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VALUE REGIONS OF UNIVALENT SELF-MAPS WITH TWO BOUNDARY FIXED POINTS

Pavel Gumenyuk[†] **and Dmitri Prokhorov**[‡]

University of Stavanger, Department of Mathematics and Natural Sciences
N-4036 Stavanger, Norway; pavel.gumenyuk@uis.no

Petrozavodsk State University, Lenina 33, 185910 Petrozavodsk, Russia
and Saratov State University, Department of Mathematics and Mechanics
Astrakhanskaya 83, 410012 Saratov, Russia; ProkhorovDV@info.sgu.ru

Abstract. In this paper we find the exact value region $\mathcal{V}(z_0, T)$ of the point evaluation functional $f \mapsto f(z_0)$ over the class of all holomorphic injective self-maps $f\colon \mathbf{D} \to \mathbf{D}$ of the unit disk $\mathbf{D}$ having a boundary regular fixed point at $\sigma = -1$ with $f'(-1) = e^T$ and the Denjoy–Wolff point at $\tau = 1$.

1. Introduction

Since the seminal paper [11] by Cowen and Pommerenke, the study of holomorphic functions with finite angular derivative at prescribed boundary points has been an active field of research in complex analysis, see, e.g., [2, 3, 10, 15, 17, 33, 38], just to mention some works in the topic.

Given a holomorphic function f in the unit disk $\mathbf{D} := \{z\colon |z| < 1\}$ and a point $\sigma \in \partial\mathbf{D}$ such that there exists finite angular limit $f(\sigma) := \angle\lim_{z\to\sigma} f(z)$, the *angular derivative* at σ is $f'(\sigma) := \angle\lim_{z\to\sigma} \big(f(z) - f(\sigma)\big)/(z - \sigma)$.

On the one hand, for univalent (i.e., holomorphic and injective) functions f, existence of the angular derivative $f'(\sigma)$ different from 0 and ∞ is closely related to the geometry of $f(\mathbf{D})$ near $f(\sigma)$; moreover, if there exists $f'(\sigma) \neq 0, \infty$, then the behaviour of f at the boundary point σ resembles conformality, see, e.g., [32, §§4.3, 11.4].

On the other hand, for the dynamics of a holomorphic (but not necessarily univalent) self-map $f\colon \mathbf{D} \to \mathbf{D}$, a crucial role is played by the points $\sigma \in \partial\mathbf{D}$ for which $f(\sigma) = \sigma$ (or, more generally, $f(\sigma) \in \partial\mathbf{D}$) and the angular derivative $f'(\sigma)$ is finite, see, e.g., [5–7, 8, 9, 14, 16, 31]. Such points σ are called *boundary regular fixed points*, see Section 2 for precise definitions and some basic theory. In particular, a classical result due to Wolff and Denjoy asserts that if $f \in \mathsf{Hol}(\mathbf{D}, \mathbf{D})$ has no fixed points in $\mathbf{D}$, then it possesses the so-called *(boundary) Denjoy–Wolff point*, i.e., a unique boundary regular fixed point τ such that $f'(\tau) \leqslant 1$.

In this paper we study *univalent* self-maps $f\colon \mathbf{D} \to \mathbf{D}$ with a given boundary regular fixed point $\sigma \in \partial\mathbf{D}$ and the Denjoy–Wolff point $\tau \in \partial\mathbf{D} \setminus \{\sigma\}$. Using

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automorphisms of $\mathbf{D}$, we may suppose that $\tau = 1$ and $\sigma = -1$. Our main result is the *sharp value region* of $f \mapsto f(z_0)$ for all such self-maps of $\mathbf{D}$ with $f'(-1)$ fixed. To give a detailed statement, fix $z_0 \in \mathbf{D}$, $T > 0$ and let $\zeta_0 = x_1^0 + ix_2^0 := \ell(z_0)$, where

$$\ell\colon \mathbf{D} \to \mathbf{S}; \quad z \mapsto \log\big((1+z)/(1-z)\big),$$

is a conformal map of $\mathbf{D}$ onto the strip $\mathbf{S} := \big\{\zeta\colon -\pi/2 < \operatorname{Im}\zeta < \pi/2\big\}$. Define

$$a_\pm(T) := e^{-T/2}\sin x_2^0 \pm (1 - e^{-T/2}), \quad R(a,T) := \log\frac{1-a}{1-a_+(T)}\,\log\frac{1+a}{1+a_-(T)},$$

$$V(\zeta_0, T) := \Big\{x_1 + ix_2 \in \mathbf{S}\colon a_-(T) \leqslant \sin x_2 \leqslant a_+(T),\ \big|x_1 - x_1^0 - \tfrac{T}{2}\big| \leqslant \sqrt{R(\sin x_2, T)}\Big\}.$$

Theorem 1. *Let $f \in \mathsf{Hol}(\mathbf{D},\mathbf{D}) \setminus \{\mathsf{id}_{\mathbf{D}}\}$ and $T > 0$. Suppose that*

(i) *f is univalent in $\mathbf{D}$;*
(ii) *the Denjoy–Wolff point of f is $\tau = 1$;*
(iii) *$\sigma = -1$ is a boundary regular fixed point of f and $f'(-1) = e^T$.*

Then

$$f(z_0) \in \mathcal{V}(z_0, T) := \ell^{-1}\big(V(\ell(z_0), T)\big) \setminus \{z_0\} \quad \text{for any } z_0 \in \mathbf{D}. \tag{1.1}$$

This result is sharp, i.e., for any $w_0 \in \mathcal{V}(z_0,T)$ there exists $f \in \mathsf{Hol}(\mathbf{D},\mathbf{D}) \setminus \{\mathsf{id}_{\mathbf{D}}\}$ satisfying (i)–(iii) and such that $f(z_0) = w_0$.

We can also characterize functions f delivering boundary points of $\mathcal{V}(z_0,T)$. In many extremal problems for univalent functions $f\colon \mathbf{D} \to \mathbf{C}$ normalized by $f(0) = f'(1) - 1 = 0$, the Koebe function $f_0(z) := z/(1-z)^2$ mapping $\mathbf{D}$ onto $\mathbf{C} \setminus (-\infty, \frac14]$, and its rotations $f_\theta(z) = e^{i\theta} f_0(e^{-i\theta}z)$, $\theta \in \mathbf{R}$, are known to be extremal. For bounded univalent functions $f\colon \mathbf{D} \to \mathbf{D}$ normalized by $f(0) = 0$, $f'(0) > 0$, the role of the Koebe function is played by the Pick functions $p_\alpha(z) := f_0^{-1}(\alpha f_0(z))$, $\alpha \in (0,1)$, mapping $\mathbf{D}$ onto $\mathbf{D} \setminus [-1, -r]$, $r = r(\alpha) \in (0,1)$. In our case, it would be natural to expect that some functions of the form $f = h_1 \circ p_\alpha \circ h_2$, where $h_1, h_2 \in \mathsf{Aut}(\mathbf{D})$, are extremal.

Theorem 2. *For any $w_0 \in \partial\mathcal{V}(z_0,T) \setminus \{z_0\}$, there exists a unique $f = f_{w_0}$ satisfying conditions (i)–(iii) in Theorem 1 and such that $f_{w_0}(z_0) = w_0$. If $w_0 = \ell^{-1}(\zeta_0 + T)$, then f_{w_0} is a hyperbolic automorphism of $\mathbf{D}$, namely $f_{w_0}(z) = \ell^{-1}(\ell(z) + T)$. Otherwise, f_{w_0} is a conformal mapping of $\mathbf{D}$ onto $\mathbf{D}$ minus a slit along an analytic Jordan arc γ orthogonal to $\partial\mathbf{D}$, with $f'_{w_0}(1) = 1$. Moreover, $f_{w_0} = h_1 \circ p_\alpha \circ h_2$ for some $h_1, h_2 \in \mathsf{Aut}(\mathbf{D})$ and $\alpha \in (0,1)$ if and only if $w_0 = \ell^{-1}\big(x_1^0 + \frac{T}{2} + i \arcsin a_\pm(T)\big)$.*

Remark 1.1. Note that z_0 is a boundary point of the value region $\mathcal{V}(z_0,T)$, but does not belong to $\mathcal{V}(z_0,T)$. The proof of the above theorem, given in Section 4, shows that z_0 would be included, and this would be the only modification of the value region, if we replaced the equality $f'(-1) = e^T$ in condition (iii) of Theorem 1 by the *inequality* $f'(-1) \leqslant e^T$ and removed the requirement $f \neq \mathsf{id}_{\mathbf{D}}$ assuming as a convention that $\mathsf{id}_{\mathbf{D}}$ satisfies (ii). Note also that under the conditions of Theorem 1 modified in this way, $f(z_0) = z_0$ if and only if $f = \mathsf{id}_{\mathbf{D}}$, see Remark 2.3.

If $f \in \mathsf{Hol}(\mathbf{D},\mathbf{D})$ has boundary regular fixed points at ± 1, then replacing f by $h \circ f$, where h is a suitable hyperbolic automorphism with the same boundary fixed points, we may suppose that $\tau = 1$ is the Denjoy–Wolff point. In this way, as a corollary of Theorems 1 and 2 we easily deduce a sharp estimate for $f'(-1)f'(1)$, which was obtained earlier with the help of the extremal length method in [15, Section 4].

Corollary 1. *Let $z_0 \in \mathbf{D}$ and let $f \in \mathsf{Hol}(\mathbf{D},\mathbf{D})$ be a univalent function with boundary regular fixed points at* 1 *and* -1. *Then*

$$(1.2)\quad \sqrt{f'(-1)f'(1)} \geqslant L\big(\sin \mathsf{Im}\,\ell(z_0), \sin \mathsf{Im}\,\ell(f(z_0))\big), \quad L(a,b) := \max\left\{\tfrac{1+a}{1+b}, \tfrac{1-a}{1-b}\right\}.$$

Inequality (1.2) *is sharp. The equality can occur only for hyperbolic automorphisms and functions of the form $f = h_1 \circ p_\alpha \circ h_2$, $h_1, h_2 \in \mathsf{Aut}(\mathbf{D})$, $\alpha \in (0,1)$.*

Recently, the sharp value regions of $f \mapsto f(z_0)$ have been determined for other classes of univalent self-maps [23, 24, 35, 37]. One of the main instruments is the classical parametric representation of univalent functions, going back to the seminal work by Loewner [29]. In this paper, we use a new variant of Loewner's parametric method, which is specific for functions satisfying conditions of Theorem 1. This variant of parametric representation was discovered quite recently, see [20, 21]. We discuss it in Section 3.

It is also worth mentioning that in [17], using another specific variant of the parametric representation, Goryainov obtained the sharp value region of $f \mapsto f'(0)$ in the class of all univalent $f \in \mathsf{Hol}(\mathbf{D},\mathbf{D})$, $f(0) = 0$, having a boundary regular fixed point at $\sigma = 1$ with a given value of $f'(1)$.

To complete the Introduction, we recall another related result obtained by Goryainov [18, 19]. Dropping the univalence requirement, one can study holomorphic self-maps $f\colon \mathbf{D} \to \mathbf{D}$ satisfying conditions (ii) and (iii) in Theorem 1 by using relationships between boundary regular fixed points and the Alexandrov–Clark measures. In particular, according to [18, 19], the value region $\mathcal{D}(0,T)$ of $f \mapsto f(0)$ over all such self-maps f is the closed disk whose diameter is the segment $\big[0, \ell^{-1}(T)\big]$, with the boundary point $z_0 = 0$ excluded. Analyzing the functions delivering the boundary points of $\mathcal{D}(0,T)$, one can conclude that $\partial\mathcal{D}(0,T) \bigcap \partial\mathcal{V}(0,T) = \{0, \ell^{-1}(T)\}$.

2. Holomorphic self-maps of the unit disk

In this section we cite some basic theory of holomorphic self-maps of $\mathbf{D}$. More details can be found, e.g., in the monograph [1].

Let $f \in \mathsf{Hol}(\mathbf{D},\mathbf{D})$ and $\sigma \in \partial\mathbf{D}$. According to the classical Julia–Wolff–Carathéodory Theorem, see, e.g., [1, Theorem 1.2.5, Proposition 1.2.6, Theorem 1.2.7], if

$$(2.1)\qquad \alpha_f(\sigma) := \liminf_{\mathbf{D}\ni z\to\sigma} \frac{1-|f(z)|}{1-|z|} < +\infty,$$

then

$$(2.2)\qquad \exists \angle\lim_{z\to\sigma} f(z) =: f(\sigma) \in \partial\mathbf{D}, \quad \exists \angle\lim_{z\to\sigma} \frac{f(z)-f(\sigma)}{z-\sigma} =: f'(\sigma) = \alpha_f(\sigma)\frac{f(\sigma)}{\sigma},$$

and

$$(2.3)\qquad \frac{|f(z)-f(\sigma)|^2}{1-|f(z)|^2} \leqslant |f'(\sigma)|\,\frac{|z-\sigma|^2}{1-|z|^2} \quad \text{for all } z \in \mathbf{D},$$

with the equality sign if and only if $f \in \mathsf{Aut}(\mathbf{D})$. Note that in its turn, existence of the limits in (2.2) satisfying $f(\sigma) \in \partial\mathbf{D}$ and $f'(\sigma) \neq \infty$ immediately implies (2.1).

Definition 2.1. Points $\sigma \in \partial\mathbf{D}$ satisfying (2.2) are referred to as *regular contact points* of f. If in addition to (2.2), $f(\sigma) = \sigma$, then σ is said to be a *regular fixed point* of f. The number $f'(\sigma)$ is called the *angular derivative* of f at σ.

Among all fixed points (boundary and internal) of a self-map $f \neq \mathsf{id}_{\mathbf{D}}$, there is one point of special importance for dynamics. On the one hand, if $f(\tau) = \tau$ for some $\tau \in \mathbf{D}$, then by the Schwarz Lemma, τ is the only fixed point of f in $\mathbf{D}$. If in addition, f is not an elliptic automorphism, then $|f'(\tau)| < 1$ and hence the sequence of iterates $(f^{\circ n})$, $f^{\circ 1} := f$, $f^{\circ(n+1)} := f \circ f^{\circ n}$, converges (to the constant function equal) to τ locally uniformly in $\mathbf{D}$. On the other hand, if f has no fixed points in $\mathbf{D}$, then by the Denjoy–Wolff Theorem, see, e.g., [1, Theorem 1.2.14, Corollary 1.2.16, Theorem 1.3.9], f has a unique boundary regular fixed point $\tau \in \partial\mathbf{D}$ such that $f'(\tau) \leqslant 1$ and moreover, $f^{\circ n} \to \tau$ locally uniformly in $\mathbf{D}$ as $n \to +\infty$.

Definition 2.2. The point τ above is referred to as the *Denjoy–Wolff point* of f.

Remark 2.3. Since the strict inequality holds in (2.3) unless $f \in \mathsf{Aut}(\mathbf{D})$, a self-map f can have a fixed point in $\mathbf{D}$ and a boundary regular fixed point σ with $f'(\sigma) \leqslant 1$ only if $f = \mathsf{id}_{\mathbf{D}}$.

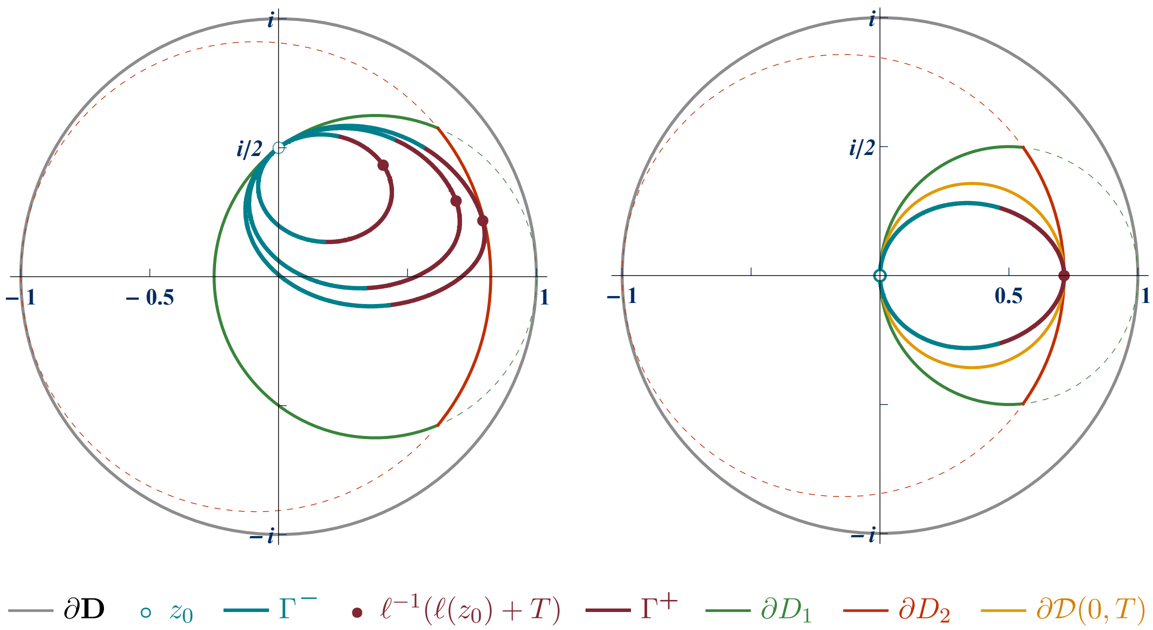

Figure 1. The value region $\mathcal{V}(z_0, T)$ and the disks D_1, D_2 for $z_0 := i/2$, $T \in \{\log 2, \log 4, \log 6\}$ and for $z_0 := 0$, $T := \log 6$. The right picture also shows the disk $\mathcal{D}(0, T)$. Notation $\Gamma^{\pm}$ is explained in Section 4.

Remark 2.4. Let $f_n(z) := -\sigma H^{-1}\big(\alpha/H(z/\sigma) + \beta/(H(z/\sigma)+n)\big)$, where $n \in \mathbf{N}$, $\alpha, \beta > 0$, and $H(z) := (1+z)/(1-z)$. Note that $f_n(\mathbf{D}) \subset \mathbf{D}$ for all $n \in \mathbf{N}$ and that $f_n(z) \longrightarrow f(z) := (z+c)/(1+\bar{c}z)$, $c := \sigma(1-\alpha)/(1+\alpha)$, locally uniformly in $\mathbf{D}$ as $n \to +\infty$. Moreover, $f_n(\sigma) = f(\sigma) = \sigma$ and $f_n'(\sigma) = \alpha + \beta$ for all $n \in \mathbf{N}$, but $f'(\sigma) = \alpha$. This example shows that the map $f \mapsto f'(\sigma)$ is not continuous. However, it turns out to be semicontinuous in the following sense. Suppose that $f_n(z) \to f(z)$ as $n \to +\infty$ and that $\sigma \in \partial\mathbf{D}$ is a boundary regular fixed point of $f_n \in \mathsf{Hol}(\mathbf{D}, \mathbf{D})$ for all $n \in \mathbf{N}$ with $\alpha := \liminf_{n\to+\infty} f_n'(\sigma) < +\infty$. Then passing in Julia's inequality (2.3) applied for functions f_n to the limit, we conclude that f satisfies (2.3) with $|f'(\sigma)|$ replaced by α. It follows that $\alpha_f(\sigma) \leqslant \alpha < +\infty$. Therefore, either $f \equiv \sigma$ or $f \in \mathsf{Hol}(\mathbf{D}, \mathbf{D})$ and σ is a regular boundary fixed point of f with $f'(\sigma) \leqslant \alpha$. As a consequence, the set of all $f \in \mathsf{Hol}(\mathbf{D}, \mathbf{D})$ sharing two different boundary regular fixed points σ_1 and σ_2 and satisfying $f'(\sigma_j) \leqslant \alpha_j < +\infty$, $j = 1, 2$, is compact.

According to inequality (2.3), the value region $\mathcal{V}(z_0,T)$ in Theorem 1 lies in the intersection of two closed disks $D_1,D_2\subset\overline{\mathbf{D}}$ whose boundaries pass through z_0 and $\tau=1$ and through $\ell^{-1}(\ell(z_0)+T)$ and $\sigma=-1$, respectively. Comparison of $\mathcal{V}(z_0,T)$ with $D_1\cap D_2$ is show in Figure 1. On the right picture, for which $z_0=0$, we also place the value range $\mathcal{D}(0,T)$ of $f\mapsto f(0)$ over all holomorphic but not necessary injective maps $f:\mathbf{D}\to\mathbf{D}$ satisfying conditions (ii) and (iii) in Theorem 1, see [18, 19].

3. Parametric representation

Denote the class of all $f\in\mathsf{Hol}(\mathbf{D},\mathbf{D})$ satisfying conditions (i)–(iii) in Theorem 1 by $\mathfrak{U}(T)$. The following theorem, proved in [21], gives a parametric representation for $\mathfrak{U}(T)$ in terms of a Loewner–Kufarev-type ODE.

Theorem 3. [21, Corollary 1.2] *The class $\mathfrak{U}(T)$ coincides with the set of all functions representable in the form $f(z)=w_z(T)$ for all $z\in\mathbf{D}$, where $w_z(t)$ is the unique solution to the initial value problem*

$$\frac{\mathrm{d}w_z}{\mathrm{d}t}=\tfrac{1}{4}(1-w_z)^2(1+w_z)q(w_z,t),\quad t\in[0,T],\quad w_z(0)=z, \tag{3.1}$$

with some function $q\colon\mathbf{D}\times[0,T]\to\mathbf{C}$ satisfying the following conditions:

(i) *for every $z\in\mathbf{D}$, $q(z,\cdot)$ is measurable on $[0,T]$;*
(ii) *for a.e. $t\in[0,T]$, $q(\cdot,t)$ has the following integral representation*

$$q(z,t)=\int_{\partial\mathbf{D}\setminus\{1\}}\frac{1-\kappa}{1+\kappa z}\,\mathrm{d}\nu_t(\kappa), \tag{3.2}$$

where ν_t is a probability measure on $\partial\mathbf{D}\setminus\{1\}$.

Remark 3.1. A related parametric representation for a class of univalent self-maps of a strip was considered in [13].

Remark 3.2. In many cases, it is more convenient to deal with the the union $\mathfrak{U}'(T):=\bigcup_{0\leqslant T'\leqslant T}\mathfrak{U}(T')$, where we define $\mathfrak{U}(0):=\{\mathsf{id}_{\mathbf{D}}\}$. Indeed, it is evident from the argument of Remark 2.4 that in contrast to $\mathfrak{U}(T)$, the class $\mathfrak{U}'(T)$ is compact. Moreover, it is easy to see that Theorem 3 gives representation of $\mathfrak{U}'(T)$ if all probability measures ν_t in (3.2) are replaced with all positive Borel measures ν_t satisfying

$$\nu_t(\partial\mathbf{D}\setminus\{1\})\ \in\ [0,1]. \tag{3.3}$$

Note that the possibility of $\nu_t=0$ is not excluded.

Remark 3.3. Obviously, the right-hand side of (3.1) can be written as $G(w_z,t)$, where $G(z,t):=\frac{1}{4}(1-z)^2(1+z)q(z,t)$ with q satisfying conditions (i) and (ii) in Theorem 3. By [20, Theorem 1], $G(\cdot,t)$ is an infinitesimal generator in $\mathbf{D}$ for each $t\in[0,T]$. For simplicity, extend G to all $t\geqslant0$ by setting $G(z,t)\equiv0$ for any $t>T$. Then according to the general theory of Loewner–Kufarev-type equations, see [4, Sections 3–5], for any $s\geqslant0$ and any $z\in\mathbf{D}$, the initial value problem $\mathrm{d}w/\mathrm{d}t=G(w,t)$, $t\geqslant s$, $w(s)=z$, has a unique solution $w=w_{z,s}(t)$ defined for all $t\geqslant s$ and the functions $\varphi_{s,t}(z):=w_{z,s}(t)$, $z\in\mathbf{D}$, $t\geqslant s\geqslant0$, form an evolution family, see [4, Definition 3.1].

Proposition 1. *Let $\vartheta\colon[0,T]\to(-\pi,\pi)\setminus\{0\}$, $T>0$, be a C^1-smooth function. Suppose that in the conditions of Theorem 3, $\mathrm{d}\nu_t(e^{i\theta})=\delta\big(\theta-\vartheta(t)\big)\,\mathrm{d}\theta$ for all $t\in[0,T]$, where δ stands for the Dirac delta function. Then f maps $\mathbf{D}$ onto $\mathbf{D}\setminus\gamma$, where γ is a slit in $\mathbf{D}$, i.e. γ is the image of a homeomorphism $\gamma\colon[0,1]\mapsto\overline{\mathbf{D}}$ with $\gamma\big([0,1)\big)\subset\mathbf{D}$ and $\gamma(1)\in\partial\mathbf{D}$. Moreover,*

(i) *if* ϑ *is a real-analytic function on* $[0,T]$*, then* γ *is a real-analytic Jordan arc orthogonal to* $\mathbf{D}$*;*

(ii) γ *is a circular arc or a straight line segment orthogonal to* $\partial\mathbf{D}$ *if and only if*

$$\lambda(t) := i\frac{1+e^{i\vartheta(t)}}{1-e^{i\vartheta(t)}} = C_1 e^{-t/2}\left(C_2 e^{t/2} + \sqrt{C_2^2(e^t-1)+1}\right)^3 \tag{3.4}$$

for all $t\in[0,T]$ *and some constants* $C_1, C_2 \in \mathbf{R}$, $C_1 \neq 0$.

Proof. In the conditions of the proposition, (3.1) takes the following form:

$$\frac{\mathrm{d}w_z}{\mathrm{d}t} = \tfrac14(1-w_z)^2(1+w_z)\frac{1-e^{i\vartheta(t)}}{1+e^{i\vartheta(t)}w_z}, \quad t\in[0,T], \quad w_z(0)=z. \tag{3.5}$$

The change of variables $\omega_z := H(w_z)$, where $H(w) := i(1+w)/(1-w)$ maps $\mathbf{D}$ conformally onto $\mathbf{H} := \{\omega\colon\ \mathsf{Im}\,\omega > 0\}$, transforms the above problem to

$$\frac{\mathrm{d}\omega_z}{\mathrm{d}t} = \frac{\omega_z}{1-\lambda(t)\omega_z}, \quad t\in[0,T], \quad \omega_z(0)=H(z), \tag{3.6}$$

where $\lambda(t) := H(e^{i\vartheta(t)})$ for all $t\in[0,T]$. Making further change of variables

$$\hat\omega_z(t) := \omega_z(t) + \int_0^t \frac{\mathrm{d}s}{\lambda(s)}, \quad \xi(t) := \frac{1}{\lambda(t)} + \int_0^t\frac{\mathrm{d}s}{\lambda(s)}, \quad \tau = v(t) := \frac12\int_0^t \frac{\mathrm{d}s}{\lambda(s)^2},$$

we obtain the chordal Loewner equation

$$\frac{\mathrm{d}\hat\omega_z}{\mathrm{d}\tau} = \frac{2}{\xi-\hat\omega_z}, \quad \tau\in[0,v(T)], \quad \hat\omega_z(0)=H(z). \tag{3.7}$$

The geometry of solutions to (3.7) is well-studied, see, e.g., [27, 30, 36, 22, 39]; see also [25]. In particular, since the function $s\mapsto \xi(v^{-1}(s))$ is C^1-smooth, it follows that $z\mapsto\hat\omega_z(T)$ maps $\mathbf{D}$ onto $\mathbf{H}$ minus a slit along some Jordan arc γ_0. Taking into account that $w_z(T) = H^{-1}\big(\hat\omega_z(T) - C\big)$, where $C := \int_0^T \lambda(t)^{-1}\,\mathrm{d}t$, this proves the first part of the proposition.

If ϑ is real-analytic, then $s\mapsto\xi(v^{-1}(s))$ is real-analytic on $[0,T]$ as well, and hence by [28, Theorem 1.4], γ_0 is a real-analytic Jordan arc. Moreover, the argument of [28, Section 6.1] shows that in such a case, γ_0 is orthogonal to $\mathbf{R}$. This proves (i).

It remains to prove (ii). Suppose that γ is a circular arc or a straight line segment orthogonal to $\partial\mathbf{D}$. Then we can find a linear-fractional transformation H_* of $\mathbf{D}$ onto $\mathbf{H}$ such that $H_*(\gamma) = [0,i]$. Let $(\varphi_{s,t})$ be the evolution family associated with equation (3.5), see Remark 3.3. Note that $\varphi_{t,T}(\mathbf{D}) \supset \varphi_{t,T}(\varphi_{0,t}(\mathbf{D})) = \varphi_{0,T}(\mathbf{D}) = f(\mathbf{D})$ for any $t\in[0,T]$. It follows that the intersection of a sufficiently small neighbourhood of $H_*^{-1}(\infty)$ with $\partial\varphi_{t,T}(\mathbf{D})$ is an open arc of $\partial\mathbf{D}$ containing $H_*^{-1}(\infty)$. Therefore, for each $t\in[0,T]$, there exists a unique $h_t\in\mathsf{Aut}(\mathbf{D})$ such that $g_t := H_*\circ\varphi_{t,T}\circ h_t\circ H_*^{-1} \in \mathsf{Hol}(\mathbf{H},\mathbf{H})$ satisfies the Laurent expansion $g_t(z) = z - c(t)/z + \ldots$ at ∞ with some $c(t)\in\mathbf{R}$.

Denote $H_t := H_*\circ h_t^{-1}$ for all $t\in[0,T]$. By construction, $\mathbf{H}\setminus[0,i] = g_0(\mathbf{H}) \subset g_t(\mathbf{H}) \subset g_T(\mathbf{H}) = \mathbf{H}$ for all $t\in[0,T]$. Thanks to continuity of ϑ, the function $t\mapsto c(t)$ is C^1-smooth. Therefore, according to the classical result [26] by Kufarev et al, see also [12], for any $z\in\mathbf{D}$, $\tilde\omega_z(t) := g_t^{-1}\circ g_0(H_0(z))$, $t\in[0,T]$, is the unique solution to the initial value problem $\mathrm{d}\tilde\omega_z/\mathrm{d}t = -c'(t)/\tilde\omega_z$, $\tilde\omega_z(0) = H_0(z)\in\mathbf{H}$.

By construction, $\tilde{\omega}_z(t) = H_t(w_z)$ for all $t \in [0,T]$ and all $z \in \mathbf{D}$. Comparing the differential equations for $\tilde{\omega}_z$ and w_z, one can conclude that for all $t \in [0,T]$,

$$H_t(w) := \frac{\lambda(t)H(w) - 1}{a(t)\big(\lambda(t)H(w) - 1\big) + b(t)} \tag{3.8}$$

with real coefficients $a(t)$ and $b(t)$ satisfying

$$\mathrm{d}a/\mathrm{d}t = a^3/b^2, \quad \mathrm{d}b/\mathrm{d}t = -3a + b + 3a^2/b, \quad t \in [0,T], \tag{3.9}$$

and such that $\lambda'(t)/\lambda(t) = 1 - 3a(t)/b(t)$ and $b(t)\lambda(t) > 0$ for all $t \in [0,T]$. System (3.9) can be solved by introducing a new unknown function $k(t) := a(t)/b(t)$. In this way, one can easily check that λ must be of the form (3.4).

Conversely, if λ is given by (3.4), then system (3.9) has a real-valued solution satisfying $\lambda'(t)/\lambda(t) = 1 - 3a(t)/b(t)$ and $b(t)\lambda(t) > 0$ for all $t \in [0,T]$. It follows that for any $z \in \mathbf{D}$, the function $\tilde{\omega}_z(t) := H_t(w_z(t))$, where H_t is given by (3.8), is a solution to $\mathrm{d}\tilde{\omega}_z/\mathrm{d}t = -1/\big(b(t)^2\,\tilde{\omega}_z\big)$, $t \in [0,T]$, $\tilde{\omega}_z(0) = H_0(z) \in \mathbf{H}$. Solving the latter initial value problem for $\tilde{\omega}_z$, we conclude that the image of the map $\mathbf{D} \ni z \mapsto \tilde{\omega}_z(T)$ is the domain $\mathbf{H} \setminus [0, i\sqrt{Q_T}\,]$, $Q_T := 2\int_0^T b(t)^{-2}\,\mathrm{d}t$. Thus, $\gamma = H_T^{-1}\big([0, i\sqrt{Q_T}\,]\big)$ is a circular arc or a straight line segment orthogonal to $\partial\mathbf{D}$. The proof is now complete. □

4. Proof of the main results

In this section we prove Theorems 1 and 2. Fix $T > 0$. We start by considering the problem to determine the compact value region $\{f(z_0)\colon f \in \mathfrak{U}'(T)\}$. Thanks to Theorem 3 and Remark 3.2, it coincides with the reachable set $\{w_{z_0}(T)\}$ of the controllable system (3.1) in which the measure-valued control $t \mapsto \nu_t$ satisfies (3.3). The change of variables

$$\zeta = \ell(w), \quad \lambda = i\frac{1+\kappa}{1-\kappa},$$

reduces our problem to finding the reachable set $\Omega'_T := \{\zeta(T)\}$ for the following controllable system

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \int_{\mathbf{R}} \frac{\mathrm{d}\mu_t(\lambda)}{1 - i\lambda e^{\zeta}}, \quad t \in [0,T]; \quad \zeta|_{t=0} = \zeta_0 := \ell(z_0), \tag{4.1}$$

where μ_t's are positive Borel measures on $\mathbf{R}$ with $\mu_t(\mathbf{R}) \leqslant 1$. By using the prime in the notation Ω'_T we emphasize that this reachable set corresponds to the class $\mathfrak{U}'(T)$.

Denote $x_1 := \mathsf{Re}\,\zeta$ and $x_2 := \mathsf{Im}\,\zeta$. Note that $x_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. For any fixed $\zeta = x_1 + ix_2 \in \mathbf{S}$, the range of the right-hand side in (4.1), regarded as a function of the measure μ_t, is the disk

$$\left\{\omega \in \mathbf{C}\colon \left|\omega - \frac{e^{-ix_2}}{2\cos x_2}\right| \leqslant \frac{1}{2\cos x_2}\right\}.$$

Therefore, replacing the measure-valued control $t \mapsto \mu_t$ with the complex-valued control

$$u(t) := 2e^{ix_2}\cos x_2 \int_{\mathbf{R}} \frac{\mathrm{d}\mu_t(\lambda)}{1 - i\lambda e^{x_1 + ix_2}},$$

we can rewrite (4.1) in the following form

$$(4.2)\qquad \frac{\mathrm{d}x_1}{\mathrm{d}t} = \operatorname{Re}\frac{u(t)e^{-ix_2}}{2\cos x_2} = \frac{1}{2}\operatorname{Re}u(t) + \frac{\operatorname{tg}x_2}{2}\operatorname{Im}u(t), \qquad x_1(0) = x_1^0 := \operatorname{Re}\zeta_0,$$

$$(4.3)\qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \operatorname{Im}\frac{u(t)e^{-ix_2}}{2\cos x_2} = \frac{1}{2}\operatorname{Im}u(t) - \frac{\operatorname{tg}x_2}{2}\operatorname{Re}u(t), \qquad x_2(0) = x_2^0 := \operatorname{Re}\zeta_0,$$

where $u\colon [0,T] \to U := \{u\colon |u-1| \leqslant 1\}$ is an arbitrary measurable function.

Introduce the Hamilton function

$$\mathcal{H}(x_1, x_2, \Psi_1, \Psi_2, u) := \Psi_1 \operatorname{Re}\frac{ue^{-ix_2}}{2\cos x_2} + \Psi_2 \operatorname{Im}\frac{ue^{-ix_2}}{2\cos x_2} = \operatorname{Re}\frac{ue^{-ix_2}(\Psi_1 - i\Psi_2)}{2\cos x_2},$$

where Ψ_1, Ψ_2 satisfy the adjoint system of ODEs

$$(4.4)\qquad \frac{\mathrm{d}\Psi_1}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial x_1} = 0, \quad \frac{\mathrm{d}\Psi_2}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial x_2} = -\operatorname{Im}\frac{u(t)(\Psi_1 - i\Psi_2)}{2\cos^2 x_2}.$$

Boundary points of the reachable set Ω'_T, forming a dense subset of $\partial\Omega'_T$, are generated by the driving functions u^* satisfying the necessary optimal condition in the form of Pontryagin's maximum principle,

$$(4.5)\qquad \max_{u\in U}\mathcal{H}\big(x_1(t), x_2(t), \Psi_1(t), \Psi_2(t), u\big) = \mathcal{H}\big(x_1(t), x_2(t), \Psi_1(t), \Psi_2(t), u^*(t)\big)$$

for all $t \in [0,T]$, see, e.g., [34]. Trajectories $(x_1(t), x_2(t))$ in (4.5) are optimal in the reachable set problem, and $(\Psi_1(t), \Psi_2(t))$ satisfy the adjoint system (4.4) with the optimal trajectories. In particular, $(\Psi_1(t), \Psi_2(t))$ does not vanish, and hence the maximum in (4.5) is attained at the unique point $u^* = 1 + e^{i(x_2+\varphi)}$, where $\varphi := \arg(\Psi_1 + i\Psi_2)$. Therefore, from (4.2)–(4.4) for the optimal trajectories we obtain

$$(4.6)\qquad \frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\cos\varphi + \cos x_2}{2\cos x_2}, \quad x_1(0) = x_1^0,$$

$$(4.7)\qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\sin\varphi - \sin x_2}{2\cos x_2}, \quad x_2(0) = x_2^0,$$

$$(4.8)\qquad \frac{\mathrm{d}\Psi_1}{\mathrm{d}t} = 0,$$

$$(4.9)\qquad \frac{\mathrm{d}\Psi_2}{\mathrm{d}t} = \frac{\sin\varphi - \sin x_2}{2\cos^2 x_2}\,\big|\Psi_1 - i\Psi_2\big|.$$

System (4.6)–(4.9) is invariant w.r.t. multiplication of (Ψ_1, Ψ_2) by a positive constant. Therefore, we may assume that either $\Psi_1 \equiv 0$, or $\Psi_1 \equiv 1$, or $\Psi_1 \equiv -1$.

If $\Psi_1 \equiv 0$, then $\varphi = \pm\pi/2$ and we easily get that for all $t \geqslant 0$,

$$(4.10)\qquad x_1(t) = x_1(0) + t/2, \quad \sin x_2(t) = a_\pm(t) := e^{-t/2}\sin x_2(0) \pm (1 - e^{-t/2}).$$

Now let $\Psi_1 \equiv 1$. Then $\varphi \in (-\pi/2, \pi/2)$ and equation (4.9) takes the following form

$$(4.11)\qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\sin\varphi - \sin x_2}{2\cos^2 x_2}\cos\varphi = \frac{\cos\varphi}{\cos x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t}.$$

System (4.7), (4.11) admits the first integral

$$I(x_2, \varphi) := \frac{1 - \sin\varphi}{1 + \sin\varphi}\,\frac{1 + \sin x_2}{1 - \sin x_2} \,>\, 0,$$

and as a result it can be integrated in quadratures. Namely, if $C := I\big(x_2(0), \varphi(0)\big) \neq 1$, we obtain the following identities

$$B_1(t) - CB_2(t) = (C-1)t/2, \tag{4.12}$$

$$x_1(t) - x_1(0) = \frac{B_1(t) - \sqrt{C}B_2(t)}{\sqrt{C} - 1}, \tag{4.13}$$

where

$$B_1(t) := \log\frac{1 - \sin x_2(t)}{1 - \sin x_2(0)}, \quad B_2(t) := \log\frac{1 + \sin x_2(t)}{1 + \sin x_2(0)}.$$

Excluding C from (4.12), (4.13) and setting $t = T$ gives

$$\begin{aligned} x_1(T) &= x_1(0) + \frac{1}{2}\Big(T + \sqrt{\big(T + 2B_1(T)\big)\big(T + 2B_2(T)\big)}\Big) \\ &= x_1(0) + \frac{T}{2} + \sqrt{R\big(\sin x_2(T), T\big)}, \end{aligned} \tag{4.14}$$

where we took into account that according to (4.12),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(t + 2B_1(t)\Big) = \frac{2C}{1 + \sin x_2(t) + C\big(1 - \sin x_2(t)\big)} > 0$$

and therefore, $T + 2B_1(T) > 0$.

For $C = 1$, we have $\varphi(t) = x_2(t)$ and hence $\mathrm{d}\varphi/\mathrm{d}t = \mathrm{d}x_2/\mathrm{d}t = 0$, $\mathrm{d}x_1/\mathrm{d}t = 1$. Therefore, if $C = 1$, then (4.12) and (4.14) hold as well. Since $C > 0$, from (4.12) we obtain that $x_2(T) \in J(T) := (\arcsin a_-(T), \arcsin a_+(T))$. On the other hand, for any $x \in J(T)$ there exists a unique $C = C(x) > 0$ that verifies (4.12) with T and x substituted for t and $x_2(t)$, respectively. Solving $I(x_2(0), \varphi(0)) = C(x)$ provides us with the initial condition in equation (4.11) for which $x_2(T) = x$.

Investigating the case $\Psi_1 \equiv -1$ in a similar way, we conclude that $\partial\Omega'_T$ is the union of the two Jordan arcs

$$\Gamma^{\pm}(T) := \Big\{x_1 + ix_2 \in \mathbf{S}\colon a_-(T) \leqslant \sin x_2 \leqslant a_+(T),\ x_1 = x_1^0 + \tfrac{T}{2} \pm \sqrt{R(\sin x_2, T)}\Big\},$$

which do not intersect except for the common end-points $\omega^{\pm} := x_1^0 + T/2 + i\arcsin a_{\pm}(T)$, delivered by solutions (4.10). Taking into account that by the very definition, $\mathfrak{U}'(T') \subset \mathfrak{U}'(T)$ for any $T' \in [0, T]$, it follows that $\Omega'_T = V(\zeta_0, T)$.

The next step in the proof is to pass from the class $\mathfrak{U}'(T)$ to the class $\mathfrak{U}(T)$. In the problem of finding the value region of the functional $f \mapsto f(z_0)$, this is equivalent to replacing the range U of the admissible controls u in (4.2)–(4.3) by $U \setminus \{0\}$. Denote by Ω_T the corresponding reachable set. By re-scaling the time, the problem to find $\Omega_{T'}$, $T' \in (0, T)$, can be restated as the reachable set problem at the same time T and for the same controllable system, but with the value range of admissible controls restricted to $\alpha\big(U \setminus \{0\}\big)$, $\alpha := T'/T$. Note also that $\Gamma^+(T) \cup \Gamma^-(T) \setminus \{\zeta_0\} \subset \Omega_T$ for any $T > 0$. Since $\alpha\big(U \setminus \{0\}\big) \subset U \setminus \{0\}$ for any $\alpha \in (0, 1)$, it follows that

$$\Gamma^+(T') \cup \Gamma^-(T') \setminus \{\zeta_0\} \subset \Omega_{T'} \subset \Omega_T \quad \text{for any } T \in (0, T].$$

Thus $\Omega_T = V(\zeta_0, T) \setminus \{\zeta_0\}$, which completes the proof of Theorem 1.

To prove Theorem 2, we have to identify the functions delivering the boundary points of $\mathcal{V}(z_0, T)$. They correspond to the controls u^* satisfying Pontryagin's maximum principle (4.5). It is easy to see from the above argument that every point $\omega \in \partial\Omega'_T \setminus \{\zeta_0\}$ corresponds to a unique control, which is C^1-smooth and takes values on $\partial U \setminus \{0\}$. It follows that the corresponding measures μ_t in (4.1) and the measures

ν_t in the Loewner-type representation (3.1), (3.2) are also unique. They are probability measures concentrated at one point that moves smoothly with t. Namely, $d\mu_t(\lambda) = \delta\big(\lambda - \lambda^*(t)\big)\, d\lambda$, where

$$\lambda^*(t) := \frac{1 - 2\cos x_2(t)/\big(e^{-ix_2(t)} + e^{i\varphi(t)}\big)}{ie^{x_1(t)+ix_2(t)}} = e^{-x_1(t)}\,\frac{\sin\frac{\varphi(t)-x_2(t)}{2}}{\cos\frac{\varphi(t)+x_2(t)}{2}}. \tag{4.15}$$

The point $\omega = \omega_0 := \zeta_0 + T \in \Gamma^+$ corresponds to $C = 1$, in which case $\varphi(t) = x_2(t)$ for all $t \in [0, T]$ and hence $\lambda^*(t) \equiv 0$. Therefore, from (4.1) we see that the unique $f \in \mathfrak{U}(T)$ delivering the boundary point $\ell^{-1}(\omega_0)$ of $\mathcal{V}(z_0, T)$ is the hyperbolic automorphism

$$f(z) = \frac{z + c(T)}{1 + c(T)z}, \quad c(T) := \frac{e^T - 1}{e^T + 1}, \quad \text{for all } z \in \mathbf{D}.$$

For the common end-points $\omega^\pm$ of Γ^+ and Γ^-, which correspond to $\varphi = \pm\pi/2$, formula (4.15) simplifies to $\lambda^*(t) = \pm e^{-x_1(t)}$. In view of (4.10), the latter expression coincides with $\lambda(t)$ given by (3.4) if we set $C_1 := \pm e^{-x_1^0}$ and $C_2 := 0$. Taking into account the correspondence between μ_t and ν_t and applying Proposition 1, we conclude that the unique functions $f \in \mathfrak{U}(T)$ delivering the points $\ell^{-1}(\omega^\pm)$ map $\mathbf{D}$ onto $\mathbf{D}$ minus a slit along a circular arc or a segment of a straight line orthogonal to $\partial\mathbf{D}$.

It remains to compare $\lambda^*(t)$ given by (4.15) with $\lambda(t)$ given by (3.4) for the case $\omega \in \partial\Omega_T \setminus \{\zeta_0, \omega_0, \omega^+, \omega^-\}$. Suppose $\omega \in \Gamma^+ \setminus \{\omega_0, \omega^+, \omega^-\}$. Using equations (4.6), (4.7), (4.11) and taking into account the first integral $I(x_2, \varphi) = C$, we find that

$$\Big(1 + 2\frac{d}{dt}\log\lambda^*(t)\Big)^2 = \left(\frac{\cos\varphi(t)}{\cos x_2(t)}\right)^2 = \frac{C\big(1 - a^2\big)}{\big((1 + C)a + (1 - C)a^2\big)^2}, \quad a := \sin x_2(t),$$

while $\big(1 + 2(d/dt)\log\lambda(t)\big)^2 = 9C_2^2\, e^t/\big(1 + C_2^2(e^t - 1)\big)$. However, according to (4.12), e^t cannot be expressed as a rational function of $\sin x_2(t)$. This shows that λ^* is not of the form (3.4) and hence, by Proposition 1, the unique function $f \in \mathfrak{U}(T)$ that delivers the boundary point $\ell^{-1}(\omega)$ maps $\mathbf{D}$ onto $\mathbf{D}$ minus a slit along a real-analytic arc γ orthogonal to $\partial\mathbf{D}$ but different from a circular arc or a segment of a straight line. A similar argument applied to the case $\omega \in \Gamma^- \setminus \{\zeta_0, \omega^+, \omega^-\}$ completes the proof of Theorem 2. □

References

[1] Abate, M.: Iteration theory of holomorphic maps on taut manifolds. - Research and Lecture Notes in Mathematics, Complex Analysis and Geometry, Mediterranean, Rende, 1989.

[2] Anderson, J. M., and A. Vasil'ev: Lower Schwarz–Pick estimates and angular derivatives. - Ann. Acad. Sci. Fenn. Math. 33:1, 2008, 101–110.

[3] Bolotnikov, V., M. Elin, and D. Shoikhet: Inequalities for angular derivatives and boundary interpolation. - Anal. Math. Phys. 3:1, 2013, , 63–96.

[4] Bracci, F., M. Contreras, and S. Díaz-Madrigal: Evolution families and the Loewner equation I: the unit disc. - J. Reine Angew. Math. 672, 2012, 1–37.

[5] Bracci, F., M. D. Contreras, S. Díaz-Madrigal, and P. Gumenyuk: Boundary regular fixed points in Loewner theory. - Ann. Mat. Pura Appl. (4) 194:1, 2015, 221–245.

[6] Bracci, F., and P. Gumenyuk: Contact points and fractional singularities for semigroups of holomorphic self-maps of the unit disc. - J. Anal. Math. 130:1, 2016, 185–217.

[7] Contreras, M. D., and S. Díaz-Madrigal: Analytic flows on the unit disk: angular derivatives and boundary fixed points. - Pacific J. Math. 222, 2005, 253–286.

[8] Contreras, M. D., S. Díaz-Madrigal, and Ch. Pommerenke: Fixed points and boundary behaviour of the Koenigs function. - Ann. Acad. Sci. Fenn. Math. 29:2, 2004, 471–488.

[9] Contreras, M. D., S. Díaz-Madrigal, and Ch. Pommerenke: On boundary critical points for semigroups of analytic functions. - Math. Scand. 98:1, 2006, 125–142.

[10] Contreras, M. D., S. Díaz-Madrigal, and A. Vasil'ev: Digons and angular derivatives of analytic self-maps of the unit disk. - Complex Var. Elliptic Equ. 52:8, 2007, 685–691.

[11] Cowen, C. C., and C. Pommerenke: Inequalities for the angular derivative of an analytic function in the unit disk. - J. London Math. Soc. (2) 26:2, 1982, 271–289.

[12] del Monaco, A., and P. Gumenyuk: Chordal Loewner equation. - In: Complex analysis and dynamical systems VI, Part 2, Contemp. Math. 667, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2016, 63–77.

[13] Dubovikov, D. A.: An analog of the Löwner equation for mappings of strips. - Izv. Vyssh. Uchebn. Zaved. Mat. 2007:8, 2007, 77–80 (in Russian); transl. in Russian Math. (Iz. VUZ) 51:8, 2007, 74–77.

[14] Elin, M., V. Goryainov, S. Reich, and D. Shoikhet: Fractional iteration and functional equations for functions analytic in the unit disk. - Comput. Methods Funct. Theory 2:2, 2002.

[15] Frolova, A., M. Levenshtein, D. Shoikhet, and A. Vasil'ev: Boundary distortion estimates for holomorphic maps. - Complex Anal. Oper. Theory 8:5, 2014, 1129–1149.

[16] Goryainov, V. V.: Fractional iterates of functions that are analytic in the unit disk with given fixed points. - Mat. Sb. 182:9, 1991, 1281–1299 (in Russian); transl. in Math. USSR-Sb. 74:1, 1993, 29–46.

[17] Goryainov, V. V.: Evolution families of conformal mappings with fixed points and the Löwner–Kufarev equation. - Mat. Sb. 206:1, 2015, 39–68 (in Russian); transl. in Sb. Math. 206:1-2, 2015, 33–60.

[18] Goryainov, V. V.: Holomorphic self-maps of the unit disc with two fixed points. - Presentation at XXV St. Petersburg Summer Meeting in Mathematical Analysis Tribute to Victor Havin (1933–2015), June 25–30, 2016, available at http://gauss40.pdmi.ras.ru/ma25/index.php?page=presentations.

[19] Goryainov, V. V.: Holomorphic mappings of the unit disc into itself with two fixed points. - Mat. Sb. 208:3, 2017, 54–71 (in Russian); transl. in Sb. Math. 208:3, 2017, 360–376.

[20] Goryainov, V. V., and O. S. Kudryavtseva: One-parameter semigroups of analytic functions, fixed points and the Koenigs function. - Mat. Sb. 202:7, 2011, 43–74 (in Russian); transl. in Sb. Math. 202:7-8, 2011, 971–1000.

[21] Gumenyuk, P.: Parametric representation of univalent functions with boundary regular fixed points. - Constr. Approx. (to appear), https://doi.org/10.1007/s00365-017-9376-4.

[22] Ivanov, G., D. Prokhorov, and A. Vasil'ev: Non-slit and singular solutions to the Löwner equation. - Bull. Sci. Math. 136:3, 2012, 328–341.

[23] Koch, J., and S. Schleissinger: Value ranges of univalent self-mappings of the unit disc. - J. Math. Anal. Appl. 433:2, 2016, 1772–1789.

[24] Koch, J., and S. Schleissinger: Three value ranges for symmetric self-mappings of the unit disc. - Proc. Amer. Math. Soc. 145:4, 2017, 1747–1761.

[25] Kufarev, P. P.: On integrals of simplest differential equation with moving pole singularity in the righ-thand side. - Tomsk. Gos. Univ. Uchyon. Zapiski 1, 1946, 35–48 (in Russian).

[26] Kufarev, P. P., V. V. Sobolev, and L. V. Sporyševa: A certain method of investigation of extremal problems for functions that are univalent in the half-plane. - Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. 200, 1968, 142–164 (in Russian).

[27] Lind, J.: A sharp condition for the Loewner equation to generate slits. - Ann. Acad. Sci. Fenn. Math. 30, 2005, 143–158.

[28] Lind, J., and H. Tran: Regularity of Loewner curves. - Indiana Univ. Math. J. 65:5, 2016, 1675–1712.

[29] Löwner, K.: Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. - Math. Ann. 89, 1923, 103–121.

[30] Marshall, D. E., and S. Rohde: The Loewner differential equation and slit mappings. - J. Amer. Math. Soc. 18, 2005, 763–778.

[31] Poggi-Corradini, P.: Canonical conjugations at fixed points other than the Denjoy–Wolff point. - Ann. Acad. Sci. Fenn. Math. 25:2, 2000, 487–499.

[32] Pommerenke, Ch.: Boundary behaviour of conformal mappings. - Springer-Verlag, 1992.

[33] Pommerenke, Ch., and A. Vasil'ev: Angular derivatives of bounded univalent functions and extremal partitions of the unit disk. - Pacific J. Math. 206:2, 2002, 425–450.

[34] Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko: The mathematical theory of optimal processes. - Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961 (in Russian).

[35] Prokhorov, D., and K. Samsonova: Value range of solutions to the chordal Loewner equation. - J. Math. Anal. Appl. 428:2, 2015.

[36] Prokhorov, D., and A. Vasil'ev: Singular and tangent slit solutions to the Löwner equation. - In: Analysis and Mathematical Physics, Trends in Mathematics, Birkhäuser Verlag, 2009, 455–463.

[37] Roth, O., and S. Schleissinger: Rogosinski's lemma for univalent functions, hyperbolic Archimedean spirals and the Loewner equation. - Bull. Lond. Math. Soc. 46:5, 2014.

[38] Vasil'ev, A.: On distortion under bounded univalent functions with the angular derivative fixed. - Complex Var. Theory Appl. 47:2, 2002, 131–147.

[39] Wong, C.: Smoothness of Loewner slits. - Trans. Amer. Math. Soc. 366:3, 2014, 1475–1496.

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