensure proper and intuitive scaling, since data points may be negative or positive. The counting parameters h and k are associated with P_j and N_j, respectively, to run through the $P_j+N_j=759$ data points. The seven parameters $a_{pi}, a_r, a_\pi, a_y, a_m, a_v, a_u$ are the weights assigned to the seven terms, which can be positive or nonpositive. If the weight is positive, it means that the corresponding term positively impacts the interest rate i_t. If the weight is negative, it means that the corresponding term negatively impacts the interest rate i_t.

The four terms in (1), i.e. the inflation rate π_t, the equilibrium real interest rate r_t^*, the deviation $\pi_t - \pi_t^*$ in inflation rate and the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP, are originally included in the Taylor (1993, 1999) rules. The Taylor (1993, 1999) rules assign default weight one to both the inflation rate π_t and the equilibrium real interest rate r_t^*.

The first new term is the deviation $Log\left(\frac{m_t}{\bar{m}_t}\right)$ in money supply. Thus, the two variables the money supply m_t and the potential money supply $\bar{m}_t$ are introduced. We adopt the standard Hodrick and Prescott (1997) filter to estimate the potential money supply $\bar{m}_t$. The method is widely used in macroeconomics to investigate the potential GDP, especially in real business cycle theory (Furceri and Mourougane, 2012). The interest rate i_t is the price of the money supply m_t applying supply and demand considerations. As Friedman (1961) suggests, money supply m_t has a negative effect on the interest rate i_t. Conrad (2021) also points out that the interest rate i_t decreases when the money supply m_t increases. Nevertheless, central banks may choose to increase the interest rate i_t to prevent savers' extensive withdrawals when the money supply m_t increases. This is consistent with Ascari and Ropele (2013). They suggest a positive relationship between the money supply m_t and the interest rate i_t.

The second new term is the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity. The money velocity v_t and the potential money velocity $\bar{v}_t$ are introduced. The two variables are present in the Quantity Equation (Friedman, 1970). The money velocity v_t is defined as the ratio of nominal GDP to the money supply (Federal Reserve Bank of St. Louis, 2022). The potential money velocity $\bar{v}_t$ is defined as the ratio of nominal potential GDP to the potential money supply. The money velocity v_t is widely accepted to have a positive impact on the inflation rate π_t (Mendizabal, 2006). This is consistent with Taylor (1993, 1999) assuming positive correlation between the inflation rate π_t and the interest rate i_t. Thus, the money velocity v_t may affect the interest rate i_t positively.

The third new term is the deviation $\bar{u}_t - u_t$ in the unemployment rate. The unemployment rate u_t is present in the short run Phillips (1958) curve. It shows an inverse relationship between the inflation rate π_t and the unemployment rate u_t over the short run. Taylor (1993) assumes a positive correlation between the inflation rate π_t and the interest rate i_t. Hence, the unemployment rate u_t may impact the interest rate i_t negatively. Summing up, as specified in (1), the seven weights of the seven terms scale these terms relative to each other, and scale them overall relative to the interest rate i_t on the left hand side.

3. ANALYZING THE MODEL

3.1. Data Sources

This article uses the monthly US data. The data range is from January 1, 1959 to March 31, 2022, collected and estimated from the following sources. We estimate the real GDP y_t and the real potential GDP $\bar{y}_t$ from the US Bureau of Economic Analysis (2022) and the US Congressional Budget Office (2022b), respectively. We apply the quadratic interpolation method to convert quarterly data to monthly data for the real GDP y_t and the real potential GDP $\bar{y}_t$. We estimate the M2 money supply m_t and the money velocity v_t from the Board of Governors of the Federal Reserve System (US) (2022b), and the Federal Reserve Bank of St. Louis (2022), respectively. The unemployment rate u_t and the natural unemployment rate $\bar{u}_t$ are estimated from the US Bureau of Labor Statistics (2022b) and the US Congressional Budget Office (2022a), respectively. Again, we adopt the quadratic interpolation method to convert quarterly data to monthly data for the natural unemployment rate $\bar{u}_t$. The inflation rate π_t and the empirical interest rate i_t are derived from the US Bureau of Labor Statistics (2022a), and the Board of Governors of the Federal Reserve System (US) (2022a), respectively. The target inflation rate π_t^* is from several sources. We set the target inflation rate $\pi_t^*=1.5\%$ from January 1, 2000 to December 30, 2007 inspired by Shapiro and Wilson (2019). For the remaining January 1, 1959-March 31, 2022 time periods, we use the common $\pi_t^*=2\%$, as Taylor (1993) assumes for January 1, 1984 to September 31, 1992. Finally, we use the common equilibrium real interest rate $r_t^*=2\%$ from January 1, 1959 to March 31, 2022, as used by Taylor (1993) for January 1, 1984 to September 31, 1992, and consistent with the estimation of Kiley (2020).

3.2. Estimating the Parameters and Illustrating the Solutions

Table 1 shows the estimations of the seven parameter values $a_{pi}, a_r, a_\pi, a_y, a_m, a_v, a_u$ in (1), the sum S of the squared differences between the empirical interest rate i_t and the estimated interest rate i_t in (1), the number N of free choice variables for each estimation, and the specifics of each estimation.

Curve 1 assumes seven free choice variables, and represents the combination of the Taylor (1993) rule, the Quantity Equation

Table 1: Curve label, estimated parameter values $a_{pi}, a_r, a_\pi, a_y, a_m, a_v, a_u$, the sum S of the squared differences between the empirical interest rate i_t and the estimated interest rate i_t in (1), the number N of free choice variables, and the estimation specifics

Curve	a_{pi}	a_r	a_π	a_y	a_m	a_v	a_u	S	N	Estimation specifics
1	66.72	–11.44	–16.84	–3.13	1.29	2.73	2.06	0.44567	7	Combination of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve with scaling, optimizing $a_{pi}, a_r, a_\pi, a_y, a_m, a_v, a_u$
2	61.94	–9.00	–14.75	–1.12	1.25	2.98	0	0.45341	6	$a_u = 0$, combination of the Taylor (1993) rule and the Quantity Equation (Friedman, 1970) with scaling, optimizing $a_{pi}, a_r, a_\pi, a_y, a_m, a_v$
3	71.23	–12.69	–18.61	–0.73	0	0	2.19	0.45157	5	$a_m = a_v = 0$, combination of the Taylor (1993) rule and the Phillips (1958) curve with scaling, optimizing $a_{pi}, a_r, a_y, a_\pi, a_u$
4	25.72	11.32	0	0	0.41	0.94	0.92	0.45628	5	$a_\pi = a_y = 0$, combination of the Quantity Equation (Friedman, 1970) and Phillips (1958) curve with scaling, optimizing $a_{pi}, a_r, a_y, a_\pi, a_u$
5	66.14	–10.02	–16.34	1.64	0	0	0	0.46065	4	$a_m = a_v = a_u = 0$, Taylor (1993) rule with scaling, optimizing a_{pi}, a_r, a_y, a_π
6	25.82	10.77	0	0	0.77	1.88	0	0.45850	4	$a_\pi = a_y = a_u = 0$, Quantity Equation (Friedman, 1970) with scaling, optimizing a_{pi}, a_r, a_m, a_v
7	25.68	11.58	0	0	0	0	1.55	0.45780	3	$a_\pi = a_y = a_m = a_v = 0$, Phillips (1958) curve with scaling, optimizing a_{pi}, a_r, a_u
8	27.44	15.18	5.55	1.88	0	0	0	0.83070	0	Taylor (1993) rule
9	27.44	15.18	5.55	3.76	0	0	0	0.81949	0	Taylor (1999) rule
Average 1-7								0.45094	0	Average of curves 1-7
Average 1-9								0.47122	0	Average of curves 1-9

(Friedman, 1970), and the Phillips (1958) curve with scaling, where $a_{pi}, a_r, a_\pi, a_y, a_m, a_v, a_u$ are optimized. With scaling no difference exists between the two Taylor (1993, 1999) rules, so we refer to the Taylor (1993) rule with scaling in general. That leads to the lowest sum of squares S=0.44567 in Table 1. The corresponding optimal weights are $a_{pi}, a_r, a_\pi, a_y, a_m, a_v, a_u$ 66.72, –11.44, –16.84, –3.13, 1.29, 2.73 and 2.06, respectively. This indicates that the inflation rate π_t with a weight 66.72 is very explanatory to the interest rate i_t. Thereafter, in degree of explanatory power, follows the deviation $\pi_t - \pi_t^*$ In inflation rate with a negative weight –16.84, the equilibrium real interest rate π_t^* with a negative weight –11.44, and the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP with a negative weight –3.13. The three new terms have lower weights. That is, the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity has weight 2.73, the deviation $\bar{u}_t - u_t$ in unemployment rate has weight 2.06, and the deviation $Log\left(\frac{m_t}{\bar{m}_t}\right)$ in money supply has weight 1.29.

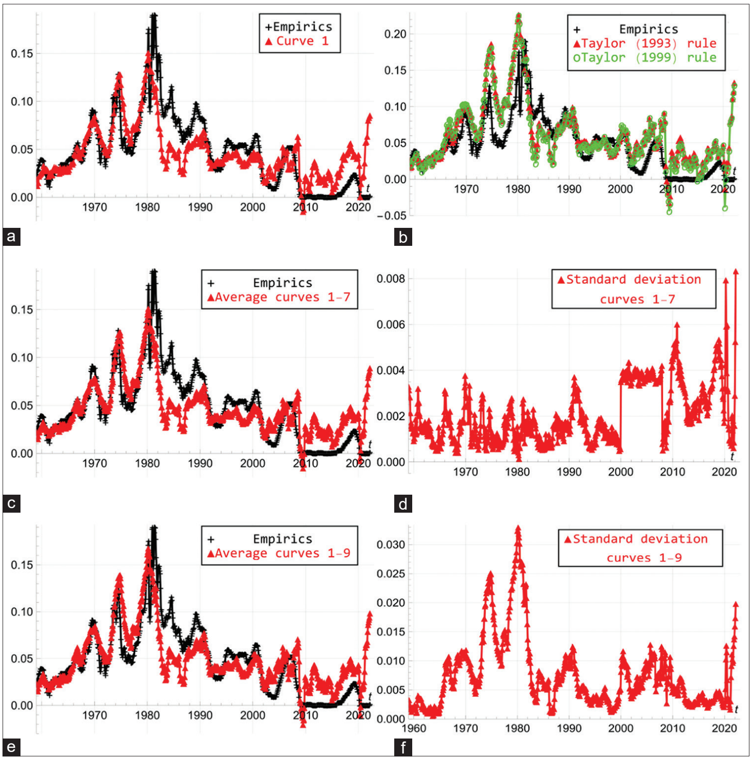
Curve 2 assumes six free choice variables, and represents the combination of the Taylor (1993) rule and the Quantity Equation (Friedman, 1970) with scaling, where $a_{pi}, a_r, a_\pi, a_y, a_m, a_v$ are optimized assuming $a_u = 0$. That leads to a slightly higher sum of squares S=0.45341 compared to curve 1 in Table 1. The corresponding optimal weights $a_{pi}, a_r, a_\pi, a_y, a_m, a_v$ are 61.94, –9.00, –14.75, –1.12, 1.25 and 2.98, respectively. Again, the inflation rate π_t has the highest weight 61.94 compared to the other five terms.

Curve 3 assumes five free choice variables, and represents the combination of the Taylor (1993) rule and the Phillips (1958) curve with scaling, where $a_{pi}, a_r, a_y, a_\pi, a_u$ are optimized assuming a_m=a_v=0. That leads to a sum of squares S=0.45157. The corresponding optimal weights $a_{pi}, a_r, a_y, a_\pi, a_u$ are 71.23, –12.69, –18.61, –0.73 and 2.19, respectively. Under the assumption a_m=a_v=0, the optimal weight assigned to the inflation rate π_t increases from 66.72 in curve 1 to 71.23 in curve 3. Meanwhile, the optimal weight assigned to the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP increases from –3.13 in curve 1 to –0.73 in curve 3.

Curve 4 assumes five free choice variables, and represents the combination of the Quantity Equation (Friedman, 1970) and Phillips (1958) curve with scaling, where $a_{pi}, a_r, a_y, a_\pi, a_u$ are optimized assuming a_π=a_y=0. That causes a sum of squares S=0.45628. The corresponding optimal weights $a_{pi}, a_r, a_y, a_\pi, a_u$ are 25.72, 11.32, 0.41, 0.94 and 0.92, respectively. Notably, under the assumption a_π=a_y=0, the weights assigned to the remaining five terms are positive. The optimal weights in curve 4 are substantially lower compared to the absolute values of the optimal weights in curve 1.

Curve 5 assumes four free choice variables, and represents the Taylor (1993) rule with scaling, where a_{pi}, a_r, a_y, a_π are optimized assuming a_m=a_v=a_u=0. That causes a sum of squares S=0.46065. The corresponding optimal weights a_{pi}, a_r, a_π, a_y are 66.14, −10.02, −16.34 and 1.64, respectively. Curve 6 assumes four free choice variables, and represents the Quantity Equation (Friedman, 1970) with scaling, where a_{pi}, a_r, a_m, a_v are optimized

Figure 1: The monthly US January 1, 1959-March 31, 2022 empirical interest rate i_t and the interest rate i_t based on (1). Panel a: a_{pi}= 66.72, a_r = –11.44, a_π = –16.84, a_y = –3.13, a_m = 1.29, a_v = 2.73, a_u=2.06. Panel b: The Taylor (1993, 1999) rules. Panel c: The average of the curves 1-7. Panel d: The standard deviation of the curves 1-7. Panel e: The average of curves 1-9. Panel f: The standard deviation of the curves 1-9



assuming a_π=a_y=a_u=0. That causes a slightly lower sum of squares S=0.45850 compared to curve 5. The corresponding optimal weights a_{pi},a_r,a_m,a_v are 25.82, 10.77, 0.77 and 1.88, respectively. Curve 7 assumes three free choice variables, and represents the Phillips (1958) curve with scaling, where a_{pi},a_r,a_u are optimized assuming a_π=a_y=a_m=a_v=0. That causes an even lower sum of squares S=0.45780 compared to curves 5 and 6. The corresponding optimal weights a_{pi},a_r,a_u are 25.68, 11.58 and 1.55, respectively. The results show that the Phillips (1958) curve with scaling explains the interest rate i_t better than the Taylor (1993) rule with scaling and the Quantity Equation (Friedman, 1970) with scaling.

Curve 8 represents the Taylor (1993) rule, assuming $a_\pi s_\pi$=$a_y s_y$=0.5, $a_{pi} s_{pi}$=$a_r s_r$=1, a_m=a_v=a_u=0. That causes a sum of squares S=0.83077. Curve 9 represents the Taylor (1999) rule, assuming $a_\pi s_\pi$=$a_y s_y$=0.5, $a_{pi} s_{pi}$=$a_r s_r$=1, a_m=a_v=a_u=0. That causes a slightly lower sum of squares S=0.81953. The sum of squares S= 0.44567 in curve 1 is 46.35% and 45.62%, respectively, lower than the Taylor (1993) rule's S=0.83077, and the Taylor (1999) rule's S=0.81953. Hence curve 1 explains the interest rate i_t better than both Taylor (1993, 1999) rules.

"Curve average 1-7" shows the average of curves 1-7. The corresponding sum of squares is S=0.45094, i.e. a 45.72% decrease and a 44.97% decrease, respectively, compared with the Taylor (1993) rule and the Taylor (1999) rule. Finally, "Curve average 1-9" shows the average of curves 1-9. The corresponding sum of squares is S=0.47122, i.e. a 43.27% decrease and a 42.50% decrease, respectively, compared with the Taylor (1993, 1999) rules.

Overall, among the curves 1-7, the weight a_{pi} assigned to the inflation rate π_t the weight a_m assigned to the deviation $Log\left(\frac{m_t}{\bar{m}_t}\right)$ in money supply, the weight a_v assigned to the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity, and the weight a_u assigned to the deviation $\bar{u}_t - u_t$ in unemployment rate are always positive. That means that the inflation rate π_t, the deviation $Log\left(\frac{m_t}{\bar{m}_t}\right)$ in money supply, the deviation $Log\left(\frac{v_t}{\bar{v}_t}\right)$ in money velocity, and the deviation $\bar{u}_t - u_t$ in unemployment rate impact the interest rate i_t positively. Notably, the weight a_π assigned to the deviation π_t– π_t^* in inflation rate is always negative. The weight a_r assigned to the equilibrium real interest rate r_t^*, and the weight a_y assigned to the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP are predominantly negative. Hence the equilibrium real interest rate r_t^*, the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP may impact the interest rate i_t negatively. These findings differ from the common wisdom, and the Taylor (1993, 1999) rules, that the deviation $\pi_t - \pi_t^*$ in inflation rate and the deviation $Log\left(\frac{y_t}{\bar{y}_t}\right)$ in real GDP impact the interest rate i_t positively.

Figure 1, panel a plots the empirical interest rate i_t with black "+", and curve 9 for the interest rate i_t in (1) with red filled triangles according to Table 1. Panel b plots the Taylor (1993; 1999) rules. Panel c plots the average interest rate i_t of the curves 1-7. Panel d plots the standard deviation of the predicted interest rate i_t of the curves 1-7. Panel e plots the average interest rate i_t of the curves 1-9. Panel f plots the standard deviation of the predicted interest rate i_t of the curves 1-9.

Panel a, curve 1 assumes seven free choice variables, where $a_{pr}, a_r, a_\pi, a_y, a_m, a_v, a_u$ are optimized. It fits the empirical interest rate i_t better than the Taylor (1993, 1999) rules, and has the lowest sum of squares S=0.44567 in Table 1. The local maximum of curve 1 in 1974 is close to the empirical interest rate i_t. Curve 1 shows an especially high interest rate i_t in 1980. In addition, it predicts negative interest rate i_t from April, 2009 to October, 2009. Panel b shows the Taylor (1993; 1999) rules. Overall, the Taylor (1999) rule predicts marginally lower interest rate i_t compared with the Taylor (1993) rule after the maximum in 1980.

Panel c, curve "Average curves 1-7" shows the average interest rate of the curves 1-7. Overall, the predicted interest rate is lower than the empirical interest rate i_t, except after 2010. Furthermore, it predicts negative interest rate i_t from April, 2009 to September, 2009. Panel d, curve "Standard deviation curves 1-7" shows the standard deviation of the interest rate i_t of the curves 1-7. In general, the standard deviation of the 1-7 curves is quite low. It shows moderately high values in 2010, 2020 and 2022.

Panel e, curve "Average curves 1-9" shows the average interest rate i_t of the curves 1-9. Panel f, curve "Standard deviation curves 1-9" shows the standard deviation of the interest rate i_t of the curves 1-9. Overall, the curve "Average curves 1-9" predicts a marginally higher interest rate i_t compared with the "Average curves 1-7". Similarly, the curve "Standard deviation curves 1-9" shows higher interest rate i_t compared to the "Standard deviation curves 1-7". This is because, overall, the Taylor (1993, 1999) rules predict higher interest rate i_t compared with curves 1-7.

4. CONCLUSION

The article establishes a generalized interest rates model by generalizing the Taylor (1993) rule from four terms to seven terms, and scaling the terms relative to each other. First, the article introduces three additional terms, i.e. the deviation in money supply, the deviation in money velocity, and the deviation in unemployment rate, which accounts for the money supply, the money velocity, and the unemployment rate, respectively. Second, the article investigates the seven combinations of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve, allowing the presence of one rule, two rules, or all three rules. Third, the article innovatively explores the scaling issue within the seven terms, i.e. the inflation rate, the equilibrium real interest rate, the deviation in inflation rate, the deviation in real GDP (Gross Domestic Product), the deviation in money supply, the deviation in money velocity, and the deviation in unemployment rate. To our best knowledge, the article investigates the scaling issue for the first time related to the Taylor (1993) rule's framework. The optimal seven weights are estimated and tested through the monthly January 1, 1959-March 31, 2022 US data. First, the two Taylor (1993, 1999) rules are evaluated against the empirics. Second, the seven combinations of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve with scaling are explored and tested.

The findings show that, first, all the seven combinations of the Taylor (1993) rule, the Quantity Equation (Friedman, 1970), and the Phillips (1958) curve with scaling give substantially better results than both the Taylor (1993, 1999) rules without scaling. The second best combination is the Taylor (1993) rule and the Phillips (1958) curve with scaling. Third best is the combination of the Taylor (1993) rule and the Quantity Equation (Friedman, 1970) with scaling. Second, when choosing only one rule with scaling, the Phillips (1958) curve is the best, followed by the Quantity Equation (Friedman, 1970), and finally the Taylor (1993, 1999) rules. Third, when choosing between two combinations with scaling, the Taylor (1993) rule and the Phillips (1958) curve is the best, followed by the Taylor (1993) rule and the Quantity Equation (Friedman, 1970), and finally, the Quantity Equation (Friedman, 1970) and the Phillips (1958) curve.

Among the seven terms, the most explanatory term to the interest rate is the inflation rate. The weights assigned to the inflation rate are always positive. Thus, it impacts the interest rate positively. The second explanatory term is the deviation in inflation rate, and the equilibrium real interest rate. Notably, the deviation in the inflation rate impacts the interest rate negatively. The weights assigned to the equilibrium real interest rate are predominantly negative. Thereafter, with decreasing degrees of negativity, followed by the deviation in real GDP, the deviation in money velocity, the deviation in unemployment rate, and the deviation in money supply. Thus, the money velocity is more explanatory for the interest rate than the money supply. The weights assigned to the deviation in real GDP are also predominantly negative. The deviation in money velocity, the deviation in unemployment rate, and the deviation in money supply impact the interest rate positively.

Future research may compare the empirics for different geographical regions, and incorporate the monetary policy changes over different time periods. Further possibilities are to account for the uncertainty and variation of the potential real GDP, the real equilibrium interest rate, and the natural unemployment rate. Alternative methods may be assessed to better estimate these three terms. Future research may also investigate the interest rate by incorporating time series approaches, or intruding broader financial theories.

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APPENDIX A

Nomenclature

Parameters

a_{pi} Weight assigned to the inflation rate, $-\infty \leq a_{pi} \leq \infty$
a_r Weight assigned to the equilibrium real interest rate, $-\infty \leq a_r \leq \infty$
a_π Weight assigned to the deviation in inflation rate, $-\infty \leq a_p \leq \infty$
a_y Weight assigned to the deviation in real GDP, $-\infty \leq a_y \leq \infty$
a_m Weight assigned to the deviation in money supply, $-\infty \leq a_m \leq \infty$
a_v Weight assigned to the deviation in money velocity, $-\infty \leq a_v \leq \infty$
a_u Weight assigned to the deviation in unemployment rate, $-\infty \leq a_u \leq \infty$
s_{pi} Scaling parameter for the inflation rate, $s_{pi} > 0$
s_r Scaling parameter for the equilibrium real interest rate, $s_r > 0$
s_π Scaling parameter for the deviation in inflation rate, $s_\pi > 0$
s_y Scaling parameter for the deviation in real GDP, $s_y > 0$
s_m Scaling parameter for the deviation in money supply, $s_m > 0$
s_v Scaling parameter for the deviation in money velocity, $s_v > 0$
s_u Scaling parameter for the deviation in unemployment rate, $s_u > 0$

Variables

i_t Interest rate at time t, $i_t \in R$
π_t Inflation rate, $\pi_t \in R$
π_t^* Target inflation rate, $\pi_t^* \in R$
r_t^* Equilibrium real interest rate, $r_t^* \in R$
y_t Real GDP (Gross Domestic Product), $y_t \geq 0$
Real potential GDP, $\bar{y}_t \geq 0$
m_t Money supply at time t, $m_t > 0$
u_t Unemployment rate, $u_t \geq 0$
$\bar{u}_t$ Natural rate of unemployment, $\bar{u}_t \geq 0$
t Time, $t \geq 0$