Generating sets for Beurling algebras

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Abstract

We characterize in terms of Beurling–Malliavin density, the generating sets for Beurling algebras $L^1_w(\mathbb{R})$, that is the sets $\Lambda \subset \mathbb{R}$ for which a function $\varphi \in L^1_w(\mathbb{R})$ exists such that the Λ-translates $\{\varphi(x-\lambda)\}$, $\lambda \in \Lambda$, span $L^1_w(\mathbb{R})$. Our main result extends a recent theorem from [J. Bruna, A. Olevskii, A. Ulanovskii, Completeness in $L^1(R)$ of discrete translates, arXiv:math.CA/0307323v1, 2003, (Revista Mathematica Iberoamericana), submitted for publication.], which describes the generating sets for $L^1(\mathbb{R})$.

Keywords: Generating set; generating function; Beurling algebra; completeness of translates

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1. Introduction and statement of the results

Let B be a Banach space of complex functions on the real line $\mathbb{R}$. A function $\varphi(x) \in B$ is called a generator for B if $\varphi(x-t) \in B$ for every $t \in \mathbb{R}$ and the set of all translates $\{\varphi(x-t)\}_{t\in\mathbb{R}}$ spans B, i.e. the set of all finite linear combinations $\sum c_j \varphi(x-t_j), c_j \in \mathbb{C}, t_j \in \mathbb{R}$, is dense in B. The space B is called translation-invariant if $f(x-t) \in B$ for every real t, provided $f(x) \in B$.

Two classical results give description of generators in the spaces $L^1 = L^1(\mathbb{R})$ and $L^2 = L^2(\mathbb{R})$. The Wiener Tauberian theorem asserts that a function φ is a generator in L^1 if and only if its Fourier transform $\hat{\varphi}$ does not vanish. Another theorem of Wiener states that φ is a generator in L^2 if and only if the measure of the zero set of $\hat{\varphi}$ is zero. No description is known for the spaces L^p, $p \neq 1, 2$.

Let w be a measurable function on $\mathbb{R}$, and set

$$L^1_w = \left\{ f : \|f\|_w = \int_{\mathbb{R}} |f(t)| e^{w(t)}\, dt < \infty \right\}.$$

Then L^1_w is a Banach space. We shall assume that w is non-negative and

$$w(x+t) \leqslant w(x) + w(t), \quad s, t \in \mathbb{R}, \quad w(tx) \geqslant w(x) \text{ for all } x \text{ and } t \geqslant 1. \tag{1}$$

Then L^1_w is a (translation-invariant) commutative Banach algebra with respect to convolution multiplication defined by the equation

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s) g(s)\, ds, \quad f, g \in L^1_w.$$

These algebras were introduced by A. Beurling in 1938 [2].

The algebra L^1_w is called non-quasianalytic if w satisfies

$$\int_{\mathbb{R}} \frac{w(t)}{1+t^2}\, dt < \infty. \tag{2}$$

It was established in [2] that the Wiener Tauberian theorem admits extension to non-quasianalytic Beurling algebras L^1_w: suppose a weight w satisfies (1) and (2). Then a function $\varphi \in L^1_w$ is a generator in L^1_w if and only if its Fourier transform $\hat{\varphi}$ does not vanish. A modern proof of this result is presented in [8] (see also [7] for a proof based on complex analysis). On the other hand, in general, the Wiener Tauberian theorem cannot be extended to L^1_w if condition (2) does not hold (see e.g. [4] and the references therein). We refer the reader to [6] for a history of results on different extensions of the Wiener Tauberian theorem.

Let us say that a set $\Lambda \subseteq \mathbb{R}$ is generating for a Banach space B if there is a function $\varphi(x) \in B$ such that $\varphi(x-\lambda) \in B$ for every $\lambda \in \Lambda$ and the set of all Λ-translates $\{\varphi(x-\lambda)\}_{\lambda \in \Lambda}$ spans B. The function φ is called a Λ-generator for B. Recently, there have been a number of papers studying generating sets and related problems for the spaces L^p (see e.g. [1,11–13,5] and the literature therein). A full description of generating sets for the space L^1 was given in a recent paper [5]. To formulate this result, we denote by E_Λ the exponential system $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$, and by $R(\Lambda)$ its completeness radius:

$$R(\Lambda) := \sup\{r > 0 : E_\Lambda \text{ is complete in } L^2(-r, r)\},$$

where one sets $R(\Lambda) = 0$ if E_Λ is not complete in $L^2(-r, r)$ for any $r > 0$.

Theorem 1 (*Bruna et al. [5]*). *A set $\Lambda \subseteq \mathbb{R}$ is generating for L^1 if and only if $R(\Lambda) = \infty$.*

The aim of this note is to extend this result to Beurling algebras L^1_w.

We start with an inclusion result for the generating sets.

Theorem 2. *Suppose $\Lambda \subseteq \mathbb{R}$.*

(i) *Suppose $1 \leqslant p < q$. If Λ is generating for L^p, then it is generating for L^q.*

(ii) *Suppose w and ω are any measurable functions such that $\omega(x) > c + w(x)$, $x \in \mathbb{R}$, where c is a constant. If Λ is generating for L^1_ω, then it is generating for L^1_w.*

An immediate corollary of this result is that (i) if Λ is generating for L^1, then it is generating for L^p, for every $p > 1$; (ii) if Λ is generating for L^1_w, where $w \geqslant 0$, then it is generating for L^1. Hence, if a weight w is non-negative, by Theorem 1, the assumption $R(\Lambda) = \infty$ is necessary for a set Λ to be generating in L^1_w. It turns out that this assumption remains sufficient for the weights w satisfying (1) and (2). Thus, similarly to the Wiener Tauberian theorem, Theorem 1 admits extension to Beurling algebras:

Theorem 3. *Suppose w is a non-negative function satisfying* (1) *and* (2). *A set $\Lambda \subseteq \mathbb{R}$ is generating for L^1_w if and only if $R(\Lambda) = \infty$.*

If the weight w is no longer non-quasianalytic (i.e. the integral (2) diverges), we conjecture that the assumption $R(\Lambda) = \infty$ is not sufficient for a set Λ to be generating for L^1_w.

Observe that condition $R(\Lambda) = \infty$ has a clear geometric meaning. In the beginning of the 1960s Beurling and Malliavin established that the completeness radius of an exponential system can be expressed in terms of a certain density: $R(\Lambda) = \pi D(\Lambda)$, where D is called Beurling–Malliavin exterior density (for definition and basic properties of D see [9]). It is easy to check that condition $R(\Lambda) = \infty$ is equivalent to the condition that there exists a family of disjoint intervals (a_k, b_k), $k \in \mathbb{N}$, $b_k - a_k \to \infty$, $k \to \infty$, with the properties that

$$\frac{\#\,(\Lambda \cap (a_k, b_k))}{b_k - a_k} \to \infty, \quad k \to \infty, \qquad \sum_{k\in\mathbb{N}} \left(\frac{b_k - a_k}{a_k}\right)^2 = \infty.$$

Here # means the number of elements.

The rest of the note is organized as follows. First we prove Theorem 3, and then we prove some auxiliary results used in the proof of Theorem 3. Theorem 2 is proved in the last section.

2. Proof of Theorem 3

(i) Necessity of $R(\Lambda) = \infty$. Suppose that Λ is generating for L^1_w. Then, by Theorem 2(ii), Λ is generating for L^1, and so, by Theorem 1, $R(\Lambda) = \infty$.

(ii) Sufficiency of $R(\Lambda) = \infty$. By Theorem 2(ii), if Λ is generating for some weighted space L^1_ω, where $\omega(x) \geqslant w(x)$, $x \in \mathbb{R}$, then it is generating for L^1_w. Hence, without loss of generality we may assume that w is smooth, even and 'large':

$$w \in C^2(\mathbb{R}), \quad w(-x) = w(x), \qquad x \in \mathbb{R}, \quad \int_{\mathbb{R}} e^{-w(x)}\, dx < \infty. \tag{3}$$

The proof is based on two fundamental theorems of Harmonic analysis: the extension of Wiener Tauberian theorem to Beurling's algebras [2], which is used in the proof of Lemma 4, and the Beurling–Malliavin multiplier theorem [3], used in the proofs of Lemmas 5 and 6.

Denote by $\check{\varphi}(x) := \varphi(-x)$, and by L^∞_w the space of all functions f satisfying $f(x) \leqslant c e^{w(x)}$ for almost all x and some $c > 0$.

A set Λ is called a uniqueness set for a class of functions if no non-trivial function of this class vanishes on Λ. It follows from (1) that $w(x) \leqslant w(x - \lambda) + w(\lambda)$, and so the convolution $\check{\varphi} * f$ exists for every $\varphi \in L^1_w$ and $f \in L^\infty_w$. We shall denote by $\check{\varphi} * L^\infty_w$ the set of all functions $\check{\varphi} * f$, $f \in L^\infty_w$.

Lemma 4. *A function $\varphi \in L^1_w$ is a Λ-generator for L^1_w if and only if $\hat{\varphi}$ does not vanish and Λ is a uniqueness set for the class $\check{\varphi} * L^\infty_w$.*

Proof. By duality, φ is a Λ-generator for L^1_w if and only if there is no non-trivial function $f \in L^\infty_w$ which is orthogonal to all translates $\varphi(x-\lambda)$:

$$(f * \check{\varphi})(\lambda) = \int_{\mathbb{R}} f(x)\varphi(x-\lambda)\,dx = 0 \quad \text{for every } \lambda \in \Lambda.$$

Suppose a function $\varphi \in L_w$ is a Λ-generator for L^1_w. Then for every non-trivial function $f \in L^\infty_w$, the convolution $\check{\varphi} * f$ cannot vanish on Λ, i.e. Λ is a uniqueness set for the class $\check{\varphi} * L^\infty_w$. Moreover, since $\varphi \in L^1_w$ is a generator for L^1_w, by the extension of Wiener Tauberian theorem to Beurling algebras, $\hat{\varphi}$ does not vanish.

Conversely, suppose $\hat{\varphi}$ does not vanish and that Λ is a uniqueness set for $\check{\varphi} * L^\infty_w$. Suppose a function $f \in L^\infty_w$ is such that $(\check{\varphi} * f)(\lambda) = 0$ for all $\lambda \in \Lambda$. Then, $\check{\varphi} * f = 0$ a.e. Now, by the extension of Wiener Tauberian theorem to Beurling algebras, $f = 0$ a.e. Hence, φ is a Λ-generator for $\check{\varphi} * L^\infty_w$, which proves the lemma. □

Let σ be a non-decreasing function defined on $(0, \infty)$. Following [5], we introduce the following classes of entire functions:

$$B(\sigma) := \{f \text{ entire function: } |f(x+iy)| \leqslant C_f e^{|y|\sigma(|y|)},\ x+iy \in \mathbb{C}\},$$

where C_f is a constant depending only on *f*. The following two steps are the main ingredients of the proof of Theorem 1 in [5]:

- For every non-decreasing function $\sigma(y) \nearrow \infty$, $y \to \infty$, there exists a function $\varphi \in L^1$ such that $\hat{\varphi}$ does not vanish and $\check{\varphi} * L^\infty \subseteq B(\sigma)$.
- For every $\Lambda \subset \mathbb{R}$ with $R(\Lambda) = \infty$ there exists a non-decreasing function $\sigma(y) \nearrow \infty$, $y \to \infty$, such that Λ is a uniqueness set for $B(\sigma)$.

It turns out that a similar approach works in the more general case of Beurling algebras. However, our proofs are quite different from the proofs in [5].

Let ω be a positive function, and σ be a non-decreasing function, where both functions are defined on $(0, \infty)$. We now introduce more general classes of entire functions:

$$A(\sigma, \omega) := \{f \text{ entire function: } |f(x+iy)| \leqslant C_f e^{|y|\sigma(|y|)+\omega(x)},\ x+iy \in \mathbb{C}\},$$

where C_f is a constant depending only on *f*.

The following lemmas are analogues of the two steps described above:

Lemma 5. *For every non-negative weight w satisfying* (1), (2) *and* (3), *and every non-decreasing function $\sigma(y) \nearrow \infty$ there exists a function $\varphi \in L^1_w$ such that $\hat{\varphi}$ does not vanish and $\check{\varphi} * L^\infty_w \subseteq A(\sigma, w)$.*

Lemma 6. *For every non-negative weight w satisfying* (1), (2) *and* (3), *and every set $\Lambda \subset \mathbb{R}$ with $R(\Lambda) = \infty$, there exists a non-decreasing function $\sigma(y) \nearrow \infty$ such that Λ is a uniqueness set for $A(\sigma, w)$.*

Lemmas 5 and 6 will be proved in the next section.

We can now complete the proof of Theorem 3. By Lemma 6, for every $\Lambda \subset \mathbb{R}$ satisfying $R(\Lambda) = \infty$, there exists $\sigma(y) \nearrow \infty$ such that Λ is a uniqueness set for $A(\sigma, w)$. By Lemma 5, there exists $\varphi \in L^1_w$ such that $\hat{\varphi}$ does not vanish and $\check{\varphi} * L^\infty_w \subseteq A(\sigma, w)$. Hence, Λ is a uniqueness set for $\check{\varphi} * L^\infty_w$. We conclude, by Lemma 4, that φ is a Λ-generator for L^1_w, so that Λ is a generating set for L^1_w.

Remark. One can easily establish the necessity of $R(\Lambda) = \infty$ without use of Theorem 1. One can show that for every $\varphi \in L^1_w$ such that $\hat{\varphi}$ does not vanish, the set $\check{\varphi} * L^\infty_w$ contains all entire functions f of finite exponential type such that $f \in L^2(\mathbb{R})$. Hence, if Λ is generating for L^1_w, then Λ is a uniqueness set for this class of functions, i.e. the exponential system $E(\Lambda)$ is complete in L^2 on every interval $(-r, r)$. This implies $R(\Lambda) = \infty$.

3. Proof of Lemmas 5 and 6

Proof of Lemma 5. Observe that if two non-decreasing functions satisfy $\sigma_1(y) \leqslant \sigma_2(y)$, $y > 0$, then $A(\sigma_1, w) \subseteq A(\sigma_2, w)$. It follows that it is enough to prove Lemma 5 for slowly increasing functions. So, we may assume that

$$\sigma(2y) \leqslant 2\sigma(y), \quad y > 0, \qquad \sigma(y) = o(\log y), \quad y \to \infty. \tag{4}$$

In what follows, for simplicity, we shall denote by c different positive constants.

Step 1: There exists an entire function h such that $\hat{h}$ is non-negative, and

$$|h(x + iy)| \leqslant e^{|y| - 8w(x)} \qquad \text{for all } x + iy \in \mathbb{C}. \tag{5}$$

We say that a non-negative measurable function W admits multipliers, if for every positive δ there exists an entire function f of exponential type $\leqslant \delta$ such that $|f(x)(1 + W(x))| \leqslant 1$ for all real x. Beurling and Malliavin [3] established, using independent proofs, two such conditions: $\int_{-\infty}^{+\infty} [\log W(x)/(1 + x^2)]\, dx < \infty$ and either (i) W is the restriction to $\mathbb{R}$ of an entire function of exponential type, or (ii) $\log W$ is uniformly Lipschitz over $\mathbb{R}$. Assumption (1) shows that the function $\exp(16w(x))$ is uniformly Lipschitz, so that, by (2), it admits multipliers. In particular, there exists an entire function h_1 of exponential type $\leqslant \frac{1}{4}$ satisfying

$$|h_1(x)| \leqslant \exp(-16w(x)), \qquad x \in \mathbb{R}. \tag{6}$$

Set

$$\psi(x) := x^{-2} \sin^2(x/8), \qquad h_2(x) := h_1(x)\psi(x), \qquad h_3(x) := h_2(x) * \bar{h}_2(-x).$$

Clearly, h_3 is of exponential type $\leqslant 1$, and the Fourier transform of h_3 satisfies $\hat{h}_3(x) = |\hat{h}_2(x)|^2$, so the function $\hat{h}_3$ is non-negative. Recall that, by (1), $-w(x - s) - w(s) \leqslant -w(x)$, and, by (3), $w(-s) = w(s)$. This and (6) give

$$\begin{aligned} |h_3(x)| &\leqslant \int_{\mathbb{R}} |h_2(x - s)\bar{h}_2(-s)|\, ds \\ &\leqslant \int_{\mathbb{R}} e^{-16w(x-s) - 16w(s)} \psi(x - s)\psi(s)\, ds \leqslant c(x) e^{-16w(x)}, \end{aligned}$$

where

$$c(x) := \int_{\mathbb{R}} \psi(x - s)\psi(s)\, ds \to 0, \qquad |x| \to \infty.$$

Clearly, if $\delta > 0$ is small enough, the function $h(x) := \delta h_3(x)$ is of exponential type $\leqslant 1$, satisfies (6) and $\hat{h}$ is non-negative.

It is well-known that if h is an entire function of exponential type $\leqslant 1$ bounded on the real line, then the function $\log|h(x+iy)| - |y|$, $y \neq 0$, is bounded from above by the Poisson integral (see [10, Chapter 5]):

$$\log|h(x+iy)| \leqslant |y| + \frac{|y|}{\pi}\int_{\mathbb{R}} \frac{\log|h(t)|}{(t-x)^2+y^2}\,dt, \quad y \neq 0.$$

Using estimate (6) and the second inequality in (1), we obtain:

$$\begin{aligned}\log|h(x+iy)| &\leqslant |y| - \frac{|y|}{\pi}\int_{\mathbb{R}} \frac{16w(t)}{(t-x)^2+y^2}\,dt = |y| - \frac{16|y|}{\pi}\int_{\mathbb{R}} \frac{w(x+t)}{t^2+y^2}\,dt \\ &\leqslant |y| - \frac{16|y|w(x)}{\pi}\int_0^\infty \frac{1}{t^2+y^2}\,dt = |y| - 8w(x), \quad y \neq 0,\end{aligned}$$

which proves (5).

Step 2: There exists a sequence $\Gamma = \{\gamma_k\}_{k=1}^\infty \subset \mathbb{N}$ and a subsequence $n = n_j \to \infty$ such that

$$\sum_{k=1}^n \frac{1}{\gamma_k} \leqslant \sigma(\gamma_n), \quad n = 1, 2, \ldots, \tag{7}$$

$$\sum_{k=1}^n \frac{1}{\gamma_k} \geqslant \sigma(\gamma_n) - 1, \quad n = n_j. \tag{8}$$

We shall construct Γ as a union of disjoint (integer) intervals:

$$\Gamma = \bigcup_{k=1}^\infty \{m_k, \ldots, m_k + l_k - 1\}.$$

Here $\{m_k\} \subset \mathbb{N}$ is any sequence satisfying

$$\sigma(m_1) \geqslant 2, \quad \sigma(m_{k+1}) \geqslant 2\sum_{j=1}^{m_k} \frac{1}{j}, \quad k = 1, 2, \ldots \tag{9}$$

and the sequence $\{l_k\}$ is uniquely defined by the following procedure: it follows from (9) and (4) that there is a unique integer l_1 such that

$$\sum_{j=0}^k \frac{1}{m_1+j} < \sigma(m_1+k), \quad 0 \leqslant k \leqslant l_1 - 1, \qquad \sum_{j=1}^{l_1} \frac{1}{m_1+j} \geqslant \sigma(m_1+l_1).$$

We set $\gamma_k := m_1 + k - 1$ for $1 \leqslant k \leqslant l_1$, and $n_1 := m_1 + l_1$. Clearly, (7) holds for $1 \leqslant n \leqslant l_1$, and (8) holds for $n = n_1$.

Observe that

$$\sigma(m_1+l_1) \leqslant \sum_{j=1}^{l_1} \frac{1}{m_1+j} \leqslant \sum_{j=1}^{m_1+l_1} \frac{1}{j}.$$

Hence, by (9), $m_2 > m_1 + l_1$. It follows from (4) that there exists $l_2 \geqslant 1$ such

that

$$\sum_{j=0}^{l_1-1} \frac{1}{m_1+j} + \sum_{j=0}^{k} \frac{1}{m_2+j} < \sigma(m_2+k), \quad 0 \leqslant k \leqslant l_2 - 1,$$

$$\sum_{j=0}^{l_1-1} \frac{1}{m_1+j} + \sum_{j=0}^{l_2} \frac{1}{m_2+j} \geqslant \sigma(m_2+l_2).$$

Then, we set $\gamma_k := m_2 + k - l_1 - 1$ for $l_1 + 1 \leqslant k \leqslant l_1 + l_2$, and $n_2 := m_2 + l_2$. We see that (7) holds for $l_1 + 1 \leqslant n \leqslant l_1 + l_2$, and (8) holds for $n = n_2$, and so on.

Step 3: Set

$$\varphi(z) := h(z) \prod_{k=1}^{\infty} \frac{\sin(z/8\gamma_k)}{z/(8\gamma_k)}, \quad z \in \mathbb{C}, \tag{10}$$

where h and γ_k have been defined in Steps 1 and 2. Then we have

$$|\varphi(x)| \leqslant e^{-8w(x)} \quad \text{for all } x \in \mathbb{R}, \tag{11}$$

$$|\varphi(iy)| \leqslant c e^{|y|\sigma(|y|)/4} \quad \text{for all } y \in \mathbb{R}. \tag{12}$$

Observe that (11) follows from (5) when $y = 0$.

To verify (12) we use the inequalities:

$$\left| \frac{\sin iy}{iy} \right| \leqslant e^{|y|}, \quad |y| \geqslant 1, \qquad \left| \frac{\sin iy}{iy} \right| \leqslant e^{y^2}, \quad 0 \leqslant |y| \leqslant 1.$$

These inequalities and (7) give

$$\left| \prod_{k=1}^{\infty} \frac{\sin(iy/8\gamma_k)}{iy/(8\gamma_k)} \right| \leqslant \prod_{8\gamma_k \leqslant |y|} e^{\frac{|y|}{8\gamma_k}} \prod_{8\gamma_k > |y|} e^{(\frac{y}{8\gamma_k})^2}$$
$$\leqslant \exp\left\{ \frac{|y|}{8} \sum_{\gamma_k \leqslant |y|} \frac{1}{\gamma_k} + \left(\frac{y}{8}\right)^2 \sum_{k \geqslant |y|/8} \frac{1}{k^2} \right\} \leqslant c \exp\left\{ \frac{|y|\sigma(|y|)}{4} \right\}.$$

Step 4. The Fourier transform $\hat{\varphi}$ is everywhere positive on $\mathbb{R}$.

Let χ_δ denote the characteristic function of $[-\delta, \delta]$. Then $(2\delta)^{-1}\hat{\chi}_\delta(x) = \sin(\delta x)/(\delta x)$. So, it follows from (10) that

$$\hat{\varphi}(x) = \left(\hat{h} * (4\gamma_1 \chi_{1/(8\gamma_1)}) * (4\gamma_2 \chi_{1/(8\gamma_2)}) * \cdots \right)(x).$$

Since, by (8), $\sum_{k=1}^{\infty} 1/\gamma_k = \infty$, we see that the infinite convolution of the characteristic functions $4\gamma_k \chi_{1/(8\gamma_k)}$ is everywhere positive. Since $\hat{h}$ is non-negative (see Step 1), we see that $\hat{\varphi}$ is everywhere positive on $\mathbb{R}$.

Step 5: We have

$$|\varphi(x+iy)| \leqslant c e^{|y|\sigma(|y|)/4} \quad \text{for all } x + iy \in \mathbb{C}. \tag{13}$$

Indeed, by (12), this is true for $x = 0$. However, since $\hat{\varphi}$ is positive for a fixed y, the function $|\varphi(x+iy)|$ attains its maximum when $x = 0$.

Step 6: Set

$$u(x+iy) := \frac{|y|}{\pi}\int_{\mathbb{R}} \frac{w(t)}{(t-x)^2+y^2}\,dt = \frac{|y|}{\pi}\int_{\mathbb{R}} \frac{w(x+t)}{t^2+y^2}\,dt. \tag{14}$$

Then $u(x+iy)$ is harmonic for $y \neq 0$ and satisfies:

$$\frac{w(x)}{2} \leqslant u(x+iy) \leqslant w(x) + c|y| + c \qquad \text{for all } x+iy \in \mathbb{C}, \quad y \neq 0. \tag{15}$$

It follows from (2) that the integral in (14) converges, so that u is harmonic for $y \neq 0$. Recall that w is even: $w(-x) = w(x)$. Hence, $u(-x+iy) = u(x+iy)$, and so it suffices to check (15) for $x \geqslant 0$. Since, by (1), $w(x+t) \geqslant w(x)$, $x, t \geqslant 0$, we obtain

$$u(x+iy) \geqslant \frac{|y| w(x)}{\pi}\int_0^\infty \frac{1}{t^2+y^2}\,dt = \frac{w(x)}{2}.$$

It also follows from (2) that $w(x+t) \leqslant w(x) + w(t)$, so that

$$\begin{aligned} u(x+iy) &\leqslant \frac{|y| w(x)}{\pi}\int_{\mathbb{R}} \frac{1}{t^2+y^2}\,dt + \frac{|y|}{\pi}\int_{\mathbb{R}} \frac{w(t)}{t^2+y^2}\,dt \\ &= w(x) + \frac{|y|}{\pi}\int_{\mathbb{R}} \frac{w(t)}{t^2+y^2}\,dt. \end{aligned}$$

Since w is smooth (see (3)), the last term is bounded when $y \to 0$, so that the right estimate in (15) follows.

Step 7: We have

$$|\varphi(x+iy)| \leqslant c e^{|y|\sigma(|y|) - 2w(x)} \qquad \text{for all } x+iy \in \mathbb{C}. \tag{16}$$

We shall verify (16) for $y = y_0$, where $y_0 > 0$ is an arbitrary number. The proof is similar for $y = y_0 < 0$. Set

$$v(x+iy) := \log|\varphi(x+iy)| + \log|\varphi(x+i(2y_0 - y))| + 8u(x+iy),$$

where u is defined in (14). Then v is subharmonic in the strip $0 < y < 2y_0$. Recall, by (4), that $\sigma(2y_0) \leqslant 2\sigma(y_0)$. Using (11), (12) and (15), we see that on the upper boundary of this strip v is bounded above by a constant:

$$\begin{aligned} v(x+2iy_0) &\leqslant 2y_0\sigma(2y_0)/4 - 8w(x) + 8w(x) + cy_0 + c \\ &= y_0\sigma(2y_0)/2 + cy_0 + c \\ &\leqslant y_0\sigma(2y_0) + c \\ &\leqslant 2y_0\sigma(y_0) + c. \end{aligned}$$

One can check that the same estimate holds on the real axis. Hence, the estimate holds for all points in the strip. In particular, by the left inequality in (15), we have for $y = y_0$ that

$$2\log|\varphi(x+iy_0)| = v(x+iy_0) - 8u(x+iy_0) \leqslant 2y_0\sigma(y_0) + c - 4w(x).$$

This implies (16) for $y = y_0$.

Now, using (16), for each function $f \in L^\infty_w$, we have

$$\begin{aligned} |(\check{\varphi} * f)(x+iy)| &= \left|\int_{\mathbb{R}} \varphi(s-x-iy) f(s)\,ds\right| \leqslant \int_{\mathbb{R}} |\varphi(t-iy) f(x+t)|\,dt \\ &\leqslant c\int_{\mathbb{R}} e^{|y|\sigma(|y|) - 2w(t)} e^{w(x+t)}\,dt. \end{aligned}$$

Since, by (1), $w(x+t) \leqslant w(x)+w(t)$, it follows from (3) that

$$|(\varphi * f)(x+iy)| \leqslant c e^{|y|\sigma(|y|)+w(x)} \int_{\mathbb{R}} e^{-w(t)}\, dt \leqslant c e^{|y|\sigma(|y|)+w(x)}.$$

Hence, $\check{\varphi} * f \in A(\sigma, w)$, which completes the proof of Lemma 5. $\square$

Proof of Lemma 6. It was established in [5] that for every $\Lambda \subseteq \mathbb{R}$ with $R(\Lambda)=\infty$ there exists a non-decreasing function $\sigma_1(y) \nearrow \infty$ such that Λ is a uniqueness set for $B(\sigma_1)$.

Set $\sigma(y)=\sigma_1(y)-1$, $y \geqslant 0$. Let h be an entire function of exponential type $\leqslant 1$ satisfying (5). Then, clearly, $fh \in B(\sigma_1)$, for every function $f \in A(\sigma, w)$. We conclude that Λ is a uniqueness set for $A(\sigma, w)$. $\square$

4. Proof of Theorem 2

First, we state two simple lemmas without proof:

Lemma 7. *Suppose a function $\varphi \in L^p$, $1 \leqslant p < \infty$, and a function ψ is bounded with compact support. Then, $\varphi * \psi \in L^P$ for every $p \leqslant P \leqslant \infty$.*

Lemma 8. *A function φ is a Λ-generator for L^p, $p \geqslant 1$, if and only if there is no non-trivial function $f \in L^P$, $1/p+1/P=1$, such that $(f * \check{\varphi})(\lambda)=0$ for all $\lambda \in \Lambda$.*

Proof of Theorem 2(i). Suppose Λ is a generating set for L^p, and that $1 \leqslant p < q \leqslant \infty$. Let φ be a Λ-generator for L^p, and let χ_1 be the characteristic function of the interval $(-1,1)$. To establish (i), we show that the function $\Phi := \varphi * \chi_1$ is a Λ-generator for L^q. Let $1 \leqslant Q < P \leqslant \infty$ be the numbers such that $1/p+1/P=1$ and $1/q+1/Q=1$. Suppose there exists $f \in L^Q$ such that $(f * \check{\Phi})(\lambda)=0$ for all $\lambda \in \Lambda$. By Lemma 8, we have to show that $f=0$ a.e. We see that $((f * \chi_1) * \check{\varphi})(\lambda)=0$, $\lambda \in \Lambda$. By Lemma 7, we have $f * \chi_1 \in L^P$. Since φ is a Λ-generator for L^p, by Lemma 8, we conclude that

$$(f * \chi_1)(x) = \int_{x-1}^{x+1} f(s)\, ds = 0 \quad \text{a.e.}$$

Since $f \in L^Q$, where $Q < \infty$, this implies $f=0$ a.e. Hence, Φ is a Λ-generator for L^q.

(ii) Let $\varphi \in L^1_\omega$ be a Λ-generator for L^1_ω. Since $\omega(x) \geqslant c + w(x)$ for all x, we have for any function $f \in L^1_\omega$ that $\|f\|_w \leqslant e^{-c}\|f\|_\omega < \infty$. It follows that $\varphi \in L^1_w$, and that every function $f \in L^1_\omega$ can be approximated in the norm of L^1_w by finite linear combinations of $\varphi(x-\lambda)$, $\lambda \in \Lambda$. However, clearly, the functions $f \in L^1_\omega$ form a dense subset in L^1_w. Hence, any function $f \in L^1_w$ can be approximated in the norm of L^1_w by finite linear combinations of $\varphi(x-\lambda)$, $\lambda \in \Lambda$, so that φ is a Λ-generator for L^1_w. $\square$

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