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MASTER´S THESIS

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Abstract

The binomial real options valuation approach using the market asset disclaimer assumption with an emphasis on state-dependent cash flows is reviewed and implemented using geometric Brownian Motion as the stochastic process for project uncertainty and the cash flows. A comprehensive analysis is conducted to identify the value drivers of options, including timing-aspects, intrinsic option value versus the value of flexibility, sensitivities of the binomial model to interest rate and volatility, and revision of volatility estimates for the BDH case.

The example case is then extended by using the mean reverting stochastic process for the project value and cash flows using the censored binomial presented by Hahn (2005) and the non-censored binomial presented by Bastian-Pinto, Brandão, and Hahn (2010).

Finally, the case is valued with a simple, European option equivalent, Monte Carlo approach with the underlying factors following geometric Brownian Motion and mean reverting models, and the results are compared.

The model files can be made available upon request to the author for anybody interested.

Acknowledgement

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'An investment in knowledge pays the best interest'

Benjamin Franklin

'Education is a progressive discovery of our own ignorance'

Will Durant

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Abbreviations and general nomenclature

The following abbreviations are used throughout the text. Additional abbreviations necessary in specific chapters will be introduced when used in the individual chapters.

BDH	Referring to binomial project value model by Brandao, Dyer and Hahn
CF	Cash flow
DCF	Discounted cash flow
GBM	Geometric Brownian motion
GCE	Generalized Conditional Expectations (approach for project volatility estimation)
M1	Model 1 of Schwartz
MAD	Market Asset Disclaimer (assuming the NPV to be a project twin-asset)
MCS	Monte Carlo simulation
MR	Mean reversion / mean reverting process
NPV	Net present value
OU	Ornstein-Uhlenbeck process
ROA	Real options analysis
ROV	Real options valuation
SDE	Stochastic differential equation

State variables

- i - # of up movements
- j - # of down movements
- n - period number $= i + j$
- t - time at period n
- Δt - time increment

Nomenclature for stochastic process definitions

- S - underlying asset / twin asset
- Y - logarithm of the underlying, $\ln(S)$
- Φ - mean reversion point of the underlying asset
- φ - logarithm of the mean reversion point, $\ln(\Phi)$
- σ - volatility of the process measured as standard deviation > 0
- μ - drift of the process (absolute measure for arithmetic, percentage drift for geometric models)
- dt - time increment
- η - mean reversion coefficient / mean reversion speed
- W_t - the Wiener process
- ε - standard normally distributed $N(0,1)$ random component

Unit variables for binomial methods

$V_{i,j}$	-	NPV of project in given state
z	-	logarithm of period return, percentage return in continuous compounding
σ	-	annual standard deviation of z
η	-	mean reversion coefficient
p	-	probability of up movement in next period for given state
Φ	-	long-term equilibrium level
$CF_{i,j}$	-	cash flow in given state
Λ	-	dynamically programmed NPV from roll-back calculation

1. Introduction

1.1. General introduction:

This paper is about real options modelling. Real options valuation (ROV) is the valuation of future actions with flexibility, where several processes and modelling methods are available. This paper will focus on the application of binomial option pricing models to projects with both geometric Brownian motion and mean reverting characteristics. These financial models are tools to evaluate different possibilities but are not the same as performing the decisions. The models are based on assumptions that must be considered when used for decision support.

Real options analysis is acknowledged to be a good tool to valuate strategic investments and investment under uncertainty, but presentation of the results is often hard to communicate to non-technical decision-makers and stakeholders. The results from closed-form solutions and Monte Carlo methods are often presented as a single number, whereby they lose part of the reason for doing the analysis – understanding the forecast of the value development with its uncertainty and corresponding optimal decision strategies. Binomial and trinomial methods have been criticized for being too simple, but we argue that lattice-based models have a high communicational value, especially when presented to non-technical decision-makers.

Generic simplified models for option pricing is becoming more accessible, but accurate valuation of real options based on several uncertainties with realistic models still require expertise in the fields of stochastic theory, market understanding and modelling skills or mathematical skills (depending on approach).

Many widely traded commodities often exhibit mean reverting characteristics. Still, projects with a value dependent upon mean reverting processes have generally been modelled as geometric Brownian Motion as an extension of the financial options theory and methods. If the underlying revenues and costs have mean reverting characteristics, we have assumed that the development of the remaining project value will also be better approximated to a mean reverting process than a geometric Brownian Motion.

First, we introduce the background of options pricing and real options valuation with definitions of the concepts and the main modelling methods. The second part of the paper focus on the theory and implementation of a binomial option pricing model for projects (as a whole) approximating geometric Brownian motion based on the discounted cash flow estimates for parameter estimation, called the marketed asset disclaimer, with focus on state-dependent cash flow estimates (building on Copeland and Antikarov, 2001, and Brandao et al., 2005). The method and results are analyzed and decomposed to identify the value drivers of the options in the model. Further, the same approach is implemented in two different mean reverting binomial lattice methods, the censored model and the non-censored model. The case example is also valued using a simplified European Monte Carlo method with sales price and cost processes modelled as geometric Brownian Motions and mean reverting, using the expected value at the term date of the options as the best estimate of the option payoff. In the end the results are analyzed and compared with concluding remarks.

1.2. Introduction to options valuation

In 1900 Louis Bachelier (Bachelier) was the first to introduce stochastic processes to finance through applying what is now called a Brownian motion to model the market noise of the Paris Bourse. The field of stochastic integration continued to develop through the work of Einstein (1905), Wiener, and most significantly Itô. In the realm of financial options Kassouf and Thorp introduced hedge ratios and dynamic hedging.

Based on these stochastic processes Black and Scholes (1973) developed their famous option-pricing formula for European financial options. By setting up and solving a partial differential equation for a risk-neutral portfolio with continuously revised delta hedging, they enabled easy calculation of the "right price" of an option. In other words, they showed how one can set up a portfolio of stocks and issued bonds (borrowings) that replicates the change in value for an option in the short term and thereby how that value is expected to change for a given range of outcomes, determined by a volatility estimate. The derivation of the Black-Scholes formula is consequently the solution of the Black-Scholes equation using Itô´s lemma. Merton (1973b) contributed the formula with the no-arbitrage argument.

Four years later Myers (1977) coined the phrase "Real-Options" as he started to gain insight into how financial option-theory can be used in valuation of real (non-financial) assets.

Cox, Ross, and Rubinstein (1979) developed the binomial option pricing model, where the underlying financial asset is modelled in a discrete-time tree or lattice. The option value is calculated from the replicating portfolio theory.

Boyl (1977) introduced Monte Carlo methods to option pricing, but not until the 90´s did they become readily available with possibilities for valuation of American options.

2. Theory

2.1. What are real options?

The name real options comes from Myers (1977) description of options on “real assets”. Real options are options on non-financial assets and can be seen as decision opportunities for a corporation or an individual. The real option is based on the uncertain value of some underlying asset, representing a right, but not obligation, to execute an action – typically an investment, at some point in time. The options may be related to the project value as a whole, like growth options and abandonment options, or to operational flexibility, like switching options on inputs and outputs for a production system. The contingent claim from a real option depends on the outcome of some uncertain events, including the effect of learning over time.

In their book *Investments under Uncertainty* Dixit and Pindyck (1994) describe how real options can capture the value of flexibility in investments with uncertainty. In their book *Real-Options: A Practitioner’s Guide* Copeland and Antikarov (2001) describe the comparison to financial options with examples of their respective financial option counterparts. Trigeorgis (1993) explain that the value of managerial flexibility is a type of real option and Luehrman (1998) state that real options theory can be used to valuate strategic decision-making, noting how business strategy is much more like a series of options than a series of static cash flows. Triantis and Borison (2001) describe three categories of interpretations of real options by practitioners as (1) a way of thinking, (2) an analytical tool, and (3) an organizational process. This thesis will mainly focus on the modelling of real options using different stochastic processes, exemplified in a case with two real options on the project value level.

To understand the dynamics of real option valuations (ROV) one must get an overview of 4 major aspects of real options:

1. Types of options
2. Stochastic processes
3. Modelling methods
4. Model inputs

2.2. Types of options

To understand the drivers of an option's value, it is first important to understand the structure of the option. The two most basic option types are call options and put options.

> **Call option:** Gives the option holder the right, but not obligation to **acquire** an asset in the future.
>
> **Put option:** Gives the option holder the right, but not obligation to **sell** an asset in the future.

The price to which the option holder can buy or sell is called the **strike price**. Further, an option can be classified as a European or an American option.

> **European option:** Can only be exercised **at** a pre-determined expiration date.
>
> **American option:** Can be exercised at **any time up to** the expiration date.

Options limited to this framework (American or European, call or put options) are called vanilla options. Two other exercise-time related financial options terms that are particularly related to ROA are:

> **Bermudan option:** Can be exercised at any time in a **set exercise interval**.
>
> **Evergreen option**: Can be exercised only after a predetermined **period of notice** (giving a lag-effect).

2.2.1. Simple options

Options that gives the right to only one action (subsequent) and are exposed to only one underlying risky asset are often called simple options. These basic option types are related to time perspectives, scaling decisions, and single start/stop decisions. Below is a list of the common simple real option types:

Real option	Financial option equivalent	Type and description
Invest	Call	Call for project CF
Abandon	Put	Put of full CF
Expand	Scale up (call)	Call for marginal expansion of CF
Contract	Scale down (put)	Put for marginal downscaling of CF
Postpone	Call	Call for project CF at a later time (learning option)
Extend	Call	Call for extended CF after original project CF

Table 1 - Simple real options

2.2.2. Non-simple options

Simple options can be combined subsequently to form non-simple options. Also, options dependent on multiple underlying processes are classified as non-simple. These include compound options, rainbow options and switching options. Examples of actions that can be modelled as non-simple options include product mix (output) options, process mix (input) options, operation options and sequencing options. Option valuation modelled dependent on the outcome of a combination of private- and market uncertainties is another example.

2.2.3. In- or out of the money

Options with an expected value of the payout at a given point in time are termed to be "in the money". For call options, this means that the price of the underlying asset is higher than the strike price, and for put options that the asset price is below the strike price. Options with expected value of payout if exercised at current time are termed to be "out of the money".

2.3. Stochastic processes and concepts

An options payoff is a function of the development of the underlying asset in time. Option pricing models estimate this uncertain development as a stochastic process. Stochastic process characteristics include arithmetic versus geometric development, processes with drift versus martingales, continuous versus discrete models, mean reversion, jump diffusions and many other factors. The most common stochastic process used in ROA is geometric Brownian motion (GBM). A general introduction to stochastic processes can be found in *Options, Futures and Other Derivatives* by Hull and Basu (2016), *Paul Wilmott Introduces Quantitative Finance* Wilmott (2007) or *Introduction to Stochastic Calculus Applied to Finance* by Lamberton and Lapeyre (2011) and others.

As an introduction to stochastic modelling some basic concepts of stochastic processes are described in the following section. The general nomenclature of for the processes are as follows:

S	-	underlying asset / twin asset
Y	-	logarithm of the underlying, $\ln(S)$
Φ	-	mean reversion point of the underlying asset
φ	-	logarithm of the mean reversion point, $\ln(\Phi)$
σ	-	volatility of the process measured as standard deviation > 0
μ	-	drift of the process (absolute measure for arithmetic, percentage drift for geometric models)
dt	-	time increment
η	-	mean reversion coefficient / mean reversion speed
W_t	-	the Wiener process
ε	-	standard normally distributed $N(0,1)$ random component

2.3.1. Random walk

A random walk is a stochastic process that starts in 0 and evolves with $+1$ or -1 with probability p and $(1-p)$ respectively over n periods. This is a discrete model.

2.3.2. Markov process

A Markov process is a memoryless process where history is irrelevant, whereby only the current value of the variable is relevant for predictions.

2.3.3. Martingale process

A Martingale is a process with expected value equal current value. This is equivalent to zero expected drift.

$$E(S_{t+1}) = S_t \qquad \mu = 0$$

2.3.4. Wiener process

A Wiener process is a standard Brownian motion for time $0 \leq s \leq t$ characterized by:

1. $W_0 = 0$

2. W_t is almost surely continous
3. Each increment is independent
4. Each increment is normally distributed with expected value $\mu = 0$ (no drift) and variance $\sigma^2 = t - s$ (written $W_t \sim N(0, t-s)$)

The stochastic differential equation (SDE) for a Wiener process can be written as

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t$$

2.3.5. Arithmetic VS Geometric

While an arithmetic change process is additive, a geometric process is multiplicative. For many processes a series of percentage-wise changes is preferred for modelling as this often reflects the underlying change better than a series of absolute changes. For example, changes in the logarithmic value limit the development to non-negative values, which is true for stock- and commodity prices.

2.3.6. Geometric Brownian motion

Geometric Brownian Motion (GBM) is the most commonly used stochastic process for option valuation in general. GBM follows the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where the increment of the wiener process is $dW_t = \sqrt{dt}\,\varepsilon_t$. In the geometric process the drift is measured as expected percentage change. Discounting of future cash flows is incorporated as drift. Itô's lemma gives the analytical solution

$$\ln(S_t) = \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}\varepsilon_t \qquad S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\varepsilon_t}$$

The conditional expected value is $E[S_t|S_{t-\Delta t}] = S_{t-\Delta t}e^{\mu\Delta t}$ where $\mu = \ln(1 + discrete\ drift)$ is the continuous drift. From time series data the model can by calibrated by $\hat{\sigma} = \sqrt{\frac{Var[z]}{\Delta t}}$ and $\hat{\mu} = \frac{E[z]}{t} + \frac{\sigma^2}{2}$ where $z = \ln\left(\frac{S_t}{S_{t-1}}\right)$ is the period return in percent for which $E[z]$ is the expected value and $Var[z]$ is the variance. One can also calibrate the process from futures data, reflecting the markets view of uncertainty, giving the risk-neutral drift. One can also calculate the expected future volatility in the market from the market prices of options on the asset (*implied volatility*).

2.3.7. Mean reverting models

In contrast to Brownian Motion processes with drift, mean reverting processes (MR) tend to revert to some long-term equilibrium value. Macroeconomic principles support the assumption of mean reversion in commodity markets (Schwartz, 1997). High prices tend to stimulate additional investments (to increase capacity), dampen demand and increase substitution where possible. Low prices tend to reduce investments in new capacity and increase demand for the commodity. These reactions all tend to normalize to a long-term supply-demand equilibrium. We will discuss whether we can categorize projects exposed to mean reverting processes to the same stochastic behavior is chapter 5.

2.3.7.1. Arithmetic Ornstein-Uhlenbeck processes

The most basic mean reversion is the arithmetic Ornstein-Uhlenbeck process. The SDE for the Ornstein-Uhlenbeck process (OU) can be written as

$$dS_t = \eta(\Phi - S_t)dt + \sigma dW_t$$

The process has the expected value $E[S_t|S_{t-\Delta t}] = S_{t-\Delta t}e^{-\eta\Delta t} + \Phi(1 - e^{-\eta\Delta t})$. The process is a Markov process, but the increments are not independent since they depend on the difference between the current price and the long run equilibrium. The three quantitative methods for calibration of an observed arithmetic OU process are least squares estimation, maximum likelihood estimation and the jackknife technique. This is covered by W. Smith (2010). The discretized solution is to the SDE is

$$S_t = S_{t-1}e^{-\eta\Delta t} + \Phi(1 - e^{-\eta\Delta t}) + \sigma\sqrt{\frac{1 - e^{-\eta\Delta t}}{2\eta}}\varepsilon_t$$

An alternative process to the OU process is a model that reverts to the slope of the starting point plus a drift component $\Phi_t = S_0 + \mu t$. The arithmetic Ornstein-Uhlenbeck process with drift can be called trend stationary.

$$dS_t = \big(\mu + \eta(\mu t - S_t)\big)dt + \sigma dW_t$$

It can be shown that the exact solution is

$$S_t = S_0 e^{-\eta\Delta t} + \mu\Delta t + \sigma\sqrt{\frac{1 - e^{-\eta\Delta t}}{2\eta}}\varepsilon_t$$

When η goes to 0 the process becomes the arithmetic Brownian motion.

2.3.7.2. Geometric mean reverting models

To restrict the values to be non-negative we can model the mean reversion geometrically. The simplest approach for this is to model the volatility geometrically, keeping the other parameters similar to the arithmetic Ornstein-Uhlenbeck process.

$$dS_t = \eta(\Phi - S_t)dt + \sigma S_t dW_t$$

This however is not practical when it comes to finding the numerical solution and calibration of the model. For these reasons it is often preferred to work with the arithmetic OU process. This has led to development of arithmetic processes of the logarithm of the price. One of these is the model of Dixit and Pindyck (1994), dubbed the DPM model for mean reversion

$$dS_t = \eta S_t(\Phi - S_t)dt + \sigma S_t dW_t$$

By Itô's lemma the process becomes

$$dY_t = \eta^*(\varphi^* - Y_t)dt + \sigma dW_t$$

Where $\varphi^* = \left(\Phi - \frac{\sigma^2}{2\eta}\right)\frac{Y_t}{S_t} = \left(\ln(\varphi) - \frac{\sigma^2}{2\eta}\right)\frac{\ln(S_t)}{S_t}$ and $\eta^* = \eta\frac{S_t}{Y_t}$ whereby the parameters are functions of S_t and not constant, making the model hard or impossible to calibrate from historical data or the derivatives market.

Schwartz (1997) proposed 3 different models for commodity pricing. Model 1 describes

$$dS_t = \eta(\varphi - \ln(S_t))S_t dt + \sigma S_t dW_t$$

From Itô's lemma on Y_t:

$$dY_t = \eta(\varphi^* - Y_t)dt + \sigma dW_t$$

Where $\varphi^* = \varphi - \sigma^2/2\eta$. The expected value is $E[Y_t] = Y_{t-\Delta t}e^{-\eta\Delta t} + \varphi^*(1 - e^{-\eta\Delta t}) + \sigma^2(1 - e^{-2\eta\Delta t})/4\eta$. Note that this process does not refer to the mean reversion of the price, but of the logarithm of the price.

The model for mean reversion of the project value modelled in chapter 5 is that of an arithmetic OU-process of the logarithmic value

$$dY_t = \eta(\varphi - Y_t)dt + \sigma dW_t$$

The mean reverting models takes the long-term equilibrium as a constant over time, and so stochastic processes far into the future for quickly reverting processes might undervalue the long-term uncertainty. E.g. for an oil field with expected life of 40 years, the uncertainty of the oil and gas prices will likely be undervalued, and the outcome heavily dependent on the long-term equilibrium.

2.3.8. Two-factor models

Schwartz and Smith (2000) proposed a two-factor model with short-term variation and long-term drift for commodity spot-price modelling, where both are stochastic processes. The model decomposes the price to a long-term drift component ξ_t modelled as a GBM, and a short-term variation component χ_t modelled as an Ornstein-Uhlenbeck process that revert to zero.

$$ln(S_t) = Y_t = \chi_t + \xi_t$$

$$d\chi_t = -\eta\chi_t dt + \sigma_\chi dz_\chi$$

$$d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi$$

When the short-term component is zero the price will be equal to the long-term equilibrium price. The increments of the two Brownian motion processes, dz_ξ and dz_χ are assumed to be correlated $\left(dz_\xi dz_\chi = \rho_{\xi\chi} dt\right)$. Possible calibration methods for the model includes (1) implied estimation, (2) sequential optimization and (3) Kalman filtering applied with the state-space approach using maximum likelihood estimators for parameters of the unobserved state variables.

2.3.9. Other processes

Other significantly relevant concepts in the field that are not covered in this article include gaussian jump diffusion processes, multifactor processes and processes with stochastic volatility and stochastic discount rate. The use of stochastic models is also widespread in interest rate modelling, including the Vasicek model, the Cox-Ingersoll-Ross model, the Ho-Lee model and the Hull-White model.

Engle (1982) developed a model for autoregressive conditional heteroscedasticity (ARCH), enabling fluctuation of the volatility according to an autoregressive function, where the geometric processes presented assume the volatility to be constant. Bollerslev (1986) further developed a generalized autoregressive constant heteroscedasticity model (GARCH) where the variance of the process is modelled as an autoregressive moving average process (ARMA). This paper will only model homoscedastic volatility.

2.3.10. Tests for determination of stochastic process

Ozorio, Bastian-Pinto, and Brandão (2012) discuss the importance of choosing the right stochastic process to approximate the uncertainty of the process in question. They suggest 5 methods to test the stochastic process type to data:

- Dickey-Fuller test
- Augmenterd DF test
- Unit roots
- Variance ratio test
- Adherence measures for a sample (e.g. Pseudo R^2, Mean quadratic error and Mean absolute percentage error)

2.4. Investment valuation principles

2.4.1. Net Present Value approach

The traditional approach to value potential capital investments is the net present value (NPV) approach. The NPV of a project is the present value of the expected future cash flows. This is set up in a discounted cash flow (DCF) model with expected future income and expenses discounted at a "risk-adjusted" rate. Riskier projects will thus be discounted more.

$$NPV = -I + \sum_{i=0}^{N} \frac{CF_i}{(1 + WACC)^{t_i}}$$

The most common method for finding the risk-adjusted rate is the weighted average cost of capital (WACC) of a firm. The WACC is weighted between the required rate of return from equity holders, calculated from Merton´s capital asset pricing model (CAPM), and the cost of debt.

$$WACC = r_E \frac{E}{D+E} + r_D \frac{D}{D+E}(1-\tau) \text{ where } r_E = r_f + \beta\left(r_m - r_f\right)$$

E is the market value of equity and D is the market value of debt, and τ is the tax rate. r_f is the risk-free rate, r_m is the expected return in the market and β indicates whether the investment is more or less volatile than the market. Discounting with the firms WACC might be appropriate when valuing projects that extends a homogeneous project portfolio. For projects that does not mimic the general riskiness of the firm, the problem is to find a discount rate that reflect the economic project riskiness.

2.4.2. Decision Analysis

Decision analysis was coined by Ron Howard in the 1960s. The decision analysis approach sets up a decision tree that describes the sequence of uncertainties and decisions. This is done in a dynamic tree or an influence diagram consisting of chance nodes, decision nodes and information nodes. The chance nodes carry subjectively assigned probabilities of each outcome of the node, where the outcomes are quantified with a utility value for general decisions. The optimal decision strategy is found as the highest certainty equivalent when solving the tree backward. The certainty equivalent is the value for which the decision maker is indifferent between taking the certainty equivalent for sure or the uncertain alternative. Qualitative decisions can be modelled by quantifying the utility of each outcome as a measure of preference. An investors utility function describes his/her preferences, where he/she can be classified as risk-loving, risk-neutral or risk-averse. For more on decision analysis see Bratvold and Begg (2010).

2.4.3. Risk-neutral valuation

The objective of the risk-adjustment of the discount rate used in NPV analysis is to compensate for uncertainty in future cash flows. If the future cash flows were certain we could discount at the risk-free rate. An alternative valuation approach to the NPV analysis thus becomes the certainty equivalent of the uncertain future cash flows discounted at the risk-free rate. The certainty equivalent is the value adjusted by the risk-neutral measure, also called the martingale equivalent.

Let's look at this for a stock. In a complete market the no-arbitrage argument state that the price calibrated with the right expectations of the value of the underlying. Thereby the expected return of holding the stock will be the risk-free rate of return.

For real options the risk-neutral process is estimated from using the risk-neutral processes of the variables affecting the project valuation. If an oil project knows its production, sells future production in the futures market, hedges its costs and adjusted the valuation for other private risks (with risk-neutral probabilities), then we can estimate the risk-neutral cash flows of the project. If the project with its rights can be bought or sold (shorted) in the market, the price must be the risk-neutral cash flows discounted at the risk-free rate. If it was not, one could buy or short the project value and pocket the difference to the risk-neutral project value.

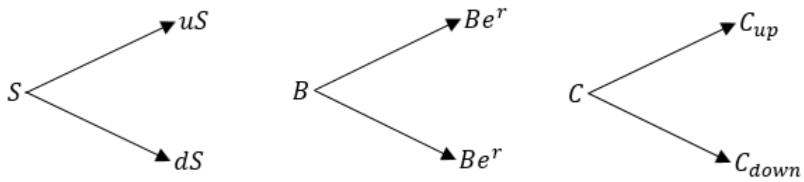
The risk-adjusted NPV method and decision analysis are not directly compatible methods because of the risk-adjusted discount-rate, but J. E. Smith and Nau (1995) showed how decision analysis is consistent with option pricing methods when using risk-neutral valuation. J. E. Smith and McCardle (1998) implement a combination of decision analysis for private risks and risk-neutral real option pricing for market risks that can be hedged, through the valuation of an oil property. The certainty equivalent of expected future cash flows is the value of the discounted cash flows. Smith and McCardle call the approach an integrated valuation procedure.

Risk-neutral processes can be estimated using the capital asset pricing model of Merton (1973a) or other methods. For widely traded commodities the risk-neutral drift can be calculated from the futures market, where futures are standardized contracts for delivery on a future date for a given price. (The spot price is the special case of a futures contract where time to term date equal null.) Thereby the futures price captures the markets expectations of the price development. The benefit or premium associated with holding the underlying asset rather than a futures contract or derivative product is known as the convenience yield. For further discussion of estimation of estimation of market price of risk, see Hull and Basu (2016).

2.4.4. Replicating portfolio theory

The most basic idea behind options pricing is to make a portfolio that replicate the payoffs of a given option, where the no-arbitrage argument (Merton, 1973b) state that the option and the replicating portfolio must at all times and in all states have the same value. The replicating portfolio is set up based on the underlying asset of the option and borrowings. In financial terms the replicating portfolio approach valuate the option based on a continually revised delta-hedge of the option using the underlying security and bonds. Black and Scholes (1973) proved that this continuous hedge removes the expected return of the underlying asset as a factor in the options value, enabling risk-neutral valuation of the option value. The expected return of the option can thus be discounted at the risk-free rate. This was the key insight behind the Black-Scholes model.

The replicating portfolio consist of m units of the underlying security with value V and B units of a risk-free bonds paying r in annual interest. The option is a contingent claim on the underlying security. The capital loss or gain from the replicating portfolio in an *up* or *down* state is calculated as the payoff of the call option *C* in the *up* and *down* state of the underlying asset.



We get two equations, one for each state, $muS + Be^r = C_{up}$ and $mdS + Be^r = C_{down}$. Solving for the unknowns, *m* and *B*, we get

$$m = \frac{C_{up} - C_{down}}{(u-d)S}, \qquad B = \frac{uC_{down} - dC_{up}}{(u-d)e^r}$$

If an option trades above or below the two perfectly hedged replication, then one would be able to sell (short) or buy the option while also constructing the hedged replicating portfolio, pocketing the difference as an arbitrage opportunity. By imposing the no-arbitrage argument we can calculate the risk-neutral probability of the replicating portfolio, where

$$S = \frac{puS + (1-p)dS}{e^r}$$

gives

$$p = \frac{e^r - d}{u - d}$$

2.4.5. Perspectives on uncertainty for real options

We use real options models as a valuation tool under conditions of uncertainty. Whenever we need to quantify uncertainty we should consider who´s uncertainty. For project evaluation we aim to represent the uncertainty of the decision-maker, where the decision-maker ultimately represent the shareholders of the company. The uncertainty is most often represented through risk-adjusted discounting, or alternatively through the utility value from decision analysis.

The breakthrough in option pricing came when option prices became independent from the expected development of the underlying based on the theory that one can replicate the payoff of the option with a delta-hedged replicating portfolio. The payoff from the option can thus be scaled up and down, and the alternative investment is the risk-free rate. The cost of synthesizing the replicated portfolio is a function of the uncertainty of the underlying asset, which is calculated from a stochastic process.

With the MAD approach real options pricing is done using replicating portfolio theory on the NPV, but a replicating project does not (necessarily) exist. Thereby it's not obvious that the drift is the risk-free rate, and the uncertainty we want to quantify will be the expected forward-looking uncertainty for the stakeholders of the company.

2.5. Real option valuation methods

2.5.1. Black-Scholes option pricing model

Black and Scholes (1973) developed the first option pricing model from the replicating portfolio approach. The Black-Scholes equation is a stochastic differential equation that captures the replicating portfolio for a European option that consist of the underlying uncertain financial asset (stock) modelled as a GBM, and borrowings. Because of the no-arbitrage argument the option value equals the cost of synthesizing the replicated portfolio. For a European option, the equation is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S} + rS\frac{\partial C}{\partial S} - rC = 0$$

Where C is the call option value and P is the put option value. The closed-form solution to the equation is the Black-Scholes formula as follows

$$C = N(d_1)S_0 - N(d_2)Ke^{-rT}, \qquad P = -N(-d_1)S_0 + N(-d_2)Ke^{-rT}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T)\right), \qquad d_2 = d_1 - \sigma\sqrt{T}$$

K strike price and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution function. The formula is extensively used in financial markets, but carry a strict set of assumptions, limiting the applicability to ROV. The model assumes:

1. Only be exercised at maturity – European options only
2. Only one source of uncertainty – no rainbow options
3. Contingent on only one underlying asset – no compound options

4. No dividends
5. Uncertainty follows geometric Brownian motion
6. Parameters are observable
7. Volatility is constant with time
8. Risk-free rate is constant with time

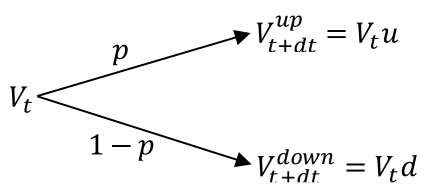
(Black & Scholes) (Copeland & Antikarov)

2.5.2. Other stochastic differential equations

Several expanded SDE-based closed-form solution models have been developed since the Black-Sholes formula was published. Examples of relaxed assumptions in other SDE-based models include correction for dividends, perpetual American option model adjustments, mean reversion characteristic of risky asset, correction for varying volatility (Heston) and models for two underlying risky assets. However, some of these models become to mathematically sophisticated to be practical for practitioners, and SDE-based models has limitations when working with high-dimensional problems or don't have an analytical solution. The closed form solutions to the basic stochastic processes are presented in chapter 2.3.

2.5.3. Binomial option pricing model

Binomial recombining trees for financial options were first developed by Cox et al. (1979) based on the replicating portfolio theory through the binomial option pricing model. The model can value American options with dividends as fractions of the asset and added educational value through visualization of discrete steps. The twin security, S, can over each discrete time step Δt develop to an up value, Su, or a down value, Sd, with respective probabilities p and $(1-p)$. By requiring that the first and second moment of the of the binomial diffusion to match that of the continuous diffusion, the up and down movements are calculated as $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}} = 1/u$. The up-probability is defined as $p = \frac{1+r_f\Delta t-d}{u-d}$, where r_f is the risk-free rate.



The probability of an up move, p is calculated as

$$p = \frac{e^{r_f\Delta t} - d}{u - d}$$

The values are calculated back from the end values to the When a rollback tree with the possible developments of S is constructed options are added in the lattice as maximum values of the *exercise* and *not exercise* values in their respective exercise times. Starting with the endpoint the nodes in the lattice are calculated backwards to time $t = 0$ as

$$S = \frac{pSu + (1-p)Sd}{e^{r_f \Delta t}}$$

The probability distribution implied by the lattice converges to a geometric Brownian motion when Δt goes to zero.

Binomial trees can be developed in several other ways, including binomial trees correcting for skewness and kurtosis, trees with probability of 0,5 for all nodes, trees approximating mean-reverting processes, and to three-dimensional trees (2 underlying uncertainties) and two-factor trees. Boyle (1986) introduced trinomial trees, much used in interest rate modelling.

Binomial trees with recombining nodes are called lattices. The original binomial option pricing model is recombining, but if the volatility is not considered constant over time (if the diffusion is heteroscedastic) or the model includes fixed dividends, the tree will not be recombining. However, heteroscedastic diffusion models can be transformed to be homoscedastic, and dividends can be modelled as a fraction of the underlying to keep the lattice form. Despite the limitations, lattices are popular with practitioners because of the computational simplicity and how they allow for ease of communication of the optimal decision strategy and identification of option value drivers without much additional analysis. Another advantage of lattices is the reduced computational burden. For n periods (coundting from 0) a lattice will have n endpoints and $n(1+n)/2$ nodes. Non-recombining trees will have 2^{n-1} and $2^n - 1$. (J. E. Smith, 2005)

Lattices can easily be modelled in excel. From personal experience the preferred tool of modelling non-recombining trees is DPL from Syncopation Software, but this can also be calculated in excel with VBA or using other programming languages.

2.5.4. Marketed Asset Disclaimer (MAD)

Previous models work well for modelling financial options based on the replicating portfolio approach, but what is the twin-security for a project? It is practically impossible to find a priced security whose cash payouts are perfectly correlated to a project. Copeland and Antikarov

(2001) suggest using the present value of the project without options. By assuming that the NPV is the best unbiased estimator for the market value of the project we can use it as the underlying asset for calculating real options on the project level. They call this the marketed asset disclaimer. Further they assume that the change in project value follow a random walk. A Monte Carlo simulation of the DCF with uncertainty in marketable parameters create a distribution of possible periodic project return. The periodic return volatility is estimated as the standard deviation of the return distribution from the simulation. Brandão, Dyer, and Hahn (2005a) point out that it's important to isolate the uncertainty in project variables to the period for which project returns are estimated. Following periods are set to conditional expected values to avoid overstating the period volatility by including uncertainty in later periods. They also stress the extraction and add-back of cash flows from the project in the development and roll-back trees to avoid the modelled volatility in later years from affecting received cash flows. The cash flow manipulation is analogous to dividends of financial options.

J. E. Smith (2005) point out that the MAD approach inconsistently use a risk-adjusted discount rate in the calculation of the NPV, but risk-free rate in the following binomial lattice of the development of the remaining project value. He suggests using a fully risk-neutral approach, adjusting the stochastic processes to risk-free development discounted at the risk-free rate in the DCF with the MCS estimation of the project process parameters. The risk-neutral MAD approach with extraction of period cash flows, coined the BDH method, will be thoroughly covered from chapter 3 and onwards.

2.5.5. Monte Carlo Method

Monte Carlo simulation (MCS) is the method of generating a probability distribution for the range of potential outcomes of an uncertain calculation by sampling a large number of iterations of the problem. MCS was first applied for option pricing by Boyle (1977). The use of MCS to value a European vanilla option is done by estimating the discounted average option payout at time T. The iterative process for a GBM process follows Itô's formula as $\ln(S_T) = (\mu - \sigma^2/2)T + \sigma\sqrt{T}\varepsilon$, where ε represent the standard wiener process, normally distributed with mean of 0, standard deviation 1, $N(0\,,1)$. Using the risk-free rate r_f as drift the value of the underlying will thus evolve to

$$S_T = S_0 e^{\left(r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\varepsilon}$$

The call option for the presented process can be calculated as

$$C_0 = e^{-rT}\text{average}(\max(S_T - K, 0))$$

The main strength of MCS for option pricing is the potential to value high-dimensional path-dependent American options, where the least squares Monte Carlo method (LSM) of Longstaff and Schwartz (2001) is the most common. These models can also incorporate many different stochastic processes into one model. However, the model is computationally intensive and less visually intuitive compared to policy trees from binomial models. Thomas and Bratvold (2015) implemented the LSM method to the switching option of a blowdown decision using the correlated two-factor models for oil and gas prices. Before 1993 MCS was only used for European options.

2.5.6. Greeks

The Greeks are quantities representing the sensitivities of options parameters to differences in the input parameters, where the first order Greeks refers to the percentage sensitivities of the options value. These are tools extensively used for understanding development of the option value and in the hedging of portfolios of financial options rather than options valuation.

	Underlying (S)	Volatility (σ)	Interest rate (r_f)	Time to expiry (T)
Option value (V)	Delta (Δ)	Vega (υ)	Rho (ρ)	Theta (θ)

Although much of the developments in financial options have been applied to real options, these risk management tools have received less focus in the ROA literature (Haug, 2006, 2007). Vega and rho has been calculated for a real options case in chapter 4.3 with subsequent discussion.

2.5.7. Local conclusion for methods

The three most used models for ROV are closed-form solutions to stochastic differential equations, binomial option pricing methods, and the least squares Monte Carlo method. Closed-form solutions are exact, but have limitations, especially for multi-dimensional problems. Binomial methods have proven robust and are often good tools for visualization of the optimal policy when limited to one or two dimensions. These can also value American options. The most widely used method is the least squares MCS method from its flexibility and accuracy.

For American mean reverting real options the main methods are least squares MCS, binomial approximations and trinomial approximations.

2.6. Input variables

The last part needed to understand the mechanisms of ROV are the input variables to the methods. The input variables required vary with the type of method and stochastic process. This paper will later cover parameter estimation for the BDH-method (GBM), and for a mean-reverting versions of the BDH, where we model the project value with cash flows (as opposed to the underlying variables; oil price and operational cost).

3. BDH method

Brandão, Dyer, and Hahn (2005b) (further BDH) describe an approach to ROV of projects building on the MAD approach of Copeland and Antikarov (2001) with a particular focus on separating the periodic project cash flows (like dividends). The concepts were demonstrated in an example case of an oil production project. Through correspondence with (J. E. Smith, 2005) the case was developed as a fully risk-neutral version, where the DCF values are estimated with risk-neutral growth and discounting. The case was first developed from a standard equilibrium DCF with wrong volatility estimates, but in subsequent versions the volatility estimate was corrected to capture isolated annual volatility. The example case given by BDH has been developed both as a lattice in excel and as a tree in DPL, including a non-recombining tree capturing the heteroscedastic diffusion results from running the MCS for each year separately and a bivariate tree of the underlying uncertainties. The method is referred to as the BDH method (though 'MAD cash flow method' might be a more describing name), and the example case is called the BDH case.

As a preparation for the development of the mean-reverting BDH method this chapter summarize the BDH method with its developments, before presenting the BDH case with extended analysis.

3.1. Model overview

The model development can be decomposed into a series of 4 main steps:
First, the risk-neutral DCF for the project is constructed. The equilibrium DCF (discounted at WACC) is standard procedure for valuation and can serve as a starting-point and as a reference.

Next, the risk-neutral stochastic behavior of the uncertain variables with corresponding correlations must be estimated and incorporated into the risk-neutral DCF. The consolidated project volatility is estimated by the standard deviation of the logarithmic return of one period in a Monte Carlo simulation modelling the isolated stochastic behavior of the uncertain variables with conditional expected values for the following periods.

Based on the project value and volatility a binomial lattice is constructed following up and down movements subtracting the cash flow proportional to the payout ratio in the given state and time. The probability of moving up from any point in the lattice is calculated from the volatility estimate and the risk-free rate. The project value can now be calculated as the discounted expected value in the next period (given by the up and down nodes) plus the cash flow in the given state and time. This is referred to as dynamic programming.

Finally, options can be added as maximum statements in the lattice in their respective term periods, where the value of the opportunity will be added to the risk-neutral NPV estimate.

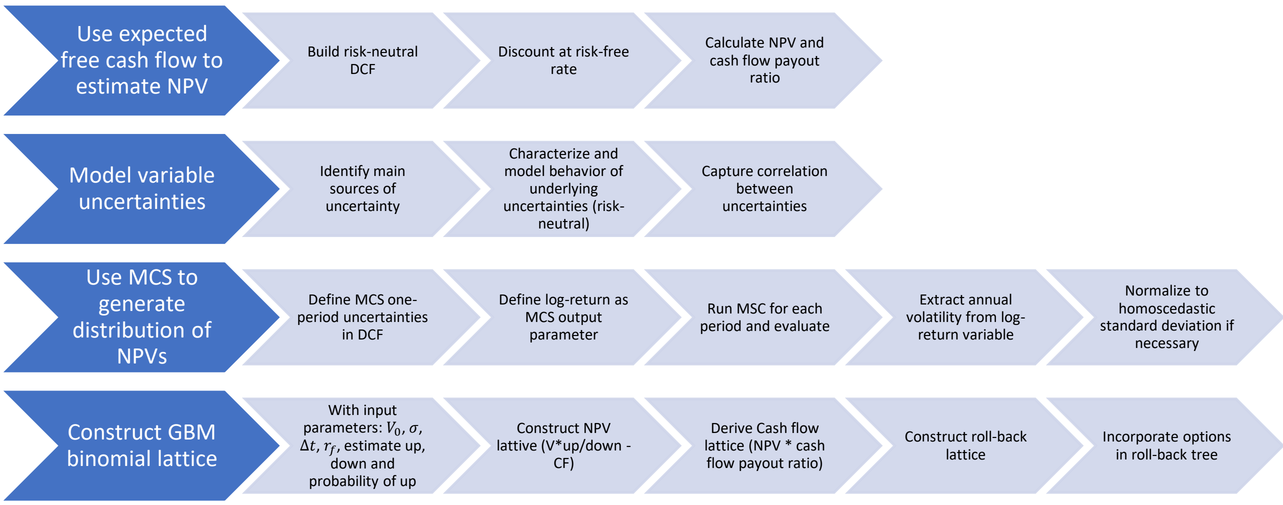


Figure 1 - MAD cash flow method overview

Note that the process consolidates the uncertainties, enabling the valuation of projects dependent on multiple uncertain processes to be modelled as simple options (of not subsequent).

3.2. Parameter estimation – calibration of the model

As financial options are based on traded securities, estimation of the parameters for financial option valuation are fairly straight forward. For ROA, where the underlying asset is generally not traded, these parameters must be estimated differently.

The binomial GBM approximation of the real option related to the project value require the following parameters with comparisons to valuation of stock options:

- Present value of expected future cash flows (NPV) (equivalent to stock price)
- Consolidated volatility of project return (equivalent to stock volatility)
- Cash flow payout ratio (equivalent to dividend yield)
- Discount rate, at risk-free rate for risk-neutral valuation (drift)
- Investment cost or selling price (strike price)

Also, the option type must be defined with the payoff function and time/time interval for exercise.

3.2.1. Project value and cash flow payout ratio

Based on the MAD assumption we use the NPV of the project without flexibility as the twin asset. Since we will develop the case fully risk neutral, we use the risk-neutral NPV with risk-adjusted processes for the uncertain project variables and discount the cash flows at the risk-free rate. Otherwise, the method is similar to the equilibrium DCF method. The cash flows of the projects are analogous to dividend yields for a financial option, where the cash flow payout ratio δ_t is used to estimate the cash flow in each year. The cash flow payout ratio is calculated as the fraction of the NPV_t in time t that comes from the cash flow, CF_t.

3.2.2. Underlying sources of uncertainty

As a starting point, the prices, quantities and costs related to the operation are often the main sources of uncertainty. Tools like tornado diagrams and sensitivity analysis can be helpful for selecting the most significant sources of uncertainty. The uncertain variables are modelled in the DCF as stochastic processes, where choosing the stochastic process that best represent the expected behavior of the given variable is important. The moments of the respective processes are estimated through historical data, implied volatility from derivative markets, a twin security, or simply through an educated guess. See (Ozorio et al., 2012) and (Ozorio, Shevchenko, & Bastian-Pinto, 2013) for more on calibration and fitness tests for stochastic processes in ROV.

3.2.3. Consolidated project volatility

The volatility, denoted by σ, is defined here as the standard deviation of the logarithmic project returns, $\sigma(z)$, for a defined time increment, dt. The logarithmic return represents the percentage change in expected NPV from period $n-1$ to n, representing time $t-\Delta t$ to t.

$$z = \ln\left(\frac{V_t}{V_{t-\Delta t}}\right)$$

While estimation of volatility for financial options is based on implied volatility from the Black Scholes formula, the volatility of the expected cash flows in ROV must be estimated in a bottom-up manner. This is done by identifying and estimating the stochastic variation for each uncertain variable in the DCF with corresponding correlations and then running Monte Carlo simulations (MCS) to collect them to a consolidated project value volatility.

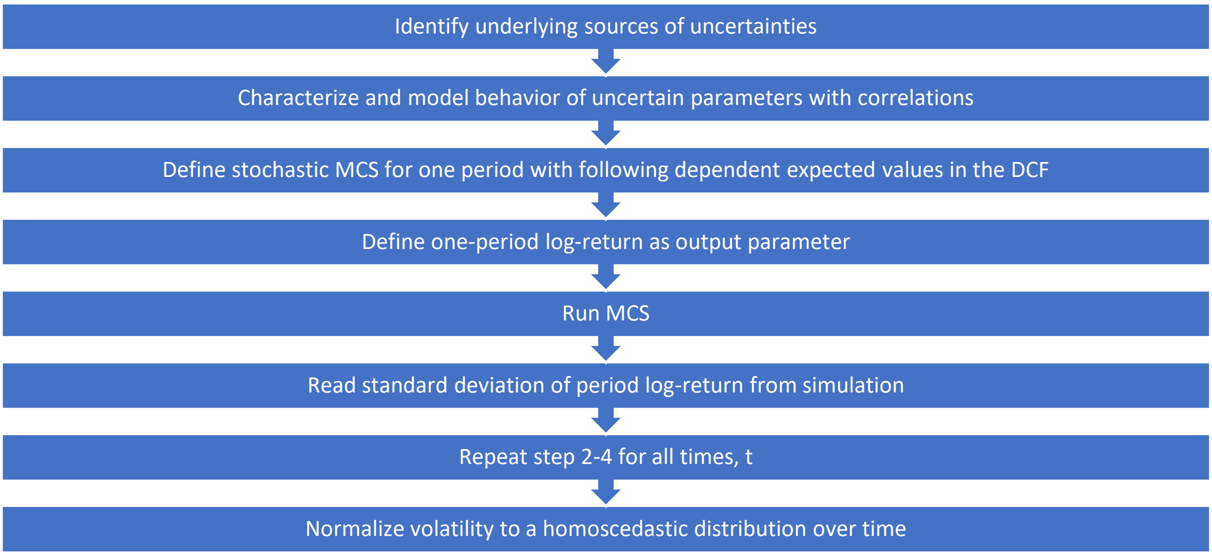


Figure 2 - Estimation procedure for consolidated project volatility

With the stochastic input variable models defined, we are almost ready to run the MCS. Recall that the volatility, $\sigma(z)$, is defined as the standard deviation of the logarithmic return **for a defined time increment** $\boldsymbol{t - \Delta t}$ **to** $\boldsymbol{t}$. Therefore, we must isolate the underlying parameter volatility to one time increment in the DCF, using conditional expected values in the following periods. Brandão, Dyer, and Hahn (2012) call this the Generalized Conditional Expectations (GCE) approach. This will give one volatility estimate for each time step, $\sigma(z_t)$, when done for all time steps.

$$z_t = \ln\left(\frac{V_t}{V_{t-\Delta t}}\right) = \ln\left(\frac{CF_t + \{NPV_{t+\Delta t}|CF_t\}}{NPV_{t-\Delta t}}\right) = \ln(\frac{CF_t + PV_t\left(E_t(CF_{t+dt}) + \cdots + E_t(CF_T)|CF_t\right)}{V_{t-\Delta t}}$$

Where

$$NPV_t = \sum_{i=t+1}^{T} \frac{CF_i}{(1+k)^{t_i}} \Bigg| CF_t$$

After having defined the stochastic inputs parameters and the output function z_t in Crystal Ball or @Risk, we are ready to run the MCS. We get the standard deviation σ from the program output for variable z_t.

The annual project volatility is defined as $\sigma_A = \frac{\sigma_{\Delta t}}{\sqrt{\Delta t}}$ where dt is the time step used in the pro forma DCF in years (normally yearly, quarterly or monthly).

As an alternative to the MCS estimation and statistical calibration procedures, Copeland and Antikarov argues that managers and industry experts working with valuation will have subjective estimates, not only for the expected values required for a DCF model, but can also give, for example a 95% confidence interval for the parameter or the project uncertainty. They argue that many professionals also have direct intuitions of the parameter volatilities. Others have used the volatility of the stock price (Dixit & Pindyck, 1994), or the traded commodity price volatility (Paddock, Siegel, & Smith, 1988) as proxies for the project volatility. Tufano (1998) showed that fixed costs and leverage affect the volatility of stocks and project value.

When more than one uncertainty is modelled, the correlation between them needs to be considered for the consolidated project volatility to be representative.

For a binomial tree to be convertible to a binomial lattice the volatility needs to be homoscedastic over time. When the simulation gives a heteroscedastic volatility distribution this can be normalized to become homoscedastic or the problem can be modelled with a (non-recombining) binomial tree.

On a side-note, MCS is in and of itself a good tool to say something about the range of outcomes with corresponding probability estimates. For other uses the MCS normally carries uncertainty in all years of the time series. Glasserman (2013) covers MCS in financial engineering at large.

3.3. Binomial trees

Let´s start with developing the BDH method binomial lattice approximating geometric Brownian Motion. The input parameters, $V_0, \sigma, r, \Delta t, \delta_t$ for all discrete times, t, define the up and down movements, u and d.

$$u = e^{+\sigma\sqrt{\Delta t}}, \qquad d = e^{-\sigma\sqrt{\Delta t}}$$

Here $u > 1$ and $d < 1$ are the multiplicative factors for each step development in the binomial model.

For each state i, j after n steps, where i is the number up movements, and j is the number of down movements, time t can be written as $t = (i + j)\Delta t = n\Delta t$. In excel the lattice is implemented with step numbers increasing in the column rightward and down-moves developing in the rows downward. The number of moves away from the expected value $i - j$ can be written $n - 2j$ for ease of implementation in excel.

3.3.1. Development tree

We can now develop the first NPV lattice from the following equations:

$$
\begin{tikzcd}
 & V_n^{up} = V_{n-1}(1-\delta_{n-1})u \\
V_{n-1} \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\
 & V_n^{down} = V_{n-1}(1-\delta_{n-1})d
\end{tikzcd}
$$

The general formula for the value after an up move or a down move at a given time t is expressed in normal values and logarithmic values, where the $\pm$ indicates up or down, as follows:

$$V_t^{\pm} = V_{t-\Delta t}(1 - \delta_{t-\Delta t})e^{\pm\sigma\sqrt{\Delta t}}$$

$$\ln\left(V_t^{\pm}\right) = \ln(V_{t-\Delta t}) + \ln(1 - \delta_{t-\Delta t}) \pm \sigma\sqrt{\Delta t}$$

3.3.2. Cash flow lattice

The cash flow for each state is given by the remaining project value in the given state and time multiplied by the cash flow payout ratio for step n.

$$CF_{i,j} = V_{i,j}\delta_n$$

These to lattices, the development lattice and the cash flow lattice, describe the value of the remaining cash flows, and the cash flows given in each state. If we were to model the project value without subtracting the cash flow as dividends from each period, the volatility would be overestimated because we would model uncertainty into the realized cash flows. Also, since the project value would not correctly estimate the remaining portion of the project value, the option would be compared to an incorrect estimate of the underlying.

3.3.3. Roll-back lattice

To calculate the roll-back lattice the probability of up movement, p is needed. For the GBM approximation p is the same for all states in the lattice. From the theory of the replicating portfolio Copeland and Antikariv showed that

$$p = \frac{1 + r\Delta t - d}{u - d}$$

Now we can calculate the NPV in a given state dependent on the CF in the period and the NPV in the next period (instead of the previous one). The last year will consist only of the cash flows in the last year, while the previous years are calculated as

$$\Lambda_{i,j} = CF_{i,j} + \frac{p\Lambda_{i+1,j} + (1-p)\Lambda_{i,j+1}}{1+r}$$

$$\Lambda_{t} = CF_{i,j} + \frac{p\Lambda_{t+\Delta \mathrm{t}}^{up} + (1-p)\Lambda_{t+\Delta \mathrm{t}}^{down}}{1+r}$$

For any GBM approximated lattice the roll-back tree will carry identical values as the development tree before options are incorporated. The last step of the lattice development is to incorporate the project options to the roll-back tree.

Options are incorporated by taking the highest value of the available alternatives for the future.

$$\Lambda_{i,j} = CF_{i,j} + max\left\{\frac{p\Lambda_{i+1,j} + (1-p)\Lambda_{i,j+1}}{1+r}, \qquad Option\right\}$$

$$Option\ value = \Lambda_0 - V_0$$

The main differences of the presented approach from the approach BDH first implemented in (Brandão et al., 2005b) is the correction of the volatility estimation by the GCE approach, risk neutralization of uncertain project variables with risk-neutral discounting, and development from a non-recombining tree into a lattice. The last two developments were suggested by J. E. Smith (2005).

3.4. The BDH case

When working with real options the devil is in the details. We present a walkthrough of the risk-neutral BDH oil case with GBM processes for anyone new to the BDH framework.

The oil article case has the goal of modelling two real options on an oil field. The owners with a stake of 75% of the field are given the opportunity to either buy out the remaining stake of the field for the fixed sum of $40 million, or the opportunity to sell the project for $100 million. The option term date is in year 5.

The oil reserves and production profile are assumed to be estimated deterministic. The field will start production at 10% of the 90 million barrels, declining with 15% annually. The initial oil price is $25/bbl, and variable cost is $10/bbl, with risk-neutral growth rates of 0% and 2% respectively. For details on estimation of risk-neutral growth rates, see (J. E. Smith, 2005). In the risk-neutral approach the cash flows are discounted at the risk-free rate, set to 5%.
The most accurate binomial model for a problem with two underlying uncertain parameters approximated by GBM or MR is the bivariate tree model, developed with a binomial process for each of the variables. The case is still developed in the BDH framework which an example, with the possibility of adding additional uncertain project variables.

The risk-neutral DCF is set up in excel:

Risk-Free Rate	5 %	r_f
Oil Reserves	90	MM bbls
Initial Production Rate	0,10	of reserves
Decline Rate	0,15	per year
Fixed Prod. Cost	5	($MM)/year
Develop Cost	180	($MM)capital
PSC Share	0,25	share

Year	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Remaining Reserves		90,0	81,0	73,4	66,8	61,3	56,6	52,6	49,2	46,3	43,9
Production Level		9,0	7,7	6,5	5,5	4,7	4,0	3,4	2,9	2,5	2,1
Variable Op Cost Rate		10,2	10,4	10,6	10,8	11,0	11,3	11,5	11,7	12,0	12,2
Oil Price		25,0	25,0	25,0	25,0	25,0	25,0	25,0	25,0	25,0	25,0
Revenues		225,0	191,3	162,6	138,2	117,5	99,8	84,9	72,1	61,3	52,1
Production Cost		(96,8)	(84,6)	(74,0)	(64,8)	(56,9)	(50,0)	(44,0)	(38,8)	(34,3)	(30,4)
Cash Flow		128,2	106,7	88,6	73,4	60,6	49,9	40,9	33,3	27,0	21,7
Profit Sharing		(32,1)	(26,7)	(22,1)	(18,3)	(15,1)	(12,5)	(10,2)	(8,3)	(6,8)	(5,4)
Net Cash Flows		96,2	80,0	66,4	55,0	45,4	37,4	30,7	25,0	20,3	16,3
PV of Cash Flows	392,0	411,6	331,2	263,8	207,3	159,9	120,1	86,9	59,0	35,8	16,3
Cash Flow Ratios		0,2336	0,2415	0,2518	0,2654	0,2842	0,3113	0,3528	0,4233	0,5664	1,0000

Figure 3 - Risk-neutral DCF for GBM BDH case

The risk-neutral project NPV is calculated to be \$392 million. The cash flow payout ratio, δ_t is calculated as CF_t/NPV_t.

Oil price GBM process			Variable cost GBM process		
Oil price	25	per bbl	Operating Cost	10	per bbl
Price growth (discrete)	0,00 %	μ_d	Cost growth (discrete)	2,00 %	μ_d
Price growth (continuous)	0,00 %	μ_c	Cost growth (continuous)	1,98 %	μ_c
Volatility	15,0 %	σ	Volatility	10,0 %	σ
Drift	-1,13 %	μ^*	Drift	1,48 %	μ^*

Figure 4 - Input parameters for the stochastic project variables

To calculate the consolidated volatility in period 1 the price and cost are evaluated as a stochastic process from step 0 to step 1:

$$Price_1 = Price_0 e^{\mu^*_{Price} + \sigma_{Price} dz_{Price}}$$

$$VarCost_1 = VarCost_0 e^{\mu^*_{VarCost} + \sigma_{VarCost} dz_{Cost}}$$

Here dz is a Wiener process normally distributed $N(0,1)$ with mean 0 and standard deviation 1, and $\mu^* = \mu_c - \sigma^2/2$. The adjusted drift is obtained from Itôs lemma (chapter 13, (Hull & Basu, 2016)). For this case the two stochastic project variables are assumed to be uncorrelated ($dz_{Price} dz_{cost} = \rho_{PriceCost} dt$, where $\rho_{PriceCost} = 0$), but correlation can easily be added when using professional MCS software like @Risk and Crystal ball. In contrast to closed-form solutions and multivariable lattices, correlation and additional stochastic variables can be added to MCS method without a heavy computational burden.

The following periods are calculated as conditional expected values, based on the value from time step 1.

$$Price_n = Price_{n-i}\left(1 + \mu_{d,Price}\right)$$

$$VarCost_n = VarCost_{n-i}\left(1 + \mu_{d,VarCost}\right)$$

To estimate the consolidated project volatility correctly the MCS should model all stochastic variables in the DCF with corresponding correlations. For this case the production profile and fixed cost are modelled deterministically, and the correlation between price and cost ignored.

Define the logarithmic period return as $z = \ln\left(V_{1,MCS}/\overline{V}_0\right)$.

Year	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Remaining Reserves		90,0	81,0	73,4	66,8	61,3	56,6	52,6	49,2	46,3	43,9
Production Level		9,0	7,7	6,5	5,5	4,7	4,0	3,4	2,9	2,5	2,1
Variable Op Cost Rate		10,1	10,4	10,6	10,8	11,0	11,2	11,4	11,7	11,9	12,1
Oil Price		24,7	24,7	24,7	24,7	24,7	24,7	24,7	24,7	24,7	24,7
Revenues		222,5	189,1	160,7	136,6	116,1	98,7	83,9	71,3	60,6	51,5
Production Cost		(96,3)	(84,2)	(73,7)	(64,5)	(56,6)	(49,7)	(43,8)	(38,6)	(34,2)	(30,3)
Cash Flow		126,1	104,9	87,1	72,1	59,5	49,0	40,1	32,7	26,5	21,2
Profit Sharing		(31,5)	(26,2)	(21,8)	(18,0)	(14,9)	(12,2)	(10,0)	(8,2)	(6,6)	(5,3)
Net Cash Flows		94,6	78,7	65,3	54,1	44,6	36,7	30,1	24,5	19,8	15,9
PV of Cash Flows	385,3	404,6	325,5	259,1	203,5	156,9	117,9	85,2	57,9	35,0	15,9
E(PV of Cash Flows)	392,0	411,6	331,2	263,8	207,3	159,9	120,1	86,9	59,0	35,8	16,3
z		0,031556	$z = \ln\left(\frac{V_1}{V_0}\right)$								
σ(z)		31,8 %									

Figure 5 - Monte Carlo simulation DCF for the GBM BDH case

Colored cells carry stochastic functions. The mean present value of the cash flows converges to the expected value, but @Risk displays the median value when set to static results. When the functions are defined we run the simulation with for example 100 000 iterations. We now have the current price of the underlying $V_0 = 392{,}02$ and the cash flow payout ratio vector δ_n from the base case DCF and obtain the volatility $\sigma = 31{,}77\%$ from the simulation. We use the volatility estimate to calculate the *up* and *down* movements of the model and set up the value development tree.

$$u = e^{\sigma\sqrt{\Delta t}} = 1{,}374\,, \qquad d = \frac{1}{u} = 0{,}728$$

Year	0	1	2	3	4	5	6	7	8	9	10
CF payout ratio	0	0,2336	0,2415	0,2518	0,2654	0,2842	0,3113	0,3528	0,4233	0,5664	1
Value without	392,02	538,6	567,1	591,0	607,6	613,2	603,1	570,7	507,5	402,1	239,5
options		285,3	300,4	313,1	321,9	324,9	319,5	302,3	268,8	213,0	126,9
			159,2	165,9	170,5	172,1	169,2	160,1	142,4	112,8	67,2
				87,9	90,3	91,2	89,7	84,8	75,4	59,8	35,6
					47,9	48,3	47,5	44,9	40,0	31,7	18,9
						25,6	25,2	23,8	21,2	16,8	10,0
							13,3	12,6	11,2	8,9	5,3
								6,7	5,9	4,7	2,8
									3,1	2,5	1,5
										1,3	0,8
											0,4

$$V_t \xrightarrow{p} V_{t+1}^{up} = V_t(1-\delta_t)u$$

$$V_t \xrightarrow{1-p} V_{t+1}^{down} = V_t(1-\delta_t)d$$

0	1	2	3	4	5	6	7	8	9	10
0,0	125,8	137,0	148,8	161,3	174,3	187,7	201,3	214,8	227,7	239,5
	66,6	72,6	78,8	85,4	92,3	99,4	106,6	113,8	120,6	126,9
		38,4	41,8	45,3	48,9	52,7	56,5	60,3	63,9	67,2
			22,1	24,0	25,9	27,9	29,9	31,9	33,9	35,6
				12,7	13,7	14,8	15,9	16,9	17,9	18,9
					7,3	7,8	8,4	9,0	9,5	10,0
						4,1	4,4	4,7	5,0	5,3
							2,4	2,5	2,7	2,8
								1,3	1,4	1,5
									0,7	0,8
										0,4

$$CF_{i,j} = V_{i,j}\delta_t$$

Figure 6 - Development lattice Cash flow lattice for GBM BDH case

Figure 7 shows the expected value of remaining project cash flows over time without options.

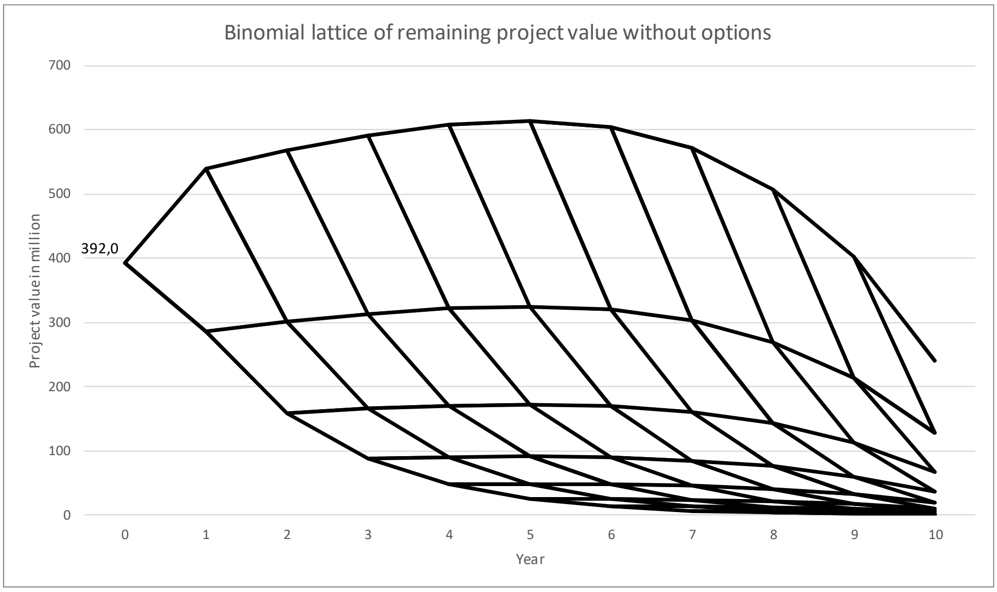


Figure 7 - Remaining project value graph for the GBM BDH case

To roll the tree back we need the probability for the remaining project value moving up. The risk-free rate is set to 5%.

$$p = \frac{1 + r\Delta t - d}{u - d} = 49{,}86\%$$

The rollback lattice is first set up without options as the discounted expected value of the subsequent period. For a GBM model like this, the rollback lattice without options will have identical values to the development lattice. Finally, the options are inserted as maximum functions of the expected value and the option. Note that rows 6-10 of the roll-back tree still stems from the development tree without options, and is not the value in the given state if an option is exercised before the project gets to that state.

	0	1	2	3	4	5	6	7	8	9	10
Value with	418,03	562,6	594,8	632,3	676,0	**719,5**	603,1	570,7	507,5	402,1	239,5
options		315,9	323,2	330,0	340,2	**362,4**	319,5	302,3	268,8	213,0	126,9
			200,6	196,8	187,6	**173,1**	169,2	160,1	142,4	112,8	67,2
				144,0	138,1	***125,9***	89,7	84,8	75,4	59,8	35,6
					117,9	***113,7***	47,5	44,9	40,0	31,7	18,9
Without options						***107,3***	25,2	23,8	21,2	16,8	10,0
							13,3	12,6	11,2	8,9	5,3
								6,7	5,9	4,7	2,8
									3,1	2,5	1,5
With option in year 5										1,3	0,8
											0,4

Without options

$$\Lambda_{n,j} = CF_{t,j} + \frac{p\Lambda_{t+1,j} + (1-p)\Lambda_{t+1,j+1}}{1+rdt}$$

With option in year 5

$$\Lambda_{5,j} = \max\left\{\underbrace{CF_{5,j} + \frac{p\Lambda_{6,j} + (1-p)\Lambda_{6,j+1}}{1+rdt}}_{Base},\ \underbrace{CF_{5,j} - 40 + \frac{4}{3}\frac{p\Lambda_{6,j} + (1-p)\Lambda_{6,j+1}}{1+rdt}}_{Buyout},\ \underbrace{CF_{5,j} + 100}_{Divest}\right\}$$

Figure 8 - Roll-back lattice for GBM BDH case

For the BDH case this yields a value of the project with options of \$418,03 million, which is \$26,02 million more than the project value without the options.

4. Option value analysis and discussion

We will analyze the first case results before we implement the case with mean reversion. This is done through decomposition of where the option value lies, looking at the sensitivities, analyzing the decision strategy with visualizing tools, and finally looking at the timing aspects of the options.

4.1. Decomposition of action value and flexibility value

Real options have extensively been referred to as "the value of flexibility". However, one should be careful not to interpret the option value as the value of flexibility before correcting for the possible value of the action in the base case scenario.

When the option is in-the-money we can distinguish between:

Value of the action – the static NPV with option minus the NPV without option

Value of the flexibility – the value added from having the right, but not the obligation to exercise the option

$$Value\ of\ action = \left(NPV_{with\ option} - NPV_{without\ option} \middle| no\ uncertainty\right)$$

$$Value\ of\ flexibility = Option\ value - Value\ of\ action$$

From this we see that the real option value can rightly be referred to as the value of flexibility when the option is *out of the money*.

For the risk-neutral BDH case the risk-neutral NPV values of the two actions are:

$$E[NPV_{no\ action}] = 392{,}016$$

$$E\left[NPV_{buyout}\right] = 392{,}052$$

$$E[NPV_{divest}] = 380{,}715$$

Without flexibility the buyout case has a marginally higher NPV by 0,036 million. We can conclude that the option value primarily stem from flexibility.

4.2. Separate option values

The option characteristic of the two options for the BDH case in year 5 are characterized in the figure below. Figures exclude the cash flow from year 5 and only consider sales value, buyout cost and expected future value.

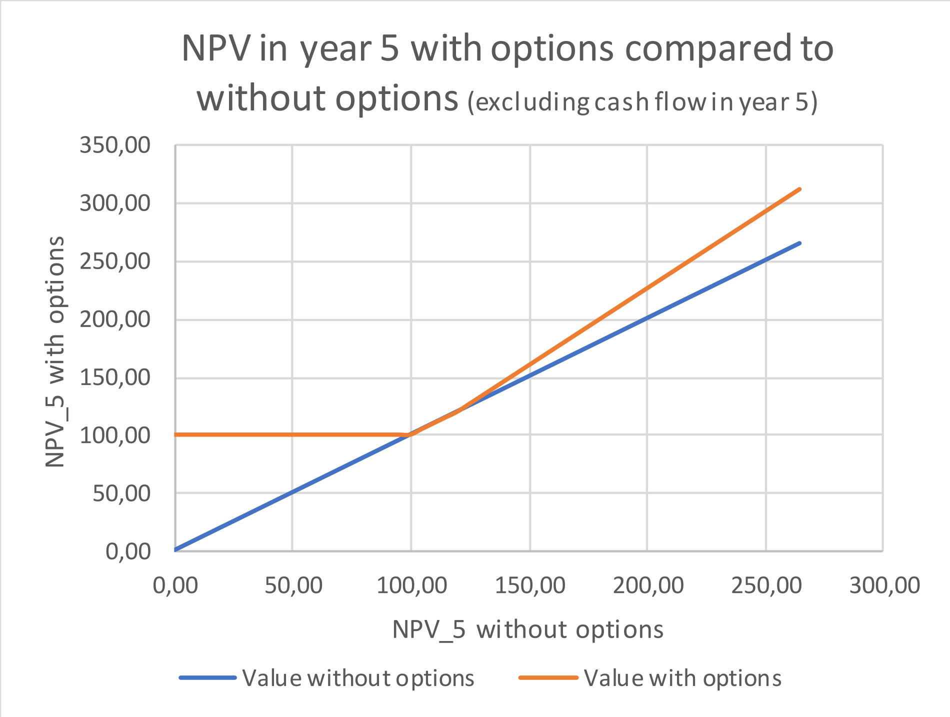


Figure 9 - Payoff graph of difference of value with options in year 5 for the GBM BDH case

The combined option value is generally not the sum of the series of option values, but since both options can only "be exercised" in year 5 and does not overlap we can add the option values.

In the decomposition of the values of the separate options we demonstrate an alternative calculation of the option values showing the explicit option values over time. A separate lattice is constructed for each option including only the value of the option. This is programmed as =IF (Buyout option = MAX(Buyout option; Divest option; Base case); Buyout option – Base case; 0), where the nodes where the option is the optimal decision will return the added value of the option in the given year. The values are rolled back as discounted expected values (without the cash flows of the base case). The combined option value is consistent with the option value calculated in chapter 3.

Buyout option value

0	1	2	3	4	5
7,38	13,4	23,8	41,2	68,4	106,3
	2,1	4,4	8,9	18,3	37,5
		0,1	0,2	0,5	1,1
			0,0	0,0	0,0
				0,0	0,0
					0,0

Divest option value

0	1	2	3	4	5
18,64	10,6	3,8	0,0	0,0	0,0
	28,5	18,4	7,9	0,0	0,0
		41,4	30,7	16,6	0,0
			56,1	47,7	34,7
				70,1	65,4
					81,7

Figure 10 - Single option values for the GBM BDH case

The same results can be obtained from only including one option in the roll-back lattice and subtracting the base case in the end.

4.3. Sensitivity analysis and the Greek equivalents

It is important to run a sensitivity analysis as the volatility often are the most sensitive parameter in the option valuation, and binomial lattices normalize the volatility. Haug (2007) gives a thorough description of the local sensitivities of financial options, called the Greeks. In *The Collector - Know your weapons 2* (Haug, 2006) covers an options vega, which gives the percentage change in the option price for each percentage change in implied volatility. In ROV the volatility quote is not from the implied volatility, but from a volatility estimate, so we will call this the vega-equivalent, denoted υ.

Running a sensitivity of the option value analysis by changing the volatility gives that a 1% increase in volatility (from 31,77%) will increase the combined option value with just under 1 million. The local vega-equivalent is $\upsilon = 0{,}041$.

The rho-equivalent ρ is the Greek of the interest rate (drift), measuring the percentage change in option value for each percentage change in the interest rate. Sensitivity analysis of the interest rate show that a 1% increase in the interest rate (from 5%) will decrease the option value by 1,8 million and the rho-equivalent $\rho = -0{,}067$. Beware that this is the rho of the binomial model, and a sensitivity analysis of interest rate referencing (as a cell link in excel) back to the base case DCF will give different values.

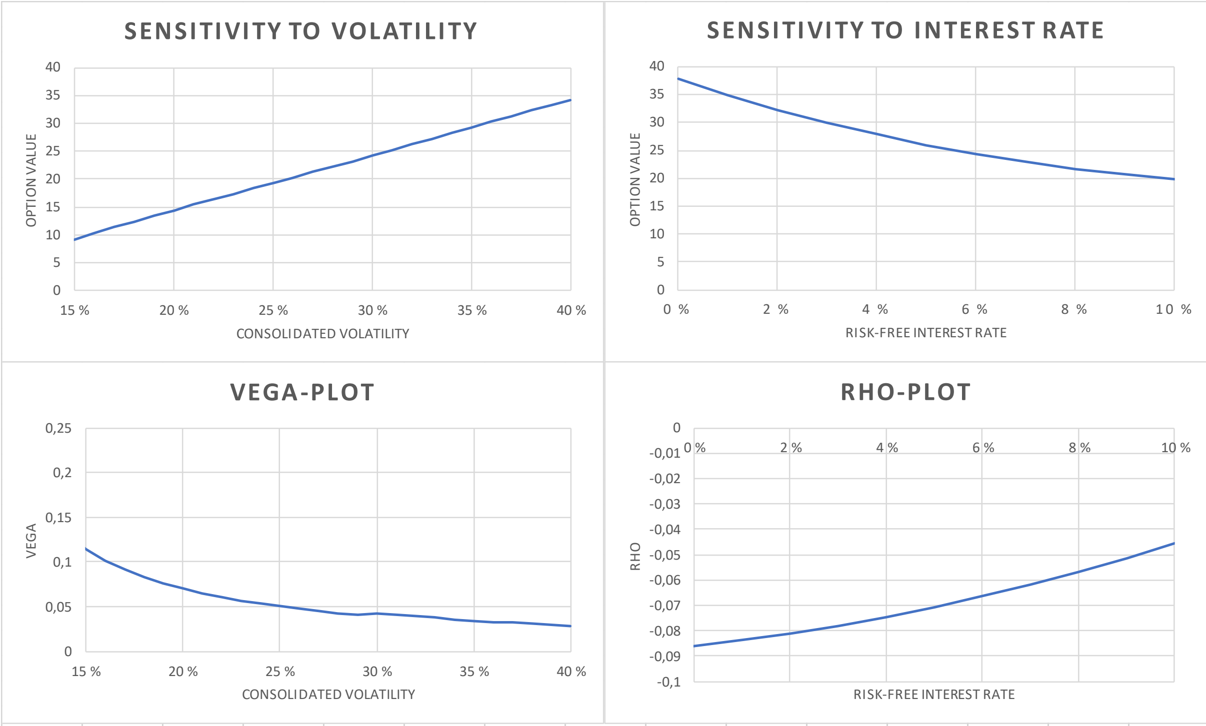


Figure 11 - Sensitivity analysis and the Greeks for the GBM BDH case

$\sigma \setminus r_f$	2 %	3 %	4 %	5 %	6 %	7 %	8 %
15 %	14,33	12,26	10,60	9,30	8,34	7,69	7,32
20 %	19,84	17,71	15,92	14,43	13,22	12,26	11,54
25 %	25,12	22,91	**20,99**	**19,35**	**17,94**	16,77	15,80
31,77 %	32,28	29,94	**27,86**	**26,02**	**24,40**	22,98	21,75
35 %	35,79	33,39	**31,23**	**29,30**	**27,59**	26,06	24,72
40 %	41,09	38,58	36,31	34,25	32,40	30,73	29,24
45 %	46,22	43,60	41,22	39,04	37,06	35,27	33,64

Figure 12 - Two-way sensitivity analysis of rf and σ of the GBM BDH case

It can be argued that the Greeks are less important for real options for three main reasons; (1) the input parameters to the options value formula are normally derived parameters, and so sensitivities to the underlying parameters are more relevant (i.e. consolidated volatility based on Monte Carlo Simulation of DCF), and (2) the illiquid nature of real options makes global sensitivities more relevant than the local sensitivities represented by the Greeks. Finally (3) project valuation is normally more concerned with sensitivities to the absolute option value than the relative sensitivities.

4.4. Normalization of volatility

Going through the input parameters for GBM models, the volatility is usually the most sensitive parameter for a given parameter confidence interval. For implementation of binomial lattices according to the BDH-method, the period volatility can be estimated from the standard deviation of the logarithmic return between any two subsequent periods, as the volatility of a GBM remain constant with time. However, the project log-returns do not follow an exact GBM process, and so the volatility will tend to change with time. For this reason, running MCS to estimate project volatility for a set of periods can give valuable information.

Just by using the average volatility over time, or alternatively a weighted average to the remaining project value, one might get estimates more that better represent the actual DCF numbers. From the table below containing the annual volatility estimates, we see that the first estimate might have been too low. This can then either be updated, or we can interpret a range of probable option values from the sensitivity analysis of the volatility.

1	2	3	4	5	6	7	8	9	10
31,77 %	32,55 %	33,58 %	34,83 %	37,02 %	36,47 %	39,97 %	40,76 %	43,41 %	45,57 %

Figure 13 - Annual volatility of the GBM BDH case from GCE

Periodic volatility can also be implemented in the model, but this will lead to non-recombing trees. (Brandão et al., 2005a) For practitioners it is still important to reflect on whether the volatility makes sense, where the other parameters for the binomial model are covered in the DCF and the option definitions as investment and term date.

4.5. Option timing (year 4-6)

To analyze the time aspect of the two options we can model the option value of equivalent options with exercise time in years 4, 5 and 6 instead of only year 5. The results are presented in figure 14 below.

0	1	2	3	4	5	6	7	8	9	10
431,38	579,7	616,7	661,2	**716,4**	**719,5**	**701,5**	570,7	507,5	402,1	239,5
	326,9	337,2	347,1	**360,7**	**362,4**	**352,8**	302,3	268,8	213,0	126,9
		209,6	209,1	**203,2**	**190,4**	**169,2**	160,1	142,4	112,8	67,2
			150,4	**148,3**	**141,5**	**127,9**	84,8	75,4	59,8	35,6
				121,2	**119,7**	**114,8**	44,9	40,0	31,7	18,9
					108,2	**107,8**	23,8	21,2	16,8	10,0
						104,1	12,6	11,2	8,9	5,3
							6,7	5,9	4,7	2,8
								3,1	2,5	1,5
									1,3	0,8
										0,4

Figure 14 - Roll-back lattice for GBM BDH lattice with extended exercise time

The new option value is $(431{,}38 - 392{,}02) = 39{,}36$ million. Expanding the exercise window one year in each direction will increase the option value by 51% ($13{,}34$ million). The highlighted optimal decision strategy also shows how the optimal exercise date for the divestiture option is as late as possible and the buyout exercise date as early as possible. If a real option with a single period exercise window is seen as a European option, then this expansion of the exercise window makes the option the equivalent of a Bermudan option.

When modelling scaling options, like the option to buyout the remaining license, the rollback of the scaled cash flow needs to reference a rollback-lattice without options to avoid double-counting the payoff function of an option (e.g. since one can only own up to 100% of the production license).

Another aspect of the binomial options method to be considered is the course time discretization. Cox et al. (1979) showed how the original binomial option pricing model for European options converges to the results of Black and Scholes when the time increment of the steps approach null. Hull and Basu (2016) report that binomial trees typically are divided into 30 or more steps in practice. Binomial models (and trinomial models and Monte Carlo methods) of ROV are normally based on DCF analysis carrying monthly, quarterly, semiannual or annual estimates. By converting the BDH case to quarterly steps the precision of the model would increase, but as we see from the sensitivities of the other aspects this is not critical.

4.6. Parameter correlation with strike value

The model creates scenarios of the development of the remaining project cash flow but does not consider stochastic behavior of the strike value (and thus, the options payout). If the selling price of $100 million in the divest option was not a contractually bound option, but an estimated asset value, then we should consider how the asset value would correlate with the market conditions. Investment costs, or the strike price of a future option to invest, might also be somewhat variable with supply and demand in the industry in question. Consider the examples of how shipyards vary prices with orderbook size which correlate to the shipping industry, or how rig costs have up to a 12-month lagged correlation component to the oil price. This aspect is especially important in the consideration of strategic options (as opposed to contractual ones, like the option of a ship or rig from a yard).

5. Mean reverting BDH approach

In cases where the underlying stochastic process fits a mean-reverting process, the problem can be modelled either with the Monte Carlo method, or by one of several different tree-building approaches.

The most commonly used method for operational options is the least-squares MCS model by Longstaff and Schwartz (2001). The model enables valuation of American style- and path dependent options by using the least squares method to estimate the conditional expected payoff to the option holder from continuation.

Hahn and Dyer (2008) implemented a mean reverting process to the binomial diffusion approximation method developed by Nelson and Ramaswamy (1990), where the probability of the value to move to the next up-state in the lattice is dependent on the difference to the mean. The model has been coined the censored model, as values for the probability must be censored for probabilities outside the defined space from 0% to 100%. The model approximates the arithmetic Ornstein-Uhlenbeck process of the logarithmic value of the underlying.

Bastian-Pinto et al. (2010) developed a binomial tree for pricing of the arithmetic Ornstein-Uhlenbeck process of the logarithmic values based on the Hull and White (1994) model. They made a lattice of the added volatility component and expected value as this development lattice is enabling rollback with non-censored probabilities.

Mean reverting models have also been developed in trinomial trees, especially for interest rate modelling, but this is considered out of scope here.

Now, we will implement the BDH method in the censored and non-censored models for mean reversion with adjustments. The model is adjusted to incorporate the drift of the discount rate. For the development of a mean-reverting BDH method, it was natural to implement it as a twist on the case originally presented by BDH, despite having to adjust the inputs for the mean reverting process subjectively which would distance the numbers from the original ones. This is mainly due to usage of the same parameters for different stochastic processes of the project variables, instead of calibrating the processes from the same data sets, which would make the models more comparable in an operational sense.

This covers instances where the underlying asset of the option, the remaining project value, follows a mean reverting process. One should however not confuse this with the different processes of the individual factors affecting the project value. The processes modelled in the MCS of the risk-neutral DCF may be approximated by a variety of different processes. The process of the value development over time should then be evaluated based on the MCS results with subjective adjustments.

The BDH case was developed with oil price and variable cost modelled as Model 1 of Schwartz and as an arithmetic OU process with geometric volatility. The presented case is that of Model 1 since this can be calibrated from market data. The project is assumed to follow the stochastic process defined from the binomial MR model implemented, namely the arithmetic OU process of the logarithmic value with an indirect adjustment for drift.

The binomial model of the remaining project value is still based on risk-neutral (to the market) uncertain future cash flows, discounted at risk-free rate. Thereby the model needs to be adjusted for drift. The GBM BDH model incorporated the drift in the probability calculation and in the discounting in the roll-back formula. The MR models will tackle the drift differently as the long-term mean level used in the model incorporate a part of the discount effect. The incorporation of drift in the roll-back is done by simply discounting over time, in the same manner as in the BDH model. This is further explained in the long-term mean estimation.

To understand why we assume that primary production projects (commodity industry projects) can be categorized as mean reverting, let´s compare them to stocks. Stocks are generally assumed to follow a geometric Brownian motion characterized by an average yield (return on common equity) and a volatility. Companies are expected to grow over time and employ their capital to optimize value generation and yield. This is typically done with a portfolio of projects in different phases. Cash flows from commodity production on the other hand, are bound by limiting conditions. This contrasts with strategic growth options, where jumps from market positioning and major technological changes will recalibrate the new mean net income of the project. Technological advances are expected to influence commodity prices over time, but for oil production projects and other capital-intensive projects these advances are hard to incorporate after the project has first been initiated (exceptions include some secondary and tertiary production technologies and plugging and abandonment).

5.1. Parameter estimation

The most likely reason for a project value to be mean reverting is that the main uncertain variables affecting the project value are mean reverting. The project parameters we need to estimate are current value V_0, cash flow payout ratio $\boldsymbol{\delta_n}$ (vector), value to which the process revert $\boldsymbol{\Phi}$ (vector), volatility σ, and mean reversion coefficient η.

5.1.1. Uncertain project variables

The reversion point for the oil price and variable cost is set to \$25 and \$12,19 respectively, equal to the expected value of the GBM at the end of the project for the sake of consistency. The variables are modelled by the Ornstein-Uhlenbeck process with geometric volatility. The price and cost values are programmed discretely in period 1 as

$$S_1 = (S_0)e^{-\eta\Delta t} + (\Phi)(1 - e^{-\eta\Delta t}) + \sigma S_0 dW_t$$

The two variables are still assumed to be uncorrelated. The mean reversion point for commodities can be approximated from the futures market. The mean reversion coefficient for the oil price was estimated to 17,24% for oil prices calibrated to Model 1 of Schwartz from the crude futures and options market as of May 2018, using the least squares method. For the variable cost a subjective estimate of 15% is assigned. The basic assumption is that variable cost reverts to the long-term mean just a little slower than the oil price.

Oil price		
Price	25,00	per bbl
Long term mean price	25,00	Φ
Standard deviation	15,0 %	σ
Mean reversion speed	17,2 %	η

Variable operating cost		
Cost	10,00	per bbl
Long term mean cost	12,19	Φ
Standard deviation	10,0 %	σ
Mean reversion speed	15,0 %	η

Oil Reserves	90	MM bbls
Initial Production Rate	0,10	of reserves
Decline Rate	0,15	per year
Discount Rate	5 %	
Fixed Prod. Cost	5	($MM)/year
PSC Share	0,25	share

Figure 15 - Project variable inputs for the MR BDH case

5.1.2. Project value

The current value V_0 and consolidated volatility σ are estimated the same way as in the GBM case. The current value of the project is calculated deterministically with expected values for the stochastic variables. The project value will change over time as parts of the project are realized in cash flows, calculated from the cash flow payout ratio vector covered in the presentation of the GBM BDH method. The risk-neutral NPV without flexibility is \$389,91 million. Also note that the expected values of the two options are both below the base case NPV in the deterministic case, so the option value arise from flexibility.

Year	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Remaining Reserves		90,0	81,0	73,4	66,8	61,3	56,6	52,6	49,2	46,3	43,9
Production Level		9,0	7,7	6,5	5,5	4,7	4,0	3,4	2,9	2,5	2,1
		2,330	2,353	2,374	2,391	2,406	2,419	2,430	2,439	2,448	2,455
Variable Op Cost Rate		10,3	10,5	10,7	10,9	11,1	11,2	11,4	11,5	11,6	11,6
		3,218	3,217	3,217	3,216	3,216	3,216	3,215	3,215	3,215	3,215
Oil Price		25,0	25,0	24,9	24,9	24,9	24,9	24,9	24,9	24,9	24,9
Revenues		224,8	191,0	162,2	137,8	117,1	99,5	84,6	71,9	61,1	51,9
Production Cost		(97,5)	(85,5)	(74,8)	(65,4)	(57,1)	(49,9)	(43,5)	(38,1)	(33,4)	(29,3)
Cash Flow		127,3	105,5	87,4	72,4	60,0	49,7	41,0	33,8	27,7	22,6
Profit Sharing		(31,8)	(26,4)	(21,9)	(18,1)	(15,0)	(12,4)	(10,3)	(8,4)	(6,9)	(5,7)
Net Cash Flows		95,5	79,1	65,6	54,3	45,0	37,2	30,8	25,3	20,8	17,0
E[PV of Cash Flows]	389,91	409,4	329,6	263,0	207,3	160,6	121,4	88,4	60,5	36,9	17,0
CF payout ratio		0,2333	0,2400	0,2493	0,2621	0,2802	0,3067	0,3479	0,4186	0,5626	1,0000
E[PV with buyout]	*388,8*	*408,2*	*328,3*	*261,7*	*205,9*	*159,2*					
E[PV with divest]	*377,7*	*396,5*	*316,1*	*248,8*	*192,4*	*145,0*					

Figure 16 - Risk-neutral DCF for the MR BDH case

5.1.3. Long-term equilibrium

The remaining project value is not constant but will change as the project progresses and cash flows are realized. Therefore, the value to which the process will revert also cannot be constant. The most basic mean reverting models revert to a calculated mean value, referenced as the *long-term mean* but in our case this value will necessarily change over time. The term still makes sense if we interpret the value as the mean project value at time t in the project life cycle if the project were to be developed an infinite number of times, as opposed to the value to which the project value will revert to as time goes to infinity. The parameter is not necessarily a mean value either, though calibration from historical data is typically calculated as a logarithmic mean value. One could argue that a more accurate name could be the value to which the process reverts at time t, or the reversion point of the process, but for consistency to other papers on mean reverting processes this paper will continue to call it the long-term mean.

By extending the MAD assumption to hold true over time we argue that the best estimator for the long-term mean vector is the base case NPV estimate at time t in the risk-neutral DCF without flexibility. The process would be expected to revert to the mean project value, approximated by the expected value in the base case at the given time, Φ_t in $\mathbf{\Phi}$ vector.

Note that the NPV estimates used as long-term mean vector are discounted, so we regard this as including the drift on the pre-option side of the option pricing. The post-option side is done similar to the GBM version, by discounting the roll-back tree. The degree to which the process will be discounted is thus to small since the value component from the last period (non-long-term mean) is not discounted. This could be corrected for by adjusting the process with a risk-premium λ corresponding to the lack of discounting mentioned. Alternatively, the long-term mean could be modelled without discounting as pure cash flows, discounting the whole process by the discount rate.

5.1.4. Project return volatility

The consolidated project volatility is calculated from the GCE approach, like what we did in the GBM approximation.

Year	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Remaining Reserves		90,0	81,0	73,4	66,8	61,3	56,6	52,6	49,2	46,3	43,9
Production Level		9,0	7,7	6,5	5,5	4,7	4,0	3,4	2,9	2,5	2,1
		2,327	2,351	2,372	2,389	2,404	2,418	2,429	2,438	2,447	2,454
Variable Op Cost Rate		10,3	10,5	10,7	10,9	11,1	11,2	11,3	11,5	11,6	11,6
		3,208	3,209	3,209	3,210	3,211	3,211	3,212	3,212	3,212	3,212
Oil Price		24,7	24,7	24,8	24,8	24,8	24,8	24,8	24,8	24,8	24,8
Revenues		222,5	189,3	161,0	137,0	116,5	99,1	84,2	71,6	60,9	51,8
Production Cost		(97,3)	(85,3)	(74,7)	(65,3)	(57,0)	(49,8)	(43,5)	(38,1)	(33,3)	(29,3)
Cash Flow		125,2	104,0	86,4	71,7	59,5	49,3	40,7	33,6	27,6	22,5
Profit Sharing		(31,3)	(26,0)	(21,6)	(17,9)	(14,9)	(12,3)	(10,2)	(8,4)	(6,9)	(5,6)
Net Cash Flows		93,9	78,0	64,8	53,8	44,6	37,0	30,5	25,2	20,7	16,9
	388,7										
PV of Cash Flows	385,4	404,6	326,2	260,7	205,7	159,5	120,6	87,9	60,2	36,8	16,9
E(PV of Cash Flows)	*389,9*	*409,4*	*329,6*	*263,0*	*207,3*	*160,6*	*121,4*	*88,4*	*60,5*	*36,9*	*17,0*
Cash flow payout ratio		*0,2321*	*0,2391*	*0,2485*	*0,2614*	*0,2796*	*0,3063*	*0,3476*	*0,4183*	*0,5624*	*1,0000*
z1		3,70 %	$z = \ln\left(\frac{V_1}{V_0}\right)$								
σ(z)		**20,36 %**									

Figure 17 - GCE volatility estimation with MCS for the MR BDH case

Note that the volatility estimate is considerably lower at $\sigma = 20{,}36\%$. Estimates for annual project return volatility grow with time for the given case, but we have not normalized the volatility estimate (to stay consistent) and will use the volatility from year 1 in the binomial models.

1	2	3	4	5	6	7	8	9	10
20,36 %	21,28 %	22,44 %	23,96 %	25,35 %	27,60 %	29,74 %	32,61 %	36,70 %	41,43 %

Figure 18 - Table of annual project return volatility with MR price and cost from GCE

It's important to note that the process volatility will be less for Model 1 than for a GBM model for a given periodic volatility. When time goes to infinity the variance of Model 1 goes to $\sigma^2/2\eta$, while the variance of the GBM model goes to infinity. This reflects the higher uncertainty in a process that does not revert to a mean, resulting in higher flexibility values. Jafarizadeh and Bratvold (2012) further discusses the potential overestimation of real option values from using GBM.

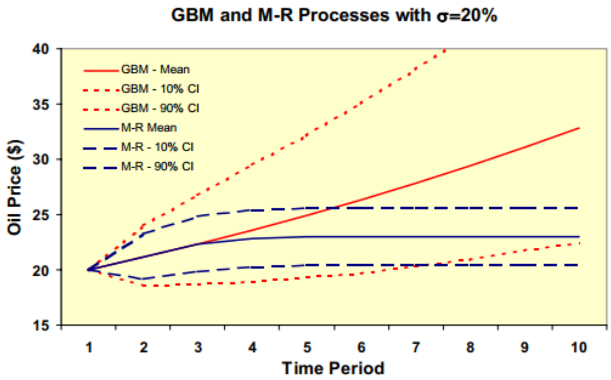


Figure 19 - GBM vs MR process variance (Hahn 2005)

5.1.5. Mean reversion coefficient

The mean reversion coefficient is a measure of the speed with which the process will revert to the long-term mean. An equivalent, more intuitive notation for the mean reversion speed is the half-life of the difference in between the current value and the long-term equilibrium. This is calculated as $\ln(2)/\eta$ and represent the time until half of the difference from the long-term mean is closed in.

Bastian-Pinto et al. (2010) presents a method of estimating the process parameters for a geometric Ornstein-Uhlenbeck process from historical time-series data. For an underlying process S, the following regression is run:

$$\ln(S_t) - \ln(S_{t-1}) = \beta_0 + \beta_1 \ln(S_{t-1}) + \varepsilon$$

Where the mean reversion coefficient, volatility and long-term mean level are obtained from the following equations

$$\eta = -\frac{\ln(\beta_1 + 1)}{\Delta t}, \qquad \sigma = \sigma_\varepsilon \sqrt{2\ln(\beta_1 + 1)/(\Delta t[(\beta_1 + 1)^2 - 1]}, \qquad \bar{S} = e^{-\frac{\beta_0}{\beta_1} + \frac{\sigma^2}{2\eta}}$$

Where σ_ε^2 is the variance of the regressions error. For widely traded commodities the coefficient can also be estimated from the futures market.

Project process parameters, however are estimated from DCF and MCS tools. Some projects, like similar type oil field projects, might have sufficient historical data to estimate parameters.

As an alternative one could assume that the cyclicality of an industry would give information about the reversion speed, but industry cycles are different processes. Such cycles are more like a sine-function description of the long-term mean, or like geometric Brownian motion with positive drift and a negative jump-diffusion component.

In the estimation of a consolidated project volatility through MCS, Copeland and Antikarov (p253-255 and p262-264) describes how to model the project value as mean reverting (in real – non-log values). The mean reversion speed for an arithmetic Ornstein-Uhlenbeck process can intuitively be estimated by asking management or industry professionals "If the uncertainty tend to revert to its average value, what percentage of the one-period deviation do you expect to be eliminated on average during the next period?" (Copeland and Antikarov, 2001, page 264)

One could also make the naïve assumption that the project is characterized by the same mean reversion speed as the uncertain project parameters. For projects with one main uncertain variable following a mean reversion, it's natural to think that the projects value development will follow the same time-line. In contrast to the naïve approach of mistaking the project volatility with the volatility of the uncertain project variable, the mean reversion coefficient is a measure of time, not magnitude. Thereby, the effects of fixed costs and leverage does not affect this parameter as they remain constant over time. However, correlations between uncertain variables will affect the mean reversion speed estimate.

Next, consider whether the half-life of the oil production project should be faster or slower compared to the half-life of the oil price. The answer to this will depend on the correlation between the uncertain project parameters (including lag-effects) and the reversion speed and/or drift of the other variables. For a project with two underlying MR variables with different mean reversion speeds the periodic cash flows will follow a mean reverting process where the project value will have a hockey-stick characteristic.

The parameter was estimated to be somewhere between that of the price and cost, at 17%. That correspond to an expectation of 17% of the difference in logarithmic value being closed each period. The corresponding half-life is 4,1 years, meaning that half the difference in logarithmic value will be closed in after 4,1 years.

5.2. Censored model

Nelson and Ramaswamy (1990) proposed an approach to model a range of different processes in a standardized way. The general form for the stochastic differential equation is given by

$$dx = \upsilon(x,t)dt + \sigma(x,t)dz$$

For implementation of a mean reverting process the problem is to find a binomial sequence with 1st moment $\upsilon(x,t)$ (expected value) and 2nd moment $\sigma(x,t)$ that converges to the given SDE.

Hahn (2005) implemented the arithmetic Ornstein-Uhlenbeck process of the logarithmic values in the Nelson and Ramaswamy approach, substituting $\upsilon(x,t) = \eta(\varphi - x_t)$ and $\sigma(x,t) = \sigma$, where x is the logarithm of the value. The end values are rolled back using a probability that reflect the mean reversion but must be censored for probabilities below and above 0% and 100%.

The development tree is modelled in the same way, here shown in logarithmic development without dividends:

$$x_t^+ = x + \sigma\sqrt{\Delta t}$$

$$x_t^- = x - \sigma\sqrt{\Delta t}$$

$$p_{x,t} = \begin{cases} 0 & if \quad \frac{1}{2} + \frac{\nu(x,t)}{2\sigma}\sqrt{\Delta t} \leq 0 \\ \frac{1}{2} + \frac{\nu(x,t)}{2\sigma}\sqrt{\Delta t} & if \quad 0 \leq \frac{1}{2} + \frac{\nu(x,t)}{2\sigma}\sqrt{\Delta t} \leq 1 \\ 1 & if \quad 0 \leq \frac{1}{2} + \frac{\nu(x,t)}{2\sigma}\sqrt{\Delta t} \leq 1 \end{cases}$$

where

$$p_{uncensored} = \frac{1}{2} + \frac{\nu(x,t)}{2\sigma} = \frac{1}{2} + \frac{\eta(\varphi - x_t)}{2\sigma}\sqrt{\Delta t}$$

Programmed as

$$p = \max\left(0; \min\left(1; \frac{1}{2} + \frac{\eta(\varphi - x_t)}{2\sigma}\sqrt{\Delta t}\right)\right)$$

Where

$x = \ln(S)$ is the natural logarithm of the stochastic process S, (from which it follows that $\Delta S^{\pm} = e^{\pm\sigma\sqrt{\Delta t}}$)

σ is the annual volatility of the stochastic process S (assumed to be homoscedastic or transformed from heteroscedastic to homoscedastic to enable recombination),

Δt is the time increment of each step,

$x_t^{\pm}$ are the up and down states of the logarithm of S in time t,

$p_{x,t}$ is the probability if an up move to the next state, and $1 - p_{x,t}$ for a down move,

η is the mean reversion coefficient.

Since the up probabilities $p_{i,j}$ are now state dependent they are set up in a lattice, whereas the probability for the GBM version is constant. The down probabilities remain $1 - p_{i,j}$. The rollback will be like the BDH-method with GBM but using the censored state-specific probabilities. Note that the roll-back lattice without options no longer has identical values to the development lattice.

The roll-back calculation references the state- and time-specific probability of moving up from the previous time step. Otherwise the roll-back calculation is identical to the GBM implementation. The expected value in the next period is still discounted, incorporating drift to the model. The drift was not factored directly into the probability estimate but indirectly from the reversion point as discussed in the parameter estimation.

$$\Lambda_{i,j} = CF_{i,j} + \frac{p_{i,j}\Lambda_{i+1,j} + (1 - p_{i,j})\Lambda_{i,j+1}}{1 + r}$$

Hahn and Dyer (2011) further developed the model to include correlated dual one-factor mean-reversion processes, and a two-factor Ornstein-Uhlenbeck process, but calculation of the probabilities of the up and down diffusions becomes computationally intensive and the tree becomes more complicated to visualize and less intuitive (e.g. 3-dimensional or non-recombining with alternating process development of the factors).

5.2.1. Censored model implementation on the BDH oil case

The real-space development lattice and cash flow lattice are developed exactly the same was as for the BDH method.

Year	0	1	2	3	4	5	6	7	8	9	10
CF payout ratio		0,233256	0,240018	0,249278	0,262084	0,280179	0,306704	0,347943	0,418562	0,562567	1
E(NPV) = Φ	389,9	409,4	329,6	263,0	207,3	160,6	121,4	88,4	60,5	36,9	17,0
	389,9	478,0	449,2	418,5	385,1	348,4	307,4	261,2	208,8	148,8	79,8
Development		318,1	299,0	278,5	256,3	231,8	204,6	173,8	139,0	99,0	53,1
tree			199,0	185,4	170,6	154,3	136,1	115,7	92,5	65,9	35,3
				123,4	113,5	102,7	90,6	77,0	61,5	43,9	23,5
					75,5	68,3	60,3	51,2	41,0	29,2	15,7
						45,5	40,1	34,1	27,3	19,4	10,4
							26,7	22,7	18,1	12,9	6,9
								15,1	12,1	8,6	4,6
									8,0	5,7	3,1
										3,8	2,0
											1,4

$$V_{n-1} \rightarrow V_n^u = V_{n-1}(1-\delta_{n-1})u$$

$$V_{n-1} \rightarrow V_n^d = V_{n-1}(1-\delta_{n-1})d$$

	0	1	2	3	4	5	6	7	8	9	10
	0,0	111,5	107,8	104,3	100,9	97,6	94,3	90,9	87,4	83,7	79,8
Cash flows		74,2	71,8	69,4	67,2	65,0	62,7	60,5	58,2	55,7	53,1
			47,8	46,2	44,7	43,2	41,8	40,3	38,7	37,1	35,3
				30,7	29,8	28,8	27,8	26,8	25,8	24,7	23,5
					19,8	19,1	18,5	17,8	17,1	16,4	15,7
						12,7	12,3	11,9	11,4	10,9	10,4
							8,2	7,9	7,6	7,3	6,9
								5,3	5,1	4,8	4,6
									3,4	3,2	3,1
										2,1	2,0
											1,4

$$CF_{i,j} = V_{i,j}\delta_t$$

Figure 20 - Development lattice and cash flow lattice for the censored MR BDH case

The probability lattice from periods 0 to $n-1$ for the BDH oil case is developed in the following figure.

0	1	2	3	4	5	6	7	8	9
50 %	44 %	37 %	31 %	24 %	18 %	11 %	5 %	0 %	0 %
	61 %	54 %	48 %	41 %	35 %	28 %	22 %	15 %	9 %
		71 %	65 %	58 %	52 %	45 %	39 %	32 %	26 %
	100 %		82 %	75 %	69 %	62 %	56 %	49 %	43 %
				92 %	86 %	79 %	73 %	66 %	60 %
					100 %	96 %	90 %	83 %	77 %
						100 %	100 %	100 %	94 %
							100 %	100 %	100 %
								100 %	100 %
									100 %

$$p_V = \max\left(0,\ \min\left(1,\ \frac{1}{2} + \frac{\eta(\Phi - \ln(V_t))\sqrt{\Delta t}}{2\sigma}\right)\right)$$

Figure 21 - Censored probabilities

	0	1	2	3	4	5	6	7	8	9	10
Without	357,668	427,2	385,4	346,8	310,8	276,4	243,0	209,2	173,9	134,3	79,8
options		323,9	290,0	258,8	229,9	202,4	175,9	149,4	121,7	90,9	53,1
			219,5	194,2	170,7	148,7	127,6	106,8	85,5	62,4	35,3
				146,4	127,3	109,6	92,7	76,4	60,0	42,8	23,5
					95,3	80,9	67,5	54,7	42,1	29,3	15,7
						59,6	49,2	39,1	29,5	20,1	10,4
							34,5	27,6	20,7	13,7	6,9
								18,4	13,8	9,2	4,6
									9,2	6,1	3,1
										4,1	2,0
											1,4

$$\Lambda_{t,j} = CF_{t,j} + \frac{p_V * \Lambda_{t+1,j} + (1-p_V)\Lambda_{t+1,j+1}}{1+r\Delta t}$$

	0	1	2	3	4	5	6	7	8	9	10
With	365,43	433,3	389,7	350,9	319,5	**296,0**	243,0	209,2	173,9	134,3	79,8
options		334,1	297,9	263,7	232,1	**208,3**	175,9	149,4	121,7	90,9	53,1
			234,6	206,6	178,4	**148,7**	127,6	106,8	85,5	62,4	35,3
				170,7	150,1	**128,8**	92,7	76,4	60,0	42,8	23,5
					132,8	**119,1**	67,5	54,7	42,1	29,3	15,7
						112,7	49,2	39,1	29,5	20,1	10,4
							34,5	27,6	20,7	13,7	6,9
								18,4	13,8	9,2	4,6
									9,2	6,1	3,1
With option in year 5										4,1	2,0
											1,4

$$\Lambda_{5,j} = CF_{5,j} + \max\left\{+\frac{p * \Lambda_{6,j} + (1-p)\Lambda_{6,j+1}}{1+r\Delta t}, \quad -40 + \frac{4}{3} * \frac{p * \Lambda_{6,j} + (1-p)\Lambda_{6,j+1}}{1+r\Delta t}, \quad +100\right\}$$

Figure 22 - Roll-back trees with and without options for the censored MR BDH case

The project value in the roll-back lattice is 8,3% less than the NPV, of which part is expected to be from the lack of drift in the model. The option values can still be calculated as the difference between the rollback tree with and without options. The combined option value is <u>$7,76 million</u>.

0	1	2	3	4	5	6	7	8	9	10
380,083	448,5	405,1	366,4	**340,7**	**296,0**	**252,5**	209,2	173,9	134,3	79,8
	349,7	314,3	280,2	**246,2**	**211,2**	**175,9**	149,4	121,7	90,9	53,1
		250,8	224,6	**198,7**	**171,8**	**141,8**	106,8	85,5	62,4	35,3
			185,4	**166,5**	**147,7**	**127,8**	76,4	60,0	42,8	23,5
				143,9	**131,2**	**118,5**	54,7	42,1	29,3	15,7
					119,7	**112,3**	39,1	29,5	20,1	10,4
						108,2	27,6	20,7	13,7	6,9
							18,4	13,8	9,2	4,6
								9,2	6,1	3,1
									4,1	2,0
										1,4

Figure 23 - Censored probabilities of up move for the censored BDH case

The bleaker project value is reflected in the increased value of the divest option, which is especially evident from the analysis of the expanded option.

Two-way sensitivity analysis of volatility, mean reversion coefficient and risk-free rate are supplied in the appendix.

5.3. Non-censored model

Bastian-Pinto et al. (2010) developed a non-censored model based on the censored model. Instead of defining the development lattice as a geometric development of the project, they isolated the volatility as suggested by Hull and White (1994) (used in trinomial Hull-White model). They start by defining an arithmetic lattice presenting the volatility in x, x^*. The lattice changes $\pm\sigma\sqrt{\Delta t}$ for each up or down step. Since the volatility is modelled separately the relative magnitude in the binomial process will remain unaltered, whereby they argue that the roll-back can be done in logarithmic values. In Excel each step is modelled as $(n-2j)\sigma\sqrt{\Delta t}$, where n and j are counted on the horizontal and vertical axis respectively. The tree will be symmetric with expected value of null. For each step Δt in the binomial process we then have that $E[x_t] = p\sigma + (1-p)(-\sigma) = pU + (1-p)D$ and $Var[x_t] = E[x_t^2] - E[x_t]^2 = p(1-p)(U-D)^2$. Since these are different expressions for the first and second moment, $\upsilon(x,t)$ and $\sigma(x,t)$, the probability is calculated differently from the censored model. By using the approximation $e^{-\eta\Delta t} \approx 1 - \eta\Delta t$ from Taylor expansion the expected value and variance of an Ornstein-Uhlenbeck process of logarithmic values can be rewritten as

$$E[x_t] = \varphi + (x_{t-\Delta t} - \varphi)e^{-\eta\Delta t} \approx x_{t-\Delta t} + (\varphi - x_{t-\Delta t})\eta\Delta t$$

$$Var[x_t] = \frac{\sigma^2}{2\eta}(1 - e^{-2\eta\Delta t}) \approx \sigma^2\Delta t$$

By setting up the moment-matching equations for the expected value and variance with starting point $x^* = 0$ and long-term mean of $\overline{x^*} = 0$ we get

$$(-x_t^*)\eta\Delta t = pU + (1-p)D, \qquad \sigma^2\Delta t = p(1-p)(U-D)^2$$

From further substituting and rewriting of the equations the probability of an up move is obtained as

$$p_{i,j} = \frac{1}{2} - \frac{\eta(-x_{i,j}^*)\sqrt{\Delta t}}{2\sqrt{\eta^2(-x_{i,j}^*)^2\Delta t + \sigma^2}}$$

This limits the volatility of the process and will be used in the roll-back. To calibrate the scale and slope to the reversion point the expected value of the underlying is added to the lattice. An intuitive way of understanding this is as the development of the uncertainty estimate as a generalized certainty equivalent band before scaling the band to the starting point and long-term equilibrium value. A time vector of the expected value of x is calculated as $E[x_t] = \varphi + (x_{t-\Delta t} - \varphi)e^{-\eta\Delta t}$. The development tree is now $x_t = E[x_t] + x_t^* = x_o e^{-\eta t} + \varphi(1 - e^{-\eta t}) + x_t^*$.

5.3.1. Non-censored model implementation on the BDH oil case

Since the model includes cash flows as dividends the model must be developed in normal values as $V_t = \left(V_{t-1}(1-\delta_{t-1})\right)^{e^{-\eta t}} \Phi_t^{1-e^{-\eta t}} e^{x_t^*}$. In Excel the logarithmic volatility lattice x^* is modelled as $(n-2j)\sigma\sqrt{\Delta t}$, where n and j are counted on the horizontal and vertical axis respectively. The nodes are programmed as $V_{i,j} = E\left[V_{i,j}\right]e^{(n-2j)\sigma\sqrt{\Delta t}}$, where $E\left[V_{i,j}\right] = (V_{t-1})^{e^{-\eta n\Delta t}} \Phi_{\mathrm{t}}^{1-e^{-\eta n\Delta t}}$ is developed in a separate vector.

δ	*0,000*	*0,233*	*0,240*	*0,249*	*0,262*	*0,280*	*0,307*	*0,348*	*0,419*	*0,563*	*1,000*
Φ	389,9	409,4	329,6	263,0	207,3	160,6	121,4	88,4	60,5	36,9	17,0
$E[V_t]$	389,9	392,9	305,5	236,8	182,1	138,2	102,6	73,6	49,8	30,1	13,7

$E[V_t] = \left(V_{t-1}(1-\delta_{t-1})\right)^{e^{-\eta\Delta t}} \Phi_t^{1-e^{-\eta\Delta t}}$

Development lattice

$j \setminus n$	0	1	2	3	4	5	6	7	8	9	10
0	389,9	481,6	459,1	436,1	411,1	382,4	348,1	306,0	253,6	187,8	104,8
1		320,5	305,5	290,2	273,6	254,5	231,7	203,7	168,8	125,0	69,8
2			203,3	193,1	182,1	169,4	154,2	135,5	112,3	83,2	46,4
3				128,5	121,2	112,7	102,6	90,2	74,8	55,4	30,9
4					80,6	75,0	68,3	60,0	49,8	36,8	20,6
5						49,9	45,4	40,0	33,1	24,5	13,7
6							30,2	26,6	22,0	16,3	9,1
7								17,7	14,7	10,9	6,1
8									9,8	7,2	4,0
9										4,8	2,7
10											1,8

$V_t = E[V_t]e^{(n-2j)\sigma\sqrt{\Delta t}}$

Cash flow lattice

	0	1	2	3	4	5	6	7	8	9	10
0	0,0	112,3	110,2	108,7	107,7	107,1	106,8	106,5	106,2	105,7	104,8
1		74,8	73,3	72,3	71,7	71,3	71,0	70,9	70,7	70,3	69,8
2			48,8	48,1	47,7	47,4	47,3	47,2	47,0	46,8	46,4
3				32,0	31,8	31,6	31,5	31,4	31,3	31,1	30,9
4					21,1	21,0	20,9	20,9	20,8	20,7	20,6
5						14,0	13,9	13,9	13,9	13,8	13,7
6							9,3	9,3	9,2	9,2	9,1
7								6,2	6,1	6,1	6,1
8									4,1	4,1	4,0
9										2,7	2,7
10											1,8

$CF_{i,j} = V_{i,j}\delta_{i+j}$

Figure 24 - Development lattice and cash flow lattice for the non-censored MR BDH case

Bastian-Pinto et al. (2010) showed that the probability of an up move for a given state and time is

$$p_{x,t} = \frac{1}{2} + \frac{\eta(-x_t^*)\sqrt{\Delta t}}{2\sqrt{\eta^2(-x_t^*)^2\Delta t + \sigma^2}}$$

By incorporating the Excel volatility lattice formula we get the simplified state dependent formula

$$p_{x,t} = \frac{1}{2} - \frac{(n-2j)\eta\Delta t}{2\sqrt{(n-2j)^2\eta^2\Delta t^2 + 1}}$$

Probability lattice

0	1	2	3	4	5	6	7	8	9
50 %	42 %	34 %	27 %	22 %	18 %	14 %	12 %	10 %	8 %
	58 %	50 %	42 %	34 %	27 %	22 %	18 %	14 %	12 %
		66 %	58 %	50 %	42 %	34 %	27 %	22 %	18 %
			73 %	66 %	58 %	50 %	42 %	34 %	27 %
				78 %	73 %	66 %	58 %	50 %	42 %
					82 %	78 %	73 %	66 %	58 %
						86 %	82 %	78 %	73 %
							88 %	86 %	82 %
								90 %	88 %
									92 %

$$p_{i,j} = \frac{1}{2} - \frac{\eta(i-j)\Delta t}{2\sqrt{\eta^2(i-j)^2\Delta t^2 + 1}}$$

Figure 25 - Probability lattice for the non-censored MR BDH case

Roll-back lattice without options

0	1	2	3	4	5	6	7	8	9	10
364,48	435,9	398,3	366,0	337,8	312,0	286,5	258,3	223,1	174,8	104,8
	329,5	298,0	269,9	244,7	221,8	199,8	177,2	150,9	117,1	69,8
		224,6	202,0	180,9	161,1	142,3	123,5	103,2	78,8	46,4
			150,7	134,3	118,5	103,0	87,6	71,5	53,4	30,9
				98,6	86,7	74,8	62,7	50,1	36,5	20,6
					62,3	53,6	44,5	35,1	25,0	13,7
						37,6	31,1	24,3	17,1	9,1
							21,4	16,6	11,5	6,1
								11,2	7,8	4,0
									5,2	2,7
										1,8

$$\Lambda_{i,j} = CF_{i,j} + \frac{p_{i,j}\Lambda_{i+1,j} + (1-p_{i,j})\Lambda_{i,j+1}}{1 + r_f\Delta t}$$

Figure 26 - Roll-back tree without options for the non-censored MR BDH case

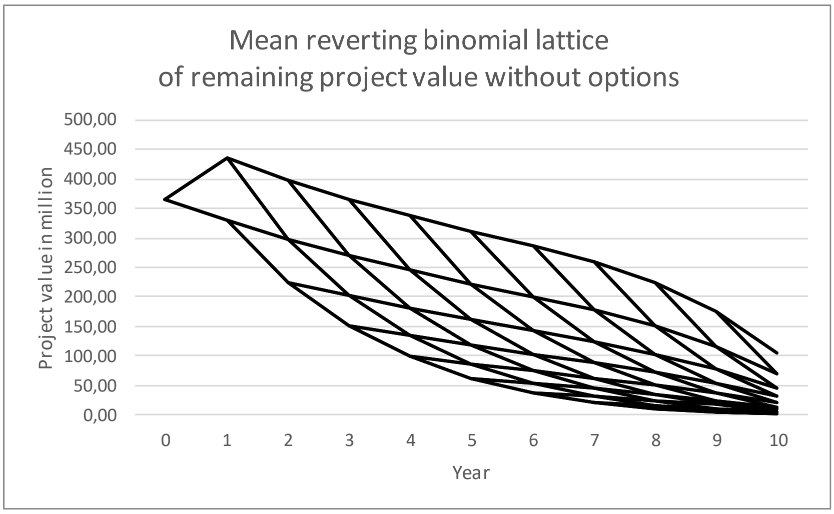


Figure 27 - Graph of remaining project value in each state without options for the non-censored MR BDH case

Roll-back lattice with options

0	1	2	3	4	5	6	7	8	9	10
372,12	442,0	403,1	371,7	351,2	**340,3**	286,5	258,3	223,1	174,8	104,8
	339,4	305,6	274,7	248,0	**231,9**	199,8	177,2	150,9	117,1	69,8
		239,0	213,1	187,1	**161,1**	142,3	123,5	103,2	78,8	46,4
			173,5	153,7	**131,6**	103,0	87,6	71,5	53,4	30,9
				134,9	**121,0**	74,8	62,7	50,1	36,5	20,6
					114,0	53,6	44,5	35,1	25,0	13,7
						37,6	31,1	24,3	17,1	9,1
							21,4	16,6	11,5	6,1
								11,2	7,8	4,0
									5,2	2,7
										1,8

Figure 28 - Roll-back tree with options for the non-censored MR BDH case

The option value is the difference from the roll-back tree without options to the one with options included, $7,63 million.

Roll-back lattice with options with extension

0	1	2	3	4	5	6	7	8	9	10
386,3	456,9	418,4	387,7	**374,4**	**340,3**	**306,4**	258,3	223,1	174,8	104,8
	354,4	321,4	290,7	**262,4**	**231,9**	**202,8**	177,2	150,9	117,1	69,8
		254,7	230,3	**205,6**	**178,9**	**147,3**	123,5	103,2	78,8	46,4
			188,7	**171,2**	**152,6**	**131,5**	87,6	71,5	53,4	30,9
				146,5	**134,4**	**120,9**	62,7	50,1	36,5	20,6
					121,7	**113,9**	44,5	35,1	25,0	13,7
						109,3	31,1	24,3	17,1	9,1
							21,4	16,6	11,5	6,1
								11,2	7,8	4,0
									5,2	2,7
										1,8

Figure 29 - Roll-back tree for the non-censored MR BDH case

The extended options from year 4-6 have a value of $21,86 million.

6. Monte Carlo method

We developed the BDH case with a MCS method for further comparison, especially against the mean reverting binomial models. In comparison to financial options these real options must be considered differently. The payoff of financial options is given by the maximum of the difference between the strike price and the market price and null at the term date. Note that for these real options, exemplified by the BDH case, the 'payoff' of exercising the option is the difference of the value from the exercised path and the not exercised path in year 10, even though the term date of the option is in year 5. Still, the decision-maker can only decide at the term date but is still exposed to uncertainty. The option is neither an American nor a European option, but best approximated by an Evergreen option when comparing to financial option equivalents (although Evergreen options don't get/pay the strike before the closing date).

If we interpret the exercise right of the option as the decision, then we only have one "term date" where the option holder can choose to exercise or not, even though the final payoff still carry uncertainty and the actual payoff is not closed yet. A naïve valuation approach to these kinds of options is to calculate the payoff using the expected values of the payoff to come at the closing date, estimated at the term date, as the best estimate of the payoff. In this way we have simplified the valuation method to the equivalent of a European option with the payoff of the difference in the expected value of the strike and the expected value of the initial base case estimates.

The mean option value calculated from the expected values are representative for the numbers we would get by running the simulation to the end in a sufficient number of iterations (assuming the proportion of type 1 errors and type 2 errors are approximately equal). This simplifies the valuation compared to American valuation techniques (like the least squares method) without being too strong an assumption.

6.1. Implementation of simplified MCS valuation on the BDH case

Periods 1 through 5 are modelled stochastic and the following periods are modelled as conditional expected values. The stochastic processes and conditional expected values are an extension of the GCE approach where volatility is isolated to one period. The value of each option is the mean of the discounted payoffs from the simulation. The payoff function in year 5 is:

$$= \text{MAX}\left(SellingPrice - \frac{NPV(6|5)}{1+r_f};\ -BuyoutCost + NewFraction * \frac{NPV(6|5)}{1+r_f} - \frac{NPV(6|5)}{1+r_f}; 0 \right)$$

The parameters of the GBM process and Model 1 process can be estimated from market data. The geometric OU process was approximated with the mean reversion coefficient from the Model 1 calibration.

Below is the Monte Carlo model where the price and cost are represented by GBM processes as specified in the original case. Beware that static cell outputs in @Risk display the median value instead of the mean. The model below displays the mean results of the option values. The frequencies of each option being in-the-money from the simulation are given in the lower right corner.

Options valuation by the Monte Carlo method for GBM price processes

Year	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Remaining Reserves		90,0	81,0	73,4	66,8	61,3	56,6	52,6	49,2	46,3	43,9
Production Level		9,0	7,7	6,5	5,5	4,7	4,0	3,4	2,9	2,5	2,1
ariable Op Cost Rate		10,1	10,3	10,5	10,6	10,8	11,0	11,2	11,4	11,7	11,9
Oil Price		24,7	24,4	24,2	23,9	23,6	23,6	23,6	23,6	23,6	23,6
Revenues		222,5	187,0	157,2	132,1	111,0	94,4	80,2	68,2	58,0	49,3
Production Cost		(96,3)	(83,8)	(73,0)	(63,6)	(55,6)	(48,9)	(43,0)	(38,0)	(33,6)	(29,8)
Cash Flow		126,1	103,2	84,2	68,5	55,4	45,5	37,2	30,2	24,4	19,5
Profit Sharing		(31,5)	(25,8)	(21,0)	(17,1)	(13,9)	(11,4)	(9,3)	(7,6)	(6,1)	(4,9)
Net Cash Flows		94,6	77,4	63,1	51,3	41,6	34,1	27,9	22,7	18,3	14,6
NPV	371,05	389,6	309,7	244,0	189,9	**145,4**	109,1	78,7	53,3	32,2	14,6
E(divest option)	**20,74**					26,46			P(Divest)		48,88 %
E(buyout option)	**8,25**					10,52			P(Buyout)		41,67 %
E(both options)	**28,98**					36,99			P(Divest or Buyout)		90,55 %

Figure 30 - Monte Carlo method for the Geometric Brownian Motion price processes

MCS valuation of the BDH case was run with the price and cost approximated by GBM, OU with geometric volatility and Model 1 of Schwartz. More than 90% of the iterations estimated one of the options to be in-the-money and thereby the optimal decision strategy in year 5 for the GBM model. The corresponding frequency for the mean reverting models were 80%. This is because the reduced volatility from modelling the price and cost as MR processes leads to a tighter distribution of project value outcomes in the mid-range between the two options. Model screenshots of the results for the MR models are attached in Appendix C. Results from 100 000 simulations are summarized in the figure below.

Process \ Option	Buyout	Divest	Both options
Geometric Brownian Motion	8,25	20,74	28,98
OU with geometric volatility	3,81	7,35	11,16
Model 1 of Schwartz	3,50	8,08	11,58

Figure 31 - Monte Carlo option valuation results

These results are compared to the other models in the next chapter.

7. Comparison of results and model differences

The project process parameters estimated from the GBM and MR models for the uncertain project parameters are presented in figures 32 through 34. The results are seen to be consistent enough to be comparable when correcting for the expected model differences explained in this chapter.

	Years		*5*		*5*	*5*	*4-6*
Method	**Process**	**Buyout**		**Divest**		**Both**	**Extended options**
BDH	GBM	28 %	7,38	72 %	18,63	26,01	39,36
MCS	GBM	28 %	8,25	72 %	20,74	28,98	
Bivariate	GBM	33 %	10,28	67 %	20,97	31,25	
Censored BDH	MR (lnOU)	11 %	0,83	89 %	6,80	7,63	22,41
Non-censored BDH	MR (lnOU)	15 %	1,38	85 %	7,56	8,94	22,88
MCS	MR (M1)	30 %	3,50	70 %	8,08	11,58	
MCS	MR (gOU)	34 %	3,81	66 %	7,35	11,16	

Figure 32 - Comparison of option value from different approaches to the BDH case

Modelling the project with the two mean reverting binomial methods results in a lower total option value compared to the conventional geometric Brownian Motion.

This is because (1) GBM processes tend to overestimate long-term uncertainty and (2) because of a partial lack of incorporation of drift in the model. The censored and non-censored models only incorporate the indirect drift of the long-term mean (estimated as the base case NPV at time t), but do not discount the continuing project value from the last period (where the drift is the discounting to reflect the time value of money).

The lower value from moving from a GBM to an MR model is expected (Jafarizadeh & Bratvold, 2012), but the underestimation of drift in the project development also underestimate the value development, resulting in an overall underestimation of the upside option (buyout). The effects are visually represented in figures 33 and 34. This is also evident from the comparison of the MR binomial models to the MR MCS models.

All other models than the mean reverting lattice models has an option value distribution between the buyout option on the upside and the divest option on the downside of 30/70 $\pm$ 4%. In contrast, the value in the MR binomial models comes mainly from divest option (85-90%).

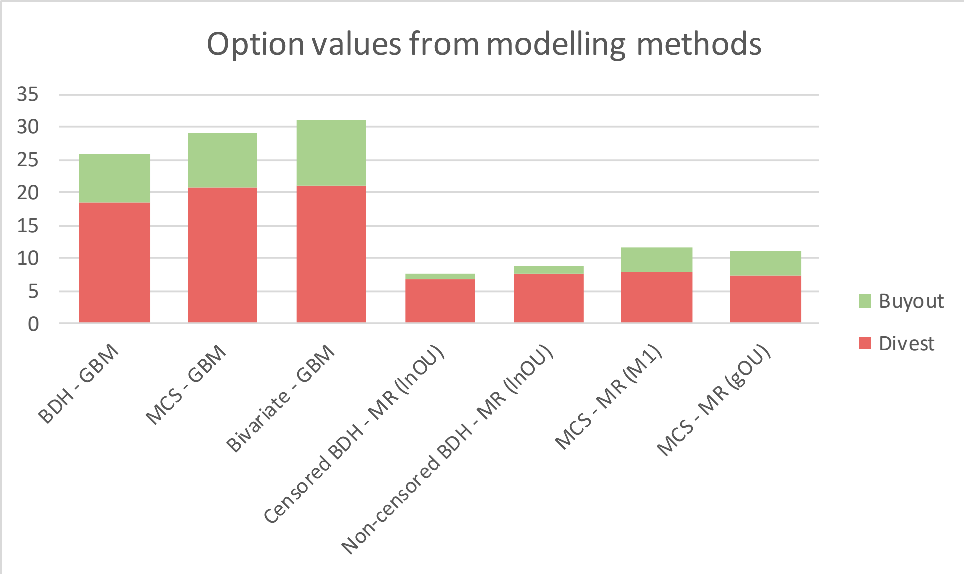


Figure 33 - Bar chart of option value estimate from each model

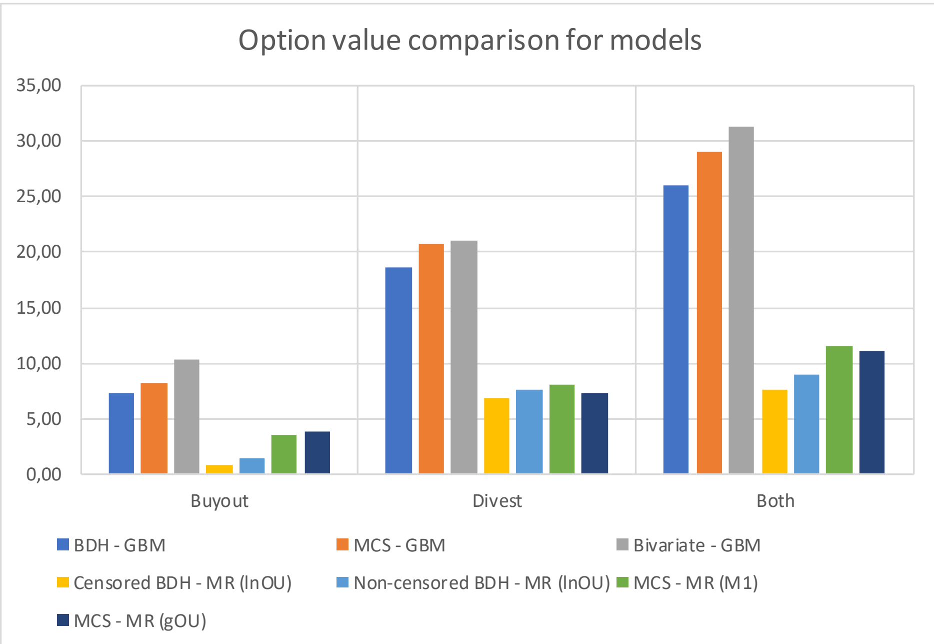


Figure 34 - Bar chart of option value estimate for each option

8. Conclusion and suggestions for further work

First, we introduce the background of options pricing and real options valuation with definitions of the concepts and the main modelling methods. The second part of the paper reviewed the BDH method with its developments and implemented the BDH case as an introduction. The method and results have been analyzed and decomposed to identify the value drivers of the options in the model. Further, the same approach has been implemented in the censored and non-censored mean reverting binomial lattice methods. We investigated the implications of the fundamental option pricing principles for project valuation, specifically reflecting on the compatibility of mean reverting project value development used for real options valuation. The case example was also valued using a simplified European Monte Carlo method with sales price and cost processes modelled as geometric Brownian Motions and mean reverting, using the expected value at the term date of the options as the best estimate of the option payoff. Finally, the case results were compared, and the model differences are explained.

Suggestions for further work in the area includes:

- Test whether the project value of projects in industries with mean reverting prices can be approximated to a mean reverting process (examples of tests listed in introduction)
- Develop the same models for another case where project variables are calibrated from the same dataset (versus using same information points with some additional information, as done in the BDH case)
- Estimation of mean reversion speed and other parameters from the Monte Carlo simulation of the discounted cash flow model.
- Develop a censored or uncensored model for a mean reverting model with drift? The challenge is not only to make a general binomial model of a mean reverting process with drift, but also to capture the right amount of drift, not captured in the mean reversion. The alternative would be to estimate the long-term equilibrium without discounting to separate the two, but this introduces other challenges.

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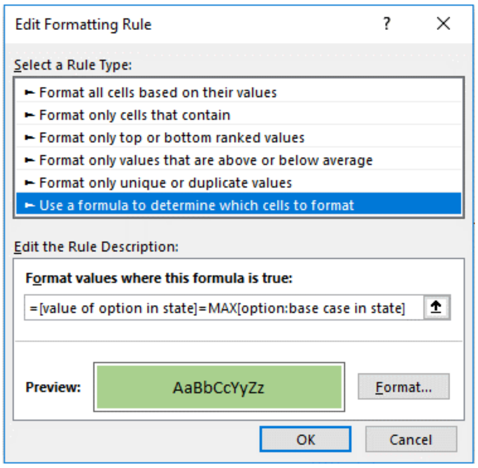
Appendix A. Practical Excel tips for lattice development

Use of Excel as a modelling tool can be very powerful, both as a computational tool and for presentation and visualization of data. The main alternative for tree development is DPL from Syncopation Software, where DPL is very good for visualizing non-recombining trees and have good analysis tools for anything from tornado diagrams and influence diagrams to optimal policy analysis, cumulative distribution functions and rainbow charts. The tree however, becomes extensively large as the number of modelled periods increase in a non-recombining tree (as discussed in chapter 2.5.3).

Since one of the arguments for using a binomial lattice is the ease of communication of the tool, additional steps to make the model more intuitive are considered to be important, especially when presenting for non-technical decision-makers.

A.1. Conditional formatting for action determination

To highligth whether the option should be excersised in a given state in the lattice one can use conditional formatting, using a formula for the maximum value to identify the preferred action and highlighting it.



						Base case	Buyout	Divest					
0	1	2	3	4	5	5	5	5	6	7	8	9	10
418,0	562,6	594,8	632,3	676,0	**719,5**	613,2	719,5	274,3	603,1	570,7	507,5	402,1	239,5
	315,9	323,2	330,0	340,2	**362,4**	324,9	362,4	192,3	319,5	302,3	268,8	213,0	126,9
		200,6	196,8	187,6	**173,1**	172,1	173,1	148,9	169,2	160,1	142,4	112,8	67,2
			144,0	138,1	**125,9**	91,2	72,9	125,9	89,7	84,8	75,4	59,8	35,6
				117,9	**113,7**	48,3	19,8	113,7	47,5	44,9	40,0	31,7	18,9
					107,3	25,6	-8,3	107,3	25,2	23,8	21,2	16,8	10,0
									13,3	12,6	11,2	8,9	5,3
										6,7	5,9	4,7	2,8
											3,1	2,5	1,5
												1,3	0,8
													0,4

Figure 35 – Optimal decision strategy highlighted using conditional formatting, GBM BDH case

The same conditional formatting technique can be used without expanding the tree at the decision point by using several conditioning rules on the same cells. Each conditional formatting rule is determined using a logic statement referencing if the cell is equal the output given by the specific option.

A.2. Figure of remaining project value or development of underlying asset

The development of the underlying asset can be visualized by a line chart or a scatter chart with straight lines in Excel. Lines connect the points in a series, so to connect both up and down development we must make a duplicate lattice with down series in the rows.

j \ *n*	0	1	2	3	4	5	6	7	8	9	10
0	389,4	508,9	514,9	522,5	530,7	537,9	540,8	533,5	505,4	436,7	291,0
1		308,7	308,8	309,2	309,4	308,5	305,0	295,9	275,8	234,6	154,2
2			186,7	184,9	182,6	179,3	174,3	165,9	151,7	126,6	81,7
3				110,9	108,4	105,0	100,4	93,9	84,1	68,6	43,3
4					64,1	61,5	58,0	53,3	46,8	37,3	22,9
5						35,7	33,3	30,2	26,0	20,3	12,1
6							18,9	17,0	14,4	11,0	6,4
7								9,4	7,9	6,0	3,4
8									4,3	3,2	1,8
9										1,7	1,0
10											0,5
											291,0
										436,7	154,2
									505,4	234,6	81,7
								533,5	275,8	126,6	43,3
							540,8	295,9	151,7	68,6	22,9
						537,9	305,0	165,9	84,1	37,3	12,1
					530,7	308,5	174,3	93,9	46,8	20,3	6,4
				522,5	309,4	179,3	100,4	53,3	26,0	11,0	3,4
			514,9	309,2	182,6	105,0	58,0	30,2	14,4	6,0	1,8
		508,9	308,8	184,9	108,4	61,5	33,3	17,0	7,9	3,2	1,0
	389,4	308,7	186,7	110,9	64,1	35,7	18,9	9,4	4,3	1,7	0,5

Figure 36 - Development lattice with up series and down series for graph development

When we have both the up series and down series the graph can be made. Figure 37 shows the mean reverting binomial rollback lattice without options, with the median remaining project value in red and the deterministic base case estimate in green.

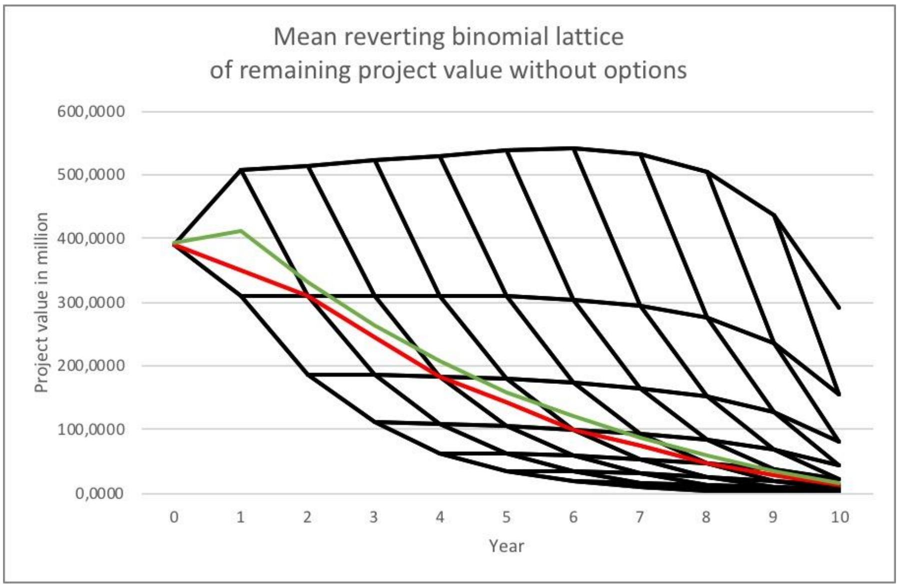


Figure 37 - Value development graph example

A.3. Conditional formatting for censoring of probabilities

For visualization of which of the state-dependent probabilities are censored in the censored model we can use the Icon sets of conditional formatting, defining the criteria for the icon to the absolute levels of 0% and 100%.

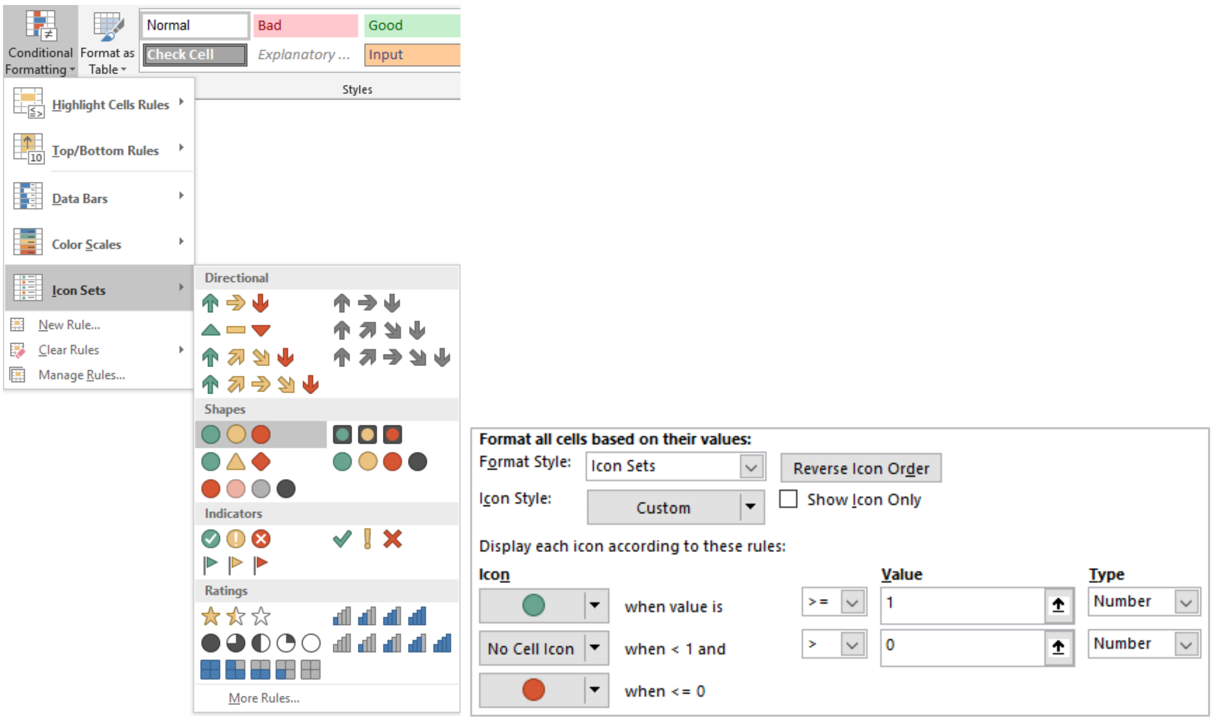


Figure 38 - Conditional formatting window for censoring traffic lights

A.4. Conditional formatting for intuition of development

When a lattice model does not include dividends, it can be modelled with color-coding of the values in each state for a much clearer intuition of the development. An example of this is how the probability tree in the non-censored model, figure 25.

A.5. VBA

Haug (2007) implement a wide variety of option pricing methods in Visual Basic for Applications (VBA), the macro programming system implemented in Microsoft Excel.

Appendix B. Two-way sensitivity analysis of mean reverting models

Two-way sensitivity analysis of volatility, risk-free interest rate and mean reversion coefficient for **the censored model** for mean reversion.

$\sigma \setminus \eta$	5 %	10 %	15 %	17 %	20 %	30 %	40 %
5 %	11,27	8,36	5,89	4,97	3,93	1,56	1,00
10 %	11,73	8,92	6,51	5,65	4,46	1,34	0,40
15 %	12,46	9,36	**6,72**	**5,92**	**4,84**	2,30	0,97
20,36 %	14,47	11,28	**8,64**	**7,71**	**6,47**	3,36	1,55
25 %	16,69	13,07	**10,08**	**9,05**	**7,64**	4,11	1,99
30 %	19,14	14,93	11,51	10,33	8,74	4,75	2,32
35 %	22,64	17,55	13,48	12,07	10,18	5,40	2,53

$\sigma \setminus r_f$	2 %	3 %	4 %	5 %	6 %	7 %	8 %
5 %	1,17	2,53	3,69	4,97	6,32	7,48	8,50
10 %	3,02	3,50	4,45	5,65	6,68	7,55	8,30
15 %	5,08	5,42	**5,70**	**5,92**	**6,73**	7,66	8,45
20,36 %	7,03	7,32	**7,55**	**7,71**	**7,83**	7,91	8,33
25 %	8,53	8,76	**8,93**	**9,05**	**9,11**	9,14	9,14
30 %	10,85	10,59	10,32	10,33	10,35	10,32	10,26
35 %	13,26	12,86	12,46	12,07	11,69	11,40	11,29

$r_f \setminus \eta$	5 %	10 %	15 %	17 %	20 %	30 %	40 %
2 %	15,15	11,29	8,12	7,03	5,57	2,06	0,32
3 %	14,95	11,33	8,35	7,32	5,94	2,57	0,72
4 %	14,72	11,33	**8,52**	**7,55**	**6,23**	3,00	1,17
5 %	14,47	11,28	**8,64**	**7,71**	**6,47**	3,36	1,55
6 %	14,19	11,20	**8,71**	**7,83**	**6,65**	3,67	1,88
7 %	13,94	11,10	8,74	7,91	6,79	3,92	2,19
8 %	14,18	11,47	9,16	8,33	7,20	4,18	2,48

Figure 39 - Two-way sensitivity analysis of volatility, risk-free rate and mean reversion coefficient for the censored MR BDH case

Two-way sensitivity analysis of volatility, risk-free interest rate and mean reversion coefficient for **the non-censored model** for mean reversion.

σ \ η	5 %	10 %	15 %	17 %	20 %	30 %	40 %
5 %	10,33	7,03	4,31	3,35	2,37	0,25	0,05
10 %	10,92	7,45	5,28	4,58	3,65	1,24	0,26
15 %	11,90	9,15	**6,88**	**6,08**	**5,01**	2,20	0,52
20,36 %	14,40	11,16	**8,49**	**7,57**	**6,32**	3,09	0,89
25 %	16,55	12,82	**9,79**	**8,75**	**7,36**	3,87	1,46
30 %	19,33	15,16	11,81	10,65	9,10	5,11	2,38
35 %	22,83	17,87	13,97	12,65	10,87	6,34	3,26

σ \ r_f	2 %	3 %	4 %	5 %	6 %	7 %	8 %
5 %	0,59	1,53	2,42	3,35	4,70	5,86	6,89
10 %	2,44	3,26	3,96	4,58	5,15	5,97	6,97
15 %	4,32	5,01	**5,60**	**6,08**	**6,49**	6,82	7,09
20,36 %	6,45	6,71	**7,18**	**7,57**	**7,88**	8,12	8,30
25 %	8,64	8,71	**8,73**	**8,75**	**8,98**	9,16	9,28
30 %	11,02	10,93	10,80	10,65	10,49	10,30	10,27
35 %	13,47	13,21	12,93	12,65	12,35	12,05	11,74

r_f \ η	5 %	10 %	15 %	17 %	20 %	30 %	40 %
2 %	14,87	10,72	7,54	6,45	4,97	2,29	1,69
3 %	14,75	10,93	7,80	6,71	5,29	1,84	1,23
4 %	14,59	11,07	**8,18**	**7,18**	**5,83**	2,33	0,83
5 %	14,40	11,16	**8,49**	**7,57**	**6,32**	3,09	0,89
6 %	14,18	11,19	**8,73**	**7,88**	**6,72**	3,74	1,71
7 %	13,94	11,18	8,91	8,12	7,05	4,29	2,41
8 %	13,69	11,13	9,03	8,30	7,32	4,77	3,02

Figure 40 - Two-way sensitivity analysis of volatility, risk-free rate and mean reversion coefficient for the non-censored MR BDH case

Appendix C. MCS valuation output

Options valuation by the Monte Carlo method for OU processes with geometric volatility

Year	0	1	2	3	4	5	6	7	8	9	10
Remaining Reserves		90,0	81,0	73,4	66,8	61,3	56,6	52,6	49,2	46,3	43,9
Production Level		9,0	7,7	6,5	5,5	4,7	4,0	3,4	2,9	2,5	2,1
Variable Op Cost Rate		10,3	10,6	10,8	11,0	11,2	11,4	11,5	11,6	11,7	11,7
Oil Price		25,0	25,0	25,0	25,0	25,0	25,0	25,0	25,0	25,0	25,0
Revenues		225,0	191,3	162,6	138,2	117,5	99,8	84,9	72,1	61,3	52,1
Production Cost		(98,0)	(86,1)	(75,5)	(66,1)	(57,7)	(50,3)	(43,9)	(38,4)	(33,6)	(29,5)
Cash Flow		127,0	105,1	87,0	72,1	59,7	49,5	40,9	33,7	27,7	22,7
Profit Sharing		(31,8)	(26,3)	(21,8)	(18,0)	(14,9)	(12,4)	(10,2)	(8,4)	(6,9)	(5,7)
Net Cash Flows		95,3	78,8	65,3	54,1	44,8	37,1	30,7	25,3	20,8	17,0
PV of Cash Flows	388,7	408,2	328,5	262,2	206,7	**160,3**	121,3	88,3	60,5	37,0	17,0
E(divest option)	**7,35**					9,38			P(Divest)		38,78 %
E(buyout option)	**3,81**					4,86			P(Buyout)		41,56 %
E(both options)	**11,16**					14,24			P(Divest or Buyout)		80,34 %

Figure 41 - Monte Carlo method for geometric volatility Ornstein-Uhlenbeck price processes

Options valuation by the Monte Carlo method for Model 1 processes

Year	0	1	2	3	4	5	6	7	8	9	10
Remaining Reserves		90,0	81,0	73,4	66,8	61,3	56,6	52,6	49,2	46,3	43,9
Production Level		9,0	7,7	6,5	5,5	4,7	4,0	3,4	2,9	2,5	2,1
		2,327	2,348	2,366	2,381	2,394	2,409	2,421	2,432	2,441	2,449
Variable Op Cost Rate		10,3	10,5	10,7	10,8	11,0	11,1	11,3	11,4	11,5	11,6
		3,208	3,198	3,191	3,184	3,179	3,184	3,189	3,193	3,196	3,199
Oil Price		24,7	24,5	24,3	24,1	24,0	24,2	24,3	24,4	24,4	24,5
Revenues		222,5	187,4	158,0	133,5	112,9	96,5	82,4	70,3	59,9	51,1
Production Cost		(97,3)	(85,1)	(74,3)	(64,8)	(56,5)	(49,4)	(43,2)	(37,8)	(33,2)	(29,1)
Cash Flow		125,2	102,3	83,7	68,7	56,4	47,1	39,2	32,5	26,8	22,0
Profit Sharing		(31,3)	(25,6)	(20,9)	(17,2)	(14,1)	(11,8)	(9,8)	(8,1)	(6,7)	(5,5)
Net Cash Flows		93,9	76,7	62,8	51,5	42,3	35,3	29,4	24,3	20,1	16,5
PV of Cash Flows	375,5	394,3	315,4	250,6	197,2	**153,0**	116,2	85,0	58,4	35,8	16,5
E(divest option)	**8,08**					10,31			P(Divest)		39,79 %
E(buyout option)	**3,50**					4,47			P(Buyout)		40,65 %
E(both options)	**11,58**					14,78			P(Divest or Buyout)		80,44 %

Figure 42 - Monte Carlo method for price processes following Model 1