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# Limit cycle oscillations at resonances

## *For systems subjected to nonlinear damping or external forces*

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**Abstract.** This paper deals with limit cycles in one degree of freedom systems. The van der Pol equation is an example of an equation describing systems with clear limit cycles in the phase space (displacement-velocity 2 dimensional plane). In this paper, it is shown that a system with nonlinear loading, representing the drag load acting on structures in an oscillatory flow (the drag term of the Morison equation), will in fact exhibit limit cycles at resonance and at higher order resonances. These limit cycles are stable, and model self-excited oscillations. As the damping in the systems is linear and constant, the drag loading will to some degree work as negative damping. The consequences of the existence of these limit cycles are that systems starting at lesser amplitudes in the phase plane will exhibit increased amplitudes until the limit cycle is obtained.

### 1. Introduction

The phase plane is a plane where the system's position and velocity are plotted for an increasing time,  $t$ . The phase plane method, which evolves around finding limit cycles, is adapted from Struble and Martin [1]. We will investigate the phase plane motions of solutions of nonlinear equations considered to model physical phenomena of importance in engineering. We are in particular interested in resonance phenomena between the natural frequency of the system and the imposed load on the system.

### 2. Limit cycles

A closed trajectory in the phase plane, where closed trajectories spiral either towards or away, is termed a *limit cycle*. The trajectories around the limit cycle are not closed. Limit cycles are either stable, half-stable or unstable. When the surrounding trajectories all spiral towards the limit cycle, it is said to be stable. If all the near trajectories spiral away, it is unstable, and if some spiral towards while other spiral away, the limit cycle is half-stable. Half-stable limit cycles are uncommon. The stable limit cycles are of importance as they model systems with self-excited vibrations or oscillations. Systems with a stable limit cycle will, independent of the starting conditions, settle into a steady trajectory in the phase plane, where there is a balance between generation and dissipation of energy.

Limit cycles only occur in systems with nonlinear terms. A linear system may have a closed trajectory, but it is then surrounded by other closed trajectories when we select other initial conditions. Nonlinear systems may be described by ordinary differential equations, containing nonlinearity in the stiffness, damping or loading terms.



For more information about limit cycles and self-excited oscillations, see Hagedorn [2] from page 117.

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x(t) = 0 \quad (1)$$

The van der Pol equation (1) is an example of a system with nonlinear damping, and is well known for exhibiting limit cycles. In the van der Pol Equation, the damping term will, for some time, work as negative damping, and pumping energy into the system. The phase plane diagram for the van der Pol equation with  $\mu=1$  is shown in Figure 1. The starting point, decided by the initial conditions (ICs) of each trajectory is marked by a square on the figure, with an arrow indicating the direction of the trajectories. All trajectories spiral towards the limit cycle. The van der Pol equation has been used in a wide variety of sciences, ranging from electricity, to engineering to medicine [3]. The burst like oscillations emerging towards the limit cycles (when starting from low initial values) have been used as a model to explain tremors in Parkinson disease. The van der Pol equation is further discussed in Bogoliubov and Mitropolsky [4] from page 186 and in Guckerheimer and Holmes [8] from page 67.

### 3. Drag loading

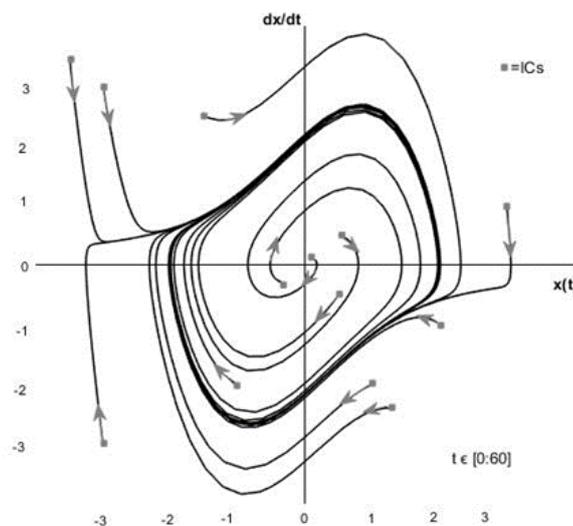
The Morison Equation is used to describe a system with a nonlinear loading term. It is used to describe the force on a cylinder in oscillatory flow. The Morison loading consists of a drag force and a mass force.

In this paper, the following equation is considered:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx(t) = F_0 \sin(\omega t) |\sin(\omega t)| \quad (2)$$

Where the loading term represents the drag force from the Morison loading.  $F_0$  is a constant,  $\omega$  is the loading frequency,  $t$  is the time, and  $m$ ,  $c$  and  $k$  represents the systems mass, damping and stiffness, respectively. Notice that the forcing term is representing a term proportional to the velocity squared, representing turbulent flow past a cylinder (an obstacle in the flow path). In this paper, only one degree of freedom systems (as given by (2)) will be investigated.

It should be noted that  $F_0$  is proportional to the absolute value of the water particle velocities in case of forces caused by fluid flow. For more discussion about the Morison equation, see Gudmestad [5].



**Figure 1:** Phase plane diagram for the unforced van der Pol equation exhibiting a clear limit cycle.

### 3.1. Resonance

Resonance occurs when the frequency of the loading is equal to the frequency of the system, i.e. when  $\omega = \omega_0$ . The systems natural frequency  $\omega_0$  is given as:

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (3)$$

During resonance, the energy from the external source is directly fed into the system, which will have amplitude growth only limited by the damping of the system.

### 3.2. Limit cycles for system with nonlinear loading

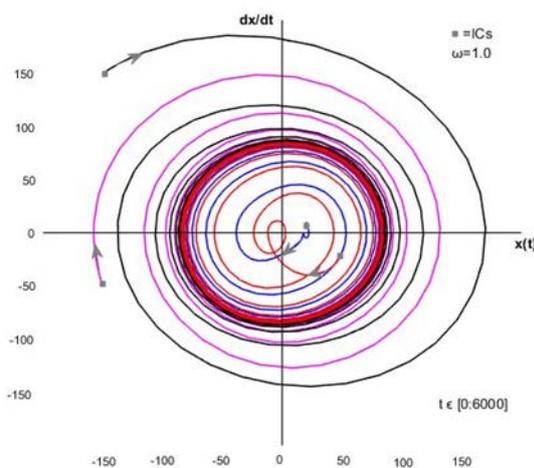
This paper considers equation (2) where examples of solutions are given for linear, constant damping,  $c = 0,5$ .

The Morison equation is not typically associated with limit cycles, and the solutions are rarely represented in the phase plane. As the drag force consists of a term proportional to the water velocity squared, and the damping term,  $c \frac{dx}{dt}$  is connected to the velocity of the structural motion, it is natural to investigate if the drag force can in fact work as damping in the system, and more importantly, if it could exhibit apparent negative damping. Note that the term  $\sin(\omega t) |\sin(\omega t)|$  can be linearized using Fourier series transformation:

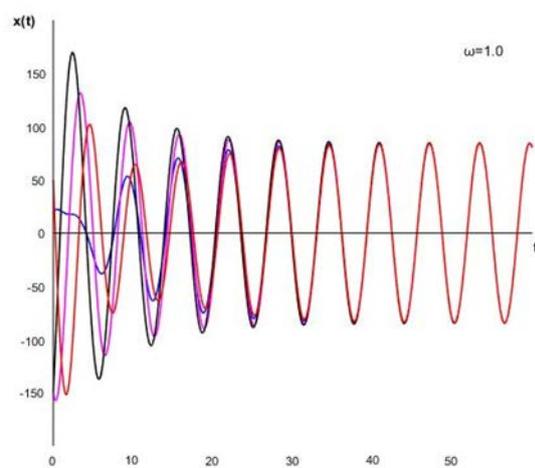
$$\sin(\omega t) |\sin(\omega t)| = c_0 + c_1 \sin(\omega t) + c_2 \cos(2\omega t) + c_3 \sin(3\omega t) + c_4 \cos(4\omega t) + \dots \quad (4)$$

In this paper, the following values are chosen (notice that other values could be chosen as well) to illustrate the behaviour at resonance):

- $F_0 = 50$
- $k = 2$
- $m = 2$  (giving  $T_0 = 2\pi$  s)
- $c = 0,5$
- $\omega = 1,0$  (at resonance)
- $\omega = 0,5$  (at second order resonance)
- $\omega = 0,333$  (at third order resonance)
- $\omega = 0,75$  (out of resonance).



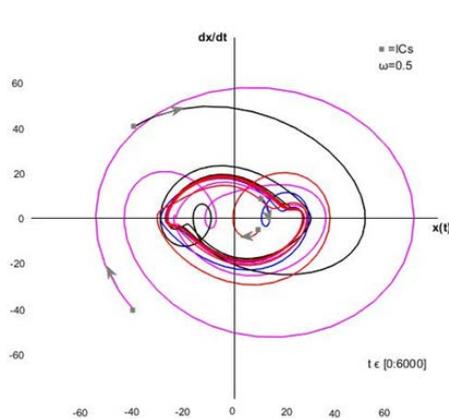
**Figure 2:** Phase plane diagram for a system with nonlinear drag force at resonance,  $\omega=1$ , Four different trajectories (with different initial conditions, IC) are shown, which all spiral towards the limit cycle marked in red.



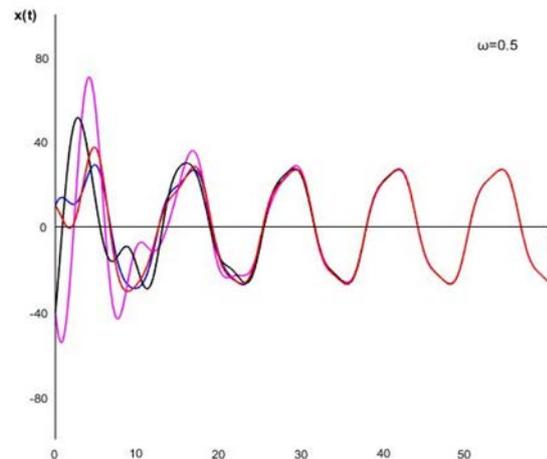
**Figure 3:** Position vs time curve for a system subjected to nonlinear drag force at resonance,  $\omega = 1,0$ . Four different trajectories are shown, overlapping after approximately 35 s.

Figure 2 shows the phase plane diagram for a system at resonance, i.e. when  $\omega = 1$ . It exhibits a clear limit cycle, marked red in the figure, and all near trajectories spiral towards it. The figure shows four different trajectories, marked by starting points and arrows indicating their directions.

Figure 3 shows the same trajectories in the traditional displacement vs time plot. The trajectories are shown for the first 50s. Around  $t = 30$  s, the trajectories are all overlapping, continuing along the same path. Notice the growth of the displacement for the system starting at low initial values.



**Figure 4:** Phase plane diagram for a system with nonlinear drag force at higher order resonance,  $\omega = 0,5$ . Four different trajectories are shown, which all spiral towards the limit cycle marked in red.



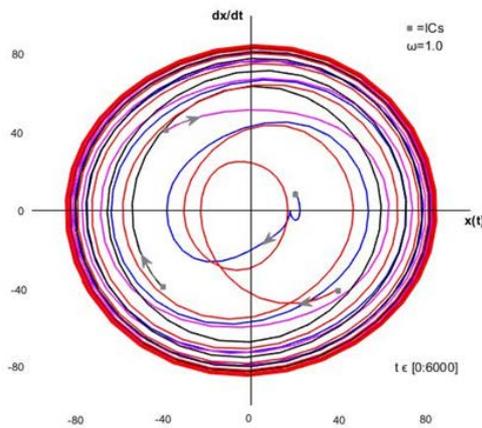
**Figure 5:** Position vs time curve for a system subjected to nonlinear drag force at higher order resonance,  $\omega = 0,5$ . Note that for this system, the trajectories overlap after 25s approximately.

Figure 4 shows the phase plane diagram for a system with  $\omega = 0,5$ , i.e. second order resonance. Four different trajectories are shown, and for this system, as well, a clear limit cycle emerges. Figure 5 shows the position vs time plot for the first 35 s for the system at higher order resonance. This system's trajectories use less time to reach the limit cycle, and are overlapping from approximately  $t = 25$  s.

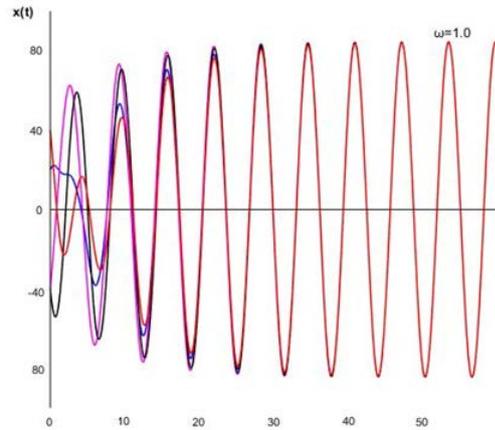
### 3.3. Trajectories experiencing apparent negative damping

Both Figure 2 and Figure 4 show limit cycles in the phase plane. The figures show four trajectories starting at different ICs, all are spiralling towards the limit cycle. Some of the trajectories have ICs making them start outside the limit cycle, spiralling inwards towards the limit cycle. However, some trajectories start on the inside of the limit cycle, experiencing a growth in amplitudes for both position and velocity until they reach the limit cycle. These trajectories are of interest as they experience apparent negative damping, “speeding the system up”.

For clarity of the negative damping effect, Figure 6 shows the phase plane diagram for the system at resonance, with trajectories starting at ICs inside the limit cycle. All the trajectories experience a growth in amplitudes until reaching the limit cycle. Figure 7 shows the same trajectories in the position vs time plane.



**Figure 6:** Phase plane diagram for the system at resonance,  $\omega=1,0$  with four different trajectories, all starting from ICs inside the limit cycle.



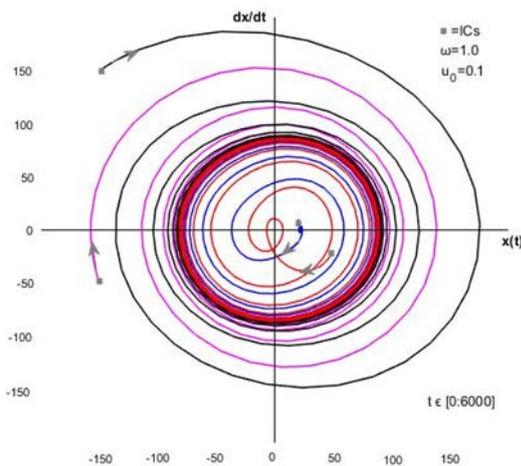
**Figure 7:** Position curve for the system at resonance,  $\omega=1,0$  with trajectories starting inside the limit cycle

### 3.4. Drag loading with current

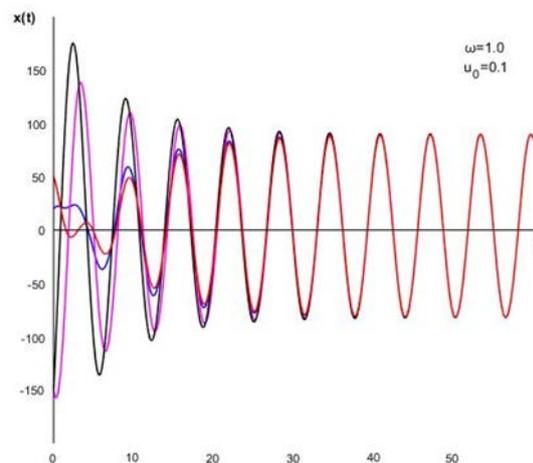
The loading term in Equation (2) is representing drag loading for water waves without current. Adding a current  $u_0$  to the system results in the following equation:

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx(t) = F_0(u_0 + a \sin(\omega t)) |(u_0 + a \sin(\omega t))| \quad (5)$$

Where  $u_0$  and  $a$  are constants. In Figure 8 and Figure 9 the values are set to  $u_0 = 0,1$  and  $a = 1,0$ , respectively, i.e. a small value of the current is added. We obtain the same form of the solutions as discussed before for the case of no current.



**Figure 8:** Phase plane diagram for the system at resonance,  $\omega = 1,0$  with current added from Equation (5).



**Figure 9:** Position vs time curve for the system at resonance,  $\omega = 1,0$ , with current from Equation (5).

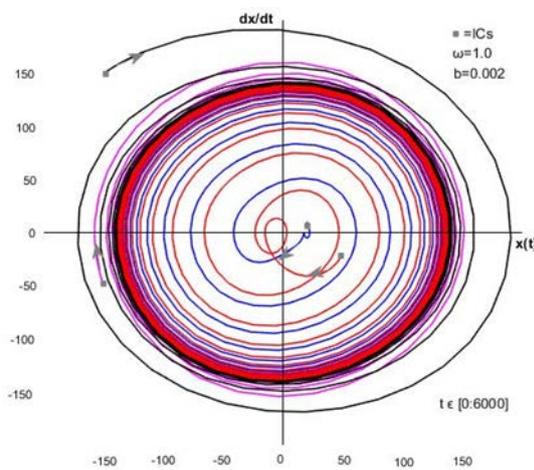
3.5. Drag loading on a flexible structure

For a flexible structure, the drag loading term in sinusoidal waves will, according to [6], take the form:

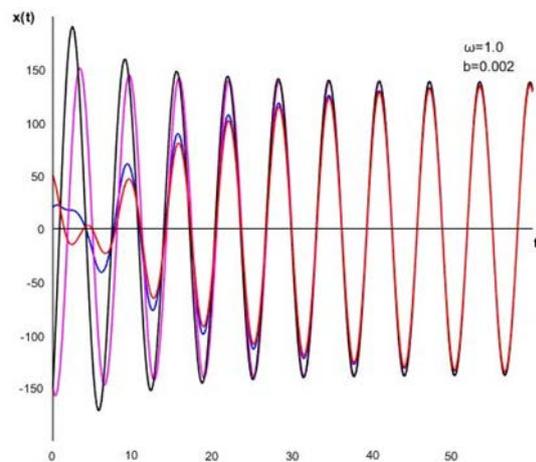
$$F_0 \left( b \frac{dx}{dt} + a \sin(\omega t) \right) \left| b \frac{dx}{dt} + a \sin(\omega t) \right| \tag{6}$$

i.e. as the structure moves, the force will be adjusted by the velocity,  $dx/dt$ , of the structural motion.

The velocity of a wave particle is here given as  $F_0 a \sin(\omega t)$  where the wave velocity amplitude is  $F_0 a$ . The velocity of the structure is given by  $F_0 b \frac{dx}{dt}$ . When the structure is fixed to the bottom, the velocity of the structure is a lot less than the velocity of the wave particles. In order to achieve this, the values of  $a$  and  $b$  in Equation (6) are set to 1,0 and 0,002 respectively. Figure 10 shows the phase plane diagram for four different trajectories, which all spiral towards a limit cycle.



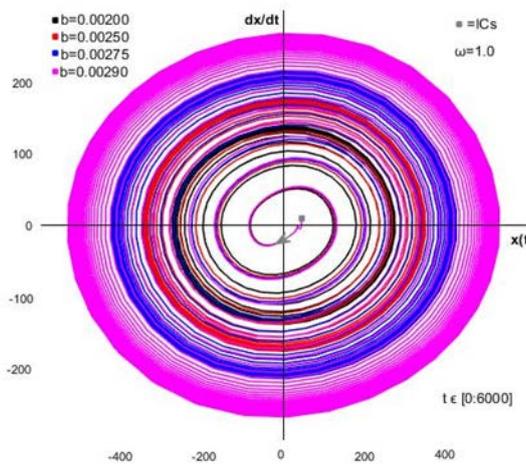
**Figure 10:** Phase plane diagram for a system at resonance subjected to loading from Equation (6). Four different trajectories are shown, all spiralling towards the limit cycle.



**Figure 11:** Position vs time plot for the system at resonance subjected to loading from Equation (6). Note that all the trajectories overlap at approximately  $t = 50$  s.

Figure 11 shows the position vs time plot for the system at resonance with forcing term from Equation (6). The trajectories overlap after approximately 50s, reaching the limit cycle.

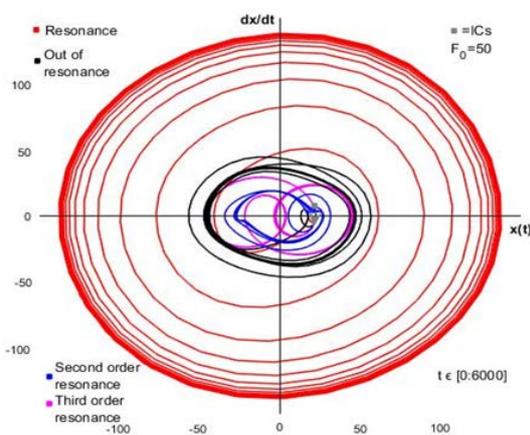
Figure 12 shows the limit cycles for different values of  $b$ . As  $b$  increases, the velocity and position amplitudes of the limit cycle increases, as the structure becomes less stiff, resulting in more structural movements. When  $b$  exceeds approximately 0,029, the phase plane trajectories spiral out from the limit cycle.



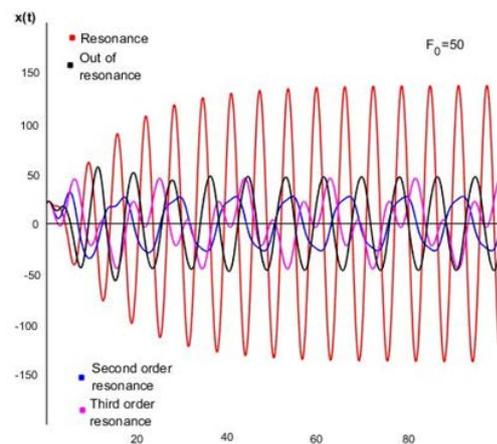
**Figure 12:** As  $b$  increases, the stiffness of the structure decreases, allowing more movement of the structure. This results in larger amplitude limit cycles.

3.6. Comparison of limit cycles for different degrees of resonance

Figure 13 shows the limit cycles for the system subjected to Equation (6) at resonance (red), at second order resonance (blue), at third order resonance (magenta) and out of resonance (black). Figure 14 shows the position vs time plot for the same trajectories.



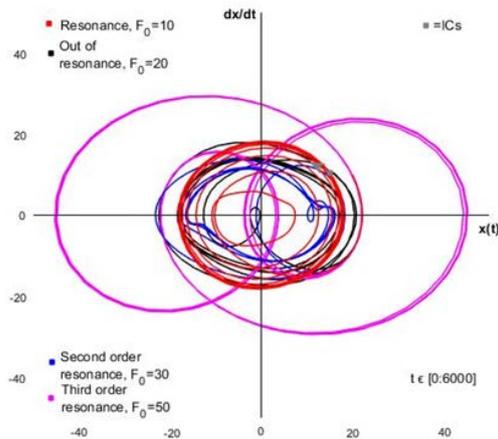
**Figure 13:** Systems with different degree of resonance subjected to loading from Equation (6) in the phase plane.



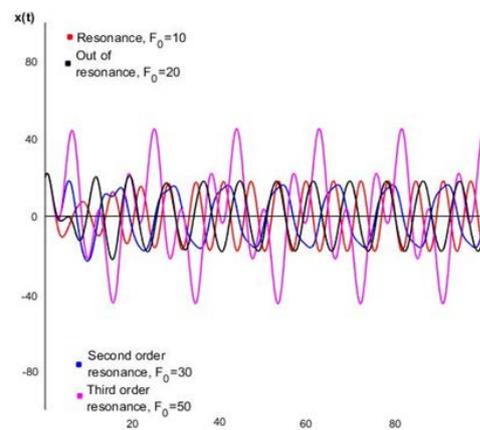
**Figure 14:** Position vs time plot for systems with different degrees of resonance, subjected to loading from Equation (6).

Figure 15 shows the phase plane diagram for systems at varying degrees of resonance, with  $F_0$  adjusted to a realistic value of the nonlinear forcing. The position vs time plot for the same systems are shown in Figure 16. It should be noted that the adjustment of  $F_0$  reflects the estimation of the relative loading from waves with the associated periods (at resonance the wave period is  $2\pi$  and at second order resonance the wave period is  $6\pi$ ).

In this case, we see that the contributions to the displacement from the higher order resonances are more pronounced than from the main resonance due to the larger absolute value of the forcing term in larger amplitude waves (normally associated with larger wave periods).



**Figure 15:** Phase plane diagram for systems at various degrees of resonance and various value of  $F_0$ , subjected to loading from Equation (6).



**Figure 16:** Position vs time plot for systems of varying degrees of resonance and various values of  $F_0$ , subjected to loading from Equation (6).

#### 4. Conclusions

A one degree of freedom system subjected to drag loading and linear, constant damping will have limit cycles near and at resonance. As the damping is constant and linear, the drag loading will to some degree work as negative damping, making the generation and dissipation of energy in the system balanced.

Limit cycles are seen in cases both with and without current, as well as for more flexible systems subjected to drag load on a flexible structure. For higher order resonances,  $\beta = 1/2$  and  $\beta = 1/3$ , similar behaviour is observed, thereby energy from waves with higher periods, for example 15 seconds and 10 seconds, respectively, are fed into a system with natural period of 5 seconds. In storm situations, the energy spectrum has high values for higher periodic waves and considerable displacements and velocities can be expected. The drag load from the Morison equation is used to describe oscillatory flow past an obstacle. Typically, the oscillatory flow is representing sea waves, and the obstacles fixed offshore structures in the sea.

It is suggested that this application may not be the only use for this type of loading model. The heart pumps blood at a certain frequency, so the oscillatory flow could be the blood flowing and the obstacle could be a blood clot. Using the drag load from the Morison equation in this regard would describe the flow in blood veins past obstacles as for example blood clots. The research question suggested is therefore if certain frequencies of the heartbeat would cause resonance in the blood system and the possible movements of the clots in case of changing heart rhythm and more forced blood circulation during physical efforts, a situation most undesirable for the patient.

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