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On the Impossibility of Deterrence in Sequential Colonel Blotto Games

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A sequential Colonel Blotto and rent seeking game with fixed and variable resources is analyzed. With fixed resources, which is the assumption in Colonel Blotto games, we show for the common ratio form contest success function that the second mover is never deterred. This stands in contrast to Powell's (*Games and Economic Behavior* **67**(2), 611–615) finding where the second mover can be deterred. With variable resources both players exert efforts in both sequential and simultaneous games, whereas fixed resources cause characteristics of all battlefields or rents to impact efforts for each battlefield. With variable resources only characteristics of a given battlefield impact efforts are to win that battlefield because of independence across battlefields. Fixed resources impact efforts and hence differences in unit effort costs are less important. In contrast, variable resources cause differences in unit effort costs to be important. The societal implication is that resource constrained opponents can be expected to engage in warfare, whereas an advantaged player with no resource constraints can prevent warfare.

Keywords: Blotto; multiple rents; fixed resources; variable resources; rent seeking.

1. Introduction

Colonel Blotto games assume that two opponents allocate fixed resources across multiple battlefields or prizes. Rent seeking for multiple prizes assumes either variable or fixed resources for each opponent. Both Colonel Blotto games and rent seeking are commonly analyzed as simultaneous move games. Since many allocation situations are sequential, e.g., protecting infrastructures against attackers or sequential bid for voters (Groseclose and Snyder, 1996), this paper seeks to understand the different implications. We first position the problem within the literature.

Within the Colonel Blotto literature, with fixed resources, Shubik and Weber (1981) account for complementarities among defended targets and determine cost trade-offs between systems defense and alternative measures. Roberson (2006) describe the equilibrium payoffs to the classic Colonel Blotto game for any number of battlefields, and any level of relative resources, assuming that each player's payoff is the proportion of battlefields to which the player sends a higher level of force. Kvasov (2007) analyzes a first-price simultaneous-move all-pay auction, where the player submitting the highest bid for a given object wins that object. Robson (2005), using a contest success function, analyzes two players' resource allocation across a collection or sequence of different contests, and how interdependencies depend on the contest success function and values of the prizes.

Within the rent seeking literature, the following research assumes budget constraints. Che and Gale (1997) assume that each bidder has finite wealth and show that less intensive (decisive) rent seeking can cause more rent dissipation. Che and Gale (1998, 2006) and Kaplan and Wettstein (2006) show that exogenous caps in all-pay auctions reduces a high valuation bidder's winning chances. Caps may also increase aggregate contributions and lower total surplus. With fixed resources, Snyder (1989) analyzes simultaneous contests for legislative seats based on campaign strategies in multiple districts. Two parties maximize either the expected number of legislative seats, or the probability of winning a majority of the seats, which leads to qualitatively different behavior.

With variable resources, the following research assumes a contest success function. Clark and Konrad (2007a) consider a model where two players exert efforts in several dimensions. The player that wins a certain number of these dimensions is awarded a prize. Clark and Konrad (2007b) analyze a defender who needs to successfully defend all fronts, and an attacker who needs to win at only one front. They show that even with defender advantage on each front, the defender's payoff is zero if the number of fronts is large. Clark and Konrad (2008) investigate how multiple simultaneous R&D contests depend on whether firms already hold relevant patents and the availability of an option to invent around. Klumpp and Polborn (2006) analyze campaign spending in sequential and simultaneous elections in single state to determine candidates for US presidential elections. In an all-pay auction assuming simultaneous and sequential distribution, Clark and Riis (1998) consider competition for multiple identical rents, where each player can only win one rent.

Assuming a contest success function and fixed resources it is easily shown that the sequence of moves does not affect the agents' choices and utilities, and no agent withdraws (i.e., exerts no effort). With variable resources we show that the second mover can be deterred. With variable resources it is easily shown that the first moving defender always prefers the sequential game, and the second moving attacker prefers the sequential game when he has a lower unit effort cost than the defender. When one agent has moved in the sequential game, the game is predictable for the second mover, in contrast to an uncertain simultaneous move game where no agent knows the other agent's effort.

One rare exception to the assumption of simultaneous moves is Powell (2009). He analyzes a sequential Colonel Blotto game, where the defender moves first and the attacker moves second. He assumes a contest success function different from the one in the current paper, and thus gets different results. He assumes that the probability that an attack on a given site succeeds depends only on the defensive resource

allocated to that site, and not on the amount of resources the attacker allocates to that site. However, the probability that the attacker attacks a given site depends on the defender's resource allocation. The defender's loss and attacker's gain depend on multiplying these two probabilities, multiplying with a loss or gain value for each site, and summing over all sites. The analysis shows that the defender min-maxes the attacker causing a pure-strategy subgame perfect equilibrium where the defender defends all sites, the attacker attacks all sites that are not well defended, and refrains from attacking sites that are well protected.

In contrast, in the current paper we use the common ratio form contest success function (Tullock 1980) to show that the attacker can never be deterred when both players have fixed resources which is the common assumption for Colonel Blotto game. However, with variable resources, the attacker can be deterred. With fixed resources linkages between battlefields or rents emerge, whereas with variable resources each battlefield is analyzed in isolation.[a]

Section 2 considers the model with fixed resources, and Sec. 3 with variable resources. Section 4 compares the results. Section 5 concludes.

2. A Model with fixed resources

2.1. *The model*

The defender has a resource r which is allocated into defense efforts $s_i \geq 0$ at unit costs $a_i > 0$ across n sites valued at $V_i > 0,\ i = 1, \ldots, n$. Analogously, the attacker has a resource R which is transformed into attack efforts $S_i \geq 0$ at unit costs $A_i > 0$ across the n sites, i.e.,

$$\sum_{i=1}^{n} a_i s_i = r, \quad \sum_{i=1}^{n} A_i S_i = R. \tag{1}$$

Using the conventional ratio form contest success function (Tullock, 1980), the defender defends site i successfully with probability

$$f_i = \begin{cases} 1/2 & \text{if } s_i = S_i = 0 \\ \dfrac{s_i}{s_i + S_i} & \text{otherwise} \end{cases}, \tag{2}$$

where $\partial f_i / \partial s_i > 0$, $\partial f_i / \partial S_i < 0$. The attacker attacks site i successfully with the remaining fraction $1 - f_i$. The agents' utilities are

$$u = \sum_{i=1}^{n} f_i V_i - r, \quad U = \sum_{i=1}^{n} (1 - f_i) V_i - R, \tag{3}$$

[a]Fixed resources can be imposed by legislation, a social planner, by agreement between the agents, and in principle also by each agent. President Barack Obama was the first to decline public funding for his 2008 campaign, though his motivation was partly to avoid the associated spending limits which candidates have become masters of circumventing, http://www.nytimes.com/2008/06/20/us/politics/20obamacnd.html, retrieved 10 December 2011.

where f_i is defined in (2). We subtract r and R to compare with the game with variable resources. We consider a two period game with complete and perfect information. In period 1 the defender chooses s_i simultaneously for all n sites to maximize his utility. In period 2 the attacker chooses S_i simultaneously for all n sites, taking as given the defender's choices s_i in period 1. The two agents' $2n-2$ free choice variables are $s_1, \ldots, s_{n-1}$ and $S_1, \ldots, S_{n-1}$ where s_n and S_n follow from (1).

2.2. *Solving the model*

Applying backward induction to determine subgame perfect Nash equilibrium, Appendix A implies

$$S_i = \frac{R}{A_i} \frac{\frac{A_i/R}{a_i/r} V_i \Big/ \left(\frac{A_i/R}{a_i/r}+1\right)^2}{\sum_{i=1}^{n} \left(\frac{A_i/R}{a_i/r} V_i \Big/ \left(\frac{A_i/R}{a_i/r}+1\right)^2 \right)}, \quad s_i = \frac{A_i/R}{a_i/r} S_i,$$
$$u = \sum_{i=1}^{n} \frac{\frac{A_i/R}{a_i/r} V_i}{\frac{A_i/R}{a_i/r}+1} - r, \quad U = \sum_{i=1}^{n} \frac{V_i}{\frac{A_i/R}{a_i/r}+1} - R. \tag{4}$$

Property 1. (a) The defender defends all sites and the attacker attacks all sites. (b) $\partial u/\partial a_i < 0$, $\partial^2 u/\partial a_i^2 > 0$, $\partial u/\partial A_i > 0$, $\partial^2 u/\partial A_i^2 < 0$, $\partial u/\partial V_i > 0$, $\partial^2 u/\partial V_i^2 = 0$, $\partial u/\partial r > 0$ when $\sum_{i=1}^{n} \frac{\frac{A_i/R}{a_i/r} V_i}{\left(\frac{A_i/R}{a_i/r}+1\right)^2} > r$, $\partial^2 u/\partial r^2 < 0$, $\partial u/\partial R < 0$, $\partial^2 u/\partial R^2 > 0$, $\partial U/\partial A_i < 0$, $\partial^2 U/\partial A_i^2 > 0$, $\partial U/\partial a_i > 0$, $\partial^2 U/\partial a_i^2 < 0$, $\partial U/\partial V_i > 0$, $\partial^2 U/\partial V_i^2 = 0$, $\partial U/\partial R > 0$ when $\sum_{i=1}^{n} \frac{\frac{A_i/R}{a_i/r} V_i}{\left(\frac{A_i/R}{a_i/r}+1\right)^2} > R$, $\partial^2 U/\partial R^2 < 0$, $\partial U/\partial r < 0$, $\partial^2 U/\partial r^2 > 0$.

Proof. Follows from differentiating (4). □

3. A Model with variable resources

3.1. *The model*

We consider the same game as in Sec. 2.1 except that the agents have variable resources. Thus r and R do not apply and the agents have expenditures $a_i s_i$ and $A_i S_i$ for each site. We thus replace (3) with

$$u = \sum_{i=1}^{n} (f_i V_i - a_i s_i), \quad U = \sum_{i=1}^{n} ((1-f_i) V_i - A_i S_i). \tag{5}$$

3.2. *Solving the model*

Applying backward induction to determine subgame perfect Nash equilibrium, Appendix B implies

$$S_i = \begin{cases} \dfrac{(2a_i - A_i)V_i}{4a_i^2} & \text{if } A_i \le 2a_i \\ 0 & \text{if } A_i \ge 2a_i \end{cases}, \quad s_i = \begin{cases} \dfrac{A_i V_i}{4a_i^2} & \text{if } A_i \le 2a_i \\ \dfrac{V_i}{A_i} & \text{if } A_i \ge 2a_i \end{cases},$$

$$u = \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{A_i V_i}{4a_i} & \text{if } A_i \le 2a_i \ \forall\, i \\ \displaystyle\sum_{i=1}^{n} \left(1 - \frac{a_i}{A_i}\right) V_i & \text{if } A_i \ge 2a_i \ \forall\, i \end{cases}, \tag{6}$$

$$U = \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{(2a_i - A_i)^2 V_i}{4a_i^2} & \text{if } A_i \le 2a_i \ \forall\, i \\ 0 & \text{if } A_i \ge 2a_i \ \forall\, i \end{cases}.$$

Property 2. (a) When $A_i \le 2a_i \ \forall i$, the attacker is not deterred and $\partial u/\partial a_i < 0$, $\partial^2 u/\partial a_i^2 > 0$, $\partial u/\partial A_i > 0$, $\partial^2 u/\partial A_i^2 = 0$, $\partial u/\partial V_i > 0$, $\partial^2 u/\partial V_i^2 = 0$, $\partial U/\partial A_i < 0$, $\partial^2 U/\partial A_i^2 > 0$, $\partial U/\partial a_i > 0$, $\partial^2 U/\partial a_i^2 > 0$ when $4a_i/3 < A_i < 2a_i$, $\partial^2 U/\partial a_i^2 < 0$ when $0 < A_i < 4a_i/3$, $\partial U/\partial V_i > 0$, $\partial^2 U/\partial V_i^2 = 0$. (b) When $A_i \ge 2a_i \ \forall i$, the attacker is deterred and $\partial u/\partial a_i < 0$, $\partial^2 u/\partial a_i^2 = 0$, $\partial u/\partial A_i > 0$, $\partial^2 u/\partial A_i^2 < 0$, $\partial u/\partial V_i > 0$, $\partial^2 u/\partial V_i^2 = 0$, $U = 0$.

Proof. Follows from differentiating (6). □

If $A_i \le 2a_i$ for $i = 1, \ldots, j$, which does not deter the attacker from these j sites, and $A_i \ge 2a_i$ for $i = j+1, \ldots, n$, which deters the attacker from the remaining $n - j$ sites, then the utilities are

$$u = \sum_{i=1}^{j} \frac{A_i V_i}{4a_i} + \sum_{i=j+1}^{n} \left(1 - \frac{a_i}{A_i}\right) V_i, \quad U = \sum_{i=1}^{j} \frac{(2a_i - A_i)^2 V_i}{4a_i^2}, \quad 0 \le j \le n. \tag{7}$$

4. Comparing properties 1 and 2

Property 1 states that an agent's utility decreases convexly in his own unit effort cost and in the other agent's resource, increases concavely in the other agent's unit effort cost, increases linearly in the site valuation, and is inverse U formed in his own resource. For the latter result, an agent with a small resource benefits from a larger resource, but as the resource exceeds the specified level, the budget constraint is no longer binding and the agent prefers the model without budget constraints.

Whereas Property 1 specifies dependence on five parameters for each agent and site (two unit effort costs, the site valuation, and the two budget constraints), Property 2 specifies dependence on three parameters for each agent and site (two unit effort costs, the site valuation, and there are no budget constraints). Property 1 retains the symmetry between the first and second mover, whereas Property 2 does not. We first consider $A_i \leq 2a_i$, which does not deter the second mover. Property 2 states that an agent's utility decreases convexly in his own unit effort cost, just as in Property 1. However, the first mover's (the defender) utility increases linearly in the second mover's unit effort cost, in contrast to increasing concavely in Property 1. The second mover's (the attacker's) utility increases concavely in the first mover's unit effort cost when $0 < A_i < 4a_i/3$, just as in Property 1, but increases convexly in the first mover's unit effort cost when $4a_i/3 < A_i < 2a_i$. This latter result means that if the second mover is disadvantaged with a large unit effort cost, below that of being deterred, then it is especially beneficial for the second mover that the first mover's unit effort cost increases. No such convex increase is present in Property 1. Both agents' utilities increase linearly in the site valuation, just as in Property 1. Second, when $A_i \geq 2a_i$, which deters the second mover, the first mover's utility decreases linearly in his own unit effort cost, increases concavely in the second mover's unit effort cost, and increases linearly in the site valuation. The deterred second mover earns no utility. Appendix C shows further distinguishing factors.

5. Conclusion

The paper analyzes a sequential Colonel Blotto and rent seeking game with fixed and variable resources relevant when a defender allocates resources across several differently valued battlefields or rents before an attacker attacks. With fixed resources, which is the assumption in Colonel Blotto games, we show for the common ratio form contest success function that the second mover cannot be deterred. The agents' choices and utilities are the same in sequential and simultaneous games. This stands in contrast to Powell's (2009) finding where the second mover can be deterred. With variable resources we show that the second mover is deterred (exerts no effort) when disadvantaged with a unit effort cost more than twice that of the first mover. In the simultaneous game no agent withdraws. The societal implication is that resource constrained opponents can be expected to engage in warfare, whereas an advantaged player with no resource constraints can prevent warfare.

Fixed resources cause characteristics of all battlefields or rents to impact efforts for each battlefield where agents substitute efforts across battlefields, with variable resources only characteristics of a given battlefield impact efforts to win that battlefield because of independence across battlefields. Fixed resources impact efforts and hence differences in unit effort costs are less important. In contrast, variable resources cause differences in unit effort costs to be important.

With fixed resources, agents earn maximum utilities for an intermediate value of their own resource, since wasting a too large resource is costly, and a too small

resource does not win a battlefield. Although an agent with a small resource prefers a larger resource, as the resource exceeds the specified level, the fixed resource which then becomes a budget constraint is no longer binding and the agent prefers the model with variable resources. With variable resources, the optimization logic prevents negative utilities and ensures intermediate optimal efforts.

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Appendix A. Solving the game in Sec. 2

In order to differentiate with respect to S_i, $i = 1, \ldots, n-1$, we write (3) as

$$U = \frac{S_i V_i}{s_i + S_i} + \sum_{\substack{j=1,\\ j\neq i}}^{n-1} \frac{S_j V_j}{s_j + S_j} + \frac{\left(\frac{R - A_i S_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j}{A_n}\right) V_n}{\frac{r - a_i s_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j}{a_n} + \frac{R - A_i S_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j}{A_n}} - R, \tag{A.1}$$

where the agents' efforts s_n and S_n for n sites are expressed as functions of their efforts for sites $1, 2, \ldots, n-1$. The attacker's first-order condition for S_i for site i is

$$\frac{\partial U}{\partial S_i} = \frac{S_i V_i}{(s_i + S_i)^2} - \frac{\left(\frac{r - a_i s_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j}{a_n}\right)\left(\frac{R - A_i S_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j}{A_n}\right) V_n}{\frac{R - A_i S_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j}{A_i}\left(\left(\frac{r - a_i s_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j}{a_n}\right) + \left(\frac{R - A_i S_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j}{A_n}\right)\right)^2} = 0 \tag{A.2}$$

which is solved to yield

$$S_i = \frac{\sqrt{A_i a_n A_n s_i \left(r - a_i s_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right) V_i V_n}}{A_i a_n \left[A_n V_n \left(r - a_i s_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right) - A_i a_n s_i V_i\right]} \times \left[A_n \left(r - a_i s_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right) + a_n \left(R + A_i s_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j\right)\right]$$

$$-\frac{A_i a_n s_i\left[a_n\left(R-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j\right)V_i + A_n\left(r-a_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right)(V_i+V_n)\right]}{A_i a_n\left[A_n V_n\left(r-a_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right)-A_i a_n s_i V_i\right]} \tag{A.3}$$

The attacker's second-order condition for S_i is

$$\frac{\partial^2 U}{\partial S_i^2}=-\frac{2s_i V_i}{(s_i+S_i)^3} - \frac{2A_i^2 a_n\left(\frac{r-a_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j}{a_n}\right)V_n}{A_n\left(\left(\frac{r-a_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j}{a_n}\right)+\left(\frac{R-A_i S_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j}{A_n}\right)\right)^3}<0 \tag{A.4}$$

which is always satisfied. Inserting (A.3) into (3) gives (after some tedious simplifications) the defender's first period utility

$$u=\sum_{\substack{j=1,\\ j\neq i}}^{n-1}\frac{s_j V_j}{s_j+S_j} + \frac{\left(\sqrt{A_i a_n s_i V_i}+\sqrt{A_n V_n\left(r-a_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right)}\right)^2}{A_n\left(r-a_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right)+a_n\left(R+A_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j\right)}-r. \tag{A.5}$$

Differentiating u with respect to s_i for site i, and equating with 0, gives

$$\frac{\partial u}{\partial s_i}=\frac{\sqrt{A_i a_n s_i V_i}+\sqrt{A_n V_n\left(r-a_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right)}}{A_n\left(r-a_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right)+a_n\left(R+A_i s_i-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j\right)} \times\left(\frac{\sqrt{A_i a_n V_i}}{\sqrt{s_i}}\left[A_n\left(r-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j\right)+a_n\left(R-\sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j\right)\right]\right.$$

$$- \frac{A_n a_n V_n \left[A_i \left(r - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j \right) + a_i \left(R - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j \right) \right]}{\sqrt{A_n V_n \left(r - a_i s_i - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j \right)}} \Bigg) = 0 \tag{A.6}$$

which is solved to yield

$$s_i = \left(r - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j \right) A_i V_i \left(A_n \left(r - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j \right) + a_n \left(R - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j \right) \right)^2 \Big/ d,$$

$$d = a_i A_i V_i \left(A_n \left(r - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j \right) + a_n \left(R - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j \right) \right)^2$$

$$+ A_n a_n V_n \left(A_i \left(r - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j \right) + a_i \left(R - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j \right) \right)^2 \tag{A.7}$$

Solving (A.7) and (A.3) gives

$$S_i = \frac{a_i \left(R - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} A_j S_j \right)}{A_i \left(r - \sum_{\substack{j=1,\\ j\neq i}}^{n-1} a_j s_j \right)} s_i = \frac{a_i (A_i S_i + A_n S_n)}{A_i (a_i s_i + a_n s_n)} s_i \Rightarrow \frac{a_i s_i}{A_i S_i} = \frac{a_n s_n}{A_n S_n}, \tag{A.8}$$

where (1) as equalities is used for the second equality. Equation (A.8) applies for $i = 1, \ldots, n$ and is rewritten as

$$\frac{a_1 s_1}{A_1 S_1} = \frac{a_2 s_2}{A_2 S_2} = \cdots = \frac{a_n s_n}{A_n S_n} \Rightarrow a_j s_j = \frac{a_i s_i}{A_i S_i} A_j S_j, \quad i, j = 1, \ldots, n. \tag{A.9}$$

Equation (A.9) expresses how the agents' efforts for site j depend on each other, and on s_i and S_i. Inserting (A.9) into (1) gives

$$s_i = \frac{A_i / R}{a_i / r} S_i. \tag{A.10}$$

Inserting (A.9) and (A.10) into (A.7) gives

$$S_1 = \frac{a_1 (A_n r + a_n R)^2 V_1}{a_n (A_1 r + a_1 R)^2 V_n} S_n \Rightarrow S_i = \frac{a_i (A_n r + a_n R)^2 V_i}{a_n (A_i r + a_i R)^2 V_n} S_n. \tag{A.11}$$

Inserting (A.11) into (1) gives

$$S_n = \frac{R}{A_n} \frac{\frac{A_n / R}{a_n / r} V_n \Big/ \left(\frac{A_n / R}{a_n / r} + 1 \right)^2}{\sum_{i=1}^{n} \left(\frac{A_i / R}{a_i / r} V_i \Big/ \left(\frac{A_i / R}{a_i / r} + 1 \right)^2 \right)} \tag{A.12}$$

which generalizes to (4). Inserting (A.10) into (3) gives the utilities in (4). It can be shown that the defender's second-order condition for s_i for site i is satisfied.

Appendix B. Solving the game in Sec. 3

Starting with the second period, the attacker's first-order condition for S_i implies

$$\frac{\partial U}{\partial S_i} = \frac{s_i V_i}{(s_i + S_i)^2} - A_i = 0 \Rightarrow S_i = \sqrt{\frac{s_i V_i}{A_i}} - s_i \tag{B.1}$$

which is inserted into (5) to yield

$$u = \sum_{i=1}^{n} \Big(\sqrt{s_i V_i A_i} - a_i s_i \Big). \tag{B.2}$$

Differentiating u with respect to s_i gives

$$\frac{\partial u}{\partial s_i} = \frac{\sqrt{A_i V_i}}{2\sqrt{s_i}} - a_i = 0 \Rightarrow s_i = \frac{A_i V_i}{4a_i^2} \tag{B.3}$$

which is inserted into (B.1) to yield S_i in (6). Inserting the efforts in (6) into (5) gives the utilities in (6). The order in which the two agents compete for the n rents is irrelevant. The attacker is deterred if $A_i \geq 2a_i$. To ensure the deterrence, it suffices for the defender to choose s_i so that the attacker earns negative utility for rent i. Using (5), this gives $\frac{S_i V_i}{s_i + S_i} - A_i S_i \leq 0$, and hence $s_i = V_i/A_i$. The agents' second-order conditions are

$$\frac{\partial^2 U}{\partial S_i^2} = -\frac{2 s_i V_i}{(s_i + S_i)^3} < 0, \quad \frac{\partial^2 u}{\partial s_i^2} = -\frac{\sqrt{A_i V_i}}{4 s_i^{3/2}} < 0 \tag{B.4}$$

which are always satisfied.

Appendix C. Five distinguishing factors for the two models

First, with fixed resources, the agents substitute efforts across the sites and hence the efforts in (4) depend on all the model's parameters. With variable resources, each site is viewed independently, and hence the efforts in (6) depend on only the parameters for that site. Second, with fixed resources, the agents' efforts in (4) are proportional to r/a_i and R/A_i, respectively, multiplied with a factor between 0 and 1, i.e., influenced by his resource divided by his unit cost. With variable resources, the defender's effort in (6) is proportional to $A_i V_i / a_i^2$, quadratically influenced by his unit cost, which reinforces the impact of differences in unit costs, whereas the

attacker's effort in (6) is proportional to V_i and decreases to 0 as his unit cost A_i increases toward $2a_i$. Unit costs are thus especially important with variable resources.

Third, inserting $a_i = A_i$ into (4) and (6) gives

$$s_{iB} = \frac{r}{R} S_{iB}, S_{iB} = \frac{R}{A_i} \frac{\frac{r}{R} V_i \Big/ \left(\frac{r}{R} + 1\right)^2}{\sum_{i=1}^{n} \left(\frac{r}{R} V_i \Big/ \left(\frac{r}{R} + 1\right)^2\right)},$$

$$u_B = \frac{\frac{r}{R}}{\frac{r}{R} + 1} \sum_{i=1}^{n} V_i - r, \quad U_B = \frac{1}{\frac{r}{R} + 1} \sum_{i=1}^{n} V_i - R,$$

$$\frac{\partial u_B}{\partial r} = \frac{R \sum_{i=1}^{n} V_i}{(r+R)^2} - 1 = 0 \Rightarrow r = \sqrt{R \sum_{i=1}^{n} V_i} - R, \quad \frac{\partial^2 u_B}{\partial r^2} = -\frac{2R \sum_{i=1}^{n} V_i}{(r+R)^3} < 0,$$

$$\frac{\partial U_B}{\partial R} = \frac{r \sum_{i=1}^{n} V_i}{(r+R)^2} - 1 = 0 \Rightarrow R = \sqrt{r \sum_{i=1}^{n} V_i} - r, \quad \frac{\partial^2 U_B}{\partial R^2} = -\frac{2r \sum_{i=1}^{n} V_i}{(r+R)^3} < 0,$$

$$s_{iW} = S_{iW} = \frac{V_i}{4A_i}, \quad u_W = U_W = \sum_{i=1}^{n} \frac{V_i}{4} \tag{C.1}$$

where subscript B denotes fixed resources and subscript W denotes no budget constraints.

Hence with fixed resources, the agents earn maximum utilities for an intermediate value of their own resource. If an agent's budget resource is large, and since he is required to use his entire budget, he earns negative utility because of the budget cost. If an agent's budget is low, he earns utility of low absolute magnitude as determined by (C.1). With variable resources, the agents always earn positive utilities since the optimization logic prevents negative utilities. The agents are prevented from incurring large costly efforts, and the logic of the ratio form contest success function ensures that they incur positive efforts since incurring no effort guarantees zero utility.

Fifth, comparing (4) and (6), the agents prefer fixed resources when

$$u_B > u_W \Rightarrow r < \begin{cases} \sum_{i=1}^{n} \left(\frac{\frac{A_i/R}{a_i/r}}{\frac{A_i/R}{a_i/r} + 1} - \frac{A_i/a_i}{4} \right) V_i & \text{if } \frac{A_i}{a_i} \leq 2 \,\forall\, i \\ \sum_{i=1}^{n} \left(\frac{\frac{A_i/R}{a_i/r}}{\frac{A_i/R}{a_i/r} + 1} - \left(1 - \frac{a_i}{A_i}\right) \right) V_i & \text{if } \frac{A_i}{a_i} \geq 2 \,\forall\, i \end{cases}$$

$$U_B > U_W \Rightarrow R < \begin{cases} \sum_{i=1}^{n} \left(\frac{1}{\frac{A_i/R}{a_i/r} + 1} - \frac{(2 - A_i/a_i)^2}{4} \right) V_i & \text{if } \frac{A_i}{a_i} \leq 2 \,\forall\, i \\ \sum_{i=1}^{n} \frac{V_i}{\frac{A_i/R}{a_i/r} + 1} & \text{if } \frac{A_i}{a_i} \geq 2 \,\forall i \end{cases} \tag{C.2}$$

which are satisfied when the agents' resources are not too large which is costly. Equation (C.2) specifies how a social planner can dictate upper bounds for the agents' resources which benefit one or both of them, and prevent resource waste which may occur when each agent maximizes utility individually without resource constraints.

Property 3. The agents collectively prefer fixed resources when the site values V_i are large, when there are many sites n, and when the resources r and R are low, as expressed in (C.2).

Proof. Follows from (C.2). □

Collectively agents prefer fixed resources when V_i and n are large, and r and R are low, to limit their expenses. Agents' desire to win many (large n) valuable (large V_i) sites drives a second desire for a large resource (large r and R) which in turn renders the budget constraints not binding, causing large expenses.

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