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Asymptotic behavior of a compressible two-phase model with well–formation interaction

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Abstract

In this work we consider a compressible gas–liquid model with a well–reservoir interaction term that is relevant for coupled wellbore-reservoir flow systems involved in e.g. drilling operations. Main focus is on deriving estimates that are independent of time. Under suitable conditions on the well–reservoir interaction term we obtain such estimates which allow prediction of the long-time behavior of the gas and liquid masses. Moreover, we also obtain a quantification of the convergence rates as a function of time and gain some insight into the role played by the rate characterizing how fast the well–reservoir interaction must die out. The model is investigated in a free boundary setting where the initial mass is a mixture of both phases, i.e. no single-phase zone exists.

Keywords:
Two-phase flow
Well-reservoir flow
Weak solutions
Asymptotic behavior
Free boundary problem

1. Introduction

Management of subsurface resources involves a system comprising the wellbore and the target reservoir. As discrete pathways through geological formations, boreholes and wells are critical to the success of many water, energy, and environmental management operations. Examples are oil and gas production, geothermal energy production, geologic carbon sequestration, subsurface remediation. Many well operations involve gas–liquid flow in a wellbore where there is some interaction with the surrounding reservoir. Equipment can be placed along the wellbore that allow for some kind of control on the flow between well and formation. For an example of such a model in the context of single-phase flow we refer to [4,5] and references therein. In this paper we consider a gas–liquid model with inclusion of well–reservoir interaction.

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The dynamics of the two-phase well flow is supposed to be dictated by a compressible gas–liquid model of the drift-flux type. More precisely, it takes the following form

$$
\begin{aligned}
&\partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] = [\alpha_g\rho_g]A,\\
&\partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] = 0,\\
&\partial_t[\alpha_l\rho_l u_l + \alpha_g\rho_g u_g] + \partial_x\big[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + P\big] = -q + \partial_x[\varepsilon\partial_x u_{mix}], \quad u_{mix} = \alpha_g u_g + \alpha_l u_l,
\end{aligned} \tag{1}
$$

where $\varepsilon \geqslant 0$. This formulation allows us to study transient flows in a well together with a possible flow of gas between well and surrounding reservoir represented by the rate term $A(x,t)$. The model is supposed under isothermal conditions. The unknowns are ρ_l, ρ_g the liquid and gas densities, α_l, α_g volume fractions of liquid and gas satisfying $\alpha_g + \alpha_l = 1$, u_l, u_g velocities of liquid and gas, P common pressure for liquid and gas, and q representing external forces like gravity and friction. Since the momentum is given only for the mixture, we need an additional closure law which connects the two-phase fluid velocities. We consider the special case where a no-slip condition is assumed, i.e., $u_g = u_l = u$. This is relevant for a flow regime corresponding to dispersed bubble flow where the gas–liquid mixture appears to be of a fairly homogeneous nature [16]. In the following we ignore external forces by setting $q = 0$. A highly relevant issue to address is related to the long-time behavior of the model. More precisely, we may ask:

- Under what conditions on the well–reservoir term $A(x,t)$ can we obtain a system that will give a stable long-time behavior? And what is the long-time behavior of masses and fluid velocity?

In this work we only give a partial answer to this question in the sense that we identify conditions on $A(x,t)$ that will ensure that the long-time behavior of (1) becomes similar to that of the model without well–reservoir interaction, i.e. $A = 0$ in (1).

Now we give more details about the framework in which the model is studied. Assuming a polytropic gas law relation $p = C\rho_g^\gamma$ with $\gamma > 1$ and incompressible liquid $\rho_l = \text{Const}$ we get a pressure law of the form

$$
P(n,m) = C\left(\frac{n}{\rho_l - m}\right)^\gamma, \tag{2}
$$

where we use the notation $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$. We consider (1) in a free boundary problem setting where the masses m and n initially occupy only a finite interval $[a,b] \subset \mathbb{R}$. That is,

$$
n(x,0) = n_0(x) > 0, \qquad m(x,0) = m_0(x) > 0, \qquad u(x,0) = u_0(x), \quad x \in [a,b],
$$

and $n_0 = m_0 = 0$ outside $[a,b]$. The viscosity coefficient ε is assumed to depend on the masses m and n, i.e. $\varepsilon = \varepsilon(n,m)$. More precisely, we assume that

$$
\varepsilon(n,m) = D\frac{(n+m)^\beta}{(\rho_l - m)^{\beta+1}}, \quad \beta \in (0, 1/3), \tag{3}
$$

for a constant D. See [8] for more information concerning the choice of the viscosity coefficient. Introducing the total mass $\rho = n + m$ and rewriting the model (1) in terms of Lagrangian variables, it was suggested in [3] to consider the following gas–liquid model:

$$\partial_t n + (n\rho)\partial_x u = nA,$$
$$\partial_t \rho + \rho^2 \partial_x u = nA,$$
$$\partial_t u + \partial_x P(n,\rho) = -u\frac{n}{\rho}A + \partial_x\big(\varepsilon(n,\rho)\rho\partial_x u\big), \quad x \in (0,1), \tag{4}$$

with pressure law

$$P(n,\rho) = \left(\frac{n}{\rho_l - [\rho - n]}\right)^{\gamma}, \quad \gamma > 1, \tag{5}$$

and viscosity coefficient

$$\varepsilon(n,\rho) = \frac{\rho^{\beta}}{(\rho_l - [\rho - n])^{\beta+1}}, \quad \beta \in (0, 1/3), \tag{6}$$

where we have set the constants C, D to be one for simplicity, whereas boundary conditions are

$$P(n,\rho) = \varepsilon(n,\rho)\rho u_x, \quad \text{at } x = 0, 1, \quad t \geqslant 0, \tag{7}$$

and initial conditions are

$$n(x,0) = n_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad u(x,0) = u_0(x), \quad x \in [0,1]. \tag{8}$$

In particular, a global existence result for weak solutions was obtained for the model problem (4)–(8). The objective of the current work is to continue the study of this model. The novelty lies in the fact that we explore under what circumstances time-independent estimates can be obtained which allow to extract information about the asymptotic behavior of the gas and liquid masses. Such results have been obtained for a gas–liquid model similar to (4), however, without any well–reservoir interaction [14,25]. In [14] such results were obtained for different initial data and different choices of the mass-dependent viscosity function. We also refer to this work for an overview of related results in the context of single-phase Navier–Stokes flow model [9,10,19,15,21,20,12].

The main impact from the well–reservoir term A, which makes the analysis in this work different from previous works on the gas–liquid model, is as follows:

- The well–reservoir interaction by $A(x,t)$ creates an additional time-dependence expressed by the fact that the variable $c = n/\rho$ becomes time-dependent and related to $A(x,t)$ by Eq. (26).
- Lemma 3.1 (energy estimate) depends on the fact that $\|A(\cdot,t)\|_\infty$ is in $L^1(0,\infty)$. Moreover, both Lemma 3.3 (boundary behavior) and Lemmas 3.2 and 3.4 (regularity of c_x and Q_x) must deal with the new interaction term $A(x,t)$ in an appropriate manner. The two latter lemmas require that $\|A_x(\cdot,t)\|_\infty$ is in $L^1(0,\infty)$. For Lemma 3.4 we derive the inequality (63) which demonstrates the role of the well–reservoir term A. This lay the foundation for obtaining the long-time behavior of the masses m and n as stated in Theorem 2.2.
- The decay rates of the masses are controlled by means of Lemma 5.1. This lower limit is required in order to control new terms that appear owing to $A(x,t)$. This is different from the result in [14]. Note also that we employ the variable transformation (91), which depends on A, in order to obtain a reformulation of the model as expressed by (97) which allows for application of the ideas of Nagasawa [18,14] to prove Theorem 2.3.

Note that the well–reservoir two-phase model (4) involves a "friction-like" term $-u(n/\rho)A$ in the momentum equation representing an acceleration effect due to influx/efflux of gas between well and reservoir. Such external force terms typically imply that smallness assumptions must be made on the initial fluid velocity in order to obtain time-independent estimates. See [7] (and references therein) for an example of this in the context of a gas–liquid flow model and [24] for an example for single-phase Navier–Stokes equations. We avoid this for the well–reservoir model by using that $\|A(\cdot,t)\|_\infty$ is in $L^1(0,\infty)$ to obtain the time-independent uniform estimate (36) of Lemma 3.1 and the time-independent estimate (48) of Corollary 3.2.

The main observations obtained through the analysis of this work concerning the long-time behavior of the model (4) is:

- In order to prove that the gas and liquid mass will vanish in the same manner as for the model without well–reservoir interaction ($A=0$), it is not necessary to use information about the flow direction of gas between well and reservoir ($A>0$ or $A<0$) or any smallness assumption on A. However, we need that $\|A(\cdot,t)\|_\infty$ and $\|A_x(\cdot,t)\|_\infty$ are in $L^1(0,\infty)$, see Theorem 2.2 and Remark 2.1.
- In particular, in order to obtain estimates of the rate at which gas and liquid masses tend to zero as time goes to infinity, the assumption on A must be strengthened in the sense that $(1+t)^{\beta+3}\|A(\cdot,t)\|_\infty$ is required to be in $L^1(0,\infty)$. There is also a corresponding sharpening of the restriction on β associated with the viscosity term (6), see Theorem 2.3 and Remark 5.1.

The rest of this paper is organized as follows. In Section 2 we state precisely the main theorems and their assumptions. In Section 3 we describe a priori estimates for the model where emphasis is on the time-independent estimates. In Section 4 it is explained how the obtained estimates lead to Theorem 2.2. Section 5 contains the proof of Theorem 2.3.

2. Main results

Below we give a precise description of the two main results of this paper, Theorem 2.2 and Theorem 2.3, and under which assumptions on initial data, parameters γ and β, and well–reservoir rate function $A(x,t)$ these results hold. Note that we do not try to optimize the parameter choice for $\beta>0$. First of all we intend to illustrate the mechanisms that give rise to limitations on this parameter.

We now recall the following (global) existence result for weak solutions that was obtained in [3].

Theorem 2.1 *(Global existence result). Assume that $\gamma>1$ and $\beta\in(0,1/3)$ respectively in (5) and (6), and that the initial data (n_0,m_0,u_0) satisfy*

(i) $\inf_{[0,1]} n_0>0$, $\sup_{[0,1]} n_0<\infty$, $\inf_{[0,1]} m_0>0$, *and* $\sup_{[0,1]} m_0<\rho_l$;
(ii) $n_0, m_0\in W^{1,2}(I)$;
(iii) $u_0\in L^{2q}(I)$, *for* $q\in\mathbb{N}$,

where $I=(0,1)$. As a consequence, the function $c_0=\frac{n_0}{n_0+m_0}$ satisfies that

$$\inf_{[0,1]} c_0>0,\qquad \sup_{[0,1]} c_0<1,\quad c_0\in W^{1,2}(I). \tag{9}$$

Moreover, the function $Q_0=\frac{n_0+m_0}{\rho_l-m_0}$ satisfies that

$$\inf_{[0,1]} Q_0>0,\qquad \sup_{[0,1]} Q_0<\infty,\quad Q_0\in W^{1,2}(I). \tag{10}$$

In addition, the well–formation flow rate function $A(x,t)$ is assumed to satisfy for all times $t\geqslant 0$

(iv) $\sup_{x\in[0,1]}|A(x,t)|\leqslant M<\infty$;
(v) $A(\cdot,t)\in W^{1,2}(I)$;
(vi) $A(0,t)=0$.

Then the initial–boundary problem (4)–(8) possesses a global weak solution (n, ρ, u) in the sense that for any $T > 0$, the following hold:

(A) *We have the estimates:*

$$n, \rho \in L^\infty\big([0,T], W^{1,2}(I)\big), \qquad n_t, \rho_t \in L^2\big([0,T], L^2(I)\big),$$
$$u \in L^\infty\big([0,T], L^{2q}(I)\big) \cap L^2\big([0,T], H^1(I)\big).$$

More precisely, $\forall (x,t) \in [0,1] \times [0,T]$ it follows that

$$0 < \inf_{x\in[0,1]} c(x,t), \qquad \sup_{x\in[0,1]} c(x,t) < 1, \qquad c := \frac{n}{\rho},$$
$$0 < \mu \inf_{x\in[0,1]} (c) \leqslant n(x,t) \leqslant \left(\frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}\right) \sup_{x\in[0,1]} (c),$$
$$0 < \mu \leqslant \rho \leqslant \frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}, \tag{11}$$

for a non-negative constant $\mu = \mu(\|c_0\|_{W^{1,2}(I)}, \|Q_0^\beta\|_{W^{1,2}(I)}, \|A\|_{W^{1,2}(I)}, \|u_0\|_{L^{2q}(I)}, \inf_{[0,1]} c_0, \sup_{[0,1]} c_0, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, M, T) > 0$.

(B) *Moreover, the following equations hold:*

$$n_t + n\rho u_x = nA, \qquad \rho_t + \rho^2 u_x = nA,$$
$$(n,\rho)(x,0) = \big(n_0(x), \rho_0(x)\big), \quad \textit{for a.e. } x \in (0,1) \textit{ and any } t \geqslant 0,$$
$$\int_0^\infty \int_0^1 \left[u\phi_t + \big(P(n,\rho) - E(n,\rho)u_x\big)\phi_x - u\frac{n}{\rho}A\phi\right] dx\,dt + \int_0^1 u_0(x)\phi(x,0)\,dx = 0 \tag{12}$$

for any test function $\phi(x,t) \in C_0^\infty(D)$, with $D := \{(x,t) \mid 0 \leqslant x \leqslant 1,\ t \geqslant 0\}$.

A uniqueness result was also given under suitable restrictions on parameters. We refer to [3] for details.

Now we focus on the long-time behavior. The first result describes under which conditions on A the masses m and n tend to zero as time goes to infinity.

Theorem 2.2 *(Asymptotic behavior of mass functions). Let (n, ρ, u) be a global weak solution as defined in Theorem 2.1. We assume that $\gamma > 1$, $\beta \in (0, 1/3)$, and $\gamma > \beta + 1$. In addition, the constraints on the well–formation flow rate function $A(x,t)$ are strengthened by requiring that*

$$\sup_{x\in[0,1]} \big|A(x,t)\big| \leqslant M(t) \in L^1\big([0,\infty)\big); \tag{13}$$
$$\sup_{x\in[0,1]} \big|A_x(x,t)\big| \leqslant N(t) \in L^1\big([0,\infty)\big). \tag{14}$$

We then have the following asymptotic behavior of the mass functions $n(x,t)$, $m(x,t)$:

$$\lim_{t\to\infty} \sup_{x\in[0,1]} n(x,t) = 0, \tag{15}$$

$$\lim_{t\to\infty} \sup_{x\in[0,1]} m(x,t) = 0. \tag{16}$$

We can also give decay rates of the mass functions $n(x,t)$, $m(x,t)$. However, for that result further restrictions on both A, represented by the function $M(t)$, and the parameter $\beta > 0$ are needed.

Theorem 2.3 *(Decay rate of the mass functions). Again let (n, ρ, u) be a global weak solution as defined in Theorem 2.1. Again, we assume that $\gamma > 1$ and $\gamma > \beta + 1$. However, we in addition assume that $\beta \in (0, 1/6)$. The constraints on the well–formation flow rate function $A(x,t)$ are strengthened by requiring that $M(t)$ obeys the following estimates independent of time $t > 0$:*

$$\int_0^t (1+s)^{\beta+3} M(s)\,ds \leqslant C, \qquad \int_0^t (1+s)^{\beta-1} \int_s^\infty M(\xi)\,d\xi\,ds \leqslant C. \tag{17}$$

For any $x \in [0,1]$, we then have the following decay rate estimates for the mass functions $n(x,t)$, $m(x,t)$

$$n(x,t), m(x,t) \leqslant C(1+t)^{-\frac{\beta}{\gamma-1+2\beta}}. \tag{18}$$

Remark 2.1. Note that we do require that $\int_0^\infty |A(x,t)|\,ds < \infty$, however, we do not require that $\int_0^\infty |A_t(x,t)|\,ds < \infty$. Hence, we may not conclude that $A(x,t) \to 0$ for all $x \in [0,1]$ as $t \to \infty$. In fact, no assumptions on continuity properties of $A(\cdot,t)$ as a function of time has been used to obtain the above results.

3. Estimates

In the following we will frequently take advantage of the fact that the model (4) can be rewritten in a more amenable form for deriving various estimates [6,22,23]. We first describe this reformulation, and then present a number of a priori estimates.

We introduce the variable

$$c = \frac{n}{\rho}, \tag{19}$$

and see that (4) corresponds to

$$\rho\partial_t c + c\partial_t\rho + \left[c\rho^2\right]\partial_x u = [c\rho]A,$$
$$\partial_t\rho + \rho^2\partial_x u = [c\rho]A,$$
$$\partial_t u + \partial_x P(c,\rho) = -ucA + \partial_x\big(E(c,\rho)\partial_x u\big),$$

that is,

$$\rho\partial_t c + c[c\rho]A = [c\rho]A,$$
$$\partial_t\rho + \rho^2\partial_x u = [c\rho]A,$$
$$\partial_t u + \partial_x P(c,\rho) = -ucA + \partial_x\big(E(c,\rho)\partial_x u\big),$$

which, in turn can be reformulated as

$$\partial_t c = c(1-c)A = ckA, \qquad k = k(x,t) := 1 - c(x,t),$$
$$\partial_t \rho + \rho^2 \partial_x u = c\rho A,$$
$$\partial_t u + \partial_x P(c,\rho) = -ucA + \partial_x\big(E(c,\rho)\partial_x u\big), \tag{20}$$

with

$$P(c,\rho) = c^\gamma \left(\frac{\rho}{\rho_l - k(x,t)\rho}\right)^\gamma, \quad k(x,t) = 1 - c(x,t),\ \gamma > 1, \tag{21}$$

and

$$E(c,\rho) = \left(\frac{\rho}{\rho_l - k(x,t)\rho}\right)^{\beta+1}, \quad 0 < \beta < 1/3. \tag{22}$$

Moreover, boundary conditions are given by

$$P(c,\rho) = E(c,\rho)u_x, \quad \text{at } x = 0, 1, \quad t \geqslant 0, \tag{23}$$

whereas initial data are

$$c(x,0) = c_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad u(x,0) = u_0(x), \quad x \in [0,1]. \tag{24}$$

Corollary 3.1. *Under the assumptions of Theorem 2.1, it follows that*

$$0 < \inf_{(x,t)\in[0,1]\times[0,\infty)} c(x,t), \qquad \sup_{(x,t)\in[0,1]\times[0,\infty)} c(x,t) < 1. \tag{25}$$

Proof. Note that from (20) we have

$$c_t = c(1-c)A(x,t),$$

which corresponds to

$$\frac{1}{c(1-c)} c_t = A(x,t), \quad c \in (0,1),$$

i.e.

$$G(c)_t = A(x,t), \qquad G(c) = \log\left(\frac{c}{1-c}\right).$$

This implies that

$$\frac{c(x,t)}{1-c(x,t)} = \frac{c_0(x)}{1-c_0(x)} \exp\left(\int_0^t A(x,s)\,ds\right). \tag{26}$$

Note also that the inverse of $h(c) = c/(1-c)$ is $h^{-1}(d) = d/(1+d)$, such that $h^{-1} : [0,\infty) \to [0,1)$ and is one-to-one. Consequently,

$$c(x,t) = h^{-1}\left(\frac{c_0(x)}{1-c_0(x)}\exp\left(\int_0^t A(x,s)\,ds\right)\right), \tag{27}$$

and $0 < c(x,t) < 1$ for $c_0(x) \in (0,1)$. In particular, we see that if

$$0 < \inf_{x\in[0,1]} c_0(x), \qquad \sup_{x\in[0,1]} c_0(x) < 1, \qquad \sup_{x\in[0,1]} \left|A(x,t)\right| \leqslant M(t) \in L^1\big([0,\infty)\big),$$

which follows from the assumptions on n_0, m_0, and A given in Theorem 2.1, we have that

$$C^{-1} \leqslant \exp\left(-\int_0^t M(s)\,ds\right) \leqslant \exp\left(\int_0^t A(x,s)\,ds\right) \leqslant \exp\left(\int_0^t M(s)\,ds\right) \leqslant C.$$

Hence, the conclusion (25) follows from (27). $\square$

We introduce the variable

$$Q(\rho,k) = \frac{\rho}{\rho_l - k(x,t)\rho}, \tag{28}$$

and observe that

$$\rho = \frac{\rho_l Q}{1+kQ}, \qquad \frac{1}{\rho} = \frac{1}{\rho_l Q} + \frac{k}{\rho_l}. \tag{29}$$

Thus, we may rewrite the model (20) in the following form

$$\begin{aligned} \partial_t c &= kcA, \\ \partial_t Q + \rho_l Q^2 u_x &= cAQ, \\ \partial_t u + \partial_x P(c,Q) &= -ucA + \partial_x\big(E(Q)\partial_x u\big), \end{aligned} \tag{30}$$

with

$$P(c,Q) = \big[cQ(\rho,k)\big]^\gamma, \quad \gamma > 1, \tag{31}$$

and

$$E(Q) = Q(\rho,k)^{\beta+1}, \quad 0 < \beta < 1/3. \tag{32}$$

This model is then subject to the boundary conditions

$$P(c,Q) = E(Q)u_x, \quad \text{at } x = 0,1, \quad t \geqslant 0. \tag{33}$$

In addition, we have the initial data

$$c(x,0) = c_0(x), \qquad Q(x,0) = Q_0(x), \qquad u(x,0) = u_0(x), \quad x = [0,1]. \tag{34}$$

3.1. A priori estimates

Now we derive a priori estimates for (c, Q, u) by making use of the reformulated model (30)–(34).

Lemma 3.1 *(Energy estimate). Let C be a constant independent of any time $T > 0$. Under the assumptions of Theorem 2.2 we then have the basic energy estimate where $t \in [0, T]$*

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma Q(\rho,k)^{\gamma-1}}{\rho_l(\gamma-1)}\right)(x,t)\,dx + \int_0^t \int_0^1 Q(\rho,k)^{\beta+1}(u_x)^2\,dx\,ds \leqslant C. \tag{35}$$

Moreover,

$$Q(\rho,k)(x,t) \leqslant C, \quad \forall (x,t) \in [0,1]\times[0,T], \tag{36}$$

and finally, for any positive integer q,

$$\int_0^1 u^{2q}(x,t)\,dx + q(2q-1)\int_0^t \int_0^1 u^{2q-2} Q(\rho,k)^{1+\beta}(u_x)^2\,dx\,dt \leqslant C. \tag{37}$$

Proof. We consider the proof in three steps.

Estimate (35): We multiply the third equation of (30) by u and integrate over $[0,1]$ in space. We apply the boundary condition (33) and the equation

$$\frac{c^\gamma}{\rho_l(\gamma-1)}\left(Q^{\gamma-1}\right)_t + c^\gamma Q^\gamma u_x = \frac{1}{\rho_l}c^{\gamma+1}Q^{\gamma-1}A, \tag{38}$$

obtained from the second equation of (30) by multiplying with $c^\gamma Q^{\gamma-2}$. This equation also corresponds to

$$\frac{1}{\rho_l(\gamma-1)}\left(c^\gamma Q^{\gamma-1}\right)_t - \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left(c^\gamma\right)_t + c^\gamma Q^\gamma u_x = \frac{1}{\rho_l}c^{\gamma+1}Q^{\gamma-1}A, \tag{39}$$

which in turn can be rewritten as

$$\frac{1}{\rho_l(\gamma-1)}\left(c^\gamma Q^{\gamma-1}\right)_t - \frac{\gamma}{\rho_l(\gamma-1)}Q^{\gamma-1}c^\gamma kA + P(c,Q)u_x = \frac{1}{\rho_l}c^{\gamma+1}Q^{\gamma-1}A, \tag{40}$$

where we have used the first equation of (30). Then, we get

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\right)dx - \int_0^1 \frac{\gamma c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}[kA]\,dx + \int_0^1 u^2[cA]\,dx \\ &\quad + \int_0^1 E(Q)(u_x)^2\,dx = \frac{1}{\rho_l}\int_0^1 c^{\gamma+1}Q^{\gamma-1}A\,dx = \frac{1}{\rho_l}\int_0^1 c^\gamma Q^{\gamma-1}[cA]\,dx.\end{aligned}$$

We can then integrate in time over $[0,t]$ and estimate as follows

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\right) dx + \int_0^t \int_0^1 E(Q)(u_x)^2\, dx\, ds$$

$$\leqslant \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{c^\gamma Q_0^{\gamma-1}}{\rho_l(\gamma-1)}\right) dx + \int_0^t \int_0^1 u^2\big[c|A|\big]\, dx\, ds + \int_0^t \int_0^1 \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\big[(\gamma-c)|A|\big]\, dx\, ds$$

$$\leqslant C + C\int_0^t M(s) \int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\right) dx\, ds, \tag{41}$$

where $M(s) \in L^1(0,\infty)$. From this and Remark 3.1 given below, (35) follows.

Estimate (36): From the second equation of (30) we deduce the equation

$$\frac{1}{\rho_l}\big(Q^\beta\big)_t + \beta Q^{\beta+1}u_x = \frac{\beta}{\rho_l} c Q^\beta A. \tag{42}$$

Integrating over $[0,t]$, we get

$$Q^\beta(x,t) = Q^\beta(x,0) - \beta\rho_l \int_0^t Q^{\beta+1}u_x\, ds + \beta\int_0^t cQ^\beta A\, ds. \tag{43}$$

Then, we integrate the third equation of (30) over $[0,x]$ and get

$$\int_0^x u_t(y,t)\, dy + P(c,Q) - P\big(c(0,t), Q(0,t)\big) + \big(E(Q)u_x\big)(0,t) + \int_0^x ucA\, dy$$

$$= E(Q)u_x = Q^{\beta+1}u_x.$$

Using the boundary condition (33) and inserting the above relation into the right-hand side of (43), we get

$$Q^\beta(x,t) = Q^\beta(x,0) - \beta\rho_l \int_0^t \left(\int_0^x u_t(y,t)\, dy + P(c,Q) + \int_0^x ucA\, dy\right) ds + \beta\int_0^t cQ^\beta A\, ds$$

$$= Q^\beta(x,0) - \beta\rho_l \int_0^x \big(u(y,t) - u_0(y)\big)\, dy - \beta\rho_l \int_0^t P(c,Q)\, ds$$

$$- \beta\rho_l \int_0^t \int_0^x u[cA]\, dy\, ds + \beta\int_0^t Q^\beta[cA]\, ds. \tag{44}$$

Now using the Cauchy and Hölder inequalities and (35) as well as the assumptions on the initial data and $A(x,t)$ given by (13), we can further estimate as follows

$$
\begin{aligned}
Q^{\beta}(x,t)+\beta\rho_l\int_0^t P(c,Q)\,ds &\leqslant Q^{\beta}(x,0)+\beta\rho_l\int_0^1 \big|u(y,t)\big|dy+\beta\rho_l\int_0^1 \big|u_0(y)\big|\,dy\\
&\quad +C\int_0^t\int_0^1 |A||u|\,dy\,ds+C\int_0^t |A|Q^{\beta}(x,s)\,ds\\
&\leqslant C+C\int_0^t\int_0^1 |A|^{\frac12}|u||A|^{\frac12}\,dy\,ds+C\int_0^t |A|Q^{\beta}(x,s)\,ds\\
&\leqslant C+C\int_0^t\int_0^1 |A||u|^2\,dy\,ds+C\int_0^t\int_0^1 |A|\,dy\,ds+C\int_0^t |A|Q^{\beta}(x,s)\,ds\\
&\leqslant C+C\int_0^t M(s)\int_0^1 |u|^2\,dy\,ds+C\int_0^t M(s)\,ds+C\int_0^t |A|Q^{\beta}(x,s)\,ds\\
&\leqslant C+C\int_0^t M(s)Q^{\beta}(x,s)\,ds. \qquad (45)
\end{aligned}
$$

Finally, after an application of Gronwall's inequality as described in Remark 3.1, the upper bound (36) follows.

Estimate (37): Multiplying the third equation of (30) by $2qu^{2q-1}$, integrating over $[0,1]\times[0,t]$ and integration by parts together with application of the boundary conditions (33), we get

$$
\begin{aligned}
&\int_0^1 u^{2q}\,dx+2q(2q-1)\int_0^t\int_0^1 Q(\rho,k)^{\beta+1}(u_x)^2u^{2q-2}\,dx\,ds\\
&\quad=\int_0^1 u_0^{2q}\,dx+2q(2q-1)\int_0^t\int_0^1 c^{\gamma}Q(\rho,k)^{\gamma}u^{2q-2}u_x\,dx\,ds-2q\int_0^t\int_0^1 [cA]u^{2q}\,dx\,ds. \qquad (46)
\end{aligned}
$$

For the second term on the right-hand side of (46) we apply Cauchy's inequality with and get

$$
\begin{aligned}
&\int_0^t\int_0^1 c^{\gamma}Q(\rho,k)^{\gamma}u^{2q-2}u_x\,dx\,ds\\
&\quad\leqslant \frac12\int_0^t\int_0^1 c^{2\gamma}Q(\rho,k)^{2\gamma-\beta-1}u^{2q-2}\,dx\,ds+\frac12\int_0^t\int_0^1 Q(\rho,k)^{\beta+1}u^{2q-2}(u_x)^2\,dx\,ds.
\end{aligned}
$$

The last term clearly can be absorbed in the second term of the left-hand side of (46). Finally, let us see how we can bound the term $\frac12\int_0^t\int_0^1 c^{2\gamma}Q(\rho,k)^{2\gamma-1-\beta}u^{2q-2}\,dx\,ds$. Following along the lines of [14] we find using Young's inequality (i.e. $ab\leqslant(1/p)a^p+(1/r)b^r$ where $1/p+1/r=1$, with the choice $p=q$ and $r=q/(q-1)$), and thereafter the Hölder inequality that

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_0^1 c^{2\gamma} Q^{2\gamma-1-\beta} u^{2q-2}\,dx\,ds \\
&= \frac{1}{2}\int_0^t\int_0^1 c^{\frac{\gamma}{q}+\gamma} Q^{\frac{\gamma}{q}+\gamma-\beta-1} c^{\frac{(q-1)\gamma}{q}} Q^{\frac{(q-1)\gamma}{q}} u^{2q-2}\,dx\,ds \\
&\leqslant \frac{1}{2q}\int_0^t\int_0^1 c^{q\gamma} Q^{q(\gamma-\beta-1)} c^{\gamma} Q^{\gamma}\,dx\,ds + \frac{q-1}{2q}\int_0^t\int_0^1 c^{\gamma} Q^{\gamma} u^{2q}\,dx\,ds, \\
&\leqslant \frac{1}{2q}\int_0^t \max_{[0,1]}\big([cQ]^{\gamma}\big)\left(\int_0^1 c^{q\gamma} Q^{q(\gamma-\beta-1)}\,dx\right) ds + \frac{q-1}{2q}\int_0^t \max_{[0,1]}\big([cQ]^{\gamma}\big)\left(\int_0^1 u^{2q}\,dx\right) ds \\
&\leqslant C + C\int_0^t \max_{[0,1]}\big([cQ]^{\gamma}\big)\left(\int_0^1 u^{2q}\,dx\right) ds,
\end{aligned}
$$

where we have used (36), the requirement $\gamma \geqslant \beta+1$, as well as Corollary 3.2 below. To sum up, we now get that

$$
\begin{aligned}
&\int_0^1 u^{2q}\,dx + q(2q-1)\int_0^t\int_0^1 Q(\rho,k)^{\beta+1}(u_x)^2 u^{2q-2}\,dx\,ds \\
&\leqslant \int_0^1 u_0^{2q}\,dx + 2q(2q-1)\left[C + C\int_0^t \max_{[0,1]}\big([cQ]^{\gamma}\big)\left(\int_0^1 u^{2q}\,dx\right) ds\right] + 2q\int_0^t\int_0^1 \big(c|A|\big)u^{2q}\,dx\,ds \\
&= C + C\int_0^t \left(\max_{[0,1]}\big([cQ]^{\gamma}\big) + M(s)\right)\left(\int_0^1 u^{2q}\,dx\right) ds.
\end{aligned}
\tag{47}
$$

Finally, in view of estimate (48) of Corollary 3.2, we can use Gronwall's inequality as described in Remark 3.1 and conclude that estimate (37) holds. □

We now state the following useful corollary, which is used extensively throughout the paper.

Corollary 3.2. *For any $(x,t) \in [0,1]\times[0,T]$, we have*

$$
\int_0^t P(c,Q)\,ds = \int_0^t [cQ]^{\gamma}\,ds \leqslant C. \tag{48}
$$

In particular, $\int_0^\infty \max_{x\in[0,1]} P(c,Q)\,dt \leqslant C$.

Proof. This follows directly from Eq. (45), since the term $\int_0^t M(s)Q^{\beta}(x,s)\,ds \leqslant C\int_0^\infty M(s)\,ds \leqslant C$, by application of estimate (36) and assumption on M given in (13). □

Remark 3.1. It follows from (41) that

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\right) dx \leqslant C_2 + \int_0^t C_1(s) \int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\right) dx\, ds, \tag{49}$$

where C_2 is a constant, and $C_1(s) \in L^1(0,\infty)$. We then define the function $\eta(t)$ such that

$$\eta(t) = \int_0^t C_1(s) \int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\right) dx\, ds. \tag{50}$$

It then follows by differentiating $\eta(t)$ and using (49) that

$$\eta'(t) = C_1(t) \int_0^1 \left(\frac{1}{2}u^2 + \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\right) dx \leqslant C_1(t)\big[C_2 + \eta(t)\big] = C_1(t)\eta(t) + \psi(t), \tag{51}$$

where $\psi(t) = C_2 C_1(t) \in L^1(0,\infty)$. Clearly, the differential form of Gronwall's inequality then let us conclude that

$$\eta(t) \leqslant e^{\int_0^t C_1(s)\,ds}\left[\eta(0) + \int_0^t \psi(s)\,ds\right] \leqslant C_2 e^{\int_0^\infty C_1(s)\,ds} \int_0^\infty C_1(s)\,ds \leqslant C, \tag{52}$$

where C is a constant independent of t and we have used that $\eta(0) = 0$.

Remark 3.2. It is instructive to compare the model (4) with the gas–liquid model studied by Fan et al. [7]. Their model contains a friction term which is of the form $-fm^2u|u|$ appearing on the right-hand side of the momentum equation. This term prevents the authors to obtain a time-independent upper bound for Q similar to (36). Instead they have to rely on other arguments that involve sufficiently small fluid velocity $\|u\|_2 \leqslant \varepsilon$. The model (4) also contains a "friction"-like term $-cAu$. It is the $L^1(0,\infty)$ control of $\|A(\cdot,t)\|_\infty$ which allows us to obtain (36) without requiring any smallness on fluid velocity u.

The next lemma describes under which conditions $c(x,t)$ is in $W^{1,2}(I)$. The new aspect here compared to [3] is that the estimate must be time-independent.

Lemma 3.2 *(Additional regularity on c). Under the assumptions of Theorem 2.2 we have the estimate*

$$\int_0^1 (\partial_x c)^2\, dx \leqslant C. \tag{53}$$

Proof. We set $w = c_x$ and derive from the first equation of (30)

$$w_t = w(1-c)A - cwA + ckA_x = w(1-2c)A + ckA_x.$$

Hence, multiplying by w and integrating over $[0,1]$ we get

$$\int_0^1 \left(\frac{1}{2}w^2\right)_t dx = \int_0^1 (1-2c)Aw^2\,dx + \int_0^1 ckA_x w\,dx. \tag{54}$$

Clearly, in view of the assumptions on the flow rate A given by (13) and (14) and the bound on c from Corollary 3.1, we see that

$$\begin{aligned}
\frac{1}{2}\int_0^1 w^2\,dx &= \frac{1}{2}\int_0^1 w_0^2\,dx + \int_0^t\int_0^1 (1-2c)Aw^2\,dx\,ds + \int_0^t\int_0^1 ckA_x w\,dx\,ds \\
&\leqslant C + C\int_0^t M(s)\int_0^1 w^2\,dx\,ds + \frac{1}{2}\int_0^t\int_0^1 |A_x|\,dx\,ds + \frac{1}{2}\int_0^t\int_0^1 |A_x|w^2\,dx\,ds \\
&\leqslant C + C\int_0^t M(s)\int_0^1 w^2\,dx\,ds + \frac{1}{2}\int_0^t N(s)\,ds + \frac{1}{2}\int_0^t N(s)\int_0^1 w^2\,dx\,ds \\
&\leqslant C + C\int_0^t \big[M(s)+N(s)\big]\int_0^1 w^2\,dx\,ds,
\end{aligned}$$

where we have used Cauchy's inequality. We conclude, by Gronwall's inequality as before, that (53) holds. $\square$

The behavior of Q at the boundaries is given in the next lemma. The obtained estimates on the mass function Q on the boundary will be required both in the proof of Lemma 3.4 and the proof of Theorem 2.2. Note that the ode equation that describes the behavior at the boundary contains an additional term due to the appearance of A. However, this term is a "good" term and an estimate is obtained similar to what was obtained in [14].

Lemma 3.3. *Let $d=0$ or 1, and $\nu(d,t)=(\gamma-\beta)\int_0^t c(d,s)A(d,s)\,ds$. We then have*

$$Q(d,t) = \left[e^{-\nu(d,t)}(\gamma-\beta)\rho_l\int_0^t e^{\nu(d,s)}c(d,s)^\gamma\,ds + e^{-\nu(d,t)}Q_0^{\beta-\gamma}\right]^{\frac{1}{\beta-\gamma}}. \tag{55}$$

Proof. From the momentum equation in (30) it follows that

$$\int_0^1 u_t\,dx = -\int_0^1 ucA\,dx, \tag{56}$$

due to the boundary conditions. From (56) it then follows using $x=0$ or $x=1$ in the first line of (44) that

$$Q^\beta(d,t) = Q_0^\beta(d) - \beta\rho_l\int_0^t [cQ]^\gamma(d,s)\,ds + \beta\int_0^t c(d,s)A(d,s)Q^\beta(d,s)\,ds. \tag{57}$$

A differentiation of (57) with respect to the time variable t gives us the following ordinary differential equation (ODE)

$$\frac{d}{dt}Q(d,t) = -\rho_l c(d,t)^{\gamma} Q(d,t)^{\gamma+1-\beta} + c(d,t)A(d,t)Q(d,t). \tag{58}$$

The ODE equation is in the form

$$y'(t) = -b(t)y^p + a(t)y, \qquad p = \gamma + 1 - \beta,$$

for suitable choices of $a(t)$, $b(t)$, and $y(t) = Q(d,t)$. This is an ODE of Bernoulli type, and its closed form solution is given by

$$y(t) = e^{\int_0^t a(s)\,ds}\left((\gamma-\beta)\int_0^t e^{(\gamma-\beta)\int_0^s a(\xi)\,d\xi} b(s)\,ds + y(0)^{\beta-\gamma}\right)^{\frac{1}{\beta-\gamma}}.$$

This implies (55). □

We will need the following useful corollary later.

Corollary 3.3. *Let $d = 0$ or 1. There exist positive constants C_1 and C_2 such that for any $t > 0$, we have*

$$C_1(1+t)^{\frac{-1}{\gamma-\beta}} \leqslant Q(d,t) \leqslant C_2(1+t)^{\frac{-1}{\gamma-\beta}}. \tag{59}$$

Proof. Using the assumptions on $A(x,t)$, it follows directly from Lemma 3.3 and Corollary 3.1 that

$$e^{-2C}\left[\left(\inf_{t>0} c(d,t)^{\gamma}\right)t + Q_0(d)^{\beta-\gamma}\right]^{\frac{-1}{\gamma-\beta}} \leqslant Q(d,t) \leqslant e^{2C}\left[Ct + Q_0(d)^{\beta-\gamma}\right]^{\frac{-1}{\gamma-\beta}}$$

since

$$\left|\nu(d,t)\right| \leqslant (\gamma-\beta)\int_0^{\infty} M(t)\,dt \leqslant C. \qquad \square$$

We want to obtain a time-independent estimate of $(Q^{\beta})_x$ in L^2, similar to the estimate of c_x in Lemma 3.2. The proof of this result is based on the approach taken in [14]. However, a new aspect compared to the analysis in [14] is the repeated use of the time-independent estimate of c_x in L^2 and the need for $\|A(\cdot,t)\|_{\infty}$ and $\|A_x(\cdot,t)\|_{\infty}$ to be in $L^1(0,\infty)$. Note that the original ideas go back to a work by Guo and Zhu [11] which in turn is based on estimates that were obtained by Kanel in [13] (1D) and Bresch, Desjardins, Lin, and Mellet, Vasseur for multi-dimensional case, [1,2,17].

Lemma 3.4 *(Additional regularity). We have the estimate*

$$\int_0^1 \left(\partial_x Q^{\beta}\right)^2 dx + \int_0^1\int_0^t \left(\partial_x (cQ)^{\frac{\gamma+\beta}{2}}\right)^2 ds\,dx \leqslant C. \tag{60}$$

Proof. Using (30) we find that

$$
\begin{aligned}
\big(Q^{\beta}\big)_{xt} &= \big(\beta Q^{\beta-1} Q_t\big)_x \\
&= \big(\beta c A Q^{\beta}\big)_x - \big(\rho_l \beta Q^{\beta+1} u_x\big)_x \\
&= \big(\beta c A Q^{\beta}\big)_x - \rho_l \beta [u_t + P_x + ucA].
\end{aligned}
\tag{61}
$$

Multiplying (61) by $(Q^{\beta})_x$ and integrating (in x and t) over $[0,1]\times[0,t]$ we get

$$
\begin{aligned}
\frac{1}{2}\int_0^1 \big(Q^{\beta}\big)_x^2\,dx &= \frac{1}{2}\int_0^1 \big(Q_0^{\beta}\big)_x^2\,dx \\
&\quad + \int_0^1\int_0^t \beta\big(cAQ^{\beta}\big)_x\big(Q^{\beta}\big)_x\,ds\,dx - \int_0^1\int_0^t \rho_l\beta\big(u_t + P(c,Q)_x + ucA\big)\big(Q^{\beta}\big)_x\,ds\,dx \\
&:= L_1 + L_2 + L_3 + L_4 + L_5.
\end{aligned}
\tag{62}
$$

After a series of manipulation and estimation of the right-hand side of the above equation (see Appendix A for details) the following inequality is obtained:

$$
\begin{aligned}
&\int_0^1 \big(Q^{\beta}\big)_x^2\,dx + C\int_0^1\int_0^t \big((cQ)^{\frac{\gamma+\beta}{2}}\big)_x^2\,ds\,dx \\
&\quad \leqslant C + \int_0^t \max_{x\in[0,1]}\big(|A| + |A_x| + [cQ]^{\gamma}\big)\left(\int_0^1 \big(Q^{\beta}\big)_x^2\,dx\right)ds.
\end{aligned}
\tag{63}
$$

Finally, application of Gronwall's inequality and the assumption that $\|A(\cdot,t)\|_{\infty}, \|A_x(\cdot,t)\|_{\infty} \in L^1(0,\infty)$, gives the estimate (60). $\square$

4. Asymptotic behavior of the mass variables

In this section we prove Theorem 2.2. A first step towards this aim is to strengthen the estimate of $[cQ]^{\gamma}$ as time goes to infinity. Following along the lines of [14], we introduce the function

$$
g(t) = \int_0^1 [cQ]^{\gamma}\,dx.
\tag{64}
$$

We find for all $t>0$ that

$$
\int_0^t g(s)\,ds = \int_0^t\int_0^1 [cQ]^{\gamma}\,dx\,ds \leqslant \int_0^t \max_{x\in[0,1]} [cQ]^{\gamma}\,ds \leqslant C,
\tag{65}
$$

due to Corollary 3.2. Moreover, we observe from the first and second equations of (30) that

$$g'(t) = \int_0^1 \big((cQ)^\gamma\big)_t\, dx = \int_0^1 \gamma A (cQ)^\gamma\, dx - \int_0^1 \rho_l \gamma c^\gamma Q^{\gamma+1} u_x\, dx. \tag{66}$$

Then it follows that

$$\int_0^\infty \big|g'(t)\big|\, dt \leqslant \int_0^\infty \int_0^1 \gamma |A| (cQ)^\gamma\, dx\, dt + \int_0^\infty \int_0^1 \rho_l \gamma c^\gamma Q^{\gamma+1} |u_x|\, dx\, dt := I_{g1} + I_{g2}. \tag{67}$$

We can now estimate I_{g1} and I_{g2} as follows.

$$I_{g1} = \int_0^\infty \int_0^1 \gamma |A| (cQ)^\gamma\, dx\, dt \leqslant C \int_0^\infty \max_{x\in[0,\infty]} \big(|A|\big)\, dt \leqslant C, \tag{68}$$

since $|P(c,Q)| \leqslant C$ and in view of assumption on A. Furthermore,

$$\begin{aligned} I_{g2} = \int_0^\infty \int_0^1 \rho_l \gamma c^\gamma Q^{\gamma+1} |u_x|\, dx\, dt &\leqslant C \int_0^\infty \int_0^1 Q^{1+\beta} u_x^2\, dx\, dt + C \int_0^\infty \int_0^1 c^{2\gamma} Q^{2\gamma+1-\beta}\, dx\, dt \\ &\leqslant C + C \int_0^\infty \max_{x\in[0,1]} \big([cQ]^\gamma\big)\, dt \leqslant C, \end{aligned} \tag{69}$$

where we have used the Cauchy inequality, Corollary 3.2 as well as (35) and (36). It is then clear that we have $g(t) \in L^1(0,\infty)$ and $g'(t) \in L^1(0,\infty)$, and we can thus conclude that

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \int_0^1 [cQ]^\gamma(x,t)\, dx = 0. \tag{70}$$

However, we can also prove a stronger result.

Lemma 4.1. *Let $0 < \lambda < \infty$. We then have that*

$$\lim_{t\to\infty} \int_0^1 [cQ]^\lambda(x,t)\, dx = 0. \tag{71}$$

Proof. For $\lambda \in (0,\gamma)$ we set $p = \gamma/\lambda > 1$ and use Hölder's inequality to conclude that

$$\int_0^1 [cQ]^\lambda\, dx \leqslant \left(\int_0^1 [cQ]^{\lambda p}\, dx\right)^{1/p} = \left(\int_0^1 [cQ]^\gamma\, dx\right)^{\lambda/\gamma} \to 0,$$

as $t \to \infty$, in view of (70). For $\lambda \geqslant \gamma$, the result has been proved already. □

Proof of Theorem 2.2. Now we are in a position where we can derive the asymptotic behavior for the gas and liquid masses. First, we choose $s > \beta > 0$, and note that

$$\big((cQ)^s\big)_x = s(cQ)^{s-\beta}(cQ)^{\beta-1}(cQ)_x = \frac{s}{\beta}(cQ)^{s-\beta}\big((cQ)^\beta\big)_x.$$

Hence, using Corollary 3.3, Lemmas 4.1 and 3.4, as well as Hölder's inequality we get

$$\begin{aligned} 0 \leqslant (cQ)^s(x,t) &= (cQ)^s(0,t) + \int_0^x \big((cQ)^s\big)_y\,dy \\ &\leqslant C(1+t)^{\frac{-s}{\gamma-\beta}} + \frac{s}{\beta}\left(\int_0^1 (cQ)^{2(s-\beta)}\,dx\right)^{\frac{1}{2}}\left(\int_0^1 \big((cQ)^\beta\big)_x^2\,dx\right)^{\frac{1}{2}} \to 0 \quad \text{as } t \to 0. \end{aligned} \tag{72}$$

Here we have employed that

$$(cQ)_x^\beta = c^\beta\big(Q^\beta\big)_x + \beta c^{\beta-1}Q^\beta c_x$$

such that

$$\int_0^1 \big((cQ)^\beta\big)_x^2\,dx \leqslant 2\int_0^1 c^{2\beta}\big(Q^\beta\big)_x^2\,dx + 2\int_0^1 \beta c^{2(\beta-1)}Q^{2\beta}c_x^2\,dx \leqslant C\int_0^1 \big(Q^\beta\big)_x^2\,dx + C\int_0^1 c_x^2\,dx \leqslant C,$$

in view of estimates (25), (36), Lemma 3.2 and Lemma 3.4. We can thus conclude (also due to Corollary 3.1) that

$$\lim_{t\to\infty} Q(x,t) = 0, \tag{73}$$

and since $Q = \frac{\rho}{\rho_l - k\rho}$, that

$$\lim_{t\to\infty} \rho(x,t) = 0. \tag{74}$$

Obviously, we then also have, since $n = c\rho$, that

$$\lim_{t\to\infty} n(x,t) = 0, \tag{75}$$

and since $m = \rho - n$, it follows that

$$\lim_{t\to\infty} m(x,t) = 0. \tag{76}$$

This proves Theorem 2.2. $\square$

5. Decay rates of the mass functions

This section is devoted to the proof of Theorem 2.3. As a preparation for this we first derive a time-dependent lower estimate of Q. The proof essentially follows along the lines of [3], but due to Lemma 3.4 we are now able to give a detailed threshold with respect to the dependence of the time variable in the estimate. This is necessary for the forthcoming result of Lemma 5.2 which again is the basis for deriving rate estimates of the masses m and n.

Lemma 5.1 *(Pointwise lower limit). Let $0 < \beta < 1/6$ and assume that $\int_0^t (1+s)M(s)\,ds \leqslant C$. Then for sufficiently large t we have a pointwise lower limit on $Q(\rho, k)$ of the form*

$$Q(\rho,k)(x,t) \geqslant C\frac{1}{(1+t)^4}, \quad \forall x \in [0,1]. \tag{77}$$

Proof. We first define

$$v(x,t) = \frac{1}{Q(x,t)}, \qquad V(t) = \max_{[0,1]\times[0,t]} v(x,s).$$

We calculate as follows:

$$\begin{aligned} v(x,t) - v(0,t) &= \int_0^x \partial_x v\,dx \leqslant \int_0^1 |\partial_x Q| v^2\,dx = \frac{1}{\beta}\int_0^1 v^{\beta+1}\left|\partial_x Q^\beta\right|dx \\ &\leqslant \frac{1}{\beta}\left(\int_0^1 \left|\partial_x Q^\beta\right|^2 dx\right)^{1/2}\left(\int_0^1 v^{2(\beta+1)}\,dx\right)^{1/2} \\ &\leqslant C\left(\int_0^1 v\,dx\right)^{1/2}\left(\left(\max_{[0,1]} v(\cdot,t)\right)^{2\beta+1}\right)^{1/2} \\ &\leqslant C\left(\int_0^1 v\,dx\right)^{1/2}\left(\max_{[0,1]} v(\cdot,t)\right)^{\beta+1/2}, \end{aligned} \tag{78}$$

where we have used (60). Next, we focus on how to estimate $\int_0^1 v\,dx$. The starting point is the observation that the second equation of (30) can be written as

$$v_t - \rho_l u_x = -[cA]v.$$

Integrating over $[0,1]\times[0,t]$ we get

$$\begin{aligned} \int_0^1 v(x,t)\,dx &= \int_0^1 v(x,0)\,dx + \rho_l\int_0^t \big[u(1,s) - u(0,s)\big]\,ds - \int_0^t\int_0^1 [cA]v\,dx\,ds \\ &\leqslant \left(\inf_{[0,1]} Q_0\right)^{-1} + 2\rho_l\int_0^t \max_{[0,1]}\left|u(\cdot,s)\right|ds + \int_0^t\int_0^1 c|A|v\,dx\,ds \end{aligned}$$

$$\leqslant \Big(\inf_{[0,1]} Q_0\Big)^{-1} + 2\rho_l\sqrt{t}\left(\int_0^t \big\|u^2(s)\big\|_{L^\infty(I)}\,ds\right)^{1/2} + \int_0^t\int_0^1 c|A|v\,dx\,ds, \tag{79}$$

where we have used Hölder's inequality. In light of Sobolev's inequality $\|f\|_{L^\infty(I)} \leqslant C\|f\|_{W^{1,1}(I)}$ it follows that the second last term of (79) can be estimated as follows:

$$\begin{aligned}
\int_0^t \big\|u^2(s)\big\|_{L^\infty(I)}\,ds &\leqslant C\int_0^t \big\|u^2(s)\big\|_{W^{1,1}(I)}\,ds \\
&= C\left(\int_0^t\int_0^1 u^2\,dx\,ds + \int_0^t\int_0^1 \big|\big(u^2\big)_x\big|\,dx\,ds\right) \\
&\leqslant Ct + 2C\int_0^t\int_0^1 Q^{\frac{1+\beta}{2}}|u||u_x|v^{\frac{1+\beta}{2}}\,ds\,ds \\
&\leqslant Ct + 2C\left(\int_0^t\int_0^1 Q^{1+\beta}u_x^2u^2\,dx\,ds\right)^{1/2}\left(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\right)^{1/2} \\
&\leqslant Ct + C\left(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\right)^{1/2},
\end{aligned} \tag{80}$$

where we have used (35) and (37) with $q=2$ and Hölder's inequality. Combining (79) and (80) we get

$$\begin{aligned}
\int_0^1 v(x,t)\,dx &\leqslant \Big(\inf_{[0,1]} Q_0\Big)^{-1} + C\sqrt{t}\left[Ct + C\left(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\right)^{1/2}\right]^{1/2} + \int_0^t\int_0^1 c|A|v\,dx\,ds \\
&\leqslant C + Ct + Ct^{1/2}\left(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\right)^{1/4} + \int_0^t\int_0^1 c|A|v\,dx\,ds \\
&= C + Ct + Ct^{1/2}\left(\int_0^t\int_0^1 v^{2\beta}v^{1-\beta}\,dx\,ds\right)^{1/4} + \int_0^t\int_0^1 c|A|v\,dx\,ds \\
&\leqslant C + Ct + V(t)^{\beta/2}Ct^{1/2}\left(\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds\right)^{1/4} + CV(t)^{\beta}\int_0^t\int_0^1 |A|v^{1-\beta}\,dx\,ds,
\end{aligned} \tag{81}$$

where the inequality $(a+b)^{\frac{1}{2}} \leqslant a^{\frac{1}{2}} + b^{\frac{1}{2}}$ and Hölder's inequality have been used. Now we focus on estimating $\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds$. For that purpose, we note that the second equation of (30), by multiplying with $Q^{\frac{\beta-1}{2}-1}$, can be written as

$$\big(Q^{\frac{\beta-1}{2}}\big)_t = \rho_l\left(\frac{1-\beta}{2}\right)Q^{\frac{\beta+1}{2}}u_x - \left(\frac{1-\beta}{2}\right)[cA]Q^{\frac{\beta-1}{2}}.$$

Integrating this equation over $[0,t]$ we get

$$Q^{\frac{\beta-1}{2}}(x,t) = Q^{\frac{\beta-1}{2}}(x,0) + \rho_l\left(\frac{1-\beta}{2}\right)\int_0^t Q^{\frac{\beta+1}{2}} u_x\,ds - \left(\frac{1-\beta}{2}\right)\int_0^t [cA]Q^{\frac{\beta-1}{2}}\,ds.$$

Consequently, using the inequality $(a+b)^2 \leqslant 2a^2+2b^2$ we get by Hölder's inequality.

$$\begin{aligned}
Q^{\beta-1}(x,t) &\leqslant 2Q^{\beta-1}(x,0) + 4\rho_l^2\left(\frac{1-\beta}{2}\right)^2\left(\int_0^t Q^{\frac{\beta+1}{2}}u_x\,ds\right)^2 + 4\left(\frac{1-\beta}{2}\right)^2\left(\int_0^t [cA]Q^{\frac{\beta-1}{2}}\,ds\right)^2 \\
&\leqslant 2Q^{\beta-1}(x,0) + Ct\int_0^t Q^{\beta+1}u_x^2\,ds + C\left(\int_0^t |A|\,ds\right)\left(\int_0^t |A|Q^{\beta-1}\,ds\right) \\
&\leqslant 2Q^{\beta-1}(x,0) + Ct\int_0^t Q^{\beta+1}u_x^2\,ds + C\int_0^t |A|Q^{\beta-1}\,ds,
\end{aligned}$$

since $|A| \in L^1(0,\infty)$. Furthermore, integrating over $[0,1]$ in space yields

$$\begin{aligned}
\int_0^1 v^{1-\beta}\,dx &= \int_0^1 Q^{\beta-1}\,dx \\
&\leqslant 2\int_0^1 v^{1-\beta}(x,0)\,dx + Ct\int_0^1\int_0^t Q^{\beta+1}u_x^2\,ds\,dx + C\int_0^1\int_0^t |A|v^{1-\beta}\,ds\,dx \\
&\leqslant C + Ct + C\int_0^t M(s)\int_0^1 v^{1-\beta}\,dx\,ds,
\end{aligned} \tag{82}$$

where we have used (35). In order to proceed, we again utilize Gronwall's inequality on differential form. Defining a function $\eta(t)$ such that

$$\eta(t) = \int_0^t M(s)\int_0^1 v^{1-\beta}\,dx\,ds, \tag{83}$$

we observe using Eq. (82) that

$$\eta'(t) \leqslant M(t)\big(C + Ct + C\eta(t)\big) = M(t)(C+Ct) + CM(t)\eta(t). \tag{84}$$

Clearly, we can then conclude that

$$\eta(t) \leqslant \exp\left(\int_0^t CM(s)\,ds\right)\int_0^t M(s)(C+Cs)\,ds \leqslant C, \tag{85}$$

since $M(t)(1+t) \in L^1(0,\infty)$. Thus, it follows from Eq. (82) that

$$\int_0^1 v^{1-\beta}\, dx \leqslant C + Ct. \tag{86}$$

Consequently, (81) and (86) imply that

$$\begin{aligned}\int_0^1 v(x,t)\, dx &\leqslant C + Ct + Ct^{1/2}\big(t+t^2\big)^{\frac{1}{4}} V(t)^{\beta/2} + CV(t)^{\beta} \\ &\leqslant C + Ct + Ct^{1/2}\big(t^{\frac{1}{4}} + t^{\frac{1}{2}}\big) V(t)^{\beta/2} + CV(t)^{\beta}. \end{aligned} \tag{87}$$

Substituting (87) into (78) we get

$$\begin{aligned} v(x,t) - v(0,t) &\leqslant C\big(C + Ct + C\big(t^{\frac{3}{4}} + t\big)V(t)^{\beta/2} + CV(t)^{\beta}\big)^{\frac{1}{2}} V(t)^{\beta+1/2} \\ &\leqslant Ct\big[1 + V(t)^{\beta/4} + V(t)^{\beta/2}\big]V(t)^{\beta+1/2} \\ &\leqslant Ct \max\big(CV(t)^{(3/2)\beta+1/2}, 3\big), \end{aligned} \tag{88}$$

for sufficiently large t, e.g., $t \geqslant 1$. Here we have also used the inequality $(1 + x^{\beta/4} + x^{\beta/2})x^{\beta+1/2} \leqslant Cx^{(3/2)\beta+1/2}$ which holds for $x \geqslant 1$ and an appropriate constant $C \geqslant 3$. This follows by observing that

$$\begin{aligned} f(x) = Cx^{(3/2)\beta+1/2} - x^{\beta+1/2}\big(1 + x^{\beta/4} + x^{\beta/2}\big) &= x^{\beta+1/2}\big((C-1)x^{\beta/2} - 1 - x^{\beta/4}\big) \\ \geqslant x^{\beta+1/2}\big((C-1)x^{\beta/2} - 1 - x^{\beta/2}\big) &= x^{\beta+1/2}\big((C-2)x^{\beta/2} - 1\big) \geqslant 0, \end{aligned}$$

for $x \geqslant 1$ and $C \geqslant 3$. In conclusion, we have from (88) and Corollary 3.3 that

$$\begin{aligned} V(t) &\leqslant C(1+t)^{1/(\gamma-\beta)} + Ct \max\big(V(t)^{(3/2)\beta+1/2}, 1\big) \\ &\leqslant C(1+t)\big[1 + \max\big(V(t)^{(3/2)\beta+1/2}, 1\big)\big], \end{aligned} \tag{89}$$

since $\gamma - \beta > 1$. From the inequality

$$x \leqslant C(1+t)\big(1 + x^{\xi}\big), \quad 0 < \xi < 1,\ x \geqslant 0,$$

we see that either $x \leqslant 2C(1+t)$ if $x \leqslant 1$ or

$$x \leqslant 2C(1+t)x^{\xi}.$$

That is,

$$x\big(1 - 2C(1+t)x^{\xi-1}\big) \leqslant 0,$$

or

$$\frac{1}{2C(1+t)} \leqslant x^{\xi-1},$$

or

$$x \leqslant C(1+t)^{1/(1-\xi)},$$

for a redefined C. For $\xi = (3/2)\beta + 1/2$ we see that $\beta \in (0, 1/6)$ implies that $1/2 < \xi < 3/4$. Consequently,

$$x \leqslant C(1+t)^{1/(1-\xi)} \leqslant C(1+t)^4. \tag{90}$$

Hence, we conclude that $V(t) \leqslant C(1+t)^4$. $\square$

We follow along the lines of [18,14], and transform the original problem using a new function $w(x,t)$. However, we choose to employ a slightly different definition of $w(x,t)$ than the one used in [14] to account for terms related to well–reservoir interaction. We let

$$\widetilde{u} = u - \int_0^1 u_0(y)\,dy + \int_0^\infty \int_0^1 cAu\,dy\,ds. \tag{91}$$

The model (30), expressed in terms of the variables $(c, Q, \widetilde{u})$, is given by

$$\begin{aligned}
&\partial_t c = kcA, \\
&\partial_t Q + \rho_l Q^2 \partial_x \widetilde{u} = cAQ, \\
&\partial_t \widetilde{u} + \partial_x P(cQ) = -[\widetilde{u} + \widetilde{K}]cA + \partial_x\big(E(Q)\partial_x \widetilde{u}\big),
\end{aligned} \tag{92}$$

where $\widetilde{K} = \int_0^1 u_0(y)\,dy - \int_0^\infty \int_0^1 cAu\,dy\,ds$ is a finite constant. For later use we also note that

$$\begin{aligned}
\int_0^1 \widetilde{u}\,dx &= \int_0^1 [u - u_0]\,dx + \int_0^t \int_0^1 cAu\,dx\,ds + \int_t^\infty \int_0^1 cAu\,dx\,ds \\
&= \int_0^t \int_0^1 u_t\,dx\,ds + \int_0^t \int_0^1 cAu\,dx\,ds + \int_t^\infty \int_0^1 cAu\,dx\,ds \\
&= \int_t^\infty \int_0^1 cAu\,dx\,ds,
\end{aligned} \tag{93}$$

where we have used the last equation of (30) combined with the boundary conditions. Now we introduce a variable w similar to the one used in [14] and given by

$$w(x,t) = \rho_l \widetilde{u}(x,t) - \frac{1}{1+t}\int_0^x \frac{1}{Q}\,dy + \frac{1}{1+t}\int_0^1 \int_0^x \frac{1}{Q}\,dy\,dx. \tag{94}$$

From (94), we observe that

$$w_x = \rho_l \widetilde{u}_x - \frac{1}{1+t}\frac{1}{Q}, \tag{95}$$

and

$$
\begin{aligned}
w_t &= \rho_l \widetilde{u}_t + \frac{1}{(1+t)^2}\int_0^x \frac{1}{Q}\,dy + \frac{1}{1+t}\int_0^x \frac{1}{Q^2}Q_t\,dy \\
&\quad - \frac{1}{(1+t)^2}\int_0^1\int_0^x \frac{1}{Q}\,dy\,dx - \frac{1}{1+t}\int_0^1\int_0^x \frac{1}{Q^2}Q_t\,dy\,dx \\
&= \rho_l \widetilde{u}_t + \frac{1}{(1+t)^2}\int_0^x \frac{1}{Q}\,dy + \frac{1}{1+t}\int_0^x \left[\frac{cA}{Q} - \rho_l \widetilde{u}_x\right]dy \\
&\quad - \frac{1}{(1+t)^2}\int_0^1\int_0^x \frac{1}{Q}\,dy\,dx - \frac{1}{1+t}\int_0^1\int_0^x \left[\frac{cA}{Q} - \rho_l \widetilde{u}_x\right]dy\,dx \\
&= \rho_l \widetilde{u}_t + \frac{1}{(1+t)^2}\int_0^x \frac{1}{Q}\,dy + \frac{1}{1+t}\int_0^x \left[\frac{cA}{Q}\right]dy - \frac{\rho_l}{1+t}\left[\widetilde{u}(x,t) - \widetilde{u}(0,t)\right] \\
&\quad - \frac{1}{(1+t)^2}\int_0^1\int_0^x \frac{1}{Q}\,dy\,dx - \frac{1}{1+t}\int_0^1\int_0^x \left[\frac{cA}{Q}\right]dy\,dx + \frac{\rho_l}{1+t}\int_0^1 \left[\widetilde{u}(x,t) - \widetilde{u}(0,t)\right]dx \\
&= \rho_l \widetilde{u}_t + \frac{1}{(1+t)^2}\int_0^x \frac{1}{Q}dy + \frac{1}{1+t}\int_0^x \left[\frac{cA}{Q}\right]dy - \frac{\rho_l}{1+t}\widetilde{u}(x,t) + \frac{\rho_l}{1+t}\int_t^\infty\int_0^1 cAu\,dx\,ds \\
&\quad - \frac{1}{(1+t)^2}\int_0^1\int_0^x \frac{1}{Q}\,dy\,dx - \frac{1}{1+t}\int_0^1\int_0^x \left[\frac{cA}{Q}\right]dy\,dx \\
&= \rho_l \widetilde{u}_t - \frac{w}{1+t} + \frac{1}{1+t}\int_0^x \left[\frac{cA}{Q}\right]dy - \frac{1}{1+t}\int_0^1\int_0^x \left[\frac{cA}{Q}\right]dy\,dx + \frac{\rho_l}{1+t}\int_t^\infty\int_0^1 cAu\,dx\,ds, \qquad (96)
\end{aligned}
$$

where we have used the second equation of (92) and (93) as well as (94). Consequently, we see that the third equation of (92) takes the following form

$$
\begin{aligned}
&w_t + \frac{w}{1+t} - \frac{1}{1+t}\int_0^x \left[\frac{cA}{Q}\right]dy + \frac{1}{1+t}\int_0^1\int_0^x \left[\frac{cA}{Q}\right]dy\,dx - \frac{\rho_l}{1+t}\int_t^\infty\int_0^1 cAu\,dx\,ds + \rho_l P(cQ)_x \\
&= -\left[w + \frac{1}{1+t}\int_0^x \frac{1}{Q}dy - \frac{1}{1+t}\int_0^1\int_0^x \frac{1}{Q}\,dy\,dx + \rho_l \widetilde{K}\right]cA + \left(E(Q)w_x + \frac{Q^\beta}{1+t}\right)_x.
\end{aligned}
$$

Thus, the system (92) can be formulated as follows in the variables (c, Q, w).

$$
c_t = kcA,
$$

$$
Q_t + Q^2 w_x + \frac{Q}{1+t} = cAQ,
$$

$$
w_t + \frac{w}{1+t} + \rho_l P(cQ)_x - \big(E(Q)w_x\big)_x - \left(\frac{Q^\beta}{1+t}\right)_x
$$

$$
= -wcA + T_A^{(1)} - \int_0^1 T_A^{(1)}\,dx - \widetilde{T}_A^{(1)} + \widetilde{T}_A^{(2)} + T_A^{(3)} - \rho_l cA\widetilde{K}, \tag{97}
$$

where

$$
T_A^{(1)}(x,t) = \frac{1}{1+t}\int_0^x \left[\frac{cA}{Q}\right] dy,
$$

$$
\widetilde{T}_A^{(1)}(x,t) = \frac{cA}{1+t}\int_0^x \frac{1}{Q}\,dy,
$$

$$
\widetilde{T}_A^{(2)}(x,t) = \frac{cA}{1+t}\int_0^1\int_0^x \frac{1}{Q}\,dy\,dx,
$$

$$
T_A^{(3)}(t) = \frac{\rho_l}{1+t}\int_t^\infty\int_0^1 cAu\,dx\,ds.
$$

In the following we will need a considerable stronger assumption on the behavior of the function $A(x,t)$ as $t \to \infty$ in order to handle the new terms associated with well–reservoir dynamics. More precisely, we shall assume that for all times $t > 0$

$$
\int_0^t (1+s)^{\beta+3} M(s)\,ds \leqslant C, \qquad \int_0^t (1+s)^{\beta-1}\int_s^\infty M(\xi)\,d\xi\,ds \leqslant C. \tag{98}
$$

Then we will show that the following energy-type of estimate for the variable w can be obtained.

Lemma 5.2. *Let (n,m,u) be a global weak solution to our problem in the sense of Theorem 2.1. If assumption (98) is satisfied as well as the assumptions of Lemma 5.1, the following estimate holds*:

$$
\frac{1}{2}(1+t)^\beta \int_0^1 w^2\,dx + \frac{(1+t)^{\beta-1}}{1-\beta}\int_0^1 Q^{\beta-1}\,dx + \frac{\rho_l(1+t)^\beta}{\gamma-1}\int_0^1 c^\gamma Q^{\gamma-1}\,dx
$$

$$
+ \left(1-\frac{\beta}{2}\right)\int_0^t (1+s)^{\beta-1}\int_0^1 w^2\,dx\,ds + \int_0^t (1+s)^\beta \int_0^1 Q^{1+\beta} w_x^2\,dx\,ds
$$

$$
+ \rho_l \frac{\gamma-1-\beta}{\gamma-1}\int_0^t (1+s)^{\beta-1}\int_0^1 c^\gamma Q^{\gamma-1}\,dx\,ds \leqslant C. \tag{99}
$$

Proof. We start by multiplying the momentum equation in (97) by w and then integrate it over $[0,1]$ with respect to x. Using integration by parts and exploiting the boundary conditions as well as Eq. (95), we then obtain the equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 w^2\,dx+\frac{1}{1+t}\int_0^1 w^2\,dx+\int_0^1 Q^{\beta+1}w_x^2\,dx\\
&=\frac{-1}{1+t}\int_0^1 Q^{\beta}w_x\,dx+\rho_l\int_0^1 (cQ)^{\gamma}w_x\,dx-\int_0^1 cAw^2\,dx+\int_0^1 wT_A^{(1)}\,dx-\int_0^1 w\widetilde{T}_A^{(1)}\,dx\\
&\quad-\left(\int_0^1 w\,dx\right)\left(\int_0^1 T_A^{(1)}\,dx\right)+\int_0^1 w\widetilde{T}_A^{(2)}\,dx+\int_0^1 wT_A^{(3)}\,dx-\rho_l\widetilde{K}\int_0^1 wcA\,dx\\
&=:\hat{L}_1+\hat{L}_2+H_3+H_4+H_5+H_6+H_7+H_8+H_9.
\end{aligned}\tag{100}$$

We can further manipulate $\hat{L}_1$ and $\hat{L}_2$ such that

$$\hat{L}_1=\frac{-1}{(1-\beta)(1+t)}\int_0^1\left(Q^{\beta-1}\right)_t dx+\frac{1}{(1+t)^2}\int_0^1 Q^{\beta-1}\,dx+H_1,\tag{101}$$

where $H_1=\frac{-1}{1+t}\int_0^1 cAQ^{\beta-1}\,dx$ and where we have used the second equation of (97). Similarly, using the first and second equations of (97), we get

$$\begin{aligned}
\hat{L}_2&=-\frac{\rho_l}{\gamma-1}\int_0^1 c^{\gamma}\left(Q^{\gamma-1}\right)_t dx-\frac{\rho_l}{1+t}\int_0^1 c^{\gamma}Q^{\gamma-1}\,dx+H_2\\
&=-\frac{\rho_l}{\gamma-1}\int_0^1\left(c^{\gamma}Q^{\gamma-1}\right)_t dx-\frac{\rho_l}{1+t}\int_0^1 c^{\gamma}Q^{\gamma-1}\,dx+H_2+H_{10},
\end{aligned}\tag{102}$$

where

$$H_2=\rho_l\int_0^1 c^{\gamma+1}Q^{\gamma-1}A\,dx,\qquad H_{10}=\frac{\rho_l\gamma}{\gamma-1}\int_0^1 kc^{\gamma}Q^{\gamma-1}A\,dx.$$

Now let θ be a real number to be determined later. Combine the results from (101) and (102) with (100), multiply the result by $(1+t)^{\theta}$ and integrate with respect to t over $[0,t]$. Using integration by parts, we then obtain the following equation.

$$\frac{1}{2}(1+t)^{\theta}\int_0^1 w^2\,dx+\frac{(1+t)^{\theta-1}}{1-\beta}\int_0^1 Q^{\beta-1}\,dx+\frac{\rho_l(1+t)^{\theta}}{\gamma-1}\int_0^1 c^{\gamma}Q^{\gamma-1}\,dx$$

$$+\left(1-\frac{\theta}{2}\right)\int_0^t (1+s)^{\theta-1}\int_0^1 w^2\,dx\,ds+\int_0^t (1+s)^{\theta}\int_0^1 Q^{1+\beta}w_x^2\,dx\,ds$$

$$+\frac{\beta-\theta}{1-\beta}\int_0^t (1+s)^{\theta-2}\int_0^1 Q^{\beta-1}\,dx\,ds+\rho_l\frac{\gamma-1-\theta}{\gamma-1}\int_0^t (1+s)^{\theta-1}\int_0^1 c^{\gamma}Q^{\gamma-1}\,dx\,ds$$

$$=\frac{1}{2}\int_0^1 w_0^2\,dx+\frac{1}{1-\beta}\int_0^1 Q_0^{\beta-1}\,dx+\frac{\rho_l}{\gamma-1}\int_0^1 c_0^{\gamma}Q_0^{\gamma-1}\,dx$$

$$+\hat{H}_1+\hat{H}_2+\hat{H}_3+\hat{H}_4+\hat{H}_5+\hat{H}_6+\hat{H}_7+\hat{H}_8+\hat{H}_9+\hat{H}_{10}, \tag{103}$$

where we have that $\hat{H}_i=\int_0^t (1+s)^{\theta}H_i\,ds$ for $i=1,\dots,9$ and

$$\hat{H}_{10}=\int_0^t \frac{\rho_l\gamma(1+s)^{\theta}}{\gamma-1}\left(\int_0^1 c^{\gamma}(1-c)Q^{\gamma-1}A\,dx\right)ds. \tag{104}$$

Furthermore, choosing $\theta=\beta$ and using Lemma 5.1, estimate (36) as well as assumption (98) and the Cauchy and Hölder inequalities, the various terms on the right-hand side of equation (103) can be estimated as follows.

$$\begin{aligned}\hat{H}_1&=-\int_0^t (1+s)^{\beta-1}\left(\int_0^1 cAQ^{\beta-1}\,dx\right)ds\\&\leqslant C\int_0^t (1+s)^{\beta-1}(1+s)^{4(1-\beta)}\left(\int_0^1 |A|\,dx\right)ds\\&\leqslant C\int_0^t (1+s)^{3(1-\beta)}M(s)\,ds\leqslant C,\end{aligned} \tag{105}$$

$$\hat{H}_2=\int_0^t (1+s)^{\beta}\rho_l\int_0^1 c^{\gamma+1}Q^{\gamma-1}A\,dx\,ds\leqslant C\int_0^t (1+s)^{\beta}M(s)\,ds\leqslant C, \tag{106}$$

$$\hat{H}_3=-\int_0^t (1+s)^{\beta}\int_0^1 cAw^2\,dx\,ds\leqslant C\int_0^t (1+s)^{\beta}M(s)\int_0^1 w^2\,dx\,ds, \tag{107}$$

$$\begin{aligned}\hat{H}_4&=\int_0^t (1+s)^{\beta}\int_0^1 wT_A^{(1)}\,dx\,ds=\int_0^t (1+s)^{\beta-1}\int_0^1 w\left(\int_0^x \frac{cA}{Q}\,dy\right)dx\,ds\\&\leqslant C\int_0^t (1+s)^{\beta+3}\int_0^1 w\sup_{x\in[0,1]}\big(|A|\big)\,dx\,ds\end{aligned}$$

$$\leqslant C\int_0^t (1+s)^{\beta+3}M(s)\,ds + C\int_0^t (1+s)^{\beta+3}M(s)\int_0^1 w^2\,dx\,ds$$

$$\leqslant C + C\int_0^t (1+s)^{\beta+3}M(s)\int_0^1 w^2\,dx\,ds, \tag{108}$$

$$\hat{H}_5 = -\int_0^t (1+s)^{\beta}\int_0^1 w\widetilde{T}_A^{(1)}\,dx\,ds = -\int_0^t (1+s)^{\beta-1}\int_0^1 \left(cAw\int_0^x \frac{1}{Q}\,dy\right)dx\,ds$$

$$\leqslant C\int_0^t (1+s)^{\beta+3}\int_0^1 w \sup_{x\in[0,1]}\big(|A|\big)\,dx\,ds$$

$$\leqslant C\int_0^t (1+s)^{\beta+3}M(s)\,ds + C\int_0^t (1+s)^{\beta+3}M(s)\int_0^1 w^2\,dx\,ds$$

$$\leqslant C + C\int_0^t (1+s)^{\beta+3}M(s)\int_0^1 w^2\,dx\,ds, \tag{109}$$

$$\hat{H}_6 = -\int_0^t (1+s)^{\beta}\left(\int_0^1 w\,dx\right)\left(\int_0^1 T_A^{(1)}\,dx\right)ds$$

$$= -\int_0^t (1+s)^{\beta-1}\left(\int_0^1 w\,dx\right)\left(\int_0^1\int_0^x \frac{cA}{Q}\,dy\,dx\right)ds$$

$$\leqslant C\int_0^t (1+s)^{\beta+3}\int_0^1 w \sup_{x\in[0,1]}\big(|A|\big)\,dx\,ds$$

$$\leqslant C\int_0^t (1+s)^{\beta+3}M(s)\,ds + C\int_0^t (1+s)^{\beta+3}M(s)\int_0^1 w^2\,dx\,ds$$

$$\leqslant C + C\int_0^t (1+s)^{\beta+3}M(s)\int_0^1 w^2\,dx\,ds, \tag{110}$$

$$\hat{H}_7 = \int_0^t (1+s)^{\beta}\int_0^1 w\widetilde{T}_A^{(2)}\,dx\,ds$$

$$= \int_0^t (1+s)^{\beta-1}\int_0^1 wcA\left(\int_0^1\int_0^x \frac{1}{Q}\,dy\,dx\right)dx\,ds$$

$$\leqslant C\int_0^t (1+s)^{\beta+3}\int_0^1 w \sup_{x\in[0,1]}\big(|A|\big)\,dx\,ds$$

$$\leqslant C\int_0^t (1+s)^{\beta+3}M(s)\,ds + C\int_0^t (1+s)^{\beta+3}M(s)\int_0^1 w^2\,dx\,ds$$

$$\leqslant C + C\int_0^t (1+s)^{\beta+3}M(s)\int_0^1 w^2\,dx\,ds, \tag{111}$$

$$\hat{H}_8 = \int_0^t (1+s)^{\beta}\int_0^1 wT_A^{(3)}\,dx\,ds$$

$$= \rho_l\int_0^t (1+s)^{\beta-1}\left(\int_s^\infty\int_0^1 cAu\,dx\,ds\right)\left(\int_0^1 w\,dx\right)ds$$

$$\leqslant C\int_0^t (1+s)^{\beta-1}\left(\int_s^\infty M(\xi)\,d\xi\right)\left(\int_0^1 w\,dx\right)ds$$

$$\leqslant C\int_0^t (1+s)^{\beta-1}\left(\int_s^\infty M(\xi)\,d\xi\right)\left(\int_0^1 w^2\,dx\right)^{1/2}ds$$

$$\leqslant C\int_0^t (1+s)^{\beta-1}\int_s^\infty M(\xi)\,d\xi\,ds + C\int_0^t (1+s)^{\beta-1}\int_s^\infty M(\xi)\,d\xi\int_0^1 w^2\,dx\,ds$$

$$\leqslant C + C\int_0^t (1+s)^{\beta-1}\int_s^\infty M(\xi)\,d\xi\int_0^1 w^2\,dx\,ds, \tag{112}$$

$$\hat{H}_9 = -\rho_l\widetilde{K}\int_0^t (1+s)^{\beta}\int_0^1 wcA\,dx\,ds$$

$$\leqslant C\int_0^t (1+s)^{\beta}M(s)\left(\int_0^1 w^2\,dx\right)^{1/2}ds$$

$$\leqslant C\int_0^t (1+s)^{\beta}M(s)\,ds + C\int_0^t (1+s)^{\beta}M(s)\int_0^1 w^2\,dx\,ds$$

$$\leqslant C + C\int_0^t (1+s)^{\beta}M(s)\int_0^1 w^2\,dx\,ds, \tag{113}$$

$$\hat{H}_{10} = \int_0^t \frac{\rho_l \gamma (1+s)^\beta}{\gamma - 1} \left(\int_0^1 c^\gamma (1-c) Q^{\gamma-1} A \, dx \right) ds$$

$$\leqslant C \int_0^t (1+s)^\beta M(s) \, ds \leqslant C. \tag{114}$$

Finally, employing the above estimates in combination with (103), we get

$$\mathrm{LHS}_{(103)} \leqslant C + C \int_0^t \left[(1+s)^3 M(s) + M(s) + (1+s)^{-1} \int_s^\infty M(\xi) \, d\xi \right] (1+s)^\beta \int_0^1 w^2 \, dx \, ds.$$

In particular, it follows that

$$\frac{1}{2}(1+t)^\beta \int_0^1 w^2 \, dx \leqslant C + C \int_0^t k(s)(1+s)^\beta \int_0^1 w^2 \, dx \, ds,$$

where, in view of assumption (98)

$$k(s) = \left[(1+s)^3 M(s) + (1+s)^{-1} \int_s^\infty M(\xi) \, d\xi \right] \in L^1(0, \infty).$$

Thus, application of Gronwall's inequality gives

$$\int_0^t \left[(1+s)^3 M(s) + (1+s)^{-1} \int_s^\infty M(\xi) \, d\xi \right] (1+s)^\beta \int_0^1 w^2 \, dx \, ds \leqslant C.$$

Hence, the result (99) follows. $\square$

Proof of Theorem 2.3. We can now give a proof of Theorem 2.3. First we choose a constant $k = \frac{\gamma-1}{2} + \beta$. We can then write that

$$(cQ)^k(x,t) = (cQ)^k(0,t) + \int_0^x \left((cQ)^k \right)_y dy$$

$$\leqslant C(1+t)^{\frac{-k}{\gamma-\beta}} + C \int_0^x \left((cQ)^{k-\beta} (cQ)^\beta_y \right) dy$$

$$\leqslant C(1+t)^{\frac{-k}{\gamma-\beta}} + C \left(\int_0^1 (cQ)^{2k-2\beta} \, dx \right)^{\frac{1}{2}} \left(\int_0^1 \left((cQ)^\beta_x \right)^2 dx \right)^{\frac{1}{2}}$$

$$\leqslant C(1+t)^{\frac{-k}{\gamma-\beta}} + C\left(\int_0^1 c^\gamma Q^{\gamma-1}\,dx\right)^{\frac{1}{2}}$$

$$\leqslant C(1+t)^{\frac{-k}{\gamma-\beta}} + C(1+t)^{-\frac{\beta}{2}} \leqslant C(1+t)^{-\frac{\beta}{2}}, \tag{115}$$

where we have used Lemmas 3.4, 5.1 and 5.2, Corollaries 3.1 and 3.3, and Hölder's inequality. From this it follows for any $x \in [0,1]$ that

$$(cQ)(x,t) \leqslant C(1+t)^{-\frac{\beta}{2k}} = C(1+t)^{-\frac{\beta}{\gamma-1+2\beta}}, \tag{116}$$

and moreover, due to Eqs. (19) and (28) that $\rho \leqslant CQ \leqslant C(1+t)^{-\frac{\beta}{\gamma-1+2\beta}}$. Hence,

$$n(x,t) \leqslant C(1+t)^{-\frac{\beta}{\gamma-1+2\beta}}, \tag{117}$$

and

$$m(x,t) = [\rho - n](x,t) \leqslant C(1+t)^{-\frac{\beta}{\gamma-1+2\beta}}. \tag{118}$$

This completes the proof of Theorem 2.3. $\square$

Remark 5.1. There seems to be a direct link between the restriction on $\beta \in (0, 1/6)$ and the time decay rate specified in (98). Choosing β to be higher than 1/6 implies that a corresponding faster decay rate appears in the estimate (77) of Lemma 5.1. Consequently, a stronger assumption on time decay rate is required for $M(t)$ in Lemma 5.2, as expressed by (98), in order to get the time-independent estimates. More precisely, from the inequality (90) we see that if $\beta \to 1/3^-$, then $\xi \to 1^-$ and the exponent $1/(1-\xi)$ blows up and we can no longer control terms in the proof of Lemma 5.2 by estimates similar to (98).

Appendix A

In this appendix we estimate the quantities L_1, L_2, L_3, L_4 and L_5, which are used in the proof of Lemma 3.4. First, it is clear from properties of the initial data that

$$L_1 = \frac{1}{2}\int_0^1 \left(Q_0^\beta\right)_x^2\,dx \leqslant C. \tag{119}$$

Estimation of L_2.

$$L_2 = \int_0^1\int_0^t \beta\left(cAQ^\beta\right)_x\left(Q^\beta\right)_x\,ds\,dx = \int_0^1\int_0^t \beta\left((cA)_x Q^\beta + (cA)\left(Q^\beta\right)_x\right)\left(Q^\beta\right)_x\,ds\,dx$$

$$\leqslant C\int_0^t\int_0^1 |c_x||A|\left(Q^\beta\right)_x\,dx\,ds + C\int_0^t\int_0^1 |A_x|\left(Q^\beta\right)_x\,dx\,ds + C\int_0^t\int_0^1 |A|\left(Q^\beta\right)_x^2\,dx\,ds$$

$$\leqslant C\int_0^t\int_0^1 |A|c_x^2\,dx\,ds + C\int_0^t\int_0^1 |A|\left(Q^\beta\right)_x^2\,dx\,ds + C\int_0^t\int_0^1 |A_x|\,dx\,ds + C\int_0^t\int_0^1 |A_x|\left(Q^\beta\right)_x^2\,dx\,ds$$

$$\leqslant C + C\int_0^t\int_0^1 |A|\big(Q^{\beta}\big)_x^2\,dx\,ds + C\int_0^t\int_0^1 |A_x|\big(Q^{\beta}\big)_x^2\,dx\,ds, \tag{120}$$

where we have used the Cauchy inequality as well as Lemma 3.1, Corollary 3.1, and Lemma 3.2.

Estimation of L_3. This estimate is rather comprehensive, and we thus split it into several steps. First, by using integration by parts and (61), we get

$$\begin{aligned} L_3 &= -\int_0^1\int_0^t \rho_l\beta u_t\big(Q^{\beta}\big)_x\,ds\,dx \\ &= -\beta\rho_l\int_0^1\int_0^t \big(u\big(Q^{\beta}\big)_x\big)_t\,ds\,dx + \beta\rho_l\int_0^1\int_0^t u\big(Q^{\beta}\big)_{xt}\,ds\,dx \\ &:= L_{31} + L_{32} + L_{33} + L_{34} + L_{35} + L_{36}, \end{aligned} \tag{121}$$

where we have that

$$L_{31} = -\rho_l\beta\int_0^1 u\big(Q^{\beta}\big)_x\,dx \leqslant C\int_0^1 u^2\,dx + \epsilon\int_0^1 \big(Q^{\beta}\big)_x^2\,dx \leqslant C + \epsilon\int_0^1 \big(Q^{\beta}\big)_x^2\,dx, \tag{122}$$

and

$$L_{32} = \rho_l\beta\int_0^1 u_0\big(Q_0^{\beta}\big)_x\,dx \leqslant C\int_0^1 u_0^2\,dx + C\int_0^1 \big(Q_0^{\beta}\big)_x^2\,dx \leqslant C. \tag{123}$$

Furthermore, by using the equation

$$\big(Q^{\beta}\big)_{tx} = \beta\big(cAQ^{\beta}\big)_x - \beta\rho_l\big(Q^{\beta+1}u_x\big)_x = \beta\big(cAQ^{\beta}\big)_x - \beta\rho_l\big[u_t + P(c,Q)_x + ucA\big],$$

we see that the remaining terms L_{33}, L_{34}, L_{35}, and L_{36} are treated as follows:

$$\begin{aligned} L_{33} &= \rho_l\beta^2\int_0^1\int_0^t u\big(cAQ^{\beta}\big)_x\,ds\,dx \\ &= \rho_l\beta^2\int_0^1\int_0^t u\big[c_xAQ^{\beta} + cA_xQ^{\beta} + cA\big(Q^{\beta}\big)_x\big]\,ds\,dx \\ &\leqslant C\int_0^1\int_0^t |u|\big(|c_x||A| + |A_x| + |A|\big|\big(Q^{\beta}\big)_x\big|\big)\,ds\,dx \\ &\leqslant C\int_0^t\int_0^1 |A|u^2\,dx\,ds + C\int_0^t\int_0^1 |A|c_x^2\,dx\,ds + C\int_0^t\int_0^1 |A_x|u^2\,dx\,ds \end{aligned}$$

$$
+C\int_0^t\int_0^1 |A_x|\,dx\,ds + C\int_0^t\int_0^1 |A|\,dx\,ds + C\int_0^t\int_0^1 |A|\big(Q^{\beta}\big)_x^2\,dx\,ds
$$

$$
\leqslant C + C\int_0^t\int_0^1 |A|\big(Q^{\beta}\big)_x^2\,dx\,ds, \tag{124}
$$

by application of Cauchy's inequality, Corollary 3.1, (35), (36), (53), and the assumptions that $|A|, |A_x| \in L^1(0,\infty)$.

$$
L_{34} = -(\rho_l\beta)^2\int_0^1\int_0^t u u_t\,ds\,dx \leqslant C\int_0^1\int_0^t \big(u^2\big)_t\,ds\,dx = C\int_0^1\big[u^2 - u_0^2\big]\,dx \leqslant C. \tag{125}
$$

We now write

$$
\begin{aligned}
L_{35} &= -(\rho_l\beta)^2\int_0^1\int_0^t u\big([cQ]^{\gamma}\big)_x\,ds\,dx \\
&= (\rho_l\beta)^2\int_0^1\int_0^t u_x[cQ]^{\gamma}\,ds\,dx - (\rho_l\beta)^2\int_0^1\int_0^t \big(u[cQ]^{\gamma}\big)_x\,ds\,dx = L_{351} + L_{352},
\end{aligned} \tag{126}
$$

where

$$
\begin{aligned}
L_{351} &= (\rho_l\beta)^2\int_0^t\int_0^1 (cQ)^{\gamma}u_x\,dx\,ds \\
&\leqslant C\int_0^t\int_0^1 Q^{\beta+1}u_x^2\,dx\,ds + C\int_0^t\int_0^1 c^{\gamma}Q^{\gamma-\beta-1}(cQ)^{\gamma}\,dx\,ds \\
&\leqslant C + C\int_0^t \max_{x\in[0,1]}\big([cQ]^{\gamma}\big)\int_0^1 c^{\gamma}Q^{\gamma-\beta-1}\,dx\,ds \leqslant C,
\end{aligned} \tag{127}
$$

where we have used the Cauchy inequality, Corollary 3.1, Corollary 3.2, (35), (36), and the assumption $\gamma \geq 1+\beta$. Moreover, we get for L_{352} that

$$
\begin{aligned}
L_{352} &= -(\rho_l\beta)^2\int_0^t \big[u(cQ)^{\gamma}\big](1,s)\,ds + (\rho_l\beta)^2\int_0^t \big[u(cQ)^{\gamma}\big](0,s)\,ds \\
&= -(\rho_l\beta)^2\int_0^t (cQ)^{\gamma-\beta}(1,s)\big((cQ)^{\beta}u\big)(1,s)\,ds + (\rho_l\beta)^2\int_0^t (cQ)^{\gamma-\beta}(0,s)\big((cQ)^{\beta}u\big)(0,s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\int_0^t \left|(cQ)^{(\gamma-\beta)\frac{n}{n-1}}(1,s)\right| ds + C\int_0^t \left|(cQ)^{(\gamma-\beta)\frac{n}{n-1}}(0,s)\right| ds \\
&\quad + C\int_0^t \left|\left((cQ)^{n\beta}u^n\right)(1,s)\right| ds + C\int_0^t \left|\left((cQ)^{n\beta}u^n\right)(0,s)\right| ds \\
&\leqslant C + C\int_0^t \left\|(cQ)^{n\beta}u^n\right\|_{L^\infty[0,1]} ds \\
&\leqslant C + C\int_0^t\int_0^1 \left|(cQ)^{n\beta}u^n\right| dx\,ds + C\int_0^t\int_0^1 \left|\left((cQ)^{n\beta}u^n\right)_x\right| dx\,ds \\
&\leqslant C + C\int_0^t\int_0^1 \left|(cQ)^{n\beta}u^n\right| dx\,ds + C\int_0^t\int_0^1 \left|(cQ)^{n\beta-1}(cQ)_x u^n\right| dx\,ds \\
&\quad + C\int_0^t\int_0^1 \left|(cQ)^{n\beta}u^{n-1}u_x\right| dx\,ds \\
&\leqslant C + C\int_0^t\int_0^1 (cQ)^{\gamma}u^{2n}\,dx\,ds + C\int_0^t\int_0^1 (cQ)^{2n\beta-\gamma}\,dx\,ds \\
&\quad + C\int_0^t\int_0^1 (cQ)^{2n\beta-\gamma-\beta}u^{2n}\,dx\,ds + \epsilon\int_0^1\int_0^t \left((cQ)^{\frac{\beta+\gamma}{2}}\right)_x^2 ds\,dx \\
&\quad + C\int_0^t\int_0^1 (cQ)^{\beta+1}u^{2n-2}u_x^2\,dx\,ds + C\int_0^1\int_0^t (cQ)^{2n\beta-\beta-1}\,ds\,dx \\
&\leqslant C + C\int_0^t \max_{[0,1]}\left([cQ]^{\gamma}\right)\left(\int_0^1 u^{2n}\,dx\right) ds + C\int_0^t \max_{[0,1]}\left([cQ]^{\gamma}\right)\left(\int_0^1 (cQ)^{2n\beta-2\gamma}\,dx\right) ds \\
&\quad + C\max_{[0,1]}\left((cQ)^{2n\beta-2\gamma-\beta}\right)\int_0^t \max_{[0,1]}\left([cQ]^{\gamma}\right)\left(\int_0^1 u^{2n}\,dx\right) ds + \epsilon\int_0^1\int_0^t \left((cQ)^{\frac{\beta+\gamma}{2}}\right)_x^2 ds\,dx \\
&\quad + C\int_0^t\int_0^1 Q^{\beta+1}u^{2n-2}u_x^2\,dx\,ds + C\int_0^t \max_{[0,1]}\left([cQ]^{\gamma}\right)\int_0^1 \left((cQ)^{2n\beta-\beta-1-\gamma}\right) dx\,ds.
\end{aligned}
$$

To sum up, we thus have

$$
L_{352} \leqslant C + \epsilon\int_0^1\int_0^t \left((cQ)^{\frac{\beta+\gamma}{2}}\right)_x^2 ds\,dx, \tag{128}
$$

for a sufficiently large integer n. In the above argument, we have used Young's inequality (i.e. $ab \leqslant (1/p)a^p + (1/r)b^q$ where $1/p + 1/r = 1$, with the choice $p = \frac{n}{n-1}$ and $q = n$), the Cauchy (standard version and ϵ-version), Corollary 3.2, Corollary 3.3, estimates (35), (36), (37), and the Sobolev embedding theorem $W^{1,1}(I) \hookrightarrow L^\infty(I)$.

Finally, we get for L_{36} that

$$L_{36} = -(\rho_l \beta)^2 \int_0^1 \int_0^t u^2 c A \, ds \, dx \leqslant C \int_0^1 \int_0^t |A| u^2 \, ds \, dx \leqslant \int_0^t \max_{x \in [0,1]} \big(|A|\big) \int_0^1 u^2 \, dx \, ds \leqslant C, \quad (129)$$

using the assumptions on A and the energy estimate.

Estimation of L_4. By doing some algebraic manipulations we find that

$$\begin{aligned}
L_4 &= -\int_0^1 \int_0^t \rho_l \beta P(c, Q)_x \big(Q^\beta\big)_x \, ds \, dx \\
&= -\rho_l \beta^2 \gamma \int_0^1 \int_0^t c^{\gamma-1} Q^{\gamma+\beta-2} (cQ)_x Q_x \, ds \, dx \\
&= -\rho_l \beta^2 \gamma \int_0^1 \int_0^t c^{\gamma-2} Q^{\gamma+\beta-2} (cQ)_x^2 \, ds \, dx + \rho_l \beta^2 \gamma \int_0^1 \int_0^t c^{\gamma-2} Q^{\gamma+\beta-2} [cQ_x + c_x Q] Q c_x \, ds \, dx \\
&=: L_{41} + L_{42} + L_{43},
\end{aligned} \quad (130)$$

where

$$\begin{aligned}
L_{41} &= -\rho_l \beta^2 \gamma \int_0^1 \int_0^t c^{\gamma-2} Q^{\gamma+\beta-2} (cQ)_x^2 \, ds \, dx \\
&= -\rho_l \beta^2 \gamma \left(\frac{2}{\beta+\gamma}\right)^2 \int_0^1 \int_0^t \frac{1}{c^\beta} \big((cQ)^{\frac{\beta+\gamma}{2}}\big)_x^2 \, ds \, dx.
\end{aligned} \quad (131)$$

Note that this term possesses a constant sign and will appear on the right-hand side of the inequality (60). Moreover,

$$\begin{aligned}
L_{42} &= \rho_l \beta \gamma \int_0^1 \int_0^t c^{\gamma-1} c_x Q^\gamma \big(Q^\beta\big)_x \, ds \, dx = \rho_l \beta \gamma \int_0^1 \int_0^t c^{-1} [cQ]^\gamma c_x \big(Q^\beta\big)_x \, ds \, dx \\
&\leqslant C \int_0^t \int_0^1 [cQ]^\gamma c_x^2 \, dx \, ds + C \int_0^t \int_0^1 [cQ]^\gamma \big(Q^\beta\big)_x^2 \, dx \, ds \\
&\leqslant C + C \int_0^t \max_{x \in [0,1]} \big([cQ]^\gamma\big) \int_0^1 \big(Q^\beta\big)_x^2 \, dx \, ds,
\end{aligned} \quad (132)$$

due to Corollaries 3.1 and 3.2 and Lemma 3.2. Finally,

$$L_{43} = \rho_l \beta^2 \gamma \int_0^1 \int_0^t c^{\gamma-2} c_x^2 Q^{\gamma+\beta} \, ds \, dx$$
$$= \rho_l \beta^2 \gamma \int_0^t \int_0^1 c^{-2} c_x^2 [cQ]^\gamma Q^\beta \, dx \, ds \leqslant C \int_0^t \max_{x\in[0,1]} \left([cQ]^\gamma\right) \int_0^1 c_x^2 \, dx \, ds \leqslant C, \tag{133}$$

due to Corollaries 3.1 and 3.2, Lemma 3.2, and (36).

Estimation of L_5.

$$L_5 = -\int_0^1 \int_0^t \rho_l \beta u c A \left(Q^\beta\right)_x ds \, dx \leqslant C \int_0^t \int_0^1 |A| u^2 \, dx \, ds + C \int_0^t \int_0^1 |A| \left(Q^\beta\right)_x^2 dx \, ds$$
$$\leqslant C \int_0^t \max_{x\in[0,1]} \left(|A|\right) \int_0^1 u^2 \, dx \, ds + C \int_0^t \int_0^1 |A| \left(Q^\beta\right)_x^2 dx \, ds$$
$$\leqslant C + C \int_0^t \int_0^1 |A| \left(Q^\beta\right)_x^2 dx \, ds, \tag{134}$$

due to the Cauchy inequality, the energy estimate and the assumptions on A.

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