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Value at Risk Analysis on Equity Portfolios by Means of Random Orthogonal Matrix Simulation

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Abstract

Value at Risk analysis is a widespread measure in the banking sector and other financial institutions for controlling risk exposure in the market. The historic simulation approach is the market leader in the banking sector, because of the simplicity of the model and the accuracy. Although there exist many more sophisticated models for estimating Value at Risk, most of them fail appropriate test criteria. When even the historical simulation approach fails the criteria, no better alternatives existed, until just recently.

In this thesis we investigate and seek to validate this new Value at Risk simulation approach, named ROM simulation. We differ from 2 models and they are termed historical ROM and deterministic ROM, and we test whether or not they pass the test given by Christoffersen (1998). Furthermore we test for different significance levels and on different portfolios. We also include the standard historical simulation which serves as a benchmark. The result from the backtesting favors the deterministic method over both the historical ROM and the historical simulation. The deterministic approach also proves to be more flexible than the other examined models.

Preface

This thesis marks the end of a 5 year study in industrial economics at University of Stavanger. This master program is divided into a bachelor's degree in science followed by 2 years of economics, contract management, and risk management, resulting in a Master of Science degree.

My genuine interest in financial markets and macro economics combined with relevant skills obtained throughout the master program made it easy to accept this master thesis scope proposed by Ph.D Roy Endre Dahl. In addition, the current relevancy to the real world applications of this work where inspiring.

Indeed, Roy Endre Dahl has provided good help and insights to the banking sector, in what seems to be a tight schedule on his side, therefore a thank you is appropriate.

I would also like to express my gratitude to Daniel Ledermann and Carol Alexander for providing clarifications when asked for.

Varhaug, 24 May 2013

Per Kristian Skretting

Contents

1. Introduction

In the aftermath of the financial crisis led by the bankruptcy of Lehman brothers in 2008, there has been much focus on the banks responsibility to control its risk, thereby securing financial stability in the banking sector and thus the general market. In fact, Carol Alexander states that

"The main factor underlying this (2008) financial crisis is the intrinsic instability in the banking system resulting from the lack of unified and intelligent principles for the accounting, regulation, and risk management of financial institutions." (Preface xxxi Market risk analysis vol.IV)

The Basel-committee provides guidelines and regulations for the banks to promote risk control, and the latest edition, Basel III, are at the time of writing being implemented in Norwegian and EU-banks. The most widespread measure for banks to control risk is simulating scenarios under which Value at Risk (VaR) is being calculated. The Basel accord gives some room for the institutions to predict the future in their own way. However, in a survey of large commercial banks by Perignon and Smith (2010), 73% of the corresponding banks replied that they used historical simulation to risk factor returns, 22% used Monte Carlo simulation (MC), and the remainder used some hybrid simulation method for computing the VaR for market risk capital requirements.

In 2009 Ledermann et al. introduced a new simulation methodology called Random Orthogonal Matrix (ROM) simulation, which is much faster than conventional MC simulation, and which applies directly to an historical sample or a parametric distribution. Each simulation matches the first four multivariate sample moments, to those of the observed samples or of the target distribution, thus simulated samples will be more realistic when applied on financial markets, compared to MC simulation which assumes a normal distribution and therefore neglects the highly leptokurtic nature of financial markets.

This paper aims to explore and validate this new simulation method on several equity portfolios and backtests are performed using Christoffersens (1998) criteria. The paper proceeds as follows: first off a general introduction to financial markets and specifically their distributions which are relevant for the work. Then a brief overview of VaR models is provided, before ROM simulation and ROM VaR is presented. Also a description of Christoffersens test criteria is presented before we finally lay out the empirical results. All calculations are performed in matlab and m-files can be found in the cd corresponding to this work.

In order to obtain results that is significant and not subject to errors from a statistical view we collected daily prices spanning over 2 decades, in detail 01.03.1993-01.03.2013, a total of 5220 data samples. We chose to evaluate 3 different portfolios with each consisting of similar assets; the first portfolio is a strictly indices portfolio consisting of 30 Morgan Stanley Capital International (MSCI) indices, secondly a commodities portfolio consisting of 12 raw materials like gold, wheat and oil etc, and lastly a currency portfolio consisting of 8 randomly taken forex pairs. In addition, a portfolio made up from merging the 3 portfolios was examined.

2. Characteristics of financial markets

2.1 Some stylized facts

In the context of financial time series data, and specifically daily data series, there are empirical observations and inferences drawn from these observations that have held so long that it has been given the status of facts, Embrechts et al. (2005). These stylized facts apply to the majority of daily series of risk factor changes e.g. commodity prices and indexes. Furthermore these stylized facts also, in some cases, continue to hold for other time series, say intraday or weekly time series. Let us for the sake of illustration consider 22 years of daily returns series of the BXLT index, shown in figure 1.[1]

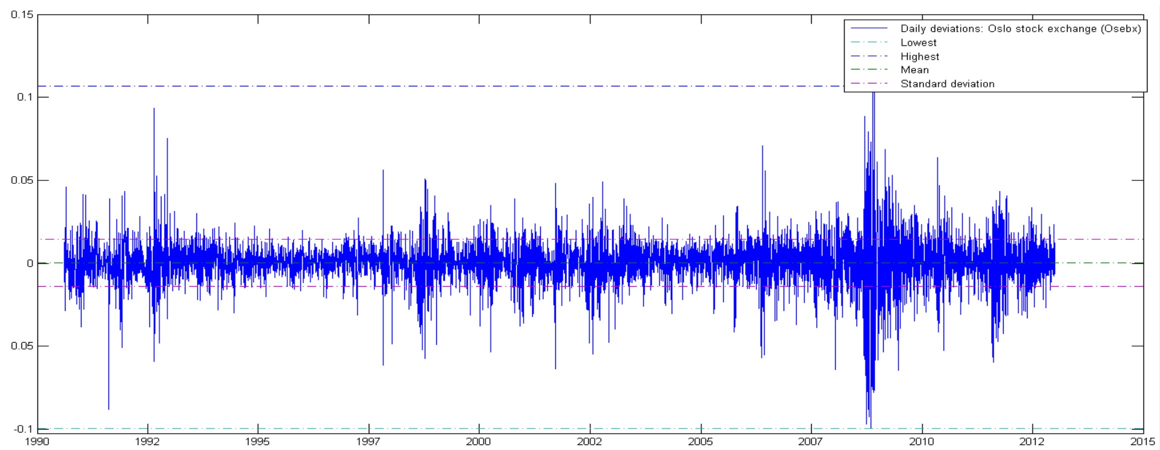

Figure 1: Return series for BXLT

Return series appear to vary over time. One may argue that the reason for varying returns is that much of what controls the market is psychology driven, i.e. fear and greed, and the resulting action by the investors due to world events. For instance consider the tsunamis impact on the nuclear power plant at Okuma Fukushima Japan Friday 11.March 2011. After the initial shock which pushed prices down sharply, the market reacted on the increasing uncertainty of the outcome of the situation. The

[1] BXLT index is an extension of the OSEBX (Oslo Stock Exchange Benchmark Index).

fifteenth of March we witnessed the third largest decline in history of the TOPIX (Tokyo Price Stock Price Index) yielding more than negative 9%. What may not seem based on rationale, perhaps, is that the BXLT also were affected, the whole world for that matter. On the fifteenth we experienced a drop of -1.8%, and yet just two days later a positive return of 2.3%. Although there is not statistical evidence, when considering this event isolated, to claim that the Oslo stock exchange was influenced by the tsunami, it does deserve some attention. It is widely accepted that indexes have some correlations.

The volatility seen in the subsequent week following 11.March is also an example of a stylized fact; extreme returns are oftentimes followed by several other extreme returns, though not necessarily with the same sign. This is known as *volatility clustering.* In figure 1, we clearly see various times where clustering is present, in particular the clustering in volatility in the period after the recent financial crisis is present.

In both figure 1 and 2 we see that the *conditional expected return are close to zero.* For this particular example the mean turns out to be 0.024%

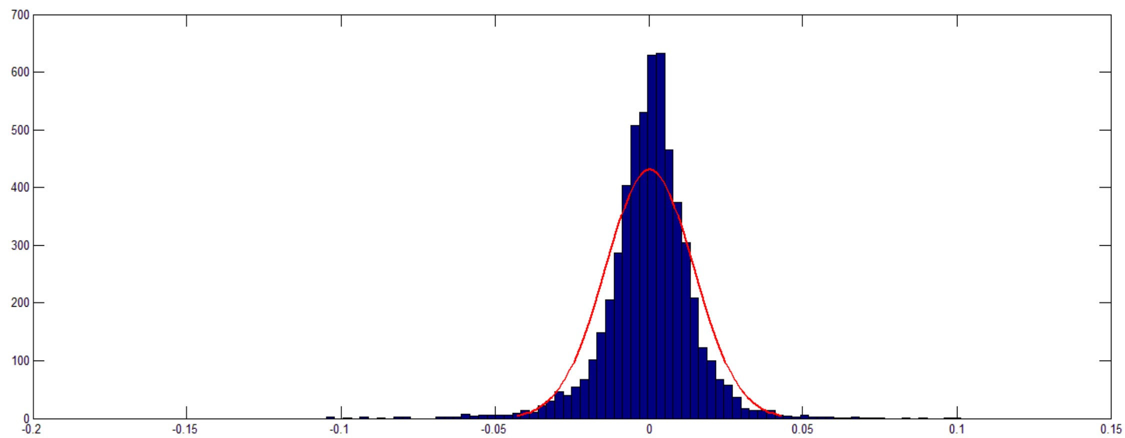

Figure 2: Histogram of BXLT

Furthermore, by the nature of figure 2, we state that also the fact *return series are leptokurtic or heavy-tailed,* holds. This represents a distribution that is narrower around the mean but has longer and heavier tails with respect to the normal distribution. This fact is important in financial engineering because simplifications to the Gaussian curve will yield too naïve results.

Without further due, the last important stylized facts are:

- *Return series are not iid. although they show little serial correlation.*
- *Series of absolute or squared returns show profound serial correlation.*

Now, in financial risk management we are most of the time more interested in multiple series of risk factor changes, rather than a single financial time series. And in a similar manner there exist stylized facts for multivariate time series, they are:

- *Multivariate return series show little evidence of cross correlation, except for contemporaneous returns.*
- *Multivariate series of absolute returns show great evidence of cross correlation.*
- *Correlations between series vary over time.*
- *If there is an extreme return in one series, it often coincides with extreme returns with several other series. As per our TOPIX-BXLT example above.*

2.2 Higher Moments

A basic goal of statistics is to organize and summarize data, and the mean and variance are introduced as summary measures of the location and variability of a distribution. However, such measures do not say anything about the shape of a distribution and in order to better describe the personality, if you will, of a distribution, skewness and kurtosis are introduced. The mean, variance, skewness and kurtosis are respectively given by:

$$\mu = E[X] \tag{1}$$

$$\sigma^2 = E[(X-\mu)^2)] \tag{2}$$

$$\tau = \frac{E[(X-\mu)^3)]}{\sigma^3} \tag{3}$$

$$\kappa = \frac{E[(X-\mu)^4)]}{\sigma^4} \tag{4}$$

The 3rd equation yields the skewness for the distribution. If the bulk of the data is at the left and the right tail is longer, with reference to the Gaussian curve, we say that the distribution is skewed right or positively skewed i.e $\tau > 0$. If the peak is toward the right and the left tail is longer, we say that the distribution is skewed left or negatively skewed.

By definition, the bell curve has a kurtosis of 3. A value above this number means that we have excess kurtosis implying that the curve has a higher peak and heavier tails than the normal distribution and the series are said to be leptokurtic. If the kurtosis is less than 3 then we have the more uncommon platykurtic series.

Now, higher moments are perhaps most applied when dealing with univariate distributions, that is, only one time series. Suppose that we have a portfolio of p assets and we wish to know whether or not we have multivariate normality. One such test was derived by Mardia (1970). He suggest that we may test for normality, given large samples, by testing the statistics $\beta_{1,p} = 0$ and $\beta_{2,p} = p(p+2)$, separately. The betas represent respectively skewness and kurtosis.

In the following, let $x_1, \ldots, x_n$ be a random sample from a p-variate population with random vector X having mean vector μ and covariance matrix S.

To test for $\beta_{1,p} = 0$, Mardia reports that;

$$A = \frac{1}{6} n b_{1,p} \tag{5}$$

has a χ^2distribution with $\frac{p(p+1)(p+2)}{6}$ degrees of freedom.

And equally, to test $\beta_{2,p} = p(p+2)$ he utilizes that;

$$B = \frac{b_{2,p} - \beta_{2,p}}{\left(\frac{8p(p+2)}{n}\right)^{\frac{1}{2}}} \tag{6}$$

is distributed as N(0,1).

The statistics $b_{1,p}$ and $b_{2,p}$ is given by:[2]

$$b_{1,p} = ave_{i,j}\left\{(x_i - \mu)^T S^{-1}\left(x_j - \mu\right)\right\}^3 \tag{7}$$

$$b_{2,p} = ave_i\left\{(x_i - \mu)^T S^{-1}\left(x_j - \mu\right)\right\}^2 \tag{8}$$

Which is invariant under affine transformations.

[2] In the univariate case, these statistics reduce to the usual univariate skewness and kurtosis

Let's illustrate with an example: consider a portfolio of 9 risky assets and n=5624 daily returns, where the statistics $b_{1,p}$ and $b_{2,p}$ are calculated to be 5.5 and 276.2. With these data as input, we obtain that A=5183.5 and B=472.3

We reject the null hypothesis that the distribution is multivariate normal if the test statistics A and B are significant. The value of χ^2 at 1% significance is 210.2 and since the value of A=5183.5 is greater we conclude that $b_{1,p}$ are significant. Likewise, since B=472.3 is greater than 2.57[3], we conclude that $b_{2,p}$ is significant. Therefore we reject the null hypothesis, and state that the portfolio is not multivariate normal at 1% significance.

As a minor note on Mardia's higher moments, it should be pointed out that comparisons between different portfolios multivariate skewness and kurtosis are not straightforward, as it is in the univariate case, because these measures depend on both the length of the matrix and more importantly the number of assets. Therefore, two portfolios that make up a pretty similar return time series may show completely different values of multivariate higher moments as defined by Mardia.

Figure 3 shows the evolution of skewness and kurtios on a portfolio in the time frame 1995 to 2013 where Mardia's higher moments are calculated from 500 data points of daily returns, using a rolling window approach, and we clearly see the correlation between the two measures. In fact, this particular series of skewness and kurtosis is 95.2% correlated. We take advantage of this correlation feature later in the paper.

[3] In the case of 5% significance; 1.96 applies.

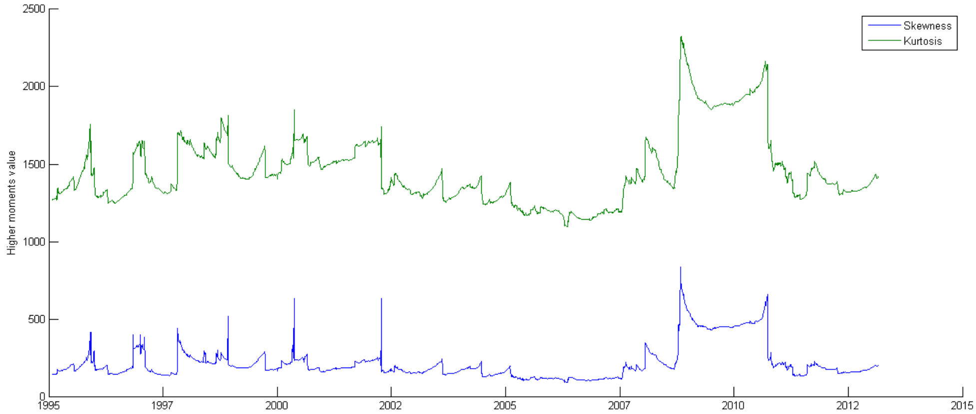

Figure 3: The evolution of skewness and kurtosis using a rolling window of 500 samples

We also see the impact of the recent financial crisis where kurtosis jumped by 60% from its mean, 1450, taken on the overall sample, to over 2300. This supports the idea that simplifications to the normal curve in financial markets are not adequate, and hence should not be utilized for estimating VaR.

3. Value at Risk models

Value at risk (VaR) is a well-known and widely used method for estimating risk in portfolios. To explain VaR, consider a portfolio of risky assets and that the portfolio has a known distribution of profit and losses (P&L). Assuming that the portfolio is left unmanaged over the risk horizon, the VaR of the portfolio at the confidence level α is given by the smallest number l such that the probability that the loss L exceeds l is no larger than $(1 - \alpha)$, Embrechts et al. (2005). More formally this may be written as:

$$VaR = (\{l \in \mathbb{R}: P(L > l) \leq 1 - \alpha\}) \qquad (9)$$

Where L is the realized loss. A vast amount of VaR models exist, but in general it is common to distinguish between 3, and we will elaborate on these in the following subsections.

3.1 Normal linear VaR

In the construction of the parametric VaR model, it is required that the returns or P&L is a linear function of its risk factors returns, Alexander (2008). Because of this, the normal linear VaR are also known as analytic VaR even though analytic expressions exist for some non-linear portfolios as well. The most basic assumption in the normal linear VaR model is that P&L are normally distributed, and that the joint distribution is multivariate normal. The covariance matrix of the risk factor returns are thus all that is required to capture the dependencies between the asset returns.

Suppose now that we seek a measure of a portfolio VaR, without attributing the VaR to different risk factors. Let X denote the returns and assume that X are i.i.d such that

$$X \sim N(\mu, \sigma^2) \qquad (10)$$

We wish to derive a formula for the α quantile return, x_α, i.e the return such that $P(X < x_\alpha) = \alpha$. The $100\%\alpha$ VaR are then, expressed as a percentage of portfolio value, given by minus this α quantile.

By standard normal transformations we have

$$P(X < x_\alpha) = P\left(\frac{X-\mu}{\sigma} < \frac{x_\alpha - \mu}{\sigma}\right) = P\left(Z < \frac{x_\alpha - \mu}{\sigma}\right) \tag{11}$$

Where $Z \sim N(0,1)$. Since $P(X < x_\alpha) = \alpha$, then

$$P\left(Z < \frac{x_\alpha - \mu}{\sigma}\right) = \alpha \tag{12}$$

By definition, $P\big(Z < \phi^{-1}(\alpha)\big) = \alpha$, so that

$$\frac{x_\alpha - \mu}{\sigma} = \phi^{-1}(\alpha) \tag{13}$$

Where ϕ is the standard normal distribution function. Now, remember that $x_\alpha = -VaR_\alpha$ and that $\phi^{-1}(\alpha) = -\phi^{-1}(1-\alpha)$ by the symmetry of the standard normal distribution. If we substitute these into eq.9 we obtain an analytic solution for the VaR for a portfolio with an i.i.d normal return. We get:

$$VaR_\alpha = \phi^{-1}(1-\alpha)\sigma - \mu \tag{14}$$

Finally, taking into consideration the time horizon of the VaR estimate:

$$VaR_{h,\alpha} = \phi^{-1}(1-\alpha)\sigma_h - \mu_h \tag{15}$$

Equation 15 is a simple formula for the $100\alpha\%$ h-day VaR, as a percentage of the portfolio value. To get to monetary terms we simply multiply by the current value of the portfolio.

3.2 Monte Carlo simulation

A Monte Carlo (MC) VaR model, in its most basic forms, uses the same assumptions as the normal linear VaR model, that is; that the asset returns are i.i.d with a multivariate normal distribution, Alexander (2008). It also assumes that the covariance matrix is able to capture all possible dependencies between the asset returns. The greatest advantage of MC models is that they are extremely flexible. For instance, we could use a copula to model the dependence and specify any type of marginal risk factor return distributions that we like.

In the MC VaR model we simulate independent standard normal vectors, which are then transformed to correlated multivariate normal vectors using the Cholesky decomposition of the covariance matrix. Then the portfolio mapping is applied to each vector of simulated risk factor changes to obtain a simulated portfolio value at the end of the risk horizon. In order to reduce the sampling error it is necessary to generate very large numbers of simulations, say 10 000 simulations or more.

If we generate 10 000 simulations, then we have 10 000 simulated portfolio values at the risk horizon in h-days time, and thus also 10 000 simulated returns on the portfolio. These are expressed in value terms and then the $100\alpha\%$ h-day VaR is taken as minus the lower α quantile of the return distribution.

Indeed, the normal MC VaR and the normal linear VaR models are very similar, the reason being that they make the same fundamental assumptions about the risk factor distribution. In fact, it is a waste of time to apply normal MC simulation when there is an analytic solution available. Both the time consumption and the sampling error which are likely to arise, favors the normal linear. Nonetheless, there are reasons to apply MC VaR to a linear portfolio, and this is the fact that MC VaR can be based on virtually any multivariate distribution for risk factor returns, whereas closed-form solutions for parametric linear VaR only exist for a few select distributions.

3.3 Historical simulation

In historic VaR models, we simply assume that what happened in the past is due to happen again in the future. This builds up a very simple, yet as it turns out, often very good estimation to VaR.

Figure 4 shows a histogram of the daily P&L for a portfolio in the past 20 years. Now, for the purpose of VaR estimation, such a long time horizon might easily be argued to be too long for several good reasons. For one, there is hardly any credibility in assuming that one have held on to this particular portfolio for so long, which is one of the fundamental ideas behind the historic simulation approach. Another challenge is that as time goes by, new assets are born and some assets disappear. If the portfolio consists among other a newly listed stock, say, then it would be impossible to use a longer time period than the most recent stock. In general we often use a period of 2-4 years daily data, find the α% worst return, and then the one day ahead VaR estimate is this value.

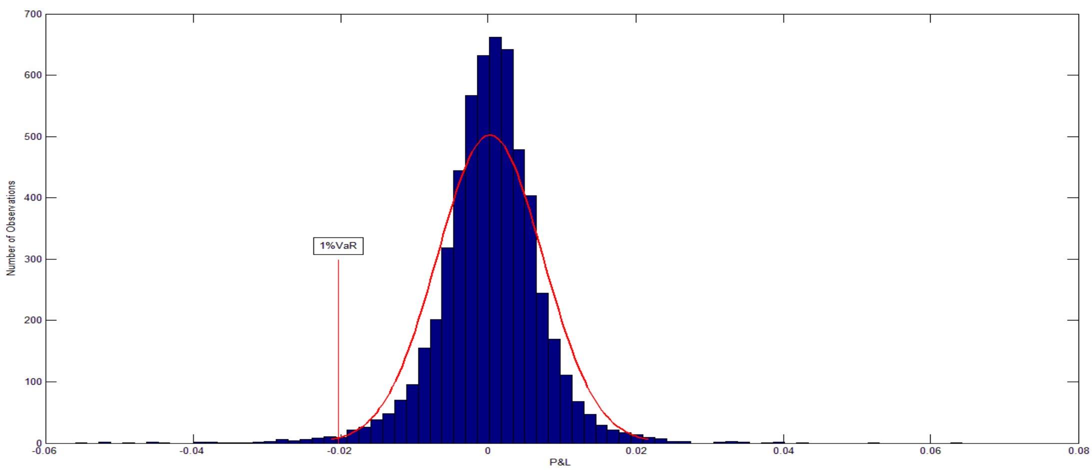

Figure 4: Historic P&L density showing 1%VaR

As for the regulatory purposes the Basel committee recommends a period of only 1 year as a minimum, Ledermann et al. (2012). Indeed, one might easily argue that such a time horizon is too short. However there are some reasons for choosing a shorter time horizon. If the purpose for estimating a reasonable and credible 1% VaR value for financial institutions, then although the historic simulation approach is said to be unaffected by individuals or the estimators beliefs, there must be a choosing of the time period which reflect the current market risk. For example, the novice might be tempted to use the period from today and back to the financial crash of 2008 to estimate a 1%VaR, which for the professional is ridiculous because he doesn't believe that the current market are so bad, and thus he avoids the pitfall of underestimating the VaR. This is overcome by simply specifying for instance 2- 4 years data are to be used for historical simulation estimations.

4. ROM simulation

This section provides an overview of the new simulation method, namely simulation via Random Orthogonal Matrices (ROM). For extensive proof, the reader is referred to Ledermann et al. (2009)(2011)(2012).

4.1 Overview

In the following discussion, let X_{mn} be a random sample of size m with n dimensions. ROM is most easily understood by considering how to adjust this random sample so that the sample mean vector and sample covariance matrix exactly matches some target mean vector μ_n and covariance matrix S_n. The latter matrix must be positive semi-definitive and hence we may always obtain a decomposition of the form $S_n = A'_n A_n$. We may transform X_{mn} in the following way:

$$L_{mn} = m^{-\frac{1}{2}}(X_{mn} - 1_m \mu'_n)A_n^{-1} \qquad (16)$$

The mean of X_{mn} is μ_n and its covariance matrix is S_n, if and only if:

$$L'_{nm} L_{mn} = I_n \quad with \quad 1'_m L_{mn} = 0'_n \qquad (17)$$

Any $m \times n$ matrix satisfying the condition (17), is called an L-matrix. There exist three distinct classes of L-matrices. Ledermann et al (2009) introduced these classes as parametric, deterministic and data-spesific.

By solving for X_{mn} in (16) we obtain:

$$X_{mn} = 1_m\mu_n' + \sqrt{m}L_{mn}A_n \qquad (18)^4$$

In a historical simulation scenario, the L matrix is made up by transforming a historical sample with m observations and n dimensions through the Gram-Schmidt (GS) process, i.e orthonormalizing the vectors.

One of the most crucial and fundamental idea behind ROM simulation is the introduction of random matrices. This follows from the fact that if L_{mn} satisfies (17), then so will $Q_mL_{mn}R_n$ where Q_m is a $m \times m$ random permutation matrix and R_n is a general $n \times n$ random orthogonal matrix. Introducing these random elements in (18) lets us simulate new samples via:

$$X_{mn} = 1_m\mu_n' + \sqrt{m}Q_mL_{mn}R_nA_n \qquad (19)$$

Where A_n may be the cholesky decomposition of the covariance matrix S_n.[5]

ROM simulation preserves the multivariate skewness and kurtosis as defined by Mardia (1970). However, the marginal distributions skewness and kurtosis are changed. Ledermann et al. (2012) explain how we can alter the properties by selecting different matrices for R_n. For example; random Cayley and random exponential matrices induce positive skew when upper Hessenberg is negative, and conversely. They show that

[4] The matrix notation used in this paper are: A_{mn} , a matrix A with dimensions $m \times n$. Where there is only one subscript it means that the matrix is quadratic, i.e $B_n = B_{nn}$.
[5] We could also set $S_n=Q_nB_nQ_n'$ where B_n is the diagonal matrix of eigenvalues, and Q_n is the orthogonal matrix of eigenvectors of S_n, so that $A_n=B_n^{1/2}Q_n'$

exponential matrices reduce the central mass of the marginals, relative to that of the basic L matrix, and upper Hessenberg and Cayley matrices tend to increase the central mass.

A permutation matrix is a square matrix, Q_m, which has exactly 1 entry in every row and every column, zeros elsewhere. Multiplying any m×n matrix L_{mn} with Q_m shuffles the rows of L_{mn} , e.g.:

$$Q_m L_{mn} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A & D \\ B & E \\ C & F \end{pmatrix} = \begin{pmatrix} B & E \\ C & F \\ A & D \end{pmatrix} \quad (20)$$

There are also some options when considering the permutation matrix Q_m. If the permutation matrix is completely arbitrary and random, then any clustering from the original sample, say the increase in volatility experienced in the financial market during the 2008 crisis, would be spread out, and thus it is highly unlikely we obtain new samples containing these clustering features. However, by choosing a cyclic random permutation we may alter the timing of clusters, if desirable.

4.2 Historical ROM simulation

At first glance, one might wonder on what is really the advantage of generating many new samples on a portfolio P&L. Indeed, no new information is provided, in the sense that the multivariate moments are they same. Moreover, a VaR estimate obtained from the historical simulation and the corresponding historical ROM simulation should yield approximately the same results. A care for the value selection on the percentile estimate should be made here. On the historical simulation, we find the 2 nearest points yielding a α% percentile of a distribution and then interpolate between the two. We do the exact same procedure on the ROM simulated samples, however, after simulating many more new samples, the distribution will be smoother and thus the interpolation error are less. From this we reason that a ROM simulated historical approach should yield more accurate VaR estimates, especially for lower quantiles. Furthermore, although Mardias measures are the same, different random matrices, R_n, produces different marginals, and hence different VaR estimates are produced. In this lie several possibilities to obtain good results in terms of a backtest.

4.3 Deterministic ROM simulation

Stressed VaR is typically computed from an historical sample using the Duffie-Pan methodology, where the sample is transformed to have a stressed covariance matrix taken on a shorter period, in particular, a period where the market were tranquil. The Duffie-Pan approach to obtain stressed samples $\tilde{X}_{mn}$ may be written as:

$$\tilde{X}_{mn} = 1_m \mu_n' + (X_{mn} - 1_m \mu_n') A_n^{-1} \tilde{A}_n \qquad (21)$$

With $\tilde{A}'_n \tilde{A}_n = \tilde{S}_n \ and \ A_n' A_n = S_n$, where $S_n$is the covariance matrix and $\mu_n$is the mean vector of X_{mn}.

Since it is widely accepted that the higher moments increases in times of crisis, and hence should be taken into account, Ledermann et al. (2012) provides the deterministic ROM framework that lets us stress, not merely the covariance matrix, but the higher moments as well. Although we have the option to stress both skewness and kurtosis, it is recognized that the two measures are highly dependent on each other, see figure 3, therefore, targeting either one of these measures should capture the relevant stress characteristic. We now implement a deterministic ROM approach which can be used to stress test a portfolio with the targeting of a stressed kurtosis.

In the context of historical ROM simulation, we first transform the historical sample X_{mn} into a data specific rectangular orthogonal matrix L_{mn}^D via the the Gram-Schmidt (GS) process, i.e $L_{mn}^D = GS(X_{mn})$. This matrix is then used to construct a stressed random sample by the transformation:

$$\tilde{X}_{mn} = 1_m \mu_n' + m^{\frac{1}{2}} L_{mn}^D R_n \tilde{A}_n \qquad (22)$$

Where $\tilde{A}_n$ is the cholesky decomposition of the stressed covariance matrix $\tilde{S}_n$. One feature which should be noted here is that when $R_n = I_n$, that is the random matrix is replaced by the identity matrix, then eq.22 reduces to the standard Duffie-Pan given in eq.21.

In order to increase kurtosis, Professor Walter Ledermann proposed the first solution to the exact covariance simulation problem, eq.17, and it is given by: $L_{mn} = (l_1, \ldots, l_n)$ where

$$l_j = \left[((m-n+j-1)(m-n+j)^{-\frac{1}{2}} * f_j\right]' \tag{23}$$

where

$$f_j = \begin{cases} 1, & for: m-n+j-1 \\ 0, & for:\ n-j \\ -(m-n+j-1), & else \end{cases}$$

For $1 \leq j \leq n$.

Expressed as a matrix it becomes:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{j(j+1)}} & \cdots & \cdots & \frac{1}{\sqrt{(m-1)m}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & & \vdots & & & \vdots \\ 0 & \frac{-2}{\sqrt{6}} & & \vdots & & & \vdots \\ \vdots & 0 & \ddots & \vdots & & & \vdots \\ \vdots & \vdots & \ddots & \frac{-j}{\sqrt{j(j+1)}} & & & \vdots \\ \vdots & \vdots & & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \frac{-(m-1)}{\sqrt{(m-1)m}} \end{pmatrix} \tag{24}$$

Source: Ledermann et al. 2009

By the introduction of a further parameter k, allows ROM simulation to target both multivariate skewness and kurtosis. Ledermann et. al (2012) introduce this parameter in three ways, thus defining type I, type II and type III L matrices. However, such considerations are beyond the scope of this paper, and since the multivariate skewness and kurtosis are assumed to be driven by a common factor, we assume that targeting either one of them should capture the relevant characteristics. Thus we focus on targeting multivariate kurtosis which is done by the use of the Ledermann matrix.

The Ledermann matrix given in equation 23 takes into account only the size of the matrix, meaning that, for instance, a 3×3 matrix will always be the same. Therefore we need to determine the length of the L_{mn}, since the portfolio asset column is typically fixed, in order to obtain a determined kurtosis, which is the objective. The length of the matrix is given by the following equation:

$$p^* = \frac{\left(2n + \tilde{k}\right) + \sqrt{\left(2n + \tilde{k}\right)^2 + 4mnr^{-1}\left(\tilde{k} - k\right)}}{2n} \qquad (25)$$

Where k=kurtosis and $\tilde{k}$ is kurtosis taken over a stressed period. We are only interested in the positive solution of this quadratic, and it should be pointed out that as long as $\tilde{k} > k$ eq.25 will be positive. The parameter r is an augmentation factor. By using the integer $p = int(p^*)$, in the following equation, we can force the distribution to a desired level of stressed kurtosis. The equation is:

$$\tilde{X}_{(m+rp)n} = \begin{pmatrix} m^{\frac{1}{2}} & L_{mn} & \\ p^{\frac{1}{2}} & L_{pn} & R_n^1 \\ \vdots & \vdots & \vdots \\ p^{\frac{1}{2}} & L_{pn} & R_n^r \end{pmatrix} \tilde{A}_n \qquad (26)$$

This particular equation provides a distribution that matches the stressed target kurtosis. For a better understanding, consider figure 5.

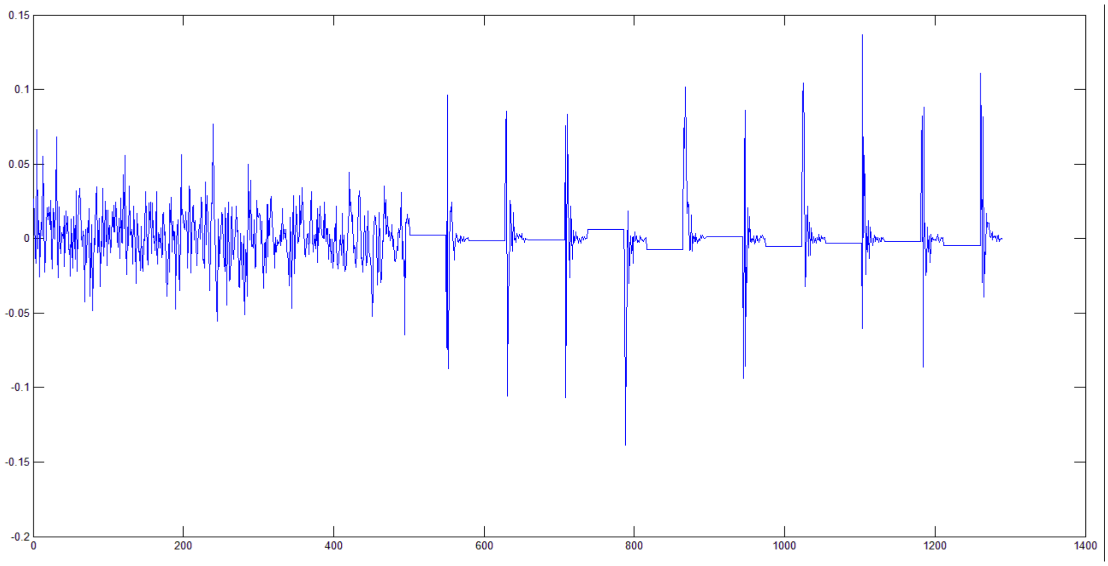

Figure 5: Stressed concatenation by using the Ledermann matrix

As an example, our objective here is to transform the initial 500 data, seen as the 500 first plot in the graph, into a stressed distribution so that the kurtosis is approximately equivalent to the kurtosis experienced in financial stress. As for this particular example, the kurtosis in the original sample is $k = 1273$. If we also include the Ledermann matrix, seen as the second part of the plot, from 500 to the end, then we find that $\tilde{k} = 1908$. The target kurtosis, kurtosis around the financial crisis, was 1902, indeed reasonably close.

A note on the parameter r is necessary. We are free to choose whatever augmentation factor r we like. The Ledermann matrix and the construction in equation 26, are designed such that for any r the target kurtosis are reached, however different VaR estimates are produced. To see this, consider two values for r; r=1 and r=10. In the case were r=1, the Ledermann matrix produces one high peak whereas for r=10, as in figure 5, 10 peaks are produced. Both distributions will approximately obtain the target level of kurtosis, $k = 1902$, but the α% taken are less extreme in the latter case since there are more ROM simulations relative to the historical. Hence the parameter r plays a minor but still a vital role in these types of simulation. This paper chose to fix the parameter to r=15, and the results in the empirical study are based on this. We have also incorporated a more detailed investigation of r in chapter 6.3.

5. Backtesting

5.1 In general

A backtest of a VaR model is of high interest to risk managers, as it reveals the accuracy of the model and is used for validation, Alexander (2008). The results of a backtest are dependent on the portfolio composition, as well as the evolution of the risk factors and the assumptions made about risk factor return distributions when building the model. Hence it is possible for a portfolio X to pass a backtest while portfolio Y fails, even though they consist of similar assets. A minor note here, and maybe an ironic one, is that when we make minimal assumptions about the distribution i.e historical simulation, it often yields the best results in terms of a backtest. In order to ensure that a backtest will have the power to reject poor models, a large collection of data must be used, say, 10 years or more. It is customary to use daily data, and the more data available, the more significant the results will be. Historical simulation tends to need more data than the parametric linear VaR and the MC VaR.

To backtest a VaR model, we employ a rolling window approach. The rolling window may consist of 500 daily data, termed in-sample, and in particular the first 1day ahead VaR estimate will be based on the 1% percentile of the first 500 data, thus leaving behind the first 500 data as background material. Next we roll the in-sample by one day such that the 2nd VaR estimate is based on the data 2-501 and we continue this way until all data are exhausted. By now we have obtained a series of VaR estimates and we have the corresponding return series, and we now check whether or not the model pass the test by observing the number of violations.

Figure 6 shows the results of the return series and the VaR estimates for the historical simulation. If we have 4700 comparable observations in the two series, we expect with a 1% VaR that there will be approximately 47 violations or exceedances. Exceedances occur when the portfolio loses more than the VaR that was predicted at the start of the risk horizon. In figure 6, this is when the blue line crosses the black line.

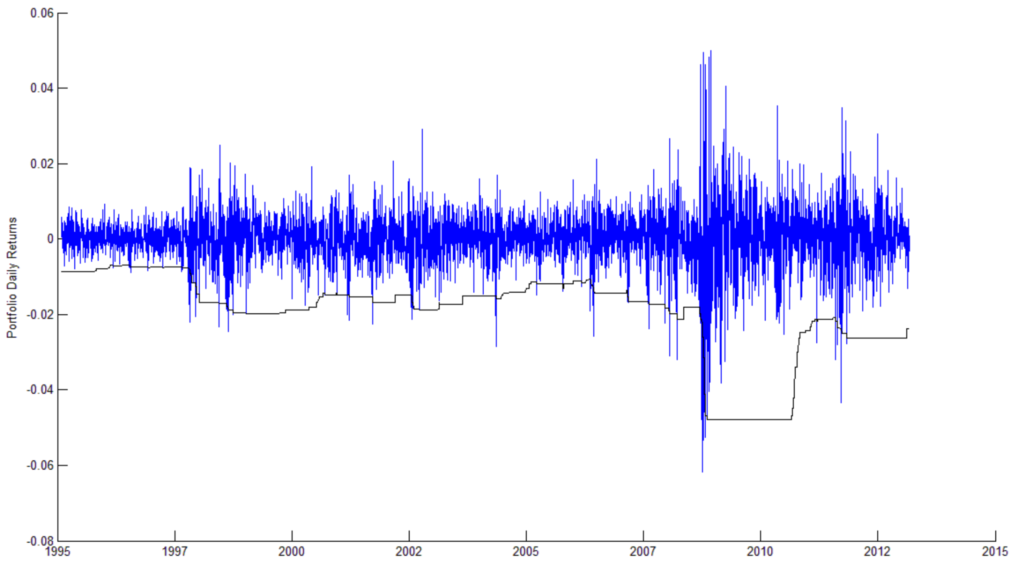

Figure 6: Illustration of historical backtesting of 4700 (18 years) daily returns.

Most backtests based on daily VaR are based on the assumption that the P&L are generated by an i.i.d Bernoulli process. A Bernoulli variable may only take two values, 0 or 1, and we denote the event that P&L loses more than VaR as success and therefore giving it the value of 1. Thus we may define an indicator function $I_{\alpha,t}$, on the time series of daily P&L relative to the 100α% daily VaR by:

$$I_{\alpha,t+1} = \begin{cases} 1, & if\ Y_{t+1} < -VaR_{1,\alpha,t} \\ 0, & Otherwise \end{cases} \qquad (27)$$

Where Y_{t+1} is the hypothetical realized return, so that the VaR estimate made at time t corresponds to a realized return at time t+1.

The probability for success at any time t is α if the model is accurate and $I_{\alpha,t}$ follows a Bernoulli process, thus the expected number of successes in a test sample with n observations is $n\alpha$.

5.2 Christoffersens test

Christoffersen (1998) formalized and generalized tests on the independence of exceedances, i.e the exceedances arrive in clusters, and the conditional coverage test which combine the unconditional coverage test and the independence test. That is

$$LR_{cc} = LR_{uc} + LR_{ind} \qquad (28)$$

The test statistic for an unconditional coverage test is given by:

$$LR_{uc} = \frac{\pi_{exp}^{n_1}\left(1-\pi_{exp}\right)^{n_0}}{\pi_{obs}^{n_1}(1-\pi_{obs})^{n_0}} \qquad (29)$$

And it is a test of the null hypothesis that the indicator function (27) has a constant success probability equal to the significance level of the VaR. Where:

- π_{exp}: the expected proportion of exceedances.
- π_{obs}: the observed proportion of exceedances.
- n_1: the observed number of exceedances.
- $n_0 = n - n_1$: where n is the sample size of the backtest.

So n_0 is the number of returns with indicator 0, i.e non-violations. We see that $\pi_{exp}=\alpha$ and that $\pi_{obs}=n_1/n$.

The asymptotic distribution of $-2Ln\ LR_{uc}$ is chi-squared with one degree of freedom.[6] The 1% critical value of a chi-squared distribution with one degree of freedom is 6.6349,

[6] The degrees of freedom is dependent on the possible outcome. In general the degrees of freedom is equal to the amount outcomes less 1.

implying that we reject the null hypothesis, that the VaR model is accurate, if we obtain a number greater than 6.6349.

Now, Christoffersen goes further and develops an independence test designed to see beyond the simple unconditional test, because he realizes that the probability of having a violation tomorrow, given that there was a violation today, is no longer α. In addition to the bulleted list above, let n_{ij} be the number of returns with indicator value i followed by indicator value j. e.g n_{00} is the number of times a non-violation is followed by another non-violation, n_{01} is the number of times a non-violation is followed by an exceedance and so on. Furthermore let π_{01} be the proportion of exceedances, given that the last return was a non-violation, and π_{11} be the proportion of violations given the last return was a violation. From this Christoffersen derives:

$$LR_{ind} = \frac{\pi_{obs}^{n_1}(1-\pi_{obs})^{n_0}}{\pi_{01}^{n_{01}}(1-\pi_{01})^{n_{00}}\pi_{11}^{n_{11}}(1-\pi_{11})^{n_{10}}} \tag{30}$$

Where the asymptotic distribution $-2\ln LR_{ind}$ is χ^2 with one degree of freedom.

The independence test serve as a measure as to how quickly a VaR model is to adapt to changing market conditions. For example, if we have a poor model, then a backtest would possibly yield a high value of LR_{ind}, because there was a period of extreme losses and specifically many consecutive days where the loss exceeded the VaR estimate. On the other hand, one drawback from the independence measure is that it is based on a first order Markov chain only; therefore it is unable to recognize the event of a period of many violations as long as there is a day between them.

6. Presentation of data and empirical results

We now turn to evaluate the ROM methodology. For this purpose we chose to evaluate 4 portfolios, one a strictly indices portfolio consisting of 30 Morgan Stanley Capital International (MSCI) indices, second a commodities portfolio consisting of 12 raw materials like gold, wheat and oil etc, thirdly a currencies portfolio, and at last a combined portfolio of the three[7]. Daily prices were obtained by DataFeed 2.0 provided by Thomson Reuters, and the data spans over 2 decades, in detail 01.03.1993-01.03.2013, a total of 5220 data samples.

Figure 7 shows the daily P&L for the MSCI portfolio and the cumulative returns shows a somewhat typical pattern for the period. The volatility clustering arising in the period of crisis is clearly visible; both the dot.com bubble of 2000 and especially the recent banking crisis of 2007-2008 are clearly evident.

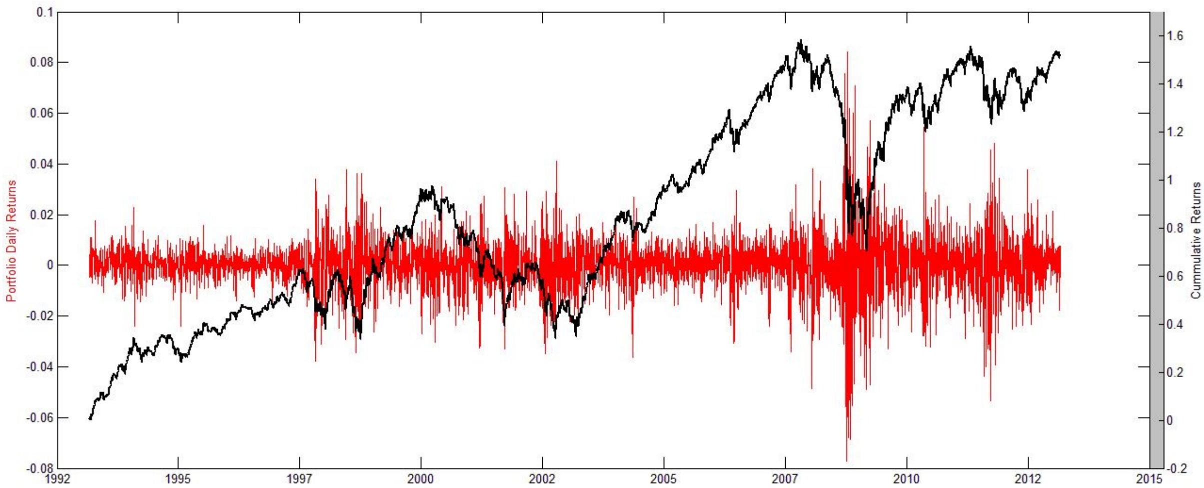

Figure 7: Historic daily (cumulative) returns of the portfolio in the period 1993-2013.

[7] The reader is referred to the appendix for detailed information of the various assets which make up the portfolios.

6.1 Comparison of historic returns and simulated historic returns

Let us first recall the main components of the historic ROM simulation process. After the application of the Gram-Schmidt process to a historic sample to obtain L_{mn}, and obtaining the Cholesky decomposition, A_n of the covariance matrix of the sample, then all there is left is the repeated multiplication of a random permutation matrix Q_m and random upper hessenberg matrix, R_n, to form as many new simulated historic samples as we like. Let us not forget to multiply with the square root of M, and add the mean vector.

Let us first take look on the volatility clustering.

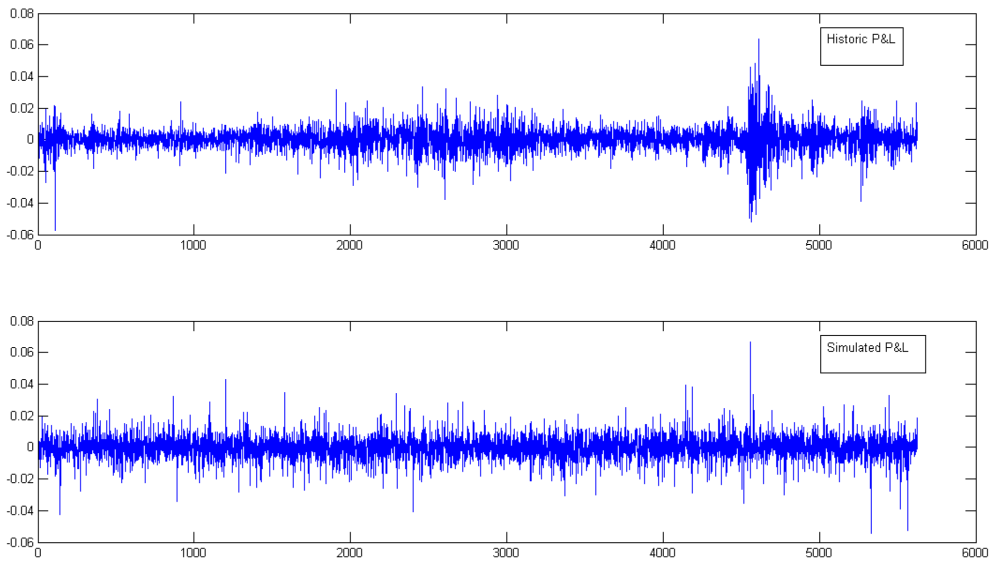

Figure 8: Historic and Simulated P&L

The upper plot in figure 8 is essentially the same as in figure 7. In the lower plot we have processed the samples through a ROM simulation with a random permutation matrix and a random upper hessenberg matrix. As a result the clustering is pretty much gone due to the shuffling by Q_n.

As the financial markets are known not to be normal distributed, and in particular not the portfolio at hand, then so should not the simulated samples be. Therefore, a crucial part of the ROM simulation is that it preserves the higher moments of the distribution. Hence the distribution should yield the same properties. Figure 9 shows the 20 years of historic returns, and the equal amount of simulated samples, and they do indeed appear very similar. In fact, a calculation of the 4[8] moments, see table 1, reveals that they are exactly the same, only the mean differs marginally.

	Mean	Variance	Skewness	Kurtosis
Original sample	1.9171E-04	1.0324E-04	5.7298E+01	2.6106E+03
Simulated sample	3.1827E-04	1.0324E-04	5.7298E+01	2.6106E+03

Table 1: Comparing the 4 moments

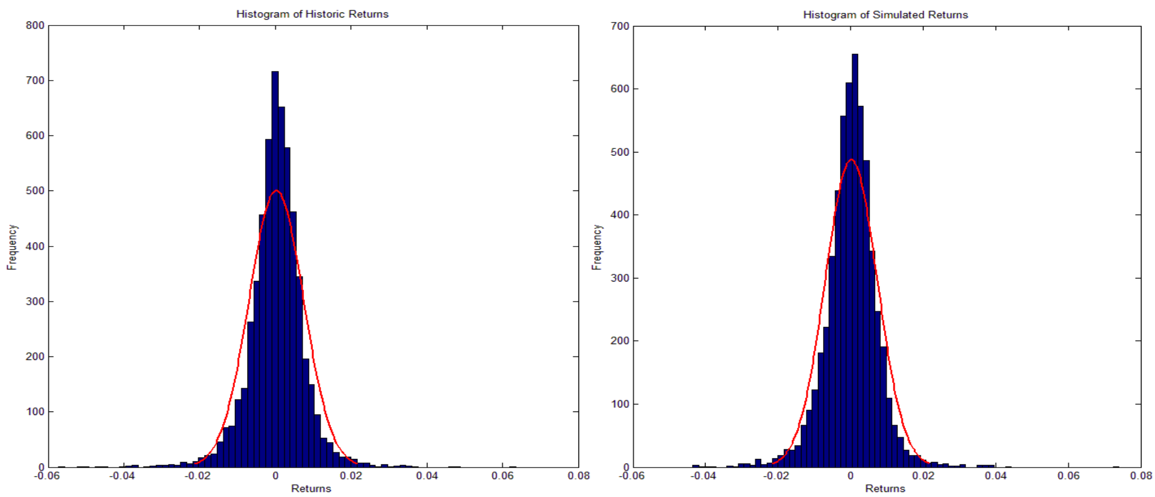

Figure 9: Histogram of the Historic and Simulated P&L

[8] To avoid confusion: The mean and variance are calculated as a portfolio, i.e we have applied portfolio mapping with equal weight on each asset. The skewness and kurtosis are calculated as per Mardia higher moments approach.

6.2 Backtesting

We now turn to validation of the models and perform various backtests using Christoffersens criteria. All along we assume the investment is held constant throughout the period and moreover the portfolio is equally weighted meaning that the investor has invested an equally amount of cash in each asset. 500 and 1000 rolling windows are employed and we test for both 99% and 99.9% significance levels. Banks are to report these numbers along with 99.97% VaR, i.e 3 violations of the VaR estimate every 40 years (2 years =500), so this figure is hard to intelligently backtest due to lack of data. From a risk perspective 99% means that we are 99% certain that the VaR estimate is not violated at any given day, whereas from a statistical point of view we expect 1 violation out of every 100 day or 47 violations out of 4700 observed samples. The dataset used consist of 5220 data samples, but the first VaR estimate is based upon the first 500 (or 1000) data, so the total market data we have for the validation purposes are 4720(or 4220).

For the deterministic backtest, equation 26 was employed in order to add kurtosis to the samples. Now, there are many ways in order to “fit the curve” so that the model is verified by a backtest. In reality, if we start at the beginning of the period in 1993, we should imagine that we did not know what the future was, having in mind the extremeness of the financial crisis of 2008. Nevertheless the advantage of the deterministic ROM simulation lies in the feature of adding to the higher moments. Therefore we now assume that we did know the Mardia kurtosis of our portfolio for the period around 2008, and for all our 500 in-sample used to estimate 1-day ahead VaR, we match the kurtosis so that it equals this stressed kurtosis via equation 26, however replacing the stressed cholesky factor,$\tilde{A}_n$,with the current cholesky factor, A_n of the 500 in-sample data. This procedure is exactly what was done in figure 5, and the VaR estimate is based on this. When the 500 in-samples reaches the crisis, the A_n naturally turns into a stressed $\tilde{A}_n$, thus we obtain extremely good results with this method. Actually, looking at figure 10 we notice that there wasn’t any violation of the 0.1%VaR around the dot.com bubble, which we perhaps would expect, and the reason for this must be that the cholesky factor is reflecting more extreme times, and therefore the overall VaR estimates are lowered. Hence this method is able to detect and adjust VaR according to general market uncertainty but off course not sudden extraordinary world events in calm times.

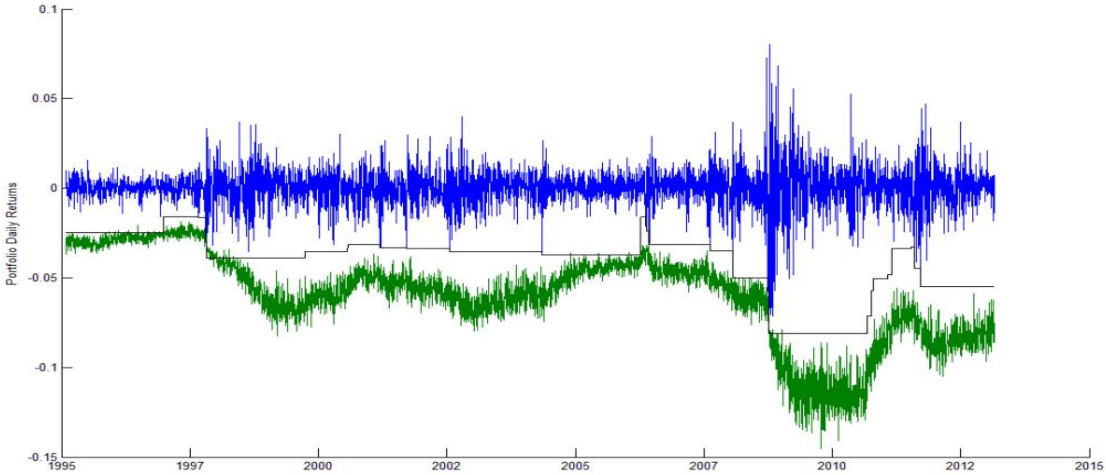

Figure 10: 99.9% VaR on the MSCI portfolio. 3 violations occurred on this particular backtest using the prescribed algorithm. Expected number of violations was 4.7, and the historical backtest resulted in 18 violations.

Let's now go on to present the empirical results. Table 2 shows the result from the backtesting of a portfolio consisting of 30 MSCI indices. We consider the historical ROM, the historical and the deterministic simulation approach and backtest for $\alpha = 1\%$ and $\alpha = 0.1\%$ using rolling windows of 500 and 1000. For every historical ROM simulation 10 000 new samples were generated via equation 19 using a random upper hessenberg matrix. For the deterministic simulation we used the value 15 for the augmentation factor r. Also here is random upper hessenberg matrix employed, and VaR are based on equation 26.

30 MSCI							
Window=500	Type	Violations	Expected violations	Empirical frequency of violations	Unconditional	Independence	Conditional
α=1%	Historical ROM	105	47	2.23 %	53.07	52.09	105.16
	Historical	67		1.42 %	7.44	16.99	24.43
	Deterministic	39		0.83 %	1.52	4.09	5.61
α=0.1%	Historical ROM	23	4.7	0.49 %	36.37	8.07	44.44
	Historical	18		0.38 %	21.68	3.59	25.27
	Deterministic	3		0.06 %	0.72	0.004	0.724
Window=1000	Type	Violations	Expected violations	Empirical frequency of violations	Unconditional	Independence	Conditional
α=1%	Historical ROM	104	42	2.47 %	64.89	48.94	113.83
	Historical	74		1.74 %	19.79	35.31	55.1
	Deterministic	58		1.37 %	5.36	29.7	35.06
α=0.1%	Historical ROM	23	4.2	0.55 %	40.54	2.47	43.01
	Historical	9		0.21 %	4.08	0.04	4.12
	Deterministic	6		0.14 %	0.66	0.017	0.68

Table 2: Backtest of the 30 MSCI portfolio

Whether or not a method passes the Christoffersens test criteria are such that for the unconditional and independence test a value greater than 6.63 means fail and for the conditional the value 9.21 applies, and is color coded with green or red. From table 2 we clearly see that the historical and the historical ROM fail most of the test whereas the deterministic is superior and pass all the tests except for one case. We generally desire to obtain violations as close as possible as expected violations or put it another way if $\alpha = 0.1\%$ we desire the empirical frequency of violations as close to this number as possible. Moreover we want the violations to appear randomly so that it also pass the independence test, which is the reason that the deterministic fails in one instance, although the further portfolio backtesting reveals that it is generally good at this point.

The next portfolio consists of 12 commodities.

12 Commodities							
Window=500	Type	Violations	Expected violations	Empirical frequency of violations	Unconditional	Independence	Conditional
α=1%	Historical ROM	56	47	1.19 %	1.57	1.8	3.37
	Historical	60		1.27 %	3.24	1.43	4.67
	Deterministic	34		0.72 %	4.11	5.07	9.18
α=0.1%	Historical ROM	7	4.7	0.15 %	0.96	0.02	0.98
	Historical	14		0.30 %	11.9	0.08	11.98
	Deterministic	5		0.06 %	0.016	0.01	0.027
Window=1000	Type	Violations	Expected violations	Empirical frequency of violations	Unconditional	Independence	Conditional
α=1%	Historical ROM	49	42	1.16 %	1.06	0.27	1.33
	Historical	52		1.23 %	2.15	0.17	2.32
	Deterministic	39		0.92 %	0.25	0.78	1.03
α=0.1%	Historical ROM	9	4.2	0.21 %	4.08	0.038	4.118
	Historical	7		0.17 %	1.53	0.02	1.55
	Deterministic	5		0.12 %	0.14	0.01	0.015

Table 3: Backtest of the 12 commodities portfolio

The commodities portfolio makes up a different result where most of the tests pass the criteria, except for one where surprisingly the historical struggles. One explanation is that the historical uses observed tail samples directly. One extra observation can therefore alter the VaR estimate greatly. But when the in-sample is greater this becomes less likely and hence the historical pass the $\alpha = 0.1$ test when 1000 rolling windows are employed. Also noticeable is that the historic ROM yields good results and passes all tests. However it seems to end on this particular commodities portfolio, for all other portfolios tested it comes out as the weakest one.

The third portfolio is a foreign exchange (Forex) portfolio consisting of 8 pairs. Even though there are a number of pairs to choose from, currency pairs are known to be heavy correlated with each other, the reason being that many of the pairs are made up from each other. For instance, the pound sterling against Japanese yen, GBP/JPY are actually made up from the multiplication of $\frac{GBP}{USD} \times \frac{USD}{JPY}$. In addition, if we were to go long in (buy) both e.g GBP/USD and USD/EUR, with equal weight, it would be the equivalent of going long in the pair GBP/EUR, and hence such a portfolio will quickly be reduced.

Let us investigate this a little further. Our Forex portfolio consists of the following randomly chosen pairs;

USD/EUR , GBP/USD , EUR/GBP , JPY/GBP , AUD/GBP, USD/GBP, EUR/USD, GBP/EUR. If an investor were to invest an equally amount in each pair, then we deduce that he would be better off buying the right hand side of the equivalent $\frac{\text{USD}}{\text{EUR}} \times \frac{\text{EUR}}{\text{USD}} \times \frac{\text{GBP}}{\text{USD}} \times \frac{\text{USD}}{\text{GBP}} \times \frac{\text{EUR}}{\text{GBP}} \times \frac{\text{GBP}}{\text{EUR}} \times \frac{\text{JPY}}{\text{GBP}} \times \frac{\text{AUD}}{\text{GBP}} = \frac{\text{JPY}}{\text{GBP}} \times \frac{\text{AUD}}{\text{GBP}}$, taken into considerations like spreads in his buying. Indeed, it would be ridiculous to invest in such a portfolio, since the right hand side of the equation reveals that the investor is just betting on that the Japanese Yen and the Australian Dollar will outperform the British Pound. Nevertheless we chose to process this portfolio of 8 pairs, and we see from the result table that both the historic and the determinist method are satisfactory for all tests. Also noteworthy is it that the deterministic and the historical results coincide for $\alpha = 0.1\%$ in the 1000 window case.

8 Forex pairs							
Window=500	Type	Violations	Expected violations	Empirical frequency of violations	Unconditional	Independence	Conditional
α=1%	Historical ROM	69	47	1.46 %	8.92	5.3	11.33
	Historical	54		1.14 %	0.95	1.25	2.2
	Deterministic	41		0.87 %	0.85	0.79	1.64
α=0.1%	Historical ROM	16	4.7	0.34 %	16.45	0.1	16.55
	Historical	10		0.21 %	4.46	0.04	4.5
	Deterministic	11		0.23 %	6.06	0.05	6.11
Window=1000	Type	Violations	Expected violations	Empirical frequency of violations	Unconditional	Independence	Conditional
α=1%	Historical ROM	65	42	1.54 %	10.7	8.59	19.29
	Historical	54		1.28 %	3.07	1.69	4.76
	Deterministic	50		1.20 %	1.38	2.13	3.51
α=0.1%	Historical ROM	13	4.2	0.31 %	11.71	0.08	11.79
	Historical	8		0.19 %	2.68	0.03	2.71
	Deterministic	8		0.19 %	2.68	0.03	2.71

Table 4: Backtest of the 8 Currencies portfolio

As a grand finale we now merge the 3 portfolio to yield a greater and more mixed portfolio. For this portfolio, where the other methods fail the unconditional and conditional tests, the deterministic approach passes all tests:

Portfolio 1 + 2 + 3							
Window=500	Type	Violations	Expected violations	Empirical frequency of violations	Unconditional	Independence	Conditional
α=1%	Historical ROM	92	47	1.95 %	33.67	9.26	42.93
	Historical	71		1.50 %	10.51	15.49	26
	Deterministic	35		0.74 %	3.48	4.86	8.34
α=0.1%	Historical ROM	22	4.7	0.47 %	33.24	2.83	36.07
	Historical	19		0.40 %	24.41	0.15	24.56
	Deterministic	3		0.06 %	0.72	0.0038	0.7238
Window=1000	Type	Violations	Expected violations	Empirical frequency of violations	Unconditional	Independence	Conditional
α=1%	Historical ROM	94	42	2.30 %	47.65	24.32	71.97
	Historical	67		1.60 %	12.5	15.6	28.1
	Deterministic	52		1.21 %	2.15	4.76	6.91
α=0.1%	Historical ROM	26	4.2	0.62 %	51.12	6.68	57.8
	Historical	11		0.26 %	7.53	0.05	7.58
	Deterministic	5		0.12 %	0.1369	0.0119	0.1488

Table 5:Backtest of the combined portfolio

6.3 Optimized r?

The augmentation factor r, given in equation 26, is fixed to a value of choice. It is free of choice because the Mardia kurtosis is the same for all r. In the backtests in the previous chapter the r was set to r=15. In this section we wish to explore the idea of an optimized r, that is we seek a value of r which yields the best backtest results. A backtest are performed in order to validate a model. We now extend this thinking and look for which value of r is better and hence one might claim that we manipulate the backtests. Philosophically, however, optimizing an r and using this value for future VaR simulations are merely an extension of the thinking that what happens in the past are due to happen again, and hence such approach should provide good results also in the future.

For this purpose it was chosen to evaluate the 30 MSCI portfolio using rolling windows of 1000 at 1% significance, which was the only test that did not satisfy the Christoffersen's test. We did a backtest of 20 values for r, from 1 to 101 with increments of 5. The results for each Christoffersen parameter are given in figure 12 whereas the number of violations as a function of r is seen in figure 11:

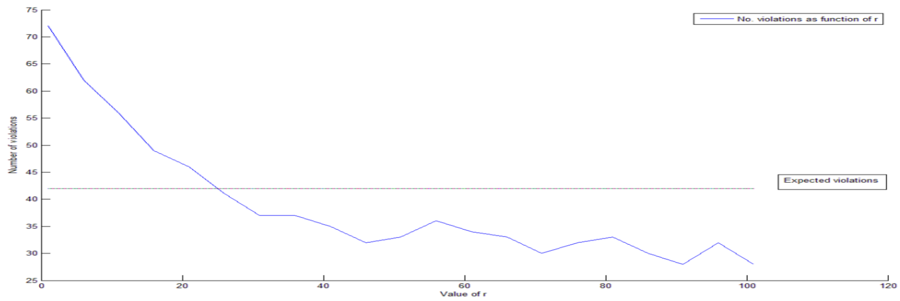

Figure 11: Plot of various values of r and the corresponding number of violations

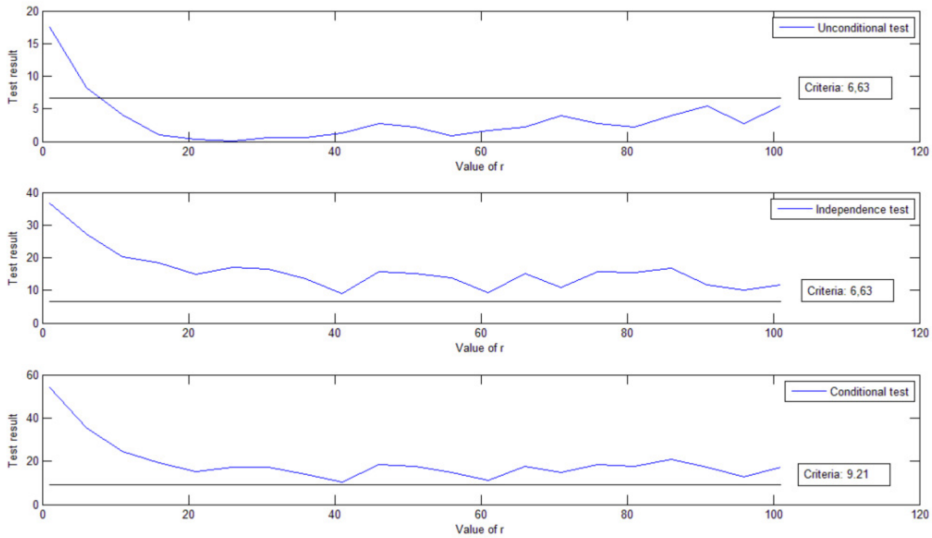

Figure 12: Results from the backtesting for increasing values of r

Upon studying the above figures we see that none value for r provides satisfactory results. There is an optimum for the unconditional test, however the exceedance followed by another exceedance i.e the indepence test, fail. Still if we were to use this method for future estimations of VaR, then the conditional test is minimized for r=41 where it takes the value 10,33. The number of violations was 35 compared to expected number of violations of 42.

Now, there seems to be a trend that the greater number of r used, the fewer overall exceedances, see figure 11. However, we did a simulation for r=1000 and no significant change was seen; total number of violations was 29, unconditional test equaled 4.67 and the independence test turned out to be 11.23. And thus we conclude that the test converges to a fixed value, whatever they might be for the specific portfolios, for increasing value of r. This reasoning, however, suggest that there are no optimum value for r, and therefore the results which led us to believe that r=41 are an optimum point must be described to as an element of chance. It is also true that by using higher values of r increases the precision of the VaR estimate, and is therefore recommended. For this particular portfolio we should use an approximate value of $r \geq 20$, because this is the point where the simulated violations crosses the expected violations and moreover from r=20 an onwards, see the conditional test, the results seems more or less arbitrary and perhaps one might claim that the convergence are sufficient. We must also remember that for increasing values of r, a significant increase in simulation time follows. On the other hand the volatility in succeeding VaR estimates is reduced. Therefore there is a need for balancing the practicality and the exactness of the simulations. In general we conclude that the greater value of r the better the VaR estimates becomes.

We also want to highlight an interesting feature of the r parameter. Consider figure 13. Now, if we use a rolling window of 1000 and make a backtest of the MSCI portfolio at 0.1% significance using a value of r in the area around 10, then we obtain a chart that looks like some error has been done in the coding. See the period in the later 2008 to late 2012 in figure 13. For comparison we also included the responding historical simulation.

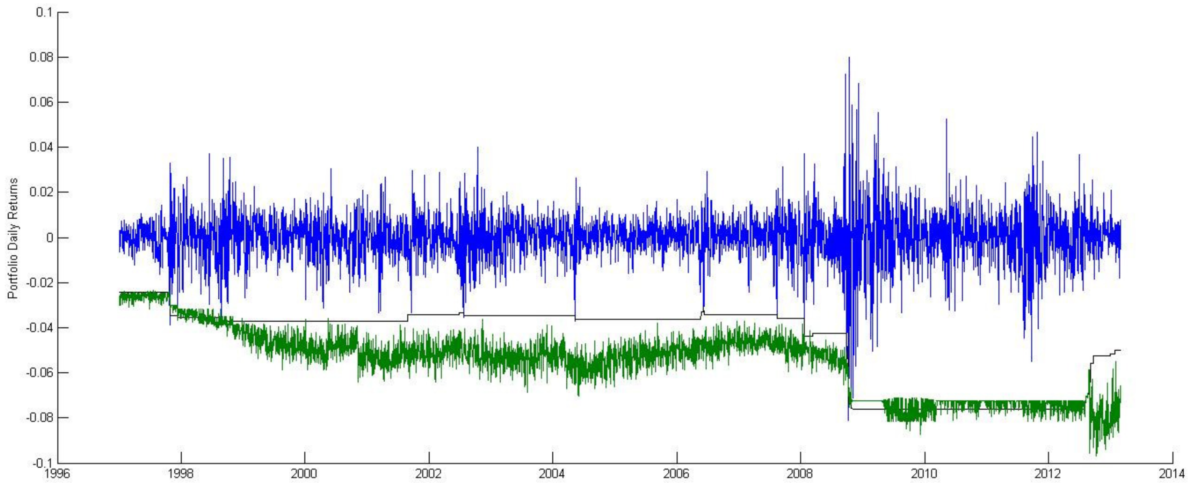

Figure 13: The effect of the cholesky matrix and low value of r

However, and from intuition, we may reason that stressed concatenation given by equation 26 sets an upper bound for the VaR estimate from its first term, ie

$$X = \sqrt{m} L_{mn}^{D} A_n$$

This is because the Mardia kurtosis is already obtained, so in reality there is no need for the construction given in eq.26, in that specific time frame, because kurtosis is naturally targeted. Nevertheless we clearly see some alterations in the VaR estimates in this time frame, and it may be explained by the volatility in the Cholesky matrix. Indeed, the Cholesky matrix are reflecting high insecurity and instability in the markets causing the VaR estimates to sometimes be lower than a previous estimate, but overall not higher than a set limit. This limit is governed by the great spikes experienced in the most extreme part of the financial crisis.

By using a higher number of r, more simulations, this feature of equation 26 are hidden, because the random elements, R_n, smoothens out the curve.

6.4 More on stress concatenation

We now visualize the contents of the sample concatenation given by eq. 26, by using both kurtosis and covariance matrix taken from the 2008 crisis and then backtest on the overall sample. Figure 14 consists of one return series and 5 corresponding VaR estimates, using window of 1000 in-samples.

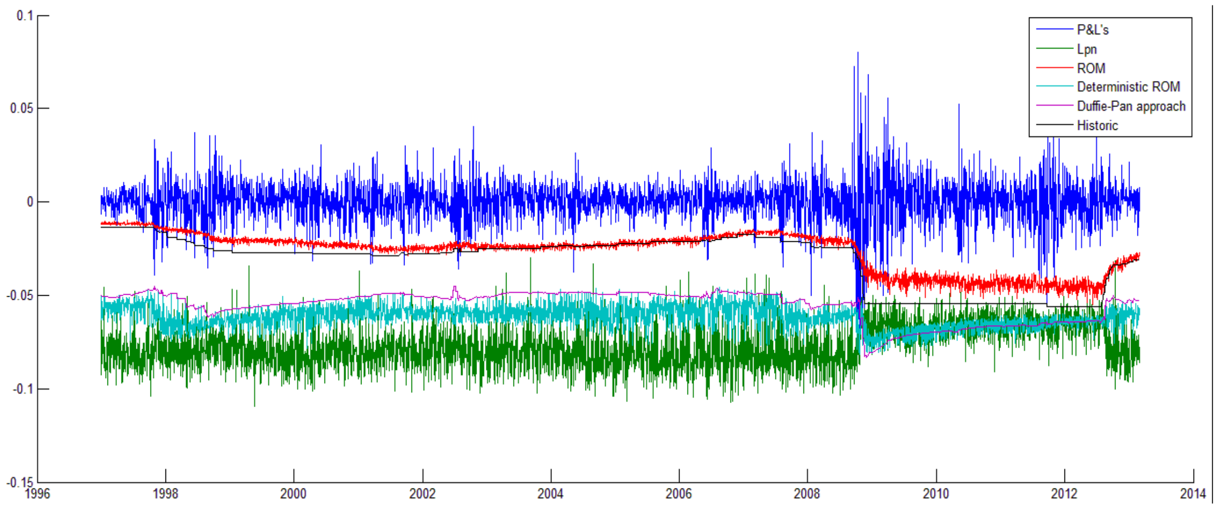

Figure 14: Visualization of the breakdown of equation 26, using both kurtosis and covariance matrix taken at 2008 levels. The historic and the historic ROM simulation are also displayed.

For a comparison, historic ROM simulated VaR estimates and simple historic VaR are included. Let us discuss the VaR series, starting from the top:

The red line is the ROM simulated historic approach were look back 1000 days and then 10 000 new samples are simulated by eq. 19 using an upper hessenberg matrix as R_n, and 1% VaR is estimated based on these new samples.

Next is the simple and basic historic approach and it goes without saying that we simply look 1000 days backwards, find the 1% worst return, and thus the estimate one day ahead is this value.

The three next lines is a breakup of equation 26. The smooth line is the first term of this equation, and expressed like this, it yields a Duffie-Pan estimate of stressed scenarios, using the period of the financial crisis of 2008 to estimate the covariance matrix, and thus the cholesky factorization $\tilde{A}_n$. Now, such approach only yields one estimate and moreover it doesn't take into account the increasing kurtosis experienced in times of financial stress.

The green line at bottom of the graph shows only the effect the Ledermann matrix, L_{pn}, i.e the increasing of kurtosis, have as estimates of VaR estimates. This is eq.26 less the first term.

And finally, in the middle of the two last points, lies the stressed sample concatenation as given by equation 26, and yields stressed VaR, where both kurtosis and the covariance is taken on the period around 2008, and we clearly see the effect of stressing not only the covariance matrix, but also the kurtosis. Now, upon looking at figure 14, it is clear that by using the 2008 levels, estimates are far too extreme, so such estimates are not appropriate in normal times, in fact, throughout this particular backtest the deterministic ROM simulation yielded only one violation of the stressed VaR estimate, which obviously happened during the spoken of crisis. And this suggests that the crisis of 2008 were indeed extraordinary.[9]

[9] To be clear at this point: In the stresstesting here we used both covariance and kurtosis from the 2008 crisis period. In the general backtesting, chapter 6.2, only kurtosis were taken on this period.

6.5 Critique of the deterministic method

There are mainly two drawbacks from utilizing deterministic ROM as compared to the historical. For one, it can be criticized for being too conservative. Generally we desire to estimate VaR as close to the realized returns without actually breaking, the reason being that for a banker, say, no unnecessary money is being kept out of the market for the sake of being solid as demanded by the authorities. From figure 10 we see that there was a period of approximately 9 years between two violations (1997-2006), where although the historical has many violations, most of the violations were made from returns that barely crossed the VaR estimate, and thus one might claim that historical serves more accurate forecast even though it does not pass the test criteria. The other drawback from the deterministic approach is the volatility in the succeeding VaR estimates. If a trader or a portfolio manager were to use this approach it would cause him to make adjustments on the portfolio more frequently than from the historical. This may be overcome by using a higher number of the augmentation factor r, although the simulation time is increased. Nevertheless, such drawbacks are perhaps just a necessary tradeoff we must appreciate in order to obtain such astounding results as demonstrated in the backtesting results.

This work targets the kurtosis identical to the kurtosis observed in the 2008 crisis, in detail 22.01.2008-22.12.2009. Intuitively this is done in order to make rolling distributions which has these empirically observed extreme tails. For the MSCI portfolio this implied a Mardia kurtosis of 1902. Now, although not quite as intuitively one might easy target even higher/ or lower values of kurtosis, and such manipulate or “fit-the-curve” in a backtest. From this we clearly see the flexibility of the deterministic ROM approach.

7. Conclusion

There are indeed many challenges on the topic of estimating good and reasonable Value at Risk levels. There exist many sophisticated models trying to yield better results than the basic historic approach, many more than we have mentioned in this paper. Monte Carlo simulation has a tradition in the banking industry to be the strongest competitor to the historic approach, but it suffers from severe drawbacks; it is tedious and time consuming in addition to often fail backtestings because it doesn't account for the highly leptokurtic nature of financial markets.

As we enter into an era of stronger and stricter banking supervision with Basel III, and even the traditional historic simulation approach fails the backtests, there is a need for better simulation procedures, which until now was not heard off. Throughout this paper we have explored some of the features of the Random Orthogonal Matrix (ROM) simulation provided by Ledermann et al. (2009). They introduce 3 general simulation approaches, data-specific, parametric and the deterministic ROM. We chose to focus on the first and the latter, thus leaving the parametric to further investigations.

We find that the data-spesific approach, what we have termed historic ROM simulation, in general does not perform better than the historic approach, and it would be unwise to utilize this method for VaR reporting, since it is both time consuming and it yields poorer results based on our portfolios backtesting results. However, we have only explored the upper hessenberg matrix as the random element in the simulation, and perhaps a random Cayley matrix would provide better results. Also from the empirical study the historic ROM simulation yielded sufficient results on the commodities portfolio whereas the historic did not. Therefore we cannot neglect the data-spesific technique for VaR purposes, and this might give motivation for further work.

But the groundbreaking feature of ROM simulation lies in the deterministic type which allows us to target higher multivariate moments as defined by Mardia (1970). In this way we may either stress test a portfolio by not just simply stressing the covariance matrix, as per Duffie-Pan approach, but also stressing skewness and kurtosis, thereby obtaining more accurate stressed VaR estimates. And moreover, since the Duffie-Pan approach only yields one estimate it suffers from single sample bias, which are overcome by ROM simulation by simply repeating the simulation and new estimates are provided by the random elements in R_n.

Our results in this paper are based on a market sample spanning over 2 decades of daily returns, and we chose to let each in-sample target the kurtosis experienced in the financial crisis of 2008. We chose to target only the kurtosis, although ROM simulation lets us target both skewness and kurtosis. We did so since it is believed that targeting either one of them should capture the relevant market characteristic.

The empirical results demonstrate the power of the deterministic approach over both the historic ROM and the historical simulation. The Christoffersen (1998) test criteria was applied, and where the other methods failed, the deterministic provided excellent results with only one exception, see chapter 6.2. In addition it seems that the deterministic approach is able to detect and adjust the VaR estimate in changing circumstances. The exactness of the VaR estimate is governed by the augmentation factor r in equation 26, and we conclude that in general; the higher value of r, the better the VaR estimate.

The results according to this paper conclude that the deterministic ROM simulation technique is superior to other two examined simulation tools, and should be applied by banks and other financial institutions, on equity portfolios, for better and more accurate VaR estimation. However it has not been tested for other assets, which typically builds up a bank's portfolio, such as bonds and home lending. Such work should be investigated.

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Appendix

Daily data from DataFeed 2.0 by Thomson Reuters			
for the period:			
Start	01.03.1993		
End	01.03.2013		
Portfolio	30 MSCI	12 Commodities	8 Forex pairs
	MSCI USA - PRICE INDEX (~U$)	Gold Bullion LBM U$/Troy Ounce	US $ TO EURO (WMR&DS) - EXCHANGE RATE
	MSCI JAPAN - PRICE INDEX (~U$)	Crude Oil-WTI Spot Cushing U$/BBL	US $ TO UK £ (WMR) - EXCHANGE RATE
	MSCI UK - PRICE INDEX (~U$)	Natural Gas, Henry Hub U$/MMBTU	EURO TO US $ (WMR&DS) - EXCHANGE RATE
	MSCI GERMANY - PRICE INDEX (~U$)	Silver Fix LBM Cash Cents/Troy ounce	UK £ TO EURO (WMR&DS) - EXCHANGE RATE
	MSCI ITALY - PRICE INDEX (~U$)	Corn No.2 Yellow Cents/Bushel	UK £ TO US $ (WMR) - EXCHANGE RATE
	MSCI AUSTRALIA - PRICE INDEX (~U$)	Wheat No.2,Soft Red Cts/Bu	EURO TO UK £ (WMR&DS) - EXCHANGE RATE
	MSCI KOREA - PRICE INDEX (~U$)	Raw Sugar-ISA Daily Price c/lb	JAPANESE YEN TO UK £ (WMR) - EXCHANGE RATE
	MSCI FRANCE - PRICE INDEX (~U$)	Coffee-Brazilian (NY) Cents/lb	AUSTRALIAN $ TO UK £ (WMR) - EXCHANGE RATE
	MSCI G7 U$ - PRICE INDEX (~U$)	Corn US No.2 South Central IL $/BSH	
	MSCI AC AMERICAS U$ - PRICE INDEX (~U$)	Wheat US HRS 14% Del Mineapolis/Dulut	
	MSCI AC WORLD EX AU U$ - PRICE INDEX (~U$)	Crude Oil North Sea BFO FOB U$/BBL	
	MSCI EAFE + CANADA U$ - PRICE INDEX (~U$)	Jet Kerosene FOB US Gulf U$/MT	
	MSCI EAFE EX UK U$ - PRICE INDEX (~U$)		
	MSCI EMU EX GERMANY U$ - PRICE INDEX (~U$)		
	MSCI FAR EAST U$ - PRICE INDEX (~U$)		
	MSCI NORDIC U$ - PRICE INDEX (~U$)		
	MSCI AC ASIA U$ - PRICE INDEX (~U$)		
	MSCI AC WORLD EX US U$ - PRICE INDEX (~U$)		
	MSCI EMU U$ - PRICE INDEX (~U$)		
	MSCI PACIFIC EX JP U$ - PRICE INDEX (~U$)		
	MSCI WORLD EX JP ($) - PRICE INDEX (~U$)		
	MSCI AC EUROPE U$ - PRICE INDEX (~U$)		
	MSCI AC FAR EAST EX JP U$ - PRICE INDEX (~U$)		
	MSCI EM EUROPE U$ - PRICE INDEX (~U$)		
	MSCI SINGAPORE F - PRICE INDEX (~U$)		
	MSCI EM U$ - PRICE INDEX (~U$)		
	MSCI AC ASIA PAC EX JP U$ - PRICE INDEX (~U$)		
	MSCI EM ASIA U$ - PRICE INDEX (~U$)		
	MSCI EM LATIN AMERICA U$ - PRICE INDEX (~U$)		
	MSCI NORTH AMERICA U$ - PRICE INDEX (~U$)		