Master's Thesis

Paramagnetism Shielding in Drilling Fluid

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Preface

This thesis has been carried out, benefited from the overall guidance of Per A. Amundsen who is professor in the University of Stavanger at the department of Science and Technology. He has also served as the contact and has guided in all phases of the analysis.

Abstract

In drilling operations, drilling fluid containing magnetic materials is used when drilling a well. The materials can significantly shield the Earth's magnetic field as measured by magnetic sensors inside the drilling strings. The magnetic property of the drilling fluid is one of the substantial error sources for the determination of magnetic azimuth for wellbores. Both the weight material, cuttings, clay and other formation material plus metal filings from the tubular wear may distort the magnetometer readings. This effect is obviously linked to the amount and kind of magnetic material in the drilling fluid, and the development of corrective means has therefore highlighted the drilling fluid.

The influence on directional Measurement While Drilling (MWD) from drilling fluids has been studied using finite element modeling techniques. The simulations have been performed for several cases with realistic representations of MWD tool geometries and varying location of Bottom Hole Assembly (BHA) versus the wellbore. The wellbore, tools and magnetic fields were modeled by finite element methods. The wellbore is modelled as a perfect circular cylinder. With unsymmetrical geometry the placement of the magnetic MWD sensors may become more sensitive to magnetic shielding effects, and small position changes may result in significant errors in the measured magnetic field components, both attenuation and amplification [1]. One important result is that for situations with perfect axial symmetry, the magnetometer readings are attenuated proportionally to the square of the magnetic susceptibility. Since the magnetic susceptibility is a small number, this means that the effect on magnetometer readings is generally negligible. However, if the symmetry is broken, the distortion on the magnetometer readings can be increased significantly. This means that segregation of cuttings, metal filings or weight material can strongly influence the strength of the measured magnetic fields [1].

It has been shown sometimes to cause significant errors in the accuracy of drilling hole positioning using magnetic surveying. Here we present a general physical approach for correcting the measured magnetic fields from *paramagnetism* by the paramagnetic material for such *in drilling fluid*. Based on information of the paramagnetic properties of the drilling fluid and the well geometry, applied to a sufficiently long straight section of a well [2] this paper will show how the magnetic field in a cylindrical wellbore can be calculated analytically by using conformal mapping .

Chapter 1

1 Introduction

1.1 Introduction to magnetism

Magnetism is the force of attraction or repulsion in and around a material that respond to an applied magnetic field. All materials responds differently to magnetic fields :

- attracted to a magnetic field *paramagnetism*;
- repulsed by a magnetic field *diamagnetism*;
- *ferromagnetism* is a strong form of paramagnetism. Ferromagnetic materials may form permanent magnets.
- *spin glass and antiferromagnetism* are complex forms of strong diamagnetic behaviour.

Certain materials such as *magnetite, iron, steel, nickel, cobalt and alloys of rare earth elements*, exhibit magnetism at levels that are easily detectable [11]. Substances that are negligibly affected by magnetic fields are known as non-magnetic substances. They include *copper, aluminium, gases, and plastic. Pure oxygen* exhibits magnetic properties when cooled to a liquid state. The magnetic state of a material depends on *temperature, pressure and applied magnetic field* so that a material may exhibit more than one form of magnetism depending on its *temperature*, etc [3].

1.2 History

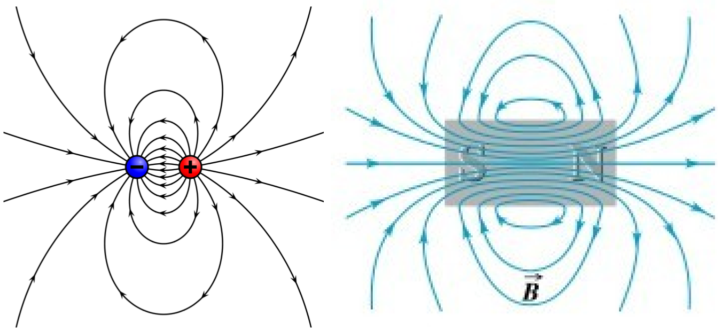
The word *magnetic* comes from *Magnesia*, the name of the district and Greece where the mineral magnetite was found.The ancient Greeks observed eletric and magnetic phenomena possibly as early as $700BC$. The Greeks knew about magnetic forces from observations that the naturally occurring mineral *magnetite* Fe_3O_4, *called synthetic magnetite*, is attracted to iron. There are documents in ancient China, between $481BC$ and $403BC$, in books named after its author, The Master of Demon Valley (*Guiguzi*) [4]: "The lodestone makes iron come or it attracts it." By the $12th$ century the Chinese were known to use the lodestone compass for navigation. In 1819 with work by Hans Oersted, a professor at the University of Copenhagen, it was discovered that an electric current could influence a compass needle. In 1831 Michael Faraday and, almost simultaneously, Joseph Henry found further links between magnetism and electricity. In 1873, James Clerk Maxwell synthesized and expanded for formulating the laws of electromagnetism in *Maxwell's equations*. His work is as important as Newton's work on the laws of motion and the theory of gravitation. In 1905, Einstein used these laws in motivating his theory of special relativity [5], requiring that the laws held true in all inertial reference frames [6].

Chapter 2

2 Basic concepts

Before we try to understand the force between magnets , it is useful to define the **B**-magnetic field and the **H**-magnetic field . Examine the magnetic pole model given the following:

2.1 Magnetism



The magnetic field vector **B**, also called *the magnetic flux density* , or *the magnetic induction* , is usually defined by the vector product force equation. The electric field **E** is here assumed to be zero. The equations are in the international System of Units, which will be used through out the thesis.

$$F_M = q\mathbf{v} \times \mathbf{B} \tag{2.1}$$

Here **v** is the velocity of the electric charge q and F_M is the resulting force on the moving charge [2].
In most cases it is more convenient to measure **B** by the magnetic torque **T** on a magnetic dipole of magnetic moment **m** (e.g. a compass needle or a current loop):

$$\mathbf{T} = m \times \mathbf{B} \tag{2.2}$$

The Maxwell's equations [7] , in a medium are the so-called *macroscopic Maxwell equations* , which are obtained from the *microscopic equations* averaging over a large number of particles.
Here

- **E** is the electric field. $\nabla E = \frac{\rho}{\epsilon}$, **D** = ϵE,
- ρ is the free electric charge density.
- **J** is total current density (including both free and bound current).

Figure 1: Maxwell's equations

Differential formulation

Name	"Microscopic" equations	"Macroscopic" equations
Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\nabla \cdot \mathbf{D} = \rho_f$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$

- $\mathbf{J_f}$ is free current density (not including bound current).
- ϵ_0 is the electric permittivity [10], and μ_0 is the magnetic permeability, with $\epsilon_0\mu_0 = \frac{1}{c^2}$, c being the velocity of light.
- $\mathbf{B} = \mu\mathbf{H}$
  The auxiliary magnetic field H, where the vacuum permeability is a proportionality constant. The above formulas remain valid in the presence of magnetic materials, except that the relation between **B** and **H** is not necessarily simple.

The relation has important consequences. Because ε_0, μ_0 can be measured in any frame, the velocity of light is the same in and frame. The magnetic field **B** (also called magnetic flux density or magnetic induction) in a system is caused by local current distributions, in addition to possible superimposed external fields (caused by currents external to the system under discussion). The effect of *a current density* **j** on the magnetic field is conventionally expressed through *an auxiliary field*, **H**, traditionally - and unfortunately - called the magnetic field strength (or magnetic intensity). In a region with no charges ($\rho = 0$) and no currents ($j = 0$), such as in a vacuum, Maxwell's equations reduce to:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \quad \texttt{in vacuum} \\
\nabla \cdot \mathbf{B} &= 0 \quad \texttt{in vacuum} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad \texttt{Faraday's law of induction} \\
\nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \quad \texttt{Displacement current}
\end{aligned}$$

2.2 The Magnetization

As we shall see shortly, the magnetization of a substance is the average magnetic moment . The magnetic properties of matter are caused by molecules possessing a magnetic dipole moment. A number, N, of such molecular dipoles $m_i \ \ (i = 1 \ldots N)$ contained in a macroscopically small volume V will act as single dipole of strength $\sum_i \mathbf{m_i}$. The average combined dipole moment is the *magnetization*, **M**:

$$\mathbf{M} = \frac{1}{V}\sum_i \mathbf{m_i} = \frac{N}{V} < m >= n < m > \tag{2.3}$$

where <. . .> denotes the space average (assumed equal to the time average in the case of fluctuations), and n is the number density of the dipoles.
In the presence of a magnetizable medium the relation between **B** and **H** is modified to:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{2.4}$$

In this formula **H** has the *same* value, given by *Ampère's law*, as it would have in the absence of the microscopic dipoles, assuming the same current distribution **j**. Thus **H** can be interpreted as the external magnetic *forcing* of the material, causing a magnetic field **B**, or, equivalently, as the magnetic moment per unit volume of the external macroscopic electric currents.

2.3 Magnetic permeability

In general, there is no simple relation between **M** and **H**, and hence not between **B** and **j**. Indeed, for permanent magnets **M** can have an *an arbitrary direction*, with magnitude up to a certain maximum, even in the absence of an external field. However, in most materials the molecular magnetic dipoles are randomly oriented with a vanishing average, so $\mathbf{M} = 0$ if $\mathbf{H} = 0$, and they respond only weakly and practically linearly to an external field. If the magnetic medium is also isotropic (no preferred direction), this leads to the relations:

$$\mathbf{M} = \chi \mathbf{H} \tag{2.5}$$

$$\Updownarrow$$

$$\mathbf{B} = \mu_0(1+\chi)\mathbf{H} = \mu \mathbf{H} \tag{2.6}$$

Here χ is *a dimensionless number*, called *the magnetic susceptibility* [1], which is a thermodynamic material property. The **permeability** of the material - μ defined as:

[1]Determination of the susceptibility entails evaluation of the magnetization produced by an applied magnetic field.

$$\mu = \mu_0(1+\chi) = \mu_0 K_m \quad (2.7)$$

is called *the magnetic permeability*.

Table 1: Material's magnetic susceptibility

χ[dimensionless]	Material
$\chi \gg 1$	ferromagnetic, depends on **H** in a rather non-trivial way(hysteresis)
$\chi > 0$	*paramagnetic*
$0 < \chi \ll 1$	can in most cases be treated as a temperature dependent material constant
$\chi < 0$	diamagnetic

Since χ is a material property, so is μ, and in inhomogeneous systems μ is generally position dependent, $\mu = \mu(r)$. In physical data tabulations one often does not tabulate χ directly, as most experimental set ups instead measure *the mass susceptibility indexmass susceptibility* $[\frac{m^3}{kg}]$ in SI or in $[\frac{cm^3}{g}]$ in *Centimetre-gram-second system* (CGS):

$$\chi^m = \frac{\chi}{\rho} \quad (2.8)$$

where ρ is the mass density $[\frac{kg}{m^3}]$ (SI) or $[\frac{g}{cm^3}]$ (CGS) of the substance.
Also *the molar susceptibility* $[\frac{m^3}{mol}]$ (SI) or $[\frac{cm^3}{mol}]$ (CGS):

$$\chi^A = A\chi^m = \frac{\chi A}{\rho} \quad (2.9)$$

is often tabulated, where A is *the molecular mass*(molecular weight) $[\frac{kg}{mol}]$ (SI) or $[\frac{g}{mol}]$ (CGS) of the substance.
If there are mixed two volumes, V_1 and V_2, of different materials with different susceptibilities, χ_1 and χ_2, and it can be assumed that the two materials do not interact chemically or magnetically, the relation (M) leads to *Wiedemann's law* for the susceptibility of a mixture:

$$\chi = \frac{\chi_1 V_1 + \chi_2 V_2}{V_1 + V_2} \quad (2.10)$$

with an obvious generalization to more complex mixtures. For χ^m one correspondingly has:

$$\chi^m = \frac{\chi_1^m M_1 + \chi_2^m M_2}{M_1 + M_2} \tag{2.11}$$

Because the average magnetic dipole moment <**m**> is often quite sensitive to the molecular surrounding, *Wiedemann's law* is not always very accurate in practice. Since tables give CGS units, we see also after these descriptions, that we could find that the magnetic field at any point in a paramagnetic material rises when replaced by vacuum, by a factor named the **relative permeability $\mathbf{K_m}$** [dimensionless]. The factor's worth depends on the kind of materials. In generally paramagnetic solids and liquids at room temperature, $\mathbf{K_m}$ is normally from *1.00001* to *1.003* in the CGS). The susceptibility is defined differently in the SI and the CGS systems.

$$\begin{aligned} \chi_m^{CGS} &= \frac{\chi_v^{CGS}}{\rho^{CGS}}, \quad \rho^{CGS}[\frac{g}{cm^3}] \\ \chi^{CGS} &= 4\pi\chi^{SI} \end{aligned}$$

For the mass susceptibility, χ_m, one must also take into account the different units for the density, thus many use lists with χ_m^{CGS} to convert to SI units. We will show how to change the CGS to SI system. For example for water in 20 $^\circ C$,

$$\chi_m^{CGS} = \frac{\chi_v^{CGS}}{\rho^{CGS}} = \frac{-7,190 \cdot 10^{-7}}{0,9982[g/cm^3]} = -7,203 \cdot 10^{-7}[\frac{cm^3}{g}]$$
$$\chi_m^{SI} = 4\pi \cdot 10^{-3} \cdot \chi_m^{CGS} = -9,051 \cdot 10^{-9}[\frac{m^3}{kg}]$$

Examples are given magnetic susceptibility, denoted by:

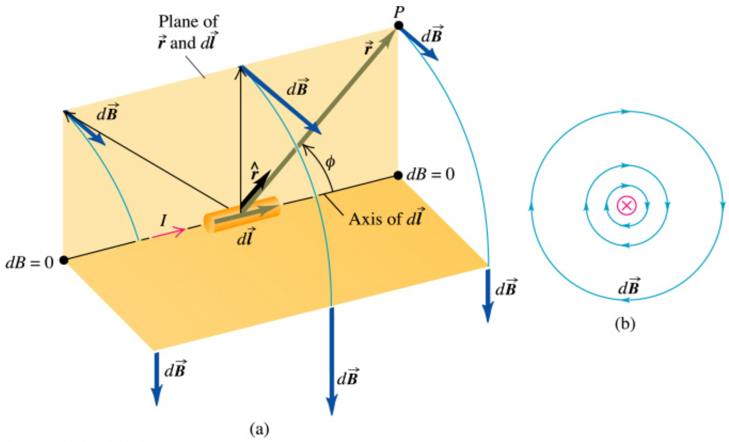
$$\chi[\text{dimensionless quantities}] = K_m - 1$$

Water is a diamagnetic material. Put a layer of water on a powerful magnet, then the magnet field significantly repels the water by its reflection, a slight dimple in the water's surface [22] [8].

2.4 Paramagnetism

Paramagnetic matters have a weakly magnetism resulting from the presence of atoms (or ions) that have permanent magnetic moments. The moments are tiny and randomly oriented in the absence of an external magnetic field. When a paramagnetic substance is placed in an external magnetic field, its atomic moments tend to line up with the field. The process must compete with temperature, which tends to randomize the magnetic moment orientations [19].
Paramagnetism is a form of magnetism whereby the paramagnetic material is only attracted when in the presence of an externally applied magnetic field [15]. Some materials exhibit a magnetization which is proportional to the applied magnetic field in which the material is placed. Right-hand rule for the magnetic field vectors due to current element $d\overrightarrow{l}$:

Figure 2: Right-hand rule for the magnetic field



Both equations named **the law of Biot and Savart**. From $I = n|q|\upsilon_d A$ the magnitude of the magnetic field $d\overrightarrow{\mathbf{B}}$ at any field point P is:

$$\begin{aligned} d\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{Idl\sin\emptyset}{r^2} \\ d\overrightarrow{\mathbf{B}} &= \frac{\mu_0}{4\pi}\frac{Id\overrightarrow{l}\times\widehat{r}}{r^2} \quad \text{(magnetic field of a current element)} \end{aligned}$$

where $d\overrightarrow{l}$ is a vector with length dl, direction is the same as the current in the conductor. With the same method, assumed the total magnetic moment $\overrightarrow{\mu}_{total}$, per unit volume V in the material, we can denote by $\overrightarrow{\mathbf{M}} = \overrightarrow{\mu}_{total}/V$ [A/m].

- the vector quantity $\overrightarrow{\mathbf{M}}$ is **the magnetization** of the material.

- The **additional** magnetic field $\overrightarrow{\mathbf{B}}_1$ is produced by a current loop which is proportional to the loop's magnetic dipole moment per unit volume in this material:

$$\overrightarrow{\mathbf{B}}_1 = \mu_0 \overrightarrow{\mathbf{M}} \quad \text{(SI unit T)}. \tag{2.12}$$

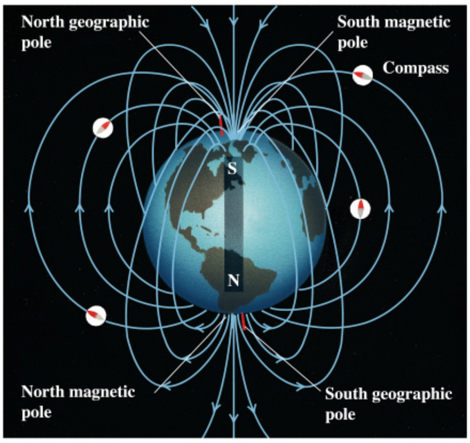
- The equation to the **total** magnetic field $\overrightarrow{\mathbf{B}}_{total}$:

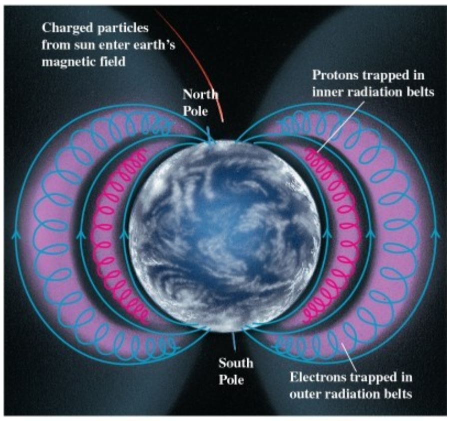
$$\overrightarrow{\mathbf{B}}_{total} = \overrightarrow{\mathbf{B}}_0 + \overrightarrow{\mathbf{B}}_1 \tag{2.13}$$

  where $\overrightarrow{\mathbf{B}}_0$ comes from **the current** magnetic field in the conductor.

All atoms have inherent sources of magnetism because electron spin contributes a magnetic moment and electron orbits act as current loops which produce a magnetic field. In most materials the magnetic moments of the electrons cancel, but in materials which are classified as paramagnetic, the cancellation is incomplete [23]. Paramagnetic materials have a relative magnetic moment with a rather weak positive magnetic susceptibility.

2.5 Induced magnetism





(a)

In this thesis it is assumed the simplest analytical situation - *that the Earth's magnetic field is time-independent*. The Earth's magnetic field is the magnetic field that extends from the Earth's inner core to where it meets the solar wind. A stream of energetic particles are emanating from the Sun, and magnetic position measurements are discontinued during geomagnetic storms. Data from the Time History of Events and Macroscale Interactions during Substorms$THEMIS$[12] show that the magnetic field, which interacts with the solar wind, is reduced when the magnetic orientation is aligned between Sun and Earth. In this thesis we assume that the Earth's magnetic field is constant since any change is of such a short duration and in this time the drilling operation is paused. It is obvious that the drill pipes are under the influence of the Earth's magnetic field. The time scale for magnetic field variations to penetrate inside the drilling pipe is in the order of

$$\tau = R_0^2 \mu \sigma \tag{2.14}$$

where $\mathbf{R_0}$ is the borehole radius and σ the (maximum) electric conductivity of the pipe [16]. Under realistic operating conditions τ will be some fraction of a second, and the field inside the borehole will adjust itself practically instantaneously to variations much slower than this. Another assumption that is present is that there are no macroscopic electric fields, and no electric currents inside the system, so only magnetization is the external field [2]. From Figure 1, $\nabla H = j$ and Ampère's law then $j = 0$. From *Helmholtz* theorem then we can find that H can be derived from a *scalar magnetic potential*, $\mathbf{\Phi}$:

$$\mathbf{H} = -\nabla \Phi$$

In an inhomogeneous system of known geometry Φ is then determined by the remaining **Maxwell equation**, $\nabla \mathbf{B} = 0$, which becomes $\nabla[\mu(r)\nabla\Phi] = 0$. In any region of constant material composition this reduces to:

$$\nabla^2 \Phi = 0 \quad \textit{Laplace's equation} \tag{2.15}$$

The above equation must be solved in each region of different magnetic permeabilities. These solutions must then be joined via implementing the appropriate boundary conditions on the interfaces between the regions. The joining relations can be found using both equations: If I and II are two regions of permeability μ^I and μ^{II}, and n is the unit vector perpendicular to their mutual interface at some point, the conditions on the field components normal and parallel to the interface at this point are [30] [31]:

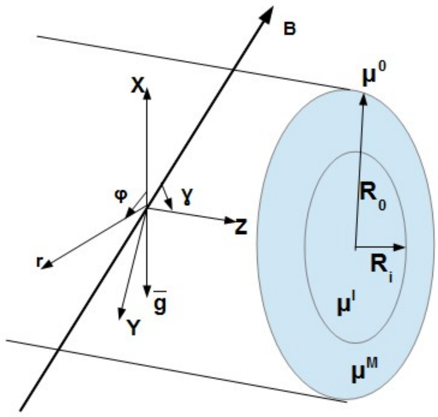
$$B_\perp^I = \mu^I H_\perp^I = \mu^I \mathbf{H}^I n = B_\perp^{II} = \mu^{II} \mathbf{H}_\perp^{II} = \mu^{II} \mathbf{H}^{II} n \tag{2.16}$$

and for the auxiliary magnetic field,

[2] any currents flowing in the borehole would effect the magnetic measurements independently of the magnetic properties of the drilling fluid

$$\mathbf{H}^I_{\parallel} = \mathbf{H}^I \times n = \mathbf{H}^{II}_{\parallel} = \mathbf{H}^{II} \times n \tag{2.17}$$

Figure 3: Wellbore geometry and coordinates.

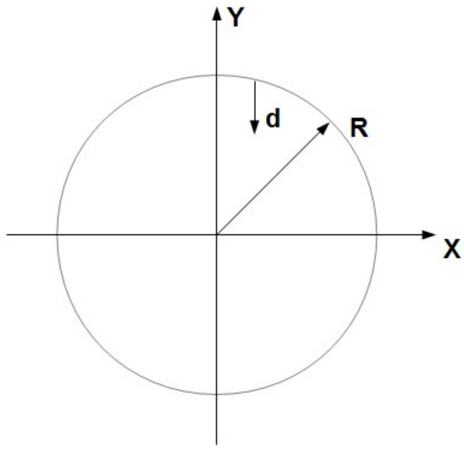


The borehole is centered on the z-axis, and the external magnetic field **B** is in the **XY**-plane. One important result is that for situations with perfect axial symmetry, the magnetometer readings are attenuated proportionally to the square of the magnetic susceptibility. Because the magnetic susceptibility is a small number, the effect on magnetometer readings is generally negligible. However, if the symmetry is broken, the distortion on the magnetometer readings can be increased significantly. This means that form and characters of material can have a strong influence on the strength of the measured magnetic fields [1]. For more-complex geometries, one must resort to numerical modelling. The remaining boundary conditions are that **H** must be everywhere finite, and that far outside the borehole we have the asymptotic behavior:

$$\mathbf{H} \to \frac{1}{\mu_0}\mathbf{B_0} \Longleftrightarrow \Phi \to -\frac{1}{\mu_0}\mathbf{B_0}r; \quad r \to \infty \tag{2.18}$$

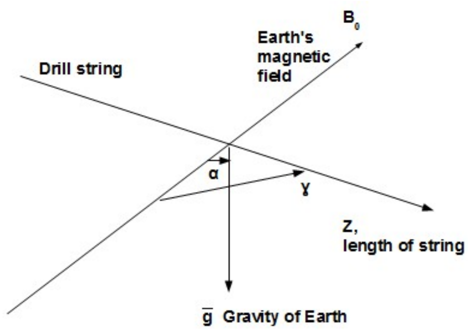
where r is the radial distance from the center of the borehole and $\mathbf{B_0}$ is the external magnetic field. It should be noted that in practice there may be some uncertainty in the asymptotic conditions for the magnetic field inside or close to the borehole. The Earth's "radius" is nearly $6,384$ km , denoted by R in the drawing. The depth of drilling, $\triangle d$, is from Earth's surface to the bottom. The Earth's magnetism changes, $4B$, depending by distance $d \backsim (\triangle d/R)$. Additionally, the dipole magnetic field is reduced after comparison to the depth of the borehole with Earth's radius.

Figure 4: Earth's Cross-section



As record in the world the depth of the borehole at 13 km is still "shallow" compared to the Earth's radius of 6384 km. In the non-magnetic drill collar housing the magnetic sensors only have a finite length, and are connected to the steel drill pipe and the bottom hole assembly, with poorly specified magnetic properties. Therefore we don't need to take into account how the Earth's magnetic field changes when the borehole becomes deeper.

Figure 5: The drill string's angles with gravity and B_0



B_0 is the magnetic field of Earth. If this drill collar is too short, one can have additional stray magnetic fields influencing the measurements [24]. We want to consider how the pipe's field is influenced by the Earth's magnetic field. $\mathbf{B_0}$, Earth's magnetic field can be found by measurements.

There are two estimated angles, one is between the drill string and gravity of Earth, α. The other angle is between the drill string and Earth's magnetic field, γ. The field could have some rest magnetization on the drill string when we are trying to measure it. This could be indicated for if there are no other sources of error.

Below is a list of the three deepest of all the wells in the world:

- Finished in 2011 and $12,345$ meters long in Sakhalin $-I$ Odoptu $OP-11$ Well (offshore the Russian island Sakhalin) [29].
- Finished in 2008 and $12,289$ m long Al Shaheen oil well in Qatar.
- Finshed in 1989, $SG-3$, and $12,262$ meters in Russia, project was named The Kola Superdeep Borehole [25].

Chapter 3

3 Case

3.1 The general case

Measurement While Drilling **(MWD)** in the oil and gas industry is measurements of the Earth's magnetic field. In fact the Earth's Magnetospheric field changes all the time due to the Solar Wind Interaction [26], charged electrical particles from the sun wind. Solutions to analyze the magnetic *MWD* - measurement after high solar activity are missing. The influence on directional *MWD* from drilling fluids has been performed for several cases with realistic representations of *MWD*-tool geometries and varying location of the *Bottom Hole Assembly* (*BHA*) vs. the wellbore. Components and contamination in drilling fluids shield the Earth's magnetic field is as recorded by the magnetic sensors in *MWD* equipment used for directional surveying of oil wells. This shielding can cause azimuth errors of 1 to 2°, and even lager errors may occur for certain wellbore directions under unfavorable conditions. There effects reduce the borehole-position accuracy sufficiently to increase the costs of hitting the planned target [1]. Our geometry is shown in Figure 3. We chose coordinates so that the drill string is along the Z - axis. The X - axis is selected so that the local gravitational field $\overrightarrow{g}$ is left in the XZ - plane, it means that the X - axis is pointed towards the upside of the drill string. The total $\overrightarrow{g}$ is summarized by three vectors measurements. The Earth's magnetic field, B_0, is separated by three components.

$$\mathbf{B}_0 = B_0(\sin\gamma, 0, \cos\gamma) \tag{3.1}$$

so B_0 is in the XZ-plane.
In cylindrical coordinates r, φ, Z, defined by $x = r\cos\varphi, y = r\sin\varphi$, by equation $H = -\nabla\varphi$ takes the form:

$$\mathbf{H_r} = -\frac{\partial\Phi}{\partial r}, \quad H_\phi = -\frac{1}{r}\frac{\partial\Phi}{\partial\varphi}, \quad \mathbf{H_Z} = -\frac{\partial\Phi}{\partial Z} \tag{3.2}$$

By denoting the solution in the region outside the borehole by the superscript $'O'$, the asymptotic condition (2.18) becomes:

$$\Phi^O(r,\varphi,Z) \to -\frac{B_0}{\mu_0}(r\sin\gamma\cos\varphi + Z\cos\gamma) \tag{3.3}$$

$$H_r^O(r,\varphi,Z) \to \frac{B_0}{\mu_0}\sin\gamma\sin\varphi \tag{3.4}$$

$$H_\varphi^O(r,\varphi,Z) \to -\frac{B_0}{\mu_0}r\sin\gamma\cos\varphi \tag{3.5}$$

$$H_z^O(r,\varphi,Z) \to \frac{B_0}{\mu_0}\cos\gamma \tag{3.6}$$

Laplace's equation (2.15) in cylindrical coordinates takes the form:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial Z^2} = 0 \tag{3.7}$$

It can be solved by the standard method of separation of variables. It is convenient first to split off the z-dependency by searching for solutions of the from $\Phi(r,\varphi,z) = \Psi(r,z)Z(z)$. Inserting this in above eq., finds:

$$\frac{1}{H}(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2}) = -\frac{1}{Z}\frac{\partial^2\Phi}{\partial Z^2} = -\kappa^2 \tag{3.8}$$

where κ is a (possibly complex) separation constant. The most general solution of (3.8) is then a linear superposition of such solutions. The solution of the z-dependent part of (3.8) is simply [27].

$$Z(z) = \begin{cases} C_\kappa^I e^{\kappa z} + C_\kappa^{II} e^{-\kappa z}, & \text{if}\kappa \neq 0 \\ C^I + C^{II} z, & \text{if}\kappa = 0 \end{cases} \tag{3.9}$$

where the C's are integration constants. In the region external to the borehole (O) these solutions for $\kappa \neq 0$ lead to asymptotically non-vanishing fields inconsistent with (3.6) in at least one of the limits $z \to \pm\infty$. Thus the only possible solutions for Φ^O are those with $\kappa = 0$. Furthermore, since the field along the borehole is continuous at the interfaces, according to (2.17), the $z-$ dependency of the potential must be of the form (3.9) also inside the borehole. Hence all solutions with $\kappa \neq 0$ are disallowed throughout space. We put the equations left side with a general solution of the form

$$\Phi(r,\varphi,z) \quad = \quad \Psi_I(r,\varphi) + \Psi_{II}(r,\varphi)Z \tag{3.10}$$

where $\Psi_I(r,\varphi)$ and $\Psi_{II}(r,\varphi)$ partly met the two-dimensional Laplace equation:

$$\nabla_2^2\Psi_\kappa = (\frac{\partial^2\Psi_\kappa}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi_\kappa}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi_\kappa}{\partial\varphi^2}) = 0 \quad \kappa = I, II \qquad (3.11)$$

Here ∇_2^2 is the two-dimensional Laplacian in the $XY-$plane, therefore $\Psi_I(r,\varphi)$ and $\Psi_{II}(r,\varphi)$ are harmonic functions. Because both the above equations are derived from the Z-axis dependence of the boundary conditions Φ^O, they are correct with any geometry with translation invariance along the Z-axis. In this thesis we will consider transformation of the magnetic field from one plane to another . Ferromagnetic materials will not be considered. The simple models are chosen to represent idealized situations: A semicircle represents segregated fluids in a horizontal wellbore, while slots represent shielding by struts or fasteners. The MWD tools are places in the wellbore eccentrically to represent tools without centralizers, and the tools are even allowed to touch the wellbore walls in order to model broken shielding. This is done to highlight how undesired situations may influence the MWD readings. In real situations, the stabilizers will in most cases prevent the tool from direct contact with the wellbore wall [1]. Supposing one knows the position of the magnetic measurement tool inside the drilling pipe, and the magnetic properties of the drilling fluid etc., it is possible to calculate the magnetic shielding by using strong and effective analytical and numerical techniques to solve the two - dimensional Laplace equation (3.10).

3.2 The coaxial case

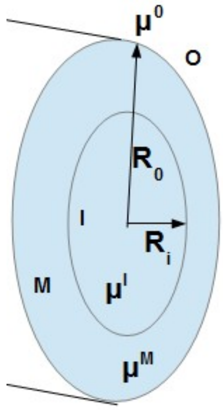
A magnetometer is a measuring instrument used to measure the strength and the direction of magnetic fields. Normally the magnetometer is placed inside a narrow non-magnetic air-filled cylinder, which ideally is in a centralized position in the drill pipe. If we take for granted that the drilling fluid is magnetically uniform, we can pattern the borehole, as far as magnetic properties are concerned, as comprising effectively of three coaxial [3] cylindrical regions:

- an internal circle with radius R_i bordering the measurement tool, graphic symbol by the I.
- drilling fluid in the intermediate region, graphic symbol by the M.

[3]Coaxial in geometry means that two or more forms share a common axis; it is the three dimensional linear analogue of concentric.

- the external circle with radius R_0, to describe the structure surrounding the borehole, graphic symbol by the O.

Figure 6: Coaxial Cylinder



The method is readily generalized to any number of coaxial regions. This gives more detailed and realistic modelling e.g. in order to treat the zone treatment between the magnetic tool housing and the drill collar, the drill collar itself, and the zone between the borehole wall individually. For less complexity it is assumed that the central and outer sections have equivalent magnetic permeability, $\mu_I = \mu_O = \mu_0$ - hence neglecting possible magnetic properties of the surrounding rock. The permeability of the drilling fluid in the intermediate region will be denoted by $\mu_M = \mu$. With these assumptions eqs. (2.16) and (2.17), $H(r, \varphi, z)$ on the two interfaces:

$$B_r^O(R_o, \varphi, z) = \mu_0 H_r^O(R_o, \varphi, z) = B_r^M(R_o, \varphi, z) = \mu H_r^M(R_o, \varphi, z) \quad (3.12)$$
$$B_r^M(R_i, \varphi, z) = \mu H_r^M(R_i, \varphi, z) = B_r^I(R_i, \varphi, z) = \mu_0 H_r^I(R_i, \varphi, z) \quad (3.13)$$
$$H_z^O(R_0, \varphi, z) = H_z^M(R_0, \varphi, z) \quad (3.14)$$
$$H_z^M(R_i, \varphi, z) = H_z^I(R_i, \varphi, z) \quad (3.15)$$
$$H_\phi^O(R_0, \varphi, z) = H_\varphi^M(R_0, \varphi, z) \quad (3.16)$$
$$H_\phi^M(R_i, \varphi, z) = H_\varphi^I(R_i, \varphi, z) \quad (3.17)$$

By noting that the Laplacian (3.11) and the boundary conditions (3.17) and (3.3)-(3.6) have well defined properties with the mirror symmetry $\varphi \leftrightarrow (-\varphi)$ that must also to the case for the solutions. So one has the symmetry relations (in the following, we shall drop the superscripts O, M, I for formulas valid in all three regions):

$$\begin{aligned}
\Phi(r,-\varphi,z) &= \Phi(r,\varphi,z) && (3.18)\\
H_r(r,-\varphi,z) &= H_r(r,\varphi,z) && (3.19)\\
H_\varphi(r,-\varphi,z) &= -H_\varphi(r,\varphi,z) && (3.20)\\
H_z(r,-\varphi,z) &= H_z(r,,z) && (3.21)
\end{aligned}$$

Solving the two - dimensional Laplace equation in cylinder coordinates, eq.(3.11), by the method of separation of variables, $H(r,\varphi) = R(r)\cdot\Phi(\varphi)$. We using the complex form $\Phi(\varphi) = ce^{\pm im\varphi}$, the real part $R(r) = ar^m + br^{-m}$, we can get the expansion solution

$$\Phi(\varphi) = A\cos(m\varphi) + B\sin(m\varphi), \quad \texttt{where A and B are constants.}$$

Since magnetic potential works on the well periodically, and must have $\Phi(\varphi) = \Phi(\varphi + 2m\pi)$, where m is an integer, by the method of Fourier series, is well known [39].

$$\Phi(\varphi) = \sum_{m=0}^{\infty} a_m\cos(m\varphi) + b_m\sin(m\varphi)$$

After considering the mirror symmetry, from (3.18), $\Phi(r,-\varphi,z) = \Phi(r,\varphi,z)$ $\Rightarrow$ $\mathrm{b}_m = 0$.

$$\Phi(\varphi) = \sum_{m=0}^{\infty} a_m\cos(m\varphi) = a_0 + \sum_{m=1}^{\infty} a_m\cos(m\varphi)$$

By Euler method to find the solution of $R(r)$, when $m = 0$ is

$$R_0(r) = a_0 + b_0\ln(r), \quad \texttt{here C and D are constants,} \qquad (3.22)$$

General real solution:

$$\begin{aligned}
\Phi(\varphi) &= \sum_{m=0}^{\infty}[a_m r^m\cos(m\varphi) + b_m r^m\cos(m\varphi)\\
&+ c_m r^{-m}\sin(m\varphi) + d_m r^{-m}\sin(m\varphi)] && (3.23)
\end{aligned}$$

The **Laplacian equation** (3.11) can be solved by the method of separation of variables. So the harmonic equations finds (the indices $J = (I, M, O)$ refer to the three regions):

$$\begin{aligned}
\Psi_I^J(r,\varphi) &= a_0^J + c_0^J lnr + \sum_{m=1}^{\infty}(a_m^J r^m + \frac{c_m^J}{r^m})\cos(m\varphi) && (3.24)\\
\Psi_{II}^J(r,\varphi) &= b_0^J + d_0^J lnr + \sum_{m=1}^{\infty}(b_m^J r^m + \frac{d_m^J}{r^m})\cos(m\varphi) && (3.25)
\end{aligned}$$

The absence of terms involving $sin(m\varphi)$ in these expressions is a consequence of the mirror symmetry (3.18). As a matter of fact eg.3.2 and eq.3.20, we see that

$$\Psi_{II} = -H_Z^O \quad (3.26)$$

Chapter 4

4 Conformal mapping

By previous work [20] it is shown that *conformal mapping* can be used to solve the specific tasks in this thesis. The main reason to use complex analysis is because the solutions has both real part and imaginary parts which satisfy the *Laplacian equation*. In physics, the *Laplacian equation* is the most important partial differential equation which applies to two and three dimensions. The second partial derivative is continuous in a range for example in fluid flow, gravity, thermal conductivity and electrical statistics. It can handle a situation for two dimensions using complex analysis, since it is known that the imaginary and real part of the analytic function is harmonic.

4.1 Cauchy-Riemann Equations

A complex equation, z, can be analyzed in an open region, D. (3.24) and (3.25) are continuous in the borehole. The primary concern is trying to calculate them with complex analysis. Supposed that

$$\begin{aligned} \text{z} &= x + iy \\ \text{f(z)} &= u(x,y) + i\nu(x,y) \text{(function of a complex number z)} \end{aligned}$$

Cauchy - Riemann equations [37] in coordinate systems, we could get a differentiable pair of functions u and ν, then so

$$u_x = \nu_y = \frac{\partial u}{\partial x}, \quad u_y = -\nu_x = \frac{\partial u}{\partial y}.$$

By polar representation, $z = r(\cos\varphi + i\sin\varphi) = r \cdot e^{i\varphi}$, the equation's form becomes:

$$u_r = \frac{1}{r}\nu_\varphi, \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial \nu}{\partial \varphi}, \quad \nu_r = -\frac{1}{r}u_\varphi, \quad \frac{1}{r}\frac{\partial u}{\partial \varphi} = -\frac{\partial \nu}{\partial r}.$$

If u and v satisfy both the *Cauchy-Riemann* equations and *Laplace's* equation (continuity equations) in two dimensions:

$$
\begin{aligned}
\nabla^2 u &= u_{x\cdot x} + u_{y\cdot y} = 0 \\
\nabla^2 v &= v_{x\cdot x} + v_{y\cdot y} = 0 \\
&\Rightarrow \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) + \frac{\partial}{\partial y}(-\frac{\partial v}{\partial x}) = 0 \\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial x}(-\frac{\partial u}{\partial y}) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) = 0
\end{aligned}
$$

The analytical equation $f(z)$ has two harmonic equations (u and v), here v is a harmonic function of u in zone D. *Laplace's Equations*: $\nabla^2\psi = 0$ is the most important partial differential equation.

4.2 Möbius Transformation

Complex analysis mathematics can be used when considering the drill string in an asymmetrical position in the bore hole. By conformal image we can see equations under the group of *Möbius function*[4], a rational function [37]:

Definition `Möbius(or Moebius) transformation`

$$
w = f(z) = \frac{az+b}{cz+d}
$$

where $|a| + |c| > 0, ad \neq bc$, here the coefficients a, b, c, d are complex or real numbers, $z \in \forall\mathbb{C}$ (all complex numbers), so that w is not a constant function.

Notice that since

$$
f' = \frac{\partial f}{\partial z} = \frac{dw-b}{-cw+a}
$$

does not vanish, the Möbius transformation $f(z)$ is conformal at every point except its pole $z = -\frac{d}{c}$. The inverse function $z = f^{-1}(w)$, (that is $f \circ f^{-1} \equiv I$, $I-$ the identity), can be computed directly:

$$
f^{-1}(w) = \frac{dw-b}{-cw+a}
$$

[4] sometimes known as a homographic transformation, or linear fractional transformation, bilinear transformations, or fractional linear transformations

Theorem 4.1. (Möbius transformation) *A Möbius transformation is uniquely determined by three points $z_i, i = 1, 2, 3, z_i \neq z_j, i, j = 1, 2, 3$.*

Let z_i, and w_i, be given, $z_i \neq z_j, w_i \neq w_j, i.j = 1, 2, 3$. We are looking for a transformation $w = f(z)$ such that

$$f(z_i) = w_i \tag{4.1}$$

Consider the cross-ratio (z, z_1, z_2, z_3) of the points $z, z_i, i = 1, 2, 3$, that is

$$T(z) = (z, z_1, z_2, z_3) := \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}. \tag{4.2}$$

The function $T(z)$; $z \in \mathbb{C}$; z_i−fixed maps $\mathbb{C}$ in a one-to-one way onto itself. Notice that the desired transformation (4.1) is given by the composition

$$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3).$$

It remains only to equate w. $w = f(z)$ is the only transformation with (4.1).

Proposition 4.2. *Möbius transformations form the group of transformations $\tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ generated (under composition) by:*

- ***translations*** *â maps of the form $z \longmapsto z + k$ where $k \in \tilde{\mathbb{C}}$;*
- ***scalings or dilations*** *â maps of the form $z \longmapsto kz$ where non-zero $k \in \tilde{\mathbb{C}}$ and $k \neq 0$;*
- ***inversion*** *â the map $z \longmapsto \frac{1}{z}$. (Note this map is not an actual inversion in the sense of inverting in a circle.)*

4.3 Decomposition and elementary properties

A Möbius transformation is equivalent to a sequence of simpler transformations. Let f be any *Möbius transformation*. Then

- $f_1(z) = z + \frac{d}{c}$(translation by $\frac{d}{c}$)
- $f_2(z) = \frac{1}{z}$(inversion and reflection with respect to the real axis)
- $f_3(z) = -\frac{(ad-bc)}{c^2} \cdot z$ (dilation and rotation)
- $f_4(z) = z + \frac{a}{c}$ (translation by $\frac{a}{c}$)

then these functions can be composed, giving

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) \quad = \quad z = f(w) = \frac{aw+b}{cw+d} \tag{4.3}$$

The inverse Möbius transformation is obviously derived by, that is, define function g_1, g_2, g_3, g_4 so each g_i is the inverse of f_i. We can get inverse formula is

$$g_4 \circ g_3 \circ g_2 \circ g_1(z) \quad = \quad w = f^{-1}(z) = \frac{dz-b}{-cz+a} \tag{4.4}$$

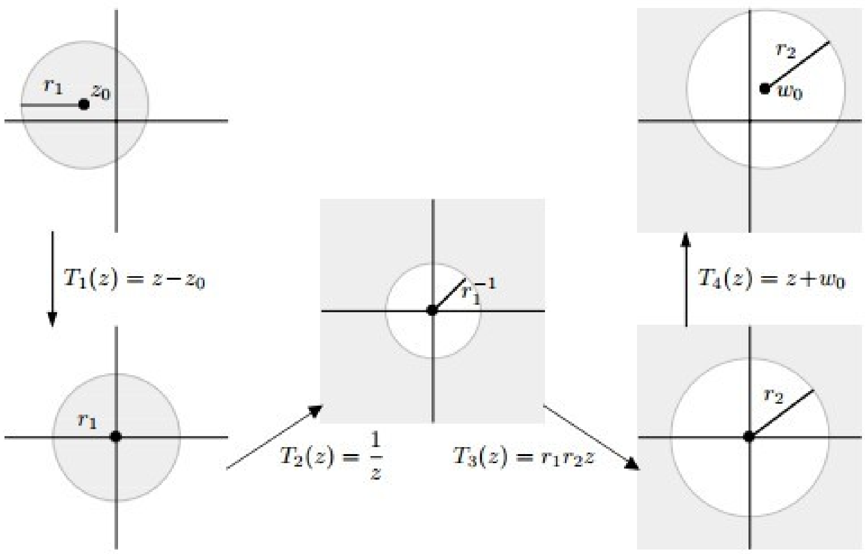
Definition We will use the term **circline** to denote anything which is a circle or a line in the complex plane.

Proposition 4.3. *The Möbius transformations map circlines to circlines.*

Example Supposed [21] that $z_0, w_0 \in \mathbb{C}$ and $r_1, r_2 > 0$ are fixed, and that we are required to find a Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which maps the disc $\{z :| z - z_0 |< r_1 \}$ to the annulus $\{w :| w - w_0 |> r_2 \}$. This can be achieved by taking $T = T_4 \circ T_3 \circ T_2 \circ T_1$, where

$$T_1(z) = z - z_0 \quad \text{and} \quad T_2(z) = \frac{1}{z} \quad \text{and} \quad T_3(z) = r_1 r_2 z \quad \text{and} \quad T_4(z) = z + w_0$$

We have the picture below



Note that T_1 is a translation which takes the centre of the disc $\{z :\mid z - z_0 \mid < r_1 \}$. Then the inversion T_2 turns into an annulus. We now apply a magnification T_3 and then use the translation T_4 to position the disc so that its centre is at w_0. It shows that

$$\text{T(z)} \quad = \quad \frac{r_1 r_2}{z - z_0} + w_0 = \frac{w_0 z + (r_1 r_2 - w_0 z_0)}{z - z_0}$$

We first observe that $z_0 = 0, w_0 = 0, T(z) = \frac{r_1\, r_2}{z}$,

$$\begin{aligned} T(0) &= \infty \\ T(\infty) &= 0 \\ T(r_1) &= r_2 \\ T(r_2) &= r_1 \end{aligned}$$

Further, we see that

$$\begin{aligned} T(z_0) &= \infty \\ T(0) &= w_0 - \frac{r_1 r_2}{z_0} \\ T(\infty) &= w_0 \end{aligned}$$

As a result of our previous deliberations, we can summarize some properties of Möbius transformations [32].

Theorem 4.4. *Let f be any Möbius transformation. Then*

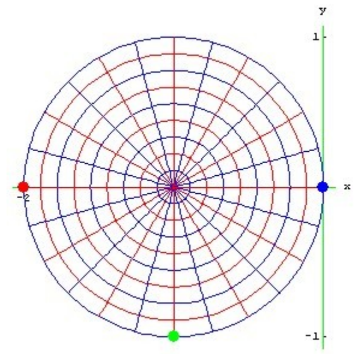
- *f can be expressed as the composition of a finite sequence of translations, magnifications, rotations, and inversions.*
- *f maps the extended complex plane one-to-one itself.*
- *f maps the class of circles and lines to itself.*
- *f is conformal at every point except its pole.*

The third property is distinguished as follows. If a line or circle

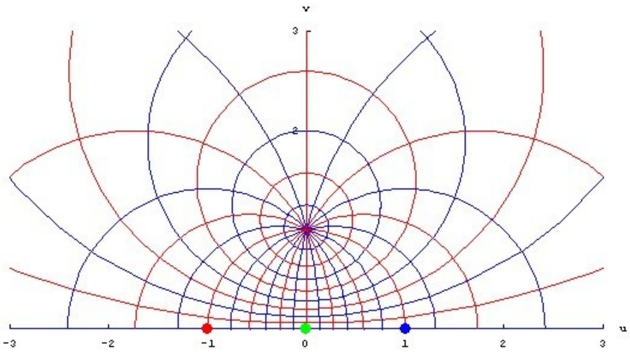
- passes through the pole($z = -\frac{d}{c}$) of the Möbius transformation, it gets mapped to an unbounded figure. Hence its image is a straight line.

- that avoids the pole is mapped to a circle.

Example The bilinear mapping $w = s(z)$ maps the disk D



The disk |z+1| < 1.



The image half-plane Im[w] > 0,

$$\text{under the mapping } w = s(z) = \frac{2 + (1 - i)z}{2 + (1 + i)z}$$

$$s[-2] = -1, s[-1 - i] = 0, s[0] = 1$$

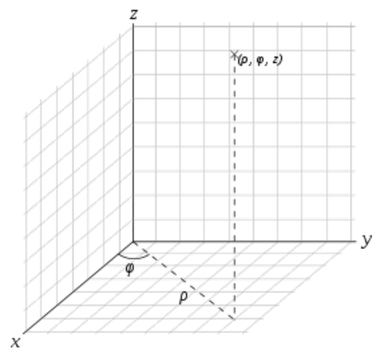
We try to transfer the original drill pipe in the original plane to the transformed plane, but the outside drill pipe, here we say it is the big circle cannot change, but the small drill string, the small circle, can be transformed with a new center. Therefore we can try to show the method which is dependent on the complex analysis.

Theorem 4.5. *(Symmetry Principle [37]) Let C_2 be a line or circle in the $z-$plane, and let $= f(z)$ be any Mòbius transformation. Then two points z_1 and z_2 are symmetric with respect to C_z if and only if their images $_1 = f(z_1)$, $w_2 = f(z_2)$ are symmetric with respect to the image of C_z under f .*

Chapter 5

5 Cylindrical coordinate system

Cylindrical coordinates (cylindrical coordinate system) is the extension of the two-dimensional polar coordinates to the Z-axis which is in a three-dimensional coordinate system. After we added the third coordinate that could designed to indicate the height of the point P from the XY-plane. Three factors: the radial distance, azimuth (or the angular position) and height (by the International Organization for Standardization ($ISO31-11$)) are labelled in the figure, below,



Definition The three coordinates (ρ, φ, z) of a point P are defined as:

- The radial distance ρ is the Euclidean distance from the z axis to the point P.
- The azimuth φ is the angle between the reference direction on the chosen plane and the line from the origin to the projection of P on the plane.
- The height z is the signed distance from the chosen plane to the point P [35].

5.1 Laurent's Theorem

Definition Let $f : D \to \mathbb{C}$ be differentiable on the annulus $A_r^R(z_0) = \{z : r <| z - z_0 |< R\} \subset D$ (where $r \geq 0$ and R may be ∞). Then $f(z)$ can be expressed uniquely by

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \tag{5.1}$$

on $A_r^R(z_0)$. For any choice of simple closed contour $\mathbb{C} \subset A_r^R(z_0)$ with $n_{\mathbb{C}}(z_0) = 1$, the coefficients a_j and b_j are given by

$$a_j = \frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{f(\xi)}{(\xi - z_0)^{j+1}} d\xi \quad \text{for} \quad j \geq 0 \tag{5.2}$$

and

$$b_j = \frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{f(\xi)}{(\xi - z_0)^{-j+1}} d\xi \quad \text{for} \quad j \geq 1 \tag{5.3}$$

Figure 7: Laurent's theorem

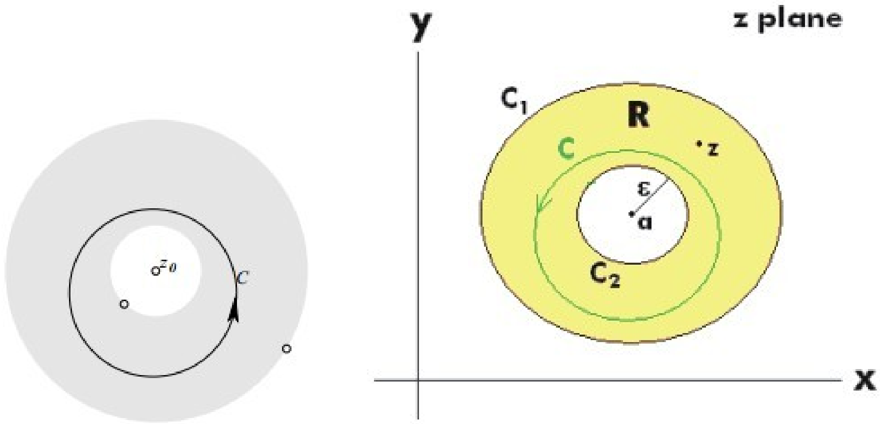


Figure 7 is satisfying $r < r_2 <| z |< r_1 < R$.

Theorem 5.1. *(Laurent's Theorem [13]) Let $f(z)$ be analytic throughout the closed annular region R bounded by two concentric circles, C_1 and C_2, centered at point a and let z be a point in R. Then $f(z)$ can be represented by*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \tag{5.4}$$

where $a_k = \frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{f(w)}{(w-a)^{k+1}} dw, (k = 0, \pm 1, \cdots)$ is a constant and each integral is taken in the counter clockwise direction around any closed curve C in the annular region that encircles the inner boundary.

5.2 Complex analysis

In complex form $\Phi(w) = \Phi_{Re} + i\Phi_{Im}$, that $\Phi^O(r_w, \theta, Z) \to -\frac{B_0}{\mu_0}(r_w \sin\gamma \cos\theta + Z\cos\gamma)$, since $\Phi^O(z)$ is analytical function, then it should satisfy the Cauchy-Riemann equations, $i\frac{\partial \Phi_{Re}}{\partial r_w} = \frac{\partial \Phi_{Im}}{\partial \theta}$, we assume

$$\begin{aligned}
\Phi_{Re} &= -\frac{B_0}{\mu_0}(r_w \sin\gamma\cos\theta + Z\cos\gamma) \\
\frac{\partial \Phi_{Re}}{\partial r_w} &= -\frac{B_0}{\mu_0}\sin\gamma\cos\theta \\
\rho\frac{\partial \Phi_{Re}}{\partial r_w} &= \frac{\partial \Phi_{Im}}{\partial \theta} \quad \textit{Cauchy-Riemann} \\
&= -r_w\frac{B_0}{\mu_0}\sin\gamma\cos\theta \qquad (5.5)
\end{aligned}$$

We try to solve this

$$\Phi_I = \int(-r_w\frac{B_0}{\mu_0}\sin\gamma\cos\theta)d\theta = -r_w\frac{B_0}{\mu_0}\sin\gamma\sin\theta + g(r_w) \qquad (5.6)$$

$g(r_w)$ is a function of parameter r_w. So we can find that

$$\frac{d(g(r_w))}{dr_w} = 0 \qquad (5.7)$$

to

$$\begin{aligned}
\Phi_{Imr_w} &= -\frac{1}{r_w}\Phi_{Re\theta} \quad \textit{Cauchy-Riemann} \qquad (5.8) \\
&= -r_w\frac{B_0}{\mu_0}\sin\gamma\sin\theta + \frac{dg(r_w)}{dr_w} \\
\Phi_{Im} &= -\frac{B_0}{\mu_0}r_w\sin\gamma\sin\theta \qquad (5.9)
\end{aligned}$$

Since we have found Φ_{Re}, Φ_{Im} in $\Phi = \Phi_{Re} + i\Phi_{Im}$, we can write the following:

$$\begin{aligned}
\Phi &= \Phi_{Re} + i\Phi_{Im} \\
&= -\frac{B_0}{\mu_0}(r_w \cdot sin\gamma \cdot \cos\theta + Z\cdot\cos\gamma) + i(-\frac{B_0}{\mu_0}\cdot r_w \cdot \sin\gamma\sin\theta) \\
&= -\frac{B_0}{\mu_0}[r_w \cdot \sin\gamma(\cos\cdot\theta + i\cdot\sin\theta) + Z\cdot\cos\gamma] \\
&= -\frac{B_0}{\mu_0}(\sin\gamma\cdot r_w\cdot e^{i\theta} + Z\cdot\cos\gamma) \\
&= -\frac{B_0}{\mu_0}(\sin\gamma\cdot w + Z\cdot\cos\gamma) \qquad (5.10)
\end{aligned}$$

We assume that

$$\begin{aligned}\Phi^O(r,\theta,z) &= \Psi_I^J(r,\theta) + \Psi_{II}^J(r,\theta)\cdot Z \\ &= -\frac{B_0}{\mu_0}(\sin\gamma\cdot w + Z\cdot\cos\gamma)\end{aligned}$$

here $w = X + iY = r_w \cdot e^{i\theta}, r_w = \sqrt{arctan(\frac{Y}{X})}$, J in areas I, O, M. We see here the solution of Ψ_I is real part of Φ, the solution of Ψ_{II} is the imaginary part of Φ. From equations (3.2), we get in the outside H- field,

$$H_r^O \rightarrow -\frac{B_0}{\mu_0}\sin\gamma\cdot e^{i\theta} \tag{5.11}$$

$$H_\theta^O \rightarrow \frac{B_0}{\mu_0}\sin\gamma\cdot i\cdot e^{i\theta} \tag{5.12}$$

$$H_Z^O \rightarrow -\frac{B_0}{\mu_0}\cos\gamma \tag{5.13}$$

We will now map the "physical" $w = X + iY$- plane conformally onto another plane, $z = x + iy$, in such a manner these the inner and outer pipe walls are mapped onto two concentric circles. This is achieved by the **Möbius** transformation. $z = r \cdot e^{i\varphi}$ with inner $w = \dfrac{z+s}{zs+1}$.
See Figure 8. Under this mapping the center of the outer pipe wall remain at the circle [20].

For example. In this big circle

$$w = \frac{az+b}{cz+d}, \quad c \neq 0$$

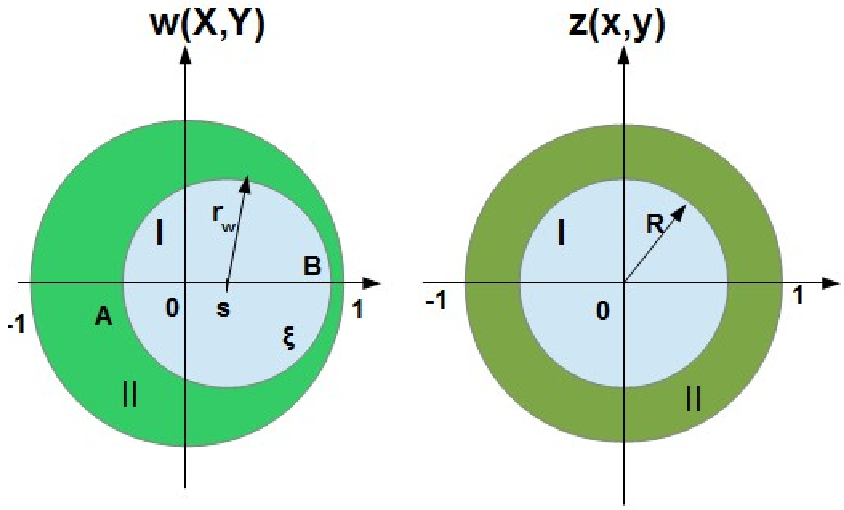
Here $c \neq 0$ for linear transformation. Furthermore, there becomes

$$w = \frac{\frac{a}{c}z + \frac{b}{c}}{z + \frac{d}{c}} = \frac{A'z + B'}{z + D'}$$

We try to hold the same points on the new plane $w(z)$, it means transforming these points at the center $(0,0)$, and two limit points $(1,0)$, $(-1,0)$ in Z -plane $\rightarrow$ to $(s,0)$, $(1,0)$, $(-1,0)$ in $w(z)$ -plane. We get that the result is $A = D$, $B = 1$. The point $(\frac{1}{s}, 0)$ in $w(z)$-plane,

$$w(z) = \frac{\frac{1}{s}z + 1}{z + \frac{1}{s}} = \frac{z+s}{zs+1} \tag{5.14}$$

Figure 8: Conformal mapping between two of coordinates.



- if $s \to 0, w(z) \to z$, then we will get the unchanged mapping.
- if $s \to 1, w(z) \to 1$, the mapping in the $w(z)-$ plane must be inside the big circle. The value is, $s < 1$.
- So $s \neq \pm i$ is singular point.

From Figure 8 we can see the original point $(-R, 0)$ in $f(z)$-plane to $(A, 0)$ in $w(z)$-plane,

$$w(-R) = \frac{s-R}{1-Rs} = A \tag{5.15}$$

and the original point $(R, 0)$ in $f(z)$-plane to $(B, 0)$ in $w(z)$-plane, then,

$$w(R) = \frac{s+R}{1+Rs} = B \tag{5.16}$$

the radius ρ to the new small circle is

$$r_w = \frac{B-A}{2} = \frac{R(1-s^2)}{1-(Rs)^2} \quad \texttt{when} \quad s \to 0, r_{w(s\to 0)} = R \tag{5.17}$$

in the circle ξ with the center is $(s, 0)$ to the new little circle is

$$\xi = \frac{A+B}{2} = \frac{S(1-R^2)}{1-(Rs)^2} \quad \texttt{when} \quad s \to 0, \xi_{s\to 0} = 0 \tag{5.18}$$

In this mapping, the point at $|\ w\ | = \infty$ is mapped to $z_\infty = -\frac{1}{s}$. The transformed potential

$$\Phi(w) = \Phi(w(z)) \tag{5.19}$$

is still a solution for the **Laplace's** equation $\nabla^2\Phi = 0$. Furthermore this transformation also concerns all boundary conditions.

5.3 Mapping

By mapping an analytical function, image angles between curves are preserved. An analytical function is differentiable and the derivative is continuous and can never be zero. We get conformal mapping except in the critical point where the derivative is zero, $f'(z) = 0$. Conformal mapping is a good tool if we are to solve the boundary problem for the *Laplacian equation*. This is because the analytical function comprising two harmonic functions remains harmonic functions under conformal mapping. An example of this is the electrical statistics where the potential can be written as a function comprising a real and an imaginary part [34]. In a magnetostatic system (with no macroscopic currents) having translational symmetry along some axis which shall be taken as the Z-axis, takes the form from (3.10) and the components of the magnetic field strength, H in the equations (3.2), and then

$$\mathbf{H_{r_w}} = -\frac{\partial\Phi}{\partial r_w} = -\frac{\partial\Psi_I}{\partial r_w} - Z\frac{\partial\Psi_{II}}{\partial r_w} \quad (5.20)$$

$$H_\theta = -\frac{1}{r_w}\frac{\partial\Phi}{\partial\theta} = -\frac{1}{r_w}\frac{\partial\Psi_I}{\partial\theta} - \frac{Z}{r_w}\frac{\partial\Psi_{II}}{\partial\theta} \quad (5.21)$$

$$\mathbf{H_Z} = -\frac{\partial\Phi}{\partial Z} = -H_{II} \quad (5.22)$$

If the regions have permeabilities μ_0 and μ_m, the boundary conditions for the fields at their common boundary are:

$$B^I_\perp = B^{\|}_\perp \Longleftrightarrow \mu_0 H'_\perp = \mu_m H''_\perp \quad (5.23)$$

$$H'_\| = H''_\| \quad (5.24)$$

$$H'_Z = H''_Z \quad (5.25)$$

Here the subscripts $\perp$ and $\|$ denotes the directions in the XY-plane normal and parallel to the interface, respectively. This means that the angle α^j ($j = 1, 2$) between the magnetic field strength in the xy-plane and the surface normal, $\mathbf{n}$, is simply given by:

$$\tan\alpha = \frac{H_\|}{H_\perp} \quad (5.26)$$

The relation between the direction angles of the magnetic fields at a boundary point is thus, from equations (5.23):

$$\tan\alpha^I = \frac{H'_\|}{H''_\perp} = \frac{H''_\|}{\frac{\mu_m}{\mu_0}H''_\perp} = \frac{\mu_0}{\mu_m}\tan\alpha^{II} \quad (5.27)$$

If one performs a conformal mapping of the xy-plane, regarded as a function of the complex variable $x + iy$, into another complex plane, these boundary conditions remain unchanged, since a conformal mapping conserves angles. If the field component in the z-direction, H_Z, is left unchanged by the mapping, any solution of the equation $\nabla^2\Phi = 0$ remains a solution after the mapping, and with the **same** boundary conditions as before, as given in equation (5.23).
There are two regions, both the inner pipe and between two pipes. **In the outside area**, from equations (3.2), we get in the outside H- field, $z > 1$, $z \to \infty$, $H \to \infty$.

$$\Psi^{O}_{II\,z\to-\frac{1}{s}} \quad \to \quad L_{II}, \quad \textit{there} \quad L_{II} = -\frac{B_0}{\mu_0}\cos\gamma \tag{5.28}$$

to compare the solution of the Laplace equations. Now we can try with the point $z_0 = 0$.

$$\begin{aligned} w(z) &= \frac{z+s}{zs+1} = \frac{z+\frac{1}{s}}{zs+1} + \frac{s-\frac{1}{s}}{zs+1} = \frac{1}{s} + \frac{s-\frac{1}{s}}{zs+1} \\ &= \frac{1}{s} + \frac{s-\frac{1}{s}}{zs}\cdot\frac{1}{1+\frac{1}{zs}} \end{aligned}$$

since $z < \frac{1}{s}$, we can write $\frac{1}{1+\frac{1}{zs}}$ in a geometric series [17],

$$\frac{1}{1+\frac{1}{zs}} = \sum_{m=0}^{\infty}(-\frac{1}{zs})^m$$

$$\begin{aligned} \Psi^{O}_{I\,z\to 0}(W(z)) &= L_I[\frac{1}{s} + \frac{s-\frac{1}{s}}{zs}\sum_{m=0}^{\infty}(-\frac{1}{zs})^m] \\ &= L_I\cdot\frac{1}{s} - L_I\cdot(s-\frac{1}{s})\sum_{m=0}^{\infty}(-\frac{1}{zs})^{m+1} \end{aligned}$$

there

$$\begin{aligned} b_0^O &= L_1 \cdot \frac{1}{s} = -\frac{1}{s} \cdot \frac{B_0}{\mu_0} \sin \gamma \\ d_0^O ln(z) &= 0 \\ d_m^O &= -L_I(s - \frac{1}{s}) = \frac{B_0}{\mu_0} \sin \gamma (s - \frac{1}{s}) \\ b_m^O &= 0 \end{aligned}$$

This is a result of the asymptotic effect of magnetic field number with negative potencies, but we transfer the equation with positive potencies, then we can get a series that is

$$\begin{aligned} \Psi^O_{I\ z\to 0}(z) &= L_I \cdot \frac{1}{s} - L_I \cdot (s - \frac{1}{s}) \sum_{m=0}^{\infty} (-\frac{1}{zs})^{m+1} + \sum_{m=1}^{\infty} g_m \cdot (zs)^m \\ &\equiv a_0^O + c_0^O \cdot ln(z) + \sum_{m=1}^{\infty} (a_m^O \cdot z^m + c_m^O \cdot z^{-m}) \\ \texttt{there} \quad L_I &= -\frac{B_0}{\mu_0} \cdot \sin \gamma \end{aligned}$$

coefficients for the series with positive potencies, we try to find the solution of $\sum_{m=1}^{\infty} g_m \cdot (zs)^m$. Both functions $\Psi^O_{I\ z\to -\frac{1}{s}}(z) = \Psi^O_{I\ z\to 0}(z)$ applies for the area in the figure.

$$\begin{aligned} \Psi^J_{II}(r, \varphi) &= b_0^J + d_0^J lnz + \sum_{m=1}^{\infty} [(b_m^J z_m + \frac{d_m^J}{z_m}) \cos(m\varphi)] \\ \texttt{When} \quad \Psi^O_{IIz\to 0} &\to L_{II} \to 1 \end{aligned}$$

From equations we can get potentials, where $z = r \cdot e^{i\varphi}$, z′ is showing in Figure 7. Earlier it gives $H'_z = -\Psi_{II}$, here we see now $\Psi^O_I(z)$. Generally in the area J, applies the equation,

$$\Psi^J_I(z) = a_0^O + c_0^O ln(z) + \sum_{m=1}^{\infty} (a_m^O z^m + c_m^O z^{-m}), \quad z = re^{i\varphi}.$$

$$\Psi^J_I(z) = a_0^O + c_0^O ln(z) + ic_0^O + \sum_{m=1}^{\infty} (a_m^O r^m e^{im\varphi} + c_m^O r^{-m} e^{-im\varphi})$$

Expressed in the transformed variable $z = x + iy$, the problem to be solved in thus the same as in [2], with the same geometry, except that the boundary conditions at infinity (3.3) ,

$$\Psi_I^O(w)_{w\to\infty} \to L_I \cdot w \tag{5.29}$$

is placed by

$$\Psi_I^O(z)_{z\to -\frac{1}{s}} \to L_I \cdot w(z) = L_I \frac{z+s}{zs+1} \tag{5.30}$$

This is the series describing the asymptotic behaviour of the magnetic field, but it also has a contributions nearby well, as shown in this contribution must also be considered. First of all conditions, $w \to \infty$, $z_0 = -\frac{1}{s}$,

$$w = \frac{z+s}{z \cdot s + 1} = \frac{z + \frac{1}{s} + s - \frac{1}{s}}{s(z + \frac{1}{s})} = \frac{1}{s} + \frac{1 - \frac{1}{s^2}}{z + \frac{1}{s}} \to \frac{1 - \frac{1}{s^2}}{z + \frac{1}{s}} \tag{5.31}$$

In region I,

$$\Psi_I^I(w) = \sum_{m=0}^{\infty} a_m^I \cdot z^m \tag{5.32}$$

The sum starts at $m = 0$ because there are no magnetic poles. Between the two pipes

$$\Psi_I^M(w) = \sum_{m=-\infty}^{\infty} a_m^M z^m + a_0^{m'} lnZ \quad m \neq 0 \tag{5.33}$$

From eq.(3.10),

$$\begin{aligned}
\Phi^I(r,\varphi,Z) &= \Phi^M(r,\varphi,Z) \qquad (5.34)\\
&\Rightarrow \\
\Psi_I^I(r,\varphi) + \Psi_{II}^I(r,\varphi)Z &= \Psi_I^M(r,\varphi) + \Psi_{II}^M(r,\varphi)Z, \quad \texttt{for all Z} \\
&\Longleftrightarrow \\
\Psi_I^I(r,\varphi) &= \Psi_I^M(r,\varphi) \\
\Psi_{II}^I(r,\varphi) &= \Psi_{II}^M(r,\varphi)
\end{aligned}$$

We find similar in (3.15) and (3.17), and $\Psi_z = H_{II}$,

$$\begin{aligned}
H_r &= \Psi^{I,M}_{I,r}(r,\varphi) + \Psi^{I,M}_{II,r}(r,\varphi)Z && (5.35)\\
H_{I,r}(r,\varphi) &= -\frac{\partial \Psi_I}{\partial r}, \quad H_{II,r}(r,\varphi) = -\frac{\partial \Psi_{II}}{\partial r}\\
H_\varphi &= \Psi^{I,M}_{I,\varphi}(r,\varphi) + \Psi^{I,M}_{II,\varphi}(r,\varphi)Z && (5.36)\\
H_{I,\varphi}(r,\varphi) &= -\frac{1}{r}\frac{\partial \Psi_I}{\partial r}, \quad \Psi_{II,\varphi}(r,\varphi) = -\frac{1}{r}\frac{\partial H_{II}}{\partial r}\\
\mu_I H^I_{I,r}(r,\varphi) &= \mu_M H^M_{I,r}(r,\varphi), \quad \mu^M = \mu\\
\mu_I H^I_{II,r}(r,\varphi) &= \mu_M H^M_{II,r}(r,\varphi), \quad \mu^I = \mu_0\\
H^I_{I,\varphi} &= H^M_{I,\varphi}\\
H^I_{II,\varphi} &= H^M_{II,\varphi}
\end{aligned}$$

Inner wall is at $\mid z \mid = R$, yielding the boundary condition (3.2). Here $z = re^{i\varphi} = Re^{i\varphi}$

$$H_r = -\frac{\partial \Phi}{\partial r} = -\frac{\partial Z}{\partial r}\cdot\frac{\partial \Phi}{\partial Z} \tag{5.37}$$

$$H_\varphi = -\frac{1}{r}\cdot\frac{\partial \Phi}{\partial \varphi} = -\frac{1}{r}\cdot\frac{\partial Z}{\partial \varphi}\cdot\frac{\partial \Phi}{\partial Z} = -\frac{i}{r}\cdot re^{i\varphi}\cdot\frac{\partial \Phi}{\partial Z} \tag{5.38}$$

$$H_Z = -\frac{\partial \Phi}{\partial Z} = -\Psi_{II} \tag{5.39}$$

From (3.13), we got that

$$\mu_M H^M_r = \mu_I H^I_r \tag{5.40}$$

And similarly in the area $I \to II$-region. In the inner area, $m \geqslant 0$. So we get the new equation,

$$\Psi^I_I = \sum_{m=0}^{\infty} a^I_m Z^m \tag{5.41}$$

then we try to find two solutions $H^I_Z = -\frac{\partial \Psi^I_I}{\partial Z}$, and $H^M_Z = -\frac{\partial \Psi^M_I}{\partial Z}$.

$$H^I_Z = -\frac{\partial \Psi^I_I}{\partial Z} = \sum_{m=1}^{\infty} m a^I_m Z^{m-1} \tag{5.42}$$

$$H^M_Z = -\frac{\partial \Psi^M_I}{\partial Z} = \sum_{m=1}^{\infty} m a^M_m Z^{m-1} \tag{5.43}$$

Put these two equations in (5.40), $Z^{m-1} = re^{i(m-1)\varphi}$. $\mid Z \mid = R$ radius of the inner circle. From (5.33), $a_{m=0} = 0$.

$$\begin{aligned}
H^I_{I,r} &= -\sum_{m=1}^{\infty} m a^I_m e^{im\varphi} r^{m-1}\\
H^M_{I,r} &= -\sum_{m=1}^{\infty} m a^M_m e^{im\varphi} r^{m-1} - a^{M'}_0 \frac{1}{r}
\end{aligned}$$

$\mu_I H^I_{I,r}(R,\varphi) = \mu_M H^M_{I,r}(R,\varphi) \Rightarrow$

$$-\mu_I \sum_{m=1}^{\infty} m a^I_m e^{im\varphi} R^{m-1} \quad = \quad -\mu_M \sum_{m=-\infty}^{\infty} m a^M_m e^{im\varphi} R^{m-1} - \mu_M \frac{a^{M'}_0}{R} \quad m \neq 0$$

- $m < 0 \Rightarrow a^M_m = 0$ for $m < 0$;
- $m > 0$, $\mu_I m a^I_m R^{m-1} = \mu_M m a^M_m R^{m-1} \Rightarrow \mathrm{a}^M_m = \dfrac{\mu_I}{\mu_M} a^I_m$
- $m = 0$, $\mu_M \dfrac{a^{M'}_0}{R} = 0 \Rightarrow \mathrm{a}^{M'}_0 = 0$.

$a^{M'}$ is a contribution of the magnetic field near the well, and we must also determine the coefficients by boundary conditions. These coefficients for the nearby magnetic field must be decided on the basis of boundary conditions. Similarly for H_{II},

$$\begin{aligned} H^I_{II,r} &= -\frac{\partial \Psi^I_{II}}{\partial r} = -\sum_{m=1}^{\infty} m b^I_m e^{im\varphi} R^{m-1} \\ H^M_{II,r} &= -\frac{\partial \Psi^M_{II}}{\partial r} = -\sum_{m=-\infty}^{\infty} m b^M_m e^{im\varphi} R^{m-1} + b^{M'}_0 \frac{1}{R} + b^M_0 \end{aligned}$$

if and only if two equations expressed as Fourier series are the same,

$$-\mu_I(\sum_{m=1}^{\infty} m b^I_m R^{m-1} e^{im\varphi} + b^I_0) = -\mu_M(\sum_{m=-\infty}^{\infty} m b^M_m R^{m-1} e^{im\varphi} + b^{m'}_0 \frac{1}{R} + b^m_0)$$

- $m < 0 \Rightarrow \mathrm{b}^M_m = 0$.
- $m > 0$, $\mu^I m b^I_m R^{m-1} = \mu_M m b^M_m R^{m-1} \Rightarrow \mathrm{b}^M_m = \dfrac{\mu_I}{\mu_M} b^I_m$.
- $m = 0$, $b^I_0 = \mu_M b^{M'}_0 \dfrac{1}{R} = 0 \Rightarrow \mathrm{b}^{M'}_0 = 0$.

Using the same method from (3.13),

$$\begin{aligned} H^I_{I,\varphi} &= -\frac{1}{r}\frac{\partial \Psi^I_I}{\partial \varphi} = -\frac{1}{R}\sum_{m=1}^{\infty} i m a^I_m R^m e^{im\varphi} \\ H^M_{I,\varphi} &= -\frac{1}{r}\frac{\partial \Psi^M_I}{\partial \varphi} = -\frac{1}{R}(\sum_{m=-\infty}^{\infty} i m a^M_m R^m e^{im\varphi} + i a^{iM'}_0) \\ H^I_{II,\varphi} &= -\frac{1}{r}\frac{\partial \Psi^I_{II}}{\partial \varphi} = -\frac{1}{R}(\sum_{m=1}^{\infty} i m b^I_m R^m e^{im\varphi} + 0) \\ H^M_{II,\varphi} &= -\frac{1}{r}\frac{\partial \Psi^M_{II}}{\partial \varphi} = -\frac{1}{R}(\sum_{m=-\infty}^{\infty} i m b^M_m R^m e^{im\varphi} + i b^{iM'}_0) \end{aligned}$$

When two of the Fourier-series are equal, it is if and only if they have the same coefficients.

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} X_m e^{im\varphi} &= \sum_{m=-\infty}^{\infty} Y_m e^{im\varphi} \\
\sum_{m=-\infty}^{\infty} (X_m - Y_m) e^{im\varphi} &= 0 \\
\sum_{m=-\infty}^{\infty} (X_m - Y_m) \int_0^{2\pi} e^{i(m-m')\varphi} d\varphi &= 0
\end{aligned}$$

If $m \neq m' \rightarrow \int_0^{2\pi} e^{i(m-m')\varphi} d\varphi = 0$.

$$(X_{m'} - Y_{m'}) \cdot 2\pi = 0 \quad \Leftrightarrow \quad X_m = Y_m$$

$H^M_{I,\varphi} = H^I_{I,\varphi}$, And similarly in the area $I \rightarrow II$-region $H^M_{II,\varphi} = H^I_{II,\varphi} \Rightarrow$

$$\sum_{m=0}^{\infty} m a^I_m e^{im\varphi} R^m = \sum_{m=-\infty}^{\infty} m a^M_m e^{im\varphi} R^m$$

- $m < 0 \Rightarrow \mathrm{a}^M_m = b^M_m = 0$.
- $m > 0 \Rightarrow \mathrm{a}^I_m = a^M_m, b^I_m = b^M_m$.
- $m = 0 \Rightarrow \mathrm{a}^{iM'}_0 = b^{iM'}_0 = 0$.

So we can begin with Ψ_{II},

$$\Psi^I_{II} = \sum_{m=-\infty}^{\infty} b^I_m Z^m + b^{I'}_0 lnZ, \quad m \neq 0 \tag{5.44}$$

H is a measurable magnetic field , so there is no singularity point. It means that $\Psi^I_{II}(0)$ has to find a final place (no monopoly),

$$\Psi^I_{II}(0) = \Psi^I_z(0) \tag{5.45}$$

here $b^I_m = 0$, for $m < 0$. So we get the following,

$$\Psi^I_{II}(Z) = \sum_{m=0}^{\infty} b^I_m Z^m \tag{5.46}$$

Corresponding equations in the two areas,

$$\Psi^M_{II}(Z) = \sum_{m=-\infty}^{\infty} b^M_m Z^m + b^M_o + b^{M'}_0 lnZ \tag{5.47}$$

$H_Z^I = Re(\Psi_{II}^I)$ and $H_Z^M = Re(\Psi_{II}^M(Z))$ is physically measurable. ($Z = Re^{i\varphi}, lnZ = lnR + i\varphi$.)

$$\begin{aligned} H_{II}^I &= \sum_{m=1}^{\infty} b_m^I R^m e^{im\varphi} + b_0^I \\ H_{II}^M &= \sum_{m=-\infty}^{\infty} b_m^M R^m e^{im\varphi} + b_0^M + b_0'^M (lnR + i\varphi) \end{aligned} \tag{5.48}$$

Here we can try to use the following method - *Fourier-series* to find the coefficients of $ie^{im\varphi}$.

$$\int_0^{2\pi} e^{i(m-m')\varphi} d\varphi = \begin{cases} 2\pi, & \text{if} \quad m = m' \\ 0, & \text{if} \quad m \neq m' \end{cases} \tag{5.49}$$

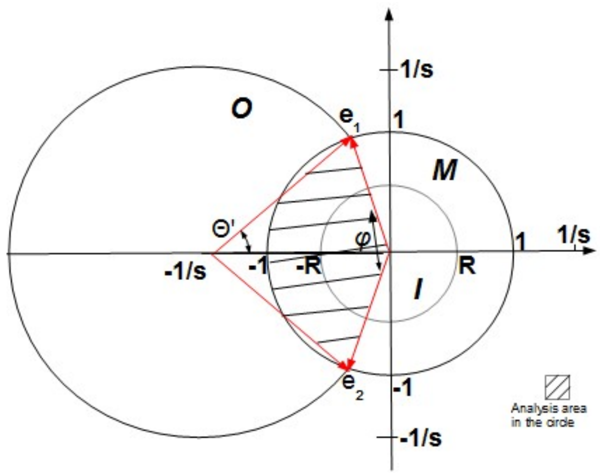
Two Fourier- series are the same when if and only if coefficients of $ie^{im\varphi}$ are equal. $\Rightarrow$

for $m < 0 : b_m^I = 0$.

for $m > 0 : b_m^I = b_m^M$.

for $m = 0 : b_0^M + b_0^{M'}(lnR + i\varphi) = b_0^I$. Since the left side of the equation is independent of $\varphi \Rightarrow$ $b_0^{M'} = 0$. So $b_0^I = b_0^M$.

Figure 9: Analytical region in wellbore.



The area is applicable [20]. To show the series solutions about the point $-\frac{1}{s}$ is valid. Conformal mapping in to regions, inner region I and outer region M. Ψ_I of Ψ_{II} is analytical inside the circles, $0 \leq z \leq R$ and $R \leq z \leq 1$, and along the ranges of them. Between points e_1 and e_2 in the red region, indicates where the series solution about the point $-\frac{1}{s}$ can set equal the series solution in region M. From the above figure, we try to find θ', θ' in a isosceles triangle, so

$$\sin\frac{\theta'}{2} = \frac{1/2}{1/s} = \frac{s}{2}, 0 \leq s \leq 1.$$

Solution to $\theta'(s \to 1) = \pi/3 \Rightarrow$ in the circle with $\mid z \mid \leq 1$ for $\varphi(s \to 1) = 2\theta'$.

- In the inner region, **I**, $0 \leq z \leq R$.

- In the outer region, **M**, $R \leq z \leq 1$.

The below equations are valid in the striped area (Analysis area in Figure 5.3).

$$\Psi^{O}_{I\ z\to -1/s}(z) = \Psi^{O}_{I\ z\to 0}(z)$$

From (3.16), $H^O_Z = H^M_Z$ we got

$$H^O_{II}(z = re^{i\varphi}) = H^M_{II}(z = re^{i\varphi}), \quad r = 1. \tag{5.50}$$

Series expansion in the point $z_0 = 0$ and $z \to -1/s$ is valid for H_I. As binomial-series, on the point $z_0 = -1/s$,

$$(z + \frac{1}{s})^m = \sum_{k=0}^{\infty} z^k \binom{m}{k} (\frac{1}{s})^{m-k} = s^k \sum_{k=0}^{m} \binom{m}{k} e^{ik\varphi} s^{-m}$$

$$\sum_{m=0}^{\infty} X_m \sum_{k=0}^{m} Y_{m,k} = \sum_{k=0}^{\infty}\sum_{m=k}^{m} X_m Y^k_m \tag{5.51}$$

By choosing the coefficients in the boundary conditions we should find the solutions,

$$\sum_{m=0}^{\infty} d^O_m (z + \frac{1}{s})^m = \sum_{k=0}^{\infty}\sum_{k=m}^{m} d^O_m \binom{m}{k} e^{ik\varphi} s^{k-m} \tag{5.52}$$

After $\sum_{m=0}^{\infty} d^O_m (e^{i\varphi} + \frac{1}{s})^m = \sum_{m=0}^{\infty} b^M_m e^{im\varphi}$, we assumed that $n = m - k$,

$$b^M_k = \sum_{n=0}^{\infty} d^O_{k+n} \binom{n+k}{k} \frac{1}{s^n} \tag{5.53}$$

Theorem 5.2 (Maximum Modulus Principle). *If f is analytic in a domain D and $\mid f(z) \mid$ achieves its maximum value at a point z_0 in D, then f is constant in D.*

We knew already $\Psi_{II}^{O} = H_Z$,

$$H_Z^O = \sum_{m=0}^{\infty} d_m (z + \frac{1}{s})^m \tag{5.54}$$

Since the magnetic field inevitably will fade away between two drilling pipes so by *Maximum Modulus Principle*, function (5.54) is analytical (possibly by analytical continuation) for all $\mid z \mid > 1$. H_Z is constant which we already know $H_{II} = H_Z^O$. Then the resulting solution is in two parts $\Rightarrow$

- for $m > 0 \Rightarrow d_m = 0$,
- or rest part is (5.53), $d_m = \sum_{n=0}^{\infty} d_{k+n}^{O} \binom{n+k}{k} \frac{1}{s^n}$

get further, in the $I-$area, and (5.32), $Z = re^{i\varphi}$,

$$\begin{aligned}
H_r^{O,M} &= -\frac{\partial \Phi}{\partial r} = \Psi_{I,r} + \Psi_{II,r} Z = -(\frac{\partial \Psi_I}{\partial r} + Z \frac{\partial \Psi_{II}}{\partial r}) \\
H_\varphi^{O,M} &= -\frac{1}{r}\frac{\partial \Phi}{\partial \varphi} = \Psi_{I,\varphi} + \Psi_{II,\varphi} Z = -\frac{1}{r}(\frac{\partial \Psi_I}{\partial \varphi} + Z \frac{\partial \Psi_{II}}{\partial \varphi}) \\
H_{I,\varphi}^{O} &= -\frac{1}{r}\frac{\partial \Psi_I}{\partial \varphi} = -\frac{1}{r}\frac{\partial Z}{\partial \varphi}\frac{\partial \Psi_I}{\partial Z} = -\frac{1}{r} rie^{i\varphi}\frac{\partial \Psi_I}{\partial Z} \\
&= -ie^{i\varphi}\frac{\partial \Psi_I}{\partial Z} \qquad (5.55) \\
H_{I,r}^{O} &= -\frac{\partial \Psi_I}{\partial r} = -\frac{\partial Z}{\partial r}\frac{\partial \Psi_I}{\partial Z} = -e^{i\varphi}\frac{\partial \Psi_I}{\partial Z}
\end{aligned}$$

we can find that

$$\frac{\partial \Psi_I}{\partial Z} = \sum_{m=1}^{\infty} m a_m^M r^{m-1} e^{im\varphi} \tag{5.56}$$

put $r = 1$ and 5.56 in the equations 5.55 and 5.56, we got

$$H_{I,r}^{M} = -\sum_{m=1}^{\infty} m a_m^M e^{im\varphi} \tag{5.57}$$

$$H_{I,\varphi}^{M} = -i\sum_{m=1}^{\infty} m a_m^M e^{im\varphi} \tag{5.58}$$

Here we can obviously see that $H^M_{I,r}$ is the real part and $H^M_{I,\varphi}$ is the imaginary part of the solutions. From earlier $z \to 1/s$ in (5.29),

$$\begin{aligned}\Psi^O_I &= \sum_{m=0}^{\infty} c^O_m (z+\frac{1}{s})^m \\ &= c^O_0 + c^O_1(z+\frac{1}{s}) + \sum_{m=2}^{\infty} c^O_m(z+\frac{1}{s})^m \end{aligned} \quad (5.59)$$

The reason that the magnetic field impossibly can be infinitely large is because it has to end somewhere. So the last subsection in the function must to be zero.

$$\sum_{m=2}^{\infty} c^O_m(z+\frac{1}{s})^m = 0 \quad (5.60)$$

and at the same time $c^O_0 = 0$. From (3.12) and $z = re^{i\varphi}, r = 1$

$$\begin{aligned}\mu_0 H^O_{I,r} &= \mu_m H^M_{I,r} \\ \mu_0 c^O_1 e^{i\varphi} &= \mu_m \sum_{m=1}^{\infty} m a^M_m e^{im\varphi}, \quad a^M_m = 0 \quad for \quad m \geq 2. \end{aligned} \quad (5.61)$$

$\Longrightarrow$

$$\begin{aligned}\mu_0 c^O_1 &= \mu_m a^M_1 \\ \mu_0 c^O_0 &= \mu_I c^I_0 = \mu_m c^M_0\end{aligned}$$

From (5.3) $\Rightarrow \mu_0 c^O_1 = \mu_1 a^I_1$

In additional of (5.35), $H^O_{II} = \sum_{m=0}^{\infty} b^I_m z^m$, $H^M_{II} = \sum_{m=-\infty}^{\infty} d_m(z+\frac{1}{s})^m z^m$

$$\begin{aligned}\mu_M H^M_{II,r} &= \mu_O H^O_{II,r} & (5.62)\\ \mu_M H^M_{II,r} &= -\mu_m \sum_{m=-\infty}^{\infty} b^M_m m r^{m-1} e^{im\varphi} - b^M_0 - b^{M'}_0 \frac{1}{R} & (5.63)\\ \mu_O H^O_{II,r} &= -\mu_O \sum_{m=-\infty}^{\infty} d^O_m m(z+\frac{1}{s})^{m-1} r^{m-1} e^{im\varphi} & (5.64)\end{aligned}$$

Following the conditions (3.12), and together with (3.13) $\Longrightarrow$

- $m = 0 \Rightarrow b^M_0 + b^{M'}_0 \frac{1}{R} = 0$
- $m = 1 \Rightarrow \mu_m b^M_1 = \mu_0 d^O_1$
- $m \geq 2 \Rightarrow b^M_m = 0$

And (5.3) this leads to:

$$H^O_\varphi = H^O_{I,\varphi} + H^O_{II,\varphi} Z \quad (5.65)$$

$$H^O_\varphi = -\frac{i}{R}\sum_{m=-\infty}^{\infty} m a^O_m R^m e^{im\varphi} - a^{iO'}_0 \frac{i}{R} - \frac{Zi}{R}\sum_{m=-\infty}^{\infty} m b^O_m R^m e^{im\varphi} - \frac{Zi}{R} b^{O'}_0$$

$$H^M_\varphi = H^M_{I,\varphi} + H^M_{II,\varphi} Z \quad (5.66)$$

$$H^M_\varphi = -\frac{i}{R}\sum_{m=0}^{\infty} m c^M_m (z+\frac{1}{s})^{m-1} e^{i\varphi} - \frac{Zi}{R}\sum_{m=-\infty}^{\infty, m\neq 0} m d^M_m r(z+\frac{1}{s})^{m-1} e^{i\varphi}$$

$$H^O_{I,\varphi} = H^M_{I,\varphi} \quad (5.67)$$

$$H^O_{II,\varphi} = H^M_{II,\varphi} \quad (5.68)$$

$\Longrightarrow$

- $m < 0 \Rightarrow a^O_m = 0$,
- For $m = 0 \Rightarrow a^O_0 = c^O_0 = a^{O'}_0 = b^{O'}_0 = 0$,
- $m = 1 \Rightarrow c^M_1 = a^O_1, b^O_1 = d^M_1$,
- For $m \geq 2 \Rightarrow c^M_m = a^O_m = 0$ and $b^O_m = d^M_m = 0$.

The same as $\mu_o H^O_r = \mu_m H^M_r = \mu_I H^I_{,r}$ in the area II and I,

$$\mu_0 d_1 = \mu_m b^M_1 = \mu_I b^I_1 \quad (5.69)$$

then according $H^O_{I,\varphi} = H^M_{I,\varphi} = H^I_{I,\varphi}$ when $r = 1$,

$$ic_1 e^{i\varphi} = i\sum_{m=0}^{\infty} a^O_m m e^{im\varphi} \quad (5.70)$$

as the similar requirement as (5.61), $H_{I,\varphi} \Rightarrow$

$$ic_1 e^{i\varphi} = ia^O_1 e^{im\varphi} \quad \Rightarrow \quad c_1 = a^O_1 \quad \texttt{when} \quad m = 1 \quad (5.71)$$

Kind of relative to $H_{II,\varphi} \Rightarrow$

$$d_1 r i e^{i\varphi} = ib^O_1 e^{i\varphi} \quad \Rightarrow \quad d_1 = b^O_1 \quad \texttt{when} \quad m = 1 \quad (5.72)$$

also $H_{I,z}$ and $H_{II,z}$ is that we trying to find out, with (5.56),
$H_{I,II,z} = -\frac{\partial \Psi_{I,II}}{\partial z}$

$$d^M_1 = a^O_1 \quad (5.73)$$

$$d^O_1 = b^{M'}_0 + b^I_1 \quad (5.74)$$

In inner area I,

$$\begin{aligned} H &= H_I + H_{II} Z && (5.75) \\ H_I^i &= \sum_{m=-\infty}^{\infty} a_m^i Z^m + a'^i ln Z && (5.76) \\ H_{II}^i &= \sum_{m=-\infty}^{\infty} b_m^i Z^m + b'^i ln Z && (5.77) \\ H_r^i &= H_{I,r}^i + H_{II,r}^i Z && (5.78) \\ H_\varphi^i &= H_{I,\varphi}^i + H_{II,\varphi}^i Z && (5.79) \end{aligned}$$

in outer area II, $H_{II} = -H_Z$, will not change in $Z-$ axis, $H_{Z,inner} = H_{Z,outer}$ on both surfaces, (5.13) $H_Z = -\frac{B_0}{\mu_0} cos\gamma$,

$$H = H_I - H_z \cdot Z \quad (5.80)$$

H_z does not contain $r, \varphi \Rightarrow$,

$$H_{II,r} = H_{II,\varphi} = 0 \quad b'_m = 0 (m \neq 0) \quad (5.81)$$

On the basis of these factors we can discuss this problem in area I,

$$\begin{aligned} H_r^i &= -\frac{\partial H_I^i}{\partial r} = -e^{i\varphi} H_I^{i'} \\ &= -\sum_{m=1}^{\infty} a_m^i m r^{m-1} e^{im\varphi} - a'^i \frac{1}{r} && (5.82) \\ H_\varphi^i &= -\frac{1}{r}\frac{\partial H_I^i}{\partial \varphi} = -Z\frac{i}{r} H_I^{i'} = i H_r^i && (5.83) \\ i &= I, M, O. && (5.84) \end{aligned}$$

$a_m^I = 0$ for $m < 0, a'^I = 0, a_0^I$ is irrelevant.

$$\begin{aligned} H_r^I &= -\sum_{m=1}^{\infty} m a_m^I r^{m-1} e^{im\varphi} && (5.85) \\ H_r^M &= -\sum_{m\neq 0}^{\infty} m a_m^M r^{m-1} e^{im\varphi} - \frac{a'^M}{r} && (5.86) \end{aligned}$$

Here if

$$\begin{aligned} H_r &= A + iB \\ iH_r &= iA - B \quad \texttt{A, B is real number.} && (5.87) \end{aligned}$$

Boundary demands gives the following

$$\mu_0 Re[H_r^O(R_i,\varphi,z)] \quad = \quad \mu Re[H_r^M(R_i,\varphi,z)] \tag{5.88}$$

We considered Re indicating the real part of the $H-$ field,
if $R_i = 1$ in considered physics. Boundary demands will be satisfied from symmetry, and lead to

$$Re[H_r(r,-\varphi,z)] \quad = \quad Re[H_r(r,\varphi,z)] \tag{5.89}$$

according to (3.19), $H_r(r,-\varphi) - H_r(r,\varphi) = 0$.

$$\begin{aligned} Re\sum_{m=1}^{\infty} mr^{m-1}a_m^I(e^{im\varphi} - e^{-im\varphi}) &= Re\sum_{m=1}^{\infty} mr^{m-1}a_m^I 2i\sin(m\varphi) = 0 \\ \Rightarrow a_m^I &= a_m^{I*} \quad \texttt{is real number} \end{aligned}$$

we can check with φ component,

$$\begin{aligned} ReH_\varphi(r,-\varphi) &= -ReH_\varphi(r,\varphi) \\ -ImH_r^I(r,-\varphi) &= ImH_r^I(r,\varphi) \\ Im\sum_{m=1}^{\infty} mr^{m-1}a_m^I(-e^{-im\varphi} - e^{-im\varphi}) &= Im\sum_{m=1}^{\infty} mr^{m-1}a_m^I 2\cos(m\varphi) = 0 \end{aligned}$$

It means

$$\begin{aligned} H_r^I + H_\varphi^I &= Re\sum_{m=1}^{\infty} ma_m^I r^{m-1}e^{im\varphi} - Im\sum_{m=1}^{\infty} ma_m^I r^{m-1}e^{im\varphi} \\ \Rightarrow a_m^I &= a_m^{I*} \end{aligned} \tag{5.90}$$

From (3.20) $H_\varphi^M(r,\varphi) - H_\varphi^M(r,-\varphi) = 0$,

$$\begin{aligned} Re\sum_{m=-\infty}^{\infty} mr^{m-1}a_m^I(e^{im\varphi} - e^{-im\varphi}) = Re\sum_{m=1}^{\infty} mr^{m-1}a_m^M 2i\sin(m\varphi) = 0 \\ Re\sum_{m=1}^{\infty}[a_m^M r^{m-1} + \frac{a_{-m}^M}{r^{m+1}}]2i\sin(m\varphi) = 0 \\ \Rightarrow a_m^M r^{m-1} + \frac{a_{-m}^M}{r^{m+1}} \quad \texttt{is real number for} \quad \forall r \in [R_I, R_0] \end{aligned}$$

Here a_m^M and a_{-m}^M is real number. then

$$Im\{H_\varphi^M(r,-\varphi)+H_\varphi^M(r,\varphi)\}=0$$

$$Im\sum_{m=1}^{\infty} m(a_m^M r^{m-1}-\frac{a_{-m}^M}{r^{m+1}})2\cos(m\varphi)+2Im\frac{a'^m}{r}=0$$

$$\texttt{if and only if} a'^M = a'^{M*} \tag{5.91}$$

Solving the equations for $m=1$, all coefficients with $m\geq 2$ vanishes. $\Rightarrow Im\sum_{m=1}^{\infty}(a_1^M-\frac{a_{-1}^M}{r^2})2cos(2\varphi)=0$, and $Im\frac{a'^M}{r}=0$. In the inner pipe with radius with R_I,

$$\begin{aligned}
H_r^I &= -\sum_{m=1}^{\infty} mr^{m-1}a_m^I\cos(m\varphi)\\
H_r^M &= -\sum_{m=1}^{\infty} m(a_m^M r^{m-1}-\frac{a_{-m}^M}{r^{m+1}})\cos(m\varphi)-m\frac{a'^M}{r},\quad a'^M=0
\end{aligned}$$

Following (3.13), $\mu_I H_r^I=\mu_m H_r^M, r=R_i=R$ inner, got we

$$\begin{aligned}
m\mu_I R_I^{m-1}a_m^I &= m\mu_M(a_m^M R_i^{m-1}-\frac{a_{-m}^M}{R_i^{m+1}})\\
a_m^I &= \frac{\mu_m}{\mu_I}(a_m^M-\frac{a_{-m}^M}{R_i^{2m}})
\end{aligned} \tag{5.92}$$

use the same way,

$$\begin{aligned}
H_\varphi^I &= \sum_{m=1}^{\infty} mr^{m-1}a_m^I\sin(m\varphi)\\
H_\varphi^M &= \sum_{m=1}^{\infty} m(a_m^M r^{m-1}+\frac{a_{-m}^M}{r^{m+1}})\sin(m\varphi)
\end{aligned}$$

Following (3.13), $H_\varphi^I=H_\varphi^M, r=R_i=R$ inner , we got

$$a_m^I = a_m^M+\frac{a_{-m}^M}{R_i^{2m}} \tag{5.93}$$

when $m\geq 1$,

$$a_m^M = \frac{a_m^I}{2}(1+\frac{\mu_I}{\mu_m}) \tag{5.94}$$

$$a_{-m}^M = \frac{a_m^I}{2}(1-\frac{\mu_I}{\mu_m})R_I^{2m} \tag{5.95}$$

In $O-$area,

$$H_r^O = \sum_{m=1}^{\infty} ma_{-m}^O\frac{1}{r^{m+1}}e^{im\varphi}-a_1^O e^{i\varphi}$$

$H_r^O(r,-\varphi) = H_r^O(r,\varphi) \Rightarrow a_{-m}^O$ is real number

$$a_1^O = -\frac{B_0}{\mu_0}\sin\gamma = a_1^{O*} \tag{5.96}$$

when $m = 1$,

$$\begin{aligned} H_r^j &= (a_1^j - \frac{a_{-1}^j}{r^2})\sin\varphi + \sum_{m=2}^{\infty} m a_{-m}^j \frac{1}{r^{m+1}}\sin(m\varphi) \\ H_\varphi^j &= (a_1^j + \frac{a_{-1}^j}{r^2})\sin\varphi + \sum_{m=2}^{\infty} m a_{-m}^j \frac{1}{r^{m+1}}\sin(m\varphi) \end{aligned}$$

$\mu_0 H_r^O = \mu_m H_r^M$, when $r = R_o = R$ outer, when $m = 1$,

$$\mu_0(a_1^O - \frac{a_{-1}^O}{R_0^2}) = \mu_M(a_1^M - \frac{a_{-1}^M}{R_0^2}) \tag{5.97}$$

we know already that $H_\varphi^O = H_\varphi^M$,

$$a_1^O + \frac{a_{-1}^O}{R_0^2} = a_1^M + \frac{a_{-1}^M}{R_0^2} \tag{5.98}$$

when $m \geq 2$,

$$\begin{aligned} \mu_0 \frac{a_{-m}^O}{R_0^{m+1}} &= \mu_m(-a_m^M R_0^{m-1} + \frac{a_{-m}^M}{R_0^{m+1}}) \\ \frac{a_{-m}^O}{R_0^{m+1}} &= a_m^M R_0^{m-1} + \frac{a_{-m}^M}{R_0^{m+1}} \end{aligned}$$

For the present purpose we are mostly interested in the magnetic field inside the inner cylinder, where the magnetometers are located. Going back to Cartesian coordinates and the $B-$ field, which is what is actually being measured, by help of Maple to find the solution B_x^I.

Magnetic field B_x in horizontal when z->0 in Maple.

> restart;

> $eq1 := aI[1] = ri\cdot\left(aM[1] - \frac{aM[-1]}{RI^2}\right);$

$$eq1 := aI_1 = ri\left(aM_1 - \frac{aM_{-1}}{RI^2}\right)$$

ri=µm/µI=µ/µ0 =µr = 1+χ;

> $eq2 := aI[1] = aM[1] + \frac{aM[-1]}{RI^2};$

$$eq2 := aI_1 = aM_1 + \frac{aM_{-1}}{RI^2}$$

> $solve(\{eq1, eq2\}, \{aM[1], aM[-1]\});$

$$\left\{aM_{-1} = \frac{1}{2}\,\frac{aI_1\,RI^2\,(ri-1)}{ri},\ aM_1 = \frac{1}{2}\,\frac{aI_1\,(1+ri)}{ri}\right\}$$

> $eq3 := ro\cdot\left(aO[1] - \frac{aO[-1]}{RO^2}\right) = aM[1] - \frac{aM[-1]}{RO^2};$

$$eq3 := ro\left(aO_1 - \frac{aO_{-1}}{RO^2}\right) = aM_1 - \frac{aM_{-1}}{RO^2}$$

ro=µO/µm=µO/µ=1/(1+χ);

> $eq4 := aO[1] + \frac{aO[-1]}{RO^2} = aM[1] + \frac{aM[-1]}{RO^2};$

$$eq4 := aO_1 + \frac{aO_{-1}}{RO^2} = aM_1 + \frac{aM_{-1}}{RO^2}$$

> $solve(\{eq1, eq2, eq3, eq4\}, \{aI[1], aM[1], aM[-1], aO[-1]\});$

$$\Big\{aI_1 = \left(4\,ri\,ro\,aO_1\,RO^2\right) \Big/ \left(-ri\,RI^2 + RI^2 + RO^2 + RO^2\,ri + ro\,ri\,RI^2 - ro\,RI^2 + ro\,RO^2 + ro\,RO^2\,ri\right),\ aM_{-1} = \left(2\,RI^2\,ro\,aO_1\,RO^2\,(ri-1)\right) \Big/ \left(-ri\,RI^2 + RI^2 + RO^2 + RO^2\,ri + ro\,ri\,RI^2 - ro\,RI^2 + ro\,RO^2 + ro\,RO^2\,ri\right),\ aM_1 = \left(2\,(1+ri)\,ro\,aO_1\,RO^2\right) \Big/ \left(-ri\,RI^2 + RI^2 + RO^2 + RO^2\,ri + ro\,ri\,RI^2 - ro\,RI^2 + ro\,RO^2 + ro\,RO^2\,ri\right),\ aO_{-1} = \left(aO_1\,RO^2\,\left(ro\,ri\,RI^2 - ro\,RI^2 + ro\,RO^2 + ro\,RO^2\,ri + ri\,RI^2 - RI^2 - RO^2 - RO^2\,ri\right)\right) \Big/ \left(-ri\,RI^2 + RI^2 + RO^2 + RO^2\,ri + ro\,ri\,RI^2 - ro\,RI^2 + ro\,RO^2 + ro\,RO^2\,ri\right)\Big\}$$

> $res := \%[1];$

$$res := aI_1 = \left(4\, ri\, ro\, aO_1\, RO^2\right) \Big/ \left(-ri\, RI^2 + RI^2 + RO^2 + RO^2\, ri + ro\, ri\, RI^2 - ro\, RI^2 + ro\, RO^2 + ro\, RO^2\, ri\right)$$

> $\%\%[4];$

$$aO_{-1} = \left(aO_1\, RO^2 \left(ro\, ri\, RI^2 - ro\, RI^2 + ro\, RO^2 + ro\, RO^2\, ri + ri\, RI^2 - RI^2 - RO^2 - RO^2\, ri\right)\right) \Big/ \left(-ri\, RI^2 + RI^2 + RO^2 + RO^2\, ri + ro\, ri\, RI^2 - ro\, RI^2 + ro\, RO^2 + ro\, RO^2\, ri\right)$$

> $subs\left(\left\{ri = 1 + \mathrm{chi}, ro = \frac{1}{(1 + \mathrm{chi})}\right\}, res\right);$

$$aI_1 = \left(4\, aO_1\, RO^2\right) \Big/ \left(-(1 + \chi)\, RI^2 + 2\, RI^2 + 2\, RO^2 + RO^2\, (1 + \chi) - \frac{RI^2}{1 + \chi} + \frac{RO^2}{1 + \chi}\right)$$

> $simplify(\%);$

$$aI_1 = -\frac{4\, aO_1\, RO^2\, (1 + \chi)}{RI^2\, \chi^2 - 4\, RO^2 - 4\, RO^2\, \chi - RO^2\, \chi^2}$$

>

>

Magnetic field B_y in horizontal when a->0 in Maple

> $restart;$

> $eq1 := aI[m] = ri\cdot\left(aM[m] - \frac{aM[-m]}{RI^{2\,m}}\right);$

$$eq1 := aI_m = ri\left(aM_m - \frac{aM_{-m}}{RI^{2\,m}}\right)$$

ri=µm/µI=µ/µ0 =µr = 1+χ;

> $eq2 := aI[m] = aM[m] + \frac{aM[-m]}{RI^{2\,m}};$

$$eq2 := aI_m = aM_m + \frac{aM_{-m}}{RI^{2\,m}}$$

> $solve(\{eq1, eq2\}, \{aM[m], aM[-m]\});$

$$\left\{aM_m = \frac{1}{2}\,\frac{aI_m\,(1+ri)}{ri},\ aM_{-m} = \frac{1}{2}\,\frac{aI_m\,RI^{2\,m}\,(ri-1)}{ri}\right\}$$

> $eq3 := ro\cdot\left(\frac{aO[-m]}{RO^{m+1}}\right) = -aM[-m]RO^{m+1} + \frac{aM[-m]}{RO^{m+1}};$

$$eq3 := \frac{ro\,aO_{-m}}{RO^{m+1}} = -aM_{-m}RO^{m+1} + \frac{aM_{-m}}{RO^{m+1}}$$

ro=µO/µm=µO/µ=1/(1+χ);

> $eq4 := aM[m]RO^{m-1} + \frac{aM[-m]}{RO^{m+1}} = \frac{aO[-m]}{RO^{m+1}};$

$$eq4 := aM_m RO^{m-1} + \frac{aM_{-m}}{RO^{m+1}} = \frac{aO_{-m}}{RO^{m+1}}$$

> $solve(\{eq1, eq2, eq3, eq4\}, \{aI[m], aM[-m], aM[m], aO[-m]\});$

$$\{aI_m = 0, aM_m = 0, aM_{-m} = 0, aO_{-m} = 0\}$$

> $res := \%[1];$

$$res := aI_m = 0$$

> $\%\%[4];$

$$aO_{-m} = 0$$

> $subs\left(\left\{ri = 1 + \text{chi}, ro = \frac{1}{(1+\text{chi})}\right\}, res\right);$

$$aI_m = 0$$

> $simplify(\%);$

$$aI_m = 0$$

>

>

We finally have:

$$\Phi^I = -\frac{B_0}{\mu_0}(z\cos\gamma + \frac{4\chi(1+\chi)\sin\gamma}{(2+\chi)^2 - (\frac{R_i}{R_o})^2\chi^2}) \quad (5.99)$$

$$B_x^I = -\mu_0\frac{\partial\Phi^I}{\partial x} = \frac{4B_0(1+\chi)\sin\gamma}{(2+\chi)^2 - (\frac{R_i}{R_o})^2\chi^2} \quad (5.100)$$

$$B_y^I = -\mu_0\frac{\partial\Phi^I}{\partial y} = 0 \quad (5.101)$$

$$B_z^I = -\mu_0\frac{\partial\Phi^I}{\partial z} = B_0\cos\gamma \quad (5.102)$$

From this it is showed that the magnetic field strength along the borehole, Z-axis, is not influenced at all by the magnetic shielding [33],and neither is B_y. However, the field transverse to the borehole axis is damped by a factor [28].

$$\begin{aligned} D &= \frac{B_x^I}{B_x^O(\infty)} = \frac{4(1+\chi)}{(2+\chi)^2 - (\frac{R_i}{R_0})^2\chi^2} \\ &= 1 - \frac{1}{4}[(1-\frac{R_i}{R_0})^2](\chi^2 - \chi^3) + O(\chi^4) \end{aligned} \quad (5.103)$$

This equation is the main result of this section.

5.4 Numerical in the second boundary

When the drilling pipe moves from the centre $(0,0)$ to $(s,0)$, we assume that the magnetic field should be ending on the second boundary $z \to -\frac{1}{s}$,

$$\begin{aligned} \Psi_O^O &= \sum_{m=-1}^{\infty} c_m^O(\frac{1}{z}+s)^m \\ H_r^O &= Re - e^{i\varphi}\sum_{m=1}^{\infty} mc_m^O(\frac{1}{z}+s)^{m-1}(-\frac{1}{z^2}) \end{aligned}$$

- $m > 1 : (Z + \frac{1}{s})^m|_{Z \to -\frac{1}{s}} \to 0$. Validity condition.
- $m = 1 : H_r^O \to a_1^O \frac{e^{i\varphi}}{z^2}$ when $z \to -\frac{1}{s} \to s^2 e^{i\varphi} a_1^O$ The solution shows that is constant, not depend on the radius or angles.
- $m = 0 : H_r^O = 0$.
- $m = -1 : \frac{e^{i\varphi}}{z^2} a_{-1}^O (\frac{1}{z} + s)^{-2} = \frac{-e^{i\varphi} a_1^O}{(1+zs)^2}$, there is a singular point: $z = -\frac{1}{s}$

Then we could use the function

$$\Phi_0^O = \sum_{m=-1}^{-\infty} a_{-m}^O (\frac{1}{z} + s)^{-m} = \sum_{m=1}^{\infty} a_m^O (\frac{1}{z} + s)^m$$

The magnetic field will disappear in some place when $Z \to -\frac{1}{s}$, with the same method that

$$H_r^O = Re(-e^{i\varphi} \sum_m^{\infty} m a_m^O (\frac{1}{z} + s)^{m-1})$$

i.e.

$$\begin{aligned} H_r^O &= Re(-e^{-i\varphi} \sum_{m=1}^{\infty} m a_m^O (\frac{1}{z} + s)^{m-1})|_{|Z|=1} \\ &= Re(-e^{-i\varphi} \sum_{m=1}^{\infty} m a_m^O (\frac{1}{e^{i\varphi}} + s)^{m-1})|_{|Z|=1} \end{aligned}$$

suppose $m = n + 1$

$$\begin{aligned} H_r^O &= Re(e^{i\varphi} \sum_{n=0}^{\infty} (n+1) a_{n+1}^O \sum_{k=0}^{n} \binom{n}{k} e^{-ik\varphi} s^{n-k}) \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1}^O \sum_{k=0}^{n} \binom{n}{k} (\cos(k+1)\varphi) s^{n-k} \end{aligned} \quad (5.104)$$

The reason of this thesis ending here is because of the parameter in 5.104 that shows that we should use numerical methods to solve 5.104. The most useful information of this equation is that there is a chance to use mathematical methods to find out the relationship between magnetic field in drilling pipe and effects of differences conditions. In geophysical logging, the measured parameters generally include Earth's magnetic field, the magnetic susceptibility of the drilling fluid material, and the angles between the force vectors that can be measured.

Chapter 6

6 Conclusion

In practice, it can be some uncertainty in the asymptotic condition for the magnetic field inside the wellbore and near wellbore. The steel drill pipe containing the magnetic sensors is of a non-magnetic material, but can get magnetic properties by friction from solids/particles in the drilling mud. If the section of the magnetic sensors are too short, this can in addition to magnetic field caused by mud, get a field from above also gives a contribution to influence the magnetic sensors [2].
For the simplest case in this thesis the *first* assumption is that there is no electrical flow in any regions of the borehole ($j = 0$), mentioned in Chapter 2. We have learned the definition of *the auxiliary magnetic field H* and try to find the method of solution in drilling pipe. It is only the Earth's magnetic field which induces magnetism in the borehole. Since the technology improves, the components in the wells get more developed ("intelligent wells") and this leads to increased electronics. Hydraulic steering vs. electrically controlled components in the well must be considered, as this may affect the magnetic measurements in the well (you get a field in additional to that induced by Earth's magnetic field). It is assumed that the drilling fluid does not change in its magnetic characteristics, but in practice this is not the case. After a certain period of time steel will wear away from the drill pipe and therefore steel particles will be included in the drilling mud and this will increase their magnetic properties (susceptibility increases). To make the drilling mud as little magnetic as possible, procedures must be implemented to replace and clean the drilling mud as often as possible. Drilling mud viscosity is also a factor to be taken into account, where there has been made a study on that low viscosity mud gives a maximum erosion rate (cuttings), and therefore the quicker more magnetic due to increasing wear on the drill pipe [2]. After *Maxwell's equations*, and through a series of definitions and theorem of magnetism the results showed a clear trend: The magnitude of magnetic susceptibility, and thereby potential distorted magnetic azimuth, is correlated with the magnetic contamination of the drilling fluid [1].
As mentioned in the introduction of the thesis, there are many factors that can affect the magnetic field in a well. Shielding of magnetic fields in the case where the string is centric in annulus has shown that this contribution has been too small to describe the maximum deviation observed from the Earth's magnetic field in MWD [1]. Therefore, we here considered an asymmetric situation as a source of the greatest

deviation in magnetic measurements in MWD. The results forming the task is given by the transformation in Figure 8. The thesis has shown, in Chapter 3, that in order to calculate the shielding of the magnetic field inside a drill pipe, we find that the ideal test form is are perfect cylinder with the circular cross section(*the second assumption*), and filled with drilling fluid of known susceptibility. The inner drilling pipe's activity is only allowed within an outer circle. The *third* assumption is show in Figure 8 representing a cross-section of the drilling pipes that the starting point is the centre point for both pipes. For a long straight well the magnetic fields along the well are not influenced by magnetic materials inside the wellbore.*Fourth*, after the correlation between the radius of Earth and the depth of the well, there is a general presumption that we can ignore the depth of the drilling hole on the vertical axis.

It has been shown that it is possible to use complex mathematics to formulate boundary conditions to the magnetic field in a well when the drill string is in a asymmetrical annulus [1]. In this thesis, Chapter 4, through the *Cauchy- Riemann equations* and *Möbius transformation* it is therefore sufficient to solve two two-dimensional Laplacian equations with well defined boundary conditions (2.16), (2.17) and (3.3)-(3.6), only to consider the transverse magnetic field and is given explicitly by equation (5.103) in a plane, taken to be the xy-plane, perpendicular to the well axis. The magnetic sensors are on the axis of the borehole, the shielding of the transverse factor is quadratic in the susceptibility of the drilling fluid.

Combining complex analysis and conformal mapping presented in Chapter 5, we could like to show how these main methods can be used to solve the case in this thesis. As a lucky result, the solution of magnetic field in the drilling fluid has been found in the singularity point $z \to 0$. The solution is 5.99-5.102 proved that magnetic shielding is changed on the vertical X-axis, while is on the horizontal Z-axis is constant when the Earth's magnetic field is not changing, and for the Y-axis it is zero. Amundsen et al. [41] [42] have shown that a magnetic sensor placed at the centre of a cylindrical wellbore filled with a fluid magnetic susceptibility in SU units, χ, will be reduced by a factor of $(1 - 1/4\chi^2)$ plus higher order terms. With the fields off centre in a cylindrical wellbore this can be calculated analytically using conformal mapping. For more complex geometries one must resort to numerical modeling. We see from (5.103) that since in realistic cases $R_i << R_o$, the magnetic shielding is not very sensitive to the geometry of the situation, at least not as long as the magnetometers are positioned on the cylinder axis. With enough iron residues in the drilling fluid, the susceptibility χ may exceed valued of 0.1 [43], which is significant for the accuracy of magnetic surveys.

Unfortunately, we did not found out the magnetic shielding on $z \to -1/s$. One should here instead resort to numerical analysis methods. Numerical models have also been made of more realistic geometries like collar and probe based tools, where the shielding material is distributed in a more complicated manner.

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