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Killing superalgebras for Lorentzian four-manifolds

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Abstract: We determine the Killing superalgebras underpinning field theories with rigid unextended supersymmetry on Lorentzian four-manifolds by re-interpreting them as filtered deformations of $\mathbb{Z}$-graded subalgebras with maximum odd dimension of the $N = 1$ Poincaré superalgebra in four dimensions. Part of this calculation involves computing a Spencer cohomology group which, by analogy with a similar result in eleven dimensions, prescribes a notion of Killing spinor, which we identify with the defining condition for bosonic supersymmetric backgrounds of minimal off-shell supergravity in four dimensions. We prove that such Killing spinors always generate a Lie superalgebra, and that this Lie superalgebra is a filtered deformation of a subalgebra of the $N = 1$ Poincaré superalgebra in four dimensions. Demanding the flatness of the connection defining the Killing spinors, we obtain equations satisfied by the maximally supersymmetric backgrounds. We solve these equations, arriving at the classification of maximally supersymmetric backgrounds whose associated Killing superalgebras are precisely the filtered deformations we classify in this paper.

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Contents

1 Introduction

A number of impressive exact results [1–9] obtained in recent years via supersymmetric localisation have motivated a more systematic exploration of quantum field theories with rigid supersymmetry in curved space. A critical feature in many of these calculations is the non-trivial rôle played by certain non-minimal curvature couplings which regulate correlation functions, so a clear understanding of the general nature of such couplings would be extremely useful.

Several isolated examples of curved backgrounds which support rigid supersymmetry, like spheres and anti-de Sitter spaces (also various products thereof), have been known for some time [10, 11]. Beyond these examples, the most systematic strategy for identifying curved backgrounds which support some amount of rigid supersymmetry has hereto been that pioneered by Festuccia and Seiberg in [12]. In four dimensions, they described how a large class of rigid supersymmetric non-linear sigma-models in curved space can be obtained by taking a decoupling limit (in which the Planck mass goes to infinity) of the corresponding locally supersymmetric theory coupled to minimal off-shell supergravity. In this limit, the gravity supermultiplet is effectively frozen out, leaving only the fixed bosonic supergravity fields as data encoding the geometry of the supersymmetric curved background. Following this paradigm, several other works explored the structure of rigid supersymmetry for field theories in various dimensions on curved manifolds in both Euclidean and Lorentzian signature [13–19].

A well-established feature of supersymmetric supergravity backgrounds is that they possess an associated rigid Lie superalgebra [20–34] that we shall refer to as the Killing superalgebra of the background. Indeed, with respect to an appropriate superspace formalism, the construction described in ([20], section 6.4) (and reviewed in [34]) explains how this Killing superalgebra may be construed in terms of the infinitesimal rigid superisometries of a given background supergeometry. The even part of the Killing superalgebra contains the Killing vectors which preserve the background, whereas the odd part is generated by the rigid supersymmetries supported by the background. The image of the odd-odd bracket for the Killing superalgebra spans a Lie subalgebra of Killing vectors which preserve the background. This Lie subalgebra, together with the rigid supersymmetries, generate an ideal of the Killing superalgebra, which we call the Killing ideal of the background. The utility of this construction is that it often allows one to infer important geometrical properties of the background directly from the rigid supersymmetry it supports. For example, in dimensions six, ten and eleven, it was proved in [35, 36] that any supersymmetric supergravity background possessing more than half the maximal amount of supersymmetry is necessarily (locally) homogeneous.

As a rule, the interactions in a non-linear theory with a local (super)symmetry may be constructed unambiguously by applying the familiar Noether procedure to the linearised version of the theory. Indeed, this is the canonical method for deriving interacting gauge theories in flat space, supergravity theories and their locally supersymmetric couplings to field theory supermultiplets. However, depending on the complexity of the theory in question, it may not be the most wieldy technique and it is sometimes preferable to proceed with some inspired guesswork, perhaps based on the assumption of a particular kind of symmetry (e.g., conformal coupling in a conformal field theory). Either way, the guiding principle is to deform (in some sense) the free theory you know in the most general way that is compatible with the symmetries you wish to preserve.

One way to motivate the construction we shall describe in this paper is as an attempt to streamline the procedure for deducing which curved backgrounds support rigid supersymmetry directly in terms of their associated Killing superalgebras. Instead of applying the Noether method to obtain some complicated local supergravity coupling, taking a rigid limit, looking for supersymmetric backgrounds and then computing the Killing superalgebras of those backgrounds, our strategy will be to simply start with the unextended Poincaré superalgebra (without R-symmetry) and obtain all the relevant Killing superalgebras directly as filtered deformations (see below for the definition) of its subalgebras. As expected for the deformation problem of an algebraic structure, there is a cohomology theory which governs the infinitesimal deformations. In this case this is a generalised Spencer cohomology theory, studied in a similar context by Cheng and Kac in [37, 38]. In the present work, we shall apply this philosophy to the unextended Poincaré superalgebra on $\mathbb{R}^{1,3}$, following a similar analysis on $\mathbb{R}^{1,10}$ pioneered in [39, 40] which yielded what might be considered a Lie-algebraic derivation of eleven-dimensional supergravity.

Let us describe more precisely the problem we set out to solve. Let (V,η) denote the Lorentzian vector space on which four-dimensional Minkowski space is modelled, $\mathfrak{so}(V)$ the Lie algebra of the Lorentz group and S its spinor representation. The associated $N{=}1$ Poincaré superalgebra $\mathfrak{p}$ has underlying vector space $\mathfrak{so}(V)\oplus S\oplus V$ and Lie brackets, for all $A,B\in\mathfrak{so}(V)$, $s\in S$ and $v,w\in V$, given by

$$[A,B]=AB-BA \qquad [A,s]=\sigma(A)s \qquad [A,v]=Av \qquad \text{and} \qquad [s,s]=\kappa(s,s)\,, \tag{1.1}$$

where σ is the spinor representation of $\mathfrak{so}(V)$ and $\kappa:\odot^2 S\to V$ is such that $\kappa(s,s)\in V$ is the Dirac current of s. (This and other relevant notions are defined in the appendix.) The Poincaré superalgebra is $\mathbb{Z}$-graded by assigning degrees 0, -1 and -2 to $\mathfrak{so}(V)$, S and V, respectively and the $\mathbb{Z}_2$ grading is compatible with the $\mathbb{Z}$ grading, in that the parity is the degree mod 2. More precisely, the even subalgebra is the Poincaré algebra $\mathfrak{p}_{\bar 0}=\mathfrak{so}(V)\oplus V$ and the odd subspace is $\mathfrak{p}_{\bar 1}=S$. By a $\mathbb{Z}$-graded subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ we mean a Lie subalgebra $\mathfrak{a}=\mathfrak{a}_0\oplus\mathfrak{a}_{-1}\oplus\mathfrak{a}_{-2}$, with $\mathfrak{a}_i\subset\mathfrak{p}_i$.

Now recall that a Lie superalgebra $\mathfrak{g}$ is said to be *filtered*, if it is admits a vector space filtration

$$\mathfrak{g}^\bullet \ : \qquad \cdots\supset\mathfrak{g}^{-2}\supset\mathfrak{g}^{-1}\supset\mathfrak{g}^{0}\supset\cdots\,,$$

with $\cup_i\mathfrak{g}^i=\mathfrak{g}$ and $\cap_i\mathfrak{g}^i=0$, which is compatible with the Lie bracket in that $[\mathfrak{g}^i,\mathfrak{g}^j]\subset\mathfrak{g}^{i+j}$. Associated canonically to every filtered Lie superalgebra $\mathfrak{g}^\bullet$ there is a graded Lie superalgebra $\mathfrak{g}_\bullet=\bigoplus_i\mathfrak{g}_i$, where $\mathfrak{g}_i=\mathfrak{g}^i/\mathfrak{g}^{i+1}$. It follows from the fact that $\mathfrak{g}^\bullet$ is filtered that $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$, hence $\mathfrak{g}_\bullet$ is graded.

We say that a Lie superalgebra $\mathfrak{g}$ is a *filtered deformation* of $\mathfrak{a}<\mathfrak{p}$ if it is filtered and its associated graded superalgebra is isomorphic (as a graded Lie superalgebra) to $\mathfrak{a}$. If we do not wish to mention the subalgebra $\mathfrak{a}$ explicitly, we simply say that $\mathfrak{g}$ is a *filtered subdeformation* of $\mathfrak{p}$.

The problem we address in this note is the classification of filtered subdeformations $\mathfrak{g}$ of $\mathfrak{p}$ for which $\mathfrak{g}_{-1} = S$ (and hence $\mathfrak{g}_{-2} = V$).

This paper is organised as follows. In section 2 we define and calculate the Spencer cohomology group $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ of the Poincaré superalgebra. This is the main cohomological calculation upon which the rest of our results are predicated. In particular we use it to extract the equation satisfied by the Killing spinors, recovering in this way the form of the (old minimal off-shell) supergravity Killing spinor equation. We will also use this cohomological calculation as a first step on which to bootstrap the calculation of infinitesimal subdeformations of the Poincaré superalgebra. We give two proofs of the main result in section 2 (Proposition 3): a traditional combinatorial proof using gamma matrices and a representation-theoretic proof exploiting the equivariance under $\mathfrak{so}(V)$. In section 3 we prove that the (minimal off-shell) supergravity Killing spinors generate a Lie superalgebra, and that this Lie superalgebra is a filtered subdeformation of $\mathfrak{p}$. These results are contained in Theorem 7 in section 3.2 and Proposition 8 in section 3.3, respectively. In section 4 we classify, up to local isometry, the geometries admitting the maximum number of Killing spinors. We do this by solving the zero curvature equations for the connection relative to which the Killing spinors are parallel, and this is done by first solving for the vanishing of the Clifford trace of the curvature: this simplifies the calculation and might be of independent interest. Section 4.4 contains the result of the classification of maximally supersymmetric backgrounds up to local isometry: apart from Minkowski space and AdS_4, we find the Lie groups admitting a Lorentzian bi-invariant metric. In section 5 we finish the determination of maximally supersymmetric filtered subdeformations of $\mathfrak{p}$ and recover in this way the Killing superalgebras of the maximally supersymmetric backgrounds found in section 4.4. In the case of a Lie group with bi-invariant metric, we note that the Killing ideal is a filtered deformation of $\mathfrak{a} = S \oplus V$ and also explicitly describe all other associated maximally supersymmetric filtered subdeformations of $\mathfrak{p}$. The main result there is Theorem 14 in section 5.4. Finally, in section 6, we offer some conclusions.

Given the nature of this problem, it is inevitable that we shall recover some known results and observations which it would be remiss of us not to contextualise. In particular, in addition to $\mathbb{R}^{1,3}$, our classification of Killing superalgebras for maximally supersymmetric backgrounds yields, up to local isometry, the following conformally flat Lorentzian geometries:

- AdS_4;
- $\mathrm{AdS}_3 \times \mathbb{R}$, with AdS_3 identified with $\mathrm{SL}(2,\mathbb{R})$ with its bi-invariant metric;
- $\mathbb{R} \times \mathrm{S}^3$, with S^3 identified with $\mathrm{SU}(2)$ with its bi-invariant metric; and
- NW_4, a symmetric plane wave isometric to the Nappi-Witten group with its bi-invariant metric.

We prove that the geometries above are indeed realised as the maximally supersymmetric backgrounds of minimal off-shell supergravity in four dimensions, in Lorentzian signature. That is, we do *not* assume the form of the supergravity Killing spinor equation from the

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outset — we actually derive it via Spencer cohomology! It therefore follows that the first three geometries above are precisely the maximally supersymmetric backgrounds obtained in [12]. Indeed, the classification of maximally supersymmetric backgrounds of minimal off-shell supergravity in four dimensions has been discussed in various other contexts in the recent literature, e.g., see [18] (section 2.1), [41] (sections 4.2–3), [34], [42] (p. 2), [43] (pp. 12–13). The NW_4 background is rarely mentioned explicitly — perhaps because, unlike the other maximally supersymmetric Lorentzian backgrounds, it has no counterpart in Euclidean signature — but it is noted in ([42] p. 2) as a plane wave limit, albeit in the context of $N = 2$ supergravity backgrounds. It is also worth pointing out that ([18] section 2.1) contains several useful identities (e.g., integrability conditions and covariant derivatives of Killing spinor bilinears) that we also encounter in our construction of the Killing superalgebra for minimal off-shell supergravity backgrounds.

2 Spencer cohomology

In this section we define and calculate the (even) Spencer cohomology of the Poincaré superalgebra. This calculation has two purposes. The first is to serve as a first step in the classification of filtered subdeformations of the Poincaré superalgebra which is presented in section 5. The second is to derive the equation satisfied by the Killing spinors which, as we show in section 3, generate the filtered subdeformation. The main result, whose proof takes the bulk of the section, is Proposition 3.

2.1 Preliminaries

Let $\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$, where $\mathfrak{p}_{-2} = V$, $\mathfrak{p}_{-1} = S$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$, be the Poincaré superalgebra and $\mathfrak{p}_- = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1}$ the negatively graded part of $\mathfrak{p}$. We will now determine some Spencer cohomology groups associated to $\mathfrak{p}$. We recall that the cochains of the Spencer complex of $\mathfrak{p}$ are linear maps $\wedge^p \mathfrak{p}_- \to \mathfrak{p}$ or, equivalently, elements of $\wedge^p \mathfrak{p}_-^* \otimes \mathfrak{p}$, where $\wedge^\bullet$ is meant here in the super sense, and that the degree in $\mathfrak{p}$ is extended to the space of cochains by declaring that $\mathfrak{p}_p^*$ has degree $-p$. The spaces in the complexes of even cochains of small degree are given in table 1, although for $d = 4$ there are cochains also for $p = 5, 6$ which we omit.

Let $C^{d,p}(\mathfrak{p}_-, \mathfrak{p})$ be the space of p-cochains of degree d. The Spencer differential

$$\partial : C^{d,p}(\mathfrak{p}_-, \mathfrak{p}) \to C^{d,p+1}(\mathfrak{p}_-, \mathfrak{p})$$

is the Chevalley-Eilenberg differential for the Lie superalgebra $\mathfrak{p}_-$ relative to its module $\mathfrak{p}$ with respect to the adjoint action. For $p = 0, 1, 2$ and $d \equiv 0 \pmod 2$ it is explicitly given by the following expressions:

$$\begin{aligned} \partial : C^{d,0}(\mathfrak{p}_-, \mathfrak{p}) &\to C^{d,1}(\mathfrak{p}_-, \mathfrak{p}) \\ \partial\zeta(X) &= [X, \zeta]\,, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \partial : C^{d,1}(\mathfrak{p}_-, \mathfrak{p}) &\to C^{d,2}(\mathfrak{p}_-, \mathfrak{p}) \\ \partial\zeta(X, Y) &= [X, \zeta(Y)] - (-1)^{xy}[Y, \zeta(X)] - \zeta([X, Y])\,, \end{aligned} \tag{2.2}$$

	p				
deg	0	1	2	3	4
0	$\mathfrak{so}(V)$	$S \to S$ $V \to V$	$\odot^2 S \to V$		
2		$V \to \mathfrak{so}(V)$	$\wedge^2 V \to V$ $V \otimes S \to S$ $\odot^2 S \to \mathfrak{so}(V)$	$\odot^3 S \to S$ $\odot^2 S \otimes V \to V$	$\odot^4 S \to V$
4			$\wedge^2 V \to \mathfrak{so}(V)$	$\odot^2 S \otimes V \to \mathfrak{so}(V)$ $\wedge^2 V \otimes S \to S$ $\wedge^3 V \to V$	$\odot^4 S \to \mathfrak{so}(V)$ $\odot^3 S \otimes V \to S$

Table 1. Even p-cochains of small degree.

$$\begin{aligned}
\partial : C^{d,2}(\mathfrak{p}_-, \mathfrak{p}) &\to C^{d,3}(\mathfrak{p}_-, \mathfrak{p}) \\
\partial\zeta(X,Y,Z) &= [X, \zeta(Y,Z)] + (-1)^{x(y+z)}[Y, \zeta(Z,X)] + (-1)^{z(x+y)}[Z, \zeta(X,Y)] \\
&\quad - \zeta([X,Y],Z) - (-1)^{x(y+z)}\zeta([Y,Z],X) - (-1)^{z(x+y)}\zeta([Z,X],Y)\,,
\end{aligned} \tag{2.3}$$

where $x, y, \dots$ are the parity of elements $X, Y, \dots$ of $\mathfrak{p}_-$ and $\zeta \in C^{d,p}(\mathfrak{p}_-, \mathfrak{p})$ with $p = 0, 1, 2$ respectively.

In this section we shall be interested in the groups $H^{d,2}(\mathfrak{p}_-, \mathfrak{p})$ with $d > 0$ and even. We first recall some basic definitions. A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{a} = \bigoplus \mathfrak{a}_p$ with negatively graded part $\mathfrak{a}_- = \bigoplus_{p<0} \mathfrak{a}_p$ is called *fundamental* if $\mathfrak{a}_-$ is generated by $\mathfrak{a}_{-1}$ and *transitive* if for any $X \in \mathfrak{a}_p$ with $p \geq 0$ the condition $[X, \mathfrak{a}_-] = 0$ implies $X = 0$.

Lemma 1. *The Poincaré superalgebra $\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$ is fundamental and transitive. Moreover $H^{d,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$ for all even $d > 2$.*

Proof. The first claim is a direct consequence of the fact that $\kappa(S,S) = V$ and that the natural action of $\mathfrak{so}(V)$ on V is faithful. For any $\zeta \in C^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = \mathrm{Hom}(\wedge^2 V, \mathfrak{so}(V))$ one has

$$\begin{aligned}
\partial\zeta(s_1, s_2, v_1) &= -\zeta(\kappa(s_1, s_2), v_1) \\
\partial\zeta(v_1, v_2, s_1) &= -\sigma(\zeta(v_1, v_2))s_1 \\
\partial\zeta(v_1, v_2, v_3) &= -\zeta(v_2, v_3)v_1 - \zeta(v_3, v_1)v_2 - \zeta(v_1, v_2)v_3
\end{aligned}$$

where $s_1, s_2 \in S$ and $v_1, v_2, v_3 \in V$. The first equation implies $\mathrm{Ker}\, \partial|_{C^{4,2}(\mathfrak{p}_-,\mathfrak{p})} = 0$, since $\mathfrak{p}$ is fundamental, and therefore $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$. Finally $C^{d,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$ and $H^{d,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$ for degree reasons, for all even $d > 4$. □

Note that the space of cochains $C^{d,p}(\mathfrak{p}_-, \mathfrak{p})$ is an $\mathfrak{so}(V)$-module and the same is true for the spaces of cocycles and coboundaries, as ∂ is $\mathfrak{so}(V)$-equivariant. This implies that each cohomology group $H^{d,p}(\mathfrak{p}_-, \mathfrak{p})$ is an $\mathfrak{so}(V)$-module, in a natural way. It remains to compute

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \frac{\ker \partial : C^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \to C^{2,3}(\mathfrak{p}_-, \mathfrak{p})}{\partial C^{2,1}(\mathfrak{p}_-, \mathfrak{p})}$$

and, in particular, to describe its $\mathfrak{so}(V)$-module structure. We consider the decomposition

$$C^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \mathrm{Hom}(\wedge^2 V, V) \oplus \mathrm{Hom}(V \otimes S, S) \oplus \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V))$$

into the direct sum of $\mathfrak{so}(V)$-submodules and write any $\zeta \in C^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ accordingly; i.e., $\zeta = \alpha + \beta + \gamma$ with

$$\begin{aligned} &\alpha \in \mathrm{Hom}(\wedge^2 V, V) \\ &\beta \in \mathrm{Hom}(V \otimes S, S) \\ \text{and} \qquad &\gamma \in \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V)) \,. \end{aligned}$$

We denote the associated $\mathfrak{so}(V)$-equivariant projections by

$$\begin{aligned} &\pi^\alpha : C^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \to \mathrm{Hom}(\wedge^2 V, V) \\ &\pi^\beta : C^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \to \mathrm{Hom}(V \otimes S, S) \\ \text{and} \qquad &\pi^\gamma : C^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \to \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V)) \,. \end{aligned} \tag{2.4}$$

Lemma 2. *The component* $\partial^\alpha = \pi^\alpha \circ \partial : \mathrm{Hom}(V, \mathfrak{so}(V)) \longrightarrow \mathrm{Hom}(\wedge^2 V, V)$ *of the Spencer differential* ∂ *is an isomorphism. In particular,* $\ker \partial|_{C^{2,2}(\mathfrak{p}_-, \mathfrak{p})} = \partial \mathrm{Hom}(V, \mathfrak{so}(V)) \oplus \mathscr{H}^{2,2}$, *where* $\mathscr{H}^{2,2}$ *is the kernel of* ∂ *acting on* $\mathrm{Hom}(V \otimes S, S) \oplus \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V))$, *and every cohomology class* $[\alpha + \beta + \gamma] \in H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ *has a unique cocycle representative with* $\alpha = 0$.

Proof. The image of $\psi \in \mathrm{Hom}(V, \mathfrak{so}(V))$ under ∂^α is given by

$$\partial^\alpha \psi(v_1, v_2) = \psi(v_1) v_2 - \psi(v_2) v_1$$

where $v_1, v_2 \in V$ and the first claim of the lemma follows from classical arguments (see [44]; see also e.g., [39, 45]).

Now for any given $\alpha \in \mathrm{Hom}(\wedge^2 V, V)$, there is a unique $\psi \in \mathrm{Hom}(V, \mathfrak{so}(V))$ such that $\partial \psi = \alpha + \widetilde{\beta} + \widetilde{\gamma}$, for some $\widetilde{\beta} \in \mathrm{Hom}(V \otimes S, S)$ and $\widetilde{\gamma} \in \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V))$. Hence, given any cocycle $\zeta = \alpha + \beta + \gamma$, we may add the coboundary $\partial(-\psi)$ without changing its cohomology class and resulting in the cocycle $(\beta - \widetilde{\beta}) + (\gamma - \widetilde{\gamma})$, which has no component in $\mathrm{Hom}(\wedge^2 V, V)$. This proves the last claim of the lemma. The decomposition $\ker \partial|_{C^{2,2}(\mathfrak{p}_-, \mathfrak{p})} = \partial \mathrm{Hom}(V, \mathfrak{so}(V)) \oplus \mathscr{H}^{2,2}$ is clear. □

2.2 The cohomology group $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$

Lemma 2 gives a canonical identification $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \mathscr{H}^{2,2}$ of $\mathfrak{so}(V)$-modules. Furthermore it follows from equation (2.3) that $\beta + \gamma$ is an element of $\mathscr{H}^{2,2}$ if and only if the following pair of equations are satisfied:

$$\gamma(s, s) v = -2\kappa(s, \beta(v, s)) \qquad \forall\, s \in S, v \in V \,, \tag{2.5}$$

and

$$\sigma(\gamma(s, s)) s = -\beta(\kappa(s, s), s) \qquad \forall\, s \in S \,. \tag{2.6}$$

Note that (2.5) fully expresses γ in terms of β, once the integrability condition that γ takes values in $\mathfrak{so}(V)$ has been taken into account. The solution of the integrability condition and of equation (2.6) is the content of the following

Proposition 3. *Let* $\beta + \gamma \in \mathrm{Hom}(V \otimes S, S) \oplus \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V))$*. Then* $\partial(\beta + \gamma) = 0$ *if and only if there exist* $a, b \in \mathbb{R}$ *and* $\varphi \in V$ *such that*

(i) $\beta(v, s) = v \cdot (a + b\,\mathrm{vol}) \cdot s - \frac{1}{2}(v \cdot \varphi + 3\varphi \cdot v) \cdot \mathrm{vol} \cdot s,$

(ii) $\gamma(s, s)v = -2\kappa(s, \beta(v, s)),$

for all $v \in V$ *and* $s \in S$*. In particular there is a canonical identification*

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \mathscr{H}^{2,2} \simeq 2\mathbb{R} \oplus V$$

of $\mathfrak{so}(V)$*-modules.*

Proof. We find it convenient to work relative to an η-orthonormal basis $(\boldsymbol{e}_\mu)$ for V. In particular the formalism of section A.2.1 is in force, as is the Einstein summation convention.

Let us contract the cocycle condition (2.5) with $w \in V$. The left-hand side becomes

$$\eta(w, \gamma(s, s)(v)) = \gamma(s, s)_{\mu\nu} w^\mu v^\nu \,, \tag{2.7}$$

whereas the right-hand side becomes

$$-2\eta(w, \kappa(s, \beta(v, s))) = -2\,\langle s, w \cdot \beta(v, s)\rangle = -2 w^\mu v^\nu \overline{s}\Gamma_\mu \beta_\nu s\,, \tag{2.8}$$

where we have introduced $\beta_\mu = \beta(\boldsymbol{e}_\mu, -)$. In summary, the first cocycle condition becomes

$$w^\mu v^\nu \left(\gamma(s, s)_{\mu\nu} + 2\overline{s}\Gamma_\mu \beta_\nu s\right) = 0\,, \tag{2.9}$$

which must hold for all $v, w \in V$, so that they can be abstracted to arrive at

$$\gamma(s, s)_{\mu\nu} + 2\overline{s}\Gamma_\mu \beta_\nu s = 0\ . \tag{2.10}$$

Symmetrising $(\mu\nu)$ we obtain the "integrability condition"

$$\overline{s}\Gamma_{(\mu}\beta_{\nu)} s = 0\,, \tag{2.11}$$

whereas skew-symmetrising $[\mu\nu]$ and using that $\gamma(s, s)_{\mu\nu} = -\gamma(s, s)_{\nu\mu}$, we arrive at

$$\gamma(s, s)_{\mu\nu} = -2\overline{s}\Gamma_{[\mu}\beta_{\nu]} s\ . \tag{2.12}$$

Notice that, as advertised, this last equation simply expresses γ in terms of β. Acting on $s \in S$,

$$\begin{aligned} \sigma(\gamma(s, s))s &= -\tfrac{1}{4}\gamma(s, s)_{\mu\nu}\Gamma^{\mu\nu} s \\ &= \tfrac{1}{2}(\overline{s}\Gamma_\mu \beta_\nu s)\Gamma^{\mu\nu} s\,, \end{aligned} \tag{2.13}$$

and inserting this equation into the second cocycle condition (2.6), we arrive at

$$(\overline{s}\Gamma^\mu s)\beta_\mu s + \tfrac{1}{2}(\overline{s}\Gamma_\mu \beta_\nu s)\Gamma^{\mu\nu} s = 0\ . \tag{2.14}$$

So we must solve equations (2.11) and (2.14) for β.

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Since $\mathrm{End}(S) \cong C\ell(V) \cong \wedge^\bullet V$ (where the first isomorphism is one of algebras and the second one of vector spaces), we may write

$$\beta_\mu = \beta^{(0)}_\mu \mathbb{1} + \beta^{(1)}_{\mu\nu}\Gamma^\nu + \tfrac{1}{2}\beta^{(2)}_{\mu\nu\rho}\Gamma^{\nu\rho} + \beta^{(3)}_{\mu\nu}\Gamma^\nu\Gamma_5 + \beta^{(4)}_\mu\Gamma_5\,, \tag{2.15}$$

with $\beta^{(i)}_\mu \in \wedge^i V$ so that

$$\bar{s}\Gamma_\mu\beta_\nu s = \beta^{(0)}_\nu \bar{s}\Gamma_\mu s + \beta^{(1)\rho}_\nu \bar{s}\Gamma_{\mu\rho}s - \beta^{(2)}_{\nu\mu\rho}\bar{s}\Gamma^\rho s + \tfrac{1}{2}\epsilon_{\mu\sigma\tau\rho}\beta^{(3)\rho}_\nu \bar{s}\Gamma^{\sigma\tau}s\,, \tag{2.16}$$

where we have used the last of the duality equations (A.19) and the symmetry relations (A.11).

Inserting this into equation (2.11), which must be true for all $s \in S$, we get that the terms which depend on $\bar{s}\Gamma^\rho s$ and $\bar{s}\Gamma^{\rho\sigma}s$ must vanish separately and we arrive at two equations:

$$\beta^{(0)}_\mu\eta_{\nu\rho} + \beta^{(0)}_\nu\eta_{\mu\rho} - \beta^{(2)}_{\mu\nu\rho} - \beta^{(2)}_{\nu\mu\rho} = 0\,, \tag{2.17}$$

and

$$\eta_{\mu\rho}\beta^{(1)}_{\nu\sigma} + \eta_{\nu\rho}\beta^{(1)}_{\mu\sigma} - \eta_{\mu\sigma}\beta^{(1)}_{\nu\rho} - \eta_{\nu\sigma}\beta^{(1)}_{\mu\rho} + \beta^{(3)\tau}_\mu\epsilon_{\nu\tau\rho\sigma} + \beta^{(3)\tau}_\nu\epsilon_{\mu\tau\rho\sigma} = 0\;. \tag{2.18}$$

Tracing this last equation with $\eta^{\mu\nu}$, we learn that

$$\beta^{(1)}_{[\rho\sigma]} = -\tfrac{1}{2}\beta^{(3)\mu\nu}\epsilon_{\mu\nu\rho\sigma}\,, \tag{2.19}$$

whereas tracing (2.18) with $\eta^{\nu\sigma}$ and using (2.19), results in

$$\beta^{(1)}_{\mu\rho} = a\eta_{\mu\rho} \qquad \text{and} \qquad \beta^{(3)}_{[\mu\rho]} = 0\,, \tag{2.20}$$

for $a = \frac{1}{4}\eta^{\mu\nu}\beta^{(1)}_{\mu\nu} \in \mathbb{R}$.

Substituting the expressions above back into equation (2.18), we find

$$\beta^{(3)\tau}_\mu\epsilon_{\nu\tau\rho\sigma} + \beta^{(3)\tau}_\nu\epsilon_{\mu\tau\rho\sigma} = 0\;. \tag{2.21}$$

Multiplying by $\frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}$, and using the identities (A.20), we obtain

$$-\delta^\alpha_\mu\beta^{(3)\beta}_\nu + \delta^\beta_\mu\beta^{(3)\alpha}_\nu - \delta^\alpha_\nu\beta^{(3)\beta}_\mu + \delta^\beta_\nu\beta^{(3)\alpha}_\mu = 0\;. \tag{2.22}$$

Tracing the expression above with $\eta^{\nu\beta}$, we arrive at

$$\beta^{(3)}_{\mu\alpha} = b\eta_{\mu\alpha}\,, \tag{2.23}$$

for $b = \frac{1}{4}\eta^{\mu\nu}\beta^{(3)}_{\mu\nu} \in \mathbb{R}$.

Tracing equation (2.17) with $\eta^{\mu\nu}$ gives

$$2\beta^{(0)}_\rho - 2\eta^{\mu\nu}\beta^{(2)}_{\mu\nu\rho} = 0\,, \tag{2.24}$$

while tracing it with $\eta^{\nu\rho}$ gives

$$5\beta^{(0)}_\mu + \eta^{\nu\rho}\beta^{(2)}_{\nu\rho\mu} = 0\;. \tag{2.25}$$

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These two equations together imply

$$\beta^{(0)}_\mu = 0\,, \tag{2.26}$$

which, when inserted into equation (2.17), yields

$$\beta^{(2)}_{(\mu\nu)\rho} = 0\ . \tag{2.27}$$

This implies $\beta^{(2)}_{\mu\nu\rho} = \beta^{(2)}_{[\mu\nu\rho]}$ (i.e., $\beta^{(2)} \in \wedge^3 V$), so that it can be parametrised by $\varphi \in V$ such that

$$\beta^{(2)}_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\varphi^\sigma\ . \tag{2.28}$$

In summary, the general solution of equation (2.11) is

$$\beta_\mu = \Gamma_\mu(a + b\Gamma_5) + \varphi^\nu\Gamma_{\mu\nu}\Gamma_5 + \beta^{(4)}_\mu\Gamma_5\,, \tag{2.29}$$

where we have used the the last of the identities (A.19).

Next we solve the second cocycle condition (2.14). Using the expression for β_μ given in equation (2.29), we can rewrite the first term of equation (2.14) as follows:

$$(\bar{s}\Gamma^\mu s)\left(\Gamma_\mu(a + b\Gamma_5) + \varphi^\nu\Gamma_{\mu\nu}\Gamma_5 + \beta^{(4)}_\mu\Gamma_5\right)s, \tag{2.30}$$

where, using that the Dirac current of s Clifford annihilates s (see Proposition 15), the first term vanishes. Similarly, using $\Gamma_{\mu\nu} = -\Gamma_\nu\Gamma_\mu - \eta_{\mu\nu}$ and again the fact that $(\bar{s}\Gamma^\mu s)\Gamma_\mu s = 0$, the first term in equation (2.14) becomes

$$(\bar{s}\Gamma^\mu s)\left(\beta^{(4)}_\mu - \varphi_\mu\right)\Gamma_5 s\ . \tag{2.31}$$

We now rewrite the second term in equation (2.14) by inserting the expression for β_ν in equation (2.29) into equation (2.16) to obtain

$$\tfrac{1}{2}(\bar{s}\Gamma_\mu\beta_\nu s)\Gamma^{\mu\nu}s = \tfrac{1}{2}(\bar{s}\Gamma_{\mu\nu}(a + b\Gamma_5)s)\Gamma^{\mu\nu}s - (\bar{s}\Gamma^\mu s)\varphi_\mu\Gamma_5 s\,, \tag{2.32}$$

where we have again used $\Gamma_{\mu\nu} = -\Gamma_\nu\Gamma_\mu - \eta_{\mu\nu}$ and the fact that $(\bar{s}\Gamma^\mu s)\Gamma_\mu s = 0$. The first term on the right-hand side vanishes by virtue of the fact that the Dirac 2-form of s and its dual both Clifford annihilate s (see Proposition 15). In summary, equation (2.14) becomes

$$(\bar{s}\Gamma^\mu s)\left(\beta^{(4)}_\mu - 2\varphi_\mu\right)\Gamma_5 s = 0\,, \tag{2.33}$$

for all $s \in S$, whose general solution is

$$\beta^{(4)}_\mu = 2\varphi_\mu\ . \tag{2.34}$$

Inserting this into equation (2.29), we arrive at

$$\beta_\mu = \Gamma_\mu(a + b\Gamma_5) + \varphi^\nu\Gamma_{\mu\nu}\Gamma_5 + 2\varphi_\mu\Gamma_5\,,$$

which can be rewritten as

$$\beta_\mu = \Gamma_\mu(a + b\Gamma_5) - \tfrac{1}{2}\varphi^\nu\left(\Gamma_\mu\Gamma_\nu + 3\Gamma_\nu\Gamma_\mu\right)\Gamma_5\,,$$

from where the result follows. □

p	$\mathrm{Hom}(V, \wedge^p V)$
0	$\wedge^1 V$
1	$\wedge^0 V \oplus \wedge^2 V \oplus (V \otimes \wedge^1 V)_0$
2	$2 \wedge^1 V \oplus (V \otimes \wedge^2 V)_0$
3	$\wedge^0 V \oplus \wedge^2 V \oplus (V \otimes \wedge^1 V)_0$
4	$\wedge^1 V$

Table 2. Irreducible components of $\mathrm{Hom}(V, \wedge^p V)$ for $p = 0, \ldots, 4$.

Alternative proof. It may benefit some readers to see an alternative proof of this result, which exploits the equivariance under $\mathfrak{so}(V)$.

Let us consider the first cocycle condition (2.5). Given $\beta \in \mathrm{Hom}(V, \mathrm{End}(S))$ and any $v \in V$ we let $\beta_v \in \mathrm{End}(S)$ to be defined by $\beta_v s = \beta(v, s)$ and rewrite (2.5) as $\gamma(s, s)v = -2\kappa(s, \beta_v s)$. Taking the inner product with v and using (A.13) and (A.9) we arrive at

$$0 = \langle s, v \cdot \beta_v s \rangle \ , \tag{2.35}$$

for all $s \in S$, $v \in V$. In other words, for all $v \in V$, the endomorphism $v \cdot \beta_v$ of S is in $\wedge^2 S = \wedge^0 V \oplus \wedge^3 V \oplus \wedge^4 V$ or, equivalently, it is fixed by the anti-involution ς defined by the symplectic form on S. We claim that the solution space of equation (2.35) is an $\mathfrak{so}(V)$-submodule of $\mathrm{Hom}(V, \mathrm{End}(S))$. To see this, it is convenient to consider the $\mathfrak{so}(V)$-equivariant map

$$\Upsilon : \mathrm{Hom}(V, \mathrm{End}(S)) \to \mathrm{Hom}(\odot^2 V, \mathrm{End}(S))$$

which sends β to $\Upsilon(\beta)$ given by

$$\Upsilon(\beta)(v, w) = v \cdot \beta_w + w \cdot \beta_v \ ,$$

for all $v, w \in V$. We consider also the natural decompositions into $\mathfrak{so}(V)$-submodules

$$\begin{aligned} \mathrm{Hom}(V, \mathrm{End}(S)) &\cong \bigoplus_{p=0}^{4} \mathrm{Hom}(V, \wedge^p V) \ , \\ \mathrm{Hom}(\odot^2 V, \mathrm{End}(S)) &\cong \bigoplus_{q=0}^{4} \mathrm{Hom}(\odot^2 V, \wedge^q V) \ , \end{aligned} \tag{2.36}$$

which are induced by the usual identification $\mathrm{End}(S) = \bigoplus_{p=0}^4 \wedge^p V$. This allows us to write any elements $\beta \in \mathrm{Hom}(V, \mathrm{End}(S))$ and $\theta \in \mathrm{Hom}(\odot^2 V, \mathrm{End}(S))$ as $\beta = \beta_0 + \cdots + \beta_4$ and $\theta = \theta_0 + \cdots + \theta_4$, where $\beta_p \in \mathrm{Hom}(V, \wedge^p V)$ and $\theta_q \in \mathrm{Hom}(\odot^2 V, \wedge^q V)$. The claim then follows from the fact that equation (2.35) is equivalent to $\Upsilon(\beta)_q = 0$ for $q = 1, 2$.

In table 2 above we list the decomposition of $\mathrm{Hom}(V, \wedge^p V)$ for $p = 0, 1, \ldots, 4$ into irreducible $\mathfrak{so}(V)$-modules, with $(V \otimes \wedge^p V)_0$ denoting the kernel of Clifford multiplication $V \otimes \wedge^p V \to \wedge^{p-1} V \oplus \wedge^{p+1} V$.

From the first decomposition in (2.36) we immediately infer that $\mathrm{Hom}(V, \mathrm{End}(S))$ is the direct sum of five different isotypical components, namely

$$2 \wedge^0 V \ , \qquad 4 \wedge^1 V \ , \qquad 2 \wedge^2 V \ , \tag{2.37}$$

and

$$2(V\otimes\wedge^1 V)_0\ ,\qquad (V\otimes\wedge^2 V)_0\ . \tag{2.38}$$

Note now that for any $\Theta,\Theta'\in\wedge^2 S$ the element $\beta\in\mathrm{Hom}(V,\mathrm{End}(S))$ defined by

$$\beta_v s = v\cdot\Theta\cdot s+\Theta'\cdot v\cdot s$$

satisfies

$$\begin{aligned}\varsigma(v\cdot\beta_v) &= -\eta(v,v)\varsigma(\Theta)+\varsigma(v\cdot\Theta'\cdot v)\\ &= -\eta(v,v)\Theta+v\cdot\varsigma(\Theta')\cdot v\\ &= -\eta(v,v)\Theta+v\cdot\Theta'\cdot v\\ &= v\cdot\beta_v\end{aligned}$$

and it is therefore a solution of (2.35). If instead $\Theta,\Theta'\in\odot^2 S=\wedge^1 V\oplus\wedge^2 V$ a similar computation yields $\varsigma(v\cdot\beta_v)=-v\cdot\beta_v$. In summary we get that the solution space of equation (2.35) contains an $\mathfrak{so}(V)$-module isomorphic to

$$\wedge^0 V\oplus 2\wedge^3 V\oplus\wedge^4 V\ , \tag{2.39}$$

where, say, $\Theta\in\wedge^0 V\oplus\wedge^3 V\oplus\wedge^4 V$, $\Theta'\in\wedge^3 V$ and that there exists another submodule which is isomorphic to

$$2\wedge^1 V\oplus 2\wedge^2 V \tag{2.40}$$

and formed by elements which do not satisfy (2.35). Note that the direct sum of (2.39) and (2.40) gives all the isotypical components (2.37) in $\mathrm{Hom}(V,\mathrm{End}(S))$.

We now turn to the remaining isotypical components (2.38). We first recall that $\mathrm{Hom}(V,\mathrm{End}(S))$ contains a single irreducible submodule of type $(V\otimes\wedge^2 V)_0$. We fix an orthonormal basis $(\boldsymbol{e}_\mu)$ of V, consider the element

$$\beta=\boldsymbol{e}_1^\flat\otimes\boldsymbol{e}_2\wedge\boldsymbol{e}_3+\boldsymbol{e}_2^\flat\otimes\boldsymbol{e}_1\wedge\boldsymbol{e}_3\in(V\otimes\wedge^2 V)_0$$

and evaluate

$$\begin{aligned}\tfrac{1}{2}\Upsilon(\beta)(\boldsymbol{e}_1+\boldsymbol{e}_2,\boldsymbol{e}_1+\boldsymbol{e}_2) &= (\boldsymbol{e}_1+\boldsymbol{e}_2)\cdot\beta_{\boldsymbol{e}_1+\boldsymbol{e}_2}\\ &= -(\boldsymbol{e}_1+\boldsymbol{e}_2)\cdot(\boldsymbol{e}_2\wedge\boldsymbol{e}_3)-(\boldsymbol{e}_1+\boldsymbol{e}_2)\cdot(\boldsymbol{e}_1\wedge\boldsymbol{e}_3)\\ &= \imath_{\boldsymbol{e}_2}(\boldsymbol{e}_2\wedge\boldsymbol{e}_3)+\imath_{\boldsymbol{e}_1}(\boldsymbol{e}_1\wedge\boldsymbol{e}_3)\\ &= -2\boldsymbol{e}_3\ .\end{aligned}$$

In other words $\Upsilon(\beta)_1\neq 0$, which implies that $(V\otimes\wedge^2 V)_0$ is not included in the solution space of equation (2.35). Finally any irreducible submodule in $\mathrm{Hom}(V,\mathrm{End}(S))$ isomorphic to $(V\otimes\wedge^1 V)_0$ is given by the image into $\mathrm{Hom}(V,\wedge^1 V)\oplus\mathrm{Hom}(V,\wedge^3 V)$ of an $\mathfrak{so}(V)$-equivariant embedding $\xi\mapsto(r_1\xi,r_2\xi)$, $\xi\in(V\otimes\wedge^1 V)_0$, where $r_1,r_2\in\mathbb{R}$. For instance the image of $\xi=\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1\in(V\otimes\wedge^1 V)_0$ is

$$\begin{aligned}\beta &= r_1(\boldsymbol{e}_1^\flat\otimes\boldsymbol{e}_2+\boldsymbol{e}_2^\flat\otimes\boldsymbol{e}_1)+r_2(\boldsymbol{e}_1^\flat\otimes\star\boldsymbol{e}_2+\boldsymbol{e}_2^\flat\otimes\star\boldsymbol{e}_1)\\ &= r_1(\boldsymbol{e}_1^\flat\otimes\boldsymbol{e}_2+\boldsymbol{e}_2^\flat\otimes\boldsymbol{e}_1)+r_2(-\boldsymbol{e}_1^\flat\otimes\boldsymbol{e}_0\wedge\boldsymbol{e}_1\wedge\boldsymbol{e}_3+\boldsymbol{e}_2^\flat\otimes\boldsymbol{e}_0\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_3)\end{aligned}$$

and we have

$$\begin{aligned}\tfrac{1}{2}\Upsilon(\beta)(\boldsymbol{e}_1,\boldsymbol{e}_1) &= \boldsymbol{e}_1\cdot\beta_{\boldsymbol{e}_1}\\ &= -r_1(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)+r_2(\boldsymbol{e}_1\cdot\boldsymbol{e}_0\wedge\boldsymbol{e}_1\wedge\boldsymbol{e}_3)\\ &= -r_1\boldsymbol{e}_1\wedge\boldsymbol{e}_2 - r_2\boldsymbol{e}_0\wedge\boldsymbol{e}_3\ .\end{aligned}$$

It follows that $\Upsilon(\beta)_2\neq 0$ unless $r_1=r_2=0$ and that the solution space of (2.35) does not contain any submodule isomorphic to $(V\otimes\wedge^1 V)_0$ either.

In summary we just showed that $\beta\in\mathrm{Hom}(V,\mathrm{End}(S))$ solves (2.35) if and only if there exist reals a,b and vectors φ_1,φ_2 such that

$$\beta_v s = v\cdot(a+b\,\mathrm{vol})\cdot s + (v\cdot\varphi_1+\varphi_2\cdot v)\cdot\mathrm{vol}\cdot s\ , \tag{2.41}$$

for all $v\in V$ and $s\in S$.

We now turn to equation (2.6), with β as in (2.41) and γ expressed in terms of β using (2.5). We remark that from the above discussion we already know that $\mathscr{H}^{2,2}$ is identified with an $\mathfrak{so}(V)$-submodule of $2\wedge^0 V\oplus 2\wedge^1 V$.

At this point it is convenient to fix an η-orthonormal basis $(\boldsymbol{e}_\mu)$ of V and use the Einstein summation convention on indices as in appendix A.2.1.

We first introduce

$$\gamma(s,s)_{\mu\nu} = \eta(\boldsymbol{e}_\mu,\gamma(s,s)\boldsymbol{e}_\nu)$$

and note that (2.5) is equivalent to $\gamma(s,s)_{\mu\nu} = -2\overline{s}\Gamma_\mu\beta_\nu s$ where we set $\beta_\mu=\beta_{\boldsymbol{e}_\mu}$. In particular,

$$\begin{aligned}\sigma(\gamma(s,s))s &= -\tfrac{1}{4}\gamma(s,s)_{\mu\nu}\Gamma^{\mu\nu}s\\ &= \tfrac{1}{2}(\overline{s}\Gamma_\mu\beta_\nu s)\Gamma^{\mu\nu}s\ ,\\ \beta(\kappa(s,s),s) &= (\overline{s}\Gamma^\mu s)\beta_\mu s\ ,\end{aligned}$$

and equation (2.6) is equivalent to

$$\tfrac{1}{2}(\overline{s}\Gamma_\mu\beta_\nu s)\Gamma^{\mu\nu}s + (\overline{s}\Gamma^\mu s)\beta_\mu s = 0\ . \tag{2.42}$$

We first show that $\mathscr{H}^{2,2}$ includes the whole isotypical component $2\wedge^0 V$. Indeed if $\beta_v s = av\cdot s$ for some real a then the left-hand side of equation (2.42) is $a(\frac{1}{2}(\overline{s}\Gamma_{\mu\nu}s)\Gamma^{\mu\nu}s+(\overline{s}\Gamma^\mu s)\Gamma_\mu s)$ and both terms are zero separately since $\omega^{(2)}(s,s)\cdot s=\omega^{(1)}(s,s)\cdot s=0$ (see Proposition 15). If $\beta_v s = bv\cdot\mathrm{vol}\cdot s$, for some real b, we also get $b(\frac{1}{2}(\overline{s}\Gamma_{\mu\nu}\Gamma_5 s)\Gamma^{\mu\nu}s+(\overline{s}\Gamma^\mu s)\Gamma_\mu\Gamma_5 s)=0$ since $\star\omega^{(2)}(s,s)\cdot s=\star\omega^{(1)}(s,s)\cdot s=0$ (see again Proposition 15).

Finally, we consider the irreducible submodule in $2\wedge^1 V$ determined by (2.41) and the image of the $\mathfrak{so}(V)$-equivariant embedding $\varphi\mapsto(\varphi_1,\varphi_2)=(r_1\varphi,r_2\varphi)$, where $r_1,r_2\in\mathbb{R}$. In other words, we consider $\beta_v s=(r_1 v\cdot\varphi+r_2\varphi\cdot v)\cdot\mathrm{vol}\cdot s$ and note that equation (2.42) gives

$$\tfrac{1}{2}r_1(\overline{s}\Gamma_\mu\Gamma_\nu\varphi\Gamma_5 s)\Gamma^{\mu\nu}s+\tfrac{1}{2}r_2(\overline{s}\Gamma_\mu\varphi\Gamma_\nu\Gamma_5 s)\Gamma^{\mu\nu}s+r_1(\overline{s}\Gamma^\mu s)\Gamma_\mu\varphi\Gamma_5 s+r_2(\overline{s}\Gamma^\mu s)\varphi\Gamma_\mu\Gamma_5 s=0\ . \tag{2.43}$$

The last term vanishes because $(\overline{s}\Gamma^\mu s)\Gamma_\mu\Gamma_5 s=-\omega^{(3)}\cdot s=0$ (see Proposition 15). The third term is

$$r_1\kappa\cdot\varphi\Gamma_5 s = -r_1\varphi\cdot\kappa\Gamma_5 s - 2r_1\eta(\kappa,\varphi)\Gamma_5 s = -2r_1\eta(\kappa,\varphi)\Gamma_5 s\,,$$

again, using that $\omega^{(3)} \cdot s = 0$. Using equation (A.19) repeatedly, the Clifford relation and Proposition 15 again, we can rewrite the first two terms of (2.43) as

$$-r_1\eta(\kappa,\varphi)\Gamma_5 s + r_2\eta(\kappa,\varphi)\Gamma_5 s\,,$$

turning equation (2.43) into

$$(r_2 - 3r_1)\eta(\kappa,\varphi)\Gamma_5 s = 0\ .$$

Since this must hold for all $\varphi \in \wedge^1 V$ and $s \in S$, it follows that $r_2 = 3r_1$. □

3 Killing superalgebras

In analogy with the results [39, 40] in eleven dimensions, we define a notion of Killing spinor from the component β of the cocycle in Proposition 3. In this section we prove that these Killing spinors generate a Lie superalgebra.

3.1 Preliminaries

Let (M, g, a, b, φ) be a four-dimensional Lorentzian spin manifold (M, g) with spin bundle $S(M)$ which is, in addition, endowed with two functions $a, b \in \mathscr{C}^\infty(M)$ and a vector field $\varphi \in \mathfrak{X}(M)$. The main aim of this section is to construct a Lie superalgebra $\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$ naturally associated with (M, g, a, b, φ).

Motivated by (i) of Proposition 3 we introduce the connection

$$D_X\varepsilon := \nabla_X\varepsilon - X\cdot(a + b\,\mathrm{vol})\cdot\varepsilon + (\varphi\wedge X)\cdot\mathrm{vol}\cdot\varepsilon - 2g(\varphi, X)\,\mathrm{vol}\cdot\varepsilon \tag{3.1}$$

on $S(M)$, where ∇ is the Levi-Civita connection of (M, g), $X \in \mathfrak{X}(M)$, $\varepsilon \in \Gamma(S(M))$.

Definition 4. *A section ε of $S(M)$ is called a* Killing spinor *if $D_X\varepsilon = 0$ for all $X \in \mathfrak{X}(M)$.*

Note that any non-zero Killing spinor is nowhere vanishing since it is parallel with respect to a connection on the spinor bundle. We set

$$\begin{aligned}\mathfrak{k}_{\bar 0} &= \{X \in \mathfrak{X}(M) \mid \mathscr{L}_X g = \mathscr{L}_X a = \mathscr{L}_X b = \mathscr{L}_X\varphi = 0\}\ ,\\ \mathfrak{k}_{\bar 1} &= \{\varepsilon \in \Gamma(S(M)) \mid D_X\varepsilon = 0\ \text{ for all }\ X \in \mathfrak{X}(M)\}\ ,\end{aligned} \tag{3.2}$$

and consider the operation $[-,-] : \mathfrak{k}\otimes\mathfrak{k} \to \mathfrak{k}$ compatible with the parity of $\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$ and determined by the following maps:

- $[-,-] : \mathfrak{k}_{\bar 0}\otimes\mathfrak{k}_{\bar 0} \to \mathfrak{k}_{\bar 0}$ is given by the usual commutator of vector fields,
- $[-,-] : \mathfrak{k}_{\bar 1}\otimes\mathfrak{k}_{\bar 1} \to \mathfrak{k}_{\bar 0}$ is a symmetric map, with $[\varepsilon,\varepsilon] = \kappa(\varepsilon,\varepsilon)$ given by the Dirac current of $\varepsilon \in \mathfrak{k}_{\bar 1}$,
- $[-,-] : \mathfrak{k}_{\bar 0}\otimes\mathfrak{k}_{\bar 1} \to \mathfrak{k}_{\bar 1}$ is given by the spinorial Lie derivative of Lichnerowicz and Kosmann (see [46] and also, e.g., [47]).

The fact that $[-,-]$ actually takes values in $\mathfrak{k}$ is a consequence of Theorem 7 below, where we show that $[-,-]$ is the bracket of a Lie superalgebra structure on $\mathfrak{k}$. Assuming that result for the moment we make the following

Definition 5. *The pair* $(\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}, [-,-])$ *is called the* Killing superalgebra *associated with* (M, g, a, b, φ).

We recall that the spinorial Lie derivative of a spinor field ε along a Killing vector field X is defined by $\mathscr{L}_X\varepsilon = \nabla_X\varepsilon + \sigma(A_X)\varepsilon$, where $\sigma : \mathfrak{so}(TM) \to \mathrm{End}(S(M))$ is the spin representation and $A_X = -\nabla X \in \mathfrak{so}(TM)$. It enjoys the following basic properties, for all Killing vectors X, Y, spinors ε, functions f and vector fields Z:

(i) $\mathscr{L}_X$ is a derivation:

$$\mathscr{L}_X(f\varepsilon) = X(f)\varepsilon + f\mathscr{L}_X\varepsilon \ ;$$

(ii) $X \mapsto \mathscr{L}_X$ is a representation of the Lie algebra of Killing vector fields:

$$\mathscr{L}_X(\mathscr{L}_Y\varepsilon) - \mathscr{L}_Y(\mathscr{L}_X\varepsilon) = \mathscr{L}_{[X,Y]}\varepsilon \ ;$$

(iii) $\mathscr{L}_X$ is compatible with Clifford multiplication:

$$\mathscr{L}_X(Z \cdot \varepsilon) = [X, Z] \cdot \varepsilon + Z \cdot \mathscr{L}_X\varepsilon \ ;$$

(iv) $\mathscr{L}_X$ is compatible with the Levi-Civita connection:

$$\mathscr{L}_X(\nabla_Z\varepsilon) = \nabla_{[X,Z]}\varepsilon + \nabla_Z(\mathscr{L}_X\varepsilon) \ .$$

We note for later use that, from property (iii) and the fact that $\odot^2 S = \wedge^1 V \oplus \wedge^2 V$, we have for any Killing vector X, spinor ε and vector field Z,

$$\begin{aligned} g([X, \kappa(\varepsilon,\varepsilon)], Z) &= X(g(\kappa(\varepsilon,\varepsilon), Z)) - g(\kappa(\varepsilon,\varepsilon), [X, Z]) \\ &= X(\langle\varepsilon, Z\cdot\varepsilon\rangle) - \langle\varepsilon, [X,Z]\cdot\varepsilon\rangle \\ &= 2\,\langle\nabla_X\varepsilon, Z\cdot\varepsilon\rangle + \langle\varepsilon, \nabla_Z X\cdot\varepsilon\rangle \\ &= 2\,\langle\nabla_X\varepsilon, Z\cdot\varepsilon\rangle + 2\,\langle\varepsilon, Z\cdot\sigma(A_X)\varepsilon\rangle \\ &= 2g(\kappa(\mathscr{L}_X\varepsilon, \varepsilon), Z) \ , \end{aligned}$$

which yields the following additional property of the spinorial Lie derivative:

(v) the Dirac current is equivariant under the action of Killing vector fields:

$$[X, \kappa(\varepsilon,\varepsilon)] = 2\kappa(\mathscr{L}_X\varepsilon, \varepsilon) \ .$$

We first collect a series of important auxiliary results, which will be needed in the proof of the main Theorem 7.

Proposition 6. *Let* ε *be a non-zero section of the spinor bundle* $S(M)$ *of* (M, g, a, b, φ), *with associated differential forms*

- $\omega^{(1)} \in \Omega^1(M)$, *where* $\omega^{(1)}(X) = \langle \varepsilon, X \cdot \varepsilon \rangle$,
- $\omega^{(2)} \in \Omega^2(M)$, *where* $\omega^{(2)}(X,Y) = \langle \varepsilon, (X \wedge Y) \cdot \varepsilon \rangle$,
- $\widetilde{\omega}^{(2)} = - \star \omega^{(2)} \in \Omega^2(M)$, *where* $\widetilde{\omega}^{(2)}(X,Y) = \langle \varepsilon, (X \wedge Y) \cdot \mathrm{vol} \cdot \varepsilon \rangle$,
- $\omega^{(3)} = - \star \omega^{(1)} \in \Omega^3(M)$, *where* $\omega^{(3)}(X,Y,Z) = \langle \varepsilon, (X \wedge Y \wedge Z) \cdot \mathrm{vol} \cdot \varepsilon \rangle$,

for all $X, Y, Z \in \mathfrak{X}(M)$. *If* ε *is a Killing spinor then*

(i) $d\omega^{(1)} = -4a\omega^{(2)} - 4b\widetilde{\omega}^{(2)} - 4\imath_\varphi \omega^{(3)}$,

(ii) $d\omega^{(2)} = 6b\omega^{(3)}$,

(iii) $d\widetilde{\omega}^{(2)} = -6a\omega^{(3)}$,

(iv) $d\omega^{(3)} = 0$.

In particular the Dirac current $K = \kappa(\varepsilon, \varepsilon)$ *of* ε *is a Killing vector field satisfying*

$$\mathscr{L}_K a = \mathscr{L}_K b = \mathscr{L}_K \omega^{(1)} = \mathscr{L}_K \omega^{(2)} = \mathscr{L}_K \widetilde{\omega}^{(2)} = \mathscr{L}_K \omega^{(3)} = 0 \tag{3.3}$$

and

$$\begin{aligned} 0 = &-2\widetilde{\omega}^{(2)}(Z,X) g(\mathscr{L}_K\varphi, Y) + 2\widetilde{\omega}^{(2)}(Z,Y) g(\mathscr{L}_K\varphi, X) - 2\widetilde{\omega}^{(2)}(\mathscr{L}_K\varphi, Y) g(Z,X) \\ &+ 2\widetilde{\omega}^{(2)}(\mathscr{L}_K\varphi, X) g(Z,Y) + 4\widetilde{\omega}^{(2)}(X,Y) g(\mathscr{L}_K\varphi, Z) \ , \end{aligned} \tag{3.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proof. For any Killing spinor ε and $X, Y, Z \in \mathfrak{X}(M)$ we compute

$$\begin{aligned} (\nabla_Z \omega^{(1)})(X) &= 2 \langle \varepsilon, X \cdot \nabla_Z \varepsilon \rangle \\ &= 2a \langle \varepsilon, X \wedge Z \cdot \varepsilon \rangle + 2b \langle \varepsilon, X \wedge Z \cdot \mathrm{vol} \cdot \varepsilon \rangle + 2 \langle \varepsilon, \varphi \wedge X \wedge Z \cdot \mathrm{vol} \cdot \varepsilon \rangle \ , \end{aligned}$$

and

$$\begin{aligned} (\nabla_Z \omega^{(2)})(X,Y) &= 2 \langle \varepsilon, X \wedge Y \cdot \nabla_Z \varepsilon \rangle \\ &= 2a \langle \varepsilon, X \wedge Y \cdot Z \cdot \varepsilon \rangle + 2b \langle \varepsilon, X \wedge Y \cdot Z \cdot \mathrm{vol} \cdot \varepsilon \rangle \\ &\quad - 2 \langle \varepsilon, X \wedge Y \cdot \varphi \wedge Z \cdot \mathrm{vol} \cdot \varepsilon \rangle + 4g(\varphi, Z) \langle \varepsilon, X \wedge Y \cdot \mathrm{vol} \cdot \varepsilon \rangle \\ &= 2ag(Z,X) \langle \varepsilon, Y \cdot \varepsilon \rangle - 2ag(Z,Y) \langle \varepsilon, X \cdot \varepsilon \rangle + 2b \langle \varepsilon, Z \wedge X \wedge Y \cdot \mathrm{vol} \cdot \varepsilon \rangle \\ &\quad + 2g(\varphi, Y) \langle \varepsilon, X \wedge Z \cdot \mathrm{vol} \cdot \varepsilon \rangle - 2g(\varphi, X) \langle \varepsilon, Y \wedge Z \cdot \mathrm{vol} \cdot \varepsilon \rangle \\ &\quad - 2g(Z,Y) \langle \varepsilon, X \wedge \varphi \cdot \mathrm{vol} \cdot \varepsilon \rangle + 2g(X,Z) \langle \varepsilon, Y \wedge \varphi \cdot \mathrm{vol} \cdot \varepsilon \rangle \\ &\quad + 4g(\varphi, Z) \langle \varepsilon, X \wedge Y \cdot \mathrm{vol} \cdot \varepsilon \rangle \ , \end{aligned}$$

where, in both cases, the last equality follows from equation (A.11) or, equivalently, that $\odot^2 S = \wedge^1 V \oplus \wedge^2 V$. In other words we have

$$\nabla_Z \omega^{(1)} = -2a\imath_Z \omega^{(2)} - 2b\imath_Z \widetilde{\omega}^{(2)} - 2\imath_Z \imath_\varphi \omega^{(3)} \ , \tag{3.5}$$

$$\nabla_Z \omega^{(2)} = 2aZ \wedge \omega^{(1)} + 2b\imath_Z \omega^{(3)} - 2\imath_Z \widetilde{\omega}^{(2)} \wedge \varphi - 2Z \wedge \imath_\varphi \widetilde{\omega}^{(2)} + 4g(\varphi, Z)\widetilde{\omega}^{(2)} \ , \tag{3.6}$$

and applying $\star$, which is a parallel endomorphism of $\Omega^\bullet(M)$, on both sides of these identities we also get

$$\nabla_Z \widetilde{\omega}^{(2)} = 2bZ \wedge \omega^{(1)} - 2a\imath_Z\omega^{(3)} + 2Z \wedge \imath_\varphi \omega^{(2)} + 2\imath_Z\omega^{(2)} \wedge \varphi - 4g(\varphi, Z)\omega^{(2)} \ , \tag{3.7}$$

$$\nabla_Z \omega^{(3)} = 2aZ \wedge \widetilde{\omega}^{(2)} - 2bZ \wedge \omega^{(2)} + 2Z \wedge \varphi \wedge \omega^{(1)} \ . \tag{3.8}$$

Claims (i)-(iv) follows then immediately from the fact that for any $\omega \in \Omega^p(M)$ we have

$$d\omega = \sum_{\mu=0}^{3} e^\mu \wedge \nabla_{e_\mu}\omega \qquad \text{and} \qquad \sum_{\mu=0}^{3} e^\mu \wedge \imath_{e_\mu}\omega = p\omega \ ,$$

where (e_μ) is a fixed local orthonormal frame field of (M, g).

Now, for any Killing spinor ε and $X, Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned} g(\nabla_X K, Y) &= 2\langle \varepsilon, Y \cdot \nabla_X \varepsilon\rangle \\ &= 2a\langle \varepsilon, Y \cdot X \cdot \varepsilon\rangle + 2b\langle \varepsilon, Y \cdot X \cdot \mathrm{vol} \cdot \varepsilon\rangle - 2\langle \varepsilon, Y \cdot (\varphi \wedge X) \cdot \mathrm{vol} \cdot \varepsilon\rangle \\ &\quad + 4g(\varphi, X)\langle \varepsilon, Y \cdot \mathrm{vol} \cdot \varepsilon\rangle \\ &= 2a\langle \varepsilon, (Y \wedge X) \cdot \varepsilon\rangle + 2b\langle \varepsilon, (Y \wedge X) \cdot \mathrm{vol} \cdot \varepsilon\rangle + 2\langle \varepsilon, (\varphi \wedge Y \wedge X) \cdot \mathrm{vol} \cdot \varepsilon\rangle \end{aligned}$$

where the last equality follows from equation (A.11). Since the last term is manifestly skewsymmetric in X and Y we have that K is a Killing vector. From $d\omega^{(3)} = 0$, we also have

$$0 = d(d\omega^{(2)}) = 6db \wedge \omega^{(3)} = -6db \wedge \star\omega^{(1)} = -6(\imath_K db)\,\mathrm{vol} \ ;$$

i.e., $\mathscr{L}_K b = 0$. One shows $\mathscr{L}_K a = 0$ in a similar way. If $\omega = \omega^{(1)}, \omega^{(2)}, \widetilde{\omega}^{(2)}$, or $\omega^{(3)}$, then $\imath_K\omega = 0$ by Proposition 15 and from (i)-(iv) we get

$$\mathscr{L}_K\omega = d\imath_K\omega + \imath_K d\omega = 0 \ .$$

This proof of (3.3) is thus completed.

In order to show (3.4) we use that K is a Killing vector and $\mathscr{L}_K\omega^{(2)} = 0$ so that for all $X, Y, Z \in \mathfrak{X}(M)$:

$$\begin{aligned} 0 &= (\mathscr{L}_K \nabla_Z \omega^{(2)})(X, Y) - (\nabla_{[K,Z]}\omega^{(2)})(X, Y) \\ &= \mathscr{L}_K((\nabla_Z\omega^{(2)})(X, Y)) - (\nabla_{[K,Z]}\omega^{(2)})(X, Y) - \nabla_Z\omega^{(2)}([K, X], Y) - \nabla_Z\omega^{(2)}(X, [K, Y]) \\ &= -2\widetilde{\omega}^{(2)}(Z, X)g(\mathscr{L}_K\varphi, Y) + 2\widetilde{\omega}^{(2)}(Z, Y)g(\mathscr{L}_K\varphi, X) - 2\widetilde{\omega}^{(2)}(\mathscr{L}_K\varphi, Y)g(Z, X) \\ &\quad + 2\widetilde{\omega}^{(2)}(\mathscr{L}_K\varphi, X)g(Z, Y) + 4\widetilde{\omega}^{(2)}(X, Y)g(\mathscr{L}_K\varphi, Z) \ , \end{aligned}$$

where the last identity follows from a direct computation using (3.6) and (3.3). $\square$

3.2 The Killing superalgebra

We state and prove the main result of section 3.

Theorem 7. *Let $X, Y \in \mathfrak{k}_{\bar{0}}$ and $\varepsilon \in \mathfrak{k}_{\bar{1}}$. Then $[X, Y] \in \mathfrak{k}_{\bar{0}}$, $\kappa(\varepsilon, \varepsilon) \in \mathfrak{k}_{\bar{0}}$ whereas $\mathscr{L}_X\varepsilon \in \mathfrak{k}_{\bar{1}}$. Moreover, $[-, -]$ defines a Lie superalgebra on $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$.*

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Proof. The fact that $[\mathfrak{k}_{\bar 0},\mathfrak{k}_{\bar 0}]\subset\mathfrak{k}_{\bar 0}$ follows from basic properties of Lie derivatives of vector fields. On the other hand for any $X\in\mathfrak{k}_{\bar 0}$ and $Z\in\mathfrak{X}(M)$ we have that

$$[\mathscr{L}_X, D_Z] = D_{[X,Z]}\,,$$

since D depends solely on the data (g,a,b,φ) which is preserved by $X\in\mathfrak{k}_{\bar 0}$. This shows that $\mathscr{L}_X\varepsilon$ is a Killing spinor or, in other words, that $[\mathfrak{k}_{\bar 0},\mathfrak{k}_{\bar 1}]\subset\mathfrak{k}_{\bar 1}$.

We already know from Proposition 6 that $K=\kappa(\varepsilon,\varepsilon)$ is a Killing vector field which satisfies $\mathscr{L}_K a=\mathscr{L}_K b=0$. To prove $K\in\mathfrak{k}_{\bar 0}$ we still need to show $\mathscr{L}_K\varphi=0$. From Proposition 6 we have

$$\begin{aligned}0=-\frac{1}{4}d(d\omega^{(1)}) &= da\wedge\omega^{(2)}+6ab\omega^{(3)}+db\wedge\widetilde{\omega}^{(2)}-6ab\omega^{(3)}+d\imath_\varphi\omega^{(3)}\\ &= da\wedge\omega^{(2)}+db\wedge\widetilde{\omega}^{(2)}-\mathscr{L}_\varphi\star\omega^{(1)}\end{aligned}$$

and hence, for any $\vartheta\in\Omega^1(M)$,

$$\begin{aligned}\vartheta\wedge\star\mathscr{L}_\varphi\omega^{(1)} &= \vartheta\wedge\mathscr{L}_\varphi\star\omega^{(1)}-\operatorname{div}(\varphi)\vartheta\wedge\star\omega^{(1)}-(\mathscr{L}_\varphi g)(\vartheta,\omega^{(1)})\,\mathrm{vol}\\ &= \vartheta\wedge da\wedge\omega^{(2)}+\vartheta\wedge db\wedge\widetilde{\omega}^{(2)}+\operatorname{div}(\varphi)\vartheta\wedge\omega^{(3)}-(\mathscr{L}_\varphi g)(\vartheta,\omega^{(1)})\,\mathrm{vol}\;.\end{aligned}\tag{3.9}$$

In the special case where $\imath_K\vartheta=0$ the first three terms of the right-hand side of the above identity are degenerate 4-forms and hence zero. Then equation (3.9) becomes

$$\begin{aligned}0&=\vartheta\wedge\star\mathscr{L}_\varphi\omega^{(1)}+(\mathscr{L}_\varphi g)(\vartheta,\omega^{(1)})\,\mathrm{vol}\\ &=-g(\mathscr{L}_\varphi\vartheta,\omega^{(1)})\,\mathrm{vol}\\ &=-(\mathscr{L}_\varphi\vartheta)(K)\,\mathrm{vol}\\ &=\vartheta(\mathscr{L}_\varphi K)\,\mathrm{vol}\\ &=-\vartheta(\mathscr{L}_K\varphi)\,\mathrm{vol}\,,\end{aligned}$$

so that $\mathscr{L}_K\varphi=fK$, for some $f\in\mathscr{C}^\infty(M)$. From this fact, equation (3.4) and $\omega^{(1)}\wedge\widetilde{\omega}^{(2)}=0$ we finally get

$$\begin{aligned}0&=f(2\widetilde{\omega}^{(2)}(X,Y)\omega^{(1)}(Z)+\widetilde{\omega}^{(2)}(X,Z)\omega^{(1)}(Y)+\widetilde{\omega}^{(2)}(Z,Y)\omega^{(1)}(X))\\ &=3f\widetilde{\omega}^{(2)}(X,Y)\omega^{(1)}(Z)\;,\end{aligned}$$

for all $X,Y,Z\in\mathfrak{X}(M)$, hence $f=0$. This proves $\mathscr{L}_K\varphi=0$ and $[\mathfrak{k}_{\bar 1},\mathfrak{k}_{\bar 1}]\subset\mathfrak{k}_{\bar 0}$.

We finally show that $[-,-]:\mathfrak{k}\otimes\mathfrak{k}\to\mathfrak{k}$ satisfies the axioms of a Lie superalgebra. This is a direct consequence of the following observations:

(i) $\mathfrak{k}_{\bar 0}$ is a Lie algebra: this is just the Jacobi identity of the Lie bracket of vector fields;

(ii) $\mathfrak{k}_{\bar 0}$ acts on $\mathfrak{k}_{\bar 1}$, by property (ii) of the spinorial Lie derivative;

(iii) the Dirac current is a symmetric $\mathfrak{k}_{\bar 0}$-equivariant map, by property (v) of the spinorial Lie derivative;

(iv) for any $\varepsilon \in \mathfrak{k}_{\bar{1}}$, with associated Dirac current $K = \kappa(\varepsilon, \varepsilon)$, we have from the definition of Killing spinor and (3.5) that

$$\begin{aligned}\mathscr{L}_K \varepsilon &= \nabla_K \varepsilon + \sigma(A_K)\varepsilon \\ &= -(\varphi \wedge K)\operatorname{vol}\cdot\varepsilon + 2g(\varphi, K)\operatorname{vol}\cdot\varepsilon + \imath_\varphi \omega^{(3)} \cdot \varepsilon \\ &= g(\varphi, K)\operatorname{vol}\cdot\varepsilon + \imath_\varphi \omega^{(3)} \cdot \varepsilon \\ &= -\varphi \cdot \omega^{(3)} \cdot \varepsilon \\ &= 0\,,\end{aligned}$$

where the last equality holds by Proposition 15. This is equivalent to the component of the Jacobi identity for $\mathfrak{k}$ with three odd elements.

The proof is thus completed. □

3.3 The Killing superalgebra is a filtered deformation

We now show that the Killing superalgebra $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ is a filtered deformation of a $\mathbb{Z}$-graded subalgebra of the Poincaré superalgebra $\mathfrak{p}$. To this aim, it is convenient to denote the triple (a, b, φ) collectively by Φ and to abbreviate the Killing spinor equation as $\nabla_Z \varepsilon = \beta^\Phi_Z \varepsilon$, where β^Φ is the $\operatorname{End}(S(M))$-valued one-form defined by

$$\beta^\Phi_Z \varepsilon = Z \cdot (a + b\operatorname{vol}) \cdot \varepsilon - (\varphi \wedge Z) \cdot \operatorname{vol}\cdot\varepsilon + 2g(\varphi, Z)\operatorname{vol}\cdot\varepsilon\,, \tag{3.10}$$

for all $Z \in \mathfrak{X}(M)$ and $\varepsilon \in \Gamma(S(M))$. The notation is chosen to make contact with that of Proposition 3. The reason for the superscript Φ is to distinguish β^Φ from the more general component β of the filtered Lie brackets in (3.15) below. For a similar reason we also introduce the $\mathfrak{so}(TM)$-valued symmetric bilinear tensor γ^Φ on $S(M)$ given by

$$\gamma^\Phi(\varepsilon, \varepsilon)(Z) = -2k(\beta^\Phi_Z \varepsilon, \varepsilon)\,,$$

for all $Z \in \mathfrak{X}(M)$ and $\varepsilon \in \Gamma(S(M))$.

Let $\mathscr{E} = \mathscr{E}_{\bar{0}} \oplus \mathscr{E}_{\bar{1}}$ be the super vector bundle with

$$\mathscr{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM) \qquad \text{and} \qquad \mathscr{E}_{\bar{1}} = S(M)$$

and (even) connection $\mathscr{D}$ defined on $\mathscr{E}_{\bar{0}}$ by [48, 49]

$$\mathscr{D}_Z \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_Z \xi + A(Z) \\ \nabla_Z A - R(Z, \xi) \end{pmatrix} \tag{3.11}$$

and on $\mathscr{E}_{\bar{1}}$ by the connection D in (3.1). A section (ξ, A) of $\mathscr{E}_{\bar{0}}$ is parallel if and only if ξ is a Killing vector and $A = -\nabla\xi$, whereas a section ε of $\mathscr{E}_{\bar{1}}$ is parallel if and only if it is a Killing spinor. Therefore $\mathfrak{k}$ is a subspace of the parallel sections of $\mathscr{E}$: $\mathfrak{k}_{\bar{1}}$ are precisely the parallel sections of $\mathscr{E}_{\bar{1}}$, whereas $\mathfrak{k}_{\bar{0}}$ are the parallel sections of $\mathscr{E}_{\bar{0}}$ which in addition leave invariant the scalars a and b and the vector field φ.

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Parallel sections ζ of a vector bundle with connection are uniquely determined by their value $\zeta|_o$ at any given point $o \in M$. (We tacitly assume that M is connected.) Let us introduce the following notation

$$(V, \eta) = (T_oM, g|_o) \qquad \mathfrak{so}(V) = \mathfrak{so}(T_oM) \qquad S = S_o(M) \ .$$

Therefore $\mathfrak{k}$ determines a subspace of $\mathscr{E}_o = V \oplus \mathfrak{so}(V) \oplus S$, which is the underlying vector space of the Poincaré superalgebra $\mathfrak{p}$. We recall that $\mathfrak{p}$ is a $\mathbb{Z}$-graded Lie superalgebra with Lie brackets given in equation (1.1) and that the $\mathbb{Z}$ and $\mathbb{Z}_2$ gradings are compatible.

Let (ξ, A_ξ), with $A_\xi = -\nabla\xi$, and (ζ, A_ζ) belong to $\mathfrak{k}_{\bar{0}}$. Their Lie bracket is given by

$$[(\xi, A_\xi), (\zeta, A_\zeta)] = (A_\xi\zeta - A_\zeta\xi, [A_\xi, A_\zeta] + R(\xi, \zeta)) \,, \tag{3.12}$$

where the bracket on the right-hand side is the commutator in $\mathfrak{so}(TM)$. We see that the Riemann curvature measures the failure of $\mathfrak{k}_{\bar{0}}$ to be a Lie subalgebra of the Poincaré algebra $\mathfrak{p}_{\bar{0}}$. If now $\varepsilon \in \mathfrak{k}_{\bar{1}}$, then the Lie bracket with (ξ, A_ξ) is given by

$$[(\xi, A_\xi), \varepsilon] = \nabla_\xi\varepsilon + \sigma(A_\xi)\varepsilon = \beta^\Phi_\xi\varepsilon + \sigma(A_\xi)\varepsilon \,, \tag{3.13}$$

where $\sigma : \mathfrak{so}(TM) \to \mathrm{End}(S(M))$ is the spinor representation. Finally, the Dirac current of a Killing spinor $\varepsilon \in \mathfrak{k}_{\bar{1}}$ is given by

$$[\varepsilon, \varepsilon] = (\kappa(\varepsilon, \varepsilon), A_{\kappa(\varepsilon,\varepsilon)}) \,,$$

where

$$A_{\kappa(\varepsilon,\varepsilon)}(Z) = -\nabla_Z\kappa(\varepsilon, \varepsilon) = -2\kappa(\nabla_Z\varepsilon, \varepsilon) = -2\kappa(\beta^\Phi_Z\varepsilon, \varepsilon) \ .$$

We now show that $\mathfrak{k}$ defines a graded subspace of $\mathfrak{p} = \mathscr{E}_o$. Define $\mathrm{ev}^{\bar{0}}_o : \mathfrak{k}_{\bar{0}} \to V$ to be evaluation at o and projection onto $V = T_oM$. More precisely,

$$\mathrm{ev}^{\bar{0}}_o(\xi, A_\xi) = \xi|_o \ .$$

Similarly, let $\mathrm{ev}^{\bar{1}}_o : \mathfrak{k}_{\bar{1}} \to S$ be the evaluation at o. We set $S' = \mathrm{im}\,\mathrm{ev}^{\bar{1}}_o$ and $V' = \mathrm{im}\,\mathrm{ev}^{\bar{0}}_o$.

Let $\mathfrak{h} = \ker \mathrm{ev}^{\bar{0}}_o$. These are the Killing vectors in $\mathfrak{k}_{\bar{0}}$ which take the form $(0, A) \in V \oplus \mathfrak{so}(V)$ at $o \in M$. Therefore $\mathfrak{h}$ defines a subspace of $\mathfrak{so}(V)$, but from equation (3.12), we see that it is also a Lie subalgebra:

$$[(0, A), (0, B)] = (0, [A, B]) \ .$$

In addition, the conditions $\mathscr{L}_\xi a = \mathscr{L}_\xi b = \mathscr{L}_\xi\varphi = 0$ that are satisfied by the Killing vectors $\xi \in \mathfrak{k}_{\bar{0}}$, when evaluated at $o \in M$, imply that if $(0, A) \in \mathfrak{h}$ then

$$A \in \mathfrak{so}(V) \cap \mathfrak{stab}(a|_o) \cap \mathfrak{stab}(b|_o) \cap \mathfrak{stab}(\varphi|_o) \,,$$

and the Lie bracket (3.13) at $o \in M$ implies that

$$[(0, A), \varepsilon] = \sigma(A)\varepsilon \ .$$

In particular, $\mathfrak{h}$ acts on S by restricting the action of $\mathfrak{so}(V)$, and this action preserves S'.

The Lie subalgebra $\mathfrak{h} < \mathfrak{k}_{\bar{0}}$ defines a short exact sequence

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k}_{\bar{0}} \xrightarrow{\mathrm{ev}_o^{\bar{0}}} V' \longrightarrow 0\,, \tag{3.14}$$

which yields a vector space isomorphism $\mathfrak{k}_{\bar{0}} \cong \mathfrak{h} \oplus V'$, and therefore *as graded vector spaces*, a (non-canonical) isomorphism

$$\mathfrak{k} \cong \mathfrak{h} \oplus S' \oplus V' \subset \mathfrak{so}(V) \oplus S \oplus V \cong \mathfrak{p}\ .$$

We now wish to express the Lie superalgebra structure on $\mathfrak{k}$ in terms of a Lie bracket on the graded vector space $\mathfrak{h} \oplus S' \oplus V'$. This requires a choice of splitting of the short exact sequence (3.14). Geometrically, this amounts to choosing for every $v \in V'$ a Killing vector field $\xi \in \mathfrak{k}_{\bar{0}}$ with $\xi|_o = v$. Such a choice gives an embedding of V' into $V \oplus \mathfrak{so}(V)$ as the graph of a linear map $\Sigma : V' \to \mathfrak{so}(V)$; that is, by sending $v \in V'$ to (v, Σ_v), where $\Sigma_v \in \mathfrak{so}(V)$ is the image of v under Σ. Any other choice of splitting would result in (v, Σ'_v) for some other linear map $\Sigma' : V' \to \mathfrak{so}(V)$, but where the difference $\Sigma - \Sigma' : V' \to \mathfrak{h}$.

The Lie bracket of $(0, A) \in \mathfrak{h}$ and $(v, \Sigma_v) \in \mathfrak{k}_{\bar{0}}$ is given by

$$[(0,A),(v,\Sigma_v)] = (Av, [A,\Sigma_v]) = (Av, \Sigma_{Av}) + (0, [A,\Sigma_v] - \Sigma_{Av})\ .$$

Similarly, if $\varepsilon \in \mathfrak{k}_{\bar{1}}$, then

$$[(v,\Sigma_v),\varepsilon] = \beta_v^{\Phi}\varepsilon + \Sigma_v\varepsilon\,,$$

whereas

$$[\varepsilon,\varepsilon] = (\kappa(\varepsilon,\varepsilon), A_{\kappa(\varepsilon,\varepsilon)}) = (\kappa(\varepsilon,\varepsilon), \Sigma_{\kappa(\varepsilon,\varepsilon)}) + (0, A_{\kappa(\varepsilon,\varepsilon)} - \Sigma_{\kappa(\varepsilon,\varepsilon)})\ .$$

Finally, if $v, w \in V'$,

$$\begin{aligned}[(v,\Sigma_v),(w,\Sigma_w)] &= (\Sigma_v w - \Sigma_w v, [\Sigma_v,\Sigma_w] + R(v,w)) \\ &= (\Sigma_v w - \Sigma_w v, \Sigma_{\Sigma_v w - \Sigma_w v}) + (0, [\Sigma_v,\Sigma_w] + R(v,w) - \Sigma_{\Sigma_v w - \Sigma_w v})\end{aligned}$$

This allows us to read off the Lie bracket on $\mathfrak{h} \oplus S' \oplus V'$. We will let $v, w \in V'$, $s \in S'$ and $A, B \in \mathfrak{h}$. Then we have

$$\begin{aligned}
[A,B] &= AB - BA \\
[A,s] &= \sigma(A)s \\
[A,v] &= Av + \underbrace{[A,\Sigma_v] - \Sigma_{Av}}_{\lambda(A,v)} \\
[s,s] &= \kappa(s,s) + \underbrace{\gamma^{\Phi}(s,s) - \Sigma_{\kappa(s,s)}}_{\gamma(s,s)} \\
[v,s] &= \underbrace{\beta_v^{\Phi}s + \Sigma_v s}_{\beta(v,s)} \\
[v,w] &= \underbrace{\Sigma_v w - \Sigma_w v}_{\alpha(v,w)} + \underbrace{[\Sigma_v,\Sigma_w] + R(v,w) - \Sigma_{\alpha(v,w)}}_{\delta(v,w)}\,,
\end{aligned} \tag{3.15}$$

which define maps $\lambda : \mathfrak{h} \otimes V' \to \mathfrak{h}$, $\gamma : \odot^2 S' \to \mathfrak{h}$, $\beta : V' \otimes S' \to S'$, $\alpha : \wedge^2 V' \to V'$ and $\delta : \wedge^2 V' \to \mathfrak{h}$.

Notice that all the under-braced terms have positive filtration degree: λ, α, β and γ have degree 2, whereas δ has degree 4. If we set those maps to zero, which is equivalent to passing to the associated graded superalgebra, then we are left with the $\mathbb{Z}$-graded subalgebra $\mathfrak{a} < \mathfrak{p}$ given by the Lie brackets

$$\begin{aligned} [A, B] &= AB - BA & \qquad [s, s] &= \kappa(s, s) \\ [A, s] &= \sigma(A)s & [v, s] &= 0 \\ [A, v] &= Av & [v, w] &= 0\ . \end{aligned} \tag{3.16}$$

Moreover, it follows from Lemma 16 in the appendix that if $\dim S' > \frac{1}{2}\dim S = 2$, then $V' = V$.

Therefore we have proved the following

Proposition 8. *The Killing superalgebra $\mathfrak{k}$ in equation (3.15) is a filtered deformation of the $\mathbb{Z}$-graded subalgebra $\mathfrak{a} < \mathfrak{p}$ defined on $\mathfrak{h} \oplus S' \oplus V'$ by the Lie brackets in (3.16). Moreover if $\dim S' > \frac{1}{2}\dim S = 2$ then the Lie algebra $\mathfrak{k}_{\bar{0}}$ of infinitesimal automorphisms of (M, g, a, b, φ) acts locally transitively around any point $o \in M$.*

4 Zero curvature equations

In this section we calculate the curvature of the connection D on the spinor bundle and solve the zero curvature equations for the metric g and the fields a, b, φ. We do this in two steps. In the first step we arrive at a first set of equations obtained by setting the Clifford trace of the curvature to zero. We perform this first step for two reasons. The first reason is by analogy with eleven-dimensional supergravity, where the vanishing of the Clifford trace of the curvature is equivalent to the bosonic field equations (and the Bianchi identity). The second reason is that this first set of equations is easier to solve and already imposes strong constraints on the geometric data which simplify the solution of the zero curvature equations. The second step is the solution of the zero curvature equations, which will yield the maximally supersymmetric backgrounds. The Killing superalgebras of these maximally supersymmetric backgrounds should (and do) agree with the maximally supersymmetric filtered deformations which we classify in section 5.

With regard to the first reason for performing the first step, we must stress that any relation in four dimensions between the equation obtained by setting to zero the Clifford trace of the curvature and the bosonic field equations of minimal off-shell supergravity remains to be seen. If we were to identify (up to constants of proportionality) the fields a, b and φ in the connection D in (3.1) with the bosonic fields in the minimal off-shell gravity supermultiplet in four dimensions (as described, say, in [50] section 16.2.3), and identify (up to an overall constant of proportionality) $D\varepsilon$ with the supersymmetry variation of the gravitino Ψ in the gravity supermultiplet, evaluated at $\Psi = 0$, one finds that the purely bosonic terms in the off-shell supergravity Lagrangian density must be proportional to $R + 24(a^2 + b^2 + |\varphi|^2)$, where R is the scalar curvature of g. The Einstein equations for this supergravity Lagrangian are $R_{\mu\nu} = -12(a^2+b^2)g_{\mu\nu} - 24\varphi_\mu\varphi_\nu$ which, after integrating out the auxiliary fields a, b and φ, imply that that g must be Ricci-flat. As we will see, the equations obtained by setting to zero the Clifford trace of the curvature are similar but different.

4.1 The curvature of the superconnection

Let us write the Killing spinor condition for $\varepsilon \in \Gamma(S(M))$ as $\nabla_Z\varepsilon = \beta^\Phi_Z\varepsilon$ for all vector fields Z, and where the $\mathrm{End}(S(M))$-valued one-form β^Φ was defined in equation (3.10). In other words, $D_Z = \nabla_Z - \beta^\Phi_Z$. The curvature R^D of D is defined by

$$\begin{aligned} R^D_{X,Y} &= D_{[X,Y]} - [D_X, D_Y] \\ &= R_{X,Y} + (\nabla_X\beta^\Phi)_Y - (\nabla_Y\beta^\Phi)_X - [\beta^\Phi_X, \beta^\Phi_Y]\,, \end{aligned}$$

where R is the curvature 2-form of ∇ on the spinor bundle. An explicit calculation shows that

$$\begin{aligned} R^D_{X,Y} = R_{X,Y} &+ X(a)Y + X(b)Y\cdot\mathrm{vol} - Y(a)X - Y(b)X\cdot\mathrm{vol} - (\nabla_X\varphi\wedge Y)\cdot\mathrm{vol} \\ &+ (\nabla_Y\varphi\wedge X)\cdot\mathrm{vol} + 2g(\nabla_X\varphi, Y)\,\mathrm{vol} - 2g(\nabla_Y\varphi, X)\,\mathrm{vol} + 2(a^2+b^2-|\varphi|^2_g)X\wedge Y \\ &+ 4a(\varphi\wedge X\wedge Y)\cdot\mathrm{vol} + 4ag(\varphi,Y)X\cdot\mathrm{vol} - 4ag(\varphi,X)Y\cdot\mathrm{vol} - 4b\varphi\wedge X\wedge Y \\ &- 4bg(\varphi,Y)X + 4bg(\varphi,X)Y + 2g(\varphi,X)\varphi\wedge Y - 2g(\varphi,Y)\varphi\wedge X\,. \end{aligned} \tag{4.1}$$

From this expression we will be able to read off a set of equations by demanding that the Clifford trace of the curvature $\mathrm{Ric}^D : TM \to \mathrm{End}(S(M))$, defined by

$$\mathrm{Ric}^D(X) = \sum_\mu e^\mu \cdot R^D_{X,e_\mu}\,, \tag{4.2}$$

vanishes. Here e^μ and e_μ are g-dual local frames of TM. Another explicit calculation shows that

$$\begin{aligned} \mathrm{Ric}^D(X) = \mathrm{Ric}(X) &- 3X(a) - 3X(b)\,\mathrm{vol} - da^\sharp\wedge X - (db^\sharp\wedge X)\cdot\mathrm{vol} + 6(a^2+b^2)X \\ &- 4|\varphi|^2_g X - 4a(\varphi\wedge X)\cdot\mathrm{vol} + 4b\varphi\wedge X + 12ag(\varphi,X)\,\mathrm{vol} - 12bg(\varphi,X) \\ &+ 4g(\varphi,X)\varphi + \big(\nabla_\mu\varphi_\nu X_\rho\Gamma^{\mu\nu\rho} - \nabla_\mu\varphi^\mu X - 2g(\nabla\!\!\!/\varphi, X)\big)\cdot\mathrm{vol}\,, \end{aligned} \tag{4.3}$$

where Ric stands for the Ricci operator and we have introduced the shorthand $g(\nabla\!\!\!/\varphi, X) = \Gamma^\rho\nabla_\rho\varphi_\mu X^\mu$.

4.2 The vanishing of the Clifford trace of the curvature

We now describe the equations arising by demanding that the Clifford trace of the curvature of the spinor connection D vanishes; in other words, that for all vector fields X, $\mathrm{Ric}^D(X) = 0$. This is a system of equations with values in $\mathrm{End}(S(M))$, which is isomorphic as a vector bundle to $\bigoplus_{p=0}^4 \wedge^p TM$. This means that the components of these equations in each summand have to be satisfied separately. The $p=1$ component relates the Ricci tensor to the data (a,b,φ), whereas the $p\neq 1$ components constrain (a,b,φ). We start with these first.

4.2.1 The $p = 0$ component

The $p = 0$ component of the equation $\mathrm{Ric}^D(X) = 0$ is given by

$$-3X(a) - 12bg(\varphi, X) = 0\,,$$

which, after abstracting X, is equivalent to

$$da^\sharp = -4b\varphi\ . \tag{4.4}$$

4.2.2 The $p = 4$ component

The $p = 4$ component of $\mathrm{Ric}^D(X) = 0$ is given by

$$-3X(b)\,\mathrm{vol} + 12ag(\varphi, X)\,\mathrm{vol} = 0$$

which is equivalent to

$$db^\sharp = 4a\varphi\ . \tag{4.5}$$

4.2.3 The $p = 2$ component

The $p = 2$ component of $\mathrm{Ric}^D(X) = 0$ is given by

$$-da^\sharp \wedge X - (db^\sharp \wedge X)\cdot\mathrm{vol} - 4a(\varphi\wedge X)\cdot\mathrm{vol} + 4b\varphi\wedge X = 0\,,$$

which using equations (4.4) and (4.5) becomes

$$(\varphi\wedge X)\cdot(b + a\,\mathrm{vol}) = 0\ .$$

Multiplying by $b - a\,\mathrm{vol}$ and since this has to be true for all X, we arrive at

$$(a^2 + b^2)\varphi = 0\ . \tag{4.6}$$

It follows from this equation that there are three branches of solutions:

(I) $a = b = \varphi = 0$,

(II) $a^2+b^2 > 0$ and $\varphi = 0$, in which case a and b are constant by equations (4.4) and (4.5), and

(III) $a = b = 0$ and $\varphi \neq 0$.

4.2.4 The $p = 3$ component

The $p = 3$ component of $\mathrm{Ric}^D(X) = 0$ is given by

$$-(\nabla_\mu\varphi^\mu X + 2g(\nabla\!\!\!/\varphi, X))\cdot\mathrm{vol} = 0\,,$$

which, abstracting X, can be written as

$$\nabla_\mu\varphi^\mu\Gamma^\nu + 2\nabla_\mu\varphi^\nu\Gamma^\mu = 0\ . \tag{4.7}$$

Multiplying with Γ_ν on both left and right we arrive at the pair of equations:

$$-4\nabla_\mu\varphi^\mu + 2\nabla_\mu\varphi_\nu\Gamma^\mu\Gamma^\nu = 0$$
$$-4\nabla_\mu\varphi^\mu + 2\nabla_\mu\varphi_\nu\Gamma^\nu\Gamma^\mu = 0\ .$$

Adding the two equations, and using the Clifford relations,

$$-8\nabla_\mu\varphi^\mu - 4\nabla_\mu\varphi^\mu = 0 \implies \nabla_\mu\varphi^\mu = 0\ .$$

Plugging this back into equation (4.7), we arrive at

$$\nabla_\mu\varphi^\nu\Gamma^\mu = 0\,,$$

which says that φ is parallel:

$$\nabla\varphi = 0\ . \tag{4.8}$$

4.2.5 The $p = 1$ component

Finally we arrive at the $p = 1$ component of $\mathrm{Ric}^D(X) = 0$:

$$\mathrm{Ric}(X) + 6(a^2+b^2)X - 4|\varphi|_g^2 X + 4g(\varphi, X)\varphi + \nabla_\mu\varphi_\nu X_\rho\Gamma^{\mu\nu\rho}\cdot \mathrm{vol} = 0\ .$$

The last term vanishes because φ is parallel, so that we are left with

$$\mathrm{Ric}(X) + 6(a^2+b^2)X - 4|\varphi|_g^2 X + 4g(\varphi, X)\varphi = 0\ .$$

We can abstract X and leave it as an equation on the Ricci operator itself:

$$\mathrm{Ric} = -12(a^2+b^2)\,\mathrm{Id} + 8|\varphi|_g^2\,\mathrm{Id} - 8\varphi\otimes\varphi^\flat\,, \tag{4.9}$$

which, in terms of the symmetric Ricci tensor, becomes

$$R_{\mu\nu} = -12(a^2+b^2)g_{\mu\nu} + 8|\varphi|_g^2 g_{\mu\nu} - 8\varphi_\mu\varphi_\nu\ . \tag{4.10}$$

4.3 The solutions

Let us analyse the type of solutions to these equations. We have seen that there are three branches of solutions stemming from the $p = 2$ component equation (4.6).

(I) $a = b = \varphi = 0$. In this case, the $p = 1$ component equation simply says that g is Ricci-flat. In this background, Killing spinors are parallel and therefore the supersymmetric backgrounds are the Ricci-flat manifolds whose holonomy is contained in the isotropy of a spinor. Since the Dirac current of a parallel spinor is null and parallel, these metrics are Ricci-flat Brinkmann metrics. See, e.g., ([51] section 3.2.3) for a discussion of these geometries.

(II) $a^2 + b^2 \neq 0$ and $\varphi = 0$. Putting $\varphi = 0$, we see from equations (4.4) and (4.5) that $da = db = 0$, so they are constant and the Ricci tensor is given by

$$R_{\mu\nu} = -12(a^2+b^2)g_{\mu\nu}\,,$$

so that g is Einstein with negative cosmological constant. The Killing spinors are (up to an R-symmetry which allows us to set $b = 0$, say) geometric Killing spinors. Such geometries are reviewed in ([52] sections 6–7) and discussed in [53].

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(III) $a = b = 0$ and $\varphi \neq 0$. Then φ is a parallel vector field and the Ricci tensor, given by

$$R_{\mu\nu} = -8\left(\varphi_\mu \varphi_\nu - |\varphi|_g^2 g_{\mu\nu}\right) , \tag{4.11}$$

is also parallel. This is a kind of fluid solution. Ricci-parallel geometries have been studied in [54]. The determining factor is the algebraic type of the Ricci endomorphism. In this case, this depends on the causal type of φ, which is constant because φ is parallel. If φ is timelike or spacelike, so that (in our mostly minus conventions) $|\varphi|_g^2$ is positive or negative, respectively, then the Ricci endomorphism is diagonalisable and the geometry decomposes (up to coverings) into a product $M = \mathbb{R} \times N$ of a line and a three-dimensional Einstein space N, hence a space form. Moreover, upon identifying the spin bundle of M with (an appropriate number of copies of) the spin bundle of N, it is not difficult to see that Killing spinors in these backgrounds correspond to geometric Killing spinors on N (up to an R-symmetry). If φ is null, then the Ricci endomorphism is two-step nilpotent and the geometry is Ricci-null. The subbundle of TM of orthogonal vectors to φ is also in this case integrable in the sense of Frobenius but the above simple interpretation of Killing spinors is missing since the associated integrable submanifolds N have a degenerate induced metric.

4.4 Maximally supersymmetric backgrounds

Maximally supersymmetric backgrounds are those for which the spinor connection D is flat. The zero curvature condition $R^D_{X,Y} = 0$ for all vector fields X, Y becomes a system of equations with values in $\mathrm{End}(S(M))$ and therefore, just as for the vanishing of the Clifford trace of the curvature, the different components of the curvature must vanish separately. We can reuse our calculations above, since if D is flat, the Clifford trace of the curvature certainly vanishes. This means that we can consider the three branches described above. We will meet the geometries we are about to discuss again in the next section, where we classify the maximally supersymmetric filtered subdeformations of the Poincaré superalgebra.

4.4.1 Maximally supersymmetric backgrounds with $a = b = \varphi = 0$

If $a = b = \varphi = 0$, the connection D agrees with the Levi-Civita spin connection and hence D-flatness means flatness and every such background is locally isometric to Minkowski spacetime.

4.4.2 Maximally supersymmetric backgrounds with $\varphi = 0$ and $a^2 + b^2 > 0$

If $\varphi = 0$, then a, b are constant and not both zero and hence the D-flatness condition is

$$R_{X,Y} = -2(a^2 + b^2) X \wedge Y \,,$$

as an equation in $\mathrm{End}(S(M))$. This is equivalent to

$$R_{\mu\nu\rho\sigma} = 4(a^2 + b^2)(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \,,$$

which says that g is locally isometric to AdS_4.

4.4.3 Maximally supersymmetric backgrounds with $a = b = 0$ and $\varphi \neq 0$

If $a = b = 0$, and using that φ is parallel, the D-flatness condition is

$$R_{X,Y} = 2|\varphi|_g^2 X \wedge Y - 2g(\varphi, X)\varphi \wedge Y + 2g(\varphi, Y)\varphi \wedge X\,,$$

again as an equation in $\mathrm{End}(S(M))$. The corresponding Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = -4|\varphi|_g^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) - 4\varphi_\nu\varphi_\rho g_{\mu\sigma} + 4\varphi_\nu\varphi_\sigma g_{\mu\rho} + 4\varphi_\mu\varphi_\rho g_{\nu\sigma} - 4\varphi_\mu\varphi_\sigma g_{\nu\rho}\,. \quad (4.12)$$

Since φ and g are parallel, so is the Riemann tensor and hence this corresponds to a locally symmetric space. Furthermore, it is conformally flat. Indeed, in four dimensions, the Weyl tensor is given in terms of the Riemann tensor, the Ricci tensor $R_{\mu\nu} = g^{\rho\sigma}R_{\mu\rho\sigma\nu}$ and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ by

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \tfrac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) - \tfrac{1}{6}R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\,.$$

Inserting the above expression for $R_{\mu\nu\rho\sigma}$ into the Weyl tensor we see that it vanishes, so that the geometry is conformally flat. The corresponding Ricci tensor is given by equation (4.11) and the Ricci scalar is $R = 24|\varphi|_g^2$.

This geometry corresponds to a Lorentzian Lie group with a bi-invariant metric. Indeed, the equation (4.12) satisfied by the Riemann tensor is equivalent to the vanishing of the curvature of a metric connection with parallel totally skewsymmetric torsion proportional to the Hodge dual of φ. As shown, for instance, in [55, 56], the existence of a flat metric connection with closed skewsymmetric torsion is equivalent to the manifold being locally isometric to a Lie group with a bi-invariant metric.

Since φ is parallel, its g-norm is constant and in a Lorentzian manifold this can be of three types:

1. $|\varphi|_g^2 > 0$. This is timelike in our conventions. The background is locally isometric to $\mathbb{R} \times \mathrm{S}^3$, where we identify the round S^3 with the Lie group $\mathrm{SU}(2)$ with its bi-invariant metric.

2. $|\varphi|_g^2 < 0$. This is spacelike and hence the background is locally isometric to $\mathrm{AdS}_3 \times \mathbb{R}$, where we identify AdS_3 with $\mathrm{SL}(2, \mathbb{R})$ with its bi-invariant metric.

3. $|\varphi|_g^2 = 0$. This is the null case and hence the background is locally isometric to the Nappi-Witten group [57] with its bi-invariant metric.

5 Maximally supersymmetric filtered deformations

We now resume the analysis of filtered subdeformations of the Poincaré superalgebra by classifying the filtered deformations with maximal odd dimension. We will show that they correspond precisely to the Killing superalgebras of the maximally supersymmetric backgrounds classified in section 4.4.

More precisely, let $\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0$ be a $\mathbb{Z}$-graded subalgebra of the Poincaré superalgebra $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ with $\mathfrak{a}_{-1} = S$. By Lemma 16, we also have that $\mathfrak{a}_{-2} = V$, so that $\mathfrak{a}$ differs from $\mathfrak{p}$ only in zero degree, where $\mathfrak{a}_0 = \mathfrak{h}$ is a subalgebra of $\mathfrak{so}(V)$. The aim of this section is to classify, for any possible given $\mathfrak{h}$, the filtered deformations $\mathfrak{g}$ of $\mathfrak{a}$. We will see that they are essentially governed by the $\mathfrak{h}$-invariant elements $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})^{\mathfrak{h}}$ of the Spencer group $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ of $\mathfrak{a}$, where $\mathfrak{a}_- = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1}$ is the negatively graded part of $\mathfrak{a}$.

In section 5.1 we set up the calculation of $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$, which will be described in section 5.2. This result will then be used in section 5.3 to classify the filtered deformations. The results are summarised in Theorem 14 in section 5.4.

5.1 Preliminaries

Here we set up the calculation of the Spencer cohomology $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$. We introduce the Spencer complex of $\mathfrak{a}$ in complete analogy to the Spencer complex of $\mathfrak{p}$ (cf. section 2): one has simply to replace $\mathfrak{so}(V)$ with $\mathfrak{h}$ in the definitions. For instance any element $\zeta \in C^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ can be uniquely written as the sum $\zeta = \alpha + \beta + \gamma$, where

$$\alpha \in \operatorname{Hom}(\wedge^2 V, V)\,, \qquad \beta \in \operatorname{Hom}(V \otimes S, S) \qquad \text{and} \qquad \gamma \in \operatorname{Hom}(\odot^2 S, \mathfrak{h}) \tag{5.1}$$

and the Lie brackets of a general filtered deformations of $\mathfrak{a}$ take the form

$$\begin{aligned} [A,B] &= AB - BA & \qquad [s,s] &= \kappa(s,s) + \gamma(s,s) \\ [A,s] &= \sigma(A)s & [v,s] &= \beta(v,s) \\ [A,v] &= Av + \lambda(A,v) & [v,w] &= \alpha(v,w) + \delta(v,w)\,, \end{aligned} \tag{5.2}$$

for some maps $\lambda : \mathfrak{h} \otimes V \to \mathfrak{h}$ and $\delta : \wedge^2 V \to \mathfrak{h}$, where $A, B \in \mathfrak{h}$, $s \in S$, $v, w \in V$.

We recall that a transitive and fundamental $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{a} = \bigoplus \mathfrak{a}_p$ with negatively graded part $\mathfrak{a}_- = \bigoplus_{p<0} \mathfrak{a}_p$ is called a *full prolongation of degree k* if $H^{d,1}(\mathfrak{a}_-, \mathfrak{a}) = 0$ for all $d \geq k$.

Lemma 9. *Let $\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0$ be a $\mathbb{Z}$-graded subalgebra of the Poincaré superalgebra which differs only in zero degree. Then $\mathfrak{a}$ is fundamental, transitive and $H^{d,2}(\mathfrak{a}_-, \mathfrak{a}) = 0$ for all even $d > 2$. Furthermore it is a full prolongation of degree $k = 2$.*

Proof. We only show the last claim, the others follow as in the proof of Lemma 1. Any $\zeta \in C^{2,1}(\mathfrak{a}_-, \mathfrak{a})$ satisfies $\zeta(V) \subset \mathfrak{h}$, $\zeta(S) \subset \mathfrak{a}_1 = 0$ and

$$\begin{aligned} \partial\zeta(s_1, s_2) &= -\zeta(k(s_1, s_2)) \\ \partial\zeta(s_1, v_1) &= -\sigma(\zeta(v_1))s_1 \\ \partial\zeta(v_1, v_2) &= \zeta(v_1)v_2 - \zeta(v_2)v_1 \end{aligned}$$

for all $s_1, s_2 \in S$, $v_1, v_2 \in V$. The first equation directly implies that $\zeta = 0$ is the only cocycle and hence $H^{2,1}(\mathfrak{a}_-, \mathfrak{a}) = 0$. If $d > 2$ then $C^{d,1}(\mathfrak{a}_-, \mathfrak{a}) = 0$ for degree reasons. $\square$

Remark. One can actually prove that $\mathfrak{a}$ is a full prolongation of degree $k = 1$, based on the non-trivial fact that the so-called "maximal prolongation" $\mathfrak{g}^\infty$ of $\mathfrak{a}_- = V \oplus S$ is a simple Lie superalgebra of type $\mathfrak{sl}(1|4)$ with a special $\mathbb{Z}$ grading of the form $\mathfrak{g}^\infty = \mathfrak{g}^\infty_{-2} \oplus \cdots \oplus \mathfrak{g}^\infty_2$, cf. [58]; but the simpler result of Lemma 9 suffices for our purposes.

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To state the main first intermediate result on filtered deformations $\mathfrak{g}$ of $\mathfrak{a}$ we recall that the Lie brackets of $\mathfrak{g}$ have components of nonzero degree: the sum $\mu : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a}$ of all components of degree 2 and the unique component $\delta : \wedge^2 V \to \mathfrak{h}$ of degree 4.

Proposition 10. *Let $\mathfrak{a} = V \oplus S \oplus \mathfrak{h}$ be a $\mathbb{Z}$-graded subalgebra of the Poincaré superalgebra $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ which differs only in zero degree. If $\mathfrak{g}$ is a filtered deformation of $\mathfrak{a}$ then:*

1. *$\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}$ is a cocycle in $C^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ and its cohomology class*
$$[\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$$
is $\mathfrak{h}$-invariant (that is, the cocycle $\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}$ is $\mathfrak{h}$-invariant up to coboundaries); and
2. *if $\mathfrak{g}'$ is another filtered deformation of $\mathfrak{a}$ such that $[\mu'|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}] = [\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}]$ then $\mathfrak{g}'$ is isomorphic to $\mathfrak{g}$ as a filtered Lie superalgebra.*

Proof. The first claim follows directly from Proposition 2.2 of [37]. Let now $\mathfrak{g}$ and $\mathfrak{g}'$ be filtered deformations of $\mathfrak{a}$ such that $[\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}] = [\mu'|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}]$. Then $(\mu - \mu')|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}$ is a Spencer coboundary and we may first assume without any loss of generality that $\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-} = \mu'|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}$ by Proposition 2.3 of [37]. Moreover, since $\mathfrak{a}$ is a fundamental and transitive full prolongation of degree $k = 2$ by Lemma 9, Proposition 2.6 of [37] applies and we may also assume $\mu = \mu'$ without any loss of generality. In other words we just showed that $\mathfrak{g}'$ is isomorphic as a filtered Lie superalgebra to another filtered Lie superalgebra $\mathfrak{g}''$ which satisfies $\mu'' = \mu$.

Now, given any two filtered deformations $\mathfrak{g}$ and $\mathfrak{g}'$ of $\mathfrak{a}$ with $\mu = \mu'$ it is easy to see that $\delta - \delta' = (\delta - \delta')|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}$ is a Spencer cocycle (use e.g., [37] equation 2.6). However $H^{4,2}(\mathfrak{a}_-, \mathfrak{a}) = \ker \partial|_{C^{4,2}(\mathfrak{a}_-,\mathfrak{a})} = 0$ by Lemma 9 and hence $\delta = \delta'$. This proves that any two filtered deformations $\mathfrak{g}$ and $\mathfrak{g}'$ of $\mathfrak{a}$ with $[\mu'|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}] = [\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}]$ are isomorphic. □

In other words, filtered deformations are determined by the space $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})^{\mathfrak{h}}$ of $\mathfrak{h}$-invariant elements in $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$. In particular the components of non-zero filtration degree $\lambda = \mu|_{\mathfrak{h} \otimes V} : \mathfrak{h} \otimes V \to \mathfrak{h}$ and $\delta : \wedge^2 V \to \mathfrak{h}$ are completely determined by the class $[\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})^{\mathfrak{h}}$, up to isomorphisms of filtered Lie superalgebras.

We will now describe $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$. We recall that this group has already been determined in Proposition 3 when $\mathfrak{a} = \mathfrak{p}$ is the Poincaré superalgebra. Therein we also described the kernel $\mathscr{H}^{2,2}$ of the Spencer operator acting on $\mathrm{Hom}(V \otimes S, S) \oplus \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V))$: it consists of the maps $\beta + \gamma \in \mathrm{Hom}(V \otimes S, S) \oplus \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V))$ which are of the form given by Proposition 3. To avoid confusion with the general components (5.1) we will denote these maps by $\beta^\Phi + \gamma^\Phi$ from now on, that is we set $\Phi = (a, b, \varphi) \in 2\mathbb{R} \oplus V$ and

$$\begin{aligned} \beta^\Phi(v, s) &= v \cdot (a + b \,\mathrm{vol}) \cdot s - \frac{1}{2}(v \cdot \varphi + 3\varphi \cdot v) \cdot \mathrm{vol} \cdot s \,, \\ \gamma^\Phi(s, s)v &= -2\kappa(s, \beta(v, s)) \,, \end{aligned}$$

for all $v \in V$ and $s \in S$, according to Proposition 3. In addition we set

$$\begin{aligned} \gamma^\varphi(s, s)v &= 2\kappa(s, (\varphi \wedge v) \cdot \mathrm{vol} \cdot s) \,, \\ \gamma^{(a,b)}(s, s)v &= -2a\kappa(s, v \cdot s) - 2b\kappa(s, v \cdot \mathrm{vol} \cdot s) \,, \end{aligned}$$

for all $v \in V$, $s \in S$.

We will also determine the $\mathfrak{h}$-invariant classes in $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$, the Lie subalgebras $\mathfrak{h} \subset \mathfrak{so}(V)$ for which $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})^{\mathfrak{h}} \neq 0$, hence the graded subalgebras $\mathfrak{a} = V \oplus S \oplus \mathfrak{h}$ of $\mathfrak{p}$ admitting nontrivial filtered deformations. The condition $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})^{\mathfrak{h}} \neq 0$ has strong consequences and, as we will now see, gives rise to a dichotomy: either $\varphi = 0$ and $a^2 + b^2 \neq 0$ or $\varphi \neq 0$ and $a = b = 0$.

5.2 The cohomology group $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$

We start with the following

Proposition 11. *Let $\mathfrak{a} = V \oplus S \oplus \mathfrak{h}$ be a $\mathbb{Z}$-graded subalgebra of the Poincaré superalgebra $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ which differs only in zero degree. Then*

$$H^{2,2}(\mathfrak{a}_-, \mathfrak{a}) = \frac{\left\{\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi} \,\middle|\, \Phi \in 2\mathbb{R} \oplus V,\ \widetilde{\psi} : V \to \mathfrak{so}(V) \text{ s.t. } \gamma^\Phi(s,s) - \widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h}\right\}}{\{\partial\psi \,|\, \psi : V \to \mathfrak{h}\}}$$

and

(i) the cohomology class $[\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}]$ is trivial if and only if $\Phi = 0$;

(ii) the condition $\gamma^\Phi(s,s) - \widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h}$ is satisfied for all $s \in S$ if and only if separately

$$\gamma^\varphi(s,s) - \widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h}\ , \tag{5.3}$$

$$\gamma^{(a,b)}(s,s) \in \mathfrak{h}\ , \tag{5.4}$$

for all $s \in S$;

(iii) if $[\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}]$ is an $\mathfrak{h}$-invariant cohomology class then $\mathfrak{h}$ leaves φ invariant, that is $\mathfrak{h} \subset \mathfrak{h}_\varphi$ where $\mathfrak{h}_\varphi = \mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$ and $\mathfrak{stab}(\varphi)$ is the Lie algebra of the stabiliser of φ in $\mathrm{GL}(V)$.

In particular if $[\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ is a nontrivial and $\mathfrak{h}$-invariant cohomology class then exactly one of the following two cases occurs:

(1) if $\varphi = 0$ then $a^2 + b^2 \neq 0$, $\gamma^\Phi(s,s) = \gamma^{(a,b)}(s,s) \in \mathfrak{h}$ for all $s \in S$ and the cohomology class $[\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}] = [\beta^\Phi + \gamma^\Phi]$;

(2) if $\varphi \neq 0$ then $a = b = 0$ and

$$\gamma^\varphi(s,s) \in \mathfrak{h}_\varphi\ , \tag{5.5}$$

$$\widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h}_\varphi\ , \tag{5.6}$$

for all $s \in S$.

Proof. From Lemma 2 we know that given any $\alpha \in \mathrm{Hom}(\wedge^2 V, V)$, there is a unique $\widetilde{\psi} \in \mathrm{Hom}(V, \mathfrak{so}(V))$ such that $\partial\widetilde{\psi} = \alpha + \widetilde{\beta} + \widetilde{\gamma}$, for some $\widetilde{\beta} \in \mathrm{Hom}(V \otimes S, S)$ and $\widetilde{\gamma} \in \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V))$. Any cochain $\alpha + \beta + \gamma \in C^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ may be therefore uniquely written as

$$\alpha + \beta + \gamma = (\alpha + \beta + \gamma - \partial\widetilde{\psi}) + \partial\widetilde{\psi} = (\beta - \widetilde{\beta}) + (\gamma - \widetilde{\gamma}) + \partial\widetilde{\psi}\,,$$

where $\beta - \widetilde{\beta} \in \mathrm{Hom}(V \otimes S, S)$ and $\gamma - \widetilde{\gamma} \in \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V))$. If $\alpha + \beta + \gamma$ is a cocycle, then so is $(\beta - \widetilde{\beta}) + (\gamma - \widetilde{\gamma})$, so that by Proposition 3, $\beta - \widetilde{\beta} = \beta^\Phi$ and $\gamma - \widetilde{\gamma} = \gamma^\Phi$ for some $\Phi \in 2\mathbb{R} \oplus V$ or, in other words,

$$\ker \partial\big|_{C^{2,2}(\mathfrak{a}_-,\mathfrak{a})} \subset \mathscr{H}^{2,2} \oplus \partial\mathrm{Hom}(V, \mathfrak{so}(V)) \ . \tag{5.7}$$

Conversely equation (2.2) tells us that $\partial\widetilde{\psi}(s,s) = -\widetilde{\psi}(\kappa(s,s))$ for all $s \in S$ so that an element $\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}$ is in $C^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ if and only if

$$\gamma^\Phi(s,s) - \widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h} \ , \tag{5.8}$$

for all $s \in S$. This fact together with (5.7) yield immediately the claim on $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$.

If $\Phi = 0$, then $\widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h}$ for all $s \in S$ and $\partial\widetilde{\psi}$ is in the image of $C^{2,1}(\mathfrak{a}_-, \mathfrak{a}) = \mathrm{Hom}(V, \mathfrak{h})$, proving one implication of claim (i). The other implication is trivial.

We will now have a closer look at condition (5.8), using that $\odot^2 S = \wedge^1 V \oplus \wedge^2 V$. From (ii) of Proposition 3 we have

$$\begin{aligned}\gamma^\Phi(s,s)v &= -2\kappa(s, \beta^\Phi(v,s)) \\ &= -2a\kappa(s, v \cdot s) - 2b\kappa(s, v \cdot \mathrm{vol} \cdot s) + 2\kappa(s, (\varphi \wedge v - 2\eta(\varphi, v)) \cdot \mathrm{vol} \cdot s) \\ &= -2a\kappa(s, v \cdot s) - 2b\kappa(s, v \cdot \mathrm{vol} \cdot s) + 2\kappa(s, (\varphi \wedge v) \cdot \mathrm{vol} \cdot s) \ ,\end{aligned}$$

with the first two terms (resp. last term) in the r.h.s. of the above equation acting on the component $\wedge^2 V$ (resp. $\wedge^1 V$) of $\odot^2 S$ but trivially on the other component $\wedge^1 V$ (resp. $\wedge^2 V$). In particular condition (5.8) splits into (5.3) and (5.4), proving claim (ii).

Let now $[\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ be an $\mathfrak{h}$-invariant class; i.e., for any $x \in \mathfrak{h}$ there is a $\psi \in \mathrm{Hom}(V, \mathfrak{h})$ such that $x \cdot (\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}) = \partial\psi$. In other words, in terms of the $\mathfrak{so}(V)$-equivariant projections (2.4), we have:

$$x \cdot (\pi^\alpha(\partial\widetilde{\psi})) = \pi^\alpha(\partial\psi) \ , \tag{5.9}$$

$$x \cdot (\beta^\Phi + \pi^\beta(\partial\widetilde{\psi})) = \pi^\beta(\partial\psi) \ , \tag{5.10}$$

$$x \cdot (\gamma^\Phi + \pi^\gamma(\partial\widetilde{\psi})) = \pi^\gamma(\partial\psi) \ . \tag{5.11}$$

Equation (5.9) and the $\mathfrak{so}(V)$-equivariance of π^α and ∂ imply

$$(\pi^\alpha \circ \partial)(\psi) = (\pi^\alpha \circ \partial)(x \cdot \widetilde{\psi})$$

so that $x \cdot \widetilde{\psi} = \psi$, by Lemma 2. Equation (5.10) yields therefore

$$\begin{aligned}\pi^\beta(\partial\psi) &= x \cdot \left(\beta^\Phi + \pi^\beta(\partial\widetilde{\psi})\right) = x \cdot \beta^\Phi + x \cdot \pi^\beta(\partial\widetilde{\psi}) \\ &= x \cdot \beta^\Phi + \pi^\beta(\partial(x \cdot \widetilde{\psi})) = x \cdot \beta^\Phi + \pi^\beta(\partial\psi)\end{aligned}$$

from which $\beta^{x \cdot \varphi} = x \cdot \beta^\varphi = x \cdot \beta^\Phi = 0$. This proves claim (iii).

We now prove the last claims. Let $[\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}]$ be a nontrivial $\mathfrak{h}$-invariant class. If $\varphi = 0$ then $a^2 + b^2 \neq 0$ by (i) and $\partial\widetilde{\psi}$ is in the image of $C^{2,1}(\mathfrak{a}_-, \mathfrak{a}) = \mathrm{Hom}(V, \mathfrak{h})$ by (5.3). This fact together with (5.4) immediately gives case (1).

If $\varphi \neq 0$ then (ii) and (iii) imply

$$\gamma^{\varphi}(s,s) - \widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h}_{\varphi} \,, \tag{5.12}$$

$$\gamma^{(a,b)}(s,s) \in \mathfrak{h}_{\varphi} \,, \tag{5.13}$$

for all $s \in S$. We fix an orthonormal basis $\{e_\mu\}$ of V, use the Einstein summation convention and note that equation (5.13) gives

$$\begin{aligned} 0 &= \gamma^{(a,b)}(s,s)\varphi \\ &= 2\varphi^{\mu}(\bar{s}\Gamma_{\mu}\Gamma_{\nu}(a + b\,\mathrm{vol})s)e^{\nu} \\ &= 2a\varphi^{\mu}(\bar{s}\Gamma_{\mu}\Gamma_{\nu}s)e^{\nu} + 2b\varphi^{\mu}(\bar{s}\Gamma_{\mu}\Gamma_{\nu}\,\mathrm{vol}\,s)e^{\nu} \\ &= 2a\varphi^{\mu}(\bar{s}\Gamma_{\mu\nu}s)e^{\nu} + 2b\varphi^{\mu}(\bar{s}\Gamma_{\mu\nu}\,\mathrm{vol}\,s)e^{\nu} \\ &= 2\bar{s}(\varphi^{\mu}(a\Gamma_{\mu\nu} + b\Gamma_{\mu\nu}\,\mathrm{vol}))se^{\nu} \,, \end{aligned}$$

for all $s \in S$, hence $\varphi^{\mu}(a\Gamma_{\mu\nu} + b\Gamma_{\mu\nu}\,\mathrm{vol}) = 0$ for every $0 \leq \nu \leq 3$. Since $\varphi \neq 0$ this readily implies $a = b = 0$. Similarly

$$\begin{aligned} \gamma^{\varphi}(s,s)\varphi &= -\varphi^{\mu}(\bar{s}\Gamma_{\mu}(\Gamma_{\nu}\varphi + 3\varphi\Gamma_{\nu})\,\mathrm{vol}\,s)e^{\nu} \\ &= -2\varphi^{\mu}\varphi^{\rho}(\bar{s}\Gamma_{\mu}\Gamma_{\rho\nu}\,\mathrm{vol}\,s)e^{\nu} \\ &= -2\varphi^{\mu}\varphi^{\rho}(\bar{s}\Gamma_{\mu\rho\nu}\,\mathrm{vol}\,s)e^{\nu} \\ &= 0 \,, \end{aligned}$$

so that $\gamma^{\varphi}(s,s) \in \mathfrak{h}_{\varphi}$ for all $s \in S$ automatically and, from equation (5.12), we infer that $\widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h}_{\varphi}$ for all $s \in S$ too. This is case (2). $\square$

By the results of Proposition 10 and Proposition 11, we need only to consider the filtered deformations associated to $\mathfrak{h}$-invariant cohomology classes in $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ with $\Phi \neq 0$. Indeed if $\Phi = 0$ then $[\mu|_{\mathfrak{a}_-\otimes\mathfrak{a}_-}] = 0$ and the associated filtered Lie superalgebras are just the $\mathbb{Z}$-graded subalgebras of the Poincaré superalgebra.

We now investigate separately the cohomology classes in family (1) and (2) of Proposition 11.

Lemma 12. *Let $[\beta^{\Phi} + \gamma^{\Phi}] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ be a nontrivial and $\mathfrak{h}$-invariant cohomology class with $\varphi = 0$. Then $\mathfrak{h} = \mathrm{Im}(\gamma^{\Phi}) = \mathfrak{so}(V)$.*

Proof. First of all, as $a^2 + b^2 \neq 0$ by Proposition 11, we have that right multiplication by $a + b\,\mathrm{vol}$ in $C\ell(V)$ is a linear isomorphism. In particular it restricts to a linear isomorphism of $\wedge^2 V \subset C\ell(V)$. On the other hand, from Proposition 3:

$$\begin{aligned} \eta(w, \gamma^{\Phi}(s,s)v) &= -2\eta(w, \kappa(s, v \cdot (a + b\,\mathrm{vol}) \cdot s)) \\ &= -2\,\langle s, w \cdot v \cdot (a + b\,\mathrm{vol}) \cdot s \rangle \\ &= -2\,\langle s, w \wedge v \cdot (a + b\,\mathrm{vol}) \cdot s \rangle \,, \end{aligned}$$

for all $w \wedge v \in \wedge^2 V \subset C\ell(V)$ and $s \in S$. Since $\gamma^{\Phi}(s,s) \in \mathfrak{h}$ for all $s \in S$ from (1) of Proposition 11 and $\odot^2 S = \wedge^1 V \oplus \wedge^2 V$, the claim follows. $\square$

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To proceed further, we need to consider the case where $\varphi \neq 0$, $a = b = 0$. It is however sufficient to consider φ up to the action of $\mathrm{CSO}(V) = \mathbb{R}^\times \times \mathrm{SO}(V)$. To see it, we note that the group CSpin(V) with Lie algebra $\mathfrak{co}(V)$ is a double-cover of $\mathrm{CSO}(V)$ and it naturally acts on the Poincaré superalgebra $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ by 0-degree Lie superalgebra automorphisms ($t\,\mathrm{Id} \in$ CSpin(V) acts with eigenvalues $0, e^{-t}$ and e^{-2t} on, respectively, $\mathfrak{so}(V)$, S and V). In particular any element $c \in$ CSpin(V) sends a $\mathbb{Z}$-graded subalgebra $\mathfrak{a} = V \oplus S \oplus \mathfrak{h}$ of $\mathfrak{p}$ into an (isomorphic) $\mathbb{Z}$-graded subalgebra $\mathfrak{a}' = c \cdot \mathfrak{a} = V \oplus S \oplus (c \cdot \mathfrak{h})$ of $\mathfrak{p}$ and, if $\mathfrak{g}$ is a filtered deformation of $\mathfrak{a}$ associated with φ then $\mathfrak{g}' = c \cdot \mathfrak{g}$ is a filtered deformation of $\mathfrak{a}'$, which is associated with $\varphi' = c \cdot \varphi$.

We will distinguish φ according to whether it is spacelike, timelike or lightlike and denote by $\Pi \subset V$ the line defined by the span of φ. In the first two cases we can decompose $V = \Pi \oplus \Pi^\perp$ into an orthogonal direct sum and $\mathfrak{h}_\varphi = \mathfrak{so}(\Pi^\perp) \subset \mathfrak{so}(V)$. If φ is lightlike, we choose an η-Witt basis for V such that $V = \mathbb{R}\langle \boldsymbol{e}_+, \boldsymbol{e}_-\rangle \oplus W$ and $\varphi = \boldsymbol{e}_+$. Our plane is $\Pi = \mathbb{R}\langle \boldsymbol{e}_+\rangle$ and $\mathfrak{h}_\varphi = \mathfrak{so}(W) \Subset (\boldsymbol{e}_+ \wedge W) \subset \mathfrak{so}(V)$, where $\boldsymbol{e}_+ \wedge W$ is the abelian Lie subalgebra of $\mathfrak{so}(V)$ consisting of null rotations fixing $\boldsymbol{e}_+$. In this case we decompose any $v \in V$ into

$$v = v_+ + v_- + v_\perp \,,$$

where $v_+ \in \Pi$, $v_- \in \mathbb{R}\langle \boldsymbol{e}_-\rangle$ and $v_\perp \in W$.

Lemma 13. *Let $[\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ be a nontrivial and $\mathfrak{h}$-invariant cohomology class with $\varphi \neq 0$ and $a = b = 0$. Then $\mathrm{Im}(\gamma^\Phi) = \mathfrak{h}_\varphi$ and there exists a unique cocycle representative $\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}$ for which $\gamma^\Phi(s,s) - \widetilde{\psi}(\kappa(s,s)) = 0$ for all $s \in S$.*

Proof. We already know from (2) of Proposition 11 that $\mathrm{Im}(\gamma^\Phi) \subset \mathfrak{h}_\varphi$. In addition:

$$\begin{aligned}
\eta(w, \gamma^\Phi(s,s)v) &= \eta(w, \kappa(s, (v \cdot \varphi + 3\varphi \cdot v) \cdot \mathrm{vol} \cdot s)) \\
&= -2\eta(w, \kappa(s, v \cdot \varphi \cdot \mathrm{vol} \cdot s)) \\
&= 2\,\langle s, w \cdot v \cdot \imath_\varphi \mathrm{vol} \cdot s\rangle \\
&= 2\,\langle s, \imath_w \imath_v(\imath_\varphi \mathrm{vol}) \cdot s\rangle \,,
\end{aligned} \tag{5.14}$$

for all $v, w \in V$. Using (5.14) and $\odot^2 S = \wedge^1 V \oplus \wedge^2 V$, we first see that $\gamma^\Phi(\wedge^2 V) = 0$. We now break our arguments into two cases, depending on whether or not the line Π corresponding to φ is degenerate.

If φ is spacelike or timelike then from (5.14) we see that $\gamma^\Phi(\Pi) = 0$ whereas

$$\gamma^\Phi|_{\Pi^\perp} : \Pi^\perp \subset \wedge^1 V \longrightarrow \mathfrak{so}(\Pi^\perp)$$

is an $\mathfrak{so}(\Pi^\perp)$-equivariant monomorphism, hence an isomorphism by dimensional reasons. If φ is lightlike we decompose

$$\begin{aligned}
\eta(w, \gamma^\Phi(s,s)v) =& 2\,\langle s, \imath_w \imath_v(\imath_\varphi \mathrm{vol}) \cdot s\rangle \\
=& 2\,\langle s, \imath_{w_\perp} \imath_{v_-}(\imath_\varphi \mathrm{vol}) \cdot s\rangle + 2\,\langle s, \imath_{w_-} \imath_{v_\perp}(\imath_\varphi \mathrm{vol}) \cdot s\rangle + 2\,\langle s, \imath_{w_\perp} \imath_{v_\perp}(\imath_\varphi \mathrm{vol}) \cdot s\rangle \,,
\end{aligned} \tag{5.15}$$

which readily gives $\gamma^\Phi(\Pi) = 0$, $\gamma^\Phi(e_-)$ is a generator of $\mathfrak{so}(W)$ and, finally, $\gamma^\Phi(W) = e_+ \wedge W$. In this case γ^Φ is an $\mathfrak{h}_\varphi$-equivariant isomorphism from $\mathbb{R}\langle e_-\rangle \oplus W$ to $\mathfrak{h}_\varphi$.

To prove the last statement, we recall that $\gamma^\Phi(s,s) - \widetilde{\psi}(\kappa(s,s)) \in \mathfrak{h}$ for all $s \in S$, by Proposition 11. On the other hand we just saw that the operator $\gamma^\Phi - \widetilde{\psi}(\kappa(-,-))$ acts trivially on $\wedge^2 V \subset \odot^2 S$ and possibly non-trivially only on $\wedge^1 V \subset \odot^2 S$. In other words it is an operator of the form $\psi(\kappa(-,-)) : \odot^2 S \to \mathfrak{h}$ for some $\psi \in C^{2,1}(\mathfrak{a}_-, \mathfrak{a}) = \mathrm{Hom}(V, \mathfrak{h})$ and such a ψ is clearly unique, since $\mathfrak{a}$ is fundamental. Subtracting the coboundary $\partial\psi$ to the cocycle $\beta^\Phi + \gamma^\Phi + \partial\widetilde{\psi}$ gives the last claim. □

We collect here for later use different equivalent characterizations of the map $\widetilde{\psi} : V \to \mathfrak{so}(V)$ associated to the unique cocycle representative of Lemma 13:

(i) $\widetilde{\psi}(\kappa(s,s)) = \gamma^\varphi(s,s)$ for all $s \in S$;

(ii) $\widetilde{\psi}(u) = 2\imath_u \imath_\varphi \mathrm{vol}$ for all $u \in V$;

(iii) $\widetilde{\psi}(u)v = 2\imath_v \imath_u \imath_\varphi \mathrm{vol}$ for all $u, v \in V$;

(iv) $\eta(w, \widetilde{\psi}(u)v) = 2\imath_w \imath_v \imath_u \imath_\varphi \mathrm{vol}$ for all $u, v, w \in V$;

(v) $\widetilde{\psi}(u)s = -(\varphi \wedge u) \cdot \mathrm{vol} \cdot s$ for all $u \in V$ and $s \in S$;

(vi) $(\widetilde{\psi}(u)v) \cdot s = 2(\varphi \wedge u \wedge v) \cdot \mathrm{vol} \cdot s$ for all $u, v \in V$ and $s \in S$.

We also remark that $\widetilde{\psi}$ is an $\mathfrak{h}_\varphi$-equivariant map with the kernel Π and image $\mathfrak{h}_\varphi$.

5.3 Integrability of the infinitesimal deformations

In this section we construct a filtered deformation $\mathfrak{g}$ for any of the nontrivial $\mathfrak{h}$-invariant elements in $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$. Our description of $\mathfrak{g}$ will be very explicit and rely on a direct check of the Jacobi identities. To describe the Lie superalgebra structure of $\mathfrak{g}$, it is convenient to introduce a formal parameter t which keeps track of the order of the deformation. In particular, the original graded Lie superalgebra structure on a subalgebra $\mathfrak{a} = V \oplus S \oplus \mathfrak{h}$ of the Poincaré superalgebra $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ has order t^0 whereas the infinitesimal deformation has order t.

From Proposition 10, Proposition 11, and Lemma 12, Lemma 13 we know that there are two different families of non-trivial filtered deformations $\mathfrak{g}$. The first family has $\varphi = 0$, $a^2 + b^2 \neq 0$ and $\mathfrak{h} = \mathfrak{so}(V)$, that is $\mathfrak{a} = \mathfrak{p}$. In this case $\gamma^\Phi : \odot^2 S \to \mathfrak{so}(V)$ is surjective and by (2) of Proposition 11 the filtered Lie superalgebra $\mathfrak{g}$ has the brackets of the form

$$\begin{aligned} [A, v] &= Av + t\lambda(A, v) & [v, w] &= t^2\delta(v, w) \\ [A, s] &= \sigma(A)s & [v, s] &= t\beta^\Phi(v, s) = tv \cdot (a + b\,\mathrm{vol}) \cdot s \\ [A, B] &= AB - BA & [s, s] &= \kappa(s, s) + t\gamma^\Phi(s, s)\,, \end{aligned} \tag{5.16}$$

where $A, B \in \mathfrak{so}(V)$, $s \in S$, $v, w \in V$, for some maps $\lambda : \mathfrak{so}(V) \otimes V \to \mathfrak{so}(V)$ and $\delta : \wedge^2 V \to \mathfrak{so}(V)$ to be determined. In other words the brackets on $V \otimes S$ and $\odot^2 S$ are respectively given by β^Φ and γ^Φ and we can always assume $\alpha = 0$ without any loss of generality.

The second family has $\varphi \neq 0$, $a = b = 0$ and $\mathfrak{h}$ is a Lie subalgebra of the stabiliser $\mathfrak{h}_\varphi = \mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$, see (iii) of Proposition 11. We recall that $\varphi \in \wedge^1 V$ can be either spacelike, timelike or lightlike. In this case the bracket on $\odot^2 S$ is simply given by the Dirac current and the filtered Lie superalgebra $\mathfrak{g}$ has the form

$$\begin{aligned} [A, v] &= Av + t\lambda(A, v) & [v, w] &= t\alpha(v, w) + t^2\delta(v, w) = t\pi^\alpha(\partial\widetilde{\psi})(v, w) + t^2\delta(v, w) \\ [s, s] &= \kappa(s, s) & &= t\widetilde{\psi}(v)w - t\widetilde{\psi}(w)v + t^2\delta(v, w) \\ [A, s] &= \sigma(A)s & [v, s] &= t\beta(v, s) = t\beta^\Phi(v, s) + t\pi^\beta(\partial\widetilde{\psi})(v, s) \\ [A, B] &= AB - BA & &= -\tfrac{1}{2}t(v \cdot \varphi + 3\varphi \cdot v) \cdot \mathrm{vol} \cdot s + t\widetilde{\psi}(v)s\,, \end{aligned} \tag{5.17}$$

where $A, B \in \mathfrak{h}$, $s \in S$, $v, w \in V$, for some maps $\lambda : \mathfrak{h} \otimes V \to \mathfrak{h}$ and $\delta : \wedge^2 V \to \mathfrak{h}$ to be determined.

To go through all the Jacobi identities systematically, we use the notation $[ijk]$ for $i, j, k = 0, 1, 2$ to denote the identity involving $X \in \mathfrak{a}_{-i}$, $Y \in \mathfrak{a}_{-j}$ and $Z \in \mathfrak{a}_{-k}$. We first consider the second case (5.17), which is slightly more involved, and claim that the Jacobi identities are satisfied if we set both λ and δ to be zero. To show this, it is first convenient to note that $[V, V] \subset V$, $[V, S] \subset S$ and $[S, S] \subset V$ and prove that the putative bracket operations restricted on $V \oplus S$ satisfy the Jacobi identities. We have:

- the [112] identity is satisfied by virtue of the characterization (i) of $\widetilde{\psi}$, the $\mathfrak{h}_\varphi$-equivariance of the Dirac current and the first cocycle condition (2.5);
- the [111] identity is satisfied by virtue of the characterization (i) of $\widetilde{\psi}$ and the second cocycle condition (2.6);
- the [122] identity is satisfied provided
$$[\beta_v, \beta_w]s - \beta_{\alpha(v,w)}s = 0\,, \tag{5.18}$$
for all $v, w \in V$ and $s \in S$;
- the [222] identity is satisfied provided
$$\mathfrak{S}(\alpha(u, \alpha(v, w))) = 0\,, \tag{5.19}$$
where $\mathfrak{S}$ is the cyclic sum on $u, v, w \in V$.

Now using characterization (iv) of $\widetilde{\psi}$ one can check that

$$\eta(x, \widetilde{\psi}(u)\widetilde{\psi}(v)w) = 4\eta(\imath_v\imath_w\imath_\varphi \mathrm{vol}, \imath_x\imath_u\imath_\varphi \mathrm{vol}) = \eta(x, \widetilde{\psi}(\widetilde{\psi}(w)v)u)$$

for all $u, v, w, x \in V$, from which

$$\begin{aligned} \alpha(u, \alpha(v, w)) &= \widetilde{\psi}(u)\widetilde{\psi}(v)w - \widetilde{\psi}(u)\widetilde{\psi}(w)v + \widetilde{\psi}(\widetilde{\psi}(w)v)u - \widetilde{\psi}(\widetilde{\psi}(v)w)u \\ &= 2\widetilde{\psi}(u)\widetilde{\psi}(v)w - 2\widetilde{\psi}(u)\widetilde{\psi}(w)v \\ &= 4\widetilde{\psi}(u)\widetilde{\psi}(v)w \end{aligned}$$

and $\mathfrak{S}(\alpha(u,\alpha(v,w))) = 4\mathfrak{S}(\widetilde{\psi}(u)\widetilde{\psi}(v)w) = 0$ by characterization (iii) of $\widetilde{\psi}$. This is the [222] Jacobi identity (5.19). On the other hand, for all $v, w \in V$ and $s \in S$ we have

$$\begin{aligned}\beta_v\beta_w s &= -\tfrac{1}{2}\beta_v((w\cdot\varphi + 3\varphi\cdot w)\cdot \mathrm{vol}\cdot s) + \beta_v(\widetilde{\psi}(w)(s))\\ &= -(\varphi\wedge v - 2\eta(\varphi,v))\cdot(\varphi\wedge w - 2\eta(\varphi,w))\cdot s - (\varphi\wedge v - 2\eta(\varphi,v))\cdot \mathrm{vol}\cdot\widetilde{\psi}(w)s\\ &\quad - \mathrm{vol}\cdot\widetilde{\psi}(v)(\varphi\wedge w - 2\eta(\varphi,w))\cdot s + \widetilde{\psi}(v)\widetilde{\psi}(w)s\end{aligned}$$

and therefore, repeatedly using equations (A.4) and the fact that $\widetilde{\psi}(u)\varphi = 0$ for all $u \in V$, also

$$\begin{aligned}[\beta_v,\beta_w]s &= -[\varphi\wedge v,\widetilde{\psi}(w)]\,\mathrm{vol}\cdot s + [\varphi\wedge w,\widetilde{\psi}(v)]\,\mathrm{vol}\cdot s\\ &\qquad - [\varphi\wedge v,\varphi\wedge w]s + [\widetilde{\psi}(v),\widetilde{\psi}(w)]s\\ &= (\varphi\wedge\widetilde{\psi}(w)v)\cdot\mathrm{vol}\cdot s - (\varphi\wedge\widetilde{\psi}(v)w)\cdot\mathrm{vol}\cdot s\\ &\qquad - 2\eta(\varphi,\varphi)v\wedge w\cdot s + 2\eta(\varphi,v)\varphi\wedge w\cdot s - 2\eta(\varphi,w)\varphi\wedge v\cdot s\\ &\qquad + [\widetilde{\psi}(v),\widetilde{\psi}(w)]s\ .\end{aligned} \tag{5.20}$$

In a similar way we can prove:

$$\beta_{\alpha(v,w)}s = (\varphi\wedge\widetilde{\psi}(w)v)\cdot\mathrm{vol}\cdot s - (\varphi\wedge\widetilde{\psi}(v)w)\cdot\mathrm{vol}\cdot s + \widetilde{\psi}(\widetilde{\psi}(v)w)s - \widetilde{\psi}(\widetilde{\psi}(w)v)s\ . \tag{5.21}$$

In summary we use (5.20) and (5.21), together with characterizations (v) and (vi) of $\widetilde{\psi}$, to arrive at:

$$\begin{aligned}[\beta_v,\beta_w]s - \beta_{\alpha(v,w)}s &= -2\eta(\varphi,\varphi)v\wedge w\cdot s + 2\eta(\varphi,v)\varphi\wedge w\cdot s - 2\eta(\varphi,w)\varphi\wedge v\cdot s\\ &\qquad + [\widetilde{\psi}(v),\widetilde{\psi}(w)]s - \widetilde{\psi}(\widetilde{\psi}(v)w)s + \widetilde{\psi}(\widetilde{\psi}(w)v)s\\ &= -2\eta(\varphi,\varphi)v\wedge w\cdot s + 2\eta(\varphi,v)\varphi\wedge w\cdot s - 2\eta(\varphi,w)\varphi\wedge v\cdot s\\ &\qquad - [\varphi\wedge v,\varphi\wedge w]s - 2\varphi\cdot(\varphi\wedge v\wedge w)s - 2(\varphi\wedge v\wedge w)\cdot\varphi\cdot s\\ &= 0\ ,\end{aligned}$$

proving the [122] Jacobi identity (5.18).

Let $\mathfrak{g}_- = (V\oplus S, [-,-])$ be the filtered Lie superalgebra structure on $V\oplus S$ we have just described. Note that the Lie bracket of $\mathfrak{g}_-$ is defined in terms of $\widetilde{\psi}$, Clifford multiplication, Dirac current of spinors and the vector φ, so that the stabilizer $\mathfrak{h}_\varphi = \mathfrak{so}(V)\cap\mathfrak{stab}(\varphi)$ of φ in $\mathfrak{so}(V)$ acts naturally on $\mathfrak{g}_-$ by outer derivations. It is then clear from (5.17) that, for any subalgebra $\mathfrak{h}$ of $\mathfrak{h}_\varphi$, the semidirect sum $\mathfrak{g} = \mathfrak{h}\ltimes\mathfrak{g}_-$ is the required filtered deformation of $\mathfrak{a} = V\oplus S\oplus\mathfrak{h}$.

We now consider the first case (5.16) and set λ to be zero. We have:

- the [000] identity is satisfied since $\mathfrak{so}(V)$ is a Lie algebra;
- the [001] and [002] identities are satisfied because S and V are $\mathfrak{so}(V)$-modules;
- the [011] and [012] identities are satisfied because the $[SS]$, $[SV]$ Lie brackets are $\mathfrak{so}(V)$-equivariant;

- the [111] identity is satisfied by virtue of the second cocycle condition (2.6);
- the [022] identity requires $\delta : \wedge^2 V \to \mathfrak{so}(V)$ to be $\mathfrak{so}(V)$-equivariant;
- the [222] identity is satisfied provided

$$\mathfrak{S}(\delta(v,w)u) = 0 \ , \tag{5.22}$$

where $\mathfrak{S}$ is the cyclic sum on $v, w, u \in V$;
- the [122] identity is satisfied provided

$$\delta(v,w)s = [\beta^\Phi_v, \beta^\Phi_w]s \ , \tag{5.23}$$

for all $v, w \in v$ and $s \in S$;
- the [112] identity has a component of order t, which is satisfied by virtue of the first cocycle condition (2.5) and one of order t^2, which reads

$$\delta(v, \kappa(s,s)) = 2\gamma^\Phi(\beta^\Phi_v s, s) \ , \tag{5.24}$$

for all $v \in V$ and $s \in S$;

Since $\wedge^2 V$ is an irreducible $\mathfrak{so}(V)$-representation of complex type, we have that the [022] Jacobi identity is satisfied if and only if there exist $r, r' \in \mathbb{R}$ such that

$$\delta(v,w)u = r(\eta(v,u)w - \eta(w,u)v) + r' \star (v \wedge w \wedge u) \ ,$$

for all $v, w, u \in V$. However it is easy to see that (5.22) implies $r' = 0$.

We will now show that (5.23) and (5.24) hold true for $r = 4(a^2 + b^2)$. Indeed:

$$\begin{aligned}
\delta(v,w)s &= \tfrac{r}{4}(v \cdot w \cdot s - w \cdot v \cdot s) \ , \\
[\beta^\Phi_v, \beta^\Phi_w]s &= v \cdot (a + b\,\mathrm{vol}) \cdot w \cdot (a + b\,\mathrm{vol}) \cdot s - w \cdot (a + b\,\mathrm{vol}) \cdot v \cdot (a + b\,\mathrm{vol}) \cdot s \\
&= (a^2 + b^2)(v \cdot w \cdot s - w \cdot v \cdot s) \ ,
\end{aligned}$$

for all $v, w \in V$, $s \in S$, whereas

$$\begin{aligned}
\eta(\delta(v, \kappa(s,s))u, w) &= r(\eta(v,u)\eta(\kappa(s,s), w) - \eta(\kappa(s,s), u)\eta(v,w)) \\
&= r(\eta(v,u) \langle s, w \cdot s\rangle - \eta(v,w) \langle s, u \cdot s\rangle) \ , \\
2\eta(\gamma^\Phi(\beta^\Phi_v s, s)u, w) &= -2\eta(\kappa(\beta^\Phi_v s, \beta^\Phi_u s), w) - 2\eta(\kappa(s, \beta^\Phi_u \beta^\Phi_v s), w) \\
&= 2(a^2 + b^2)(\langle s, v \cdot w \cdot u \cdot s\rangle - \langle s, w \cdot u \cdot v \cdot s\rangle) \\
&= 4(a^2 + b^2)(\eta(v,u) \langle s, w \cdot s\rangle - \eta(v,w) \langle s, u \cdot s\rangle) \ ,
\end{aligned}$$

for all $v, w, u \in V$, $s \in S$.

5.4 Summary

We summarise the results of sections 5.1, 5.2 and 5.3 in the following

Theorem 14. *There are exactly two families of nontrivial filtered deformations* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *of* $\mathbb{Z}$*-graded subalgebras* $\mathfrak{a} = V \oplus S \oplus \mathfrak{h}$ *of the Poincaré superalgebra* $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$*, which we now detail:*

1. *In this case* $\mathfrak{h} = \mathfrak{so}(V)$*, there exist* $a, b \in \mathbb{R}$ *such that* $a^2 + b^2 \neq 0$ *and the Lie brackets of* $\mathfrak{g}$ *are given by*

$$\begin{aligned} [A, v] &= Av & [v, w] &= 4(a^2 + b^2) v \wedge w \\ [A, s] &= \sigma(A)s & [v, s] &= v \cdot (a + b\,\mathrm{vol}) \cdot s \\ [A, B] &= AB - BA & [s, s] &= \kappa(s, s) + \gamma^{(a,b)}(s, s)\,, \end{aligned} \tag{5.25}$$

   where $v, w \in V$*,* $s \in S$*,* $A, B \in \mathfrak{so}(V)$ *and* $\gamma^{(a,b)}(s, s) \in \mathfrak{so}(V)$ *is defined by*

$$\gamma^{(a,b)}(s, s)v = -2\kappa(s, v \cdot (a + b\,\mathrm{vol}) \cdot s)\ ;$$

2. *In this case there exists a nonzero* $\varphi \in V$*,* $\mathfrak{h}$ *is any Lie subalgebra of the stabiliser* $\mathfrak{h}_\varphi = \mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$ *of* φ *in* $\mathfrak{so}(V)$ *and the Lie brackets of* $\mathfrak{g}$ *are given by*

$$\begin{aligned} [A, v] &= Av & [v, w] &= \widetilde{\psi}(v)w - \widetilde{\psi}(w)v \\ [A, s] &= \sigma(A)s & [v, s] &= -\frac{1}{2}(v \cdot \varphi + 3\varphi \cdot v) \cdot \mathrm{vol} \cdot s + \widetilde{\psi}(v)s \\ [A, B] &= AB - BA & [s, s] &= \kappa(s, s)\,, \end{aligned} \tag{5.26}$$

   where $v, w \in V$*,* $s \in S$*,* $A, B \in \mathfrak{h}$ *and* $\widetilde{\psi}(v) \in \mathfrak{h}_\varphi$ *is defined by* $\widetilde{\psi}(v) = 2\imath_v \imath_\varphi \mathrm{vol}$*. In particular* $\mathfrak{g}_- = V \oplus S$ *is an ideal of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{g}_-$ *is the semidirect sum of* $\mathfrak{h}$ *and* $\mathfrak{g}_-$ *(*$\mathfrak{h}$ *acts on* $\mathfrak{g}_-$ *by restricting the vector and spinor representations of* $\mathfrak{so}(V)$*).*

Note that the associated homogeneous Lorentzian manifolds $(M = G/H, g)$*,* $Lie(G) = \mathfrak{g}_{\bar{0}}$*,* $Lie(H) = \mathfrak{h}$ *always admit a reductive decomposition* $\mathfrak{g}_{\bar{0}} = \mathfrak{h} \oplus V$*. In the first family* (M, g) *is locally isometric to* AdS_4 *whereas in the second family the geometry is that of a Lie group with a bi-invariant metric, more precisely:*

- *(i) If* φ *is spacelike then* $\mathfrak{h}_\varphi \simeq \mathfrak{so}(1, 2)$ *and* (M, g) *is locally isometric to* $\mathrm{AdS}_3 \times \mathbb{R}$*;*
- *(ii) If* φ *is timelike then* $\mathfrak{h}_\varphi \simeq \mathfrak{so}(3)$ *and* (M, g) *is locally isometric to* $\mathbb{R} \times \mathrm{S}^3$*;*
- *(iii) If* φ *is lightlike then* $\mathfrak{h}_\varphi \simeq \mathfrak{so}(2) \ltimes \mathbb{R}^2$ *and we have the so-called Nappi-Witten group [57], a central extension of the Lie group of Euclidean motions of the plane. Explicitly, if we choose an* η*-Witt basis for* V *with* $\varphi = e_+$*, then the only nonzero Lie brackets of the Lie algebra of the Nappi-Witten group are:*

$$[e_-, e_1] = 4e_2\ , \qquad [e_-, e_2] = -4e_1\ , \qquad [e_1, e_2] = -4e_+\ .$$

6 Conclusions

In this paper, we have considered the supersymmetries of rigid supersymmetric field theories on Lorentzian four-manifolds from the viewpoint of their Killing superalgebras.

We showed that the relevant Killing spinor equations, which we identify with the defining condition for bosonic supersymmetric backgrounds of minimal off-shell supergravity in four dimensions, admit a cohomological interpretation in terms of the Spencer group $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ of the $N{=}1$ Poincaré superalgebra $\mathfrak{p}$ in four dimensions. This result is in analogy with a similar result in eleven dimensions [39, 40].

We then gave a self-contained proof of the fact that supergravity Killing spinors generate a Lie superalgebra, and that this Lie superalgebra is a filtered subdeformation of $\mathfrak{p}$. Finally we classified, up to local isometry, the geometries admitting the maximum number of Killing spinors: Minkowski space, AdS_4 and the nonabelian Lie groups with a Lorentzian bi-invariant metric, namely $\mathrm{AdS}_3 \times \mathbb{R}$, $\mathbb{R} \times \mathrm{S}^3$ and the Nappi-Witten group NW_4. Our approach here is based on two independent arguments. In section 4 we solved the flatness equations for the connection defining the Killing spinor equations and described the corresponding Lorentzian geometries. In section 5 we used again Spencer cohomology techniques to describe the filtered subdeformations of $\mathfrak{p}$ with maximum odd dimension and recovered in this way the Killing superalgebras of the maximally supersymmetric backgrounds.

None of the geometries in our classification are new. The novelty in this paper lies in our approach, which *systematises* the search for backgrounds on which one can define rigid supersymmetric field theories by mapping it to an algebraic problem on which we can bring to bear representation-theoretic techniques. In forthcoming work, we shall apply these techniques to a broader class of field theories with rigid supersymmetry in higher dimensions.

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A Conventions and spinorial algebraic identities

In this appendix we define our conventions for Clifford algebras, spinors and derive a number of useful algebraic identities we will have ample opportunity to apply in the bulk of the paper.

A.1 Clifford algebra conventions

Let (V, η) be a four-dimensional Lorentzian vector space, by which we mean that η has signature -2 ("mostly minus"). We may choose an η-orthonormal basis $\boldsymbol{e}_\mu = (\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ with $\eta_{\mu\nu} = \eta(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = \mathrm{diag}(+1, -1, -1, -1)$. Such a basis defines an isomorphism $(V, \eta) \cong \mathbb{R}^{1,3}$.

The Clifford algebra $C\ell(V)$ associated to (V,η) is the real, associative, unital algebra generated by V (and the identity $\mathbb{1}$) subject to the Clifford relation (please notice the sign!)

$$v^2 = -\eta(v,v)\mathbb{1} \qquad \forall\, v \in V\ . \tag{A.1}$$

As a vector space, $C\ell(V) \cong \Lambda V = \bigoplus_{p=0}^{4} \wedge^p V$. If $v \in V$ and $\phi \in \wedge^p V$, their Clifford product, denoted by $\cdot$, is given by

$$v \cdot \phi = v \wedge \phi - \iota_{v^\flat}\phi\,, \tag{A.2}$$

where $v^\flat \in V^*$ is the dual covector defined by the inner product: $v^\flat(w) = \eta(v,w)$, for all $w \in V$. We will often drop the superscript $\flat$ if it is unambiguous to do so. The Clifford algebra is not commutative:

$$\phi \cdot v = (-1)^p \left(v \wedge \phi + \iota_v \phi\right)\ . \tag{A.3}$$

Continuing in this way we may derive the Clifford product of $\phi \in \wedge^p V$ with bivectors:

$$\begin{aligned}(v \wedge w) \cdot \phi &= v \wedge w \wedge \phi + \iota_v \iota_w \phi - v \wedge \iota_w \phi + w \wedge \iota_v \phi \\ (v \wedge w) \cdot \phi &= v \wedge w \wedge \phi + \iota_v \iota_w \phi + v \wedge \iota_w \phi - w \wedge \iota_v \phi\ .\end{aligned} \tag{A.4}$$

Let us introduce the volume element $\mathrm{vol} = \boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \in \wedge^4 V$. It obeys

$$\mathrm{vol}^2 = -\mathbb{1} \qquad \text{and} \qquad \mathrm{vol} \cdot \phi = (-1)^p \phi \cdot \mathrm{vol}\,,$$

for $\phi \in \wedge^p V$. In particular, it is not central. Clifford multiplication by the volume element agrees (up to a sign) with Hodge duality:

$$\star\, \phi = (-1)^{p(p+1)/2} \phi \cdot \mathrm{vol}\,, \tag{A.5}$$

for $\phi \in \wedge^p V$. It follows that $\star^2 = (-1)^{p+1}$ on $\wedge^p V$. In particular, it is a complex structure on bivectors, as expected.

The Lie algebra $\mathfrak{so}(V)$ of η-skewsymmetric endomorphisms of V is isomorphic, as a vector space, to $\wedge^2 V$. If $v \wedge w \in \wedge^2 V$, then the corresponding endomorphism is defined by

$$(v \wedge w)(u) = \iota_{u^\flat}(v \wedge w) = \eta(u,v)w - \eta(u,w)v\ . \tag{A.6}$$

We embed $\mathfrak{so}(V)$ in $C\ell(V)$ by sending

$$v \wedge w \mapsto \tfrac{1}{4}[v,w] = \tfrac{1}{4}(v \cdot w - w \cdot v)\ . \tag{A.7}$$

Indeed, one checks that the Clifford commutator

$$\left[\tfrac{1}{4}[v,w], u\right] = \eta(u,v)w - \eta(u,w)v\,,$$

agrees with equation (A.6).

A.2 Clifford module conventions

The Clifford algebra $Cl(V)$ is isomorphic, as a real associative algebra, to the algebra $\mathrm{Mat}_4(\mathbb{R})$ of 4×4 real matrices. Being simple, this algebra has a unique (up to isomorphism) nontrivial irreducible module, which is real and four-dimensional. Let S denote the unique (up to isomorphism) irreducible $C\ell(V)$-module, so that $C\ell(V) \cong \mathrm{End}S$. Restricting to $\mathfrak{so}(V) \subset C\ell(V)$, we obtain a representation σ of $\mathfrak{so}(V)$ on S:

$$\sigma(v \wedge w)s = \tfrac{1}{4}[v, w] \cdot s \ . \tag{A.8}$$

On S we have a symplectic structure $\langle -, - \rangle$ realising one of the canonical anti-involutions of the Clifford algebra:

$$\langle v \cdot s_1, s_2 \rangle = - \langle s_1, v \cdot s_2 \rangle \ , \tag{A.9}$$

for all $v \in V$ and $s_1, s_2 \in S$. It follows that it is also $\mathfrak{so}(V)$-invariant:

$$\langle \sigma(A)s_1, s_2 \rangle = - \langle s_1, \sigma(A)s_2 \rangle \ , \tag{A.10}$$

for all $A \in \mathfrak{so}(V)$. More generally, it follows from repeated application of equation (A.9), that if $\phi \in \wedge^p V$, then

$$\langle \phi \cdot s_1, s_2 \rangle = (-1)^{p(p+1)/2} \langle s_1, \phi \cdot s_2 \rangle \ . \tag{A.11}$$

We can therefore decompose $\mathrm{End}S \cong \odot^2 S \oplus \wedge^2 S$ into representations of $\mathfrak{so}(V)$ as

$$\odot^2 S \cong \wedge^1 V \oplus \wedge^2 V \cong V \oplus \mathfrak{so}(V) \qquad \text{and} \qquad \wedge^2 S \cong \wedge^0 V \oplus \wedge^3 V \oplus \wedge^4 V \cong 2\mathbb{R} \oplus V \ . \tag{A.12}$$

Associated with $s \in S$ there is a vector κ, called the *Dirac current* of s, that is defined by

$$\eta(\kappa, v) = \langle s, v \cdot s \rangle \ , \tag{A.13}$$

for all $v \in V$. There is also a *Dirac 2-form* $\omega^{(2)}$ defined by

$$\omega^{(2)}(v, w) = \langle s, v \cdot w \cdot s \rangle \ . \tag{A.14}$$

(One checks that indeed $\omega^{(2)}(v, w) = -\omega^{(2)}(w, v)$.) In addition we have a second 2-form $\widetilde{\omega}^{(2)}$ and a 3-form $\omega^{(3)}$ defined by

$$\widetilde{\omega}^{(2)}(v, w) = \langle s, v \cdot w \cdot \mathrm{vol} \cdot s \rangle \qquad \text{and} \qquad \omega^{(3)}(u, v, w) = \langle s, u \cdot v \cdot w \cdot \mathrm{vol} \cdot s \rangle \ . \tag{A.15}$$

It follows that

$$\widetilde{\omega}^{(2)} = - \star \omega^{(2)} \qquad \text{and} \qquad \omega^{(3)} = - \star \omega^{(1)} , \tag{A.16}$$

where $\omega^{(1)} = \kappa^\flat$ is the one-form dual to the Dirac current.

A.2.1 Gamma matrices

We denote the endomorphism of S corresponding to $\boldsymbol{e}_\mu \in V$ by Γ_μ and note that the Clifford relation (A.1) turns into the well-known

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = -2\eta_{\mu\nu}\mathbb{1}\,, \tag{A.17}$$

where we let $\mathbb{1}$ denote also the identity endomorphism of S. The vector space isomorphism $C\ell(V) \cong \Lambda V$ defines a vector space isomorphism End$S \cong \Lambda V$ and this in turns defines the standard $\mathbb{R}$-basis of End$S$:

$$\mathbb{1} \qquad \Gamma_\mu \qquad \Gamma_{\mu\nu} \qquad \Gamma_\mu\Gamma_5 \qquad \Gamma_5\,, \tag{A.18}$$

where we have introduced $\Gamma_5 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3$ as the endomorphism corresponding to the volume element and $\Gamma_{\mu\nu} = \frac{1}{2}[\Gamma_\mu, \Gamma_\nu]$. In the same way we define the totally skewsymmetric products $\Gamma_{\mu\nu\rho}$ and $\Gamma_{\mu\nu\rho\sigma}$, which obey

$$\Gamma_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\Gamma^\sigma\Gamma_5 \qquad \Gamma_{\mu\nu}\Gamma_5 = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma} \qquad \Gamma_5 = -\tfrac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\Gamma^{\mu\nu\rho\sigma}\,, \tag{A.19}$$

where $\epsilon_{0123} = +1$, we raise and lower indices with η and where the Einstein summation convention is in force. Some useful identities involving $\epsilon_{\mu\nu\rho\sigma}$ are

$$\tfrac{1}{6}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\nu\rho\sigma} = -\delta^\alpha_\mu \qquad \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho\sigma} = -\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\right)\,, \tag{A.20}$$

and

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\sigma} = -\left(\delta^\alpha_\mu\delta^\beta_\nu\delta^\gamma_\rho - \delta^\alpha_\mu\delta^\gamma_\nu\delta^\beta_\rho + \delta^\beta_\mu\delta^\gamma_\nu\delta^\alpha_\rho - \delta^\beta_\mu\delta^\alpha_\nu\delta^\gamma_\rho + \delta^\gamma_\mu\delta^\alpha_\nu\delta^\beta_\rho - \delta^\gamma_\mu\delta^\beta_\nu\delta^\alpha_\rho\right)\,, \tag{A.21}$$

whereas some useful trace-like identities involving the Γ_μ are

$$\Gamma^\nu\Gamma_\mu\Gamma_\nu = 2\Gamma_\mu \qquad \text{and} \qquad \Gamma^\rho\Gamma_{\mu\nu}\Gamma_\rho = 0\,. \tag{A.22}$$

Let $A \in \mathfrak{so}(V)$ be an η-skewsymmetric endomorphism of V. Its matrix relative to an η-orthonormal basis $\boldsymbol{e}_\mu$ has entries $A^\nu{}_\mu$ defined by

$$A\boldsymbol{e}_\mu = \boldsymbol{e}_\nu A^\nu{}_\mu\,, \tag{A.23}$$

whose corresponding skew-symmetric bilinear form has entries

$$\eta(\boldsymbol{e}_\nu, A\boldsymbol{e}_\mu) = A_{\nu\mu}\,. \tag{A.24}$$

This in turn gives rise to a bivector $\frac{1}{2}A^{\nu\mu}\boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu$ and the map $\mathfrak{so}(V) \to \wedge^2 V$ thus defined is the inverse to the one in equation (A.6). From equation (A.8), we see that the spin representation $\sigma : \mathfrak{so}(V) \to$ End$S$ sends A to

$$\sigma(A) = \tfrac{1}{4}A^{\nu\mu}\Gamma_{\mu\nu} = -\tfrac{1}{4}A^{\mu\nu}\Gamma_{\mu\nu}\,. \tag{A.25}$$

It is often convenient to introduce the notation $\overline{s}_1 s_2 = \langle s_1, s_2\rangle$ and hence to write the components of the Dirac current and the Dirac 2-form as

$$\kappa^\mu = \overline{s}\Gamma^\mu s \qquad \text{and} \qquad \omega^{(2)}_{\mu\nu} = \overline{s}\Gamma_{\mu\nu}s\,, \tag{A.26}$$

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and similarly for their (negative) duals

$$\widetilde{\omega}^{(2)}_{\mu\nu} = \overline{s}\Gamma_{\mu\nu}\Gamma_5 s \qquad \text{and} \qquad \omega^{(3)}_{\mu\nu\rho} = \overline{s}\Gamma_{\mu\nu\rho}\Gamma_5 s\,, \tag{A.27}$$

which, using the relations (A.19), can be expressed as

$$\widetilde{\omega}^{(2)}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\omega^{(2)\rho\sigma} \qquad \text{and} \qquad \omega^{(3)}_{\mu\nu\rho} = -\epsilon_{\mu\nu\rho\sigma}\kappa^\sigma\ . \tag{A.28}$$

A.3 Spinorial identities

Let $s_1, s_2 \in S$. The rank-one endomorphism $s_2\overline{s}_1$ defined by $(s_2\overline{s}_1)(s) = (\overline{s}_1 s)s_2$ can be expressed in terms of the standard basis for End$S$ via the *Fierz identity*

$$s_2\overline{s}_1 = \tfrac{1}{4}\left((\overline{s}_1 s_2)\mathbb{1} - (\overline{s}_1\Gamma^\mu s_2)\Gamma_\mu - \tfrac{1}{2}(\overline{s}_1\Gamma^{\mu\nu}s_2)\Gamma_{\mu\nu} - (\overline{s}_1\Gamma^\mu\Gamma_5 s_2)\Gamma_\mu\Gamma_5 - (\overline{s}_1\Gamma_5 s_2)\Gamma_5\right), \tag{A.29}$$

which specialises when $s_1 = s_2 = s$ to

$$s\overline{s} = -\tfrac{1}{4}\kappa - \tfrac{1}{4}\omega^{(2)} = -\tfrac{1}{4}\left(\kappa^\mu\Gamma_\mu + \tfrac{1}{2}\omega^{(2)}_{\mu\nu}\Gamma^{\mu\nu}\right)\ . \tag{A.30}$$

There are a number of algebraic identities relating a spinor s, its Dirac current and Dirac 2-form and their duals, which are collected in the following

Proposition 15. *Let $s \in S$ and κ be its Dirac current, $\omega^{(1)} = \kappa^\flat$, $\omega^{(2)}$ its Dirac 2-form, $\widetilde{\omega}^{(2)} = -\star\omega^{(2)}$ and $\omega^{(3)} = -\star(\kappa^\flat)$. Then the following identities hold:*

(a) $\kappa \cdot s = 0$

(b) $\omega^{(2)} \cdot s = 0$

(c) $\widetilde{\omega}^{(2)} \cdot s = 0$

(d) $\omega^{(3)} \cdot s = 0$

(e) $\eta(\kappa,\kappa) = 0$

(f) $\left(\omega^{(2)},\omega^{(2)}\right)_\eta = 0$

(g) $\left(\widetilde{\omega}^{(2)},\widetilde{\omega}^{(2)}\right)_\eta = 0$

(h) $\left(\omega^{(3)},\omega^{(3)}\right)_\eta = 0$

(i) $\iota_\kappa\omega^{(2)} = 0$

(j) $\iota_\kappa\widetilde{\omega}^{(2)} = 0$

(k) $\iota_\kappa\omega^{(3)} = 0$

(l) $\omega^{(1)} \wedge \omega^{(2)} = 0$

(m) $\omega^{(1)} \wedge \widetilde{\omega}^{(2)} = 0$

(n) $\omega^{(1)} \wedge \omega^{(3)} = 0$

Proof. (a) This is equivalent to $\kappa^\rho\Gamma_\rho s = 0$. Using the Fierz identity (A.30),

$$\begin{aligned}
\kappa^\rho\Gamma_\rho s &= \Gamma_\rho s\overline{s}\Gamma^\rho s \\
&= \Gamma_\rho\left(-\tfrac{1}{4}\left(\overline{s}\Gamma^\mu s\Gamma_\mu + \tfrac{1}{2}\overline{s}\Gamma^{\mu\nu}s\Gamma_{\mu\nu}\right)\right)\Gamma^\rho s \\
&= -\tfrac{1}{4}(\overline{s}\Gamma^\mu s)\Gamma_\rho\Gamma_\mu\Gamma^\rho s - \tfrac{1}{8}(\overline{s}\Gamma^{\mu\nu}s)\Gamma_\rho\Gamma_{\mu\nu}\Gamma^\rho s \\
&= -\tfrac{1}{2}(\overline{s}\Gamma^\mu s)\Gamma_\mu s \\
&= -\tfrac{1}{2}\kappa^\mu\Gamma_\mu s\,,
\end{aligned}$$

where we have used the trace identities (A.22).

(b) Using that $\overline{s}s = 0$, we see from the Fierz identity (A.30) and part (a) that

$$\omega^{(2)}_{\mu\nu}\Gamma^{\mu\nu}s = 0 \ .$$

(c) This follows from $\widetilde{\omega}^{(2)} = \mathrm{vol}\cdot\omega^{(2)}$ and part (b).

(d) This follows from $\omega^{(3)} = -\,\mathrm{vol}\cdot\kappa$ and part (a).

(e) From (a) it follows that κ is null:

$$\eta(\kappa,\kappa) = \langle s, \kappa\cdot s\rangle = 0 \ .$$

(f) Similarly, from (b) it follows that $\omega^{(2)}$ is null:

$$\left(\omega^{(2)},\omega^{(2)}\right)_\eta = \left\langle s, \omega^{(2)}\cdot s\right\rangle = 0 \ .$$

(g) This follows from the fact that $\omega^{(2)}$ is null and that Hodge duality is an isometry (up to sign).

(h) This follows from the fact that κ is null and that Hodge duality is an isometry (up to sign).

(i) This is equivalent to $\omega^{(2)}(\kappa, v) = 0$ for all v, but

$$\omega^{(2)}(v,\kappa) = \langle s, v\cdot\kappa\cdot s\rangle = 0\,,$$

where we have used (a) above.

(j) This follows from (a) and

$$\widetilde{\omega}^{(2)}(v,\kappa) = \langle s, v\cdot\kappa\cdot \mathrm{vol}\cdot s\rangle = -\,\langle s, v\cdot \mathrm{vol}\cdot\kappa\cdot s\rangle = 0 \ .$$

(k) Again this follows from (a) and

$$\widetilde{\omega}^{(2)}(u,v,\kappa) = \langle s, u\cdot v\cdot\kappa\cdot \mathrm{vol}\cdot s\rangle = -\,\langle s, u\cdot v\cdot \mathrm{vol}\cdot\kappa\cdot s\rangle = 0 \ .$$

(l) We prove the equivalent statement $\star(\omega^{(1)}\wedge\omega^{(2)}) = 0$:

$$\begin{aligned}
\epsilon_{\mu\nu\rho\sigma}\kappa^\nu\omega^{(2)\rho\sigma} &= \epsilon_{\mu\nu\rho\sigma}\kappa^\nu\overline{s}\Gamma^{\rho\sigma}s \\
&= 2\kappa^\nu\overline{s}\Gamma_{\mu\nu}\Gamma_5 s \\
&= 2\kappa^\nu\overline{s}\Gamma_\mu\Gamma_\nu\Gamma_5 s \\
&= -2\overline{s}\Gamma_\mu\Gamma_5\kappa^\nu\Gamma_\nu s = 0\,,
\end{aligned}$$

again using (a) above.

(m) Similar to the previous part, we prove that $\star(\omega^{(1)} \wedge \omega^{(2)}) = 0$:

$$\begin{aligned} \epsilon_{\mu\nu\rho\sigma}\kappa^\nu \widetilde{\omega}^{(2)\rho\sigma} &= \epsilon_{\mu\nu\rho\sigma}\kappa^\nu \overline{s}\Gamma^{\rho\sigma}\Gamma_5 s \\ &= -2\kappa^\nu \overline{s}\Gamma_{\mu\nu} s \\ &= -2\kappa^\nu \overline{s}\Gamma_\mu \Gamma_\nu s = 0\,, \end{aligned}$$

again using (a).

(n) By definition of Hodge star and (e) above,

$$\omega^{(1)} \wedge \omega^{(3)} = -\omega^{(1)} \wedge \star\omega^{(1)} = -\eta(\omega^{(1)}, \omega^{(1)})\,\mathrm{vol} = 0\ .$$

□

Two remarks are worth mentioning. The first is that from parts (l), (m) and (n) in the above proposition, it follows that $\omega^{(2)} = \omega^{(1)} \wedge \theta$, $\widetilde{\omega}^{(2)} = \omega^{(1)} \wedge \widetilde{\theta}$ and $\omega^{(3)} = \omega^{(1)} \wedge \theta^{(2)}$ for some covectors $\theta, \widetilde{\theta}$ and 2-form $\theta^{(2)}$ which are defined only modulo the ideal generated by $\omega^{(1)}$.

A second remark is that it is possible to prove the above proposition without resorting to the Fierz identity, by exploiting the representation theory of the spin group. The group $\mathrm{Spin}(V)$ sits inside the Clifford algebra $C\ell(V)$ and hence S becomes a $\mathrm{Spin}(V)$-module by restriction. The volume element defines a complex structure on S which is invariant under the spin group. The identity component of the spin group is isomorphic to $\mathrm{SL}(2,\mathbb{C})$ under which S is the fundamental 2-dimensional complex representation. The orbit structure of S under $\mathrm{Spin}(V)$ is therefore very simple; namely, there are two orbits: a degenerate orbit consisting of the zero spinor and an open orbit consisting of all the nonzero spinors. The stabiliser of a nonzero spinor s is the abelian subgroup H_s consisting of the null rotations in the direction of its Dirac current κ, and it is a subgroup of the stabiliser of any object we can construct from s in a $\mathrm{Spin}(V)$-equivariant fashion: e.g., the Dirac current and the Dirac 2-form. Now the H_s-invariant 2-forms can be seen to be of the form $\kappa \wedge \theta$, for some "transverse" 1-form θ, and hence the Dirac 2-form $\omega^{(2)}$ has this form. By equivariance under $H_s < \mathrm{Spin}(V)$, the Clifford product of $\omega^{(2)}$ on s must be again proportional to s, but by squaring we see that the constant of proportionality must be zero. Finally, Clifford multiplication by the spacelike θ is invertible, so it must be that κ Clifford-annihilates s.

A.4 A further property of the Dirac current

For completeness we discuss a further algebraic properties of the Dirac current. Recall that if $s \in S$, its Dirac current κ is defined by equation (A.13). Let us define a symmetric bilinear map $\kappa : S \otimes S \to V$ by

$$\kappa(s_1, s_2) = \tfrac{1}{2}\left(\kappa_{s_1+s_2} - \kappa_{s_1} - \kappa_{s_2}\right), \tag{A.31}$$

where κ_s denotes the Dirac current of s. It follows from the representation theory of $\mathfrak{so}(V)$ that the map κ is surjective onto V. Now consider a linear subspace $S' \subset S$ and let $V' \subset V$ denote the image of the map κ restricted to $S' \otimes S'$. For which S' do we still have that $V' = V$? The following lemma, which is a modification of the similar result in [35] for eleven dimensions, shows that this holds provided $\dim S' > 2$.

Lemma 16. *Let $S' \subset S$ be a linear subspace with $\dim S' > \frac{1}{2} \dim S$. Then the restriction of κ to $S' \otimes S'$ is surjective onto V.*

Proof. Let $S' \subset S$ have $\dim S' > \frac{1}{2} \dim S$. Let $V' = \operatorname{im} \kappa|_{S' \otimes S'}$ and let $v \in (V')^{\perp}$. We want to show that $v = 0$ so that $(V')^{\perp} = 0$ and hence $V' = V$. By definition, v is perpendicular to $\kappa(s_1, s_2)$ for all $s_1, s_2 \in S'$; equivalently, $\langle s_1, v \cdot s_2 \rangle = 0$. This means that Clifford multiplication by v maps $S' \to (S')^{\perp}$, where $\perp$ here means the symplectic perpendicular. Because of the hypothesis on the dimension of S', $\dim (S')^{\perp} < \dim S'$, so that Clifford multiplication by v has nontrivial kernel. By the Clifford relation (A.1), it follows that v is null. In other words, every vector in $(V')^{\perp}$ is null, and this means that $\dim(V')^{\perp} \leq 1$. Now for every $s \in S'$, $\kappa(s, s)$ is null and perpendicular to the null vector v, so that one of two situations must occur: either $v = 0$ or else $\kappa(s, s)$ is collinear with v. Suppose for a contradiction that $v \neq 0$. Then $\kappa(s, s)$ is collinear with v and, by polarisation, so are $\kappa(s_1, s_2)$ for all $s_1, s_2 \in S'$. But this says that V' is one-dimensional, contradicting the fact that $\dim(V')^{\perp} \leq 1$. $\square$

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