On computational linear semi-infinite programming
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Introduction

Let $S \subset R^k$ ($k < \infty$) be a compact set, $a_1, a_2, \ldots, a_n$ and b be continuous functions defined on S, $c \in R^n$ a given vector. Then we introduce the optimisation problems P_m and P_M as follows:

$P_m(a, b, c, S)$:

Minimise $c^T y$, when y varies over R^n subject to the constraints

$$a(s)^T y \geq b(s), s \in S.$$

$P_M(a, b, c, S)$:

Maximise $c^T y$, when y varies over R^n subject to the constraints

$$a(s)^T y \leq b(s), s \in S.$$

We also introduce: $P_2(a, b, S)$:

Minimise $y_0 \in R$ when y varies over R^n subject to the constraints

$$|a(s)^T y - b(s)| \leq y_0, s \in S.$$

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The first two problems $P_m(a, b, c, S)$ and $P_M(a, b, c, S)$ are examples of linear semi-infinite programs, since they call for the minimisation and maximisation of the linear form $c^T y$ subject to one condition for each point $s \in S$, which may be an infinite set, e.g. an interval. The problem $P_2(a, b, S)$ is equivalent to the task

Minimise $y_0 \in R$, when y varies over R^n subject to the constraints

$$a(s)^T y + y_0 \geq b(s), s \in S,$$

$$-a(s)^T y + y_0 \geq -b(s), s \in S.$$

We note that $P_m(a, b, c, S)$ is the task to approximate the function b from the above by a linear combination $a^T y$ which is such that the linear expression $c^T y$ is minimised. The goal of the research project CLSIP (Computational Linear Semi-Infinite Programming) is to develop computational schemes for the efficient treatment of the three problems introduced above. The following example illustrates some of the difficulties:

P_m:

Minimise $y \in R$

when

$$y \geq b(s), s \in S.$$

The optimal solution is

$$y = \sup_{s \in S} b(s),$$

provided b is bounded on S. To find the optimal value of y is a global optimisation problem, and as known, one needs to introduce assumptions on the function b and the set S in order to construct an efficient procedure for determining an optimal y.

Discretisation

A general approach to the numerical treatment of the problem $P_m(a, b, c, S)$ is to approximate it with a task, that can be solved by a finite number of arithmetic operations and logical choices. We note that these can only be carried out with a finite precision which depends on the computer and operating system used. Thus one needs to consider the influence of computational errors on the calculated result.

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One common approach is to replace the set S with a finite subset $T \in S$ and hence approximate $P_m(a, b, c, S)$ with a linear program. Provided that the functions a, b meet certain regularity constraints, such as being continuously differentiable on S, the discretisation error may be expressed as perturbations in the data a, b. In particular, if S may be represented as a real interval, whose endpoints are contained in the subset T, then the discretisation error is equivalent to the error caused by approximating a, b by the result of interpolating linearly at the points of T. We note also, that S contains only a finite set of distinct numbers which may be represented exactly in the computer, and hence $P_m(a, b, c, S)$ is represented in the computer as a linear program with a finite, albeit very large number of constraints. This calls for introducing regularity conditions on a, b to guarantee that the calculated solution of $P_m(a, b, c, S)$ is indeed a good approximation for the solution of the given LSIP. It is of interest to study other kinds of discretisation, e.g. using piecewise cubic interpolation or approximating a, b by functions with known properties, e.g. polynomials,

Semi-infinite programming in numerical analysis

Approximation with polynomials in the maximum norm

Algorithms for the classical problem of approximating a function over an interval S by polynomials in the maximum norm are attributed to Remez. These algorithms generate a sequence of vectors $y^{(r)}$ and in order to perform an iteration step one needs to solve the global minimisation problem: Maximise

$$|a(s)^T y^{(r)} - b(s)|, s \in S.$$

Hence a difficulty is to keep track of the local minima.

One-sided approximation

There is a relation between generalised Gauss quadrature rules and one-sided approximation problems when the index-set S is a real interval, since such

rules are the solutions of the duals of the one-sided problems introduced above. There are well-known algorithms for determining the abscissas and weights of these rules and using these rules one may solve the one-sided problems analytically, provided the nth derivative of b does not change sign in the interval S. This seemingly strong condition is met e.g. in problems occurring in power series summation.

Other applications

We mention here only the treatment of integral equations originating from problems in the petroleum industry and the analysis of environmental problems, like the control of air and water pollution. Several further applications are found in [3].

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