Whether to attack growing assets and enterprises today or tomorrow

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Abstract: An asset possessed by a defender grows from the first to the second period and is attacked in both periods. With large growth, there is no attack in the first period. Conflict is eliminated. The attacker postpones the attack until the second period. The attacker shows restraint in the first period in order to cash in on the fruits of her restraint in the second period. When the defender's discount parameter is at least 1/8 of the attacker's discount parameter, the defender's first period investment is inverse *U* formed in growth and eventually decreases to zero since with more growth, he eventually has to defend against a greater attack in the second period. In the second period, both actors' investments increase in growth. The defender's discount parameter does not influence whether an attack occurs in the first period. A first period attack is prevented if the attacker's discount parameter is large, and growth is above a certain value. Also, if the product of growth and the attacker's discount parameter is above one, a first period attack is prevented if the defence inefficiency is large, or the attack inefficiency is low, or the usability of appropriation is large.

Keywords: conflict; two periods; dynamics; growth; discounting; defence inefficiency; attack inefficiency; appropriative usability; contest success function; time substitution.

1 Introduction

This article analyses when to attack an asset that grows over time. The asset possessed by the defender is valuable to both agents. If the attacker appropriates the asset today, it cannot grow into the future, so the attacker may decide to farm the defender and attack the asset in the future. Dynamic conflict theory poses, as Hirshleifer (1995, p.31) points out, a 'fearful analytical problem'. Static conflict theory is well researched but needs to be supplemented with the time dimension which is essential in all conflict.[1] This article succeeds in making one new breakthrough in the dynamics of conflict. A defender is equipped with an asset. An attacker can choose to attack the asset in the first period, or the second period, or in both periods. The asset grows from the first to the second period, and the defender chooses the optimal defence in both periods. Results are generated dependent on the growth rate, different discount parameters for the defender and attacker, the defence inefficiency, the attack inefficiency, and a usability parameter for the attacker. A conventional static analysis considers the tradeoffs each actor makes between how much to invest to defend or attack the asset at a given point in time, dependent on the opponent's investment. This article additionally analyses the tradeoffs each actor makes concerning investments through time. Each actor's investment in each time period accounts for the same actor's investment in the other time period, and for the other actor's investments in both periods. The tradeoffs are crucially different for the defender and attacker in the two periods, and depend on the six parameters in sometimes unexpected ways.

A few contributions in the literature have focused on the dynamics of conflict. Sethi (1979) applies simple intertemporal optimisation, without strategic interaction, to analyse a continuous dynamic pilfering thief. Reuveny and Maxwell (2001) consider two rival groups, each dependent on a single contested renewable resource. They develop differential equations where, at each point in time, groups allocate their members between resource harvesting and resource appropriation to maximise their income. This leads to a complex non-linear dynamic interaction between conflict, the two populations, and the resource. Hausken (2005a) generalises the model to account for the within-group collective action problem, different resource stocks and efficiencies of harvesting for the two groups, and variable decisiveness of fighting between the two groups. Maxwell and Reuveny (2005) further investigate continuous conflict over renewable natural resources. They find that Hirshleifer's (1991) 'paradox of power' is self-correcting, and that if production causes damage to disputed resources, introducing conflict enhances social welfare. Related to war, Fearon (1995) and Powell (2006) consider commitment and conflicts resulting from bargaining over issues that affect future bargaining power. The literature on economic growth has to a limited degree focused on conflict, and instead on income distribution, human capital, fertility, trade development, money, etc. One example is Benhabib and Rustichini (1996) who analyse how the level of wealth and the degree of inequality affects growth.

For game theory more generally, Fudenberg and Maskin (1986) have shown that "any individually rational outcome can arise as a Nash equilibrium in infinitely repeated games with sufficiently little discounting."[2] Their result has often been used to show that cooperation rather than conflict can be sustained in long-term relationships. The prisoner's dilemma has often been used for illustration (Axelrod, 1984). For the battle of the sexes where one player values the future while the other is myopic, Hausken (2005b) shows that the first player is more inclined through conflictful behaviour to risk a conflict in the present when the future is important. Similarly, Skaperdas and Syropoulos (1996) and Garfinkel and Skaperdas (2000) show how increased importance of the future may harm cooperation. Skaperdas and Syropoulos' (1996) model is such that victory in one period puts one at an advantage in future-period conflicts. This gives growth in the contestable good regardless who holds it, and hence growth intensifies conflict. Lee and Skaperdas (1998) and Gonzalez (2005) attempt to analyse how conflict affects growth and how this feeds back into conflict. These mixed results on the literature make it quite appropriate to analyse whether a two period growth model suppresses or amplifies conflict, possibly differently in the two periods. Whether growth deters or encourages appropriation is a challenging issue which needs to be analysed thoroughly. This article provides a more nuanced view of how growth impacts conflict.

The attacker's decision in an intertemporal conflict model can also be conceptualised as how to make investment substitutions across time. A large attack in the first period is detrimental to asset growth and reduces the opportunities in the second period. Conversely, a too modest attack in the first period may also be suboptimal since the attacker's first period profit may get too low. Such substitutions have hardly been analysed in the literature, though there are a few cases with a somewhat different focus. First, Enders and Sandler (2003) suggest that a terrorist may compile and accumulate resources during times when the defender's investments are high, awaiting times when the defender may relax his efforts and choose lower investments. Second, in preventing terrorism, Keohane and Zeckhauser (2003, pp.201, 224) show that 'the optimal control of terror stocks will rely on both ongoing abatement and periodic cleanup' of 'a terrorist's 'stock of terror capacity''. Their work is influenced by Arrow et al.'s (1951) and Scarf's (1960) (S,s) model of inventory management. The optimal policy in the face of stochastic demand for a product is to replenish inventory up to a level given by S every time it falls to or below s. These contributions do not consider a strategic opponent.

Section 2 develops the two-period game. Section 3 solves the game. Section 4 discusses the conditions for neither attack nor defence in the first period. Section 5 presents simulations. Section 6 concludes.

2 A two-period game

Consider a firm, or more generally any collective or individual actor, with an asset r which it seeks to defend. The asset may be anything of value such as the firm itself, its possessions, a physical or non-physical commodity, an information set, etc. The firm is under attack by another actor which seeks to acquire the asset, or a largest possible fraction of it. The other actor may be another firm in the same or in another industry, or more generally any collective or individual actor which lawfully or unlawfully seeks to appropriate the asset, or earns profit from destroying it. The attacker's objective may be financial gain, or it may be a desire to cause maximum destruction through time for example if the attacker is a terrorist. Other possible objectives for the attacker are political gain, leisure activities, or a desire for challenges. Whether or not this other actor has assets is not under consideration in this paper. In period i, $i = 1, 2$, the firm, referred to as the defender, incurs an effort t_i to defend his asset, hereafter referred to as an investment. The effort may be hiring competent personnel such as lawyers, engineers, and security guards, or implementation of procedures to protect the asset while ensuring that it is available and accessible as the defender requires, or investment in technology such as physical barriers, firewalls, and intrusion detection systems, encryption techniques, access control mechanisms, to ascertain that the defence is optimal. The defence investment expenditure is f_i, where $\partial f_i / \partial t_i > 0$. We consider the simple case $f_i = bt_i$, where b is the inefficiency of investment. Higher b means greater inefficiency, and $1/b$ is the efficiency.[3]

Independently and simultaneously in period i, the other actor, referred to as the attacker, incurs an effort T_i to acquire the asset. The attacker seeks to be more competent than the lawyers, engineers, and security guards employed by the defender, seeks to circumvent the defence procedures, and seeks to break through the firewalls, penetrate the intrusion detection systems, encryption techniques, and access control mechanisms, thus attempting to design the optimal attack. Analogously, the attack investment expenditure is F_i, $\partial F_i / \partial T_i > 0$. We assume $F_i = BT_i$, where B is the inefficiency of investment, and $1/B$ is the efficiency. Both the expenditures bt_i and BT_i can be capital and/or labour, and both the defender and attacker are assumed risk neutral.[4]

We assume that the contest between the defender and attacker takes the form that is common in the conflict and rent seeking literature (Hirshleifer, 1995; Skaperdas, 1996). The defender gets a fraction h_i, and the attacker gets the remaining fraction $1-h_i$, where h_i is the contest success function, $\partial h_i / \partial t_i > 0, \partial h_i / \partial T_i < 0$. The fraction h_i can be thought of as an actual value, or an expected value. We use the common ratio formula (Tullock, 1980):

$$h_i = \frac{t_i}{t_i + T_i} \tag{1}$$

The defender's and attacker's profits in period i are u_i and U_i, respectively. Based on the reasoning above, the profits in the first period are:

$$u_1 = \frac{t_1}{t_1 + T_1} r - bt_1, \quad U_1 = \beta \frac{T_1}{t_1 + T_1} r - BT_1 \tag{2}$$

where β, $0 \leq \beta \leq 1$, is a usability or non-destruction parameter. It expresses that fraction of the appropriated asset that the attacker can make use of. The appropriator gains a fraction β of what the defender loses. Very few appropriations are like transfers of, say, \$1 million, from a defender to an attacker. As Grossman and Kim (1995, p.1279) point out, predation may involve violence and destruction, may have to be processed to be usable, or may be subject to deterioration during shipment. Further, if an information asset is appropriated, the attacker may not fully understand or appreciate the content. More generally, the cost of adapting an appropriated asset to one's overall portfolio of assets, an attacker generally attach different subjective utilities to assets appropriated from others compared with assets it possesses or produces itself.

After the contest in the first period the defender retains the smaller asset $rt_1 / (t_1 + T_1)$. Defenders are continuously worn down by attackers, especially if attackers are successful, and are often also worn down by the wear and tear of time. With nothing to counteract this wearing down, the assets of defenders would gradually become smaller and smaller, and eventually disappear. Usually, defenders of assets set in motion processes to ensure that assets grow over time. Such growth may be due to a blossoming economy, discovering and exploiting market opportunities, hard and skillful work, outside funding, or the benefit of being at the right place at the right time. If an asset is converted into money and placed in a bank, the asset earns an interest rate. If the asset is invested in the stock market, dividends may be paid out. Workers and consultants can be employed to grow assets, and contracts with various actors can be made to ensure further growth. Assets may grow in many different ways from one period to the next. The most straightforward method is to multiply with a growth parameter a, which corresponds to a geometric series. Another method is to raise the asset to an exponent h. Applying both these two methods, assume that the defender's asset $rt_1 / (t_1 + T_1)$ after the first period grows to $art_1 / (t_1 + T_1)$ at the start of the second period. If $a = 1$, there is no growth from the first to the second period. The defender simply starts the second period with the same asset it retained after the first period. If $a > 1$, there is positive growth. If $0 < a < 1$ the asset deteriorates which means negative growth. Finally, if $a = 0$, the asset vanishes, the defender is driven out of business, and the second period is not worth playing. The fraction $rT_1 / (t_1 + T_1)$ of the asset acquired by the attacker in the first period is not subject to growth. It is simply consumed by the attacker.

In the second period the actors invest analogously as in the first period. Independently and simultaneously, the defender incurs an effort t_2 and the attacker incurs an effort T_2. Based on the reasoning above, the profits in the second period are:

$$u_2 = \frac{t_2}{t_2 + T_2} a\left(\frac{t_1}{t_1 + T_1} r\right) - bt_2, \quad U_2 = \beta \frac{T_2}{t_2 + T_2} a\left(\frac{t_1}{t_1 + T_1} r\right) - BT_2 \tag{3}$$

Note that the term $art_1 / (t_1 + T_1)$ is present in both u_2 and U_2 since this is what is under attack in the second period, and that β is present (only once) in U_2 to express destruction also in the second period.

For the two-period game as a whole, with discounting $0 \leq \delta \leq 1$ for the defender and $0 \leq \Delta \leq 1$ for the attacker for the second period, the total profits u and U for the defender and attacker are:

$$u = u_1 + \delta u_2, \quad U = U_1 + \Delta U_2 \tag{4}$$

3 Solving the two-period game

The game is solved with backward induction, starting with the second period, assuming a subgame perfect equilibrium. We thereafter find the optimal solution in the first period, taking into account that the actors' choices in the second period must be in equilibrium. Differentiating (3), and solving the first order conditions $\partial u_2 / \partial t_2 = 0$ and $\partial U_2 / \partial T_2 = 0$ gives

$$t_2 = \frac{\frac{a\beta}{B}\left(\frac{rt_1}{t_1 + T_1}\right)}{(1 + \beta b / B)^2}, \quad T_2 = \frac{\beta bt_2}{B} \tag{5}$$

Intuitively, the ratio t_2/T_2 of the investments in the second period is inverse proportional to the ratio b/B of the inefficiencies, and inverse proportional to the usability β for the attacker. Inserting (5) into (4) gives

$$u = \frac{t_1}{t_1+T_1}\left(1+\frac{\delta a}{(1+\beta b/B)^2}\right)r - bt_1,\quad U = \frac{\beta T_1}{t_1+T_1}\left(1+\frac{t_1}{T_1}\frac{\Delta a}{(1+B/\beta b)^2}\right)r - BT_1 \tag{6}$$

When $a = 0$, or $\delta = 0$ and $\Delta = 0$, which means zero growth or discount parameters equal to zero for both actors, the total profits for the two-period game in (6) equal the profits u_1 and U_1 in (2) for the first period. The second term in each of the two brackets in (6), multiplied with the factor outside each bracket, express the additional profits the defender and attacker earn in the second period. These two terms depend on the four parameters a, b, B, β. Additionally, for the defender it depends on δ, and for the attacker it depends on Δ and t_1/T_1. The asset r operates proportionally outside the brackets. Differentiating (6), and solving the first order conditions $\partial u/\partial t_1 = 0$ and $\partial U/\partial T_1 = 0$ gives:

$$\begin{aligned} t_1 &= \frac{\beta B[(B+\beta b)^2+\delta a B^2]^2[(B+\beta b)^2-\Delta a\beta^2 b^2]}{(B+\beta b)^2[3\beta bB(B+\beta b)+(1+\delta a)B^3+(1-\Delta a)\beta^3 b^3)]^2}r, \\ T_1 &= \frac{\beta^2 b[(B+\beta b)^2+\delta a B^2][(B+\beta b)^2-\Delta a\beta^2 b^2]^2}{(B+\beta b)^2[3\beta bB(B+\beta b)+(1+\delta a)B^3+(1-\Delta a)\beta^3 b^3)]^2}r \end{aligned} \tag{7}$$

We confine attention to the interesting case $t_1 \geq 0$ and $T_1 \geq 0$, which occurs when the defender has an incentive to defend its asset in the first period. The denominators in t_1 and T_1 are equivalent. The two brackets in the numerators in t_1 and T_1 are also equivalent, but oppositely squared. The rightmost bracket in the two numerators reaches zero when, for example, growth is large, causing the corner solution $t_1 = T_1 = 0$. Inserting $a = 0$ or $\delta = \Delta = 0$ into (7) gives:

$$t_1 = \frac{\beta B r}{(B+\beta b)^2},\quad T_1 = \frac{\beta b t_1}{B} \tag{8}$$

Proposition 1. First period:

1 When $\delta > \beta b(B-\beta b)\Delta/2B^2$ (which is satisfied when $\delta > \Delta/8$), the defender's investment t_1 is inverse U formed in a so that it increases in a when $a = 0$, reaches a maximum, and decreases. It otherwise decreases directly.

2 When $\Delta > B(\beta b - B)\,\delta/2\beta^2 b^2$ (which is satisfied when $\Delta > \delta/8$), the attacker's investment T_1 decreases in a. It otherwise is inverse U formed.

3 Both investments t_1 and T_1 equal zero when $a \geq (B+\beta b)^2/\Delta\beta^2 b^2$.

Proof. See the Appendix.

When $a = 0$ or $\delta = \Delta = 0$, only the first period matters, which effectively means that the first period is the last period, which implies $t_1/T_1 = B/\beta b$. This corresponds to the general result $t_2/T_2 = B/\beta b$ or the second period in (9). The ratio of the investments is inverse proportional to the ratio b/B of the inefficiencies, and inverse proportional to the usability for the attacker.

Assume $\delta > \beta b(B - \beta b)\Delta/2B^2$ and $\Delta > B(\beta b - B)\ \delta/2\beta^2 b^2$. As a increases above zero, the ratio t_1/T_1 initially increases. The reason is that the defender seeks to defend his asset. He increases his investment t_1 so that the asset can grow to the second period without suffering too much predation by the attacker. The attacker is also interested in asset growth, and reduces the attack T_1 with the objective of launching a more substantial attack in the second period. The attacker shows restraint in the first period in order to cash in on the fruits of her restraint in the second period. He may, so to speak, farm the defender. As shown in the Appendix, the decrease of T_1 is concave when $B > 2b\beta$ and $a = 0$, concave when $B > \sqrt{3/2}b\beta$ and $a = 1/\delta = 1/\Delta$, and is always concave when $a = (B + \beta b)^2/\Delta\beta^2 b^2$. The ratio t_1/T_1 increases in a, reaches a maximum, and decreases, and thereafter t_1/T_1 decreases too. The reason is that both agents find a joint interest in asset growth. They leave a blossoming asset or enterprise partly untouched from predation. The attacker reduces T_1, and as T_1 gets low, the defender no longer needs a high defence t_1, and optimally reduces t_1. The defender's incentive to guard his asset is inverse U formed in a and thus eventually falls in the value of a because, with more growth, he has to defend against a greater attack in the second period. Eventually, both t_1 and T_1 reach zero, which for both occur when $a = (B + \beta b)^2/\Delta\beta^2 b^2$. This gives a corner solution discussed in the next section.

The uncommon event $\delta < \beta b(B - \beta b)\Delta/2B^2$ means that the defender discounts more than eight times more than the attacker. The defender is less interested in asset growth, and focuses on obtaining a good fraction of the asset in the first period through choosing an investment t_1 that initially decreases in a. The attacker is more interested in asset growth and looks forward to enjoying the fruits of asset growth in the second period. The other uncommon event $\Delta > B(\beta b - B)\ \delta/2\beta^2 b^2$ means that the attacker discounts more than eight times more than the defender. The attacker is less interested in asset growth, focuses especially on the first period, and chooses an investment T_1 that initially increases in a.

Inserting (7) into (5) gives:

$$t_2 = \frac{\beta B^2 a[(B+\beta b)^2 + \delta a B^2]}{(B+\beta b)^2[3\beta b B(B+\beta b) + (1+\delta a)B^3 + (1-\Delta a)\beta^3 b^3)]^2} r \geq 0, \quad T_2 = \frac{\beta b t_2}{B} \geq 0 \quad (9)$$

Proposition 2. Second period:

1 The defender's and attacker's investments t_2 and T_2 increase in a.

2 Attack and defence always occur in the second period. A corner solution with $t_2 = T_2 = 0$ is never possible.

Proof. See the Appendix.

The second period is the last opportunity for the attacker to attack the asset, since the game ends thereafter. Consequently the attacker chooses a large attack T_2, and an even larger attack when growth from the first to the second period is large. Naturally, the defender puts up a solid defence t_2 to counter this attack, and an even more substantial defence if growth is large. He does not want to see a promising growing enterprise predated upon.

If the attacker hypothetically were to choose zero investment $T_2 = 0$ in the second period, she would be guaranteed zero profit in the second period. The second period profits in (3), and the solutions shown in (5) and (9), show how this cannot happen.

Attack and defence may or may not occur in the first period, but always occur in the second period. If the defender were to choose an out-of-equilibrium unreasonably large second period defence t_2, there are cases where the attacker's benefit of a positive attack would not be worth the expenditure BT_2. This can be seen from the rightmost equation in (3) where the second term exceeds the first term when t_2 is large. She would then choose $T_2 = 0$. However, when both defender and attacker choose equilibrium investments t_2 and T_2, 100% deterrence of the opponent is not optimal in the second period equilibrium investments.

Inserting the first and second period investments in (7) and (9) into (2) to (4) gives the profits[5]:

$$
\begin{aligned}
u_1 &= \frac{B^2 r((B+b\beta)^2+aB^2\delta)\,((B+b\beta)^4+aB(B-b\beta)(B+b\beta)^2\delta+a^2b^3B\beta^3\delta\Delta)}{(B+b\beta)^2(3bB^2\beta+3b^2B\beta^2+B^3(1+a\delta)+b^3\beta^3(1-a\Delta))^2} \\
U_1 &= \frac{b^2 r\beta^3((B+b\beta)^6+ab\beta(B+b\beta)^2((B-2b\beta)(B+b\beta)^2+aB^3\delta)\Delta-a^2b^3\beta^3((B-b\beta)(B+b\beta)^2+aB^3\delta)\Delta^2)}{(B+b\beta)^2(3bB^2\beta+3b^2B\beta^2+B^3(1+a\delta)+b^3\beta^3(1-a\Delta))^2} \\
u_2 &= \frac{B}{\beta}t_2 = \frac{aB^3 r((B+b\beta)^2+aB^2\delta)}{(B+b\beta)^2(3bB^2\beta+3b^2B\beta^2+B^3(1+a\delta)+b^3\beta^3(1-a\Delta))} \\
U_2 &= \beta b T_2 = \frac{ab^2Br\beta^3((B+b\beta)^2+aB^2\delta)}{(B+b\beta)^2(3bB^2\beta+3b^2B\beta^2+B^3(1+a\delta)+b^3\beta^3(1-a\Delta))} \\
u &= \frac{B^2 r((B+b\beta)^2+aB^2\delta)^3}{(B+b\beta)^2(3bB^2\beta+3b^2B\beta^2+B^3(1+a\delta)+b^3\beta^3(1-a\Delta))^2} \\
U &= (b^2r\beta^3((B+b\beta)^6+a((B+b\beta)^4(B^2+2bB\beta-2b^2\beta^2)+2aB^3(B+b\beta)^3\delta+a^2B^6\delta^2)\Delta+ \\
&\quad a^2b^3\beta^3(-3bB^2\beta+b^3\beta^3-2B^3(1+a\delta))\Delta^2))/((B+b\beta)^2(3bB^2\beta+3b^2B\beta^2+B^3(1+a\delta)+b^3\beta^3(1-a\Delta))^2)
\end{aligned}
\tag{10}
$$

We define u_i/t_i and U_i/T_i as the rates of return on investments in period i. Profits are interpreted as returns. In the second period, these rates are B/β and βb, respectively, as shown in (10). Intuitively, the defender's rate of return increases if the attack inefficiency increases, or the usability for the attacker decreases, both of which are to the detriment of the attacker. Analogously, the attacker's rate of return increases in the defence inefficiency and usability. When $a = 0$ or $\Delta = \delta = 0$, the first period effectively operates as the last period, with equivalent ratios $u_1/t_1 = B/\beta$, $U_1/T_1 = \beta b$.

Whereas the second period investment ratio for the defender and attacker is $t_2/T_2 = B/\beta b$, the second period profit ratio is $u_2/U_2 = B^2/\beta^3 b^2$, which can be expressed as $u_2/U_2 = (t_2/T_2)^2/\beta$. That is, while the ratio t_2/T_2 is inverse proportional to the ratio b/B of the inefficiencies, and inverse proportional to the usability for the attacker, the ratio u_2/U_2 is inverse proportional to the square of the ratio b/B of the inefficiencies, and inverse proportional to the usability raised to the third power for the attacker. This demonstrates a magnifying effect. Consider a modest investment difference between the defender and attacker in the second period. This means that t_2/T_2 is close to one, as a result of $B/\beta b$ being close to one, which occurs for example with similar attack and defence inefficiencies and usability close to one. The modest investment difference gets magnified into a larger profit difference because of the squaring. The winner in the investment contest gets a disproportionally large fraction of the profit. Furthermore, the defender benefits additionally if the usability is low.

The first period profits u_1 and U_1 in (10) are hard to interpret. Let us exemplify these by inserting $b = B = \beta = 1$ into (7) and (10) which gives:

$$t_1 = \frac{(4+\delta a)^2(4-\Delta a)}{4[8+(\delta-\Delta)a]^2}r, \quad T_1 = \frac{(4+\delta a)(4-\Delta a)^2}{4[8+(\delta-\Delta)a]^2}r,$$
$$u_1 = \frac{(16+\delta\Delta a^2)(4+\delta a)}{4[8+(\delta-\Delta)a]^2}r, \quad U_1 = \frac{(16+\delta\Delta a^2)(4-\Delta a)}{4[8+(\delta-\Delta)a]^2}r \tag{11}$$

For this example, u_1 never equals zero, and increases throughout in δa e.g., when $\delta = \Delta$. $u_1 = r$ when $\Delta a > 0$ because of the corner solution. In contrast, U_1 equals zero when $\Delta a > 4$ and decreases throughout in Δa e.g., when $\delta = \Delta$. When Δa is large, the attacker shows restraint in the first period in order to benefit in the second period.

Proposition 3.

1 The defender's total profit increases in a.

2 The attacker's total profit increases in a unless growth is very low and the attacker is burdened by a high attack inefficiency or a low usability of appropriation. The requirement is $3b\beta\Delta > B(2\delta - \Delta)$ when $a = 0$.

3 The defender's and attacker's second period profits increase in a.

4 When $b = B = \beta = 1$, u_1 never equals zero, and increases throughout in δa e.g., when $\delta = \Delta$, while U_1 equals zero when $\Delta a > 4$ and decreases throughout in Δa e.g., when $\delta = \Delta$.

Proof. See the Appendix.

4 Conditions for neither attack nor defence in the first period

The rightmost bracket in the two numerators in (7) reaches zero when, for example, growth is large, causing $t_1 = T_1 = 0$. Then the attacker does not attack, causing no need for the defender to defend, in the first period. Conflict is entirely eliminated in the first period. The defender is left to enjoy his asset entirely undisturbed in the first period. Such lack of investment in the first period corresponds to a corner solution where $t_1 = T_1 = 0$. The defender's discount parameter δ plays no role in the rightmost bracket in the two numerators in (7), while the attacker's discount parameter Δ does play a role. Hence the attacker's time preference crucially determines whether attack and thus defence occur in the first period, though a total of five parameters play a role.

Let us analyse the conditions on the five parameters a, Δ, b, B, β that ensure neither attack nor defence in the first period, $t_1 = T_1 = 0$. This means that the defender retains his entire asset after the first period. In the second period the contest is thus over ar rather than over $art_1/(t_1 + T_1)$. Of the five parameters, growth a affects both defender and attacker directly. The defence inefficiency b is intrinsically linked to the defender. The attack inefficiency B, usability β, and discount parameter Δ are intrinsically linked to the attacker.

Proposition 4. When $(B + \beta b)^2 - \Delta a\beta^2 b^2 < 0$, neither attack nor defence occur in the first period, $t_1 = T_1 = 0$.

Proof. Follows from requiring $t_1 < 0$ or $T_1 < 0$ (equivalent requirements) in (7), which prevents an attack and does not necessitate a defence in the first period.

The inequality in Proposition 4 contains five parameters which we consider in turn. First, we rewrite the inequality as $a > (B + \beta b)^2/\Delta\beta^2 b^2$, which means that when growth is large, $t_1 = T_1 = 0$. Parameter a is the only parameter among the five parameters which can be adjusted so that the inequality is satisfied regardless of the other four parameters. It is always possible to specify sufficiently large growth that prevents an attack in the first period, which causes no need for defence in the first period. When this is the case, the moral is: Don't disturb a growing asset or enterprise. Leave it to prosper without interference.[6] Wait with the attack until another day, when growth slows down, or the enterprise is over the top. The defender and attacker develop a tacit or not so tacit truce, common understanding, mutual coordination, and joint common interest, by which peace, tranquility, and no conflict occur in the first period. The defender is left to enjoy his entire asset worth r in the first period, and the attacker is pleased earning zero profit in the first period. The attacker's reasoning is that attack on another day, in the future, may generate an even larger profit due to growth of the asset through time. Although an attack on the asset today may generate a modest instantaneous profit, it has the disadvantage that the growth of the asset gets hampered. An attack today stifles the growth of the asset since less of it is left to grow. Hampering such long term growth possibilities may severely reduce the attacker's long term profit opportunities. If the condition in Proposition 1 is met, the attacker thus chooses to show restraint in the first period, in order to gain even further profit in the second period.

To exemplify the growth requirement, assume first equal defence and attack inefficiencies $b = B$ and that $\beta = \Delta = 1$ which implies $a > 4$. This means that the asset must be quadrupled from the first to the second period to prevent an attack altogether in the first period. Second, assume a five-fold superiority of attack over defence expressed such that the attack inefficiency is 1/5 of the defence inefficiency, that is $B = b/5$, and that $\beta = \delta = 1$. This implies $a > 1.44$. Hence the asset must grow with at least 44% to deter an attack in the first period. Third, assume 50% usability for the attacker expressed as $\beta = 0.5$, and that $b = B$ and $\Delta = 1$. This implies $a > 9$. Finally, assume 50% attacker discounting, $\Delta = 0.5$, and that $b = B$ and $\beta = 1$, which implies $a > 8$. This exemplification becomes clearer as we proceed to the next parameters.

Second, we rewrite the inequality as $\Delta > (B + \beta b)^2/a\beta^2 b^2$, which means that when the attacker's discount parameter Δ is large, $t_1 = T_1 = 0$. Both large growth and a large attacker discount parameter cause higher significance for the second period. When the second period is sufficiently significant for the attacker, the attacker refrains from attacking in the first period, as discussed above. High growth can always compensate for a low attacker discount parameter, even when the discount parameter is arbitrarily close to zero, but not when it equals zero which renders the second period irrelevant for the attacker. Conversely, although a high discount parameter can compensate for low growth, the growth requirement $a > [(B + \beta b)/\beta b]^2$ has to be met. When this requirement is not satisfied, even a discount parameter $\Delta = 1$ is not sufficient to prevent an attack in the first period. As one example, at the limit when the attack inefficiency approaches zero, $B \rightarrow 0$, which means a virtually costless attack for the attacker, the requirement becomes $a > 1$, which means that the asset must grow at least arbitrarily little so that $\Delta = 1$ can deter an attack in the first period. Second, when $b = B$ and $\beta = 1$, the requirement becomes $a > 4$.

Observe in (7) how Δa and δa are both multiplicatively present twice in t_1 and T_1. The same is the case for u_1, U_1, u, U in (10). There is no individual occurrence of a or Δ or δ. This shows the linkage between growth and discounting, so that one can compensate for the other. But there is also a difference. Growth in (3) operates only on the asset, and neither on the defence investment expenditure bt_2 nor on the attack investment expenditure BT_2 in the second period. In contrast, discounting δ and Δ in (4) operate on the entire second period profits, u_2 and U_2 respectively, which means on both the asset and the expenditures. Consequently, although Δa occurs once and δa occurs twice in the second period variables t_2, T_2, u_2, U_2 in (9) and (10), growth a additionally occurs individually once and proportionally on each of these four variables.

Third, we rewrite the inequality as $b > B / \left[\beta\left(\sqrt{\Delta a} - 1\right)\right]$, which means that when the defence inefficiency is large, $t_1 = T_1 = 0$. A large defence inefficiency means that it is costly for the defender to defend himself. The attacker can easily inject a serious blow to the detriment of the defender in both the first period and the second period. She chooses to do so in the second period to take advantage of the growth of the asset. The growth requirement is $\Delta a > 1$. More specifically, without attacker discounting, $\Delta = 1$, growth must be larger than one, $a > 1$, for there to be a large b that ensures $t_1 = T_1 = 0$. With discounting $0 < \Delta < 1$, growth must be sufficiently above one, $a > 1/\Delta$, so that a large b ensures $t_1 = T_1 = 0$. This means that attacker discounting must be compensated by high growth in order to prevent an attack in the first period. With maximum attacker discounting, $\Delta = 0$, or zero growth, $a = 0$, the second period becomes irrelevant. Then even an arbitrarily large defence inefficiency does not prevent an attack in the first period. In other words, when b is large and $\Delta a > 1$, the defender enjoys his entire asset in the first period, but is not able to set up a good defence and is severely exploited by the attacker in the second period.

Fourth, we rewrite the inequality as $B < \beta b\left(\sqrt{\Delta a} - 1\right)$, which means that when the attack inefficiency is low, $t_1 = T_1 = 0$. A low attack inefficiency has an impact similar to that of a high defence inefficiency. The attacker is given an advantage in both cases. When $\Delta a > 1$, the attacker uses the advantage in the second period, leaving the asset untouched for growth in the first period.

Fifth, we rewrite the inequality as $\beta > B / \left[b\left(\sqrt{\Delta a} - 1\right)\right]$, which means that when the usability β of appropriation for the attacker is large, $t_1 = T_1 = 0$. A high usability β also gives an advantage to the attacker since much of the appropriated asset can then be utilised by the attacker. As for the third and fourth cases, the attacker uses the advantage in the second period so that the asset can grow from the first to the second period. Conversely, when β is low, although the defender may lose a significant fraction of his asset, the attacker can not utilise a significant part of the fraction of the asset that is appropriated. Hence even when $\Delta a > 1$, the attacker can not obtain sufficient advantage from the substantial asset growth to confine the attack to the second period. Consequently, the attacker attacks also in the first period. Although the usability of appropriation is low in both periods, spreading the attack over both periods causes larger profit for the attacker than confining the attack to the second period.

Let us finally solve for the corner solution when $t_1 = T_1 = 0$. With neither attack nor defence in the first period, the defender keeps his entire asset, and the attacker earns zero profit. This gives the profits $u_1 = r$ and $U_1 = 0$ in the first period. Using (3) and (4), the total profits are:

$$u = r + \delta\left(\frac{t_2}{t_2 + T_2}ar - bt_2\right), \quad U = \Delta\left(\beta\frac{T_2}{t_2 + T_2}ar - BT_2\right) \tag{12}$$

Differentiating (1), solving the first order conditions $\partial u / \partial t_2 = 0$ and $\partial U / \partial T_2 = 0$, and inserting into (3) and (4) gives:

$$t_2 = \frac{\beta Bar}{(B + \beta b)^2},\ T_2 = \frac{\beta bt_2}{B},\ u_2 = \frac{Bt_2}{\beta},\ U_2 = \beta bT_2 = \frac{\beta^3 b^2 u_2}{B^2},\ u = r + \delta u_2,\ U = \Delta U_2 \tag{13}$$

5 Simulations

This section illustrates the earlier results with simulations. Figure 1 sets $\Delta = \delta = b = B = \beta = r = 1$ and plots the investments t_1, T_1, t_2, T_2 and profits u_1, U_1, u_2, U_2, u, U as functions of growth a. t_1 is inverse U formed, T_1 decreases, and $t_2 = T_2$ increase. Division with 3, and the other divisions below, is for scaling purposes. All the profits increase except U_1 which decreases, in accordance with (10). The attacker accepts reduced profit in the first period in order to enjoy asset growth and increased profit in the second period. Neither attack nor defence occur when $a > 4$.

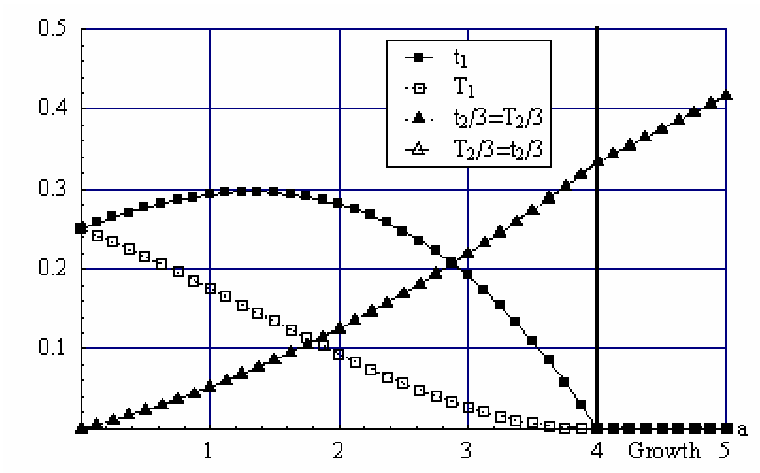
Figure 2 plots as a function of $\Delta = \delta$ and sets $a = 6$. The other parameters are as in Figure 1. Figure 2 is qualitatively similar to Figure 1. Neither attack nor defence occur when $\Delta = \delta > 2/3$.

Figure 3 plots as a function of b and sets $a = 2$. The other parameters are as in Figure 1. The defender's first and second period investments t_1 and t_2 decrease in the defence inefficiency. Neither attack nor defence occur when $b > 1/\left(\sqrt{2} - 1\right)$. As b increases and reaches this value, the attacker first increases and thereafter decreases the first period attack T_1 toward zero in order to benefit in the second period. This causes the first period profit u_1, and also the total profit u, for the defender to be U formed. However, the second period profit for the defender decreases, while the second period profit U_2 and total profit U for the attacker increase.

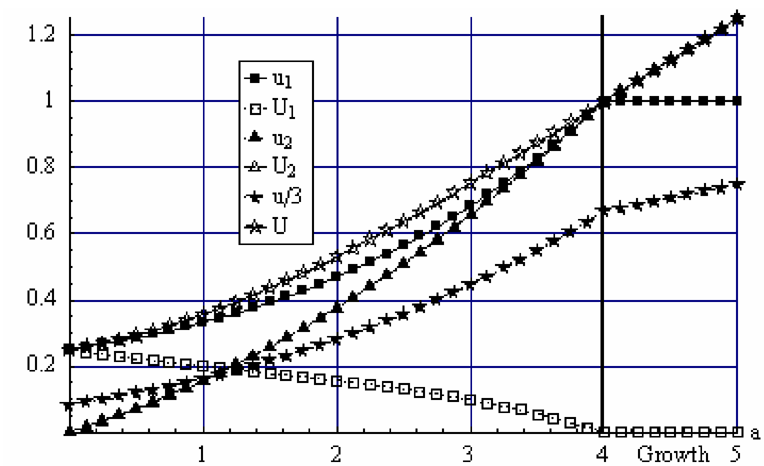
Figure 4 plots as a function of β and sets $a = 2$. The other parameters are as in Figure 1. Neither attack nor defence occur when $B < \sqrt{2} - 1$. The interpretation is analogous to that of Figure 3.

Figure 5 plots as a function of β and sets $a = 2$ and $B = 1/4$. The other parameters are as in Figure 1. Neither attack nor defence occur when $\beta > 1/\left[2\left(\sqrt{2} - 1\right)\right]$. Increasing the usability β benefits the attacker, as does increasing the defence inefficiency b. Hence Figure 5 has some characteristics similar to Figure 3.

Figure 1 (a) Investments t_1, T_1, t_2, T_2 as functions of growth α (b) profits u_1, U_1, u_2, U_2, u, U as functions of growth α (see online version for colours)

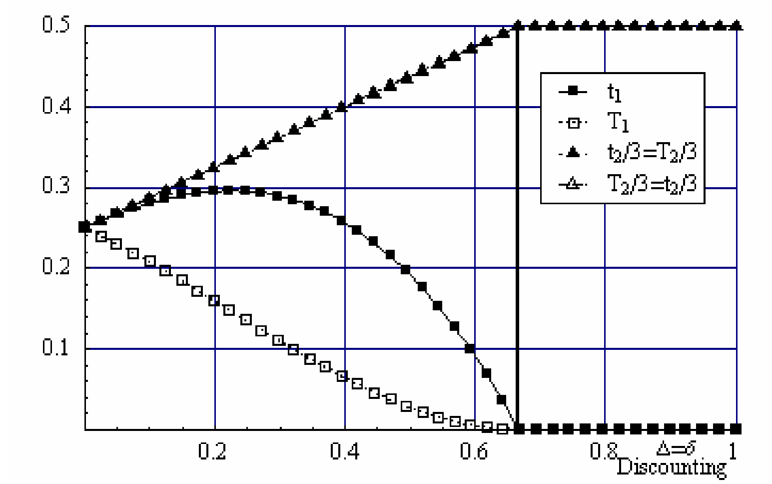


(a)

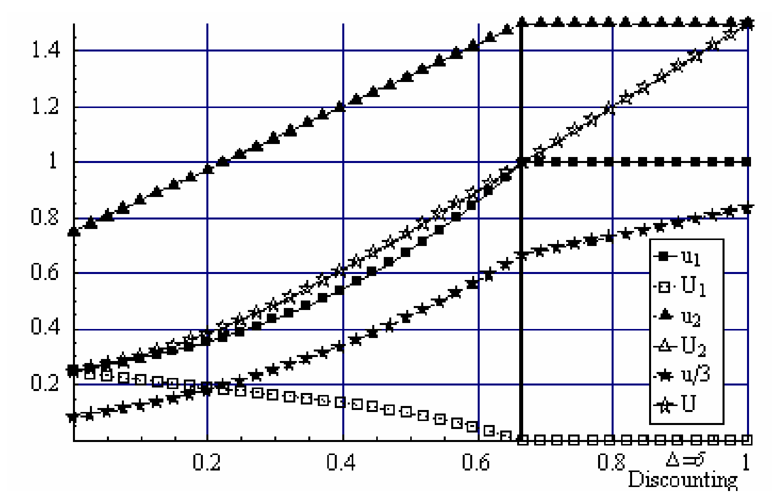


(b)

Figure 2 (a) Investments t_1, T_1, t_2, T_2 as functions of discounting $\Delta = \delta$ (b) profits u_1, U_1, u_2, U_2, u, U as functions of discounting $\Delta = \delta$ (see online version for colours)

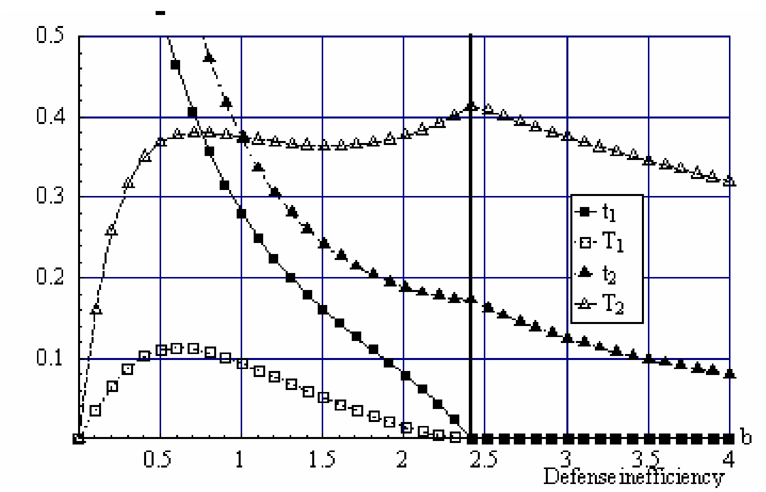


(a)

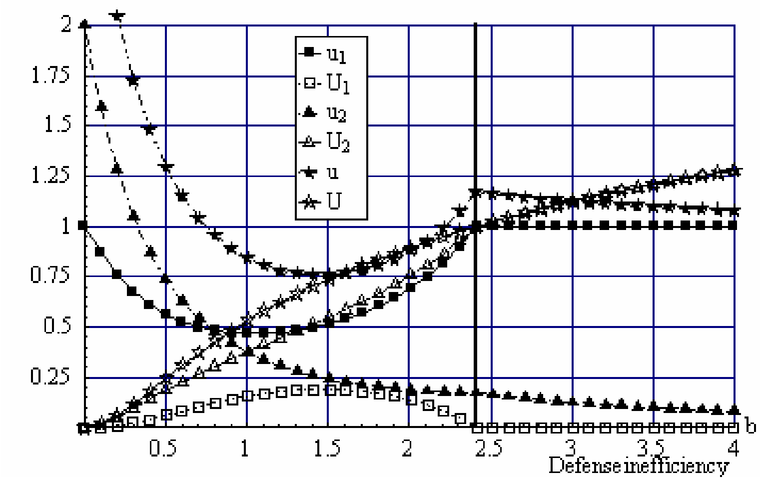


(b)

Figure 3 (a) Investments t_1, T_1, t_2, T_2 as functions of the defence inefficiency b (b) profits u_1, U_1, u_2, U_2, u, U as functions of the defence inefficiency b (see online version for colours)

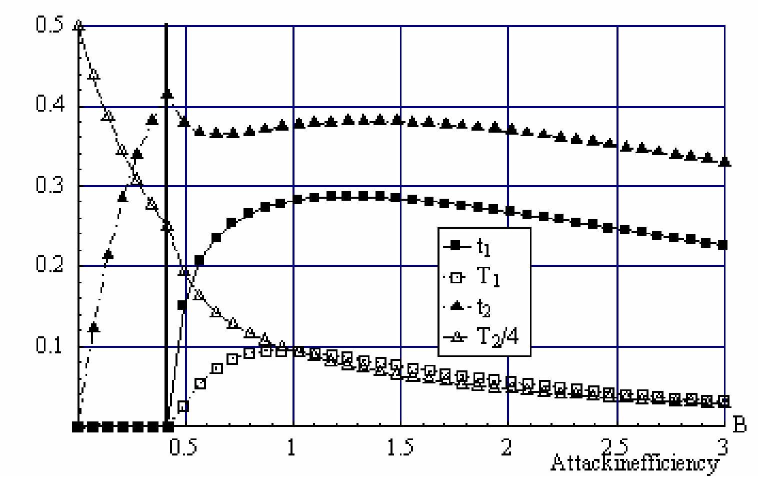


(a)

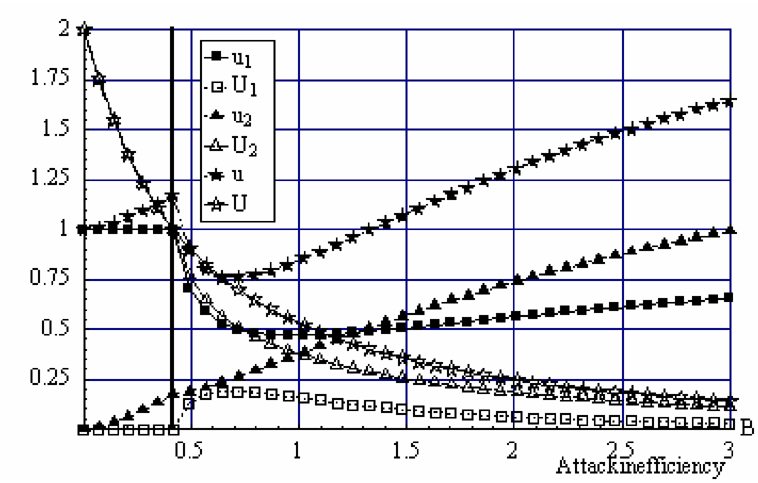


(b)

Figure 4 (a) Investments t_1, T_1, t_2, T_2 as functions of the attack inefficiency B (b) profits u_1, U_1, u_2, U_2, u, U as functions of the attack inefficiency B (see online version for colours)

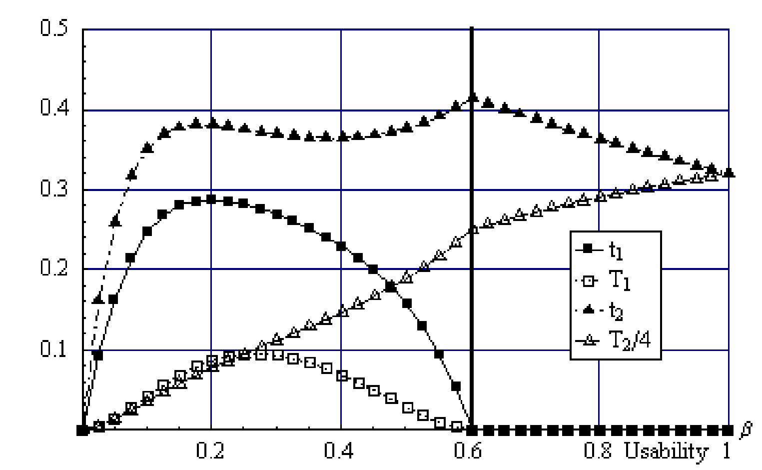


(a)

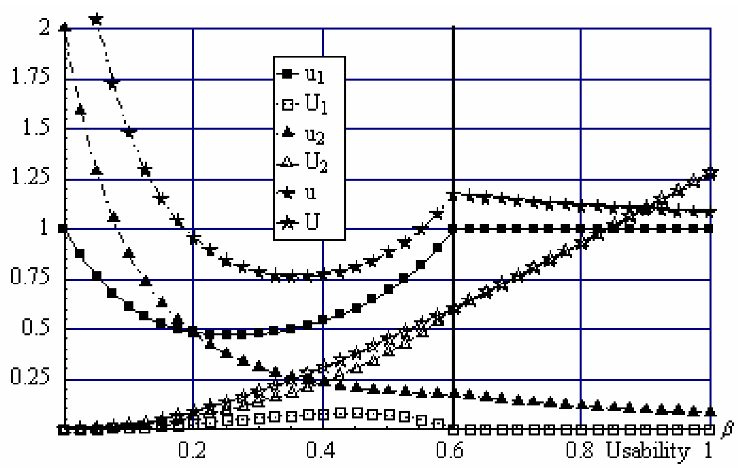


(b)

Figure 5 (a) Investments t_1, T_1, t_2, T_2 as functions of the usability β (b) profits u_1, U_1, u_2, U_2, u, U as functions of the usability β (see online version for colours)



(a)



(b)

6 Conclusions

The article succeeds in contributing to the dynamic conflict literature which is currently small due to analytical intractability. There is a hope that different approaches may change these states of affairs. A defender possesses an asset. An attacker chooses optimal attacks in each of two periods. The defender chooses optimal defences. A contest success function determines the relative fractions obtained by each actor. The defender's fraction may grow from the first to the second period. The attacker's fraction is consumed without growth.

As growth increases, the attacker becomes increasingly interested in the potentially large asset that can be appropriated in the second period. The attacker shows restraint in the first period in order to cash in on the fruits of her restraint in the second period. She may, so to speak, farm the defender. For sufficiently high growth, where four other parameters also play a role, neither attack nor defence occur in the first period. First period conflict is completely eliminated, the defender keeps his entire asset, the attacker gets nothing, and they are both pleased with the situation. This result is quite astonishing. Conflict is simply eliminated by ensuring asset growth. As an example, with equal defence and attack inefficiencies, 100% usability of appropriation, and a discount parameter one for the attacker, the asset must be quadrupled from the first to the second period to prevent an attack altogether in the first period.

When the defender's discount parameter is at least 1/8 of the attacker's discount parameter, the defender's first period investment is inverse U formed in growth and eventually decreases to zero. The defender's incentive to guard his asset eventually falls with growth because, with more growth, he has to defend against a greater attack in the second period. When the attacker's discount parameter is at least 1/8 of the defender's discount parameter, the attacker's first period investment decreases and reaches zero for the same value. In the second period both actors' investments increase in growth. Attack and defence always occur in the second period. Only if the defender were to choose an out-of-equilibrium unreasonably large second period defence, is it possible that the attacker's benefit of a positive attack would not be worth the expenditure. The second period is the last opportunity for the attacker to attack the asset. This gives substantial conflict in the second period, which can be alleviated by extending the game to three periods. The attacker can be deterred by demonstrating substantial growth into future periods.

The defender's total profit increases in growth. The attacker's total profit increases in growth unless growth is very low and the attacker is burdened by a high attack inefficiency or a low usability of appropriation. In the second period, or in the first period with either zero growth or discount parameters equal to zero for both actors, the ratio of the investments for the defender and attacker is inverse proportional to the ratio of the inefficiencies of defence and attack, and inverse proportional to the usability of appropriation for the attacker.

The growth requirement to eliminate conflict in the first period does not depend on the defender's discount parameter, but it depends on four other parameters, and can also be rewritten in terms of these. First, a first period attack is prevented if the attacker's discount parameter is large, and growth is above a certain value. Second, a first period attack is prevented if the defence inefficiency is large, and the product of growth and the discount parameter is above one. Then it is costly for the defender to defend himself. The attacker can easily inject a serious blow to the detriment of the defender in both periods.

She chooses to do so in the second period to take advantage of asset growth. Third, a first period attack is prevented if the attack inefficiency is low, and the product of growth and the discount parameter is above one. A low attack inefficiency has an impact similar to that of a high defence inefficiency. The attacker is given an advantage in both cases. Fourth, a first period attack is prevented if the usability of appropriation is large, and the product of growth and the discount parameter is above one. A high usability also gives an advantage to the attacker since much of the appropriated asset can then be utilised by the attacker. The attacker uses the advantage in the second period so that the asset can grow from the first to the second period.

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Notes

1 The military conflict literature, without production, is strongly dynamic and also stochastic. It focuses on force sizes through time, distinguishes between classical and guerrilla war, and there may be reinforcement.

2 Defining V^* as "the set of individually rational payoffs, "the Folk Theorem states: "For any $(v_1, \ldots, v_n) \in V^*$, if players discount the future sufficiently little, there exists a Nash equilibrium of the infinitely repeated game where, for all i, player i's average payoff is v_i" [Fudenberg and Maskin, (1986), p.537].

3 b can also be interpreted as the unit cost of defence investment, where t_i is generally continuous.

4 An alternative analysis may for example assume that the attacker is risk seeking while the defender is risk averse. Assuming risk neutrality simplifies the analysis. Much of the economic conflict literature related to production, appropriation, defence, and rent seeking assumes risk neutrality. See Skaperdas (1991) for an exception.

5 All calculations in this article are made using the Mathematica software package (http://www.wolfram.com).

6 Partly related notions are 'don't change a winning team' and 'if it ain't broke, don't fix it'.

Appendix

Proof of propositions

Proof of Proposition 1

1 Differentiating t_1 in (7) with respect to a and inserting $a = 0$ gives:

$$\left(\frac{\partial t_1}{\partial a}\right)_{a=0} = \frac{bBr\beta^2\left(2B^2\delta - b\beta(B-b\beta)\Delta\right)}{(B+b\beta)^5} = \frac{\left(B\sqrt{2\delta} - b\beta\sqrt{\Delta}\right)^2 + b\beta B\sqrt{\Delta}\left(2\sqrt{2\delta} - \sqrt{\Delta}\right)}{(B+b\beta)^5} \quad \text{(A1)}$$

which is positive when $\delta > \beta b(B - \beta b)\Delta / 2B^2$, which is satisfied when $\delta > \Delta/8$ (the latter inequality is a sufficient condition for the former inequality). Inserting $a = 1/\delta = 1/\Delta$ and $a = (B + \beta b)^2 / \delta\beta^2 b^2$ into $\partial t_1 / \partial a$ gives

$$\left(\frac{\partial t_1}{\partial a}\right)_{a=\frac{1}{\delta}=\frac{1}{\Delta}} = \frac{br\beta^2(4B^7 + 12bB^6\beta + 14b^2B^5\beta^2 + 4b^3B^4\beta^3 + 2b^5B^2\beta^5 + 3b^6B\beta^6 + b^7\beta^7)\delta}{B(B+b\beta)^2(2B^2 + 3bB\beta + 3b^2\beta^2)^3} > 0 \quad \text{(A2)}$$

$$\left(\frac{\partial t_1}{\partial a}\right)_{a=\frac{(B+b\beta)^2}{b^2\beta^2\Delta}} = -\frac{b^2 r\beta^3\Delta}{B(B+b\beta)^2} < 0 \quad \text{(A3)}$$

which are positive and negative, respectively. Hence when $\delta > \Delta/8$, t_1 increases to a maximum, and decreases to zero. Maximum t_1 occurs when

$$a = \frac{(B+b\beta)^2\left(-(B^2\delta + b^2\beta^2\Delta)\sqrt{8B^3\delta + b^3\beta^3\Delta} + \sqrt{b}\sqrt{\beta}\sqrt{\Delta}(B^2(2B+3b\beta)\delta + b^3\beta^3\Delta)\right)}{2\sqrt{b}B^2\sqrt{\beta}\delta\sqrt{\Delta}(-B^3\delta + b^3\beta^3\Delta)} \quad \text{(A4)}$$

and equals

$$t_1 = \frac{8B^3 r\beta\delta(B^2\delta + b^2\beta^2\Delta)}{8B^6\delta^2 + 20b^3B^3\beta^3\delta\Delta - b^6\beta^6\Delta^2 + b^{3/2}\beta^{3/2}\sqrt{\Delta}(8B^3\delta + b^3\beta^3\Delta)^{3/2}} \quad \text{(A5)}$$

2 Differentiating T_1 in (7) with respect to a and inserting $a = 0$ gives:

$$\left(\frac{\partial T_1}{\partial a}\right)_{a=0} = -\frac{bBr\beta^2(B(B-b\beta)\delta + 2b^2\beta^2\Delta)}{(B+b\beta)^5} = -\frac{\left(B\sqrt{\delta} - b\beta\sqrt{2\Delta}\right)^2 + Bb\beta\sqrt{\delta}\left(2\sqrt{2\Delta} - \sqrt{\delta}\right)}{(B+b\beta)^5} \quad \text{(A6)}$$

which is negative when $\Delta > B(\beta b - B)\delta / 2\beta^2 b^2$, which is satisfied when $\Delta > \delta/8$. Inserting $a = 1/\delta = 1/\Delta$ and $a = (B + \beta b)^2 / \delta\beta^2 b^2$ into $\partial T_1 / \partial a$ gives:

$$\left(\frac{\partial T_1}{\partial a}\right)_{a=\frac{1}{\delta}=\frac{1}{\Delta}} = -\frac{br\beta^2(B+2b\beta)(2B^6 + 5bB^5\beta + 9b^2B^4\beta^2 + 14b^3B^3\beta^3 + 16b^4B^2\beta^4 + 8b^5B\beta^5 + 2b^6\beta^6)\Delta}{B(B+b\beta)^2(2B^2 + 3bB\beta + 3b^2\beta^2)^3} < 0 \quad \text{(A7)}$$

$$\left(\frac{\partial T_1}{\partial a}\right)_{a=\frac{(B+b\beta)^2}{b^2\beta^2\Delta}} = 0 \quad \text{(A8)}$$

which specify decrease when $\Delta > \delta/8$. The second derivative is

$$\frac{\partial^2 T_1}{\partial a^2} = \frac{2bBr(B+b\beta)^2(B^2\beta\delta + b^2\beta^3\Delta)^2(-3b^2B\beta^2 + B^3(1+a\delta) + 2b^3\beta^3(-1+a\Delta))}{(3bB^2\beta + 3b^2B\beta^2 + B^3(1+a\delta) + b^3\beta^3(1-a\Delta))^4} \quad \text{(A9)}$$

which is concave when $B > 2b\beta$ and $a = 0$, and is concave for a more lenient requirements as a increases. For $a = 1/\Delta$ and $a = (B + \beta b)^2 / \Delta\beta^2 b^2$, the concavity requirements are:

$$\left(\frac{\partial^2 T_1}{\partial a^2}\right)_{a=\frac{1}{\Delta}} = \frac{2br(B+b\beta)^2\Delta^3(B^2\beta\delta + b^2\beta^3\Delta)^2(B^2(\delta+\Delta) - 3b^2\beta^2\Delta)}{B^2(3bB\beta\Delta + 3b^2\beta^2\Delta + B^2(\delta+\Delta))^4} > 0 \text{ when } B > \sqrt{\frac{3\Delta}{\delta+\Delta}}b\beta \quad \text{(A10)}$$

$$\left(\frac{\partial^2 T_1}{\partial a^2}\right)_{a=\frac{(B+b\beta)^2}{b^2\beta^2\Delta}} = \frac{2b^7 r\beta^8\Delta^3}{B^2(B+b\beta)^4(B^2\delta+b^2\beta^2\Delta)} > 0 \tag{A11}$$

3 Follows from (7).

Proof of Proposition 2

1 Differentiating t_2 in (9) with respect to a gives:

$$\left(\frac{\partial t_2}{\partial a}\right) = \frac{B^2 r\beta((B+b\beta)^5 + 2aB^2(B+b\beta)^3\delta + a^2B^2(B^3 - b^3\beta^3)\delta^2)}{((B+b\beta)^4 + a(B-b\beta)(B+b\beta)(B^2+bB\beta+b^2\beta^2)\delta)^2} > 0 \tag{A12}$$

To show that $\partial t_2 / \partial a > 0$, observe that the denominator is positive because of the quadration. To show that the numerator is positive, we set it equal to zero and solve with respect to a which gives two solutions. One solution is negative, and the other equals:

$$a = \frac{(B+b\beta)^2}{\left(\frac{\sqrt{b}\sqrt{\beta}\sqrt{B^2\delta+b^2\beta^2\Delta}}{B\sqrt{\delta}(B+b\beta)^{1/2}} - 1\right)B^2\delta} > 0 \text{ when } b\beta / B > (\delta/\Delta)^{1/3} \tag{A13}$$

We proceed to show that this value of a is always larger than $(B+\beta b)^2 / \Delta\beta^2 b^2$, which according to Proposition 1 gives the corner solution $t_1 = T_1 = 0$, which implies that a in (A13) never applies, which implies $\partial t_2 / \partial a > 0$. The requirement:

$$\frac{(B+b\beta)^2}{\left(\frac{\sqrt{b}\sqrt{\beta}\sqrt{B^2\delta+b^2\beta^2\Delta}}{B\sqrt{\delta}(B+b\beta)^{1/2}} - 1\right)B^2\delta} > \frac{(B+b\beta)^2}{b^2\beta^2\Delta} \tag{A14}$$

simplifies to:

$$1 + \frac{B^2\delta}{b^2\beta^2\Delta} > \frac{B^{3/2}\sqrt{\delta}}{b^{3/2}\beta^{3/2}\Delta} \frac{\sqrt{B^2\delta+b^2\beta^2\Delta}}{\sqrt{B}(B+b\beta)^{1/2}} \tag{A15}$$

Squaring both sides gives:

$$1 + \frac{2B^2\delta}{b^2\beta^2\Delta} + \frac{B^4\delta^2}{b^4\beta^4\Delta^2} > \frac{B^3\delta}{b^3\beta^3\Delta^2} \frac{(B^2\delta+b^2\beta^2\Delta)}{B(B+b\beta)} \tag{A16}$$

A stronger requirement is:

$$\frac{2B^2\delta}{b^2\beta^2\Delta} + \frac{B^4\delta^2}{b^4\beta^4\Delta^2} > \frac{B^3\delta}{b^3\beta^3\Delta^2} \frac{(B^2\delta+b^2\beta^2\Delta)}{B(B+b\beta)} \tag{A17}$$

where we have subtracted 1 on the LHS. When the stronger requirement is satisfied, the weaker requirement is also always satisfied. Simplifying (A17) gives:

$$2Bb^2\beta^2\Delta + B^3\delta + b^3\beta^3\Delta > 0 \tag{A18}$$

which is always satisfied.

2 The denominator in (9) is positive since it is squared, and the numerator is positive.

Proof of Proposition 3

1 Differentiating u in (10) with respect to a gives:

$$\left(\frac{\partial u}{\partial a}\right)=\frac{r(B(B+b\beta)^2+aB^3\delta)^2(B^2\delta((B+b\beta)^2(B+3b\beta)+aB^3\delta)+b^3\beta^3(2(B+b\beta)^2-aB^2\delta)\Delta)}{(B+b\,\beta)^2(3bB^2\beta+3b^2B\beta^2+B^3(1+a\delta)+b^3\beta^3(1-a\Delta))^3}>0 \tag{A19}$$

To show that the numerator is positive, we set it equal to zero and solve with respect to a which gives three solutions. Two solutions are negative (and equivalent), and the third equals:

$$a=\frac{(B+b\,\beta)^2(B^2(B+3b\beta)\delta+2b^3\beta^3\Delta)}{B^2\delta(b^3\beta^3\Delta-B^3\delta)} \tag{A20}$$

We proceed to show that this value of a is always larger than $(B+\beta b)^2/\Delta\beta^2b^2$, which according to Proposition 1 gives the corner solution $t_1 = T_1 = 0$, which implies that a in (A20) never applies. The requirement is:

$$\frac{(B+b\,\beta)^2(B^2(B+3b\beta)\delta+2b^3\beta^3\Delta)}{B^2\delta(b^3\beta^3\Delta-B^3\delta)}>\frac{(B+b\,\beta)^2}{b^2\,\beta^2\Delta} \tag{A21}$$

A stronger requirement is:

$$\frac{(B+b\,\beta)^2(B^2(B+3b\beta)\delta+2b^3\beta^3\Delta)}{B^2\delta b^3\beta^3\Delta}>\frac{(B+b\,\beta)^2}{b^2\,\beta^2\Delta} \tag{A22}$$

which simplifies to:

$$\frac{B^3+2b^3\beta^3}{B^2b^3\beta^3}+\frac{3}{b^2\,\beta^2}>\frac{1}{b^2\beta^2} \tag{A23}$$

which is always satisfied. To show that the denominator is positive, we set it equal to zero and solve with respect to a which gives:

$$a=\frac{(B+b\,\beta)^3}{b^3\,\beta^3\Delta-B^3\delta} \tag{A24}$$

We proceed to show that this value of a is always larger than $(B+\beta b)^2/\Delta\beta^2b^2$, which has the same implications as above. The requirement is:

$$\frac{(B+b\,\beta)^3}{b^3\,\beta^3\Delta-B^3\delta}>\frac{(B+b\,\beta)^2}{b^2\,\beta^2\Delta} \tag{A25}$$

A stronger requirement is:

$$\frac{(B+b\,\beta)^3}{b^3\,\beta^3\Delta}>\frac{(B+b\,\beta)^2}{b^2\,\beta^2\Delta} \tag{A26}$$

which simplifies to

$$\frac{B+b\,\beta}{b\,\beta}>1 \tag{A27}$$

which is always satisfied.

2 Differentiating U in (10) with respect to a and inserting three values of a gives:

$$\left(\frac{\partial U}{\partial a}\right)_{a=0}=\frac{b^2B^2r\,\beta^3(3\,b\,\beta\,\Delta-B(2\delta-\Delta))}{(B+b\,\beta)^5}>0 \text{ when } 3\,b\,\beta\,\Delta>B(2\delta-\Delta) \tag{A28}$$

$$\begin{aligned}\left(\frac{\partial U}{\partial a}\right)_{a=\frac{1}{\Delta}} &= (b^2r\beta^3\Delta(B^6\delta^3+3B^4(B^2+3bB\beta+3b^2\beta^2)\delta^2\Delta \\ &\quad +B^2(B^4+6bB^3\beta+18b^2B^2\beta^2+26b^3B\beta^3+15b^4\beta^4)\delta\Delta^2 \\ &\quad +(B+b\beta)^4(B^2+5bB\beta+7b^2\beta^2)\Delta^3))/((B+b\beta)^2 \\ &\quad \times(3bB\beta\Delta+3b^2\beta^2\Delta+B^2(\delta+\Delta))^3)>0\end{aligned} \tag{A29}$$

$$\left(\frac{\partial U}{\partial a}\right)_{a=\frac{(B+b\,\beta)^2}{b^2\beta^2\Delta}}=\frac{b^2r\,\beta^3\Delta}{(B+b\,\beta)^2}>0 \tag{A30}$$

3 Follows from the proof of Proposition 2 (1) since $u_2 = Bt_2/\beta$, $U_2 = \beta bT_2$.

4 Follows from (13).