SIAM J. MATH. ANAL.
Vol. 45, No. 5, pp. 2894–2923

GENUINE TWO-PHASE FLOW DYNAMICS WITH A FREE INTERFACE SEPARATING GAS-LIQUID MIXTURE FROM GAS[*]

STEINAR EVJE[†]

Abstract. In this work we deal with the no-slip drift-flux model for gas-liquid flow dynamics. We focus on a situation where there is a free interface separating the gas-liquid mixture from a pure gas region which takes a positive pressure p^*. This situation is highly relevant for gas-liquid flow in the context of wellbore operations. Previous works have assumed that there is vacuum, i.e., the pressure p^* is zero. The positive pressure $p^* > 0$ creates a boundary term that must be treated in a consistent manner throughout the analysis. We derive time-independent estimates and make some observations related to the role played by p^*. The estimates allow us to discuss the long-time behavior of the two-phase flow system. In particular, it is shown that the stationary solution connecting the gas-liquid mixture to the pure gas region with the specified pressure p^* in a continuous manner is asymptotically stable for sufficiently small initial perturbations. The analysis clearly shows how this perturbation directly depends on the size of the outer pressure p^*. A higher pressure p^* allows for larger initial perturbations from steady state. One ingredient in the analysis is the rate at which the liquid mass decays to zero at the free interface. Insight into mechanisms that control the decay rate of the liquid mass at the free interface is also of interest since such transition zones often are associated with instabilities in numerical discretizations of two-phase models.

Key words. two-phase flow, well model, gas kick, weak solutions, Lagrangian coordinates, free boundary problem, stationary solution

AMS subject classifications. 76T10, 76N10, 65M12, 35L60

DOI. 10.1137/120904317

1. Introduction.

This work is devoted to a study of a one-dimensional two-phase model of the drift-flux type. The model is frequently used in industry simulators to simulate unsteady, compressible flow of liquid and gas in pipes and wells [1, 2, 3, 6, 14, 10, 18, 21]. The model consists of two mass conservation equations corresponding to each of the two phases gas (g) and liquid (l) and one equation for the conservation of the momentum of the mixture and is given in the following form:

$$\begin{aligned} &\partial_\tau[\alpha_g\rho_g] + \partial_\xi[\alpha_g\rho_g u_g] = 0,\\ &\partial_\tau[\alpha_l\rho_l] + \partial_\xi[\alpha_l\rho_l u_l] = 0,\\ &\partial_\tau[\alpha_g\rho_g u_g + \alpha_l\rho_l u_l] + \partial_\xi[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + p] = q + \partial_\xi[\varepsilon\partial_\xi u_{mix}], \end{aligned} \tag{1}$$

where $\varepsilon \geq 0$, $u_{mix} = \alpha_g u_g + \alpha_l u_l$, and $\rho_{mix} = \alpha_g\rho_g + \alpha_l\rho_l$. The unknowns are $\rho_l(p), \rho_g(p)$ for liquid and gas densities, α_l, α_g for volume fractions of liquid and gas satisfying $\alpha_g + \alpha_l = 1$, and u_l, u_g for velocities of liquid and gas, p for the common pressure for liquid and gas, and q representing external forces like gravity and friction. In the following we set $q = 0$. We consider the model in a domain $L := \{(\xi,\tau) : 0 < \xi < l(\tau),\ \tau > 0\}$. We might think of a horizontal conduit which is closed at the left inlet whereas there is a free interface at the right outlet separating the gas-liquid mixture from a pure gas region. See Figure 1 for an illustration. The free interface is

[*]Received by the editors January 2, 2013; accepted for publication July 5, 2013; published electronically September 19, 2013.
http://www.siam.org/journals/sima/45-5/90431.html

[†]Department of Petroleum Engineering, Faculty of Science and Technology, University of Stavanger, 4036 Stavanger, Norway (steinar.evje@uis.no). This author's research was supported by A/S Norske Shell.

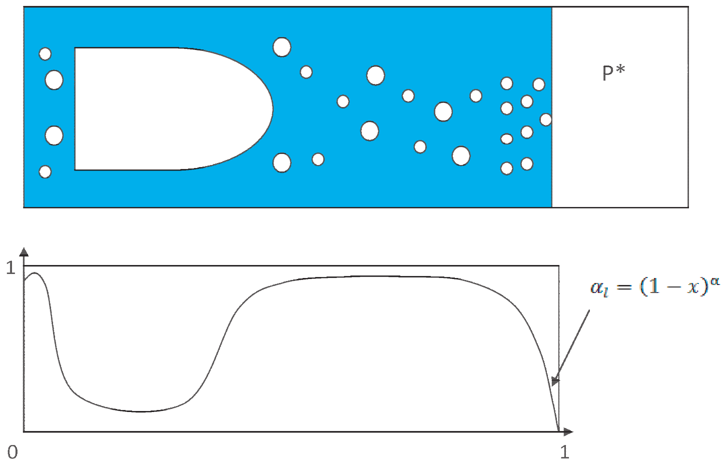

FIG. 1. *Top: Schematic figure showing gas-liquid mixture separated by a pure gas region to the right with a free boundary at the interface and a positive pressure p^* associated with the right gas region. Bottom: Description of the above gas-liquid scenario in terms of the liquid volume fraction $\alpha_l(x,\cdot)$ in Lagrangian coordinates, where $x \in [0,1]$ and the free interface corresponds to $x = 1$. Note that $\alpha_l(x,\cdot) \sim (1-x)^\alpha$, i.e., there is a decay rate $\alpha > 0$ associated with the liquid mass at the free interface.*

described by the function $l(\tau)$ which satisfies

$$l'(\tau) = u|_{\xi=l(\tau)} \text{ for } \tau > 0. \tag{2}$$

Associated with the pure gas region to the right of the moving free boundary there is a specified pressure $p^* > 0$. It is of interest to understand how the free interface and the specified outlet pressure p^* are related. Some issues we seek more understanding of are as follows.

- We deal with the two-phase nature which is different from the single-phase behavior in the sense that a liquid "vacuum" region appears at the right free interface in combination with a positive pressure p^* specified at the interface. This combination does not appear when we deal with single-phase gas flow where vacuum (zero mass) is associated with vanishing pressure. We are interested in demonstrating the well-posedness of this model as well as identifying the long-time behavior.
- In what way does the outer pressure p^* represents a force term that will stabilize the flow system? In wellbore operations which involve gas-liquid flow the ability to control p^* is exploited to stabilize and control the flow system. Can the mathematical analysis of the idealized model in this work reflect this behavior?
- How sharp is the free interface? In other words, what is the liquid mass decay rate at the the interface? Which estimates (estimates that can guarantee well-posedness and stability of the model) are sensitive to the liquid decay rate?

As a further motivation for our studies we briefly show two numerical examples obtained by using the model (1) with inclusion of friction and gravity. See [5] for more information about the numerical scheme that is employed. The examples show ascent of a gas slug initially located at the bottom of a 150-m deep well with a 100-m-high liquid column and a free gas-liquid interface at a position of 100 m (from bottom) and gas above the interface. The first example assumes that the well is open at the top with

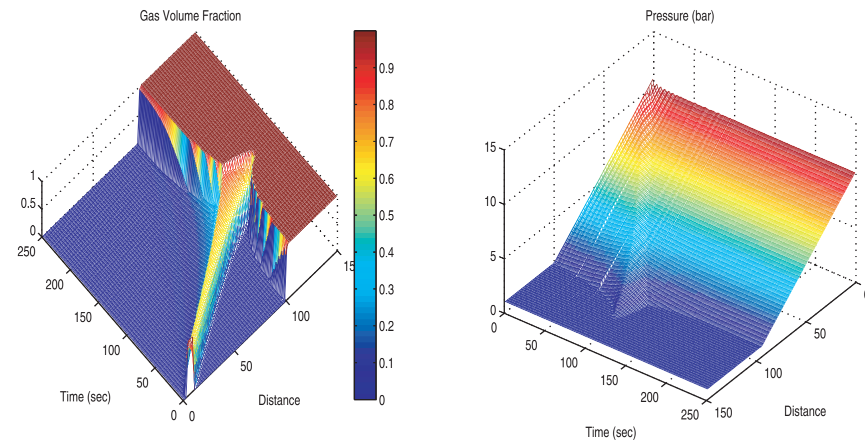

FIG. 2. *Left: The gas volume fraction reflects the strong expansion effect as the gas slug is approaching the surface where the pressure $p^* = 1$ bar. The free interface will be displaced a certain distance up before gravity outperforms the upward directed forces and drives the free interface back again. Right: The corresponding pressure behavior.*

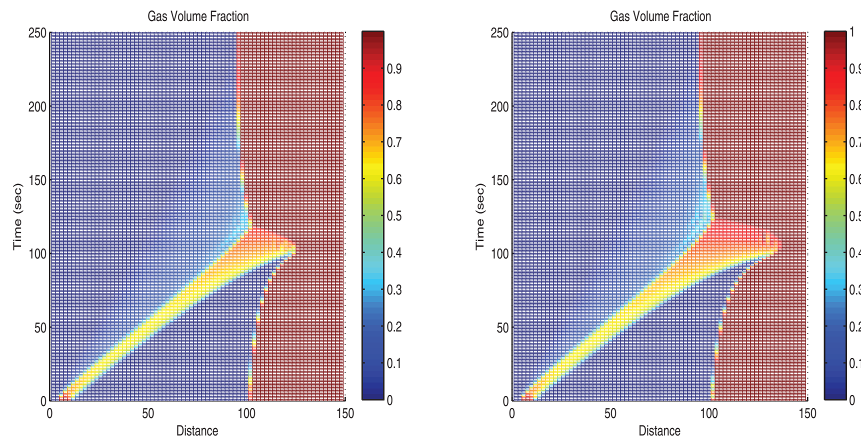

FIG. 3. *Left: Pressure $p^* = 1$ bar at the free interface. Right: Pressure $p^* = 0.5$ bar at the free interface. Clearly, the expansion of gas is much stronger at the free interface implying that this will be squeezed higher up before gravity again drives it back. The pressure p^* acts as an outer force that will have a damping effect on the solution.*

pressure $p^* = 1$ bar; see Figure 2. The second example assumes that the well is open at the top with pressure $p^* = 0.5$ bar; see Figure 3. The results clearly demonstrate the expansion effect at the free interface and how it is sensitive to the pressure p^*. A higher pressure p^* will lead to a stronger damping effect on the movement of the free interface. In other words, p^* allows one to control the characteristic behavior of the gas-liquid flow system. Motivated by this example we now want to explore this behavior in a mathematical framework.

In order to address these issues more systematically we consider the gas-liquid model for a flow regime where gas is dispersed in the liquid phase and it can be assumed that the two fluid velocities are equal, i.e., $u_g = u_l = u$. Natural initial conditions to consider are

(3)
$$n(\xi,0)=n_0(\xi),\quad m(\xi,0)=m_0(\xi),\quad u(\xi,0)=u_0(\xi)\quad \text{for } \xi\in(0,l_0) \text{ with } l|_{\tau=0}=l_0.$$

Corresponding boundary conditions, in accordance to the description given above, are

$$u|_{\xi=0}=0,\qquad n|_{\xi=l(\tau)}=n^*,\qquad m|_{\xi=l(\tau)}=0\quad \text{for } \tau>0. \tag{4}$$

This corresponds to a situation where there is gas to the right of the free gas-liquid interface $l(\tau)$ and the pressure is given by a specified pressure $p^*=p(n^*,0)$. Using the notation that $m=\alpha_l\rho_l$ and $n=\alpha_g\rho_g$, we obtain from (1) the following formulation of the model:

$$\begin{aligned} &\partial_\tau n+\partial_\xi[nu]=0,\\ &\partial_\tau m+\partial_\xi[mu]=0,\\ &\partial_\tau[(n+m)u]+\partial_\xi[(n+m)u^2]+\partial_\xi p(m,n)=\partial_\xi[\varepsilon(m,n)\partial_\xi u],\qquad \xi\in(0,l(\tau)). \end{aligned} \tag{5}$$

We also consider a polytropic gas law for the gas phase whereas liquid is assumed to be incompressible. This gives the pressure law

$$p(m,n)=C_1\left(\frac{n}{\rho_l-m}\right)^\gamma,\qquad \gamma>1. \tag{6}$$

For the viscosity we assume that it takes the following form similar to those used before [8, 26]:

$$\varepsilon(m,n)=C_2\frac{nm^{\theta-1}}{(\rho_l-m)^{\theta+1}},\qquad 0<\theta<1. \tag{7}$$

We are interested in gaining insight into how the solution of the transient model (5) will approach its stationary solution. It is convenient to study the model (5) in terms of Lagrangian variables; see section 2 for details. The model then takes the form

$$\begin{aligned} &\partial_t n+(nm)\partial_x u=0,\\ &\partial_t m+m^2\partial_x u=0,\\ &\left(\frac{n+m}{m}\right)\partial_t u+\partial_x p(n,m)=\partial_x(E(n,m)\partial_x u),\qquad x\in(0,1), \end{aligned} \tag{8}$$

with

$$p(n,m)=\left(\frac{n}{\rho_l-m}\right)^\gamma,\qquad \gamma>1 \tag{9}$$

and

$$E(n,m):=\varepsilon(n,m)m=\frac{nm^\theta}{(\rho_l-m)^{\theta+1}},\qquad 0<\theta<1. \tag{10}$$

Boundary conditions are given by

$$u(0,t)=0,\qquad n(1,t)=n^*,\qquad m(1,t)=0, \tag{11}$$

whereas initial data are

$$n(x,0)=n_0(x),\quad m(x,0)=m_0(x),\quad u(x,0)=u_0(x),\qquad x\in(0,1). \tag{12}$$

The main result of this work is summed up precisely in Theorem 2.1. It is shown that under certain conditions on the initial data a weak solution of the model problem is guaranteed to exist. Moreover, the estimates are strong enough to extract information about the long-time behavior. More precisely, it is shown that $\|p(n,m) - p^*\|_2 \to 0$ and $\|u\|_2 \to 0$ as $t \to \infty$. We may also highlight the following observations made from the analysis leading to these conclusions:

- The positive pressure p^* associated with the pure gas region at the free interface allows one to obtain uniform upper and lower estimates of $p(n,m)$ that are independent of time. This is the main result of Lemma 3.3. This result hangs on the fact that p^* is positive. As a consequence, a uniform bound on the fluid velocity can then also be obtained; see Lemma 3.7.
- The uniform bounds on $p(n,m)$ require that a sufficiently small energy is ensured for all times; see Lemma 3.1. This result is obtained by choosing initial data (n_0, m_0, u_0) sufficiently close to the stationary solution $(n_\infty, m_\infty, u_\infty)$, where $p(n_\infty, m_\infty) = p^*$ and $u_\infty = 0$; see condition (46).
- As p^* becomes larger, it is clear from the proof of Lemma 3.3 that the uniform estimate of $p(n,m)$ holds under larger disturbances on initial data from its steady state; see Remark 3.1. This is an interesting observation since in real-life wellbore flows the pressure p^* is used to control the stability of the system.
- Lemma 3.4 is essential for the L^1 estimate of Q_x, which in turn is crucial for the compactness arguments we rely on. This result is sensitive to the decay rate of the liquid mass at the free interface. The lemma makes use of the fact that $m(x,t) \sim (1-x)^{3/4}$ and $\theta \in (0, 1/3)$.

The case when p^* becomes zero (vacuum) is not covered by the analysis presented in this paper and other techniques must be employed. For various existence results for this case see [9, 27, 11, 16, 7] and references therein.

The rest of the paper is organized as follows: In section 2 we derive the model in Lagrangian coordinates and introduce a transformed version of the model which is convenient for obtaining the a priori estimates. Then we discuss the steady state behavior which clears the ground for giving a precise statement of assumptions on initial data and parameters before the main theorem is given. Section 3 deals with the a priori estimates. In section 4 the long-time behavior is discussed and convergence to the stationary solution is proved.

2. Main result. Following along the line of previous studies for the single-phase Navier–Stokes equations [19, 15, 17], it is convenient to replace the moving domain $[0, l(\tau)]$ by a fixed domain by introducing suitable Lagrangian coordinates. That is, we introduce the coordinate transformation

$$x = \int_0^{\xi} m(y,\tau)\,dy, \qquad t = \tau, \tag{13}$$

such that the free boundary $\xi = l(\tau)$ and the fixed boundary $\xi = 0$, in terms of the (x,t) coordinate system, are given by

$$x_0(t) = 0, \qquad x_{l(\tau)}(t) = \int_0^{l(\tau)} m(y,\tau)\,dy = \int_0^{l_0} m_0(y)\,dy = \text{const}, \tag{14}$$

where $\int_{a_0}^{b} m_0(y)\,dy$ is the total liquid mass initially, which we normalize to 1. Applying (13) to shift from (ξ, τ) to (x, t) in the system (5), we get

$$\begin{aligned} n_t + (nm)u_x &= 0, \\ m_t + (m^2)u_x &= 0, \\ \left(\frac{n+m}{m}\right) u_t + p(n,m)_x &= (\varepsilon(n,m)mu_x)_x, \qquad x \in (0,1), \quad t \geq 0, \end{aligned}$$

where boundary conditions are given by

$$u|_{x=0} = 0, \qquad n|_{x=1} = n^*, \qquad m|_{x=1} = 0.$$

In addition, we have the initial data

$$n(x,0) = n_0(x), \quad m(x,0) = m_0(x), \quad u(x,0) = u_0(x), \qquad x \in (0,1).$$

In other words, we have the model

$$\begin{aligned} \partial_t n + (nm)\partial_x u &= 0, \\ \partial_t m + m^2 \partial_x u &= 0, \\ \left(\frac{n+m}{m}\right) \partial_t u + \partial_x p(n,m) &= \partial_x (E(n,m)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{15}$$

with

$$p(n,m) = \left(\frac{n}{\rho_l - m}\right)^{\gamma}, \qquad \gamma > 1 \tag{16}$$

and

$$E(n,m) := \varepsilon(n,m)m = \frac{nm^{\theta}}{(\rho_l - m)^{\theta+1}}, \qquad 0 < \theta < 1. \tag{17}$$

Moreover, boundary conditions are given by

$$u(0,t) = 0, \qquad n(1,t) = n^*, \qquad m(1,t) = 0, \tag{18}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad m(x,0) = m_0(x), \quad u(x,0) = u_0(x), \qquad x \in (0,1). \tag{19}$$

Note that $p(n,m)|_{x=1} = p(n^*, 0) = (n^*/\rho_l)^{\gamma} := p^*$.

2.1. A transformed model. We introduce the variable

$$c = \frac{m}{n+m}, \tag{20}$$

and see from the first two equations of (15) that

$$\partial_t c = -\frac{m}{(n+m)^2} n_t + \left(\frac{1}{n+m} - \frac{m}{(n+m)^2}\right) m_t = \frac{nm^2}{(n+m)^2} u_x - \frac{nm^2}{(n+m)^2} u_x = 0.$$

Noting that

$$\frac{n}{m} = \frac{1-c}{c} := h(c),$$

and introducing the quantity $Q(m) = m/(\rho_l - m)$, we can deduce a reformulated model in terms of the variables (c, Q, u). That is, employing the variable

$$Q(m) = \frac{m}{\rho_l - m} \qquad \left(\text{which implies that } m = \rho_l \frac{Q}{1+Q}\right), \tag{21}$$

implicitly assuming $0 \le m < \rho_l$, it follows that

$$\begin{aligned} Q(m)_t = \left(\frac{m}{\rho_l - m}\right)_t &= \left(\frac{1}{\rho_l - m} + \frac{m}{(\rho_l - m)^2}\right) m_t \\ &= \frac{\rho_l}{(\rho_l - m)^2} m_t = -\rho_l \frac{m^2}{(\rho_l - m)^2} u_x = -\rho_l Q(m)^2 u_x, \end{aligned}$$

in view of the second equation of (15). Hence, it is seen that the model (15)–(19) can be written in terms of the variables (c, Q, u) in the form

$$\begin{aligned} &\partial_t c = 0, \\ &\partial_t Q(m) + \rho_l Q(m)^2 \partial_x u = 0, \\ &[1 + h(c)]\partial_t u + \partial_x p(c, m) = \partial_x (E(c, m)\partial_x u), \qquad x \in (0, 1), \end{aligned} \tag{22}$$

with

$$p(c, m) = [h(c)Q(m)]^\gamma, \qquad Q(m) = \frac{m}{\rho_l - m}, \qquad h(c) = \frac{1}{c} - 1, \tag{23}$$

and

$$E(c, m) = h(c)\left(\frac{m}{\rho_l - m}\right)^{\theta+1} = h(c)Q(m)^{\theta+1}, \qquad 0 < \theta < 1. \tag{24}$$

Moreover, boundary conditions are given by

$$u(0, t) = 0, \qquad c(1, t) = 0, \qquad Q(m)(1, t) = 0, \quad t \ge 0, \tag{25}$$

such that $[h(c)Q]^\gamma(1, t) = p^* = (\frac{n}{\rho_l - m})^\gamma(1, t) = (\frac{n^*}{\rho_l})^\gamma$. Initial data are

$$c(x, 0) = c_0(x), \quad Q(x, 0) = Q(m_0)(x), \quad u(x, 0) = u_0(x), \qquad x \in (0, 1). \tag{26}$$

2.2. Stationary solutions. In this section, and also in the rest of this paper, we restrict ourselves to the case where $\gamma = 2$ in the pressure function. The motivation for this is only to make the discussion more specific and we do expect that the results can be generalized to hold for $\gamma > 1$.

Let $(n_\infty, m_\infty, 0)$ be the solution of the stationary system corresponding to (22),

$$c_\infty(x) = c_0(x), \qquad u_\infty = 0, \qquad \partial_x p(h(c_\infty)Q(m_\infty)) = 0, \tag{27}$$

with boundary conditions

$$Q(m_\infty)(1) = c_\infty(1) = 0. \tag{28}$$

Integrating over $[x,1]$ we see that (27) corresponds to

$$[h(c_\infty)Q(m_\infty)] = (p^*)^{1/2} = K = \frac{n^*}{\rho_l}. \tag{29}$$

We would like to gain some understanding of a possible steady state solution $(c_\infty, Q_\infty, 0)$ and how it is related to the initial data (c_0, Q_0, u_0). From (29) we see that

$$\frac{1}{K}\frac{m_\infty}{\rho_l - m_\infty} = \frac{c_\infty}{1 - c_\infty}. \tag{30}$$

Using that the inverse of $y = x/(1-x)$ is $x = y/(1+y)$, we then get that

$$c_\infty = \frac{\frac{1}{K}\frac{m_\infty}{\rho_l - m_\infty}}{1 + \frac{1}{K}\frac{m_\infty}{\rho_l - m_\infty}} = \frac{m_\infty}{K(\rho_l - m_\infty) + m_\infty} = \frac{m_\infty}{n^* + m_\infty(1-K)}.$$

Hence, let us consider the concrete choice that $m_\infty = \delta(1-x)$. Then we find that

$$c_\infty = \frac{m_\infty}{n^* + m_\infty(1-K)} = \frac{\delta(1-x)}{n^* + \delta(1-x)(1-K)}.$$

Finally, we can also find the corresponding n_∞ by noting that $h(c_\infty) = n_\infty/m_\infty$:

$$\begin{aligned} n_\infty &= \left(\frac{1}{c_\infty} - 1\right) m_\infty = \left(\frac{n^* + m_\infty(1-K)}{m_\infty} - 1\right) m_\infty = n^* - K m_\infty \\ &= n^*(1 - {\rho_l}^{-1} m_\infty). \end{aligned}$$

For $\delta = 1$ we obtain $n_\infty = n^*(1 - {\rho_l}^{-1}(1-x)) \approx n^*$ (if $\rho_l \gg 1$) and $m_\infty = (1-x)$.

Let us also get some insight into how the choice of initial data (n_0, m_0, u_0) will define a unique stationary solution $(n_\infty, m_\infty, u_\infty = 0)$. For that purpose, we consider

$$m_0 = \delta(1-x)^\alpha, \qquad n_0 = (1-\varepsilon)x + \varepsilon, \qquad \varepsilon \approx 1, \tag{31}$$

i.e., $n^* = 1$. This gives rise to

$$c_0 = \frac{m_0}{n_0 + m_0} = \frac{\delta(1-x)^\alpha}{(1-\varepsilon)x + \varepsilon + \delta(1-x)^\alpha}, \qquad h(c_0) = \frac{n_0}{m_0} = \frac{(1-\varepsilon)x + \varepsilon}{\delta(1-x)^\alpha}.$$

Consequently, we get

$$\sqrt{p_0} = h(c_0)Q(m_0) = \frac{n_0}{\rho_l - m_0} = \frac{(1-\varepsilon)x + \varepsilon}{\rho_l - \delta(1-x)^\alpha}. \tag{32}$$

We refer to Figure 4 for a visualization of these curves. Note that $p_0 = [h(c_0)Q(m_0)]^2$ gives the initial pressure profile which takes the pressure $p^* = 1/\rho_l^2$ at the outlet and

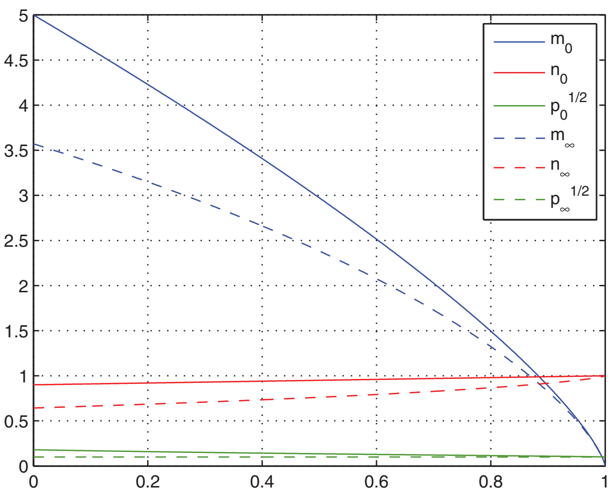

FIG. 4. *The plot shows an example of initial data* $(m_0, n_0, \sqrt{p_0})$ *versus* $(m_\infty, n_\infty, \sqrt{p_\infty})$ *with parameters as follows in* (31)*:* $\varepsilon = 0.9$*,* $\rho_l = 10$*,* $\delta = 5$*,* $\alpha = 3/4$*, and* $\sqrt{p^*} = 1/\rho_l$*.*

otherwise may not be too far away from p^* through the domain $[0, 1]$. See Remark 2.1 for more on this.

For now, we want to calculate the corresponding steady state behavior $(n_\infty, m_\infty, 0)$. First we observe that

$$c_\infty = c_0 = \frac{\delta(1-x)^\alpha}{(1-\varepsilon)x + \varepsilon + \delta(1-x)^\alpha}.$$

Moreover, we observe that m_∞ is uniquely defined from c_∞ by (30) which implies the relation

$$m_\infty = \frac{c_\infty}{1 + c_\infty(K-1)} = \frac{\delta(1-x)^\alpha}{(1-\varepsilon)x + \varepsilon + K\delta(1-x)^\alpha}, \tag{33}$$

and from (29)

$$n_\infty = (\rho_l - m_\infty)K = (1 - Km_\infty) = \frac{(1-\varepsilon)x + \varepsilon}{(1-\varepsilon)x + \varepsilon + K\delta(1-x)^\alpha}. \tag{34}$$

Consequently, as expected we get

$$\begin{aligned}\sqrt{p_\infty} &= h(c_\infty)Q(m_\infty) = \frac{n_\infty}{\rho_l - m_\infty} \\ &= \frac{(1-\varepsilon)x + \varepsilon}{\rho_l[(1-\varepsilon)x + \varepsilon + K\delta(1-x)^\alpha] - \delta(1-x)^\alpha} = \frac{1}{\rho_l} = \sqrt{p^*}.\end{aligned}$$

See Figure 4 for a comparison of initial data $(m_0, n_0, \sqrt{p_0})$ versus stationary masses $(m_\infty, n_\infty, \sqrt{p_\infty})$. Hence, we have demonstrated existence of a stationary solution by explicitly calculating it from a specified set of initial data. It also shows that in general $n_0 \neq n_\infty$ and $m_0 \neq m_\infty$. In particular, we have observed that the steady state masses depend on the outlet pressure $p^* = K^2$ and parameters that characterize the initial masses m_0, n_0 like $\delta, \alpha, \varepsilon$.

2.3. Main result.

Assumptions. The above model is subject to the following assumptions:

$$A_1(1-x)^{\frac{3}{4}} \le m_0(x) \le A_2(1-x)^{\frac{3}{4}} < \rho_l \tag{35}$$

and

$$B_1 \le n_0(x) \le B_2. \tag{36}$$

Consequently,

$$C_1(1-x)^{\frac{3}{4}} \le c_0(x) = \left[\frac{m_0}{n_0+m_0}\right](x) \le C_2(1-x)^{\frac{3}{4}} \tag{37}$$

and

$$D_1(1-x)^{-\frac{3}{4}} \le h(x) = \left[\frac{1-c_0}{c_0}\right](x) = \left[\frac{n_0}{m_0}\right](x) \le D_2(1-x)^{-\frac{3}{4}}, \tag{38}$$

such that

$$D_1(1-x)^{-\frac{7}{4}} \le \frac{dh}{dx}(x) \le D_2(1-x)^{-\frac{7}{4}}, \tag{39}$$

and

$$E_1(1-x)^{\frac{3}{4}} \le Q_0(x) = \left[\frac{m_0}{\rho_l - m_0}\right](x) \le E_2(1-x)^{\frac{3}{4}}, \tag{40}$$

and

$$F_1 \le [h(c_0)Q_0](x) = \left[\frac{n_0}{\rho_l - m_0}\right](x) \le F_2. \tag{41}$$

All the above constants are assumed to be positive. Moreover, we assume that (c_0, Q_0, u_0) satisfy the following regularity:

$$([h(c_0)Q_0]^2)_x \in L^2([0,1]), \tag{42}$$

$$u_0(x) \in H^1([0,1]), \qquad u_0(0) = 0, \tag{43}$$

$$h^{-1}(c_0)([h(c_0)Q_0]^\theta)_x^2 \in L^1([0,1]), \qquad (h(c_0)Q_0^{1+\theta}u_{0,x})_x \in L^2([0,1]). \tag{44}$$

The restriction on γ and θ is as follows:

$$\gamma = 2, \qquad \theta \in \left(0, \frac{1}{3}\right). \tag{45}$$

Then we can state the main theorem.

THEOREM 2.1 (main result). *There is a constant $\varepsilon_0 > 0$ such that if*

$$\int_0^1 \left(1 + \frac{n_0}{m_0}\right) u_0^2\, dx \le \varepsilon, \qquad \int_0^1 h(c_0)\left[p(n_0, m_0)^{1/2} - (p^*)^{1/2}\right]^2 dx \le \varepsilon \tag{46}$$

for any $\varepsilon \in [0, \varepsilon_0]$, and under the assumptions (35)–(45), *then the initial-boundary problem* (15)–(19) *possesses a global weak solution (n, m, u) in the sense that for any $T > 0$, the following statements are valid.*

(A) *We have the following regularity:*

$$\begin{aligned} n, m &\in L^\infty([0,1]\times[0,T]) \cap C^1([0,T]; L^2([0,1])),\\ u &\in L^\infty([0,1]\times[0,T]) \cap C^1([0,T]; L^2([0,1])),\\ E(n,m)u_x &\in L^\infty([0,1]\times[0,T]) \cap C^{\frac{1}{2}}([0,T]; L^2([0,1])). \end{aligned}$$

In particular, the following estimates hold for $\widetilde{A}_{1,2}$ *and* $\widetilde{B}_{1,2}$ *independent of time* $T > 0$*:*

$$\begin{gathered} \widetilde{B}_1 \le n(x,t) \le \widetilde{B}_2\\ \widetilde{A}_1(1-x)^{\frac{3}{4}} \le m(x,t) \le \widetilde{A}_2(1-x)^{\frac{3}{4}} < \rho_l \end{gathered} \tag{47}$$

$\forall (x,t) \in D_T = [0,1]\times[0,T]$. *Moreover,*

$$\begin{gathered} \sup_{t\ge 0}(\|u(\cdot,t)\|_{L^2}) \le C,\\ \sup_{t\ge 0}\|h^{-1}([hQ]^\theta)_x^2\|_{L^1} \le C(T),\\ \|u\|_{L^2(D_T)} + \|Q^2u_x^2\|_{L^2(D_T)} \le C, \quad \|u\|_{L^\infty(D_T)} \le C. \end{gathered} \tag{48}$$

(B) *The following weak formulation of* (15)–(19) *holds:*

$$\begin{aligned} &\int_0^\infty\int_0^1 [n\phi_t - nmu_x\phi]\,dx\,dt + \int_0^1 n_0(x)\phi(x,0)\,dx = 0,\\ &\int_0^\infty\int_0^1 \left[m\varphi_t - m^2u_x\varphi\right]\,dx\,dt + \int_0^1 m_0(x)\varphi(x,0)\,dx = 0,\\ &\int_0^\infty\int_0^1 \left[\left(\frac{n}{m}+1\right)u\psi_t + (p(n,m) - E(n,m)u_x)\psi_x\right]\,dx\,dt\\ &\qquad\qquad = \int_0^\infty p^*\psi(1,t)\,dt - \int_0^1 u_0(x)\psi(x,0)\,dx \end{aligned} \tag{49}$$

for any test function $\phi, \varphi, \psi \in C_0^\infty(D)$*, with* $D := \{(x,t)\,|\,0 < x \le 1,\ t \ge 0\}$.

(C) *Furthermore, the following long-time behavior holds:*

$$E := \frac{1}{2}\int_0^1 u^2(x,t)\,dx \to 0, \tag{50}$$

$$\int_0^1 (p(n,m) - p^*)^2\,dx \to 0, \ \int_0^1 \left(\sqrt{p(n,m)} - \sqrt{p^*}\right)^q\,dx \to 0 \tag{51}$$

$\forall q \in [1,\infty)$*, as time* $t \to \infty$.

Remark 2.1. Concerning the smallness assumption (46), we may consider the choice given in (31). Clearly, we then find that

$$\begin{aligned} h(c_0)Q(m_0) - (p^*)^{1/2} &= \frac{(1-\varepsilon)x + \varepsilon}{\rho_l - \delta(1-x)^\alpha} - \frac{1}{\rho_l}\frac{(\rho_l - \delta(1-x)^\alpha)}{(\rho_l - \delta(1-x)^\alpha)}\\ &= -\frac{(1-\varepsilon)(1-x)}{\rho_l - \delta(1-x)^\alpha} + \frac{C\delta(1-x)^\alpha}{\rho_l - \delta(1-x)^\alpha}. \end{aligned}$$

Consequently, since $h(c_0) \le \frac{1}{\delta(1-x)^\alpha}$ it follows that

$$\begin{aligned} &h(c_0)[h(c_0)Q(m_0) - (p^*)^{1/2}]^2 \\ &\quad\le \frac{2}{\delta(1-x)^\alpha}\left(\frac{(1-\varepsilon)(1-x)}{\rho_l - \delta(1-x)^\alpha}\right)^2 + \frac{2}{\delta(1-x)^\alpha}\left(\frac{C\delta(1-x)^\alpha}{\rho_l - \delta(1-x)^\alpha}\right)^2 \\ &\quad\le \frac{2}{\delta}\frac{(1-\varepsilon)^2(1-x)^{2-\alpha}}{(\rho_l - \delta(1-x)^\alpha)^2} + \frac{2}{\delta}\frac{C\delta^2(1-x)^\alpha}{(\rho_l - \delta(1-x)^\alpha)^2} \\ &\quad\le \frac{C}{\delta}(1-\varepsilon)^2(1-x)^{2-\alpha} + C\delta(1-x)^\alpha \end{aligned}$$

by choosing $\delta < \rho_l$. This implies that

$$\begin{aligned} \int_0^1 h(c_0)\Big[h(c_0)Q_0 - (p^*)^{1/2}\Big]^2 dx &\le C\frac{(1-\varepsilon)^2}{\delta}\int_0^1 (1-x)^{2-\alpha}dx + C\delta\int_0^1 (1-x)^\alpha dx \\ &\le C\frac{(1-\varepsilon)^2}{\delta} + C\delta. \end{aligned}$$

Obviously, we can choose the right-hand side (RHS) as small as desired by first choosing δ as small as needed, then choosing ε as close to 1 as necessary.

3. A priori estimates.

We follow along previous works and use a standard semidiscrete difference approximation to obtain the existence of the weak solution. For this purpose, we first derive some a priori estimates to obtain the desired estimates on the approximate solutions. As usual, the key point is to obtain uniform lower and upper bounds on masses. In our gas-liquid setting this means obtaining such uniform estimates on the pressure-related quantity $h(c)Q(m) = n/(\rho_l - m)$; see Lemma 3.3. The technique we rely on is similar to that used by Zhang and Fang [28] for the Navier–Stokes equations with gravity. See also the more recent works [4, 29] for related results when gravity is included in the single-phase Navier–Stokes equations. However, it is the outlet pressure p^* that represents the "external force" in our model that allows us to exploit Lemma 3.2. We have no gravity effect in our model. A consequence of relying on Lemma 3.2, is that one needs a small fluid velocity. This is ensured by the basic energy estimate, Lemma 3.1, by carefully grouping terms in such a way that the pressure term $h(c)Q$ is treated in combination with the outlet pressure p^*. This gives rise to the nonnegative term $\int_0^1 h(c)\int_{(p^*)^{1/2}}^{h(c)Q}(\frac{s^2-p^*}{s^2})\,ds\,dx$.

3.1. A priori estimates.

We are now ready to establish some important estimates. We let C and $C(T)$ denote a generic positive constant depending only on the intial data and the given time T, respectively.

LEMMA 3.1 (energy estimate). *Under the assumptions of Theorem* 2.1 *we have the basic energy estimate*

$$\begin{aligned} (52)\qquad &\int_0^1\left([1+h(c)]\frac{u^2}{2} + \frac{1}{\rho_l}h(c)\int_{(p^*)^{1/2}}^{h(c)Q}\left(\frac{s^2-p^*}{s^2}\right)ds\right)dx \\ &\qquad\qquad + \int_0^t\int_0^1 h(c)Q^{1+\theta}u_x^2\,dx\,ds \le C_1\varepsilon, \end{aligned}$$

where C_1 is independent of $t \ge 0$.

Proof. We obtain the following integral equality by multiplying the third equation of (22) by u, integrating over $[0,1]$ and using integration by parts and the boundary conditions (25):

$$\frac{d}{dt}\int_0^1 \left([1+h(c)]\frac{u^2}{2}\right)\,dx + p^* u(1,t) - \int_0^1 [h(c)Q]^2 u_x\,dx + \int_0^1 E(c,Q)u_x^2\,dx = 0. \tag{53}$$

From the second equation of (22) we get

$$u(x,t) = \frac{1}{\rho_l}\frac{d}{dt}\int_0^x \frac{1}{Q}dy$$

and

$$\frac{1}{\rho_l}\frac{d}{dt}\int_0^1 (h(c)^2 Q)\,dx + \int_0^1 [h(c)Q]^2 u_x\,dx = 0.$$

Hence, (53) takes the form

$$\begin{aligned}\frac{d}{dt}\int_0^1 \left([1+h(c)]\frac{u^2}{2}\right)\,dx + \frac{1}{\rho_l}\frac{d}{dt}\int_0^1 \frac{p^*}{Q}dx \\ + \frac{1}{\rho_l}\frac{d}{dt}\int_0^1 (h(c)^2 Q)\,dx + \int_0^1 E(c,Q)u_x^2\,dx = 0.\end{aligned} \tag{54}$$

Now we focus on the term

$$\frac{1}{\rho_l}\frac{d}{dt}\int_0^1 \left(\frac{p^*}{Q} + h(c)^2 Q\right)\,dx.$$

We have that

$$\begin{aligned}\frac{1}{\rho_l}\frac{d}{dt}\int_0^1 \left(\frac{p^*}{Q} + h(c)^2 Q\right)\,dx &= \frac{1}{\rho_l}\frac{d}{dt}\int_0^1 \left(\frac{p^*}{Q} + h(c)^2 Q - 2h(c)(p^*)^{1/2}\right)\,dx \\ &= \frac{1}{\rho_l}\frac{d}{dt}\int_0^1 h(c)\int_{(p^*)^{1/2}}^{h(c)Q}\left(\frac{s^2-p^*}{s^2}\right)\,ds\,dx.\end{aligned} \tag{55}$$

Employing (55) in combination with (54) we get after an integration in time

$$\begin{aligned}&\int_0^1 \left([1+h(c)]\frac{u^2}{2}\right)\,dx \\ &\quad + \frac{1}{\rho_l}\int_0^1 h(c)\int_{(p^*)^{1/2}}^{h(c)Q}\left(\frac{s^2-p^*}{s^2}\right)\,ds\,dx + \int_0^t\int_0^1 E(c,Q)u_x^2\,dxdt \\ &\quad = \int_0^1 \left([1+h(c_0)]\frac{u_0^2}{2}\right)\,dx + \frac{1}{\rho_l}\int_0^1 h(c_0)\int_{(p^*)^{1/2}}^{h(c_0)Q_0}\left(\frac{s^2-p^*}{s^2}\right)\,ds\,dx.\end{aligned} \tag{56}$$

Clearly, for all times $t \geq 0$

$$h(c)\int_{(p^*)^{1/2}}^{h(c)Q}\left(\frac{s^2-p^*}{s^2}\right)\,ds \geq 0.$$

For time $t = 0$ we can estimate the last term on the RHS of (56) as follows:

$$
\begin{aligned}
&\int_0^1 h(c_0) \int_{(p^*)^{1/2}}^{h(c_0)Q_0} \left(\frac{s^2 - p^*}{s^2} \right) ds\, dx \\
&\quad \le \int_0^1 h(c_0) \max \left\{ \frac{1}{p^*} \int_{(p^*)^{1/2}}^{h(c_0)Q_0} [s - (p^*)^{1/2}][s + (p^*)^{1/2}]\, ds, \right. \\
&\qquad\qquad \left. \frac{1}{[h(c_0)Q_0]^2} \int_{h(c_0)Q_0}^{(p^*)^{1/2}} [(p^*)^{1/2} - s][(p^*)^{1/2} + s]\, ds \right\} dx \\
&\quad \le C \int_0^1 h(c_0) \left[h(c_0)Q_0 - (p^*)^{1/2} \right]^2 dx,
\end{aligned} \tag{57}
$$

where

$$
C = 2 \max \left\{ \frac{h(c_0)Q_0}{p^*}, \frac{(p^*)^{1/2}}{[h(c_0)Q_0]^2} \right\}.
$$

By means of the assumptions on the smallness of u_0 and the distance between $h(c_0)Q_0$ and $h(c_\infty)Q_\infty = (p^*)^{1/2}$ as specified in (46), the estimate of (52) is obtained from (56). $\square$

Next, we seek to obtain pointwise control on masses. For that purpose we recall the following lemma that was employed in [4]. This result in turn is based on a paper by Zlotnik [30].

LEMMA 3.2. *Let $f \in C(\mathbb{R})$ and $y, b \in W^{1,1}(0,T)$. Let $y(t)$ satisfy the following equation*

$$
\frac{dy}{dt} = f(y) + \frac{db}{dt}, \qquad t \in \mathbb{R}^+, \tag{58}
$$

and $|b(t_2) - b(t_1)| \le N_0$ for any $0 \le t_1 < t_2$. Then

(1) *if $f(z) \ge 0$, for $z \le M_1$,*

$$
\min\{y(0), M_1\} - N_0 \le y(t), \qquad t \in \mathbb{R}^+; \tag{59}
$$

(2) *if $f(z) \le 0$, for $z \ge M_2$,*

$$
\max\{y(0), M_2\} + N_0 \ge y(t), \qquad t \in \mathbb{R}^+. \tag{60}
$$

We will now focus on how to control the mass quantity $[h(c)Q]$. More precisely, the following lemma is obtained.

LEMMA 3.3. *Under the assumptions of Theorem 2.1 we have the pointwise lower and upper bounds*

$$
N_1 \le [h(c)Q]^\theta(x,t) \le N_2 \qquad \forall (x,t) \in D, \tag{61}
$$

where $D = \{(x,t) : 0 \le x \le 1,\ t \ge 0\}$ and N_1, N_2 are positive constants.

Proof. We have from (22) the equation

$$
\frac{d}{dt} \int_x^1 [1+h(c)]u\, dy + p^* - [h(c)Q]^2 = -E(c,Q)u_x = -h(c)Q^{1+\theta}u_x = \frac{1}{\theta\rho_l}(hQ^\theta)_t. \tag{62}
$$

We introduce the quantity $Y(x,t) = [h(c)Q]^\theta$ and observe that (62) takes the form

$$\begin{aligned}
Y_t &= \rho_l \theta h(c)^{\theta-1} \frac{1}{\rho_l \theta}(hQ^\theta)_t \\
&= \rho_l \theta h(c)^{\theta-1} \left[\frac{d}{dt} \int_x^1 [1+h(c)]u\, dy + p^* - [h(c)Q]^2 \right] \\
&= \rho_l \theta h(c)^{\theta-1} [p^* - Y^{2/\theta}] + \frac{d}{dt}\left(\rho_l \theta h(c)^{\theta-1} \int_x^1 [1+h(c)]u\, dy \right) \\
&=: f(Y) + \frac{dB}{dt}.
\end{aligned}$$

Now, we make the following observation:

$$f(Y) \geq 0, \qquad \text{if} \quad Y \leq (p^*)^{\theta/2}. \tag{63}$$

We also note that since $\theta \in (0,1]$ it follows that $h(c)^{\theta-1} \leq D_1^{-1}(1-x)^{\frac{3}{4}(1-\theta)} \leq D_1^{-1}$. For $B = \rho_l \theta h(c)^{\theta-1} \int_x^1 [1+h(c)]u\, dy$ we have that for any $0 < t_1 < t_2$,

(64)
$$\begin{aligned}
|B(x,t_2) - B(x,t_1)| &= \rho_l \theta h(c)^{\theta-1} \int_x^1 [1+h(c)](u(y,t_2) - u(y,t_1))\, dy \\
&\leq 2\rho_l \theta D_1^{-1} \left(\int_x^1 [1+h(c)]\, dy \right)^{\frac{1}{2}} \sup_{t\geq 0} \left(\int_0^1 [1+h(c)]u^2\, dx \right)^{\frac{1}{2}} \\
&\leq 2\rho_l \theta D_1^{-1} \left(\int_x^1 [1+h(c)]\, dy \right)^{\frac{1}{2}} (2C_1\varepsilon)^{\frac{1}{2}} \\
&\leq 2\rho_l \theta D_1^{-1} \left([1-x] + C[1-x]^{\frac{1}{4}} \right)^{\frac{1}{2}} (2C_1\varepsilon)^{\frac{1}{2}} \\
&\leq C\Big([1-x]^{\frac{1}{2}} + C^{\frac{1}{2}}[1-x]^{\frac{1}{8}} \Big) (2C_1\varepsilon)^{\frac{1}{2}} \\
&\leq C_2 \varepsilon^{\frac{1}{2}},
\end{aligned}$$

since

$$\int_x^1 [1+h(c)]\, dy \leq [1-x] + C[1-x]^{\frac{1}{4}}.$$

In view of Lemma 3.2 and (63) and (64) we conclude that

$$\min\{Y(0), (p^*)^{\theta/2}\} - C_2 \varepsilon^{\frac{1}{2}} \leq Y(t), \tag{65}$$

where $Y(0) = [h(c_0)Q_0]^\theta \geq F_1^\theta > 0$. Thus, there exists ε_0 such that $\forall \varepsilon \in (0, \varepsilon_0)$, $Y(x,t) \geq N_1 > 0$.

Similarly, we have

$$f(Y) \leq 0, \qquad \text{if} \quad Y \geq (p^*)^{\theta/2}. \tag{66}$$

In view of Lemma 3.2 and (66) and (64) we conclude that

$$\max\{Y(0), (p^*)^{\theta/2}\} + C_2 \varepsilon^{\frac{1}{2}} \geq Y(t), \tag{67}$$

where $Y(0) = [h(c_0)Q_0]^\theta \leq F_2$. Hence, we can find a positive constant N_2 such that $Y(t) \leq N_2$ for all times $t \geq 0$. ∎

Remark 3.1. Note that here we cannot allow the gas mass n_0 to vanish at any point in the region at initial time because if so, then the lower constant F_1 in (41) would no longer be positive. Thus, (65) could not guarantee a positive lower limit for $Y(t)$. Similarly, we cannot allow the gas pressure p^* at the right outlet to become zero. For higher values of p^*, if $Y(0) \geq (p^*)^{\theta/2}$, it is clear that (65) gives a positive lower limit for $Y(t)$ for larger values of ε. In other words, for a large outer pressure p^*, we can allow large initial disturbances of velocity u_0 and pressure p_0 relative to the stationary solution.

COROLLARY 3.1. *We have the following pointwise control on the masses:*

$$
\begin{aligned}
&\widetilde{E}_1(1-x)^{\frac{3}{4}} \leq Q(x,t) \leq \widetilde{E}_2(1-x)^{\frac{3}{4}},\\
&\widetilde{A}_1(1-x)^{\frac{3}{4}} \leq m(x,t) \leq \widetilde{A}_2(1-x)^{\frac{3}{4}} < \rho_l,\\
&\widetilde{B}_1 \leq n(x,t) \leq \widetilde{B}_2
\end{aligned}
\tag{68}
$$

for $(x,t) \in D = \{(x,t) : 0 \leq x \leq 1, t \geq 0\}$ *and all constants are positive and independent of time.*

Proof. We use the relations

$$
m = \rho_l \frac{Q}{1+Q} \qquad \text{and} \qquad n = h(c)m
$$

in combination with estimate (61) and the upper and lower bounds on $h(c)$ stated in (38). ∎

Note that the next time-independent estimate of the fluid velocity u is crucial for obtaining the long-time behavior of u; see Lemma 4.2.

COROLLARY 3.2. *Under the assumptions of Theorem* 2.1, *we have that*

$$
\int_0^t \int_0^1 u^2(x,t)\,dx\,dt \leq C,
\tag{69}
$$

where C is independent of $t \geq 0$.

Proof. By making use of (61) and Corollary 3.1, we can estimate as follows:

$$
\begin{aligned}
|u| &\leq \int_0^x |u_y|\,dy \leq \left(\int_0^1 hQ^{\theta+1}u_x^2\,dx\right)^{1/2} \left(\int_0^1 [hQ]^{-1}Q^{-\theta}\,dy\right)^{1/2}\\
&\leq C\left(\int_0^1 hQ^{\theta+1}u_x^2\,dx\right)^{1/2}.
\end{aligned}
$$

Consequently,

$$
\int_0^t \int_0^1 u^2\,dxdt \leq C\int_0^t \int_0^1 hQ^{\theta+1}u_x^2\,dxdt \leq CC_1\varepsilon,
$$

in view of (52). ∎

The next lemma deals with the regularity of the Q_x quantity. A natural approach is to first focus on the regularity of the related variable $([h(c)Q]^\theta)_x$. Then, through a fine tuned balance between the rate of decay for the liquid mass as reflected by Corollary 3.1 which states that $m \sim (1-x)^{3/4}$, and the choice of the θ-parameter, which so far has been assumed to be in $(0,1]$, we can control $([h(c)Q]^\theta)_x$ in a weighted L^2 space.

LEMMA 3.4. *Under the assumptions of Theorem* 2.1, *we have that*

$$\int_0^1 \frac{1}{h}([hQ]^\theta)_x^2\,dx \le C(T) \tag{70}$$

for $\theta \in (0,1/3]$.

Proof. We observe that we have the following equation:

$$[1+h]u_t + ([hQ]^2)_x = -\frac{1}{\rho_l\theta}(h^{1-\theta}[hQ]^\theta)_{tx} = -\frac{1}{\rho_l\theta}\Big(h^{1-\theta}([hQ]^\theta)_x + (h^{1-\theta})_x[hQ]^\theta\Big)_t.$$

We multiply the equation by $h^{\theta-2}([hQ]^\theta)_x$ and rewrite terms on the RHS to obtain

$$\begin{aligned}
&[1+h]u_t h^{\theta-2}([hQ]^\theta)_x + 2[hQ]([hQ])_x h^{\theta-2}([hQ]^\theta)_x \\
&\quad= -\frac{1}{\rho_l\theta}\Big(h^{1-\theta}([hQ]^\theta)_x\Big)_t h^{\theta-2}([hQ]^\theta)_x - \frac{1}{\rho_l\theta}\Big((h^{1-\theta})_x[hQ]^\theta\Big)_t h^{\theta-2}([hQ]^\theta)_x \\
&\quad= -\frac{1}{2\rho_l\theta}\frac{1}{h}\Big(([hQ]^\theta)_x^2\Big)_t - \frac{1}{\rho_l\theta}(h^{1-\theta})_x(Q^\theta)_t h^{2\theta-2}([hQ]^\theta)_x \\
&\quad= -\frac{1}{2\rho_l\theta}\frac{1}{h}\Big(([hQ]^\theta)_x^2\Big)_t + (h^{1-\theta})_x Q^{\theta+1}u_x h^{2\theta-2}([hQ]^\theta)_x.
\end{aligned}$$

Then, integrating over $[0,1]$ in space we get, by using the equation $(Q^\theta)_t+\theta\rho_l Q^{\theta+1}u_x = 0$,

$$\begin{aligned}
&\int_0^1 [1+h]u_t h^{\theta-2}([hQ]^\theta)_x\,dx + \frac{1}{\theta}\int_0^1 2h^{\theta-2}[hQ]^{2-\theta}([hQ]^\theta)_x([hQ]^\theta)_x\,dx \\
&\qquad= -\frac{1}{2\rho_l\theta}\frac{d}{dt}\int_0^1 \frac{1}{h}([hQ]^\theta)_x^2\,dx + \int_0^1 (h^{1-\theta})_x Q^{\theta+1}u_x h^{2\theta-2}([hQ]^\theta)_x\,dx.
\end{aligned}$$

From this we may rewrite as follows:

$$\begin{aligned}
&\int_0^1 [1+h]u_t h^{\theta-2}([hQ]^\theta)_x\,dx + \frac{1}{\theta}\int_0^1 2h^{\theta-2}[hQ]^{2-\theta}([hQ]^\theta)_x([hQ]^\theta)_x\,dx \\
&\quad= -\frac{1}{2\rho_l\theta}\frac{d}{dt}\int_0^1 \frac{1}{h}([hQ]^\theta)_x^2\,dx + \int_0^1 (h^{1-\theta})_x Q^{\theta+1}u_x h^{2\theta-2}([hQ]^\theta)_x\,dx \\
&\quad= -\frac{1}{2\rho_l\theta}\frac{d}{dt}\int_0^1 \frac{1}{h}([hQ]^\theta)_x^2\,dx + (1-\theta)\int_0^1 h^{\theta-2}h^{-(\theta+1)}h_x[hQ]^{\theta+1}u_x([hQ]^\theta)_x\,dx \\
&\quad= -\frac{1}{2\rho_l\theta}\frac{d}{dt}\int_0^1 \frac{1}{h}([hQ]^\theta)_x^2\,dx \\
&\qquad+ (1-\theta)\int_0^1 h^{(-5+\theta)/2}h_x[hQ]^{(\theta+1)/2}\cdot h^{\frac12}Q^{(\theta+1)/2}u_x h^{-\frac12}([hQ]^\theta)_x\,dx.
\end{aligned} \tag{71}$$

We integrate (71) over $[0,t]$ in time and rearrange terms which gives

$$
\begin{aligned}
&\frac{1}{2\rho_l\theta}\int_0^1 \frac{1}{h}([hQ]^\theta)_x^2\,dx + \frac{2}{\theta}\int_0^t\int_0^1 Q^{2-\theta}([hQ]^\theta)_x^2\,dxds\\
&\quad= \frac{1}{2\rho_l\theta}\int_0^1 \frac{1}{h}([hQ_0]^\theta)_x^2\,dx - \int_0^t\int_0^1 [1+h]u_t h^{\theta-2}([hQ]^\theta)_x\,dxds\\
&\qquad+(1-\theta)\int_0^t\int_0^1 h^{(-5+\theta)/2}h_x[hQ]^{(\theta+1)/2}\cdot h^{\frac12}Q^{(\theta+1)/2}u_x h^{-\frac12}([hQ]^\theta)_x\,dxds\\
&\quad= \frac{1}{2\rho_l\theta}\int_0^1 \frac{1}{h}([hQ_0]^\theta)_x^2\,dx - \int_0^t \frac{d}{dt}\int_0^1 [1+h]^{1/2}u\cdot[1+h]^{1/2}h^{\theta-2}h^{1/2}\\
&\qquad\cdot h^{-1/2}([hQ]^\theta)_x\,dxds + \rho_l\theta\int_0^t\int_0^1 [1+h]h^{\theta-2}u(h^\theta Q^{\theta+1}u_x)_x\,dxds\\
&\qquad+(1-\theta)\int_0^t\int_0^1 h^{(-5+\theta)/2}h_x[hQ]^{(\theta+1)/2}\cdot h^{\frac12}Q^{(\theta+1)/2}u_x h^{-\frac12}([hQ]^\theta)_x\,dxds\\
&\quad= \frac{1}{2\rho_l\theta}\int_0^1 \frac{1}{h}([hQ_0]^\theta)_x^2\,dx + \int_0^1 [1+h]^{1/2}u_0\cdot[1+h]^{1/2}h^{\theta-2}h^{1/2}\\
&\qquad\cdot h^{-1/2}([hQ_0]^\theta)_x\,dx - \int_0^1 [1+h]^{1/2}u\cdot[1+h]^{1/2}h^{\theta-2}h^{1/2}\\
&\qquad\cdot h^{-1/2}([hQ]^\theta)_x\,dx + \rho_l\theta\int_0^t\int_0^1 [1+h]h^{\theta-2}u(h^\theta Q^{\theta+1}u_x)_x\,dxds\\
&\qquad+(1-\theta)\int_0^t\int_0^1 h^{(-5+\theta)/2}h_x[hQ]^{(\theta+1)/2}\cdot h^{\frac12}Q^{(\theta+1)/2}u_x h^{-\frac12}([hQ]^\theta)_x\,dxds\\
&:= A_0+A_1+A_2+A_3.
\end{aligned}
\tag{72}
$$

Here we have used the equation $([hQ]^\theta)_t + \rho_l\theta h^\theta Q^{\theta+1}u_x = 0$ such that

$$
\begin{aligned}
&\int_0^1 [1+h]h^{\theta-2}u_t([hQ]^\theta)_x\,dx\\
&\quad= \int_0^1 \Big([1+h]h^{\theta-2}u([hQ]^\theta)_x\Big)_t\,dx - \int_0^1 [1+h]h^{\theta-2}u([hQ]^\theta)_{xt}\,dx\\
&\quad= \frac{d}{dt}\int_0^1 [1+h]h^{\theta-2}u([hQ]^\theta)_x\,dx + \rho_l\theta\int_0^1 [1+h]h^{\theta-2}u(h^\theta Q^{\theta+1}u_x)_x\,dx\\
&\quad= \frac{d}{dt}\int_0^1 [1+h]^{1/2}u\cdot[1+h]^{1/2}h^{\theta-2}h^{1/2}\cdot h^{-1/2}([hQ]^\theta)_x\,dx\\
&\qquad+\rho_l\theta\int_0^1 [1+h]h^{\theta-2}u(h^\theta Q^{\theta+1}u_x)_x\,dx.
\end{aligned}
\tag{73}
$$

For the term A_0 we have

$$
\begin{aligned}
|A_0| &\le \frac{1}{2\rho_l\theta}\int_0^1 \frac{1}{h}([hQ_0]^\theta)_x^2\,dx + \int_0^1 [1+h]^{1/2}u_0\\
&\qquad\cdot[1+h]^{1/2}h^{\theta-2}h^{1/2}\cdot h^{-1/2}([hQ_0]^\theta)_x\,dx\\
&\le C + C\left(\int_0^1 [1+h]u_0^2\,dx + \int_0^1 h^{-1}([hQ_0]^\theta)_x^2\,dx\right) \le C,
\end{aligned}
$$

in view of (44). Here we also have used the following estimate:

$$[1+h]^{1/2}h^{\theta-2}h^{1/2} \sim h^{\theta-1} \le C(1-x)^{(3/4)(1-\theta)} \le C. \tag{74}$$

Similarly, we have for A_1 in (72), by using the Cauchy inequality,

$$|A_1| \le C(\delta)\int_0^1 [1+h]u^2\,dx + \delta\int_0^1 h^{-1}([hQ]^\theta)_x^2\,dx. \tag{75}$$

For the term A_2 we have (using the boundary conditions)

(76)

$$\begin{aligned}
A_2 &= \int_0^t\int_0^1 [1+h]h^{\theta-2}u(h^\theta Q^{\theta+1}u_x)_x\,dxds\\
&= \int_0^t\int_0^1 \Big([1+h]h^{\theta-2}u(h^\theta Q^{\theta+1}u_x)\Big)_x\,dxds\\
&\quad - \int_0^t\int_0^1 \Big([1+h]h^{\theta-2}u\Big)_x h^\theta Q^{\theta+1}u_x\,dxds\\
&= -\int_0^t\int_0^1 \Big([1+h]h^{\theta-2}u\Big)_x h^\theta Q^{\theta+1}u_x\,dxds\\
&= -\int_0^t\int_0^1 [1+h]h^{2\theta-2}Q^{\theta+1}u_x^2\,dxds - \int_0^t\int_0^1 \Big([1+h]h^{\theta-2}\Big)_x uh^\theta Q^{\theta+1}u_x\,dxds\\
&= -\int_0^t\int_0^1 [h^{-1}+1]h^{2\theta-2}\cdot hQ^{\theta+1}u_x^2\,dxds\\
&\quad - \int_0^t\int_0^1 \Big((\theta-2)h^{2\theta-3}+(\theta-1)h^{2\theta-2}\Big)h^{-1/2}[1+h]^{-1/2}h^{-(\theta+1)/2}[hQ]^{(\theta+1)/2}h_x\\
&\qquad \cdot [1+h]^{1/2}u\cdot h^{1/2}Q^{(\theta+1)/2}u_x\,dxds.
\end{aligned}$$

In light of Lemma 3.3 it is sufficient to have the following estimate

$$\Big((\theta-2)h^{2\theta-3}+(\theta-1)h^{2\theta-2}\Big)h^{-1/2}[1+h]^{-1/2}h^{-(\theta+1)/2}h_x \le C, \tag{77}$$

which boils down to estimating

$$h^{2\theta-2}h^{-1/2}h^{-(\theta+1)/2}h_x \sim (1-x)^{(9/4)-(9/8)\theta-(7/4)} = (1-x)^{(1/2)-(9/8)\theta} \le C$$

since $h^{(3/2)\theta-3} \sim (1-x)^{(3/4)[3-(3/2)\theta]} = (1-x)^{(9/4)-(9/8)\theta}$. Clearly, this estimate is achieved by choosing

$$\theta \le 4/9. \tag{78}$$

Consequently,

$$|A_2| \le C\int_0^t\int_0^1 hQ^{\theta+1}u_x^2\,dxds + C\int_0^t\int_0^1 [1+h]u^2\,dxds + C \le C(T). \tag{79}$$

The last term A_3 in (72) is estimated as follows:

(80)

$$\begin{aligned}|A_3| &\le (1-\theta)\int_0^t\int_0^1 h^{(-5+\theta)/2}|h_x|[hQ]^{(\theta+1)/2}\cdot h^{\frac{1}{2}}Q^{(\theta+1)/2}|u_x|h^{-\frac{1}{2}}|([hQ]^\theta)_x|\,dxds\\ &\le C\int_0^t\int_0^1 h^{\frac{1}{2}}Q^{(\theta+1)/2}|u_x|h^{-\frac{1}{2}}|([hQ]^\theta)_x|\,dxds\\ &\le C\left(\int_0^t\int_0^1 hQ^{\theta+1}u_x^2\,dxds+\int_0^t\int_0^1\frac{1}{h}([hQ]^\theta)_x^2\,dxds\right),\end{aligned}$$

where we have used the Cauchy inequality, Lemma 3.3, and that

$$h^{(-5+\theta)/2}h_x \le C \tag{81}$$

since

$$h(x)\sim(1-x)^{-3/4}, \qquad \text{i.e.,} \qquad h^{(-5+\theta)/2}\sim(1-x)^{-(3/4)(-5+\theta)/2}=(1-x)^{(15/8)-(3/8)\theta}$$

and

$$h_x(x)\sim(1-x)^{-7/4},$$

which implies that

$$h^{(-5+\theta)/2}h_x\sim(1-x)^{(1/8)-(3/8)\theta}\le C$$

if

$$\theta\le\frac{1}{3}. \tag{82}$$

To sum up, from (72) and the estimates of A_0, A_1, A_2, and A_3 we have

$$\begin{aligned}&\frac{1}{2\rho_l\theta}\int_0^1\frac{1}{h}([hQ]^\theta)_x^2\,dx+\frac{2}{\theta}\int_0^t\int_0^1 Q^{2-\theta}([hQ]^\theta)_x^2\,dxds\\ &\qquad\le C+C(\delta)\int_0^1[1+h]u^2\,dx+\delta\int_0^1\frac{1}{h}([hQ]^\theta)_x^2\,dx+C(T)\\ &\qquad\quad+C\int_0^t\int_0^1\frac{1}{h}([hQ]^\theta)_x^2\,dxds\\ &\qquad\le C(T)+\delta\int_0^1\frac{1}{h}([hQ]^\theta)_x^2\,dx+C\int_0^t\int_0^1\frac{1}{h}([hQ]^\theta)_x^2\,dxds.\end{aligned} \tag{83}$$

By an appropriate choice of $\delta>0$ and by use of Gronwall's lemma the result (70) follows. ∎

Remark 3.2. It is the estimate (81) which puts the strongest constraint on θ as expressed by (82). In particular, we observe that if we let $m_0\sim(1-x)^{2/3}$ then $h\sim(1-x)^{-2/3}$ and we may assume that $h_x\sim(1-x)^{-5/3}$. Consequently,

$$h^{(-5+\theta)/2}h_x\sim(1-x)^{-\theta/3},$$

which cannot be bounded for any $\theta>0$. By the above approach it seems that the decay rate $\alpha>0$ in $m_0\sim(1-x)^\alpha$ must be larger than 2/3 in order to ensure estimate (70).

LEMMA 3.5. *Under the assumptions of Theorem* 2.1, *we have that*

$$\int_0^1 [1+h(c)]u_t^2\,dx + \int_0^t \int_0^1 h(c)Q^{1+\theta}u_{xt}^2\,dxds \le C. \tag{84}$$

Proof. Taking the derivative of the third equation of (22), multiplying by u_t, and integrating in space over $[0,1]$ we get

$$[1+h(c)]u_{tt} + ([h(c)Q]^2)_{xt} = (h(c)Q^{1+\theta}u_x)_{xt}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [1+h(c)]u_t^2\,dx + \int_0^1 ([h(c)Q]^2)_{xt}u_t\,dx = \int_0^1 (h(c)Q^{1+\theta}u_x)_{xt}u_t\,dx.$$

Integrating in time over $[0,t]$ then gives

$$\begin{aligned}\frac{1}{2}\int_0^1 [1+h(c)]u_t^2\,dx + \int_0^t\int_0^1 ([h(c)Q]^2)_{xt}u_t\,dxds &= \frac{1}{2}\int_0^1 [1+h(c)](u_0)_t^2\,dx \\ &\quad + \int_0^t\int_0^1 (h(c)Q^{1+\theta}u_x)_{xt}u_t\,dxds.\end{aligned} \tag{85}$$

For the second term on the left-hand side of (85) we have that

$$\begin{aligned}&\int_0^t\int_0^1 ([h(c)Q]^2)_{xt}u_t\,dxds \\ &= 2\int_0^t\int_0^1 [h(c)^2QQ_t]_x u_t\,dxds = -2\rho_l\int_0^t\int_0^1 [h(c)^2Q^3u_x]_x u_t\,dxds \\ &= -2\rho_l\int_0^t \Big(h^2Q^3u_xu_t\Big)\Big|_{x=0}^{x=1} ds + 2\rho_l\int_0^t\int_0^1 [h^2Q^3u_x]u_{tx}\,dxds \\ &= 2\rho_l\int_0^t\int_0^1 [h^2Q^3u_x]u_{tx}\,dxds,\end{aligned} \tag{86}$$

where we have used the boundary conditions. For the last term on the RHS of (85) we get

$$\begin{aligned}&\int_0^t\int_0^1 (h(c)Q^{1+\theta}u_x)_{xt}u_t\,dxds \\ &= \int_0^t\int_0^1 \Big([hQ^{1+\theta}u_x]_t u_t\Big)_x\,dxds - \int_0^t\int_0^1 [hQ^{1+\theta}u_x]_t u_{xt}\,dxds \\ &= \int_0^t \Big([hQ^{1+\theta}u_x]_t u_t\Big)\Big|_{x=0}^{x=1} ds - \int_0^t\int_0^1 hQ^{1+\theta}u_{xt}^2\,dxds \\ &\quad - \int_0^t\int_0^1 h(Q^{1+\theta})_t u_xu_{xt}\,dxds \\ &= -\int_0^t\int_0^1 hQ^{1+\theta}u_{xt}^2\,dxds + \rho_l(\theta+1)\int_0^t\int_0^1 hQ^{2+\theta}u_x^2u_{xt}\,dxds.\end{aligned} \tag{87}$$

Combining (85), (86), and (87), we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 [1+h(c)]u_t^2\,dx + \int_0^t\int_0^1 hQ^{1+\theta}u_{xt}^2\,dxds \\
&= \frac{1}{2}\int_0^1 [1+h(c)](u_0)_t^2\,dx - 2\rho_l\int_0^t\int_0^1 [h^2Q^3u_x]u_{tx}\,dxds \\
&\quad + \rho_l(\theta+1)\int_0^t\int_0^1 hQ^{2+\theta}u_x^2u_{xt}\,dxds.
\end{aligned} \tag{88}$$

The first term on the RHS of (88) can be estimated in view of assumptions (42)–(44):

$$\int_0^1 (u_{0t})^2\,dx \le C\left(\int_0^1 ([h(c_0)Q_0]^2)_x^2\,dx + \int_0^1 [h(c_0)Q_0^{\theta+1}u_{0x}]_x^2\,dx\right) \le C. \tag{89}$$

For the second term on the RHS we get

$$\begin{aligned}
&\int_0^t\int_0^1 [h^2Q^3u_x]u_{tx}\,dxds \\
&\le C(\delta)\int_0^t\int_0^1 h^3Q^{5-\theta}u_x^2\,dxds + \delta\int_0^t\int_0^1 hQ^{1+\theta}u_{tx}^2\,dxds \\
&\le C\int_0^t\int_0^1 \max_{x\in[0,1]}[h^2Q^{4-2\theta}]hQ^{1+\theta}u_x^2\,dxds + \delta\int_0^t\int_0^1 hQ^{1+\theta}u_{tx}^2\,dxds \\
&\le C + \delta\int_0^t\int_0^1 hQ^{1+\theta}u_{tx}^2\,dxds,
\end{aligned} \tag{90}$$

where we have used Lemma 3.1 combined with the fact that $h^2Q^{4-2\theta} \le CQ^{2(1-\theta)} \le C$, in view of Lemma 3.3, Corollary 3.1, and $\theta \in (0,1)$. For the last term on the RHS of (88) we get

$$\begin{aligned}
&\int_0^t\int_0^1 hQ^{2+\theta}u_x^2u_{xt}\,dxds \\
&\le C(\delta)\int_0^t\int_0^1 hQ^{3+\theta}u_x^4\,dxds + \delta\int_0^t\int_0^1 hQ^{1+\theta}u_{tx}^2\,dxds.
\end{aligned} \tag{91}$$

We must check the following term in more detail:

$$\int_0^t\int_0^1 hQ^{3+\theta}u_x^4\,dxds \le \int_0^t \max_{x\in[0,1]}[Q^2u_x^2]V(s)\,ds, \tag{92}$$

with $V(s) = \int_0^1 hQ^{1+\theta} u_x^2\,dx \in L^1(0,\infty)$. In particular,

$$
\begin{aligned}
[Q^2 u_x^2] &= h^{-2} Q^{-2\theta} ([hQ^{1+\theta}] u_x)^2 \\
&= h^{-2} Q^{-2\theta} \left(-\int_x^1 [1+h(c)] u_t\,dx - (p^* - [h(c)Q]^2) \right)^2 \\
&\le h^{-2} Q^{-2\theta} \left(C + \int_x^1 [1+h(c)] u_t\,dx \right)^2 \\
&\le h^{-2+2\theta} [hQ]^{-2\theta} \left(C + \left(\int_x^1 [1+h(c)]\,dx \right)^{1/2} \left(\int_x^1 [1+h(c)] u_t^2\,dx \right)^{1/2} \right)^2 \\
&\le h^{-2(1-\theta)} \left(C + C \int_0^1 [1+h(c)] u_t^2\,dx \right) \\
&\le C + C \int_0^1 [1+h(c)] u_t^2\,dx,
\end{aligned} \tag{93}
$$

where we have used the Hölder inequality and the fact that $h(c) \sim (1-x)^{-3/4}$. Then it follows that

$$
\int_0^t \int_0^1 hQ^{3+\theta} u_x^4\,dxds \le C \int_0^t V(s)\,ds + C \int_0^t V(s) \int_0^1 [1+h(c)] u_t^2 dx\,ds. \tag{94}
$$

Combining (88)–(94) we arrive at

$$
\begin{aligned}
&\frac{1}{2} \int_0^1 [1+h(c)] u_t^2\,dx + \int_0^t \int_0^1 hQ^{1+\theta} u_{xt}^2\,dxds \\
&\qquad \le C + C \int_0^t V(s) \int_0^1 [1+h(c)] u_t^2 dx\,ds.
\end{aligned} \tag{95}
$$

Hence, the conclusion (84) follows by application of Gronwall's inequality. ∎

Corollary 3.3. *From the proof of Lemma* 3.5, *by combining* (93) *and* (84), *it follows that the following uniform estimate holds:*

$$
\|Q^2 u_x^2\|_{L^\infty([0,1]\times[0,\infty))} \le C. \tag{96}
$$

Lemma 3.6. *Under the assumptions of Theorem* 2.1, *we have that*

$$
\int_0^1 |[hQ]_x|\,dx \le C(T), \ \int_0^1 |Q_x|\,dx \le C(T), \ \int_0^1 |m_x|\,dx \le C(T), \tag{97}
$$

and

$$
\|h(c) Q^{1+\theta}\|_{L^\infty([0,1]\times[0,T])} \le C, \tag{98}
$$

and

$$
\int_0^1 |(h(c) Q^{1+\theta} u_x)_x|\,dx \le C(T). \tag{99}
$$

Proof. For estimate (97) we observe that

$$\int_0^1 |([hQ]^\theta)_x|\,dx = \int_0^1 h^{-1/2}|([hQ]^\theta)_x| \cdot h^{1/2}\,dx$$
$$\leq \left(\int_0^1 h^{-1}([hQ]^\theta)_x^2\,dx\right)^{1/2} \left(\int_0^1 h\,dx\right)^{1/2} \leq C(T),$$

in view of Lemma 3.4 and the fact that $h(c) \sim (1-x)^{-3/4}$. Consequently, we have that

$$\int_0^1 |[hQ]_x|\,dx = \frac{1}{\theta}\int_0^1 |([hQ]^\theta)_x|[hQ]^{1-\theta}\,dx \leq C(T)$$

by Lemma 3.3 and

$$\int_0^1 |Q_x|\,dx \leq \int_0^1 \frac{1}{h}|[hQ]_x|\,dx + \int_0^1 \frac{1}{h}|h_x||Q|\,dx \leq C(T) + C\int_0^1 \frac{1}{h^2}|h_x|\,dx \leq C(T).$$

From (21) and Corollary 3.1 we see that

$$\int_0^1 |m_x|\,dx = \frac{1}{\rho_l}\int_0^1 \frac{m^2}{Q^2}|Q_x|\,dx \leq C\int_0^1 |Q_x|\,dx.$$

As far as estimate (98) is concerned it suffices to note that

$$|h(c)Q^{1+\theta}u_x| \leq C + C\left(\int_x^1 [1+h(c)]\,dx\right)^{1/2}\left(\int_0^1 [1+h(c)]u_t^2\,dx\right)^{1/2} \leq C.$$

Similarly, for (99) we have

$$\int_0^1 |(hQ^{1+\theta}u_x)_x|\,dx \leq C(T). \qquad \square$$

Lemma 3.7. *Under the assumptions of Theorem* 2.1, *we have that*

$$\int_0^1 |u_x(x,t)|\,dx \leq C \tag{100}$$

and

$$\|u(x,t)\|_{L^\infty(D_T)} \leq C. \tag{101}$$

Proof. Clearly,

$$u_x = -\frac{1}{E(c,Q)}\int_x^1 [1+h]u_t\,dy - \frac{p^*}{E(c,Q)} + \frac{[h(c)Q]^2}{E(c,Q)}.$$

Using Lemma 3.3 we find

$$|u_x| \leq CQ^{-\theta}\int_x^1 [1+h]|u_t|dy + CQ^{-\theta} + CQ^{-\theta}$$

and in view of Corollary 3.1 we get

$$
\begin{aligned}
&\int_0^1 |u_x|\,dx \\
&\quad \le C\int_0^1 (1-x)^{-\frac{3}{4}\theta}\int_x^1 [1+h]u_t dy\,dx + C\int_0^1 (1-x)^{-\frac{3}{4}\theta}\,dx \\
&\quad \le C\int_0^1 (1-x)^{-\frac{3}{4}\theta}\left(\int_x^1 [1+h]dy\right)^{1/2}\left(\int_x^1 [1+h]u_t^2 dy\right)^{1/2} dx \\
&\qquad + C\int_0^1 (1-x)^{-\frac{3}{4}\theta}\,dx \\
&\quad \le C\int_0^1 (1-x)^{\frac{1}{8}-\frac{3}{4}\theta}\,dx + C \le C.
\end{aligned}
$$

This proves (100). Estimate (101) follows since

$$|u(x,t)| \le \int_0^x |u_x(x,t)|\,dx \le C. \qquad \square$$

LEMMA 3.8. *Under the assumptions of Theorem* 2.1, *we have that*

$$\int_0^1 |Q(x,t)-Q(x,s)|^2\,dx \le C|t-s|^2, \tag{102}$$

$$\int_0^1 |m(x,t)-m(x,s)|^2\,dx \le C|t-s|^2, \tag{103}$$

$$\int_0^1 |u(x,t)-u(x,s)|^2\,dx \le C|t-s|^2, \tag{104}$$

$$\int_0^1 |h(c)Q^{1+\theta}(x,t)-h(c)Q^{1+\theta}(x,s)|^2\,dx \le C|t-s|. \tag{105}$$

Proof. The proof is quite straightforward; see, for example, [11] for details. $\square$

3.2. Existence part of Theorem 2.1. In order to construct weak solutions to the initial boundary problem (IBVP) (8)–(12), we apply the line method where a system of ODEs is derived that can approximate the original model [12, 13, 19, 15, 17, 22, 20, 23, 24, 25]. Semidiscrete versions of the various lemmas can be obtained such that required estimates on (n,m,u) are obtained by means of the estimates on (c,Q,u) and the fact that $n=h(c)m$ and $m=\rho_l Q/(1+Q)$. Then, in combination with Helly's theorem, the existence part (A) and (B) of Theorem 2.1 is obtained.

4. Asymptotic behavior. In the following we discuss the asymptotic behavior of the model. More precisely, we will prove part (C) of Theorem 2.1.

LEMMA 4.1. *Suppose that* $y\in W^{1,1}_{loc}(\mathbb{R}^+)$ *satisfies*

$$y = y_1' + y_2,$$

and

$$|y_2| \le \sum_{i=1}^n \alpha_i, \quad |y'| \le \sum_{i=1}^n \beta_i, \quad \text{on } \mathbb{R}^+,$$

where $y_1 \in W^{1,1}_{loc}(\mathbb{R}^+), \lim_{s\to+\infty} y_1(s) = 0$, *and* $\alpha_i,\ \beta_i \in L^{p_i}(\mathbb{R}^+)$ *for some* $p_i \in [1,\infty),\ i = 1,\dots,n$. *Then*

$$\lim_{s\to+\infty} y(s) = 0.$$

Note that a special case of this lemma is that if $E(t) \in L^1(\mathbb{R}^+)$ and $E'(t) \in L^1(\mathbb{R}^+)$, then $E(t) \to 0$ as $t \to \infty$. This follows by setting $y_1 = 0$ in Lemma 4.1.

LEMMA 4.2. *Under the assumptions of Theorem* 2.1, *the total kinetic energy satisfies*

$$E := \frac{1}{2}\int_0^1 u^2(x,t)\,dx \to 0, \quad \text{as} \quad t \to \infty. \tag{106}$$

Proof. First, we observe that $E(t) \geq 0$ and

$$\int_0^\infty E(t)\,dt = \frac{1}{2}\int_0^\infty \int_0^1 u^2(x,t)\,dxdt \leq C$$

in view of Corollary 3.2 and

$$\begin{aligned}
|E'(t)| &= \left| \int_0^1 u\left(-[h(c)Q]^2 + [h(c)Q^{1+\theta}]u_x\right)_x\,dx \right| \\
&= \left| -u(1,t)p^* + \int_0^1 [h(c)Q]^2 u_x\,dx - \int_0^1 [h(c)Q^{1+\theta}]u_x^2\,dx \right| \\
&= \left| -p^* \int_0^1 u_x\,dx + \int_0^1 [h(c)Q]^2 u_x\,dx - \int_0^1 [h(c)Q^{1+\theta}]u_x^2\,dx \right| \\
&= \left| \int_0^1 \left([h(c)Q]^2 - p^*\right) u_x\,dx \right| + \int_0^1 [h(c)Q^{1+\theta}]u_x^2\,dx \\
&= \int_0^1 [h^{3/2}Q^{(3-\theta)/2}]E^{1/2}|u_x|\,dx + p^* \int_0^1 E^{-1/2}E^{1/2}|u_x|\,dx \\
&\quad + \int_0^1 [h(c)Q^{1+\theta}]u_x^2\,dx \\
&\leq C\left(\int_0^1 Q^{-\theta}\,dx\right)^{1/2}\left(\int_0^1 Eu_x^2\,dx\right)^{1/2} \\
&\quad + p^*\left(\int_0^1 Q^{-\theta}\,dx\right)^{1/2}\left(\int_0^1 Eu_x^2\,dx\right)^{1/2} + \int_0^1 Eu_x^2\,dx.
\end{aligned}$$

In view of Lemma 3.3 we conclude that the first two terms on the RHS are in $L^2(\mathbb{R}^+)$, the last in $L^1(\mathbb{R}^+)$. Hence, by applying Lemma 4.1 with $y_1 = 0$ we may conclude that

$$\lim_{t\to\infty} E(t) = 0. \qquad \square$$

LEMMA 4.3. *Under the assumptions of Theorem* 2.1, *the following estimate holds:*

$$\int_0^t \int_0^1 \left([h(c)Q]^2(x,s) - p^*\right)^2 dxds \leq C \tag{107}$$

for C *independent of time.*

Proof. We have

$$[hQ]^2 - p^* = \int_x^1 [1+h]u_t dy - \frac{1}{\rho_l \theta}(hQ^\theta)_t.$$

Hence,

$$\begin{aligned}
&\int_0^t \int_0^1 \Big([hQ]^2 - p^*\Big)^2 \, dxds \\
&\quad = \int_0^t \int_0^1 \left(\int_x^1 [1+h]u_t dy - \frac{1}{\rho_l\theta}(hQ^\theta)_t\right) \left([hQ]^2 - p^*\right) \, dxds \\
&\quad = \int_0^t \int_0^1 \left([hQ]^2 - p^*\right) \int_x^1 [1+h]u_t dy \, dxds - \int_0^t \int_0^1 \left([hQ]^2 - p^*\right) \frac{1}{\rho_l\theta}(hQ^\theta)_t \, dxds \\
&\quad = I_1 + I_2.
\end{aligned}$$

Clearly, integration by parts and use of the second equation of (22) give

$$\begin{aligned}
I_1 &= \int_0^t \int_0^1 \left([hQ]^2 - p^*\right) \int_x^1 [1+h]u_t dy \, dxds \\
&= \int_0^t \int_0^1 \left([hQ]^2 - p^*\right) \left(\int_x^1 [1+h]u dy\right)_t \, dxds \\
&= 2\rho_l \int_0^t \int_0^1 [hQ]^2 Q u_x \left(\int_x^1 [1+h]u dy\right) \, dxds \\
&\quad + \int_0^1 \left([hQ]^2 - p^*\right) \left(\int_x^1 [1+h]u dy\right) \, dx \\
&\quad - \int_0^1 \left([hQ_0]^2 - p^*\right) \left(\int_x^1 [1+h]u_0 dy\right) \, dx.
\end{aligned}$$

The Cauchy and Hölder inequalities combined with use of Lemmas 3.1, 3.3, and the assumptions give then

$$\begin{aligned}
I_1 &= 2\rho_l \int_0^t \int_0^1 [hQ]^2 Q u_x \left(\int_x^1 [1+h]u dy\right) \, dxds \\
&\quad + \int_0^1 \left([hQ]^2 - p^*\right) \left(\int_x^1 [1+h]u dy\right) \, dx - \int_0^1 \left([hQ_0]^2 - p^*\right) \left(\int_x^1 [1+h]u_0 dy\right) \, dx \\
&\le C \int_0^t \int_0^1 hQ^{1+\theta} u_x^2 \, dxds + C \int_0^t \int_0^1 \left(\int_x^1 [1+h]^2 dy\right) \left(\int_0^1 u^2 dy\right) \, dxds \\
&\quad + C \int_0^1 \left(\int_x^1 [1+h] dy\right)^{1/2} \left(\int_0^1 [1+h]u^2 dy\right)^{1/2} \, dx \\
&\quad + C \int_0^1 \left(\int_x^1 [1+h] dy\right)^{1/2} \left(\int_0^1 [1+h]u_0^2 dy\right)^{1/2} \, dx \\
&\le C \int_0^t \int_0^1 hQ^{1+\theta} u_x^2 \, dxds \\
&\quad + C \int_0^t \int_0^1 u^2 \, dxds + C \left(\int_0^1 [1+h]u^2 dy\right)^{1/2} + C \left(\int_0^1 [1+h]u_0^2 dy\right)^{1/2} \\
&\le C.
\end{aligned}$$

Moreover, by Lemma 3.3 and the assumptions,

$$
\begin{aligned}
I_2 &= \frac{1}{\rho_l \theta} \int_0^t \int_0^1 \left(p^* - [hQ]^2\right) (hQ^\theta)_t \, dxds \\
&= \frac{p^*}{\rho_l \theta} \int_0^1 hQ^\theta \, dx - \frac{p^*}{\rho_l \theta} \int_0^1 hQ_0^\theta \, dx \\
&\quad - \frac{1}{\rho_l(\theta+2)} \int_0^1 h^3 Q^{\theta+2} \, dx + \frac{1}{\rho_l(\theta+2)} \int_0^1 h^3 Q_0^{\theta+2} \, dxds \\
&\le C \int_0^1 h^{1-\theta} \, dx \le C.
\end{aligned}
$$

From this we can conclude that (107) holds. $\square$

4.1. Long-time behavior of Theorem 2.1. We want to prove that

$$
\int_0^1 \Big([h(c)Q]^2(x,t) - p^* \Big)^2 \, dx \to 0, \tag{108}
$$

and

$$
\int_0^1 \Big([h(c)Q](x,t) - \sqrt{p^*} \Big)^q \, dx \to 0, \qquad q \in [1, \infty), \tag{109}
$$

as $t \to \infty$.

Lemma 4.3 shows that

$$
\int_0^1 \Big([h(c)Q]^2(x,s) - p^* \Big)^2 \, dx \in L^1(\mathbb{R}^+). \tag{110}
$$

In addition, it follows that

$$
\begin{aligned}
&\left| \frac{d}{dt} \int_0^1 \left([h(c)Q]^2 - p^* \right)^2 \, dx \right| \\
&\quad = 4 \left| \int_0^1 \left([h(c)Q]^2 - p^* \right) h^2 Q Q_t \, dx \right| \\
&\quad \le 4\rho_l \left| \int_0^1 \left([h(c)Q]^2 - p^* \right) h^2 Q^3 |u_x| \, dx \right| \\
&\quad \le C \left(\int_0^1 h^3 Q^{5-\theta} \left([h(c)Q]^2 - p^* \right)^2 \, dx \right)^{1/2} \left(\int_0^1 hQ^{1+\theta} u_x^2 \, dx \right)^{1/2} \\
&\quad \le C \left(\int_0^1 hQ^{1+\theta} u_x^2 \, dx \right)^{1/2} \in L^2(\mathbb{R}^+).
\end{aligned} \tag{111}
$$

From these two estimates (110) and (111), seen in view of Lemma 4.1, we can conclude that (108) holds. Next, we see that

$$
\begin{aligned}
\int_0^1 \left([h(c)Q] - \sqrt{p^*} \right)^4 \, dx &= \int_0^1 \frac{([h(c)Q] - \sqrt{p^*})^4}{([h(c)Q]^2 - p^*)^2} ([h(c)Q]^2 - p^*)^2 \, dx \\
&\le C \int_0^1 ([h(c)Q]^2 - p^*)^2 \, dx \to 0
\end{aligned} \tag{112}
$$

as $t \to \infty$ where we have used (108). For $q \in [1,4)$ it follows, by using Hölder's inequality with $p = 4/q$ and $r = 1 - q/4 > 0$,

$$\begin{aligned}
&\int_0^1 \left([h(c)Q] - \sqrt{p^*}\right)^q \, dx \\
&\le \left(\int_0^1 ([h(c)Q] - \sqrt{p^*})^4 \, dx\right)^{q/4} \left(\int_0^1 ([h(c)Q] - \sqrt{p^*})^r \, dx\right)^{1/r} \\
&\le C \left(\int_0^1 ([h(c)Q] - \sqrt{p^*})^4 \, dx\right)^{q/4} \to 0,
\end{aligned} \tag{113}$$

by using (112). For $q \in [4, \infty)$ we have

$$\begin{aligned}
\int_0^1 \left([h(c)Q] - \sqrt{p^*}\right)^q \, dx &= \int_0^1 ([h(c)Q] - \sqrt{p^*})^4 ([h(c)Q] - \sqrt{p^*})^{q-4} \, dx \\
&\le \int_0^1 ([h(c)Q] - \sqrt{p^*})^4 \, dx \to 0.
\end{aligned} \tag{114}$$

Hence, (109) has been shown. Clearly, the results (50) and (51) of Theorem 2.1 follow from (108) and (109) by using $h(c)Q(m) = n/(\rho_l - m) = \sqrt{p(n,m)}$.

REFERENCES

[1] C. S. Avelar, P. R. Ribeiro, and K. Sepehrnoori, *Deepwater gas kick simulation*, J. Petrol. Sci. Eng., 67 (2009), pp. 13–22.

[2] C. E. Brennen, *Fundamentals of Multiphase Flow*, Cambridge University Press, New York, 2005.

[3] J. M. Delhaye, M. Giot, and M. L. Riethmuller, *Thermohydraulics of Two-Phase Systems for Industrial Design and Nuclear Engineering*, McGraw–Hill, New York, 1981.

[4] Q. Duan, *On the dynamics of Navier-Stokes equations for a shallow water model*, J. Differential Equations, 250 (2011), pp. 2687–2714.

[5] S. Evje and K.-K. Fjelde, *Hybrid flux-splitting schemes for a two-phase flow model*, J. Comput. Phys., 175 (2002), pp. 674–701.

[6] S. Evje and K.-K. Fjelde, *On a rough AUSM scheme for a one dimensional two-phase model*, Comput. & Fluids, 32 (2003), pp. 1497–1530.

[7] S. Evje, *Global weak solutions for a compressible gas-liquid model with well-formation interaction*, J. Differential Equations, 251 (2011), pp. 2352–2386.

[8] S. Evje and K. H. Karlsen, *Global weak solutions for a viscous liquid-gas model with singular pressure law*, Comm. Pure Appl. Anal., 8 (2009), pp. 1867–1894.

[9] S. Evje, T. Flåtten, and H. A. Friis, *Global weak solutions for a viscous liquid-gas model with transition to single-phase gas flow and vacuum*, Nonlinear Anal., 70 (2009), pp. 3864–3886.

[10] I. Faille and E. Heintze, *A rough finite volume scheme for modeling two-phase flow in a pipeline*, Comput. & Fluids, 28 (1999), pp. 213–241.

[11] H. A. Friis and S. Evje, *Well-posedness of a compressible gas-liquid model with a friction term important for well control operations*, SIAM J. Appl. Math., 71 (2011), pp. 2014–2047.

[12] D. Hoff, *Construction of solutions for compressible, isentropic Navier-Stokes equations in one space dimension with nonsmooth initial data*, Proc. Roy. Soc. Edinburgh Sect. A, 103 (1986), pp. 301–315.

[13] D. Hoff, *Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data*, Trans. Amer. Math. Soc., 303 (1987), pp. 169–181.

[14] M. Ishii, *Thermo-fluid Dynamic Theory of Two-Phase Flow*, Eyrolles, Paris, 1975.

[15] T.-P. Liu, Z. Xin, and T. Yang, *Vacuum states for compressible flow*, Discrete Contin. Dynam. Systems, 4 (1998), pp. 1–32.

[16] Q. Q. Liu and C. J. Zhu, *Asymptotic behavior of a viscous liquid-gas model with mass-dependent viscosity and vacuum*, J. Differential Equations, 252 (2012), pp. 2492–2519.

[17] T. Luo, Z. Xin, and T. Yang, *Interface behavior of compressible Navier–Stokes equations with vacuum*, SIAM J. Math. Anal., 31 (2000), pp. 1175–1191.

[18] J. M. Masella, Q. H. Tran, D. Ferre, and C. Pauchon, *Transient simulation of two-phase flows in pipes*, Int. J. Multiphase Flow, 24 (1998), pp. 739–755.

[19] M. Okada, *Free-boundary value problems for the equation of one-dimensional motion of viscous gas*, Japan J. Indust. Appl. Math., 6 (1989), pp. 161–177.

[20] M. Okada, S. Matusu-Necasova, and T. Makino, *Free-boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity*, Ann. Univ. Ferrara Sez VII (N.S.), 48 (2002), pp. 1–20.

[21] J. E. Romate, *An approximate Riemann solver for a two-phase flow model with numerically given slip relation*, Comput. & Fluids, 27 (1998), pp. 455–477.

[22] T. Yang, Z.-A. Yao, and C.-J. Zhu, *Compressible Navier-Stokes equations with density-dependent viscosity and vacuum*, Comm. Partial Differential Equations, 26 (2001), pp. 965–981.

[23] T. Yang and H. Zhao, *A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity*, J. Differential Equations, 184 (2002), pp. 163–184.

[24] T. Yang and C. Zhu, *Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum*, Comm. Math. Phys., 230 (2002), pp. 329–363.

[25] L. Yao and W. Wang, *Compressible Navier-Stokes equations with density-dependent viscosity, vacuum and gravitational force in the case of general pressure*, Acta Math. Sci. Ser. B (Engl. Ed.), 28 (2008), pp. 801–817.

[26] L. Yao and C. J. Zhu, *Free boundary value problem for a viscous two-phase model with mass-dependent viscosity*, J. Differential Equations, 247 (2009), pp. 2705–2739.

[27] L. Yao and C. J. Zhu, *Existence and uniqueness of global weak solution to a two-phase flow model with vacuum*, Math. Ann., 349 (2011), pp. 903–928.

[28] T. Zhang and D.-Y. Fang, *Global behavior of compressible Navier-Stokes equations with a degenerate viscosity coefficient*, Arch. Ration. Mech. Anal., 182 (2006), pp. 223–253.

[29] C. J. Zhu and R. Z. Zi, *Asymptotic behavior of solutions to 1D compressible Navier-Stokes equations with gravity and vacuum*, Discrete Contin. Dynam. Systems, 30 (2011), pp. 1263–1283.

[30] A. A. Zlotnik, *Uniform estimates and stabilization of symmetric solutions of a system of quaslinear equations*, Differential Equations, 36 (2000), pp. 701–716.