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The University of Stavanger

An Introduction to Supersymmetry

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Abstract: This thesis gives an introduction to supersymmetry. We compute the Poincaré- conformal- and Clifford algebra in any dimension. Most examples are in four dimensions, including the Wess-Zumino model and supersymmetric gauge theories. The Poincaré- and conformal superalgebra are computed in four dimensions.

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where stated otherwise by reference or acknowledgement, the work presented is entirely my own.

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Tobias Våge Henriksen

Contents

Part I
Introduction

1 The Spin-Statistics Theorem

This thesis will give an introduction to the principle called *supersymmetry*. Supersymmetry is a principle in physics relating two types of particles, *bosons* and *fermions*, which will be discussed further below. Supersymmetry, abbreviated SUSY, can, for example, be an extension of the Standard Model, which today is the model best describing the subatomic world. The Standard Model contains bosons as force carriers, for example the photon, while fermions are the fundamental particles, for example the electron. SUSY predicts that for each boson there should correspond a fermion. This is not the case in the Standard Model (as of today, without supersymmetry). This is why SUSY would be an extension of the Standard Model.

Above it was mentioned that bosons are force carriers, and fermions are fundamental particles. To give a more precise definition, bosons are particles of integer spin, while fermions carry half-integer spin. The examples mentioned above, photons and electrons, have spin 1 and spin 1/2, respectively. The spin of the particle tells us what statistics the particle have. This has been worked out in [1], [2] and [3]. We will for the most time work in one time dimension and three space dimensions. Then the **Spin-Statistics theorem** says

Theorem 1.1. *(Spin-Statistics)*

- *The exchange of two particles with integer spin is symmetric. These particles are called bosons. For two bosons B_1, B_2, this means that $B_1B_2 = B_2B_1$.*
- *The exchange of two particles with half-integer spin is anti-symmetric. These particles are called fermions. For two fermions F_1, F_2, this means that $F_1F_2 = -F_2F_1$.*

Furthermore, [2] and [3] showed that the exchange of two particles where not both are of half-integer spin is symmetric. Therefore, exchanging a boson B with a fermion F is symmetric

$$BF = FB.$$

The theorem also establishes that the only types of particles that can exist in four-dimensional space-time are bosons and fermions.

From the Spin-Statistics theorem we see that bosons satisfy **Bose-Einstein statistics**, that is, several identical bosons can occupy the same quantum state in a quantum system. Fermions, on the other hand, are particles of half-integer spin satisfying **Fermi-Dirac statistics**, that is, identical particles cannot occupy the same quantum state in a quantum system, known as the **Pauli exculsion princple**.

As mentioned above, we want to relate bosons with fermions. The relation was not realised before the in the 1970s, first by Gol'fand and Likthman, and then generalised by Haag, Łopuszański and Sohnius.

2 The Haag-Łopuszański-Sohnius Theorem

Before we look at the articles by Gol'fand and Likthman, and Haag, Łopuszański and Sohnius, we should mention another article. In 1967 Coleman and Mandula published a no-go theorem [4], stating "the impossibility of combining space-time and internal symmetries in any but a trivial way". At the time, it seemed impossible to transform bosons to fermions, and vice-versa. However, in 1971 Gol'fand and Likthman found a way to do so [5]. They extended the *Poincaré algebra*, which contains the Minkowski space-time rotations, boosts and translations (*isometries*). The Poincaré algebra only contain bosonic generators, while the algebra they created, called the Poincaré superalgebra, contains both bosonic and femrionic generators.

In 1975, Haag, Łopuszański and Sohnius showed that if one weakens the Coleman-Mandula theorem by allowing the algebra to contain both bosonic and fermionic generators, it is possible to extend the Poincaré algebra as a *superalgebra* [6]. This algebra will need to satisfy some different axioms to the algebra only containing bosonic generators.

3 Structure of Thesis

Before we can appreciate superalgebras, we need to understand the algebras describing space-time symmetries. These algebras are called *Lie algebras*, and are introduced in section 4. Using Lie derivatives, we show that *Killing vector fields* generate Lie algebras. Then we can compute the Poincaré- and the conformal algebra, which are both space-time algebras. In section 5 we consider the *Clifford algebra*. The Clifford algebra is generated by γ-matrices, which are necessary to keep the fermionc part of the theory invariant under the Lorentz algebra, which is a subalgebra of the Poincaré algebra. Next, in section 6 we will consider some example theories and check whether or not these are invariant under the Poincaré- and conformal algebra. At the end of the section, we will for the first time encounter a supersymmetry theory, in one dimension. The mathematics part of the thesis is ended in section 7, where we introduce superalgebras. We will also consider a simple superalgebra here.

Finally, in the last two sections we consider supersymmetry models in four dimensions. In section 8 the *Wess-Zumino model* will be introduced. Here we find the Poincaré- and conformal superalgebras, and show that the Wess-Zumino model is invariant under these algebras. The other type of model we will consider is a *supersymmetric gauge theory*. We show that also this is invariant under the aforementioned superalgebras, and also another typer of symmetry, namely gauge symmetry.

Let us now leave the *super* until we have acquaint ourselves with only *symmetry*.

Part II
Mathematics of Supersymmetry

4 Lie Algebras

This section has been taken from [7], chapters 2, 5 and 7.

In physics, symmetries are of major importance. A symmetry can be though of as a transformation that leaves the physical system unchanged. This can be described by the mathematical objects called groups:

Definition 4.1. A **group** is a set G, with a group multiplication, denoted $\circ$, satisfying four axioms. $\forall a, b, c \in G$, we require

1. Closure: $a \circ b \in G$,
2. Associativity: $a \circ (b \circ c) = (a \circ b) \circ c$,
3. Identity: there exist an element $\mathbb{1} \in G$ satisfying $\mathbb{1} \circ a = a \circ \mathbb{1} = a$,
4. Inverse: for every element $a \in G$, $\exists\, a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = \mathbb{1}$, where $\mathbb{1}$ is, as above, the identity in G.

Before we proceed it is convenient to introduce some maps. Let X and Y be two sets. A **map** f assigns a value $y \in Y$ to each $x \in X$, and is written $f : X \to Y$. The map is defined by $f : x \mapsto f(x)$. The map is said to be **injective** if $x \neq x'$ implies $f(x) \neq f(x')$, and **surjective** if for each y there exists at least one x such that $f(x) = y$. A map is said to be **bijective** if it is both injective and surjective. Now, let X and Y be endowed with algebraic structures, for example addition. A map is called a **homomorphism** if it preserves the algebraic structure of the set, so that $f(xx') = f(x)f(x')$, where xx' is defined by the structure in X, and $f(x)f(x')$ is defined by the structure in Y. A bijective homomorphism is called an **isomorphism**. A homomorphism $f : X \to X$ is called an **endomorphism**, and if it also is bijective it is called an **automorphism**. A map $f : X \to Y$ is a **homeomorphism** if it is continuous and has an inverse $f^{-1} : Y \to X$ which is also continuous. If both f and f^{-1} is smooth, that is, infinitely differentiable, then the map is called a **diffeomorphism**.

Let us return to our groups. Groups which contains continuous symmetries are called *Lie groups*. Before giving a definition of a Lie group it is necessary to know what a manifold is. A formal definition is not necessary for us, so only an informal definition is given: a m**-dimensional manifold** is a *topological space* which is homeomorphic to $\mathbb{R}^m$ locally. Now a definition of Lie groups can be given. This is not a concept we will have much use for in this dissertation, but is given for completeness.

Definition 4.2. A **Lie group** is a group, G, which is also a smooth manifold, with a smooth group operation $G \times G \to G : (a, b) \to a \circ b \ \forall a, b \in G$, and smooth inverse $G \to G : a \to a^{-1} \ \forall a \in G$.

The reason for not considering the Lie groups is because almost all of the information from the group is also given in its *Lie algebra*. A Lie group consists of an infinite number of elements. There is, however, a finite number of generators, and these generators form the Lie algebra.

Before defining a Lie algebra, let us recall some useful definitions:

Definition 4.3. A **vector space** V over a field $\mathbb{K}$ is a set with two operations:

- addition: $+ : V \times V \to V$, and
- multiplication: $\cdot : \mathbb{K} \times V \to V$.

Let $u, v, w \in V$ and $a, b \in \mathbb{K}$. The elements of $\mathbb{K}$ are called **scalars** and the elements of V are called **vectors**. They satisfy

1. $u + v = v + u$,
2. $(u + v) + w = u + (v + w)$,
3. 0 is the identity vector in V such that $v + 0 = 0 + v = v$,
4. for a vector u there exists an inverse $-u$ such that $u+(-u) = (-u)+u = 0$,
5. $a \cdot (u + v) = a \cdot u + a \cdot v$,
6. $(a + b) \cdot u = a \cdot u + b \cdot u$,
7. $(a \cdot b\cdot) \cdot u = a \cdot (b \cdot u)$,
8. $\mathbb{1}$ is the identity element in $\mathbb{K}$ such that $a \cdot \mathbb{1} = \mathbb{1} \cdot a = a$.

Definition 4.4. Let V be a vector space over a field $\mathbb{K}$, and let f be a linear function on V, $f : V \to \mathbb{K}$. The set of linear functions is a vector space, called the **dual vector space**, denoted V^*. An element of the dual vector space is called a **dual vector**.

From the definition it is clear that a dual vector maps a vector to scalar in $\mathbb{K}$. The space containing all tangent vectors at a point p in a manifold M, denoted T_pM, is a vector space, called the **tangent space**. It has a dual vector space, T_p^*M. Let ω be a dual vector in T_p^*M such that it is a map $\omega : T_pM \to \mathbb{R}$. Then ω is called a **one-form**. An arbitrary one-form can be written $\omega = \omega_\mu \mathrm{d}x^\mu$, where $\mathrm{d}x^\mu$ is a basis for T_p^*M, and ω_μ are the components of ω. A more general map is called a **tensor**. A tensor of type (q, r) is a mapping from q elements of T_p^*M and r elements of T_pM to a real number. The tensor is written

$$T = T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\mu_q}} \otimes \mathrm{d}x^{\nu_1} \otimes \mathrm{d}x^{\nu_r}.$$

We write the map as $T : \bigotimes^q T_p^*M \bigotimes^r T_pM \to \mathbb{R}$. The set of all tensors of type (q, r) at a point $p \in M$ defines the **tensor space** of type (q, r), and is denoted $\mathscr{T}^q_{r,p}$. Tensors of type $(q, 0)$ maps dual vectors to scalars, and are interpreted as vectors, while tensors of type $(0, r)$ maps vectors to scalars, and are interpreted as dual vectors. In particular, a a symmetric type $(0, 2)$ tensor g is called a metric. It takes two tangent vectors, $U, V \in T_pM$ as input and returns a scalar. At each point p, g satisfies $g_p(U,V) = g_p(V,U)$, where $g_p = g|_p$. We write g_p as

$$g_p = g_{\mu\nu}(p)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu,$$

where $g_{\mu\nu}(p)$ are the components of g_p. The symmetry condition secures that $g_{\mu\nu}(p) = g_{\nu\mu}(p)$. We will usually omit writing p in $g_{\mu\nu}$.

Another concept we should remind ourselves of is that of algebras.

Definition 4.5. An **algebra** $\mathscr{A}$ is a vector space over a field $\mathbb{K}$ with an additional operation which takes $\mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$. We write this operation without any sign, for example as xy for two elements $x, y \in \mathscr{A}$. For all $x, y, z \in \mathscr{A}$ and $a, b \in \mathbb{K}$

1. $(x + y)z = xy + yz$ and $x(y + z) = xy + xz$,
2. $(ax)(by) = (a \cdot b)(xy)$,

An algebra is said to be an **associative algebra** if it satisfies $(xy)z = x(yz)$. In section 5 we will introduce an example of associative algebras, namely the Clifford algebra. Now we will define Lie algebras, which are non-associative algebras, and the additional operation is the *Lie bracket*.

Definition 4.6. A **Lie algebra** is a vector space $\mathfrak{g}$, together with a map, the Lie bracket, $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying the following axioms:

1. Bilinearity: $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$, $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z]$ $\forall\ \alpha, \beta \in \mathbb{K}$, and $\forall\ X, Y, Z \in \mathfrak{g}$,
2. Skew symmetry: $[X, Y] = -[Y, X]$ $\forall\ X, Y \in \mathfrak{g}$,
3. Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ $\forall\ X, Y, Z \in \mathfrak{g}$.

We may fix a basis $\{b_i\}$ for the Lie algebra $\mathfrak{g}$. Then the Lie bracket is defined as

$$[b_i, b_j] = f_{ij}{}^k b_k,$$

where $f_{ij}{}^k$ are known as *structure constants*, and are antisymmetric in the lower indices, $f_{ij}{}^k = -f_{ji}{}^k$.

Many Lie algebras can be represented as matrices, with the Lie bracket being the commutator, $[X, Y] = XY - YX$, $X, Y \in \mathfrak{g}$. This is called a *Lie algebra representation*.

Definition 4.7. Let $\mathfrak{g}$ be a Lie algebra. A **Lie algebra representation** on a n-dimensional vector space V is a homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the set of all endomorphisms on V. If $V = \mathbb{R}^n$, $\mathrm{End}(V) = \mathrm{Mat}_n(\mathbb{R})$ is the set of real $n \times n$ matrices.

4.1 Killing Vector Fields

This section is taken from [7] and [8]. It will be clear that *Killing vector fields* form Lie algebras.

Let us begin with defining what is meant with a vector field. A **vector field** X over a m-dimensional manifold M is a smooth map from $C^\infty(M)$, the smooth functions on M, to the same space $C^\infty(M)$. Let a point $p \in M$ have local coordinates $x^\mu(p)$ such that the set $\{x^1(p), x^2(p), \dots, x^m(p)\}$ can be seen as a map from M to $\mathbb{R}^m$. Then the vector field X can be written $X = X^\mu \frac{\partial}{\partial x^\mu}$, where X^μ are the components of X in the coordinate system with coordinates $x^\mu(p)$. Since X is a map $X : C^\infty(M) \to C^\infty(M)$, for a smooth map $f : M \to \mathbb{R}$, X is a vector field if $X[f] := X^\mu \frac{\partial}{\partial x^\mu} f$ is a smooth function. We denote the set of vector fields on M as $\mathfrak{X}(M)$. A **tensor field** of type (q, r) assigns smoothly an element of $\mathcal{T}^q_{r,p}$ to each point $p \in M$. We denote the set of tensor fields of type (q, r) on M as $\mathcal{T}^q_r(M)$.

We should also introduce a new map. The set of tensors of type (q, r) on a point $p \in M$ is denoted $\mathcal{T}^q_{r,p}(M)$. A smooth map $f : M \to N$ induces a map $f^* : \mathcal{T}^0_{r,f(p)}(N) \to \mathcal{T}^0_{r,p}(M)$. f^* is called the **pullback**. In components the f^* is given by the Jacobian matrix $\partial x'^\alpha / \partial x^\mu$. If g is a smooth function, then the pullback of g by f is defined by $f^* g = g \circ f$.

A concept which will be practical when considering the upcoming Killing vector fields is the Lie derivative. The Lie derivative evaluates the change of a tensor field along the *flow* of a vector field. The **flow** generated by $X \in \mathfrak{X}(M)$ for some m-dimensional manifold M is a smooth map $\sigma : \mathbb{R} \times M \to M$. Let $t, s \in \mathbb{R}$ and $p \in M$, then σ satisfies $\sigma(0, p) = p$, $\sigma(t, \sigma(s, p)) = \sigma(t + s, p)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\sigma(t, p) = X(\sigma(t, p))$. If we fix $t, s \in \mathbb{R}$ this will instead be written as $\sigma_0(p) = p$, $\sigma_t(\sigma_s(p)) = \sigma_{t+s}(p)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_t(p) = X(\sigma_t(p))$. Let us now consider the components of the flow. The components of the flow in local coordinates x^μ is $\sigma^\mu_t(p)$. We let $t = \epsilon$ be infinitesimal. Then, a point p with coordinates $x^\mu(p)$ is mapped to

$$\begin{aligned}\sigma^\mu(\epsilon, p) &= \sigma^\mu(0, p) + \epsilon \frac{\mathrm{d}}{\mathrm{d}\epsilon}\sigma^\mu(0, p) + O(\epsilon^2) = \sigma^\mu_0(p) + \epsilon \frac{\mathrm{d}}{\mathrm{d}\epsilon}\sigma^\mu_0(p) + O(\epsilon^2) \\ &= x^\mu(p) + \epsilon X^\mu(\sigma_0(p)) + O(\epsilon^2) = x^\mu(p) + \epsilon X^\mu(p) + O(\epsilon^2), \qquad (4.1)\end{aligned}$$

where $O(\epsilon^2)$ are terms with ϵ to the power 2 or more. The **Lie derivative** of a tensor T along a flow $\sigma_t(p)$ generated by a vector field X is defined as

$$L_X T(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\sigma^*_t T(p)\right). \qquad (4.2)$$

Let now $T \in \mathcal{T}^0_1$, so that T is a smooth function, $T = f$. Then the Lie derivative is

$$\begin{aligned}L_X f(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\sigma^*_t f(p)\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\sigma_t(p)) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(f(\sigma_{t+\epsilon}(p) - f(\sigma_t(p))\right|_{t=0} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(f(\sigma_\epsilon(p) - f(\sigma_0(p))\right) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(f(\sigma_\epsilon(p)) - f(p)\right) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(f(x^\mu(p) + \epsilon X^\mu(p)) - f(x^\mu(p))\right) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(f(x^\mu(p)) + \epsilon X^\mu \frac{\partial}{\partial x^\mu} f(p) + O(\epsilon^2) - f(x^\mu(p))\right) \\ &= X^\mu \frac{\partial}{\partial x^\mu} f(p) = X[f](p). \qquad (4.3)\end{aligned}$$

That is, the Lie derivative of a smooth function is the directional derivative of that function. Another Lie derivative which is useful to us is the Lie derivative of a one-form ω along the vector field X:

$$L_X\omega = (X^\nu\partial_\nu\omega_\mu + \partial_\mu X^\nu\omega_\nu)\mathrm{d}x^\mu. \tag{4.4}$$

Furthermore, the Lie derivative satisfies the Leibniz rule:

$$L_X(t_1 \otimes t_2) = L_X(t_1) \otimes t_2 + t_1 \otimes L_X(t_2), \tag{4.5}$$

where t_1 and t_2 are tensor fields of arbitrary types (see [7], Chapter 5.3). Then, a metric, that is, a type $(0,2)$ tensor $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$, has Lie derivative

$$\begin{aligned} L_X g &= L_X(g_{\mu\nu}\mathrm{d}x^\mu) \otimes \mathrm{d}x^\nu + g_{\mu\nu}\mathrm{d}x^\mu \otimes L_X(\mathrm{d}x^\nu) \\ &= L_X(g_{\mu\nu}\mathrm{d}x^\mu) \otimes \mathrm{d}x^\nu + g_{\mu\nu}\mathrm{d}x^\mu \otimes L_X(\mathrm{d}x^\nu) \\ &\quad + \mathrm{d}x^\mu \otimes L_X(g_{\mu\nu}\mathrm{d}x^\nu) - \mathrm{d}x^\mu \otimes L_X(g_{\mu\nu}\mathrm{d}x^\nu). \end{aligned}$$

Using (4.5) on the last term gives $-L_X(g_{\mu\nu})\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - g_{\mu\nu}\mathrm{d}x^\mu \otimes L_X(\mathrm{d}x^\nu)$. Thus,

$$L_X g = L_X(g_{\mu\nu}\mathrm{d}x^\mu) \otimes \mathrm{d}x^\nu + \mathrm{d}x^\mu \otimes L_X(g_{\mu\nu}\mathrm{d}x^\nu) - L_X(g_{\mu\nu})\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu.$$

Now we are left with the Lie derivative of a smooth function $L_X(g_{\mu\nu})$, and the Lie derivative of two one-forms $L_X(\mathrm{d}x^\mu)$ and $L_X(\mathrm{d}x^\nu)$. Applying (4.3) and (4.4), we get

$$\begin{aligned} L_X g &= (X^\rho\partial_\rho g_{\mu\nu} + \partial_\mu X^\rho g_{\rho\nu})\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + (X^\rho\partial_\rho g_{\mu\nu} + \partial_\nu X^\rho g_{\mu\rho})\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \\ &\qquad - X[g_{\mu\nu}]\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \\ &= (X^\rho\partial_\rho g_{\mu\nu} + \partial_\mu X^\rho g_{\rho\nu} + \partial_\nu X^\rho g_{\rho\mu})\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \end{aligned} \tag{4.6}$$

The final Lie derivative identity we will have use for follows: For a tensor T and two vector fields X and Y, it can be shown that

$$L_{[X,Y]}T = L_X L_Y T - L_Y L_X T. \tag{4.7}$$

Now we are almost ready to introduce Killing vector fields, and to show that these form Lie algebras. The only thing we are missing is the concept of isometries:

Definition 4.8. Let M be a manifold, with metric g on M. Let $p \in M$ be a point on the manifold. An **isometry** is a diffeomorphism $f : M \to M$ which preserves the metric

$$f^* g_{f(p)} = g_p. \tag{4.8}$$

In components, f^* is given by the Jacobian, and (4.8) becomes

$$\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g_{\alpha\beta}\left(f(p)\right) = g_{\mu\nu}(p), \tag{4.9}$$

where x and x' are coordinates of p and $f(p)$, respectively.

Definition 4.9. Let again M be a manifold, with metric g. Let also $X \in \mathfrak{X}(M)$ be a vector field on M. If any set of points are displaced by ϵX, where ϵ is infinitesimal, and the displacement generates an isometry, then X is called a **Killing vector field**.

Let $f : x^\mu \to x^\mu + \epsilon X^\mu$ be an isometry. According to Definition 4.8 f satisfies

$$\begin{aligned} g_{\mu\nu}(x) &= \frac{\partial\left(x^\alpha + \epsilon X^\alpha\right)}{\partial x^\mu}\frac{\partial\left(x^\beta + \epsilon X^\beta\right)}{\partial x^\nu} g_{\alpha\beta}\left(x + \epsilon X\right) \\ &= \left(\delta^\alpha_\mu + \epsilon\partial_\mu X^\alpha\right)\left(\delta^\beta_\nu + \epsilon\partial_\nu X^\beta\right)\left(g_{\alpha\beta}(x) + \epsilon X^\gamma\partial_\gamma g_{\alpha\beta}(x) + O(\epsilon^2)\right) \\ &= g_{\mu\nu}(x) + \epsilon\left(X^\gamma\partial_\gamma g_{\mu\nu}(x) + \partial_\mu X^\alpha g_{\alpha\nu}(x) + \partial_\nu X^\beta g_{\mu\beta}(x)\right) + O(\epsilon^2), \end{aligned} \tag{4.10}$$

where $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$. We find

$$X^\gamma\partial_\gamma g_{\mu\nu} + \partial_\mu X^\alpha g_{\alpha\nu} + \partial_\nu X^\beta g_{\mu\beta} = 0. \tag{4.11}$$

This is the **Killing equation**. We recognise the left hand side of (4.11) as (4.6). Thus we can rewrite (4.11) as

$$(L_X g)_{\mu\nu} = 0. \tag{4.12}$$

Using (4.7) we find that

$$L_{[X,Y]}g = L_X L_Y g - L_Y L_X g = 0. \tag{4.13}$$

Thus, the Lie bracket of any two Killing vector fields, $[X, Y]$, is another Killing vector field. Hence, the Killing vector fields form a Lie algebra.

4.2 The Poincaré Algebra

In this section we will compute the Lie algebra of the Poincaré group by the use of Killing vector fields. The Poincaré group is a Lie group consisting of isometries in Minkowski space-time. In Minkowski space-time we use the Minkowski metric $\eta_{\mu\nu}$, which in p time dimensions and q space dimensions is $\eta_{\mu\nu} = \text{Diag}(\underbrace{-1,\ldots,-1}_{p},\underbrace{+1,\ldots,+1}_{q})$. Choosing $g_{\mu\nu} = \eta_{\mu\nu}$ in (4.11) gives

$$\begin{aligned} 0 &= X^\gamma\partial_\gamma\eta_{\mu\nu} + \partial_\mu X^\alpha\eta_{\alpha\nu} + \partial_\nu X^\beta\eta_{\mu\beta} \\ &= \partial_\mu X_\nu + \partial_\nu X_\mu. \end{aligned} \tag{4.14}$$

Differentiating (4.14) once, and then permuting indices gives

$$\partial_\gamma\partial_\mu X_\nu + \partial_\nu\partial_\gamma X_\mu = 0, \tag{4.15}$$
$$\partial_\mu\partial_\nu X_\gamma + \partial_\gamma\partial_\mu X_\nu = 0, \tag{4.16}$$
$$\partial_\nu\partial_\gamma X_\mu + \partial_\mu\partial_\nu X_\gamma = 0. \tag{4.17}$$

Adding (4.15) and (4.16), and subtracting (4.17) gives

$$\begin{aligned} 0 &= \partial_\gamma\partial_\mu X_\nu + \partial_\nu\partial_\gamma X_\mu + \partial_\mu\partial_\nu X_\gamma + \partial_\gamma\partial_\mu X_\nu - \partial_\nu\partial_\gamma X_\mu - \partial_\mu\partial_\nu X_\gamma \\ &= 2\partial_\gamma\partial_\mu X_\nu. \end{aligned} \tag{4.18}$$

We see that the second derivative vanishes. Thus X_ν must be linear in x^μ

$$X_\mu = a_\mu + b_{\mu\nu}x^\nu. \tag{4.19}$$

Substituting (4.19) in (4.14) gives

$$0 = \partial_\mu(a_\nu + b_{\nu\gamma}x^\gamma) + \partial_\nu(a_\mu + b_{\mu\sigma}x^\sigma) = b_{\nu\mu} + b_{\mu\nu}. \tag{4.20}$$

From (4.20) we find that $b_{\mu\nu}$ is antisymmetric. Then, in n dimensional space time, there are $\frac{1}{2}n(n-1)$ $b_{\mu\nu}$-matrices. In addition, there are n different a_μ-vectors. In total there are $\frac{1}{2}n(n+1)$ independent vector fields X, with components X^μ. We consider first the constant solutions, which takes $b_{\mu\nu} = 0$. Then the vector fields are

$$X = X^\mu\partial_\mu = \eta^{\mu\nu}X_\nu\partial_\mu = \eta^{\mu\nu}a_\nu\partial_\mu. \tag{4.21}$$

Now, take $a_\mu = 0$. We look at one of the antisymmetric vectors. Letting $b_{12} = -b_{21} = 1$, and the other components being zero, we get

$$\begin{aligned} X &= X^\mu\partial_\mu = \eta^{\mu\nu}X_\nu\partial_\mu = \eta^{\mu\nu}b_{\nu\gamma}x^\gamma\partial_\mu = \eta^{11}b_{12}x^2\partial_1 + \eta^{22}b_{21}x^1\partial_2 \\ &= x^2\partial_1 - x^1\partial_2. \end{aligned} \tag{4.22}$$

This is also true for the other choices of the $b_{\mu\nu}$-components. We may lower the indices on x. Thus, the vector fields are

$$X = x_\mu\partial_\nu - x_\nu\partial_\mu. \tag{4.23}$$

We give the two vector fields we found new names. We choose, as is convention, $P_\mu = \partial_\mu$ and $M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$. In Appendix B.1 the commutation relations consisting of P_μ and $M_{\mu\nu}$ has been computed. The non-vanishing commutation relations are

$$[P_\mu, P_\nu] = 0, \tag{4.24}$$
$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu, \tag{4.25}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\rho\nu} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\rho}M_{\sigma\nu}. \tag{4.26}$$

If we do not consider translations, that is, $P_\mu = 0$, then only (4.26) is left. This describes the **Lorentz algebra**.

4.3 The Conformal Algebra

In this section we compute the conformal algebra, which is the Poincaré algebra with additional generators. A conformal transformation is an angle-conserving transformation. Let M be a manifold, and let $\mathfrak{X}(M)$ be the set of vector fields on M. A diffeomorphism $f : M \to M$ is a conformal transformation if it preserves the metric up to a scale

$$f^*g_{f(p)} = \mathrm{e}^{2\sigma(p)}g_{\mu\nu}(p), \tag{4.27}$$

or in components

$$\frac{\partial x'^\alpha}{\partial x\mu}\frac{\partial x'^\beta}{\partial x^\nu}g_{\alpha\beta}\left(f(p)\right) = \mathrm{e}^{2\sigma(p)}g_{\mu\nu}(p). \tag{4.28}$$

where σ is a smooth map $\sigma : M \to M$. Let ϵ be an infinitesimal parameter. Then we can write $\sigma = \frac{1}{2}\epsilon\psi$ for ψ another map $\psi : M \to M$. Now we can rewrite the exponential as

$$\mathrm{e}^{2\sigma} = \mathrm{e}^{\epsilon\psi} = 1 + \epsilon\psi + O(\epsilon^2)$$

Let $X \in \mathfrak{X}(M)$. Then, doing the same calculation as for (4.11), but with the right-hand side of (4.28), we obtain

$$\begin{aligned} &g_{\mu\nu}(x) + \epsilon \left(X^\gamma \partial_\gamma g_{\mu\nu}(x) + \partial_\mu X^\alpha g_{\alpha\nu}(x) + \partial_\nu X^\beta g_{\mu\beta}(x) \right) + O(\epsilon^2) \\ &\quad = \left(1 + \epsilon\psi + O(\epsilon^2)\right) g_{\mu\nu}(x). \end{aligned} \tag{4.29}$$

We can rewrite this as

$$X^\gamma \partial_\gamma g_{\mu\nu} + \partial_\mu X^\alpha g_{\alpha\nu} + \partial_\nu X^\beta g_{\mu\beta} = \psi g_{\mu\nu}. \tag{4.30}$$

We should check if the (4.30), the **conformal Killing equation**, forms a Lie algebra. We again identify the left hand side as the Lie derivative of the metric, but the right hand side is this time non-zero, $(L_X g)_{\mu\nu} = \psi g_{\mu\nu}$. We use (4.7) to see if the conformal Killing vectors form a Lie algebra. ψ is not necessarily the same for all vector fields, so we should write ψ_X for ψ corresponding to the vector field X. Using (4.3), which told us that the Lie derivative of a smooth function was the ordinary directional derivative, we find

$$\begin{aligned} L_{[X,Y]}g &= L_X L_Y g - L_Y L_X g = L_X(\psi_Y g) - L_Y(\psi_X g) \\ &= L_X(\psi_Y)g + \psi_Y L_X g - L_Y(\psi_X)g - \psi_X L_Y g \\ &= (X[\psi_Y] - Y[\psi_X])g + (\psi_Y\psi_X - \psi_X\psi_Y)g =: \psi_{[X,Y]}g \end{aligned}$$

Then $[X, Y]$ is also a conformal Killing vector, because $\psi_{[X,Y]}$ is a smooth function. Thus, the conformal Killing vectors form a Lie algebra.

Now we find an expression for ψ from (4.30) by multiplying with $g^{\mu\nu}$ on both sides.

$$\begin{aligned} m\psi &= X^\gamma g^{\mu\nu}\partial_\gamma g_{\mu\nu} + \partial_\mu X^\alpha g^{\mu\nu} g_{\alpha\nu} + \partial_\nu X^\beta g^{\mu\nu} g_{\mu\beta} \\ &= X^\gamma g^{\mu\nu}\partial_\gamma g_{\mu\nu} + \partial_\mu X^\alpha \delta^\mu_\alpha + \partial_\nu X^\beta \delta^\nu_\beta \\ &= X^\gamma g^{\mu\nu}\partial_\gamma g_{\mu\nu} + \partial_\mu X^\mu + \partial_\nu X^\nu \\ &= X^\gamma g^{\mu\nu}\partial_\gamma g_{\mu\nu} + 2\partial_\mu X^\mu, \end{aligned} \tag{4.31}$$

where $m = \text{dim}M = g^{\mu\nu}g_{\mu\nu}$, and is equal to 4 in Minkowski space time. Thus, (4.30) can be written as

$$X^\gamma \partial_\gamma g_{\mu\nu} + \partial_\mu X^\alpha g_{\alpha\nu} + \partial_\nu X^\beta g_{\mu\beta} = \frac{X^\gamma g^{\rho\sigma}\partial_\gamma g_{\rho\sigma} + 2\partial_\lambda X^\lambda}{m} g_{\mu\nu}. \tag{4.32}$$

We will consider the Minkowski metric, hence we get

$$\partial_\mu X_\nu + \partial_\nu X_\mu = \frac{2}{m}\eta_{\mu\nu}\partial_\lambda X^\lambda. \tag{4.33}$$

Differentiating (4.33) once, and then permuting indices gives

$$\partial_\gamma\partial_\nu X_\mu + \partial_\gamma\partial_\mu X_\nu = \frac{2}{m}\eta_{\mu\nu}\partial_\gamma\partial_\lambda X^\lambda, \tag{4.34}$$

$$\partial_\mu\partial_\gamma X_\nu + \partial_\nu\partial_\gamma X_\mu = \frac{2}{m}\eta_{\gamma\mu}\partial_\nu\partial_\lambda X^\lambda, \tag{4.35}$$

$$\partial_\nu \partial_\mu X_\gamma + \partial_\mu \partial_\nu X_\gamma = \frac{2}{m} \eta_{\nu\gamma} \partial_\mu \partial_\lambda X^\lambda. \tag{4.36}$$

Adding (4.35) and (4.36), and subtracting (4.34) gives

$$\partial_\mu \partial_\nu X_\gamma = \frac{\eta_{\nu\gamma}\partial_\mu + \eta_{\gamma\mu}\partial_\nu - \eta_{\mu\nu}\partial_\gamma}{m} \partial_\lambda X^\lambda. \tag{4.37}$$

Furthermore, acting on (4.33) with $\partial_\mu \partial^\mu = \Box$, and then substitute $\partial_\mu X_\nu$ with (4.33) gives

$$\begin{aligned} \partial_\mu \partial^\mu \partial_\mu X_\nu + \partial_\mu \partial^\mu \partial_\nu X_\mu &= \frac{2}{m} \eta_{\mu\nu} \partial_\mu \partial^\mu \partial_\sigma X^\sigma \\ \Box \partial_\mu X_\nu + \partial_\mu \partial_\nu (\partial^\rho X_\rho) &= \frac{2}{m} \partial_\mu \partial_\nu (\partial_\sigma X^\sigma) \\ m\Box \left(\frac{2}{m} \eta_{\mu\nu} \partial_\lambda X^\lambda - \partial_\nu X_\mu \right) + m\partial_\mu \partial_\nu (\partial^\rho X_\rho) &= 2\partial_\mu \partial_\nu (\partial_\sigma X^\sigma) \end{aligned}$$

Furthermore,

$$\begin{aligned} 0 &= m\Box \left(\frac{2}{m} \eta_{\mu\nu} \partial_\lambda X^\lambda - \partial_\nu X_\mu \right) + (m-2)\partial_\mu \partial_\nu (\partial^\rho X_\rho) \\ &= \Box \left(2\eta_{\mu\nu} \partial_\lambda X^\lambda - m\partial_\nu X_\mu \right) + (m-2)\partial_\mu \partial_\nu (\partial^\rho X_\rho) \\ &= \Box \left(2\eta_{\mu\nu} \partial_\lambda X^\lambda - \eta^{\mu\nu}\eta_{\mu\nu} \partial_\nu X_\mu \right) + (m-2)\partial_\mu \partial_\nu (\partial^\rho X_\rho) \\ &= \Box \left(\eta_{\mu\nu} \partial_\lambda X^\lambda \right) + (m-2)\partial_\mu \partial_\nu (\partial^\rho X_\rho) \\ &= \left(\eta_{\mu\nu}\Box + (m-2)\partial_\mu \partial_\nu \right) (\partial^\rho X_\rho) \\ &= \left(\eta^{\mu\nu}\eta_{\mu\nu}\Box + \eta^{\mu\nu}(m-2)\partial_\mu \partial_\nu \right) (\partial^\rho X_\rho) \\ &= \left(m\Box + (m-2)\partial_\mu \partial^\mu \right) (\partial^\rho X_\rho) \\ &= (m-1)\Box(\partial_\rho X^\rho). \end{aligned} \tag{4.38}$$

From (4.38) we see that X^μ is at most quadratic in x^μ. Therefore, we can write

$$X_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\gamma} x^\nu x^\gamma \tag{4.39}$$

Plugging (4.39) into (4.37) gives

$$\partial_\mu \partial_\nu \left(a_\gamma + b_{\gamma\alpha} x^\alpha + c_{\gamma\alpha\beta} x^\alpha x^\beta \right) = \frac{\eta_{\nu\gamma}\partial_\mu + \eta_{\gamma\mu}\partial_\nu - \eta_{\mu\nu}\partial_\gamma}{m} \partial_\lambda \left(\eta^{\lambda\xi} X_\xi \right). \tag{4.40}$$

The left-hand side of (4.40) is

$$\begin{aligned} \partial_\mu \partial_\nu \left(a_\gamma + b_{\gamma\alpha} x^\alpha + c_{\gamma\alpha\beta} x^\alpha x^\beta \right) &= c_{\gamma\alpha\beta} \left(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu \right) \\ &= c_{\gamma\mu\nu} + c_{\gamma\nu\mu}, \end{aligned} \tag{4.41}$$

and the right-hand side is

$$\begin{aligned} &\frac{\eta_{\nu\gamma}\partial_\mu + \eta_{\gamma\mu}\partial_\nu - \eta_{\mu\nu}\partial_\gamma}{m} \eta^{\lambda\xi} \partial_\lambda \left(a_\xi + b_{\xi\rho} x^\rho + c_{\xi\rho\sigma} x^\rho x^\sigma \right) \\ &= \frac{1}{m} \eta^{\lambda\xi} \{ \eta_{\nu\gamma}\partial_\mu + \eta_{\gamma\mu}\partial_\nu - \eta_{\mu\nu}\partial_\gamma \} \left(b_{\xi\rho} \delta^\rho_\lambda + c_{\xi\rho\sigma} \delta^\rho_\lambda x^\sigma + c_{\xi\rho\sigma} x^\rho \delta^\sigma_\lambda \right) \\ &= \frac{1}{m} \eta^{\lambda\xi} c_{\xi\rho\sigma} \{ \eta_{\nu\gamma} (\delta^\rho_\lambda \delta^\sigma_\mu + \delta^\rho_\mu \delta^\sigma_\lambda) + \eta_{\gamma\mu} (\delta^\rho_\lambda \delta^\sigma_\nu + \delta^\rho_\nu \delta^\sigma_\lambda) - \eta_{\mu\nu} (\delta^\rho_\lambda \delta^\sigma_\gamma + \delta^\rho_\gamma \delta^\sigma_\lambda) \} \\ &= \frac{1}{m} \{ \eta_{\nu\gamma} ({c^\lambda}_{\lambda\mu} + {c^\lambda}_{\mu\lambda}) + \eta_{\gamma\mu} ({c^\lambda}_{\lambda\nu} + {c^\lambda}_{\nu\lambda}) - \eta_{\mu\nu} ({c^\lambda}_{\lambda\gamma} + {c^\lambda}_{\gamma\lambda}) \}. \end{aligned} \tag{4.42}$$

This means that (4.40) is satisfied if $c_{\mu\nu\gamma}$ is antisymmetric in its last two indices.

The constant vector fields are as P_μ in the Poincaré algebra. Next, the term linear in x is considered. Since any tensor can be written as the sum of a symmetric and an antisymmetric tensor, we can write $b_{\mu\nu} = b_{(\mu\nu)} + b_{[\mu\nu]}$. The antisymmetric part corresponds to the Lorentz algebra, found in subsection 4.2. By using (4.33), we can get more information about the symmetric part.

$$\begin{aligned}\partial_\mu(b_{\nu\rho}x^\rho) + \partial_\nu(b_{\mu\sigma}x^\sigma) &= \frac{2}{m}\eta_{\mu\nu}\partial_\lambda\left(\eta^{\lambda\xi}b_{\xi\zeta}x^\zeta\right)\\ b_{\nu\mu} + b_{\mu\nu} &= \frac{2}{m}\eta^{\lambda\xi}b_{\xi\lambda}\eta_{\mu\nu}.\end{aligned}$$

We see that when $b_{\mu\nu}$ is symmetric, it is proportional to the Minowski metric, $b_{(\mu\nu)} \propto \eta_{\mu\nu}$. We can then write $b_{\mu\nu} = \alpha\eta_{\mu\nu} + b_{[\mu\nu]}$, where α is some parameter. The vector fields are given by

$$X^\mu\partial_\mu = \eta^{\mu\nu}X_\nu\partial_\mu = \eta^{\mu\nu}\eta_{\nu\gamma}x^\gamma\partial_\mu = x^\mu\partial_\mu. \tag{4.43}$$

The vector fields of this form are called dilations, $D = x^\mu\partial_\mu$.

Now, only the term quadratic in x is left. Using (4.41) and (4.42) we write

$$c_{\gamma\mu\nu} = \frac{1}{m}\left(\eta_{\nu\gamma}c^\lambda{}_{\lambda\mu} + \eta_{\gamma\mu}c^\lambda{}_{\lambda\nu} - \eta_{\mu\nu}c^\lambda{}_{\lambda\gamma}\right).$$

Then, the vector fields are

$$\begin{aligned}X^\mu\partial_\mu &= \eta^{\mu\gamma}X_\gamma\partial_\mu\\ &= \eta^{\mu\gamma}c_{\gamma\rho\sigma}x^\rho x^\sigma\partial_\mu\\ &= \eta^{\mu\gamma}\left(\eta_{\sigma\gamma}c^\lambda{}_{\lambda\rho} + \eta_{\gamma\rho}c^\lambda{}_{\lambda\sigma} - \eta_{\rho\sigma}c^\lambda{}_{\lambda\gamma}\right)x^\rho x^\sigma\partial_\mu\\ &= \left(\delta^\mu_\sigma c^\lambda{}_{\lambda\rho} + \delta^\mu_\rho c^\lambda{}_{\lambda\sigma} - \eta_{\rho\sigma}c^\lambda{}_\lambda{}^\mu\right)x^\rho x^\sigma\partial_\mu\\ &= c^\lambda{}_{\lambda\mu}x^\mu x^\sigma\partial_\sigma + c^\lambda{}_{\lambda\mu}x^\rho x^\mu\partial_\rho - c^\lambda{}_\lambda{}^\mu x_\sigma x^\sigma\partial_\mu\\ &= c^\lambda{}_\lambda{}^\mu(2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu),\end{aligned} \tag{4.44}$$

where $x^2 = x_\sigma x^\sigma$. These vector fields are called special conformal transformations, and are denoted $K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu$.

There are, in the conformal algebra, m special conformal transformations and 1 dilation vector, in addition to the $\frac{1}{2}m(m+1)$ vectors from the Poincaré algebra. In total there are $\frac{(m+2)(m+1)}{2}$ vector fields in the conformal algebra.

In Appendix B.3 the commutation relations consisting of the generators of the conformal algebra has been computed. The non-vanishing commutation relations are

$$[P_\mu, D] = P_\mu, \tag{4.45}$$
$$[P_\mu, K_\nu] = 2\left(\eta_{\nu\mu}D + M_{\nu\mu}\right), \tag{4.46}$$
$$[D, K_\mu] = K_\mu, \tag{4.47}$$
$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu, \tag{4.48}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\rho\nu} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\rho}M_{\sigma\nu}, \tag{4.49}$$
$$[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu. \tag{4.50}$$

In Appendix B.3 we show that on $\mathbb{R}^{p,q}$ the conformal algebra is isomorphic to $\mathrm{so}(p+1,q+1)$.

Now we have found the two Lie algebras which will be considered in this thesis. In the next section we will get an understanding of the Clifford algebra, which is important in supersymmetry.

5 Real Clifford Algebras

For the Lagrangian of fermionic fields in four dimensional space-time to be invariant under Lorentz transformations we need to include γ-matrices, which generates the Clifford algebra. The **Clifford algebra** $\mathrm{C}\ell(p,q)$ over the field $\mathbb{R}$ is an associative algebra containing the identity element $\mathbb{1}$, defined by

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbb{1}, \tag{5.1}$$

where $\eta_{\mu\nu} = \mathrm{Diag}(\underbrace{-1,\ldots,-1}_{p},\underbrace{+1,\ldots,+1}_{q})$. (5.1) is referred to as the Clifford condition. We define

$$\Sigma_{\mu\nu} := \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{5.2}$$

Now, let us compute the commutator $[\Sigma_{\mu\nu},\Sigma_{\rho\sigma}]$. We begin by considering a part of the commutator

$$\begin{aligned}[\gamma_\mu\gamma_\nu,\Sigma_{\rho\sigma}] &= \gamma_\mu\gamma_\nu\Sigma_{\rho\sigma} - \Sigma_{\rho\sigma}\gamma_\mu\gamma_\nu + \gamma_\mu\Sigma_{\rho\sigma}\gamma_\nu - \gamma_\mu\Sigma_{\rho\sigma}\gamma_\nu \\ &= \gamma_\mu(\gamma_\nu\Sigma_{\rho\sigma} - \Sigma_{\rho\sigma}\gamma_\nu) + (\gamma_\mu\Sigma_{\rho\sigma} - \Sigma_{\rho\sigma}\gamma_\mu)\gamma_\nu \\ &= \gamma_\mu[\gamma_\nu,\Sigma_{\rho\sigma}] + [\gamma_\mu,\Sigma_{\rho\sigma}]\gamma_\nu\end{aligned} \tag{5.3}$$

We compute the $[\gamma_\nu,\Sigma_{\rho\sigma}]$ separately.

$$\begin{aligned}4[\gamma_\mu,\Sigma_{\rho\sigma}] &= 4(\gamma_\mu\Sigma_{\rho\sigma} - \Sigma_{\rho\sigma}\gamma_\mu) \\ &= \gamma_\mu(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho) - (\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\gamma_\mu + \gamma_\rho\gamma_\mu\gamma_\sigma - \gamma_\rho\gamma_\mu\gamma_\sigma + \gamma_\sigma\gamma_\mu\gamma_\rho - \gamma_\sigma\gamma_\mu\gamma_\rho \\ &= (\gamma_\mu\gamma_\rho + \gamma_\rho\gamma_\mu)\gamma_\sigma - (\gamma_\mu\gamma_\sigma + \gamma_\sigma\gamma_\mu)\gamma_\rho - \gamma_\rho(\gamma_\sigma\gamma_\mu + \gamma_\mu\gamma_\sigma) + \gamma_\sigma(\gamma_\rho\gamma_\mu + \gamma_\mu\gamma_\rho) \\ &= 2\eta_{\mu\rho}\gamma_\sigma - 2\eta_{\mu\sigma}\gamma_\rho - 2\gamma_\rho\eta_{\sigma\mu} + 2\gamma_\sigma\eta_{\rho\mu} = 4\eta_{\mu\rho}\gamma_\sigma - 4\eta_{\mu\sigma}\gamma_\rho\end{aligned}$$

Putting this back in (5.3)

$$[\gamma_\mu\gamma_\nu,\Sigma_{\rho\sigma}] = \gamma_\mu(\eta_{\nu\rho}\gamma_\sigma - \eta_{\nu\sigma}\gamma_\rho) + (\eta_{\mu\rho}\gamma_\sigma - \eta_{\mu\sigma}\gamma_\rho)\gamma_\nu$$

Now we are ready to find the full commutator,

$$\begin{aligned}[\Sigma_{\mu\nu},\Sigma_{\rho\sigma}] &= \frac{1}{4}[\gamma_\mu\gamma_\nu,\Sigma_{\rho\sigma}] - (\mu\leftrightarrow\nu) = \frac{1}{4}(\eta_{\nu\rho}\gamma_\mu\gamma_\sigma - \eta_{\nu\sigma}\gamma_\mu\gamma_\rho + \eta_{\mu\rho}\gamma_\sigma\gamma_\nu - \eta_{\mu\sigma}\gamma_\rho\gamma_\nu) - (\mu\leftrightarrow\nu) \\ &= \frac{1}{4}(\eta_{\nu\rho}(\gamma_\mu\gamma_\sigma + \gamma_\sigma\gamma_\mu) - \eta_{\nu\sigma}(\gamma_\mu\gamma_\rho + \gamma_\rho\gamma_\mu) + \eta_{\mu\rho}(\gamma_\sigma\gamma_\nu + \gamma_\nu\gamma_\sigma) - \eta_{\mu\sigma}(\gamma_\rho\gamma_\nu + \gamma_\nu\gamma_\rho)) \\ &= \eta_{\nu\rho}\Sigma_{\mu\sigma} - \eta_{\nu\sigma}\Sigma_{\mu\rho} + \eta_{\mu\rho}\Sigma_{\sigma\nu} - \eta_{\mu\sigma}\Sigma_{\rho\nu}\end{aligned} \tag{5.4}$$

Recalling that the Lorentz algebra is given by

$$[M_{\mu\nu},M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\rho\nu} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\rho}M_{\sigma\nu}, \tag{5.5}$$

we see that (5.4) represent (5.5).

5.1 Spinor Representations

The Clifford algebra can be classified as in Table 1. This has been worked out in Appendix A.1.

$p-q$ mod 8	$\mathrm{C}\ell(p,q)$	N
0,6	$\mathrm{Mat}_N(\mathbb{R})$	$2^{d/2}$
2,4	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-2)/2}$
1,5	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$
3	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$
7	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$

Table 1: Classification of Clifford algebras.

Even though some of these matrices contain complex and quaternionic elements, they should all be thought of as real associative algebras, in the sense of Definition 4.5.

There exists an automorphism $f : \mathrm{C}\ell(p,q) \to \mathrm{C}\ell(p,q)$ defined by $\gamma_\mu \mapsto f(\gamma_\mu) = -\gamma_\mu$, since the Clifford condition does not change: $(-\gamma_\mu)(-\gamma_\nu) + (-\gamma_\nu)(-\gamma_\mu) = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbb{1}$. It is bijective since it is both injective and surjective. We can use $f(\gamma_\mu) = -\gamma_\mu$ to find how f acts on any number of γ-matrices. For example, we need $f(\mathbb{1}) = \mathbb{1}$. We see that we also need $f(\gamma_\mu\gamma_\nu) = \gamma_\mu\gamma_\nu$, for the Clifford condition to be satisfied. We can generalise this to $f(\gamma_{\mu_1}\dots\gamma_{\mu_k}) = (-1)^k\gamma_{\mu_1}\dots\gamma_{\mu_m}$. Then, for even k the γ-matrices are mapped to themselves, while for odd k the γ-matrices are mapped to minus themselves. Thus, the automorphism lets us decompose the Clifford algebra into two subspaces, $\mathrm{C}\ell(p,q) = \mathrm{C}\ell(p,q)^0 \oplus \mathrm{C}\ell(p,q)^1$, where $\mathrm{C}\ell(p,q)^0$ is called the **even** part of the algebra, consisting of an even number of γ-matrices, including the identity, while $\mathrm{C}\ell(p,q)^1$ is called the **odd** part of the algebra, consisting of an odd number of γ-matrices. If we let 0 denote an element in the even part of the algebra, and 1 denote an element in the odd part, the multiplication rules of two elements follows

$$0 \times 0 = 0, \qquad 0 \times 1 = 1, \qquad 1 \times 1 = 0.$$

Hence, only $\mathrm{C}\ell(p,q)^0$ is closed under multiplication, and forms a subalgebra. There are two useful isomorphisms [9],

Lemma 5.1. *The following isomorphisms hold*

$$\mathrm{C}\ell(p,q)^0 \cong \mathrm{C}\ell(p-1,q), \quad p \geqslant 1 \tag{5.6}$$

$$\mathrm{C}\ell(p,q)^0 \cong \mathrm{C}\ell(q-1,p), \quad q \geqslant 1. \tag{5.7}$$

Considering $p-q = 0$ mod 8, then $p-1-q = 7$ mod 8, and $\mathrm{C}\ell(p,q)^0 \cong \mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$. As seen from Lemma 5.1, the dimension changes, $d \to d-1$, hence $N = 2^{(d-1)/2}$ when $p = q$. Going through this for all eight different values for $p-q$ mod 8, we get the classification as in Table 2.
One subgroup of the Clifford algebra is the group consisting of all invertible elements of $\mathrm{C}\ell(p,q)$, written $\mathrm{C}\ell(p,q)^\times$. Since any group needs to contain all

$p-q \bmod 8$	$\mathrm{C}\ell(p,q)^0$	N
1,7	$\mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$
3,5	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$
2,6	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-2)/2}$
4	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-4)/2}$
0	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-2)/2}$

Table 2: Classification of even Clifford algebras.

its elements inverse element, $\mathrm{C}\ell(p,q)^\times$ contains all other subgroups of $\mathrm{C}\ell(p,q)$. One of these is the **Pin group**,

$$\mathrm{Pin}(p,q) := \{v_1 v_2 \dots v_r \mid \eta^{\mu\nu} v_\mu v_\nu = \pm 1 \ , \ v_\mu \in \mathbb{R}^{p,q}\}.$$

The subgroup of even elements of $\mathrm{Pin}(p,q)$ is called the **Spin group**, and it is defined as

$$\mathrm{Spin}(p,q) := \mathrm{Pin}(p,q) \cap \mathrm{C}\ell(p,q)^0 = \{v_1 v_2 \dots v_{2k} \mid \eta^{\mu\nu} v_\mu v_\nu = \pm 1 \ , \ v_\mu \in \mathbb{R}^{p,q}\}.$$

Irreducible representations of the Pin group are called **pinor representations** $\mathscr{P}$, and irreducible representations of the Spin group are called **spinor representations** $\mathscr{S}$.

We will most of the time work in one time dimension and three space dimensions, $p = 1$ and $q = 3$. Then the Clifford algebra is isomorphic to $\mathrm{Mat}_4(\mathbb{R})$, and the even Clifford algebra is isomorphic to $\mathrm{Mat}_2(\mathbb{C})$. $\mathrm{Mat}_4(\mathbb{R})$ are acting on pinors in $\mathbb{R}^4$, while $\mathrm{Mat}_2(\mathbb{C})$ are acting on spinors in $\mathbb{C}^2$. However, $\mathbb{C}^2 \cong \mathbb{R}^4$, so by an abuse of notation, one says that the spinor representation in four-dimensional space-time is $\mathrm{Mat}_4(\mathbb{R})$. Furthermore, the real spinor representations are called **Majorana spinors**, which spinors in four-dimensional space-time are a part of.

5.2 More About the γ-matrices

Following the conventions of [10], we define the totally anitsymmetrised product of γ-matrices as

$$\gamma_{\mu_1\mu_2\dots\mu_n} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\dots\gamma_{\mu_{\sigma(n)}}, \tag{5.8}$$

where the sum is over all the permutations of the set $S_n = \{1, 2, \dots, n\}$. For example, we have $\gamma_{\mu\nu} = 2\Sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ as in (5.2). We may multiply the the totally antisymmetrised product of γ-matrices with a single γ-matrix

$$\begin{aligned}\gamma_{\mu_1\mu_2\dots\mu_n}\gamma_\nu = \gamma_{\mu_1\mu_2\dots\mu_n\nu} + \eta_{\nu\mu_n}\gamma_{\mu_1\mu_2\dots\mu_{n-1}} - \eta_{\nu\mu_{n-1}}\gamma_{\mu_1\mu_2\dots\widehat{\mu_{n-1}}\mu_n} \\ + \dots + (-1)^{n-1}\eta_{\nu\mu_1}\gamma_{\mu_2\mu_3\dots\mu_n}.\end{aligned} \tag{5.9}$$

Indices with a hat on top are omitted. A proof of this identity can be found in Appendix A.2. Now we can find a basis for the Clifford algebra. Obviously we need both $\mathbb{1}$ and γ_μ. Let us see how many other elements we need to create a basis for the Clifford algebra in n dimensions. Let us multiply two elements γ_μ and γ_ν. We use the fact that any matrix can be written as the sum of a symmetric- and an anti-symmetric matrix. Then,

$$\gamma_\mu \gamma_\nu = \gamma_{\mu\nu} + \eta_{\mu\nu} \mathbb{1}$$

We see that we have found another element necessary to complete the basis, namely $\gamma_{\mu\nu}$. Let us again multiply by γ_ρ on the right:

$$\gamma_{\mu\nu}\gamma_\rho = \gamma_{\mu\nu\rho} + \eta_{\rho\nu}\gamma_\mu - \eta_{\rho\mu}\gamma_\nu.$$

Thus, also $\gamma_{\mu\nu\rho}$ is an element in the basis. We can continue this up to there are n subscripts on the γ-matrix, but no more. If ν in (5.9) is equal to one of $\mu_1\mu_2 \dots \mu_n$, then $\gamma_{\mu_1\mu_2\dots\mu_n\nu} = 0$. Thus, a basis for the Clifford algebra is given by

$$\{\mathbb{1} \quad \gamma_\mu \quad \gamma_{\mu_1\mu_2} \quad \dots \quad \gamma_{\mu_1\mu_2\dots\mu_n}\}.$$

We define the **chirality matrix** γ, in $n = p + q$ dimensions, as

$$\gamma = \gamma_1 \cdots \gamma_n = \gamma_{1\dots n}. \tag{5.10}$$

In four-dimensional space time, with $p = 1$ and $q = 3$, γ is often denoted as $\gamma_5 = \gamma_{0123}$.

Proposition 5.2. γ satisfies the following identities:

$$\gamma_\mu \gamma = (-1)^{n-1}\gamma\gamma_\mu, \tag{5.11}$$

$$\gamma^2 = (-1)^{n(n+1)/2-q}. \tag{5.12}$$

In Minkowski space, $\gamma = \gamma_5$ satisfies

$$\gamma_{\mu\nu}\gamma_5 = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}, \tag{5.13}$$

$$\gamma_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5, \tag{5.14}$$

$$\gamma_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}\gamma_5, \tag{5.15}$$

where $\epsilon^{0123} = -\epsilon_{0123} = 1$.

This tells us that in $n = 4$ dimensional space time, with $p = 1$, $q = 3$, γ_5 anti-commutes with γ_μ, and $\gamma_5^2 = (-\mathbb{1})^{4\cdot5/2-3=7} = -\mathbb{1}$. The proposition has been proved in Appendix A.3.

A basis for the four-dimensional Clifford algebra, or in other words, a basis for real 4×4 matrices, is given by

$$\{\mathbb{1} \quad \gamma_\mu \quad \gamma_{\mu\nu} \quad \gamma_{\mu\nu\rho} \quad \gamma_{\mu\nu\rho\sigma}\}.$$

We find how many elements there are of each type:

- $\mathbb{1}$: 1 element: only $\mathbb{1}$,
- γ_μ: 4 elements: $\gamma_0, \gamma_1, \gamma_2, \gamma_3$,
- $\gamma_{\mu\nu}$: 6 elements: $\gamma_{01}, \gamma_{02}, \gamma_{03}, \gamma_{12}, \gamma_{13}, \gamma_{23}$,
- $\gamma_{\mu\nu\rho}$: 4 elements: $\gamma_{012}, \gamma_{013}, \gamma_{023}, \gamma_{123}$,
- $\gamma_{\mu\nu\rho\sigma}$: 1 element: $\gamma_{0123} = \gamma_5$.

In all there are 16 elements.

5.3 The Charge Conjugation Matrix

We continue considering the Clifford algebra in one time dimension and three space dimensions, $C\ell(1,3) \cong \mathrm{Mat}_4(\mathbb{R})$. The following theorem is useful:

Theorem 5.3. *(Skolem-Noether) Let $\mathbb{K}$ be an arbitrary field, and let $\phi : \mathrm{Mat}_n(\mathbb{K}) \to \mathrm{Mat}_n(\mathbb{K})$ be an automorphism. Then, for any $A \in \mathrm{Mat}_n(\mathbb{K})$ there exists $B \in \mathrm{Mat}_n(\mathbb{K})$, and B invertible, such that*

$$\phi(A) = BAB^{-1}.$$

In other words, every automorphism of $\mathrm{Mat}_n(\mathbb{K})$ is inner.

See [11] for a proof. The theorem can be written in a more general way, but the way it is stated above is strong enough for our purposes. The Skolem-Noether Theorem tells us that there is an automorphism defined by $\gamma_\mu \mapsto C\gamma_\mu C^{-1}$, $\gamma_\mu \in C\ell(1,3)$, $C \in \mathrm{Mat}_4(\mathbb{R})$ and C invertible. Another automorphism is given by $\gamma_\mu \mapsto \gamma_\mu^t$, since transposing the Clifford condition is also a Clifford condition

$$\gamma_\nu^t \gamma_\mu^t + \gamma_\mu^t \gamma_\nu^t = 2\eta_{\nu\mu}\mathbb{1}.$$

By the two aforementioned automorphisms there is another automorphism, which gives $\gamma_\mu^t = C\gamma_\mu C^{-1}$. We may change C to $C' := C\gamma_5$. This gives

$$\gamma_\mu^t = C'\gamma_\mu (C')^{-1} = C\gamma_5\gamma_\mu (C\gamma_5)^{-1} = -C\gamma_\mu\gamma_5\gamma_5^{-1}C^{-1} = -C\gamma_\mu C^{-1}. \quad (5.16)$$

Transposing again, we find

$$\begin{aligned}\gamma_\mu &= -(C\gamma_\mu C^{-1})^t = -(C^{-1})^t\gamma_\mu^t C^t = -(C^t)^{-1}(-C\gamma_\mu C^{-1})C^t \\ &= (C^{-1}C^t)^{-1}\gamma_\mu (C^{-1}C^t),\end{aligned}$$

leading to $(C^{-1}C^t)\gamma_\mu = \gamma_\mu(C^{-1}C^t)$. Hence, $(C^{-1}C^t)$ commutes with all elements of $C\ell(1,3)$. The **center** of an algebra is the set of elements that commutes with every element of the algebra. Hence, $(C^{-1}C^t)$ lies in the center of $C\ell(1,3)$. *Schur's lemma* gives a condition to $(C^{-1}C^t)$:

Theorem 5.4. *(Schur's Lemma) Let A be an associative algebra, and let ρ be an irreducible representation of A on a n-dimensional vector space V, that is, ρ is a homomorphism from A into $\mathrm{Mat}_n(V)$ with no non-trivial subspaces. Let $f \in \mathrm{Mat}_n(V)$ commute with $\rho(x)$ for all $x \in A$. Then $f = \lambda\mathbb{1}$, for some $\lambda \in \mathbb{R}$.*

This lemma can, as with the Skolem-Noether theorem, be stated in a more general way, which is not necessary for us. [12] gives a proof for Lie algebras, but it is similar for associative algebras. The Clifford algebra is an associative algebra and it is given by matrices γ_μ. We have shown that there are matrices $(C^{-1}C^t)$ which commute with all γ_μ. Then, by Schur's lemma $(C^{-1}C^t) = \lambda\mathbb{1}$, and $C^t = \lambda C$. Transposing, we get $C = \lambda C^t = \lambda^2 C$, thus $\lambda = \pm 1$. Then

$$C^t = \pm C. \quad (5.17)$$

To determine the sign in (5.17) we use the basis for the four-dimensional Clifford algebra, which is isomorphic to 4×4 real matrices:

$$\{\mathbb{1} \quad \gamma_\mu \quad \gamma_{\mu\nu} \quad \gamma_{\mu\nu\rho} \quad \gamma_{\mu\nu\rho\sigma}\}.$$

A 4×4 matrix has 16 elements, its dimensionality is 16. Any matrix can be written as the sum a symmetric- and an anti-symmetric matrix. A symmetric matrix consists of $n(n+1)/2 = 10$ elements, when $n = 4$. An anti-symmetric matrix consists of $n(n-1)/2 = 6$ elements, when $n = 4$. So the dimensionality of symmetric 4×4 matrices is 10, while that of anti-symmetric matrices is 6. Thus, multiplying C to the basis above should lead to 10 symmetric matrices, and 6 anti-symmetric matrices. By making use of $\gamma_\mu^t = -C\gamma_\mu C^{-1}$, and its generalisation (proof in Appendix A.4):

$$\gamma^t_{\mu_1\mu_2\ldots\mu_n} = (-1)^{n(n+1)/2} C\gamma_{\mu_1\mu_2\ldots\mu_n} C^{-1}, \tag{5.18}$$

$n > 0$, we can determine the sign of C. We find whether $C\gamma_{\mu_1\mu_2\ldots\mu_n}$ for $n = 0, 1, 2, 3, 4$ is symmetric or anti-symmetric:

$$\begin{aligned} C^t &= \pm C, \\ (C\gamma_\mu)^t = \gamma_\mu^t C^t &= -C\gamma_\mu C^{-1} C^t = \mp C\gamma_\mu, \\ (C\gamma_{\mu\nu})^t = \gamma_{\mu\nu}^t C^t &= -C\gamma_{\mu\nu} C^{-1} C^t = \mp C\gamma_{\mu\nu}, \\ (C\gamma_{\mu\nu\rho})^t = \gamma_{\mu\nu\rho}^t C^t &= C\gamma_{\mu\nu\rho} C^{-1} C^t = \pm C\gamma_{\mu\nu\rho}, \\ (C\gamma_{\mu\nu\rho\sigma})^t = \gamma_{\mu\nu\rho\sigma}^t C^t &= C\gamma_{\mu\nu\rho\sigma} C^{-1} C^t = \pm C\gamma_{\mu\nu\rho\sigma}. \end{aligned}$$

If we choose $C^t = +C$, then there will be 6 symmetric, and 10 skew-symmetric, while $C^t = -C$ gives 10 symmetric and 6 skew-symmetric. Hence, the latter option is the correct one. The C satisfying this and (5.16) is called the **charge conjugation matrix**. Introducing spinor indices on a spinor $\psi \in \mathcal{S}$, such that $(\gamma_\mu\psi)^a = (\gamma_\mu)^a{}_b\psi^b$, the charge conjugation matrix becomes

$$C_{ab} = -C_{ba}. \tag{5.19}$$

We can use C to raise and lower spinor indices, where we use the *North-West* and *South-East* conventions,

$$\psi^a = C^{ab}\psi_b \qquad\qquad \psi_a = \psi^b C_{ba}, \tag{5.20}$$

where C^{ab} is the inverse of $-C_{ab}$. After introducing indices, we write the γ-matrices as $(\gamma_\mu)^a{}_b$. When multiplying a γ-matrix with C we employ the following shorthand notation

$$(\gamma_\mu)_{ab} := (C\gamma_\mu)_{ab} = (\gamma_\mu)^c{}_b C_{ca} = -C_{ac}(\gamma_\mu)^c{}_b. \tag{5.21}$$

That is, if both spinor indices are either up or down, the γ-matrix is multiplied with C. With spinor indices we have, from the calculations below (5.18)

$$\begin{aligned} (\gamma_\mu)_{ab} &= (\gamma_\mu)_{ba}, & (\gamma_{\mu\nu\rho})_{ab} &= -(\gamma_{\mu\nu\rho})_{ba}, \\ (\gamma_{\mu\nu})_{ab} &= (\gamma_{\mu\nu})_{ba}, & (\gamma_{\mu\nu\rho\sigma})_{ab} &= -(\gamma_{\mu\nu\rho\sigma})_{ba}. \end{aligned}$$

Other useful identities are computed below.

$$(C\gamma_5)^t = \gamma_5^t C^t = -C\gamma_5 C^{-1} C = -C\gamma_5.$$

With spinor indices:

$$(\gamma_5)_{ab} = -(\gamma_5)_{ba}. \tag{5.22}$$

We will also encounter terms as $\gamma_{\mu_1\ldots\mu_n}\gamma_5$. For $n=1$ we have

$$(C\gamma_\mu\gamma_5)^t = \gamma_5^t\gamma_\mu^t C^t = \gamma_5^t(C\gamma_\mu)^t = \gamma_5^t C\gamma_\mu = -(C\gamma_5)^t\gamma_\mu = C\gamma_5\gamma_\mu = -C\gamma_\mu\gamma_5.$$

With spinor indices:

$$(\gamma_\mu\gamma_5)_{ab} = -(\gamma_\mu\gamma_5)_{ba}. \tag{5.23}$$

For $n=2,3,4$ we use Proposition 5.2:

$$(\gamma_{\mu\nu}\gamma_5)_{ab} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\gamma^{\rho\sigma})_{ab} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\gamma^{\rho\sigma})_{ba} = (\gamma_{\mu\nu}\gamma_5)_{ba}, \tag{5.24}$$

$$\begin{aligned}(\gamma_{\mu\nu\rho}\gamma_5)_{ab} &= \epsilon_{\mu\nu\rho\sigma}(\gamma^\sigma\gamma_5\gamma_5)_{ab} = -\epsilon_{\mu\nu\rho\sigma}(\gamma^\sigma)_{ab} = -\epsilon_{\mu\nu\rho\sigma}(\gamma^\sigma)_{ba} \\ &= \epsilon_{\mu\nu\rho\sigma}(\gamma^\sigma\gamma_5\gamma_5)_{ba} = (\gamma_{\mu\nu\rho}\gamma_5)_{ba}, \end{aligned}\tag{5.25}$$

$$\begin{aligned}(\gamma_{\mu\nu\rho\sigma}\gamma_5)_{ab} &= -\epsilon_{\mu\nu\rho\sigma}(\gamma_5\gamma_5)_{ab} = \epsilon_{\mu\nu\rho\sigma}C_{ab} = -\epsilon_{\mu\nu\rho\sigma}C_{ba} \\ &= \epsilon_{\mu\nu\rho\sigma}(\gamma_5\gamma_5)_{ba} = -(\gamma_{\mu\nu\rho\sigma}\gamma_5)_{ba}. \end{aligned}\tag{5.26}$$

Another combination of gamma-matrices we will encounter is

$$(\gamma_{\mu_1\ldots\mu_m})_a{}^b(\gamma_{\nu_1\ldots\nu_n})_{bc}.$$

We do not yet know what a term like $(\gamma_{\mu_1\ldots\mu_m})_a{}^b$ is. Using (5.20), we get

$$\begin{aligned}(\gamma_{\mu_1\ldots\mu_m})_a{}^b &= (\gamma_{\mu_1\ldots\mu_m})^{cb}C_{ca} = C^{bd}(\gamma_{\mu_1\ldots\mu_m})^c{}_d C_{ca} \\ &= (-C_{ac})(\gamma_{\mu_1\ldots\mu_m})^c{}_d(-C^{db}) = -(C\gamma_{\mu_1\ldots\mu_m}C^{-1})_a{}^b.\end{aligned}$$

The final minus sign comes from C^{db} being the inverse of $-C_{db}$. Then,

$$\begin{aligned}-(C\gamma_{\mu_1\ldots\mu_m}C^{-1})_a{}^b(\gamma_{\nu_1\ldots\nu_n})_{bc} &= -(C\gamma_{\mu_1\ldots\mu_m}C^{-1})_a{}^b(-(\gamma_{\nu_1\ldots\nu_n})^d{}_c C_{bd}) \\ &= (C\gamma_{\mu_1\ldots\mu_m}C^{-1}C)_{ad}(\gamma_{\nu_1\ldots\nu_n})^d{}_c \\ &= (C\gamma_{\mu_1\ldots\mu_m}\gamma_{\nu_1\ldots\nu_n})_{ac}.\end{aligned}$$

Thus,

$$(\gamma_{\mu_1\ldots\mu_m})_a{}^b(\gamma_{\nu_1\ldots\nu_n})_{bc} = (\gamma_{\mu_1\ldots\mu_m}\gamma_{\nu_1\ldots\nu_n})_{ac}. \tag{5.27}$$

Furthermore, when γ-matrices act on a spinor with lowered indices $(\gamma_{\mu_1\ldots\mu_m}\psi)_a$, we get

$$\begin{aligned}(\gamma_{\mu_1\ldots\mu_m}\psi)_a &= (\gamma_{\mu_1\ldots\mu_m}\psi)^b C_{ba} = (\gamma_{\mu_1\ldots\mu_m})^b{}_c\psi^c C_{ba} = (\gamma_{\mu_1\ldots\mu_m})_{ac}C^{cb}\psi_b \\ &= (\gamma_{\mu_1\ldots\mu_m})_a{}^b\psi_b.\end{aligned}\tag{5.28}$$

Two other identities which will be come in handy follows. We use (5.9) to prove them:

$$\begin{aligned}\gamma^\nu\gamma_\mu\gamma_\nu &= (\gamma^\nu{}_\mu + \delta^\nu_\mu)\gamma_\nu = \gamma^\nu{}_{\mu\nu} + \eta_{\nu\mu}\gamma^\nu - \delta^\nu_\nu\gamma_\mu + \gamma_\mu = \gamma_\mu - 4\gamma_\mu + \gamma_\mu \\ &= -2\gamma_\mu,\end{aligned}\tag{5.29}$$

$$\begin{aligned}\gamma^\rho\gamma_{\mu\nu}\gamma_\rho &= (\gamma^\rho{}_{\mu\nu} + \delta^\rho_\mu\gamma_\nu - \delta^\rho_\nu\gamma_\mu)\gamma_\rho \\ &= \eta_{\rho\nu}\gamma^\rho{}_\mu - \eta_{\rho\mu}\gamma^\rho{}_\nu + \delta^\rho_\rho\gamma_{\mu\nu} + \delta^\rho_\mu(\gamma_{\nu\rho} + \eta_{\nu\rho}) - \delta^\rho_\nu(\gamma_{\mu\rho} + \eta_{\mu\rho}) \\ &= \gamma_{\nu\mu} - \gamma_{\mu\nu} + 4\gamma_{\mu\nu} + \gamma_{\nu\mu} - \gamma_{\mu\nu} = 0.\end{aligned}\tag{5.30}$$

In the next section we will finally take use of the algebras we have computed in the previous two sections.

6 Field Theory

In this section we will look into some examples of physical systems, and investigate whether or not they are symmetries under both the Poincaré algebra and the Conformal algebra. At the end of the section we will consider supersymmetry for the first time. However, first we recall some necessary tools of field theory.

6.1 Principle of Least Action

This geometric part in this section has been taken from [7], chapter 7.9.1.

Let M be a m-dimensional manifold with metric $g_{\mu\nu}$. A field is a function which returns a *value* for each point on M. For example, a **scalar field** assigns a scalar to each point on M, and a **vector field** assigns a vector to each point on M. For any field φ and its derivatives, we define a function, the **Lagrangian density** $\mathscr{L}(\varphi, \partial_\mu\varphi, \partial_\mu\partial_\nu\varphi, \dots)$, where $\dots$ denotes more derivatives of the field. For simplicity we will write this only as $\mathscr{L}(\varphi) := \mathscr{L}(\varphi, \partial_\mu\varphi, \partial_\mu\partial_\nu\varphi, \dots)$, or even only $\mathscr{L}$ when it is obvious what fields we are working on. The **action** is a functional, which is a function mapping functions to numbers. We use square brackets around the input of functionals, to distinguish them from functions. The action is defined as an integral of the Lagrangian density over M with metric $g_{\mu\nu}$

$$S[\varphi] := \int_M \mathscr{L}(\varphi)\sqrt{|\det g_{\mu\nu}|}\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m, \tag{6.1}$$

where $\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m$ is an m**-form**, a totally anti-symmetric tensor of type $(0, m)$. $\wedge$ is the **wedge product**. The wedge product is a totally anti-symmetric tensor product between one-forms $\omega \in \mathscr{T}_1{}^0$, defined as

$$\mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_m} = \sum_{\sigma \in S_m} \mathrm{sign}(\sigma)\mathrm{d}x^{\mu_{\sigma(1)}} \wedge \dots \wedge \mathrm{d}x^{\mu_{\sigma(m)}}. \tag{6.2}$$

The sum is over all the permutations of the set $S_m = \{1, 2, \dots, m\}$. Then we have

$$\begin{aligned}
&\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu, \\
&\mathrm{d}x^\mu \wedge \mathrm{d}x^\mu = 0.
\end{aligned}$$

If we use another set of coordinates, $y^1, \dots, y^m$, the volume element becomes

$$\sqrt{\left|\det\left(\frac{\partial x^\mu}{\partial y^\kappa}\frac{\partial x^\nu}{\partial y^\lambda}g_{\mu\nu}\right)\right|}\mathrm{d}y^1 \wedge \dots \wedge \mathrm{d}y^m.$$

We can rewrite $\mathrm{d}y^\mu = \frac{\partial y^\mu}{\partial x^\nu}\mathrm{d}x^\nu$. Then the volume element becomes

$$\begin{aligned}
&\left|\det\left(\frac{\partial x^\mu}{\partial y^\kappa}\right)\right| \sqrt{|\det g_{\mu\nu}|}\det\left(\frac{\partial y^\lambda}{\partial x^\nu}\right)\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m \\
&\qquad\qquad = \pm\sqrt{|\det g_{\mu\nu}|}\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m
\end{aligned}$$

since $\det(AB) = \det A \det B$, and $\det A^{-1} = (\det A)^{-1}$ for two matrices A and B. Thus, when $\det\left(\frac{\partial x^\mu}{\partial y^\kappa}\right) > 0$, the volume element is invariant under the change of coordinates. We will only work in flat space, with $g_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric, which has determinant $\det \eta_{\mu\nu} = -1 \times 1 \times 1 \times 1 = -1$, and then $\sqrt{|\det \eta_{\mu\nu}|} = 1$. Then (6.1) becomes

$$S[\varphi] = \int_M \mathscr{L}(\varphi) \mathrm{d}x^1 \dots \mathrm{d}x^m. \tag{6.3}$$

Since we will almost always integrate over $\mathrm{d}x^1 \dots \mathrm{d}x^m$, we will usually omit writing this part in the future. Let $\delta\varphi$ be an infinitesimal variation of the field φ. The *Taylor-like* expansion of the action is

$$S[\varphi + \delta\varphi] = S[\varphi] + \int \delta\varphi \frac{\delta S[\varphi]}{\delta\varphi} + O((\delta\varphi)^2), \tag{6.4}$$

where $\frac{\delta S[\varphi]}{\delta\varphi}$ is the functional derivative. We find that

$$\int \delta\varphi \frac{\delta S[\varphi]}{\delta\varphi} = \lim_{\epsilon \to 0} \frac{S[\varphi + \epsilon\delta\varphi] - S[\varphi]}{\epsilon}, \tag{6.5}$$

The extrema of the action, when $\int \delta\varphi \frac{\delta S[\varphi]}{\delta\varphi} = 0$, are the equations of motion. This is the **principle of least action**.

We find the equations of motion in terms of the Lagrangian density:

$$\begin{aligned}
0 &= \int \delta\varphi \frac{\delta S[\varphi]}{\delta\varphi} = \lim_{\epsilon \to 0} \frac{S[\varphi + \epsilon\delta\varphi] - S[\varphi]}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(\int \mathscr{L}(\varphi + \epsilon\delta\varphi) - \int \mathscr{L}(\varphi) \Big) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(\int \mathscr{L}(\varphi) + \epsilon \int \frac{\partial \mathscr{L}}{\partial \varphi} \delta\varphi + \epsilon \int \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \\
&\qquad + \epsilon \int \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \varphi)} \delta(\partial_\mu \partial_\nu \varphi) + \dots - \int \mathscr{L}(\varphi) \Big) \qquad (6.6) \\
&= \int \frac{\partial \mathscr{L}}{\partial \varphi} \delta\varphi + \int \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) + \int \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \varphi)} \delta(\partial_\mu \partial_\nu \varphi) + \dots \\
&= \int \frac{\partial \mathscr{L}}{\partial \varphi} \delta\varphi - \int \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta\varphi + \int \partial_\mu \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \varphi)} \delta\varphi + \dots \\
&= \int \delta\varphi \Big(\frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} + \partial_\mu \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \varphi)} + \dots \Big).
\end{aligned}$$

In this calculation we have integrated by parts to get $\delta\varphi$ for each term. For example, for the second term we have

$$\begin{aligned}
\int_M \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) &= \int_M \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta\varphi) \\
&= - \int_M \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi. \qquad (6.7)
\end{aligned}$$

Furthermore, the divergence theorem tells us

$$\int_M \mathrm{d}x^m \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) = \int_{\partial M} \mathrm{d}x^{m-1} n_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta\varphi, \tag{6.8}$$

where n_μ is the outward pointing unit normal vector field on ∂M, which is the boundary of M. Assuming that the field vanishes at the boundary of the manifold, (6.8) also vanishes, and (6.7) becomes

$$\int_M \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) = -\int \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi. \tag{6.9}$$

In the last step of the equation where we find the equations of motion we have moved the $\delta\varphi$ to the left. We then need to be extra careful when considering fermionic fields, which are anti-commuting.

Let us now for convenience define

$$\delta S[\varphi] := \int \delta\varphi \frac{\delta S[\varphi]}{\delta\varphi}. \tag{6.10}$$

We will usually omit writing the field in $\delta S[\varphi]$, writing it only as δS. We will make extensive use of

$$\delta S = \int \delta\varphi \left(\frac{\partial \mathscr{L}}{\partial\varphi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} + \partial_\mu \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \varphi)} + \dots \right), \tag{6.11}$$

where

$$\frac{\partial \mathscr{L}}{\partial\varphi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} + \partial_\mu \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \varphi)} + \dots = 0 \tag{6.12}$$

are the **Euler-Lagrange equations**, which gives the equations of motion.

In the following we will check a few models of physics if they are symmetric under the Poincaré and conformal algebra. The model is symmetric under the algebra if the action vanishes.

6.2 Free Massless Lagrangian

We begin with the simplest model, a free scalar field, $\varphi = \phi$. Scalar fields are bosonic, and they commute with each other. The Lagrangian density of our free scalar field is

$$\mathscr{L}(\phi) = -\frac{1}{2}(\partial\phi)^2, \tag{6.13}$$

where $(\partial\phi)^2 = \partial_\mu \phi \partial^\mu \phi$. We find the equations of motion from (6.11):

$$\delta S = -\int \delta\phi \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = \int \delta\phi \frac{1}{2} \partial_\mu (2\partial^\mu \phi) = \int \delta\phi \partial_\mu \partial^\mu \phi. \tag{6.14}$$

Thus, the equations of motion for the free massless scalar field are, with $\partial_\mu \partial^\mu = \Box$, $\Box\phi = 0$.

We want to know if (6.14) is invariant under the Poincaré algebra. In subsection 4.2 we found that the generators of the Poincaré algebra are given by

$$P_\mu = \partial_\mu, \qquad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu.$$

An obvious guess on how the infinitesimal Poincaré transformations look like is

$$\delta_a \phi = a^\mu P_\mu \phi = a^\mu \partial_\mu \phi, \tag{6.15}$$
$$\delta_b \phi = b^{\mu\nu} M_{\mu\nu} \phi = b^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi, \tag{6.16}$$

for infinitesimal a^μ and $b^{\mu\nu}$. We check if these transformations keeps (6.14) invariant:

$$\delta_a S = \int \delta_a \phi \Box \phi = \int a^\mu \partial_\mu \phi \partial_\nu \partial^\nu \phi = -\int a^\mu \partial_\nu \partial^\nu \phi \partial_\mu \phi = 0,$$

$$\begin{aligned}
\delta_b S &= \int \delta_b \phi \Box \phi = \int b^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi \partial_\rho \partial^\rho \phi \\
&= -\int b^{\mu\nu}(\eta_{\rho\mu}\partial_\nu + x_\mu \partial_\rho \partial_\nu - \eta_{\rho\nu}\partial_\mu - x_\nu \partial_\rho \partial_\mu)\phi \partial^\rho \phi \\
&= \int b^{\mu\nu}\phi(\partial_\mu \partial_\nu + \delta_\mu{}^\rho \partial_\rho \partial_\nu + x_\mu \partial_\rho \partial^\rho \partial_\nu - \partial_\nu \partial_\mu - \delta_\nu{}^\rho \partial_\rho \partial_\mu - x_\nu \partial_\rho \partial^\rho \partial_\mu)\phi \\
&= -\int b^{\mu\nu}(\eta_{\mu\nu} - \eta_{\nu\mu} + x_\mu \partial_\nu - \partial_\nu \partial_\mu)\phi \partial_\rho \partial^\rho \phi = 0.
\end{aligned}$$

This shows that (6.14) is in fact Poincaré invariant.

In subsection 4.3 we found the conformal algebra. Let us check if (6.14) is invariant under conformal transformations as well. In addition to the Poincaré generators we have

$$D = x_\mu \partial_\mu, \qquad K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu.$$

We again try the obvious guess of infinitesimal conformal transformations

$$\delta_c \phi = cD\phi = cx^\mu \partial_\mu \phi, \tag{6.17}$$
$$\delta_d \phi = d^\mu K_\mu \phi = d^\mu(2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu)\phi, \tag{6.18}$$

for infinitesimal c and d^μ. Applying these transformations to (6.14) we get

$$\begin{aligned}
\delta_c S &= \int \delta_c \phi \Box \phi = \int cx^\mu \partial_\mu \phi \partial_\nu \partial^\nu \phi = -\int c(\delta_\nu{}^\mu \partial_\mu + x^\mu \partial_\mu \partial_\nu)\phi \partial^\nu \phi \\
&= \int c\phi(\partial_\nu \partial^\nu + \eta^{\nu\mu}\partial_\mu \partial_\nu + x^\mu \partial_\mu \partial_\nu \partial^\nu)\phi = -\int c(x^\mu \partial_\mu + \delta_\mu{}^\mu - 2)\phi \partial_\nu \partial^\nu \phi \\
&= -\int c(x^\mu \partial_\mu + d - 2)\phi \partial_\nu \partial^\nu \phi \neq 0,
\end{aligned}$$

$$\begin{aligned}
\delta_d S &= \int \delta_d\phi\Box\phi = \int d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu)\phi\partial_\rho\partial^\rho\phi \\
&= -\int d^\mu(2\eta_{\rho\mu}x^\nu\partial_\nu + 2x_\mu\delta^\nu_\rho\partial_\nu + 2x_\mu x^\nu\partial_\nu\partial_\rho - \eta_{\rho\nu}x^\nu\partial_\mu \\
&\quad - x_\nu\delta^\nu_\rho\partial_\mu - x_\nu x^\nu\partial_\mu\partial_\rho)\phi\partial^\rho\phi \\
&= -\int d^\mu(2\eta_{\rho\mu}x^\nu\partial_\nu + 2x_\mu\partial_\rho + 2x_\mu x^\nu\partial_\nu\partial_\rho - 2x_\rho\partial_\mu - x_\nu x^\nu\partial_\mu\partial_\rho)\phi\partial^\rho\phi \\
&= \int d^\mu\phi(2\eta_{\rho\mu}(\eta^{\rho\nu} + x^\nu\partial^\rho)\partial_\nu + 2(\delta^\rho_\mu + x_\mu\partial^\rho)\partial_\rho \\
&\quad + 2(\delta^\rho_\mu x^\nu + x_\mu\eta^{\rho\nu} + x_\mu x^\nu\partial^\rho)\partial_\nu\partial_\rho - 2(\delta^\rho_\mu + x_\rho\partial^\rho)\partial_\mu \\
&\quad - (\delta^\rho_\mu x^\nu + x_\nu\eta^{\rho\nu} + x_\nu x^\nu\partial^\rho)\partial_\mu\partial_\rho)\phi \\
&= \int d^\mu\phi((4-2d)\partial_\mu + 4x_\mu\partial_\rho\partial^\rho + 2x_\mu x^\nu\partial_\nu\partial_\rho\partial^\rho - x_\nu x^\nu\partial_\rho\partial^\rho\partial_\mu)\phi \\
&= \int d^\mu\left((4-2d)\phi\partial_\mu\phi - (2x_\mu x_\nu\partial_\nu - x_\nu x^\nu\partial_\mu - (4-2d)x_\mu)\phi\partial_\rho\partial^\rho\phi\right) \neq 0.
\end{aligned}$$

We have used $\delta^\mu_\mu = d$, d being the dimension of our manifold. Neither $\delta_c S$ nor $\delta_d S$ vanish. However, we notice that if we change each of the transformations (6.17) and (6.18) they might do. If we change the transformation (6.17) to be $\delta_c\phi = c(x^\mu\partial_\mu + \Delta)\phi$, Δ a constant, then $\delta_c S$ becomes

$$\delta_c S = \int c(x^\mu\partial_\mu + \Delta)\phi\partial_\nu\partial^\nu\phi = -\int c(x^\mu\partial_\mu + d - 2 - \Delta)\phi\partial_\nu\partial^\nu\phi.$$

This vanish when $\Delta = d - 2 - \Delta$, hence we get $\Delta = \frac{1}{2}(d-2)$. Similarly, we need a change for $\delta_d S$. Here we have an extra term $(4-2d)x_\mu$. Therefore, it seems we need a constant, κ, multiplied with x_μ. We try the transformation $\delta_d\phi = d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + \kappa x_\mu)\phi$. When we integrate by parts, the new term gives

$$\begin{aligned}
\int \kappa x_\mu\phi\partial_\rho\partial^\rho\phi &= -\int \kappa(\eta_{\rho\mu} + x_\mu\partial_\rho)\phi\partial^\rho\phi = \int \kappa(\eta_{\rho\mu}\partial^\rho + \delta_\mu{}^\rho\partial_\rho + x_\mu\partial_\rho\partial^\rho)\phi\phi \\
&= \int \kappa\phi(2\partial_\mu + x_\mu\partial_\rho\partial^\rho)\phi.
\end{aligned}$$

Plugging this into $\delta_d S$ above gives

$$\begin{aligned}
\delta_d S &= \int d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + \kappa x_\mu)\phi\partial_\rho\partial^\rho\phi \\
&= \int d^\mu\left((4-2d+2\kappa)\phi\partial_\mu\phi - (2x_\mu x_\nu\partial_\nu - x_\nu x^\nu\partial_\mu - (4-2d+\kappa)x_\mu)\phi\partial_\rho\partial^\rho\phi\right).
\end{aligned}$$

We need $4-2d+2\kappa = 0$ and $\kappa = -(4-2d+\kappa)$. These two expressions are the same, and yield $\kappa = d-2 = 2\Delta$.

We have found that the Lagrangian density $\mathscr{L}(\phi) = -\frac{1}{2}(\partial\phi)^2$ is invariant under the infinitesimal Poincaré transformations

$$\delta_a\phi = a^\mu P_\mu\phi = a^\mu\partial_\mu\phi, \tag{6.19}$$

$$\delta_b\phi = b^{\mu\nu}M_{\mu\nu}\phi = b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi, \tag{6.20}$$

and under the infinitesimal conformal transformations

$$\delta_c\phi = cD\phi = c(x^\mu\partial_\mu + \Delta)\phi, \tag{6.21}$$
$$\delta_d\phi = d^\mu K_\mu\phi = d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2\Delta x_\mu)\phi, \tag{6.22}$$

with $\Delta = \frac{1}{2}(d-2)$. In four dimensional space time, $d=4$, $\Delta = 1$.

Another type of field which will be considered is a fermionic spinor field, $\varphi = \psi$, in four dimensional space-time. We recall that we will then consider Majorana spinors, which in this case are 4×4-real matrices. We will always work in four dimensional space-time when considering fermionic fields, except when otherwise stated. Fermionic fields anti-commute, for two fermionic fields ψ_1, ψ_2 we have $\psi_1\psi_2 = -\psi_2\psi_1$. The Lagrangian for our fermionic field is

$$\mathscr{L}(\psi) = \frac{1}{2}\bar{\psi}\not{\partial}\psi = \frac{1}{2}\psi^a(\gamma^\mu)_{ab}\partial_\mu\psi^b, \tag{6.23}$$

where $\bar{\psi}$ is the **Majorana conjugate** of ψ, $\bar{\psi} = \psi^t C$, and $\not{\partial} = \gamma^\mu\partial_\mu$. The equations of motion for $\bar{\psi}$ are

$$\begin{aligned}\delta S &= \int \delta\psi^a\left(\frac{\partial\mathscr{L}}{\partial\psi^a} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^a)}\right) + \delta\psi^b\left(\frac{\partial\mathscr{L}}{\partial\psi^b} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^b)}\right)\\ &= \int \delta\psi^a\frac{\partial\mathscr{L}}{\partial\psi^a} - \delta\psi^b\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^b)} = \frac{1}{2}\int \delta\psi^a(\gamma^\mu)_{ab}\partial_\mu\psi^b + \delta\psi^b\partial_\mu\psi^a(\gamma^\mu)_{ab}\\ &= \frac{1}{2}\int \delta\psi^a(\gamma^\mu)_{ab}\partial_\mu\psi^b + \delta\psi^a\partial_\mu\psi^b(\gamma^\mu)_{ba} = \int \delta\psi^a(\gamma^\mu)_{ab}\partial_\mu\psi^b\\ &= \int \delta\bar{\psi}\not{\partial}\psi.\end{aligned} \tag{6.24}$$

We get a sign change from $\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^b)}$ since *fermionic derivatives* anitcommute with fermionic fields, $\frac{\partial}{\partial\psi^b}(\psi^a\psi^b) = \frac{\partial}{\partial\psi^b}(\psi^a)\psi^b - \psi^a\frac{\partial}{\partial\psi^b}(\psi^b)$. We investigate for Poincaré invarinace, and try the infinitesimal Poincaré transformations similar to the transformations for the bosonic scalar field:

$$\delta_a\psi = a^\mu P_\mu\psi = a^\mu\partial_\mu\psi, \tag{6.25}$$
$$\delta_b\psi = b^{\mu\nu}M_{\mu\nu}\psi = b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\psi. \tag{6.26}$$

We need the transformations on $\bar{\psi}$. However, in this case the Majorana conjugate only acst on ψ, so the transformations become

$$\delta_a\bar{\psi} = a^\mu P_\mu\bar{\psi} = a^\mu\partial_\mu\bar{\psi}, \tag{6.27}$$
$$\delta_b\bar{\psi} = b^{\mu\nu}M_{\mu\nu}\bar{\psi} = b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\bar{\psi}. \tag{6.28}$$

Then we find

$$\begin{aligned}\delta_a S &= \int \delta_a\bar{\psi}\not{\partial}\psi = \int a^\mu\partial_\mu\psi^a(\gamma^\rho)_{ab}\partial_\rho\psi^b = \int a^\mu\partial_\rho\psi^a(\gamma^\rho)_{ab}\partial_\mu\psi^b\\ &= -\int a^\mu\partial_\mu\psi^b(\gamma^\rho)_{ab}\partial_\rho\psi^a = -\int a^\mu\partial_\mu\psi^a(\gamma^\rho)_{ab}\partial_\rho\psi^b = 0\end{aligned}$$

$$\begin{aligned}
\delta_b S &= \int \delta_b \bar{\psi} \partial\!\!\!/ \psi = \int b^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\bar{\psi}\partial\!\!\!/\psi = \int b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\psi^a(\gamma^\rho)_{ab}\partial_\rho\psi^b \\
&= -\int b^{\mu\nu}\left((\eta_{\mu\nu} - \eta_{\nu\mu})\psi^a(\gamma^\rho)_{ab}\partial_\rho\psi^b + \psi^a(\gamma^\rho)_{ab}(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho\psi^b\right) \\
&= \int b^{\mu\nu}\left(\partial_\rho\psi^a(\gamma^\rho)_{ab}(x_\mu\partial_\nu - x_\nu\partial_\mu)\psi^b + \psi^a(\gamma^\rho)_{ab}(\eta_{\rho\mu}\partial_\nu - \eta_{\rho\nu}\partial_\mu)\psi^b\right) \\
&= -\int b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\psi^b(\gamma^\rho)_{ab}\partial_\rho\psi^a - 2\psi^a(\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b\right) \\
&= -\int b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\psi^a(\gamma^\rho)_{ab}\partial_\rho\psi^b - 2\psi^a(\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b\right) \neq 0. \qquad (6.29)
\end{aligned}$$

It turns out (6.23) is not invariant under (6.28). We try to continue our success of adding terms to our initial guess of transformations. In this case we need something which obeys the Lorentz algebra. In section 5 we found that $\Sigma_{\mu\nu} = \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = \frac{1}{2}\gamma_{\mu\nu}$ do obey the Lorentz algebra, so we try the following transformation:

$$\delta_b\psi = b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}\right)\psi. \qquad (6.30)$$

The transformation for $\bar{\psi}$ is then

$$\delta_b\bar{\psi} = b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\bar{\psi} + \overline{\Sigma_{\mu\nu}\psi}\right), \qquad (6.31)$$

where

$$\overline{\Sigma_{\mu\nu}\psi} = (\Sigma_{\mu\nu}\psi)^t C = \psi^t\Sigma^t_{\mu\nu}C.$$

Furthermore,

$$\Sigma^t_{\mu\nu}C = -\frac{1}{2}\gamma^t_{\mu\nu}C^t = -\frac{1}{2}(C\gamma_{\mu\nu})^t = -\frac{1}{2}C\gamma_{\mu\nu} = -C\Sigma_{\mu\nu} = C\Sigma_{\nu\mu},$$

where we have used $(C\gamma_{\mu\nu})^t = C\gamma_{\mu\nu}$. Then (6.31) is

$$\delta_b\bar{\psi} = b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\bar{\psi} + \bar{\psi}\Sigma_{\nu\mu}\right). \qquad (6.32)$$

We have

$$\int b^{\mu\nu}\bar{\psi}\Sigma_{\nu\mu}\partial\!\!\!/\psi = \int b^{\mu\nu}\psi^a(\Sigma_{\nu\mu}\gamma^\rho)_{ab}\partial_\rho\psi^b. \qquad (6.33)$$

We can write

$$\begin{aligned}
\Sigma_{\nu\mu}\gamma^\rho &= \frac{1}{2}\gamma_{\nu\mu}\gamma^\rho = \frac{1}{2}\eta^{\rho\sigma}\gamma_{\nu\mu}\gamma_\sigma = \frac{1}{2}\eta^{\rho\sigma}(\gamma_{\nu\mu\sigma} + \eta_{\sigma\mu}\gamma_\nu - \eta_{\sigma\nu}\gamma_\mu) \\
&= \frac{1}{2}\left(\gamma_{\nu\mu}{}^\rho + \delta^\rho_\mu\gamma_\nu - \delta^\rho_\nu\gamma_\mu\right).
\end{aligned}$$

Plugging this back in (6.33), we get

$$\int b^{\mu\nu}\psi^a(\Sigma_{\nu\mu}\gamma^\rho)_{ab}\partial_\rho\psi^b = \frac{1}{2}\int b^{\mu\nu}\psi^a\left(\gamma_{\nu\mu}{}^\rho + \delta^\rho_\mu\gamma_\nu - \delta^\rho_\nu\gamma_\mu\right)_{ab}\partial_\rho\psi^b.$$

Since the equations below (5.18) tells us that $(\gamma_{\mu\nu\rho})_{ab} = -(\gamma_{\mu\nu\rho})_{ba}$, the first term in the equation above vanishes:

$$\begin{aligned}\int b^{\mu\nu}\psi^a(\gamma_{\nu\mu}{}^{\rho})_{ab}\partial_\rho\psi^b &= -\int b^{\mu\nu}\partial_\rho\psi^a(\gamma_{\nu\mu}{}^{\rho})_{ab}\psi^b = \int b^{\mu\nu}\psi^b(\gamma_{\nu\mu}{}^{\rho})_{ab}\partial_\rho\psi^a \\ &= \int b^{\mu\nu}\psi^a(\gamma_{\nu\mu}{}^{\rho})_{ba}\partial_\rho\psi^b = -\int b^{\mu\nu}\psi^a(\gamma_{\nu\mu}{}^{\rho})_{ab}\partial_\rho\psi^b.\end{aligned}$$

Then we have

$$\begin{aligned}\int b^{\mu\nu}\bar{\psi}\Sigma_{\nu\mu}\partial\!\!\!/\psi &= \frac{1}{2}\int b^{\mu\nu}\psi^a\left(\delta^\rho_\mu\gamma_\nu - \delta^\rho_\nu\gamma_\mu\right)_{ab}\partial_\rho\psi^b, \\ &= \int b^{\mu\nu}\psi^a(\gamma_{[\nu})_{ab}\partial_{\mu]}\psi^b. \end{aligned} \tag{6.34}$$

Now, using the transformation (6.31),the varying action in (6.29) and (6.34), the action becomes

$$\begin{aligned}\delta_b S &= \int b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\bar{\psi} + \bar{\psi}\Sigma_{\nu\mu}\right)\partial\!\!\!/\psi \\ &= -\int b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\psi^a(\gamma^\rho)_{ab}\partial_\rho\psi^b - 2\psi^a(\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b - \psi^a(\gamma_{[\nu})_{ab}\partial_{\mu]}\psi^b\right) \\ &= -\int b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\psi^a(\gamma^\rho)_{ab}\partial_\rho\psi^b + \psi^a(\gamma_{[\nu})_{ab}\partial_{\mu]}\psi^b\right) \\ &= -\int b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\bar{\psi} + \bar{\psi}\Sigma_{\nu\mu}\right)\partial\!\!\!/\psi.\end{aligned}$$

Then the $\delta_b S$ vanishes, as desired. We have found that the Lagrangian (6.23) is invariant under the following infinitesimal Poincaré transformations:

$$\delta_a\psi = a^\mu P_\mu\psi = a^\mu\partial_\mu\psi, \tag{6.35}$$

$$\delta_b\psi = b^{\mu\nu}\left(M_{\mu\nu} + \Sigma_{\mu\nu}\right)\psi = b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}\right)\psi. \tag{6.36}$$

Let us now investigate whether or not (6.23) is invariant under conformal transformations. We try the following transformations:

$$\delta_c\psi = cD\psi = cx^\mu\partial_\mu\psi, \tag{6.37}$$

$$\delta_d\psi = d^\mu K_\mu\psi = d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu)\psi, \tag{6.38}$$

(6.37) and (6.38) applied to the varying action is

$$\begin{aligned}\delta_c S &= \int \delta_c\bar{\psi}\partial\!\!\!/\psi = \int cx^\mu\partial_\mu\psi^a(\gamma^\rho)_{ab}\partial_\rho\psi^b = -\int c(\delta^\mu_\rho + x^\mu\partial_\rho)\partial_\mu\psi^a(\gamma^\rho)_{ab}\psi^b \\ &= -\int c\partial_\rho\psi^a(\gamma^\rho)_{ab}\psi^b + \int c\left(\delta^\mu_\mu\partial_\rho\psi^a(\gamma^\rho)_{ab}\psi^b + x^\mu\partial_\rho\psi^a(\gamma^\rho)_{ab}\partial_\mu\psi^b\right) \\ &= \int c\left((d-1)\partial_\rho\psi^a(\gamma^\rho)_{ab}\psi^b + x^\mu\partial_\rho\psi^a(\gamma^\rho)_{ab}\partial_\mu\psi^b\right) \\ &= -\int c\left((d-1)\psi^b(\gamma^\rho)_{ab}\partial_\rho\psi^a + x^\mu\partial_\mu\psi^b(\gamma^\rho)_{ab}\partial_\rho\psi^a\right) \\ &= -\int c\left(x^\mu\partial_\mu + (d-1)\right)\bar{\psi}\partial\!\!\!/\psi \neq 0,\end{aligned}$$

$$
\begin{aligned}
\delta_d S &= \int \delta_d \bar{\psi} \partial\!\!\!/ \psi = \int d^\mu (2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu) \psi^a (\gamma^\rho)_{ab} \partial_\rho \psi^b \\
&= -\int d^\mu \Big(2\eta_{\rho\mu} x^\nu \partial_\nu + 2x_\mu \delta^\nu_\rho \partial_\nu + 2x_\mu x^\nu \partial_\nu \partial_\rho - \eta_{\rho\nu} x^\nu \partial_\mu \\
&\qquad - x_\nu \delta^\nu_\rho \partial_\mu - x_\nu x^\nu \partial_\mu \partial_\rho \Big) \psi^a (\gamma^\rho)_{ab} \psi^b \\
&= -\int d^\mu \left(2\eta_{\rho\mu} x^\nu \partial_\nu + 2x_\mu \delta^\nu_\rho \partial_\nu - \eta_{\rho\nu} x^\nu \partial_\mu - x_\nu \delta^\nu_\rho \partial_\mu \right) \psi^a (\gamma^\rho)_{ab} \psi^b \\
&\quad + \int d^\mu \left(2\eta_{\nu\mu} x^\nu \partial_\rho + 2x_\mu \delta^\nu_\nu \partial_\rho - \eta_{\mu\nu} x^\nu \partial_\rho - x_\nu \delta^\nu_\mu \partial_\rho \right) \psi^a (\gamma^\rho)_{ab} \psi^b \\
&\quad + \int d^\mu \partial_\rho \psi^a \left(2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu \right) \psi^b \\
&= -\int d^\mu (2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu) \psi^b (\gamma^\rho)_{ab} \partial_\rho \psi^a \\
&\quad - \int d^\mu \Big(2x^\nu \partial_\nu \psi^a (\gamma_\mu)_{ab} + 2x_\mu \partial_\nu \psi^a (\gamma^\nu)_{ab} - x^\nu \partial_\mu \psi^a (\gamma_\nu)_{ab} \\
&\qquad\qquad - x_\nu \partial_\mu \psi^a (\gamma^\nu)_{ab} - 2d x_\mu \partial_\nu \psi^a (\gamma^\nu)_{ab} \Big) \psi^b \\
&= -\delta_d S - \int d^\mu \left(2(1-d) x_\mu \partial_\nu \psi^a (\gamma^\nu)_{ab} + 2x^\nu (\partial_\nu \psi^a (\gamma_\mu)_{ab} - \partial_\mu \psi^a (\gamma_\nu)_{ab}) \right) \psi^b \\
&= -\delta_d S + \int d^\mu \left(2(1-d) \eta_{\mu\nu} \psi^a (\gamma^\nu)_{ab} + 2d \psi^a (\gamma_\mu)_{ab} - 2\delta^\nu_\mu \psi^a (\gamma_\nu)_{ab} \right) \psi^b \\
&\quad + \int d^\mu \left(2(1-d) x_\mu \psi^a (\gamma^\nu)_{ab} \partial_\nu + 2x^\nu (\psi^a (\gamma_\mu)_{ab} \partial_\nu - \psi^a (\gamma_\nu)_{ab} \partial_\mu) \right) \psi^b \\
&= -\delta_d S + \int d^\mu \left(2(1-d) x_\mu \bar{\psi} \partial\!\!\!/ \psi + 4x^\nu \psi^a (\gamma_{[\mu})_{ab} \partial_{\nu]} \psi^b \right) \\
&= -\int d^\mu \left((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu) \bar{\psi} \partial\!\!\!/ \psi - 2(1-d) x_\mu \bar{\psi} \partial\!\!\!/ \psi - 4x^\nu \psi^a (\gamma_{[\mu})_{ab} \partial_{\nu]} \psi^b \right) \neq 0.
\end{aligned}
$$

We will again try to add extra terms so that $\delta_c S$ and $\delta_d S$ might vanish. We try $\delta_c \psi = c(x^\mu \partial_\mu + \Delta')\psi$, Δ' being some constant. Since we do not know what Δ' is yet, we write the transformation for $\bar{\psi}$ as $\delta_c \bar{\psi} = c(x^\mu \partial_\mu \bar{\psi} + \overline{\Delta' \psi})$. Then

$$
\begin{aligned}
\delta_c S &= \int c(x^\mu \partial_\mu \bar{\psi} + \overline{\Delta' \psi}) \partial\!\!\!/ \psi = -\int c \left((x^\mu \partial_\mu + d - 1)\bar{\psi} - \overline{\Delta' \psi} \right) \partial\!\!\!/ \psi \\
&= -\int c \left(x^\mu \partial_\mu \bar{\psi} + \overline{(d - 1 - \Delta')\psi} \right) \partial\!\!\!/ \psi.
\end{aligned}
$$

For this to vanish we need $\Delta' = d - 1 - \Delta'$, and then $\Delta' = \frac{1}{2}(d-1)$. We do similarly for $\delta_d \psi$, but for this we need two terms, one multiplied with x_μ and one with x^ν, $\delta_d \psi = d^\mu \left(2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu + x_\mu \kappa'_1 + x^\nu \kappa'_2 \right) \psi$, with κ'_1, κ'_2 being some constants. The transformation for $\bar{\psi}$ is

$$
\delta_d \bar{\psi} = d^\mu \left((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu) \bar{\psi} + x_\mu \overline{\kappa'_1 \psi} + x^\nu \overline{\kappa'_2 \psi} \right).
$$

Then

$$\begin{aligned}
\delta_d S &= \int d^\mu \left((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu)\bar{\psi} + x_\mu \overline{\kappa_1' \psi} + x^\nu \overline{\kappa_2' \psi} \right) \partial\!\!\!/ \psi \\
&= -\int d^\mu \Big((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu)\bar{\psi}\partial\!\!\!/\psi - 2(1-d)x_\mu \bar{\psi}\partial\!\!\!/\psi - 4x^\nu \psi^a (\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b \\
&\qquad - x_\mu \overline{\kappa_1'\psi}\partial\!\!\!/\psi - x^\nu \overline{\kappa_2'\psi}\partial\!\!\!/\psi \Big) \\
&= -\int d^\mu \Big((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu)\bar{\psi}\partial\!\!\!/\psi - x_\mu \overline{(2(1-d) + \kappa_1')\psi}\partial\!\!\!/\psi \\
&\qquad - 4x^\nu \psi^a (\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b - x^\nu \overline{\kappa_2'\psi}\partial\!\!\!/\psi \Big)
\end{aligned}$$

We need $\kappa_1' = -(2(1-d) + \kappa_1')$, hence $\kappa_1' = d - 1 = 2\Delta'$. For κ_2' we use (6.34). We change $b^{\mu\nu}$ with d^μ, and the extra x^ν makes no difference, since the additional term turns out to be zero. Then we have

$$\int d^\mu x^\nu \overline{\Sigma_{\mu\nu}\psi}\partial\!\!\!/\psi = \int d^\mu x^\nu \psi^a (\gamma_{[\nu})_{ab}\partial_{\mu]}\psi^b. \tag{6.39}$$

From (6.39) we get that

$$\int d^\mu x^\nu \psi^a (\gamma_{[\mu})_{ab}\partial_{\nu]}\psi^b = \frac{1}{2}\int d^\mu x^\nu \overline{\gamma_{\nu\mu}\psi}\partial\!\!\!/\psi$$

Then

$$\begin{aligned}
\delta_d S &= \int d^\mu \left((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu)\bar{\psi} + x_\mu \overline{\kappa_1' \psi} + x^\nu \overline{\kappa_2' \psi} \right) \partial\!\!\!/ \psi \\
&= -\int d^\mu \Big((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu)\bar{\psi}\partial\!\!\!/\psi - x_\mu \overline{(2(1-d) + \kappa_1')\psi}\partial\!\!\!/\psi \\
&\qquad - 2x^\nu \overline{\gamma_{\nu\mu}\psi}\partial\!\!\!/\psi - x^\nu \overline{\kappa_2'\psi}\partial\!\!\!/\psi \Big) \\
&= -\int d^\mu \Big((2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu)\bar{\psi}\partial\!\!\!/\psi - x_\mu \overline{(2(1-d) + \kappa_1')\psi}\partial\!\!\!/\psi \\
&\qquad - x^\nu \overline{(2\gamma_{\nu\mu} + \kappa_2')\psi}\partial\!\!\!/\psi \Big).
\end{aligned}$$

Hence, we need $\kappa_2' = -(2\gamma_{\nu\mu} + \kappa_2')$, and $\kappa_2' = -\gamma_{\nu\mu}$. The transformations we have found is only valid in four-dimensional space-time, $d = 4$. Thus, the Lagrangian (6.23),

$$\mathscr{L}(\psi) = \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi = \frac{1}{2}\psi^a (\gamma^\mu)_{ab}\partial_\mu \psi^b,$$

is invariant under the infinitesimal conformal transformations

$$\delta_c \psi = c(D + \Delta')\psi = c(x^\mu \partial_\mu + \frac{3}{2})\psi, \tag{6.40}$$

$$\begin{aligned}
\delta_d \psi &= d^\mu (K + 2x_\mu \Delta' + x^\nu \kappa')\psi \\
&= d^\mu \left(2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu + 3x_\mu - x^\nu \gamma_{\nu\mu}\right)\psi.
\end{aligned} \tag{6.41}$$

6.3 Massive Lagrangian

To the free massless Lagrangian we can add a mass term, such that the Lagrangian density becomes

$$\mathscr{L}(\phi) = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{6.42}$$

We use (6.11) to find δS,

$$\delta S = \int \delta\phi \left(\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right) = \int \delta\phi \left(\partial_\mu \partial^\mu - m^2\right)\phi.$$

Thus, the equations of motion are

$$\left(\Box - m^2\right)\phi = 0. \tag{6.43}$$

This is known as the the *Klein-Gordon equation*. The equations of motion of the free massless Lagrangian found above, $\Box\phi = 0$ is a special case of the Klein-Gordon equation, with mass $m = 0$.

We want to investigate whether or not the Lagrangian (6.42) is invariant under Poincaré transformations. We recall that the infinitesimal Poincaré transformation are

$$\delta_a \phi = a^\mu P_\mu \phi = a^\mu \partial_\mu \phi, \tag{6.44}$$

$$\delta_b \phi = b^{\mu\nu} M_{\mu\nu} \phi = b^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi. \tag{6.45}$$

Only the mass term in δS needs to be considered, since we have already shown that the kinetic part is invariant in subsection 6.2. Then

$$\delta_a S = \int a^\mu \partial_\mu \phi m^2 \phi = -\int a^\mu \phi m^2 \partial_\mu \phi = 0,$$

$$\begin{aligned}\delta_b S &= \int b^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi m^2 \phi = -\int b^{\mu\nu}(\eta_{\mu\nu} + x_\mu \partial_\nu - \eta_{\nu\mu} - x_\nu \partial_\mu)\phi m^2 \phi \\ &= -\int b^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi m^2 \phi = 0.\end{aligned}$$

Hence, the massive Lagrangian (6.42) is Poincaré invariant.

The infinitesimal conformal transformations which kept the free Lagrangian invariant were given by

$$\delta_c \phi = cD\phi = c(x^\mu \partial_\mu + \Delta)\phi, \tag{6.46}$$

$$\delta_d \phi = d^\mu K_\mu \phi = d^\mu(2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu + 2\Delta)\phi, \tag{6.47}$$

with $\Delta = \frac{1}{2}(d-2)$. We investigate whether or not these transformations keeps the massive Lagrangian invariant, so that it becomes invariant under conformal transformations.

$$\begin{aligned}\delta_c S &= \int c(x^\mu \partial_\mu + \Delta)\phi m^2 \phi \\ &= \int c\left(\partial_\mu(x^\mu \phi m^2 \phi) - \partial_\mu(x^\mu)\phi m^2 \phi - x^\mu \phi m^2 \partial_\mu \phi + \Delta \phi m^2 \phi\right) \\ &= -\int c\left(\delta^\mu_\mu \phi m^2 \phi + x^\mu \phi m^2 \partial_\mu \phi - \Delta \phi m^2 \phi\right) = -\int c\left(x^\mu \partial_\mu - \Delta + \delta^\mu_\mu\right)\phi m^2 \phi.\end{aligned}$$

Then $\delta_c S = 0$ if $\Delta = -\Delta + \delta^\mu_\mu$, so that $\Delta = \frac{1}{2}d \neq \frac{1}{2}(d-2)$. Hence, the massive Lagrangian is only invariant under the conformal algebra if there is no kinetic term present in the Lagrangian. Since the action is not invariant to the dilation transformations $\delta_c\phi$, it follows that the action is not invariant to special conformal transformations $\delta_d\phi$ either. This is due to the commutation relation (4.46), which states

$$[P_\mu, K_\nu] = 2\left(\eta_{\nu\mu}D + M_{\nu\mu}\right).$$

Since the dilations do not keep the Lagrangian invariant, the right hand side is not invariant. Then, neither can the left hand side be. The translations have already been found to be an invariant, thus the special conformal transformations cannot be an invariant.

Let us do the same calculations for the spinor field, ψ. Adding a mass term to the Lagrangian, it becomes

$$\mathscr{L}(\psi) = \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}m\bar{\psi}\psi = \frac{1}{2}\psi^a(\gamma^\mu)_{ab}\psi^b + \frac{1}{2}m\psi^a C_{ab}\psi^b. \tag{6.48}$$

The equations of motion from the mass term are

$$\begin{aligned}\delta S &= \int \delta\psi^a\left(\frac{\partial\mathscr{L}}{\partial\psi^a} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^a)}\right) + \delta\psi^b\left(\frac{\partial\mathscr{L}}{\partial\psi^b} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^b)}\right)\\ &= \int \delta\psi^a\frac{1}{2}mC_{ab}\psi^b - \delta\psi^b\frac{1}{2}m\psi^a C_{ab} = \frac{1}{2}\int \delta\psi^a mC_{ab}\psi^b - \delta\psi^a mC_{ba}\psi^b\\ &= \frac{1}{2}\int \delta\psi^a mC_{ab}\psi^b + \delta\psi^a mC_{ab}\psi^b = \int \delta\bar{\psi}m\psi.\end{aligned}$$

Putting this together with the kinetic part gives the equations of motion as

$$\delta S = \int \delta\bar{\psi}(\partial\!\!\!/ + m)\psi. \tag{6.49}$$

We check if this is invariant under the same Poincaré transformations as for the massless case,

$$\delta_a\psi = a^\mu P_\mu\psi = a^\mu\partial_\mu\psi, \tag{6.50}$$
$$\delta_b\psi = b^{\mu\nu}\left(M_{\mu\nu} + \Sigma_{\mu\nu}\right)\psi = b^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}\right)\psi. \tag{6.51}$$

We will only consider the mass term in (6.49) since we already know the kinetic term is invariant under these transformations.

$$\begin{aligned}\delta_a S &= \int \delta_a\bar{\psi}m\psi = \int a^\mu\partial_\mu\bar{\psi}m\psi = \int a^\mu\partial_\mu\psi^a C_{ab}m\psi^b\\ &= -\int a^\mu\psi^a C_{ab}m\partial_\mu\psi^b = \int a^\mu\partial_\mu\psi^b C_{ab}m\psi^a = \int a^\mu\partial_\mu\psi^a C_{ba}m\psi^b\\ &= -\int a^\mu\partial_\mu\psi^a C_{ab}m\psi^b = 0,\end{aligned}$$

$$
\begin{aligned}
\delta_b S &= \int \delta_b \bar{\psi} m \psi = \int b^{\mu\nu} \left((x_\mu \partial_\nu - x_\nu \partial_\mu) \bar{\psi} + \overline{\Sigma_{\mu\nu} \psi} \right) m\psi \\
&= \int b^{\mu\nu} \left((x_\mu \partial_\nu - x_\nu \partial_\mu) \psi^a C_{ab} m \psi^b + \psi^a (\Sigma^t_{\mu\nu})_a{}^c C_{cb} m \psi^b \right) \\
&= \int b^{\mu\nu} \left[- \left((\eta_{\nu\mu} - \eta_{\mu\nu}) \psi^a C_{ab} m \psi^b + \psi^a C_{ab} m (x_\mu \partial_\nu - x_\nu \partial_\mu) \psi^b \right) + \psi^a (\Sigma_{\nu\mu})_{ab} m \psi^b \right] \\
&= \int b^{\mu\nu} \left[(x_\mu \partial_\nu - x_\nu \partial_\mu) \psi^b C_{ab} m \psi^a - \psi^b (\Sigma_{\nu\mu})_{ab} m \psi^a \right] \\
&= - \int b^{\mu\nu} \left[(x_\mu \partial_\nu - x_\nu \partial_\mu) \psi^b C_{ba} m \psi^a + \psi^b (\Sigma_{\nu\mu})_{ba} m \psi^a \right] \\
&= - \int b^{\mu\nu} \left[(x_\mu \partial_\nu - x_\nu \partial_\mu) \psi^a C_{ab} m \psi^b + \psi^a (\Sigma_{\nu\mu})_{ab} m \psi^b \right] = 0.
\end{aligned}
$$

Hence, (6.48) is invariant under the infinitesimal poincaré transformations .

Let us now see if (6.48) is invariant under the infinitesimal conformal transformations

$$
\begin{aligned}
\delta_c \psi &= c(D + \Delta')\psi = c(x^\mu \partial_\mu + \Delta')\psi, && (6.52) \\
\delta_d \psi &= d^\mu (K + 2x_\mu \Delta' + x^\nu \kappa')\psi \\
&= d^\mu \left(2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu + 2x_\mu \Delta' + x^\nu \kappa' \right) \psi. && (6.53)
\end{aligned}
$$

with $\Delta' = \frac{1}{2}(d-1)$, and $\kappa' = -\gamma_{\nu\mu}$. It is enough to check $\delta_c S$:

$$
\begin{aligned}
\delta_c S &= \int c(x^\mu \partial_\mu + \Delta') \bar{\psi} m \psi = \int c(x^\mu \partial_\mu + \Delta') \psi^a C_{ab} m \psi^b \\
&= - \int c \left(\delta^\mu_\mu \psi^a C_{ab} m \psi^b + x^\mu \psi^a C_{ab} m \partial_\mu \psi^b - \Delta' \psi^a C_{ab} m \psi^b \right) \\
&= - \int c \left(-x^\mu \partial_\mu \psi^b C_{ab} m \psi^a + (d - \Delta') \bar{\psi} m \psi \right) \\
&= - \int c \left(x^\mu \partial_\mu \psi^b C_{ba} m \psi^a + (d - \Delta') \bar{\psi} m \psi \right) \\
&= - \int c \left(x^\mu \partial_\mu + (d - \Delta') \right) \bar{\psi} m \psi.
\end{aligned}
$$

For $\delta_c S$ to vanish we need $\Delta' = d - \Delta'$ which gives $\Delta' = \frac{1}{2}d \neq \frac{1}{2}(d-1)$. Thus, (6.48) is not invariant under the infinitesimal conformal transformations, unless the kinetic term is not present.

6.4 Interacting Lagrangian

Lastly, an interaction term can be added to the Lagrangian density,

$$
\mathscr{L}(\phi) = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 - \lambda \phi^p, \tag{6.54}
$$

λ being some constant, and $p \in \mathbb{Z}$. The equations of motion are

$$
\delta S = \int \delta\phi \left(\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = \int \delta\phi \left(\Box\phi - m^2 \phi - p\lambda \phi^{p-1} \right).
$$

To investigate Poincaré invariance we again use the infinitesimal transformations

$$\delta_a\phi = a^\mu P_\mu\phi = a^\mu\partial_\mu\phi, \tag{6.55}$$
$$\delta_b\phi = b^{\mu\nu}M_{\mu\nu}\phi = b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi. \tag{6.56}$$

The only part which is yet to be checked for Poincaré invariance is the interaction part. We apply the above transformations to the equations of motion from the interaction term:

$$\begin{aligned}\delta_a S &= p\int a^\mu\partial_\mu\phi\lambda\phi^{p-1} = -p\int a^\mu\phi\lambda(p-1)\phi^{p-2}\partial_\mu\phi = -p(p-1)\int a^\mu\phi^{p-1}\lambda\partial_\mu\phi\\ &= -p(p-1)\delta_a S,\end{aligned}$$

$$\begin{aligned}\delta_b S &= p\int b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi\lambda\phi^{p-1} = -p\int b^{\mu\nu}\phi\lambda(p-1)\phi^{p-2}(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi\\ &= -p(p-1)\int b^{\mu\nu}\phi^{p-1}\lambda(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi = -p(p-1)\delta_b S.\end{aligned}$$

Thus, (6.54) is invariant under infinitesimal Poincaré transformations for any p.

We have discovered that the massive Lagrangian is not conformal invariant. However, the massless Lagrangian with interaction term might still be invariant under those transformations

$$\mathscr{L}(\phi) = -\frac{1}{2}(\partial\phi)^2 - \lambda\phi^p. \tag{6.57}$$

The infinitesimal conformal transformations are

$$\delta_c\phi = cD\phi = c(x^\mu\partial_\mu + \Delta)\phi, \tag{6.58}$$
$$\delta_d\phi = d^\mu K_\mu\phi = d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2\Delta)\phi, \tag{6.59}$$

with $\Delta = \frac{1}{2}(d-2)$. Then

$$\begin{aligned}\delta_c S &= \int c\left(x^\mu\partial_\mu + \Delta\right)\phi p\lambda\phi^{p-1}\\ &= -p\int c\left(\partial_\mu(x^\mu)\phi\lambda\phi^{p-1} + x^\mu\phi\lambda\partial_\mu\phi^{p-1} - \Delta\lambda\phi^p\right)\\ &= -p\int c\left(d\phi\lambda\phi^{p-1} + x^\mu\phi(p-1)\lambda\phi^{p-2}\partial_\mu\phi - \Delta\lambda\phi^p\right)\\ &= -\int c\left((p-1)x^\mu\partial_\mu - \Delta + d\right)\phi p\lambda\phi^{p-1}\\ &= -(p-1)\int c\left(x^\mu\partial_\mu + \Delta\right)\phi p\lambda\phi^{p-1} + \int c\left((p-1)\Delta + \Delta - d\right)\phi p\lambda\phi^{p-1}\\ &= -(p-1)\delta_c S + \int c\left(p\Delta - d\right)p\lambda\phi^p.\end{aligned}$$

Solving for $\delta_c S$,

$$p\delta_c S = \int c\left(p\Delta - d\right)p\lambda\phi^p.$$

For $\delta_c S$ to be invariant, the left hand side has to be zero. Δ is as it was found to be in the Poincaré case, $\Delta = \frac{1}{2}(d-2)$.

$$\begin{aligned} p\Delta - d &= 0 \\ \frac{p(d-2)}{2} &= d \\ p &= \frac{2d}{d-2}. \end{aligned} \tag{6.60}$$

The solutions to this Diophantine equation is found by completing the square,

$$pd - 2p - 2d = 0 \Leftrightarrow (p-2)(d-2) = 4.$$

Both $(p-2)$ and $(d-2)$ must divide 4, thus $(p-2), (d-2) \in \pm\{1, 2, 4\}$. Thus, the only solutions are

d	-2	0	1	3	4	6
p	1	0	-2	6	4	3

.

Since there can be neither negative nor zero dimensions, only the last four columns in the table above are relevant. Invariance under special conformal transformations follows from

$$[P_\mu, K_\nu] = 2\left(\eta_{\nu\mu} D + M_{\nu\mu}\right).$$

Hence

$$\mathscr{L}(\phi) = -\frac{1}{2}(\partial\phi)^2 - \lambda\phi^p \tag{6.61}$$

is invariant under conformal transformations for $p = -2, 3, 4, 6$.

We do the same checks for fermionic fields. The Lagrangian density is

$$\mathscr{L} = \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}m\bar{\psi}\psi + \lambda(\bar{\psi}\psi)^p, \tag{6.62}$$

again letting λ be some constant, and $p \in \mathbb{Z}$. However, we notice straight away that we cannot have $p > 2$. The interaction term with $p = 2$ can be written with spinor indices as

$$(\bar{\psi}\psi)^2 = C_{ab}C_{cd}\psi^a\psi^b\psi^c\psi^d,$$

where $a, b, c, d \in \{1, 2, 3, 4\}$, since we are in 4 dimensions. For $p > 2$ at least one of a, b, c, d will be repeated. Fermions anti-commute, thus the interaction term vanishes. We can also show that for $p = 2$, the interaction term is non-zero. We can write

$$(\bar{\psi}\psi)^2 = \epsilon^{abcd}C_{ab}C_{cd}\psi^1\psi^2\psi^3\psi^4.$$

$\epsilon^{abcd}C_{ab}C_{cd}$ is proportional to the *Pfaffian* of C (see [13], section 2),

$$\mathrm{pf}(C) = \epsilon^{abcd}C_{ab}C_{cd}. \tag{6.63}$$

It can also be shown that, for an anti-symmetric matrix A, the Pfaffian is $\mathrm{pf}(A) = \sqrt{\det A}$. C is an anti-symmetric matrix. Furthermore, by assumption,

C is invertible. Thus, $\det C \neq 0$, and $\mathrm{pf}(C) = \epsilon^{abcd} C_{ab} C_{cd} \neq 0$. Hence, the Lagrangian density with interaction term is

$$\mathscr{L} = \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}m\bar{\psi}\psi + \lambda(\bar{\psi}\psi)^2, \tag{6.64}$$

The equations of motion for the interaction part are

$$\begin{aligned}
\delta S_{\text{int}} &= \lambda \int \Big(\delta\psi^a C_{ab} C_{cd} \psi^b \psi^c \psi^d - \delta\psi^b C_{ab} C_{cd} \psi^a \psi^c \psi^d + \delta\psi^c C_{ab} C_{cd} \psi^a \psi^b \psi^d \\
&\qquad - \delta\psi^d C_{ab} C_{cd} \psi^a \psi^b \psi^c \Big) \\
&= \lambda \int \Big(\delta\bar{\psi}\psi\bar{\psi}\psi + \delta\psi^b C_{ba} \psi^a \bar{\psi}\psi + \delta\bar{\psi}\psi\bar{\psi}\psi + \delta\psi^d C_{dc} \bar{\psi}\psi\psi^c \Big) \\
&= 4\lambda \int \delta\bar{\psi}\psi\bar{\psi}\psi.
\end{aligned}$$

Let us see if (6.64) is Poincaré invariant. We use the infinitesimal Poincaré transformations

$$\begin{aligned}
\delta_a \psi &= a^\mu \partial_\mu \psi, \\
\delta_b \psi &= b^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu + \Sigma_{\mu\nu})\psi.
\end{aligned}$$

Then we find

$$\begin{aligned}
\delta_a S_{\text{int}} &= 4\lambda \int a^\mu \partial_\mu \bar{\psi}\psi\bar{\psi}\psi = -4\lambda \int a^\mu \Big(2\bar{\psi}\partial_\mu\psi\bar{\psi}\psi + \bar{\psi}\psi\partial_\mu\bar{\psi}\psi \Big) \\
&= -4\lambda \int a^\mu \Big(2\psi^a C_{ab} \partial_\mu \psi^b + \partial_\mu \psi^a C_{ab} \psi^b \Big) \bar{\psi}\psi \\
&= -4\lambda \int a^\mu \Big(-2\partial_\mu \psi^b C_{ab} \psi^a + \partial_\mu \psi^a C_{ab} \psi^b \Big) \bar{\psi}\psi \\
&= -4\lambda \int a^\mu \Big(2\partial_\mu \psi^b C_{ba} \psi^a + \partial_\mu \psi^a C_{ab} \psi^b \Big) \bar{\psi}\psi = -12\lambda \int a^\mu \partial_\mu \bar{\psi}\psi\bar{\psi}\psi \\
&= -3\delta_a S_{\text{int}}.
\end{aligned}$$

Then we have $4\delta_a S_{\text{int}} = 0$, so the varying action is invariant under infinitesimal translations.

For the infinitesimal Lorentz transformation we recall that

$$\delta_b \bar{\psi} = b^{\mu\nu} \left((x_\mu \partial_\nu - x_\nu \partial_\mu)\bar{\psi} + \bar{\psi}\Sigma_{\nu\mu} \right).$$

We apply this transformation to the varying action:

$$\delta_b S_{\text{int}} = 4\lambda \int b^{\mu\nu} \left((x_\mu \partial_\nu - x_\nu \partial_\mu)\bar{\psi} + \bar{\psi}\Sigma_{\nu\mu} \right) \psi\bar{\psi}\psi.$$

Let us consider the terms separately, beginning with the first:

$$\begin{aligned}
\int b^{\mu\nu} x_\mu \partial_\nu \bar{\psi}\psi\bar{\psi}\psi &= -\int b^{\mu\nu} \Big(\eta_{\nu\mu}(\bar{\psi}\psi)^2 + 2x_\mu \bar{\psi}\partial_\nu\psi\bar{\psi}\psi + x_\mu \bar{\psi}\psi\partial_\nu\bar{\psi}\psi \Big) \\
&= -\int b^{\mu\nu} \Big(\eta_{\nu\mu}(\bar{\psi}\psi)^2 + 2x_\mu \bar{\psi}\partial_\nu\psi\bar{\psi}\psi + x_\mu \bar{\psi}\psi\partial_\nu\bar{\psi}\psi \Big) \\
&= -\int b^{\mu\nu} (\eta_{\nu\mu}\bar{\psi}\psi + 3x_\mu \partial_\nu \bar{\psi}\psi)\bar{\psi}\psi.
\end{aligned}$$

The term including $x_\nu \partial_\mu$ is similar. The last term is

$$\begin{aligned}\int b^{\mu\nu}\bar{\psi}\Sigma_{\nu\mu}\psi\bar{\psi}\psi &= \int b^{\mu\nu}\psi^a(\Sigma_{\nu\mu})_{ab}\psi^b\bar{\psi}\psi = -\int b^{\mu\nu}\psi^b(\Sigma_{\nu\mu})_{ab}\psi^a\bar{\psi}\psi \\ &= -\int b^{\mu\nu}\psi^b(\Sigma_{\nu\mu})_{ba}\psi^a\bar{\psi}\psi = -\int b^{\mu\nu}\bar{\psi}\Sigma_{\nu\mu}\psi\bar{\psi}\psi.\end{aligned}$$

Hence, this term vanishes. We are left with

$$\begin{aligned}\delta_b S_{\text{int}} &= -4\lambda \int b^{\mu\nu}\Big(\eta_{\nu\mu}\bar{\psi}\psi\bar{\psi}\psi + 3x_\mu\partial_\nu\bar{\psi}\psi\bar{\psi}\psi \\ &\qquad - \eta_{\mu\nu}\bar{\psi}\psi\bar{\psi}\psi - 3x_\nu\partial_\mu\bar{\psi}\psi\bar{\psi}\psi\Big) \\ &= -12\lambda \int b^{\mu\nu}\Big((x_\mu\partial_\nu - x_\nu\partial_\mu)\bar{\psi}\psi\bar{\psi}\psi\Big).\end{aligned}$$

Thus, (6.64) is invariant under the Poincaré algebra.

Let us also check for conformal invariance. The infinitesimal conformal transformations are

$$\begin{aligned}\delta_c\psi &= c(x^\mu\partial_\mu + \frac{3}{2})\psi, \\ \delta_d\psi &= d^\mu(x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\psi.\end{aligned}$$

We only need to check one of these, and we choose δ_c, which is the simplest one. Applying δ_c to the varying action gives

$$\begin{aligned}\delta_c S_{\text{int}} &= 4\lambda \int c(x^\mu\partial_\mu + \frac{3}{2})\bar{\psi}\psi\bar{\psi}\psi \\ &= -4\lambda \int c\Big((\delta^\mu_\mu - \frac{3}{2})\bar{\psi}\psi + x^\mu\bar{\psi}\psi(\partial_\mu\bar{\psi}\psi + 2\bar{\psi}\partial_\mu\psi)\Big) \\ &= -4\lambda \int c\Big((4 - \frac{3}{2})(\bar{\psi}\psi)^2 + 3x^\mu\partial_\mu\bar{\psi}\psi\bar{\psi}\psi\Big) \\ &= -4\lambda \int c(3x^\mu\partial_\mu + \frac{5}{2})\bar{\psi}\psi\bar{\psi}\psi.\end{aligned}$$

This does not vanish. Thus, there is no pure fermionic interaction terms which are invariant under the conformal algebra in four dimensions.

6.5 Supersymmetric Quantum Mechanics

Let us, for the first time, consider a supersymmetric Lagrangian, consisting of a number of bosonic fields ϕ_i and fermionic fields ψ_i:

$$\mathscr{L}(\phi, \psi) = -\frac{1}{2}\dot{\phi}_i^2 + \frac{1}{2}\psi_i\dot{\psi}_i. \tag{6.65}$$

Now we work with only one time dimension and no space dimensions. In this case we do not need γ-matrices or the Majorana conjugate for the theory to be Lorentz invariant, which we will see. We find the equations of motion for this

Lagrangian density:

$$\begin{aligned}\delta S &= \int \delta\phi_i \left(\frac{\partial \mathscr{L}}{\partial \phi_i} - \partial_t \frac{\partial \mathscr{L}}{\partial \dot{\phi}_i}\right) + \delta\psi_i \left(\frac{\partial \mathscr{L}}{\partial \psi_i} - \partial_t \frac{\partial \mathscr{L}}{\partial \dot{\psi}}\right) \\ &= \int \delta\phi_i \partial_t \dot{\phi} + \delta\psi_i \left(\frac{1}{2}\dot{\psi}_i - \partial_t(-\frac{1}{2}\psi_i)\right) = \int \delta\phi_i \ddot{\phi}_i + \delta\psi_i \dot{\psi}_i.\end{aligned}$$

Thus, the equations of motion are $\ddot{\phi}_i = 0$ and $\dot{\psi}_i = 0$. In subsection 6.2 we showed that this is invariant under both Poincaré and conformal transformations.

In supersymmetry we want a way to be able to transform bosonic fields to fermionic fields. We try the simple transformations

$$\delta_\epsilon \phi_i = \epsilon \psi_i, \tag{6.66}$$

$$\delta_\epsilon \psi_i = \epsilon \phi_i, \tag{6.67}$$

where ϵ is infinitesimal. Let us see if these transformations keeps the action invariant:

$$\delta_\epsilon S = \int \delta_\epsilon \phi_i \ddot{\phi}_i + \delta_\epsilon \psi_i \dot{\psi}_i = \int \epsilon \psi_i \ddot{\phi}_i + \epsilon \phi_i \dot{\psi}_i.$$

We notice immediately that this will not vanish. We need should have an extra derivative in the last term. We instead try the following transformations:

$$\delta_\epsilon \phi_i = \epsilon \psi_i, \tag{6.68}$$

$$\delta_\epsilon \psi_i = \epsilon \dot{\phi}_i. \tag{6.69}$$

Under these transformations, the action gives

$$\delta_\epsilon S = \int \delta_\epsilon \phi_i \ddot{\phi}_i + \delta_\epsilon \psi_i \dot{\psi}_i = \int \epsilon \psi_i \ddot{\phi}_i + \epsilon \dot{\phi}_i \dot{\psi}_i = \int \epsilon \psi_i \ddot{\phi}_i - \epsilon \ddot{\phi}_i \psi_i = 0.$$

Hence, the transformations (6.68) and (6.69) are supersymmetry transformations keeping (6.65) invariant.

A more involving Lagrangian will be investigated next,

$$\mathscr{L}(\phi, \psi, \tilde{\psi}) = -\frac{1}{2}\dot{\phi}_i^2 + \frac{1}{2}\psi_i \dot{\psi}_i + \frac{1}{2}\tilde{\psi}_i \dot{\tilde{\psi}}_i + \frac{1}{2}\partial_i W \partial_i W + (\partial_i \partial_j W)\psi_i \tilde{\psi}_j, \tag{6.70}$$

with $\partial_i = \frac{\partial}{\partial \phi^i}$. $W = W(\phi)$ is the potential of the bosonic field. By the principle of least action,

$$\begin{aligned}\delta S &= \int \delta\phi_i \left(\frac{\partial \mathscr{L}}{\partial \phi_i} - \partial_t \frac{\partial \mathscr{L}}{\partial(\partial_t \phi_i)}\right) + \delta\psi_i \left(\frac{\partial \mathscr{L}}{\partial \psi_i} - \partial_t \frac{\partial \mathscr{L}}{\partial(\partial_t \psi_i)}\right) + \delta\tilde{\psi}_i \left(\frac{\partial \mathscr{L}}{\partial \tilde{\psi}_i} - \partial_t \frac{\partial \mathscr{L}}{\partial(\partial_t \tilde{\psi}_i)}\right) \\ &= \int \delta\phi_i \left(\ddot{\phi}_i + \partial_i \partial_j W + \partial_i \partial_j \partial_k W \psi_j \tilde{\psi}_k\right) + \delta\psi_i \left(\dot{\psi}_i + \partial_i \partial_j W \tilde{\psi}_j\right) + \delta\tilde{\psi}_i \left(\dot{\tilde{\psi}}_i - \partial_i \partial_j W \psi_j\right).\end{aligned}$$

The equations of motion are

$$\ddot{\phi}_i + \partial_j W \partial_i \partial_j W + \partial_i \partial_j \partial_k W \psi_j \tilde{\psi}_k = 0, \tag{6.71}$$

$$\dot{\psi}_i + \partial_i \partial_j W \tilde{\psi}_j = 0, \tag{6.72}$$

$$\dot{\tilde{\psi}}_i - \partial_i \partial_j W \psi_j = 0. \tag{6.73}$$

Under the transformations (6.68), (6.69) and

$$\delta_\epsilon \tilde{\psi}_i = \epsilon \partial_i W, \tag{6.74}$$

the action becomes

$$\begin{aligned}
\delta_\epsilon S = \int & \epsilon \psi_i \partial_t \partial_t \phi_i + \epsilon \partial_t \phi_i \partial_t \psi_i + \epsilon \partial_i W \partial_t \tilde{\psi}_i + \partial_i W \partial_i \partial_j W \epsilon \psi_j \\
& + \partial_i \partial_j W (\psi_i \epsilon \partial_j W + \epsilon \partial_t \phi_i \tilde{\psi}_j) + \partial_i \partial_j \partial_k W \epsilon \psi_k \psi_i \tilde{\psi}_j \\
= \epsilon \int & \psi_i \partial_t \partial_t \phi_i + \partial_t \phi_i \partial_t \psi_i + \partial_i W \partial_t \tilde{\psi}_i + \partial_i W \partial_i \partial_j W \psi_j \\
& - \partial_i \partial_j W \psi_i \partial_j W + \partial_i \partial_j W \partial_t \phi_j \tilde{\psi}_i + \partial_i \partial_j \partial_k W \psi_k \psi_i \tilde{\psi}_j \\
= \epsilon \int & \partial_t (\psi_i \partial_t \phi_i) - \partial_t \psi_i \partial_t \phi_i + \partial_t \phi_i \partial_t \psi_i \\
& + \partial_i W \partial_t \tilde{\psi}_i + \partial_i \partial_t W \tilde{\psi}_i - \partial_i \partial_j \partial_k W \psi_i \psi_k \tilde{\psi}_j \\
= \epsilon \int & + \partial_i W \partial_t \tilde{\psi}_i + \partial_t (\partial_i W \tilde{\psi}_i) - \partial_i W \partial_t \tilde{\psi}_i - \partial_k \partial_j \partial_i W \psi_i \psi_k \tilde{\psi}_j \\
= \epsilon \int & - \partial_i \partial_j \partial_k W \psi_k \psi_i \tilde{\psi}_j
\end{aligned}$$

Since the only term left is equal to itself multiplied by -1, $\delta_\epsilon S = 0$.

The Lagrangian (6.70) is symmetric under $\phi \to \phi$, $\psi \to \tilde{\psi}$ and $\tilde{\psi} \to -\psi$,

$$\mathscr{L} \to -\frac{1}{2}\dot{\phi}_i + \frac{1}{2}\tilde{\psi}_i \dot{\tilde{\psi}}_i + \frac{1}{2}\psi_i \dot{\psi}_i + \frac{1}{2}\partial_i W \partial_i W - \underbrace{\partial_i \partial_j W \tilde{\psi}_i \psi_j}_{-\partial_i \partial_j W \psi_j \tilde{\psi}_i = -\partial_i \partial_j W \psi_i \tilde{\psi}_j} = \mathscr{L}.$$

Thus, we have another set of supersymmetry transformations,

$$\delta_{\tilde{\epsilon}} \phi_i = \tilde{\epsilon} \tilde{\psi}_i, \tag{6.75}$$

$$\delta_{\tilde{\epsilon}} \tilde{\psi}_i = \tilde{\epsilon} \dot{\phi}_i, \tag{6.76}$$

$$\delta_{\tilde{\epsilon}} \psi_i = -\tilde{\epsilon} \partial_i W. \tag{6.77}$$

In the next section we will consider the algebra describing the supersymmetry. This is not described by a Lie algebra, but rather by a *Lie superalgebra*. We will see how this differs from the Lie algebra.

7 Lie Superalgebras

In this section we formally introduce Lie superalgebras, which is due to Haag, Łopuszański and Sohnius [6]. Coleman and Mandula [4] said that Lie algebras cannot relate bosons and fermions, but Haag, Łopuszański and Sohnius found that if one made some changes to the definition of the Lie algebra, one really could relate bosons and fermions. The algebra they created is now known as a Lie superalgebra. Lie superalgebras are not some kind of a Lie algebra, it is a different type of algebra, which contains the Lie algebra, as well as some more. Algebras are vector spaces with some additional operator. Superalgebras contain two types of elements, bosons and fermions. Therefore, a superalgebra has been constructed as a $\mathbb{Z}_2$-graded vector space (with an operator defined below), or a **super vector space**

$$V := V_0 \oplus V_1. \tag{7.1}$$

V_0 and V_1 are two vector spaces. The elements in V_0 and V_1 are called **homogeneous**. Let X be an element in V_i, $i \in Z_2$. The **parity** of X, denoted $|X|$, is 0 when $X \in V_0$, and 1 when $X \in V_1$. The elements of parity 0 are called even, while those of parity 1 are called odd. The even elements are bosonic generators, while the odd elements are fermionic generators. The formal definition of Lie superalgebras follows.

Definition 7.1. A **Lie superalgebra** consists of the direct sum of two vector spaces, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, together with the $\mathbb{Z}_2$-graded bracket, defined as $[-,-] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$, where i and j are added modulo 2. It satisfies

1. Bilinearity:
   $[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z]$, $[X, \alpha Y + \beta Z] = \alpha[X,Y] + \beta[X,Z]$ $\forall$ $\alpha, \beta \in \mathbb{K}$, and $\forall$ $X, Y, Z \in \mathfrak{g}$,
2. Super skew symmetry:
   $[X,Y] = -(-1)^{|X||Y|}[Y,X]$ $\forall$ $X, Y \in \mathfrak{g}$,
3. Super Jacobi identity:
   $[X,[Y,Z]] + (-1)^{|Z|(|X|+|Y|)}[Y,[Z,X]] + (-1)^{|X|(|Y|+|Z|)}[Z,[X,Y]] = 0$ $\forall$ $X, Y, Z \in \mathfrak{g}$.

From the definition we see that the bracket of two even elements X, Y is again an even element, since $[X,Y] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_{0+0} = \mathfrak{g}_0$. Doing this for each of the three different cases gives:

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \qquad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \qquad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0.$$

Furthermore, super skew symmetry tells us that $[\mathfrak{g}_0, \mathfrak{g}_0]$ and $[\mathfrak{g}_0, \mathfrak{g}_1]$ are both skew symmetric, while $[\mathfrak{g}_1, \mathfrak{g}_1]$ is symmetric. Using this, we may find other representations of the Jacobi identity. Let [000] denote that there are only even elements present, [001] denote that there are two even and one odd element, and so on. Furthermore, let Latin letters correspond to even elements, and Greek letters correspond to odd elements. Then, for any $X, Y, Z \in \mathfrak{g}_0$ and $\alpha, \beta, \gamma \in \mathfrak{g}_1$,

$$[000] : [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{7.2}$$

This is the Jacobi identity, as seen for Lie algebras.

$$\begin{aligned} [001] :&[X,[Y,\alpha]] + (-1)^{0(0+1)}[Y,[\alpha,X]] + (-1)^{1(0+0)}[\alpha,[X,Y]] \\ &= [X,[Y,\alpha]] + [Y,[\alpha,X]] + [\alpha,[X,Y]] \\ &= [X,[Y,\alpha]] - [Y,[X,\alpha]] - [[X,Y],\alpha] = 0. \end{aligned} \tag{7.3}$$

Hence, $\mathfrak{g}_1$ is a representation of of $\mathfrak{g}_0$

$$\begin{aligned} [011] :&[X,[\alpha,\beta]] + (-1)^{0(1+1)}[\alpha,[\beta,X]] + (-1)^{1(0+1)}[\beta,[X,\alpha]] \\ &= [X,[\alpha,\beta]] + [\alpha,[\beta,X]] - [\beta,[X,\alpha]] \\ &= [X,[\alpha,\beta]] - [\alpha,[X,\beta]] - [[X,\alpha],\beta] = 0. \end{aligned} \tag{7.4}$$

$$\begin{aligned}[111]: &[\alpha,[\beta,\gamma]]+(-1)^{1(1+1)}[\beta,[\gamma,\alpha]]+(-1)^{1(1+1)}[\gamma,[\alpha,\beta]] \\ &=[\alpha,[\beta,\gamma]]+[\beta,[\gamma,\alpha]]+[\gamma,[\alpha,\beta]]=0. \end{aligned} \tag{7.5}$$

Choosing $\alpha=\beta=\gamma$, we get

$$[\alpha[\alpha,\alpha]]=0. \tag{7.6}$$

Actually, this implies (7.5).

$$\begin{aligned} 0 &= [\alpha+\beta+\gamma,[\alpha+\beta+\gamma,\alpha+\beta+\gamma]] \\ &= [\alpha+\beta+\gamma,[\alpha,\alpha]+[\beta,\beta]+[\gamma,\gamma]+[\alpha,\beta]+[\alpha,\gamma] \\ &\qquad +[\beta,\alpha]+[\beta,\gamma]+[\gamma,\alpha]+[\gamma,\beta]] \\ &= [\alpha,[\alpha,\alpha]]+[\alpha,[\beta,\beta]]+[\alpha,[\gamma,\gamma]]+[\alpha,[\alpha,\beta]]+[\alpha,[\alpha,\gamma]] \\ &\quad +[\alpha,[\beta,\alpha]]+[\alpha,[\beta,\gamma]]+[\alpha,[\gamma,\alpha]]+[\alpha,[\gamma,\beta]] \\ &\quad +[\beta,[\alpha,\alpha]]+[\beta,[\beta,\beta]]+[\beta,[\gamma,\gamma]]+[\beta,[\alpha,\beta]]+[\beta,[\alpha,\gamma]] \\ &\quad +[\beta,[\beta,\alpha]]+[\beta,[\beta,\gamma]]+[\beta,[\gamma,\alpha]]+[\beta,[\gamma,\beta]] \\ &\quad +[\gamma,[\alpha,\alpha]]+[\gamma,[\beta,\beta]]+[\gamma,[\gamma,\gamma]]+[\gamma,[\alpha,\beta]]+[\gamma,[\alpha,\gamma]] \\ &\quad +[\gamma,[\beta,\alpha]]+[\gamma,[\beta,\gamma]]+[\gamma,[\gamma,\alpha]]+[\gamma,[\gamma,\beta]]. \end{aligned} \tag{7.7}$$

Noticing that

$$0=[\alpha+\beta,[\alpha+\beta,\alpha+\beta]]=[\alpha,[\beta,\beta]+2[\alpha,\beta]]+[\beta,[\alpha,\alpha]+2[\alpha,\beta]], \tag{7.8}$$

$$0=[\alpha-\beta,[\alpha-\beta,\alpha-\beta]]=[\alpha,[\beta,\beta]-2[\alpha,\beta]]-[\beta,[\alpha,\alpha]-2[\alpha,\beta]], \tag{7.9}$$

where (7.6) has been used. Subtracting (7.9) from (7.8), we find

$$[\alpha,[\alpha,\beta]]=[\alpha,[\beta,\alpha]]=-\frac{1}{2}[\beta,[\alpha,\alpha]]. \tag{7.10}$$

Substituting (7.10) in (7.7) we find

$$[\alpha,[\beta,\gamma]]+[\beta,[\gamma,\alpha]]+[\gamma,[\alpha,\beta]]=0, \tag{7.11}$$

which is (7.5). Hence, the super Jacobi identity for three odd elements can be written

$$[\alpha[\alpha,\alpha]]=0. \tag{7.12}$$

In the next section we will find a simple Lie superalgebra.

7.1 A First Superalgebra

In subsection 6.5 we found the following supersymmetry transformations ((6.68), (6.69) and (6.74)):

$$\delta_\epsilon\phi_i=\epsilon\psi_i, \qquad \delta_\epsilon\psi_i=\epsilon\dot{\phi}_i, \qquad \delta_\epsilon\tilde{\psi}_i=\epsilon\partial_i W. \tag{7.13}$$

Let Q be the generator of these transformations, with $\delta_\epsilon\varphi=\epsilon Q$. Then (7.13) gives

$$Q\phi_i=\psi_i, \qquad Q\psi_i=\partial_t\phi_i, \qquad Q\tilde{\psi}_i=\partial_i W.$$

Applying Q twice gives

$$QQ\phi_i = Q\psi_i = \partial_t\phi_i, \tag{7.14}$$
$$QQ\psi_i = Q\partial_t\phi_i = \partial_t\psi_i, \tag{7.15}$$
$$QQ\tilde{\psi}_i = Q\partial_i W = \partial_i\partial_j W Q\phi_j = \partial_t\tilde{\psi}_i, \tag{7.16}$$

where (6.73), $\dot{\tilde{\psi}}_i = \partial_i\partial_j W\psi_j$, has been used in (7.16). Thus $QQ = \partial_t = P_0 =: H$, H being the Hamiltonian. We have $Q \in \mathfrak{g}_1$ and $H \in \mathfrak{g}_0$. The brackets are

$$[H, H] = HH - HH = 0, \tag{7.17}$$
$$[Q, H] = QH - HQ = 0, \tag{7.18}$$
$$[Q, Q] = QQ + QQ = 2H. \tag{7.19}$$

This is a Lie superalgebra. Bilinearity follows as usual, and (7.17) and (7.18) are skew-symmetric, while (7.19) is antisymmetric. The super Jacobi identities are:

$$[000] : [H,[H,H]] + [H,[H,H]] + [H,[H,H]] = [H,0] + [H,0] + [H,0] = 0, \tag{7.20}$$
$$[001] : [H,[H,Q]] - [H,[H,Q]] - [[H,H],Q] = [H,0] - [H,0] - [0,Q] = 0, \tag{7.21}$$
$$[011] : [H,[Q,Q]] - [Q,[H,Q]] - [[H,Q],Q] = [H,2H] - [Q,0] - [0,Q] = 0, \tag{7.22}$$
$$[111] : [[Q,Q],Q] = [2H,Q] = 0. \tag{7.23}$$

It was also discovered that there is another set of supersymmetry transformations ((6.75), (6.76) and (6.77)):

$$\delta_{\tilde{\epsilon}}\phi_i = \tilde{\epsilon}\tilde{\psi}_i, \qquad \delta_{\tilde{\epsilon}}\tilde{\psi}_i = \tilde{\epsilon}\dot{\phi}_i, \qquad \delta_{\tilde{\epsilon}}\psi_i = -\tilde{\epsilon}\partial_i W. \tag{7.24}$$

These generate another supercharge $\tilde{Q}$, satisfying

$$\tilde{Q}\phi_i = \tilde{\psi}_i, \qquad \tilde{Q}\psi_i = -\partial_i W, \qquad \tilde{Q}\tilde{\psi}_i = \partial_t\phi_i.$$

Applying $\tilde{Q}$ twice gives

$$\tilde{Q}\tilde{Q}\phi_i = \tilde{Q}\tilde{\psi}_i = \partial_t\phi_i, \tag{7.25}$$
$$\tilde{Q}\tilde{Q}\psi_i = -\tilde{Q}\partial_i W = -\partial_i\partial_j W\tilde{Q}\phi_j = -\partial_i\partial_j W\tilde{\psi}_j = \partial_t\psi_i, \tag{7.26}$$
$$\tilde{Q}\tilde{Q}\tilde{\psi}_i = \tilde{Q}\partial_t\phi_i = \partial_t\tilde{\psi}_i, \tag{7.27}$$

Where (6.72), $\dot{\psi}_i = -\partial_i\partial_j W\tilde{\psi}_j$, has been used in (7.26). This supercharge, as with Q, satisfy $\tilde{Q}\tilde{Q} = \partial_t = H$. We have also $\tilde{Q} \in \mathfrak{g}_1$.

The vectorspace $\{H, Q, \tilde{Q}\}$ is a Lie superalgebra. The brackets are (7.17), (7.18) and (7.19) along with

$$[\tilde{Q}, H] = \tilde{Q}H - H\tilde{Q} = 0, \tag{7.28}$$
$$[\tilde{Q}, \tilde{Q}] = \tilde{Q}\tilde{Q} + \tilde{Q}\tilde{Q} = 2H. \tag{7.29}$$

We will see how $\tilde{Q}$ acts with Q on the different fields:

$$\begin{aligned} \tilde{Q}Q\phi_i &= \tilde{Q}\psi_i = -\partial_i W, & Q\tilde{Q}\phi_i &= Q\tilde{\psi}_i = \partial_i W \\ \tilde{Q}Q\psi_i &= \tilde{Q}\dot{\phi}_i = \dot{\tilde{\psi}}_i, & Q\tilde{Q}\psi_i &= -Q\partial_i W = -\dot{\tilde{\psi}}_i, \\ \tilde{Q}Q\tilde{\psi}_i &= \tilde{Q}\partial_i W = -\dot{\psi}_i, & Q\tilde{Q}\tilde{\psi}_i &= Q\dot{\phi}_i = \dot{\psi}_i. \end{aligned}$$

The last bracket can then be computed as

$$[\tilde{Q}, Q] = \tilde{Q}Q + Q\tilde{Q} = 0. \tag{7.30}$$

These brackets are all bilinear, and (7.28) is skew-symmetric, while (7.29) and (7.30) are both symmetric. The Jacobians are, in addition to (7.20) - (7.23),

$$\begin{aligned} [001] :&[H, [H, \tilde{Q}]] - [H, [H, \tilde{Q}]] - [[H, H], \tilde{Q}] = [H, 0] - [H, 0] - [0, \tilde{Q}] \\ &= 0, \qquad (7.31) \\ [011] :&[H, [\tilde{Q}, \tilde{Q}]] - [\tilde{Q}, [H, \tilde{Q}]] - [[H, \tilde{Q}], \tilde{Q}] = [H, 2H] - [\tilde{Q}, 0] - [0, \tilde{Q}] \\ &= 0, \qquad (7.32) \\ [011] :&[H, [\tilde{Q}, Q]] - [\tilde{Q}, [H, Q]] - [[H, \tilde{Q}], Q] = [H, 0] - [\tilde{Q}, 0] - [0, Q] \\ &= 0, \qquad (7.33) \\ [111] :&[[\tilde{Q}, \tilde{Q}], \tilde{Q}] = [2H, \tilde{Q}] = 0. \qquad (7.34) \end{aligned}$$

Thus, the algebra consisting of $H, Q, \tilde{Q}$ is a superalgebra.

In the final part of the thesis we consider two supersymmetry models, namely the Wess-Zumino model and supersymmetric gauge theories. In our discussion of the Wess-Zumino model, we will compute the Poincaré- and conformal superalgebra.

Part III
Supersymmetry Models

8 The Wess-Zumino Model

The Wess-Zumino model is a simple four-dimensional supersymmetric field theory, consisting of a real scalar field ϕ with mass m_1, a real *pseudo-scalar* field $\tilde{\phi}$ with mass m_2, and a real Majorana spinor ψ with mass m_3. A pseudo-scalar is a scalar which changes sign under **parity inversion**, P. That is, for $x^\mu = (t, \vec{x})$, then $x_P^\mu = (t, -\vec{x})$. $\tilde{\phi}$ will interact with γ_5, and since changing the orientation changes sign of γ_5, $\tilde{\phi}$ must also change sign to keep the action invariant, and hence $\tilde{\phi}$ must be a pseudo-scalar.

8.1 Free massless Wess-Zumino Model

We will first consider the Wess-Zumino model without mass and interactions,

$$\mathscr{L}_{\text{kin}} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\tilde{\phi})^2 - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi. \tag{8.1}$$

In subsection 6.2 we found the equations of motion for each of these terms (the pseudoscalar term is similar to the scalar term)

$$\delta S = \int \delta\phi\Box\phi + \delta\tilde{\phi}\Box\tilde{\phi} - \delta\bar{\psi}\partial\!\!\!/\psi. \tag{8.2}$$

We also found that this action is invariant under infinitesimal Poincaré transformations. We want to check if this is invariant under supersymmetry transformations as well. [10] gives us the following supersymmetry transformations:

$$\delta_\epsilon\phi = \bar{\epsilon}\psi, \qquad \delta_\epsilon\tilde{\phi} = \bar{\epsilon}\gamma_5\psi, \qquad \delta_\epsilon\psi = \partial\!\!\!/(\phi + \tilde{\phi}\gamma_5)\epsilon. \tag{8.3}$$

ϵ is a constant Majorana spinor. In (8.2) we have $\delta\bar{\psi}$ rather than $\delta\psi$. Remembering that $(C\gamma^\mu)^t = C\gamma^\mu$ and $(C\gamma^\mu\gamma_5)^t = -C\gamma^\mu\gamma_5$, we rewrite the latter supersymmetry transformation:

$$\begin{aligned}\delta_\epsilon\bar{\psi} &= \overline{\partial\!\!\!/(\phi + \tilde{\phi}\gamma_5)\epsilon} = \overline{\gamma^\mu\partial_\mu\phi\epsilon} + \overline{\gamma^\mu\partial_\mu\tilde{\phi}\gamma_5\epsilon} = \partial_\mu\phi\overline{\gamma^\mu\epsilon} + \partial_\mu\tilde{\phi}\overline{\gamma^\mu\gamma_5\epsilon}\\ &= \partial_\mu\phi(\gamma^\mu\epsilon)^tC + \partial_\mu\tilde{\phi}(\gamma^\mu\gamma_5\epsilon)^tC = -\partial_\mu\phi\epsilon^t(\gamma^\mu)^tC^t - \partial_\mu\tilde{\phi}\epsilon^t\gamma_5^t(\gamma^\mu)^tC^t\\ &= -\partial_\mu\phi\epsilon^t(C\gamma^\mu)^t - \partial_\mu\tilde{\phi}\epsilon^t(C\gamma^\mu\gamma_5)^t = -\partial_\mu\phi\epsilon^tC\gamma^\mu + \partial_\mu\tilde{\phi}\epsilon^tC\gamma^\mu\gamma_5\\ &= -\bar{\epsilon}\partial\!\!\!/\phi + \bar{\epsilon}\partial\!\!\!/\tilde{\phi}\gamma_5 = -\bar{\epsilon}\partial\!\!\!/(\phi - \tilde{\phi}\gamma_5) = -\bar{\epsilon}(\phi + \tilde{\phi}\gamma_5)\overleftarrow{\partial\!\!\!/},\end{aligned}$$

where $\overleftarrow{\partial\!\!\!/}$ means that the derivative acts on the fields to the left, $(\phi + \tilde{\phi}\gamma_5)\overleftarrow{\partial\!\!\!/} = \partial_\mu(\phi + \tilde{\phi}\gamma_5)\gamma^\mu$. The change of sign is due to $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ and $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$. Let us now apply (8.3) to (8.2):

$$\begin{aligned}\delta_\epsilon S &= \int \delta_\epsilon\phi\Box\phi + \delta_\epsilon\tilde{\phi}\Box\tilde{\phi} - \delta_\epsilon\bar{\psi}\partial\!\!\!/\psi = \int \bar{\epsilon}\psi\Box\phi + \bar{\epsilon}\gamma_5\psi\Box\tilde{\phi} + \bar{\epsilon}\partial\!\!\!/(\phi - \tilde{\phi}\gamma_5)\partial\!\!\!/\psi\\ &= \int \bar{\epsilon}\psi\Box\phi + \bar{\epsilon}\gamma_5\psi\Box\tilde{\phi} - \bar{\epsilon}\partial\!\!\!/\partial_\mu(\phi - \tilde{\phi}\gamma_5)\gamma^\mu\psi = \int \bar{\epsilon}\psi\Box\phi + \bar{\epsilon}\gamma_5\psi\Box\tilde{\phi} - \bar{\epsilon}\partial\!\!\!/\partial_\mu\gamma^\mu(\phi + \tilde{\phi}\gamma_5)\gamma^\mu\psi\\ &= \int \bar{\epsilon}\psi\Box\phi + \bar{\epsilon}\gamma_5\psi\Box\tilde{\phi} - \bar{\epsilon}\partial\!\!\!/^2(\phi + \tilde{\phi}\gamma_5)\gamma^\mu\psi.\end{aligned}$$

We notice that

$$\begin{aligned}\not{\partial}^2 &= \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu = \frac{1}{2}\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu + \partial_\nu\partial_\mu) = \frac{1}{2}\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \frac{1}{2}\gamma^\nu\gamma^\mu\partial_\mu\partial_\nu \\ &= \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu = \frac{1}{2}(2\eta^{\mu\nu}\mathbb{1})\partial_\mu\partial_\nu = \partial_\mu\partial^\mu = \Box. \end{aligned} \tag{8.4}$$

Then we see that $\delta_\epsilon S = 0$.

The supersymmetry transforamtions (8.3) are generated by a supercharge Q. We should check whether or not the Lie superalgebra consisting of P_μ, $M_{\mu\nu}$ and Q is closed. The infinitesimal transformation generated by Q is

$$\delta_\epsilon\varphi = \bar{\epsilon}Q\varphi. \tag{8.5}$$

From the supersymmetry transformation (8.3) we read off that

$$Q\phi = \psi, \qquad\qquad Q\tilde{\phi} = \gamma_5\psi.$$

It is convenient to write the spinor indices when we find how Q act on the spinor field ψ:

$$\delta_\epsilon\psi = \bar{\epsilon}Q\psi = \not{\partial}(\phi + \tilde{\phi}\gamma_5)\epsilon.$$

We rewrite the two last equalities with spinor indices:

$$\begin{aligned}\bar{\epsilon}Q\psi^a &= \epsilon^b C_{bc}Q^c\psi^a = -\epsilon^b Q^c C_{cb}\psi^a = -\epsilon^b Q_b\psi^a, \\ \not{\partial}(\phi + \tilde{\phi}\gamma_5)\epsilon^a &= (\gamma^\mu)^a{}_b\partial_\mu\phi\epsilon^b + (\gamma^\mu\gamma_5)^a{}_b\partial_\mu\tilde{\phi}\epsilon^b.\end{aligned}$$

Equating these two, and multiplying with C_{ac}, recalling that $(\gamma^\mu)_{ab} = -C_{ac}(\gamma^\mu)^c{}_b$, gives

$$\begin{aligned}-\epsilon^b Q_b\psi_c &= C_{ac}((\gamma^\mu)^a{}_b\partial_\mu\phi + (\gamma^\mu\gamma_5)^a{}_b\partial_\mu\tilde{\phi})\epsilon^b \\ &= -C_{ca}((\gamma^\mu)^a{}_b\partial_\mu\phi + (\gamma^\mu\gamma_5)^a{}_b\partial_\mu\tilde{\phi})\epsilon^b = ((\gamma^\mu)_{cb}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{cb}\partial_\mu\tilde{\phi})\epsilon^b.\end{aligned}$$

Then we have, letting $b \to a$ and $c \to b$

$$Q_a\psi_b = -(\gamma^\mu)_{ba}\partial_\mu\phi - (\gamma^\mu\gamma_5)_{ba}\partial_\mu\tilde{\phi}.$$

Remembering that $(\gamma^\mu)_{ba} = (\gamma^\mu)_{ab}$ and $(\gamma^\mu\gamma_5)_{ab} = -(\gamma^\mu\gamma_5)_{ba}$,

$$Q_a\psi_b = -(\gamma^\mu)_{ab}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{ab}\partial_\mu\tilde{\phi}. \tag{8.6}$$

All of the transformations are then

$$Q_a\phi = \psi_a, \quad Q_a\tilde{\phi} = (\gamma_5)_a{}^b\psi_b, \quad Q_a\psi_b = -(\gamma^\mu)_{ab}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{ab}\partial_\mu\tilde{\phi}. \tag{8.7}$$

The brackets defining the **Poincaré superalgebra**, in addition to (4.24), (4.25) and (4.26), are

$$[P_\mu, Q_a] = 0, \tag{8.8}$$

$$[M_{\mu\nu}, Q_a] = (\Sigma_{\mu\nu})_a{}^b Q_b, \tag{8.9}$$

$$[Q_a, Q_b] = 2(\gamma^\mu)_{ab}P_\mu. \tag{8.10}$$

They have been computed in Appendix B.2. In the $[Q_a, Q_b]$ bracket, the equations of motion for ψ has been used, hence the Poincaré superalgebra is only closed on-shell. The action (8.2) is then invariant under the Poincaré superalgebra.

In subsection 6.2 we also saw that (8.2) is invariant under the conformal algebra. We should check if the algebra generated by the generators of the conformal algebra and Q is closed. Only $[D, Q]$ and $[K_\mu, Q]$ are missing. These have been computed in Appendix B.4. There it was discovered that $[K_\mu, Q]$ does not correspond to any of the other generators we have seen so far. Therefore, we need to introduce a new fermionic operator, S_a. It is defined as

$$[K_\mu, Q_a] = (\gamma_\mu)_a{}^b S_b. \tag{8.11}$$

In Appendix B.4 we have also found how S_a acts on the fields, namely

$$\begin{aligned}
&S_a\phi = x^\mu\gamma_\mu\psi_a, \qquad S_a\tilde{\phi} = x^\mu\gamma_\mu\gamma_5\psi_a,\\
&S_a\psi_b = (x^\mu\partial_\mu + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}).
\end{aligned}$$

From this and $\delta_\zeta\varphi = \bar{\zeta}S\varphi$, ζ being an anticommuting Majorana spinor, we can find the infinitesimal supersymmetry transformations generated by S. The first two can simply be read off as

$$\delta_\zeta\phi = x^\mu\gamma_\mu\psi, \qquad\qquad \delta_\zeta\tilde{\phi} = x^\mu\gamma_\mu\gamma_5\psi.$$

For the last one we rewrite $S_a\psi_b$

$$\begin{aligned}
S_a\psi_b &= (x^\mu\partial_\mu + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab})\\
&= -(x^\mu\partial_\mu + 2)(-\phi C_{ba} + \tilde{\phi}(\gamma_5)_{ba}) - x_\mu\partial_\nu(-\phi(\gamma^{\nu\mu})_{ba} + \tilde{\phi}(\gamma^{\nu\mu}\gamma_5)_{ba})\\
&= -(x^\mu\partial_\mu + 2)(-\phi C_{ba} + \tilde{\phi}(\gamma_5)_{ba})\\
&\qquad - x_\mu\partial_\nu(-\phi(\gamma^\nu\gamma^\mu - \eta^{\mu\nu})_{ba} + \tilde{\phi}((\gamma^\nu\gamma^\mu - \eta^{\mu\nu})\gamma_5)_{ba})\\
&= -(x^\mu\partial_\mu + 2)(-\phi C_{ba} + \tilde{\phi}(\gamma_5)_{ba}) - x_\mu\partial_\nu(-\phi(\gamma^\nu\gamma^\mu)_{ba} + \tilde{\phi}(\gamma^\nu\gamma^\mu\gamma_5)_{ba})\\
&\qquad + x^\mu\partial_\mu(-\phi C_{ba} + \tilde{\phi}(\gamma_5)_{ba})\\
&= 2(\phi C_{ba} - \tilde{\phi}(\gamma_5)_{ba}) + x_\mu\partial_\nu(\phi(\gamma^\nu\gamma^\mu)_{ba} + \tilde{\phi}(\gamma^\nu\gamma_5\gamma^\mu)_{ba}).
\end{aligned}$$

If we multiply $-\zeta^a$ on both sides, the left hand side is $-\zeta^a S_a\psi_b = \delta_\zeta\psi_b$. Furthermore, raising the b index with $\psi_b = \psi^c C_{cb}$, we have

$$\delta_\zeta\psi^c C_{cb} = -2(\phi C_{ba} - \tilde{\phi}(\gamma_5)^c{}_a C_{cb})\zeta^a - x_\mu\partial_\nu(\phi(\gamma^\nu\gamma^\mu)^c{}_a C_{cb} + \tilde{\phi}(\gamma^\nu\gamma_5\gamma^\mu)^c{}_a C_{cb})\zeta^a.$$

Multiplying both sides by $(C_{cb})^{-1} = -C^{cb}$,

$$\delta_\zeta\psi^c = -2(-\phi C_{ba}C^{cb} - \tilde{\phi}(\gamma_5)^c{}_a)\zeta^a - x_\mu\partial_\nu(\phi(\gamma^\nu\gamma^\mu)^c{}_a + \tilde{\phi}(\gamma^\nu\gamma_5\gamma^\mu)^c{}_a)\zeta^a.$$

We notice that $C_{ba}C^{cb}\zeta^a = -C^{cb}\zeta^a C_{ab} = -C^{cb}\zeta_b = -\zeta^c$. Then,

$$\begin{aligned}
\delta_\zeta\psi^c &= -2(\phi\zeta^c - \tilde{\phi}(\gamma_5)^c{}_a\zeta^a) - x_\mu\partial_\nu(\phi(\gamma^\nu\gamma^\mu)^c{}_a + \tilde{\phi}(\gamma^\nu\gamma_5\gamma^\mu)^c{}_a)\zeta^a\\
&= -2(\phi(\mathbb{1})^c{}_a - \tilde{\phi}(\gamma_5)^c{}_a)\zeta^a - x_\mu(\gamma^\nu)^c{}_d\partial_\nu(\phi(\gamma^\mu)^d{}_a + \tilde{\phi}(\gamma_5)^d{}_e(\gamma^\mu)^e{}_a)\zeta^a.
\end{aligned}$$

Without indices,

$$\delta_\zeta \psi = -2(\phi - \tilde{\phi}\gamma_5)\zeta - x_\mu \gamma^\nu \partial_\nu(\phi\gamma^\mu + \tilde{\phi}\gamma_5\gamma^\mu)\zeta.$$

Thus, we have found that the infinitesimal supersymmetry transformations generated by S are

$$\delta_\zeta \phi = \bar{\zeta} x^\mu \gamma_\mu \psi, \quad \delta_\zeta \tilde{\phi} = \bar{\zeta} x^\mu \gamma_\mu \gamma_5 \psi, \quad \delta_\zeta \psi = -\partial\!\!\!/(\phi + \tilde{\phi}\gamma_5) x^\mu \gamma_\mu \zeta - 2(\phi - \tilde{\phi}\gamma_5)\zeta. \tag{8.12}$$

As always, we need $\delta_\zeta \bar{\psi}$:

$$\begin{aligned}
\delta_\zeta \bar{\psi} &= \overline{-\partial\!\!\!/(\phi + \tilde{\phi}\gamma_5) x^\mu \gamma_\mu \zeta - 2(\phi - \tilde{\phi}\gamma_5)\zeta} \\
&= -\partial_\nu \phi x^\mu \overline{\gamma^\nu \gamma_\mu \zeta} + \partial_\nu \tilde{\phi} x^\mu \overline{\gamma_5 \gamma^\nu \gamma_\mu \zeta} - 2\phi\bar{\zeta} + 2\tilde{\phi}\overline{\gamma_5 \zeta} \\
&= -\partial_\nu \phi x^\mu (\gamma^\nu \gamma_\mu \zeta)^t C + \partial_\nu \tilde{\phi} x^\mu (\gamma_5 \gamma^\nu \gamma_\mu \zeta)^t C - 2\bar{\zeta}\phi + 2\tilde{\phi}(\gamma_5\zeta)^t C \\
&= -\partial_\nu \phi x^\mu \zeta^t \gamma_\mu^t \gamma^{\nu t} C + \partial_\nu \tilde{\phi} x^\mu \zeta^t \gamma_\mu^t \gamma^{\nu t} \gamma_5^t C - 2\bar{\zeta}\phi + 2\tilde{\phi}\zeta^t \gamma_5^t C \\
&= \partial_\nu \phi x^\mu \zeta^t \gamma_\mu^t (C\gamma^\nu)^t - \partial_\nu \tilde{\phi} x^\mu \zeta^t \gamma_\mu^t \gamma^{\nu t} (C\gamma_5)^t - 2\bar{\zeta}\phi - 2\tilde{\phi}\zeta^t (C\gamma_5)^t \\
&= \partial_\nu \phi x^\mu \zeta^t \gamma_\mu^t C\gamma^\nu + \partial_\nu \tilde{\phi} x^\mu \zeta^t \gamma_\mu^t \gamma^{\nu t} C\gamma_5 - 2\bar{\zeta}\phi + 2\tilde{\phi}\zeta^t C\gamma_5 \\
&= -\partial_\nu \phi x^\mu \zeta^t (C\gamma_\mu)^t \gamma^\nu - \partial_\nu \tilde{\phi} x^\mu \zeta^t \gamma_\mu^t (C\gamma^\nu)^t \gamma_5 - 2\bar{\zeta}\phi + 2\tilde{\phi}\bar{\zeta}\gamma_5 \\
&= -\partial_\nu \phi x^\mu \zeta^t C\gamma_\mu \gamma^\nu - \partial_\nu \tilde{\phi} x^\mu \zeta^t \gamma_\mu^t C\gamma^\nu \gamma_5 - 2\bar{\zeta}\phi + 2\bar{\zeta}\tilde{\phi}\gamma_5 \\
&= -\partial_\nu \phi x^\mu \bar{\zeta}\gamma_\mu \gamma^\nu + \partial_\nu \tilde{\phi} x^\mu \zeta^t (C\gamma_\mu)^t \gamma^\nu \gamma_5 - 2\bar{\zeta}\phi + 2\bar{\zeta}\tilde{\phi}\gamma_5 \\
&= -\bar{\zeta}\partial_\nu \phi x^\mu \gamma_\mu \gamma^\nu + \partial_\nu \tilde{\phi} x^\mu \zeta^t C\gamma_\mu \gamma^\nu \gamma_5 - 2\bar{\zeta}\phi + 2\bar{\zeta}\tilde{\phi}\gamma_5.
\end{aligned}$$

Thus,

$$\delta_\zeta \bar{\psi} = -\bar{\zeta} x^\mu \gamma_\mu (\phi + \tilde{\phi}\gamma_5) \overleftarrow{\partial\!\!\!/} - 2\bar{\zeta}(\phi - \tilde{\phi}\gamma_5). \tag{8.13}$$

The varying action under these supersymmetry transformations is then

$$\begin{aligned}
\delta_\zeta S &= \int \delta_\zeta \phi \Box \phi + \delta_\zeta \tilde{\phi} \Box \tilde{\phi} - \delta_\zeta \bar{\psi} \partial\!\!\!/ \psi \\
&= \int \bar{\zeta} x^\mu \gamma_\mu \psi \Box \phi + \bar{\zeta} x^\mu \gamma_\mu \gamma_5 \psi \Box \tilde{\phi} + \bar{\zeta} x^\mu \gamma_\mu \partial\!\!\!/ (\phi - \tilde{\phi}\gamma_5) \partial\!\!\!/ \psi + 2\bar{\zeta}(\phi - \tilde{\phi}\gamma_5) \partial\!\!\!/ \psi \\
&= \int \bar{\zeta} x^\mu \gamma_\mu \psi \Box \phi + \bar{\zeta} x^\mu \gamma_\mu \gamma_5 \psi \Box \tilde{\phi} \\
&\quad - \bar{\zeta}(\delta^\mu_\nu + x^\mu \partial_\nu) \gamma_\mu \partial\!\!\!/ (\phi - \tilde{\phi}\gamma_5) \gamma^\nu \psi - 2\bar{\zeta}\partial_\nu (\phi - \tilde{\phi}\gamma_5) \gamma^\nu \psi \\
&= \int \bar{\zeta} x^\mu \gamma_\mu \psi \Box \phi + \bar{\zeta} x^\mu \gamma_\mu \gamma_5 \psi \Box \tilde{\phi} - \bar{\zeta}\gamma_\nu \partial\!\!\!/ \gamma^\nu (\phi + \tilde{\phi}\gamma_5) \psi \\
&\quad - \bar{\zeta} x^\mu \gamma_\mu \partial\!\!\!/ \gamma^\nu \partial_\nu (\phi + \tilde{\phi}\gamma_5) \psi - 2\bar{\zeta}\gamma^\nu \partial_\nu (\phi + \tilde{\phi}\gamma_5) \psi \\
&= \int \bar{\zeta} x^\mu \gamma_\mu \psi \Box \phi + \bar{\zeta} x^\mu \gamma_\mu \gamma_5 \psi \Box \tilde{\phi} - \bar{\zeta}\gamma_\nu \gamma^\mu \gamma^\nu \partial_\mu (\phi + \tilde{\phi}\gamma_5) \psi \\
&\quad - \bar{\zeta} x^\mu \gamma_\mu \Box (\phi + \tilde{\phi}\gamma_5) \psi - 2\bar{\zeta}\partial\!\!\!/ (\phi + \tilde{\phi}\gamma_5) \psi \\
&= \int 2\bar{\zeta}\gamma^\mu \partial_\mu (\phi + \tilde{\phi}\gamma_5) \psi - 2\bar{\zeta}\partial\!\!\!/ (\phi + \tilde{\phi}\gamma_5) \psi \\
&= 0,
\end{aligned}$$

where we have used (5.29) $\gamma_\nu \gamma^\mu \gamma^\nu = -2\gamma^\mu$.

We now need to check if the superalgebra generated by P_μ, $M_{\mu\nu}$, D, K_μ, Q and S is closed. This has been done in B.4, were it was discovered that the bracket $[Q, S]$ generated another bosonic operator, R, defined on-shell by

$$[Q_a, S_b] = 2DC_{ab} - 2R(\gamma_5)_{ab} - (\gamma^{\mu\nu})_{ab}M_{\mu\nu}. \tag{8.14}$$

R acts on the fields as

$$R\phi = \tilde{\phi}, \qquad R\tilde{\phi} = -\phi, \qquad R\psi = \frac{1}{2}\gamma_5\psi. \tag{8.15}$$

We find the infinitesimal transformations generated by R. They are given by $\delta_e\varphi = eR\varphi$, where e is an infinitesimal bosonic parameter.

$$\delta_e\phi = e\tilde{\phi}, \qquad \delta_e\tilde{\phi} = -e\phi, \qquad \delta_e\psi = \frac{1}{2}e\gamma_5\psi. \tag{8.16}$$

Then we find the transformation for $\bar{\psi}$,

$$\delta_e\bar{\psi} = \frac{1}{2}e\overline{\gamma_5\psi} = \frac{1}{2}e(\gamma_5\psi)^tC = -\frac{1}{2}e\psi^t\gamma_5^tC^t = -\frac{1}{2}e\psi^t(C\gamma_5)^t = \frac{1}{2}e\psi^tC\gamma_5 = \frac{1}{2}e\bar{\psi}\gamma_5.$$

The varying action, over the infinitesimal transformations generated by R, is

$$\begin{aligned}\delta_eS &= \int \delta_e\phi\Box\phi + \delta_e\tilde{\phi}\Box\tilde{\phi} - \delta_e\bar{\psi}\partial\!\!\!/\psi = \int e\tilde{\phi}\Box\phi - e\phi\Box\tilde{\phi} - \frac{1}{2}e\bar{\psi}\gamma_5\partial\!\!\!/\psi \\ &= \int e\tilde{\phi}\Box\phi - e\Box\phi\tilde{\phi} - \frac{1}{2}e\psi^a(\gamma_5\gamma^\mu)_{ab}\partial_\mu\psi^b = -\frac{1}{2}\int e\psi^a(\gamma_5\gamma^\mu)_{ab}\partial_\mu\psi^b.\end{aligned}$$

The last term vanishes:

$$\begin{aligned}\int e\psi^a(\gamma_5\gamma^\mu)_{ab}\partial_\mu\psi^b &= -\int e\partial_\mu\psi^a(\gamma_5\gamma^\mu)_{ab}\psi^b = \int e\psi^b(\gamma_5\gamma^\mu)_{ab}\partial_\mu\psi^a \\ &= \int e\psi^a(\gamma_5\gamma^\mu)_{ba}\partial_\mu\psi^b = -\int e\psi^a(\gamma_5\gamma^\mu)_{ab}\partial_\mu\psi^b = 0.\end{aligned}$$

Hence, $\delta_eS = 0$.

With the inclusion of R, the conformal superalgebra closes. The brackets defining the on-shell conformal superalgebra in four dimensions are

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu, & [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu[\rho}M_{\sigma]\nu} - \eta_{\nu[\rho}M_{\sigma]\mu}, \\
[P_\mu, D] &= P_\mu, & [K_\mu, D] &= -K_\mu, \\
[P_\mu, K_\nu] &= 2\eta_{\mu\nu}D - 2M_{\mu\nu}, & [M_{\mu\nu}, K_\rho] &= \eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu, \\
[M_{\mu\nu}, Q_a] &= -(\Sigma_{\mu\nu})_a{}^bQ_b, & [Q_a, Q_b] &= 2(\gamma^\mu)_{ab}P_\mu, \\
[K_\mu, Q_a] &= (\gamma_\mu)_a{}^bS_b, & [M_{\mu\nu}, S_a] &= (\Sigma_{\mu\nu})_a{}^bS_b, \\
[P_\mu, S_a] &= -(\gamma_\mu)_a{}^bQ_b, & [S_a, S_b] &= 2(\gamma^\mu)_{ab}K_\mu, \\
[Q_a, S_b] &= 2C_{ab}D - 2(\gamma_5)_{ab}R + (\gamma^{\mu\nu})_{ab}M_{\mu\nu}, & [R, Q_a] &= -\frac{1}{2}(\gamma_5)_a{}^bQ_b, \\
[R, S_a] &= \frac{1}{2}(\gamma_5)_a{}^bS_b, & [D, Q_a] &= \frac{1}{2}Q_a, \\
[D, S_a] &= -\frac{1}{2}S_a.
\end{aligned}$$

We have shown that the free massless Wess-Zumino model is invariant under this algebra. Let us next consider the Wess-Zumino model with mass terms.

8.2 Massive Wess-Zumino Model

Let us now add masses, so that the Lagrangian becomes

$$\mathscr{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\tilde{\phi})^2 - \bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}m_1^2\phi^2 - \frac{1}{2}m_2^2\tilde{\phi}^2 - m_3\bar{\psi}\psi. \tag{8.17}$$

We have seen that also this Lagrangian is Poincaré invariant. We also saw that is is not invariant under conformal transformations. The equations of motion were found to be

$$\delta S = \int \delta\phi(\Box - m_1^2)\phi + \delta\tilde{\phi}(\Box - m_2^2)\tilde{\phi} - \delta\bar{\psi}(\partial\!\!\!/ + m_3)\psi. \tag{8.18}$$

The supersymmetry transformations incorporating the mass are in [10] stated as

$$\delta_\epsilon\phi = \bar{\epsilon}\psi, \qquad \delta_\epsilon\tilde{\phi} = \bar{\epsilon}\gamma_5\psi, \qquad \delta_\epsilon\psi = (\partial\!\!\!/ - m_3)(\phi + \tilde{\phi}\gamma_5)\epsilon. \tag{8.19}$$

We find $\delta_\epsilon\bar{\psi}$. Recall that $(C\gamma_5)^t = -C\gamma_5$.

$$\begin{aligned}
\delta_\epsilon\bar{\psi} &= \overline{(\partial\!\!\!/ - m_3)(\phi + \tilde{\phi}\gamma_5)\epsilon} = \overline{\partial\!\!\!/(\phi + \tilde{\phi}\gamma_5)\epsilon} - \overline{m_3(\phi + \tilde{\phi}\gamma_5)\epsilon} \\
&= -\bar{\epsilon}\partial\!\!\!/(\phi - \tilde{\phi}\gamma_5) - m_3\phi\bar{\epsilon} - m_3\tilde{\phi}\overline{\gamma_5\epsilon} = -\bar{\epsilon}\partial\!\!\!/(\phi - \tilde{\phi}\gamma_5) - \bar{\epsilon}m_3\phi - m_3\tilde{\phi}(\gamma_5\epsilon)^tC \\
&= -\bar{\epsilon}\partial\!\!\!/(\phi - \tilde{\phi}\gamma_5) - \bar{\epsilon}m_3\phi + \epsilon^t m_3\tilde{\phi}(C\gamma_5)^t = -\bar{\epsilon}\partial\!\!\!/(\phi - \tilde{\phi}\gamma_5) - \bar{\epsilon}m_3\phi - \bar{\epsilon}m_3\tilde{\phi}\gamma_5 \\
&= -\bar{\epsilon}\left(\partial\!\!\!/(\phi - \tilde{\phi}\gamma_5) + m_3(\phi + \tilde{\phi}\gamma_5)\right) = -\bar{\epsilon}(\phi + \tilde{\phi}\gamma_5)(\overleftarrow{\partial\!\!\!/} + m_3).
\end{aligned}$$

Let us now apply (8.19) to (8.18).

$$\begin{aligned}
\delta_\epsilon S &= \int \delta_\epsilon\phi(\Box - m_1^2)\phi + \delta_\epsilon\tilde{\phi}(\Box - m_2^2)\tilde{\phi} - \delta_\epsilon\bar{\psi}(\partial\!\!\!/ + m_3)\psi \\
&= \int \bar{\epsilon}\psi(\Box - m_1^2)\phi + \bar{\epsilon}\gamma_5\psi(\Box - m_2^2)\tilde{\phi} + \bar{\epsilon}\left(\partial\!\!\!/(\phi - \tilde{\phi}\gamma_5) + m_3(\phi + \tilde{\phi}\gamma_5)\right)(\partial\!\!\!/ + m_3)\psi \\
&= \int \bar{\epsilon}\psi(\Box - m_1^2)\phi + \bar{\epsilon}\gamma_5\psi(\Box - m_2^2)\tilde{\phi} + \partial\!\!\!/(\phi - \tilde{\phi}\gamma_5)\partial\!\!\!/\psi - \partial\!\!\!/(\phi - \tilde{\phi}\gamma_5)m_3\psi \\
&\quad + m_3(\phi + \tilde{\phi}\gamma_5)\partial\!\!\!/\psi + m_3^2(\phi + \tilde{\phi}\gamma_5)\psi \\
&= \int \bar{\epsilon}\psi(\Box - m_1^2)\phi + \bar{\epsilon}\gamma_5\psi(\Box - m_2^2)\tilde{\phi} - \partial\!\!\!/^2(\phi + \tilde{\phi}\gamma_5)\psi - (\phi + \tilde{\phi}\gamma_5)m_3\partial\!\!\!/\psi \\
&\quad + m_3(\phi + \tilde{\phi}\gamma_5)\partial\!\!\!/\psi + m_3^2(\phi + \tilde{\phi}\gamma_5)\psi \\
&= \int \bar{\epsilon}\psi(\Box - m_1^2)\phi + \bar{\epsilon}\gamma_5\psi(\Box - m_2^2)\tilde{\phi} - \Box(\phi + \tilde{\phi}\gamma_5)\psi + m_3^2(\phi + \tilde{\phi}\gamma_5)\psi.
\end{aligned}$$

$\delta_\epsilon S$ vanishes when $m_1 = m_2 = m_3$. Thus, all fields must have the same mass, for the theory to be invariant under the Poincaré superalgebra.

8.3 Interacting Wess-Zumino Model

Lastly we add interactions. The interaction terms in the Wess-Zumino model takes the form

$$\mathscr{L}_{\text{int}} = -\lambda\left(\psi^a(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab})\psi^b + \frac{1}{2}\lambda(\phi^2 + \tilde{\phi}^2)^2 + m\phi(\phi^2 + \tilde{\phi}^2)\right), \tag{8.20}$$

so the Lagrangian describing the Wess-Zumino model is given by

$$\begin{aligned}\mathscr{L}_{\text{WZ}} =& -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\tilde{\phi})^2 - \bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}m^2\tilde{\phi}^2 - m\bar{\psi}\psi \\ &- \lambda\left(\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi + \frac{1}{2}\lambda(\phi^2+\tilde{\phi}^2)^2 + m\phi(\phi^2+\tilde{\phi}^2)\right). \end{aligned} \tag{8.21}$$

In subsection 6.4 it was discovered that bosonic fields to any power are invariant under the Poincaré algebra, while in 4 dimensions, we can only have interaction terms to power 4, for it to be conformally invariant. This is okay in this case, since the term containing ϕ^3 is connected to the mass, and we know that this is not present in theories which are invariant under conformal transformations. Therefore, only the first term needs to be checked for Poincaré and conformal invariance. Let us first find the equations of motion for the interaction part:

$$\begin{aligned}\delta S_{\text{int}} =& \int \delta\phi\frac{\partial\mathscr{L}_{\text{int}}}{\partial\phi} + \delta\tilde{\phi}\frac{\partial\mathscr{L}_{\text{int}}}{\partial\tilde{\phi}} + \delta\psi^a\frac{\partial\mathscr{L}_{\text{int}}}{\partial\psi^a} + \delta\psi^b\frac{\partial\mathscr{L}_{\text{int}}}{\partial\psi^b} \\ =& -\lambda\int \delta\phi\left(\psi^a C_{ab}\psi^b + 2\lambda(\phi^2+\tilde{\phi}^2)\phi + m(\phi^2+\tilde{\phi}^2 + 2m\phi^2\right) \\ &+ \delta\tilde{\phi}\left(-\psi^a(\gamma_5)_{ab}\psi^b + 2\lambda(\phi^2+\tilde{\phi}^2)\tilde{\phi} + 2m\phi\tilde{\phi}\right) \\ &+ \delta\psi^a(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab})\psi^b - \delta\psi^b\psi^a(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) \\ =& -\lambda\int \delta\phi\left(\bar{\psi}\psi + 2\lambda(\phi^3+\phi\tilde{\phi}^2) + m(3\phi^2+\tilde{\phi}^2)\right) \\ &+ \delta\tilde{\phi}\left(-\bar{\psi}\gamma_5\psi + 2\lambda(\phi^2\tilde{\phi}+\tilde{\phi}^3) + 2m\phi\tilde{\phi}\right) \\ &+ \delta\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi + \delta\psi^b\psi^a(\phi C_{ba} - \tilde{\phi}(\gamma_5)_{ba}) \\ =& -\lambda\int \delta\phi\left(\bar{\psi}\psi + 2\lambda(\phi^3+\phi\tilde{\phi}^2) + m(3\phi^2+\tilde{\phi}^2)\right) \\ &+ \delta\tilde{\phi}\left(-\bar{\psi}\gamma_5\psi + 2\lambda(\phi^2\tilde{\phi}+\tilde{\phi}^3) + 2m\phi\tilde{\phi}\right) \\ &+ 2\delta\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi. \end{aligned}$$

Then the varying action for the full Lagrangian density is

$$\begin{aligned}\delta S_{\text{WZ}} =& \int\delta\phi\left[\Box\phi - m^2\phi - \lambda\left(\bar{\psi}\psi + 2\lambda(\phi^2+\tilde{\phi}^2)\phi + m(3\phi^2+\tilde{\phi}^2)\right)\right] \\ &+\delta\tilde{\phi}\left[\Box\tilde{\phi} - m^2\tilde{\phi} - \lambda\left(-\bar{\psi}\gamma_5\psi + 2\lambda(\phi^2+\tilde{\phi}^2)\tilde{\phi} + 2m\phi\tilde{\phi}\right)\right] \\ &-\delta\bar{\psi}\left[\partial\!\!\!/ + m + 2\lambda(\phi - \tilde{\phi}\gamma_5)\right]\psi. \end{aligned} \tag{8.22}$$

The terms we need to check for Poincaré invariance are the ones containing different fields, for example both ϕ and $\tilde{\phi}$. The remaining terms are

$$\begin{aligned}\delta S_{\text{r}} =& -\lambda\int \delta\phi(\bar{\psi}\psi + 2\lambda\phi\tilde{\phi}^2 + m\tilde{\phi}^2) + \delta\tilde{\phi}(-\bar{\psi}\gamma_5\psi + 2\lambda\tilde{\phi}\phi^2 + 2m\phi\tilde{\phi}) \\ &+ 2\delta\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi. \end{aligned} \tag{8.23}$$

Let us first check for Poincaré invariance. We recall the infinitesimal Poincaré transformations:

$$
\begin{aligned}
\delta_a\phi &= a^\mu\partial_\mu\phi, & \delta_b\phi &= b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi,\\
\delta_a\tilde{\phi} &= a^\mu\partial_\mu\tilde{\phi}, & \delta_b\tilde{\phi} &= b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\tilde{\phi},\\
\delta_a\psi &= a^\mu\partial_\mu\psi, & \delta_b\psi &= b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\psi.
\end{aligned}
$$

Since we need $\delta_{a,b}\bar{\psi}$, we rewrite the last two equations:

$$
\delta_a\bar{\psi} = a^\mu\partial_\mu\bar{\psi}, \qquad \delta_b\bar{\psi} = b^{\mu\nu}((x_\mu\partial_\nu - x_\nu\partial_\mu)\bar{\psi} + \bar{\psi}\Sigma_{\nu\mu}),
$$

as in subsection 6.2. δS_{r} acted upon by these transformations becomes

$$
\begin{aligned}
\delta_a S_{\mathrm{r}} = -\lambda\int & a^\mu\partial_\mu\phi(\bar{\psi}\psi + 2\lambda\phi\tilde{\phi}^2 + m\tilde{\phi}^2) + a^\mu\partial_\mu\tilde{\phi}(-\bar{\psi}\gamma_5\psi + 2\lambda\tilde{\phi}\phi^2 + 2m\phi\tilde{\phi})\\
&+ 2a^\mu\partial_\mu\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi\\
= \lambda\int & a^\mu\phi\partial_\mu(\bar{\psi}\psi + 2\lambda\phi\tilde{\phi}^2 + m\tilde{\phi}^2) + a^\mu\tilde{\phi}\partial_\mu(-\bar{\psi}\gamma_5\psi + 2\lambda\tilde{\phi}\phi^2 + 2m\phi\tilde{\phi})\\
&- 2a^\mu\partial_\mu\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
\int\partial_\mu(\bar{\psi}\psi) &= \int(\partial_\mu\bar{\psi}\psi + \psi^a C_{ab}\partial_\mu\psi^b) = \int(\partial_\mu\bar{\psi}\psi - \partial_\mu\psi^b C_{ab}\psi^a)\\
&= \int(\partial_\mu\bar{\psi}\psi + \partial_\mu\psi^b C_{ba}\psi^a) = 2\int\partial_\mu\bar{\psi}\psi. \qquad (8.24)
\end{aligned}
$$

The same is true if we include γ_5, $\int\partial_\mu(\bar{\psi}\gamma_5\psi) = 2\int\partial_\mu\bar{\psi}\gamma_5\psi$, since $(\gamma_5)_{ab} = -(\gamma_5)_{ba}$. Then, we are left with

$$
\delta_a S_{\mathrm{r}} = \lambda\int a^\mu\phi\partial_\mu(2\lambda\phi\tilde{\phi}^2 + m\tilde{\phi}^2) + a^\mu\tilde{\phi}\partial_\mu(2\lambda\tilde{\phi}\phi^2 + 2m\phi\tilde{\phi}).
$$

The terms linear in λ^2 are

$$
\begin{aligned}
\int a^\mu\left(\phi\partial_\mu(\phi\tilde{\phi}^2) + \tilde{\phi}\partial_\mu(\tilde{\phi}\phi^2)\right) &= \int a^\mu\left(\phi\tilde{\phi}^2\partial_\mu\phi + 2\phi^2\tilde{\phi}\partial_\mu\tilde{\phi} + \tilde{\phi}\phi^2\partial_\mu\tilde{\phi} + 2\tilde{\phi}^2\phi\partial_\mu\phi\right)\\
&= 3\int a^\mu(\partial_\mu\phi(\phi\tilde{\phi}^2) + \partial_\mu\tilde{\phi}(\tilde{\phi}\phi^2)).
\end{aligned}
$$

Comparing this with what we had in the beginning, we see that the terms linear in λ^2 vanish. Only the terms linear in m remains:

$$
\int a^\mu(\phi\partial_\mu(\tilde{\phi}^2) + 2\tilde{\phi}\partial_\mu(\phi\tilde{\phi})) = \int a^\mu(2\phi\tilde{\phi}\partial_\mu\tilde{\phi} - 2\phi\tilde{\phi}\partial_\mu\tilde{\phi}) = 0.
$$

We also need to check $\delta_b S_{\mathrm{r}}$:

$$
\begin{aligned}
\delta_b S_{\mathrm{r}} = -\lambda\int b^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\Big(&\phi(\bar{\psi}\psi + 2\lambda\phi\tilde{\phi}^2 + m\tilde{\phi}^2)\\
&+ \tilde{\phi}(-\bar{\psi}\gamma_5\psi + 2\lambda\tilde{\phi}\phi^2 + 2m\phi\tilde{\phi}) + 2\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi\Big)\\
&+ 2b^{\mu\nu}\bar{\psi}\Sigma_{\nu\mu}(\phi - \tilde{\phi}\gamma_5)\psi.
\end{aligned}
$$

All terms in the big parenthesis vanish in the same way as for $\delta_a S_{\rm r}$. Only the last term remains:

$$\begin{aligned}\delta_b S_{\rm r} &= -2\lambda\int b^{\mu\nu}\bar{\psi}\Sigma_{\nu\mu}(\phi-\tilde{\phi}\gamma_5)\psi = -\lambda\int b^{\mu\nu}\psi^a(\phi(\gamma_{\nu\mu})_{ab}-\tilde{\phi}(\gamma_{\nu\mu}\gamma_5)_{ab})\psi^b\\ &= \lambda\int b^{\mu\nu}\psi^b(\phi(\gamma_{\nu\mu})_{ab}-\tilde{\phi}(\gamma_{\nu\mu}\gamma_5)_{ab})\psi^a = \lambda\int b^{\mu\nu}\psi^b(\phi(\gamma_{\nu\mu})_{ba}-\tilde{\phi}(\gamma_{\nu\mu}\gamma_5)_{ba})\psi^a\\ &= 2\lambda\int b^{\mu\nu}\bar{\psi}\Sigma_{\nu\mu}(\phi-\tilde{\phi}\gamma_5)\psi = 0.\end{aligned}$$

Hence, the Wess-Zumino model is Poincaré invariant.

We should next check if the Wess-Zumino model is invariant under the Poincaré superalgebra. [10] gives us the following supersymmetry transformations

$$\delta_\epsilon\phi = \bar{\epsilon}\psi, \quad \delta_\epsilon\tilde{\phi} = \bar{\epsilon}\gamma_5\psi, \quad \delta_\epsilon\psi = \left[\partial\!\!\!/ - m - \lambda(\phi+\tilde{\phi}\gamma_5)\right](\phi+\tilde{\phi}\gamma_5)\epsilon. \tag{8.25}$$

Let us first see if $\delta S_{\rm WZ}$ is invariant under these transformation. $\delta_\epsilon\bar{\psi}$ is computed:

$$\begin{aligned}\delta_\epsilon\bar{\psi} &= \overline{\left[\partial\!\!\!/ - m - \lambda(\phi+\tilde{\phi}\gamma_5)\right](\phi+\tilde{\phi}\gamma_5)\epsilon}\\ &= \overline{(\partial\!\!\!/ - m)(\phi+\tilde{\phi}\gamma_5)\epsilon} - \overline{\lambda(\phi+\tilde{\phi}\gamma_5)^2\epsilon}\\ &= -\bar{\epsilon}(\phi+\tilde{\phi}\gamma_5)(\overleftarrow{\partial\!\!\!/} + m) - \overline{\lambda(\phi^2+2\phi\tilde{\phi}\gamma_5-\tilde{\phi}^2)\epsilon}.\end{aligned}$$

The result from subsection 8.2 has been used. In the last barred term there are two terms in which the bar only do anything for ϵ. This simply becomes $-\bar{\epsilon}\lambda(\phi^2-\tilde{\phi}^2)$. The last part is

$$\begin{aligned}\overline{2\lambda\phi\tilde{\phi}\gamma_5\epsilon} &= 2\lambda\phi\tilde{\phi}\overline{\gamma_5\epsilon} = 2\lambda\phi\tilde{\phi}(\gamma_5\epsilon)^tC = -2\lambda\phi\tilde{\phi}\epsilon^t\gamma_5^tC^t\\ &= -2\epsilon^t\lambda\phi\tilde{\phi}(C\gamma_5)^t = 2\bar{\epsilon}\lambda\phi\tilde{\phi}\gamma_5.\end{aligned}$$

Hence,

$$\begin{aligned}\delta_\epsilon\bar{\psi} &= -\bar{\epsilon}(\phi+\tilde{\phi}\gamma_5)(\overleftarrow{\partial\!\!\!/} + m) - \bar{\epsilon}\lambda(\phi^2-\tilde{\phi}^2) - 2\bar{\epsilon}\lambda\phi\tilde{\phi}\gamma_5\\ &= -\bar{\epsilon}(\phi+\tilde{\phi}\gamma_5)(\overleftarrow{\partial\!\!\!/} + m) - \bar{\epsilon}\lambda(\phi+\tilde{\phi}\gamma_5)^2\\ &= -\bar{\epsilon}\left[(\phi+\tilde{\phi}\gamma_5)(\overleftarrow{\partial\!\!\!/} + m + \lambda(\phi+\tilde{\phi}\gamma_5))\right].\end{aligned}$$

Now we can apply the transformations (8.25) to (8.22):

$$\begin{aligned}\delta_\epsilon S_{\rm WZ} = \int &\bar{\epsilon}\psi\left[\Box\phi - m^2\phi - \lambda\left(\bar{\psi}\psi + 2\lambda(\phi^3+\phi\tilde{\phi}^2) + m(3\phi^2+\tilde{\phi}^2)\right)\right]\\ &+\bar{\epsilon}\gamma_5\psi\left[\Box\tilde{\phi} - m^2\tilde{\phi} - \lambda\left(-\bar{\psi}\gamma_5\psi + 2\lambda(\phi^2\tilde{\phi}+\tilde{\phi}^3) + 2m\phi\tilde{\phi}\right)\right]\\ &+\bar{\epsilon}\left[(\phi+\tilde{\phi}\gamma_5)(\overleftarrow{\partial\!\!\!/} + m + \lambda(\phi+\tilde{\phi}\gamma_5))\right]\left[\partial\!\!\!/ + m + 2\lambda(\phi-\tilde{\phi}\gamma_5)\right]\psi.\end{aligned}$$

The results from the free massive part tells us that the part containing both $\partial\!\!\!/$'s cancel the $\Box$ terms, and also that all m^2 terms cancel each other. We also saw that the terms with $m\not{\partial}$ in the last line vanishes. Let us consider the terms $(\bar{\epsilon}\psi)(\bar{\psi}\psi)$ and $(\bar{\epsilon}\gamma_5\psi)(\bar{\psi}\gamma_5\psi)$. We can write these as

$$(\bar{\epsilon}\psi)(\bar{\psi}\psi) - (\bar{\epsilon}\gamma_5\psi)(\bar{\psi}\gamma_5\psi) = \bar{\epsilon}(\psi\bar{\psi} - \gamma_5\psi\bar{\psi}\gamma_5)\psi.$$

Applying the Fierz identity (A.31), we have

$$\psi\bar{\psi} = -\frac{1}{4}(\bar{\psi}\psi)\mathbb{1} + \frac{1}{4}(\bar{\psi}\gamma_5\psi)\gamma_5 - \frac{1}{4}(\bar{\psi}\gamma^\mu\psi)\gamma_\mu + \frac{1}{4}(\bar{\psi}\gamma^\mu\gamma_5)\gamma_\mu\gamma_5 + \frac{1}{8}(\bar{\psi}\gamma^{\mu\nu}\psi)\gamma_{\mu\nu},$$

$$\begin{aligned}\gamma_5\psi\bar{\psi}\gamma_5 &= \gamma_5(-\frac{1}{4}(\bar{\psi}\psi)\mathbb{1} + \frac{1}{4}(\bar{\psi}\gamma_5\psi)\gamma_5 - \frac{1}{4}(\bar{\psi}\gamma^\mu\psi)\gamma_\mu + \frac{1}{4}(\bar{\psi}\gamma^\mu\gamma_5)\gamma_\mu\gamma_5 + \frac{1}{8}(\bar{\psi}\gamma^{\mu\nu}\psi)\gamma_{\mu\nu})\gamma_5 \\ &= -\frac{1}{4}(\bar{\psi}\psi)\mathbb{1}\gamma_5^2 + \frac{1}{4}(\bar{\psi}\gamma_5\psi)\gamma_5^3 + \frac{1}{4}(\bar{\psi}\gamma^\mu\psi)\gamma_\mu\gamma_5^2 - \frac{1}{4}(\bar{\psi}\gamma^\mu\gamma_5)\gamma_\mu\gamma_5^3 + \frac{1}{8}(\bar{\psi}\gamma^{\mu\nu}\psi)\gamma_{\mu\nu}\gamma_5^2 \\ &= \frac{1}{4}(\bar{\psi}\psi)\mathbb{1} - \frac{1}{4}(\bar{\psi}\gamma_5\psi)\gamma_5 - \frac{1}{4}(\bar{\psi}\gamma^\mu\psi)\gamma_\mu + \frac{1}{4}(\bar{\psi}\gamma^\mu\gamma_5)\gamma_\mu\gamma_5 - \frac{1}{8}(\bar{\psi}\gamma^{\mu\nu}\psi)\gamma_{\mu\nu}.\end{aligned}$$

Subtracting these, we get

$$(\bar{\epsilon}\psi)(\bar{\psi}\psi) - (\bar{\epsilon}\gamma_5\psi)(\bar{\psi}\gamma_5\psi) = \frac{1}{2}\bar{\epsilon}\left(-(\bar{\psi}\psi)\mathbb{1} + (\bar{\psi}\gamma_5\psi)\gamma_5 + \frac{1}{2}(\bar{\psi}\gamma^{\mu\nu}\psi)\gamma_{\mu\nu}\right)\psi$$

The last term vanishes:

$$\bar{\psi}\gamma^{\mu\nu}\psi = \psi^a(\gamma^{\mu\nu})_{ab}\psi^b = -\psi^b(\gamma^{\mu\nu})_{ab}\psi^a = -\psi^b(\gamma^{\mu\nu})_{ba}\psi^a = -\bar{\psi}\gamma^{\mu\nu}\psi = 0.$$

We are left with

$$\begin{aligned}(\bar{\epsilon}\psi)(\bar{\psi}\psi) - (\bar{\epsilon}\gamma_5\psi)(\bar{\psi}\gamma_5\psi) &= -\frac{1}{2}\bar{\epsilon}\left((\bar{\psi}\psi)\mathbb{1} - (\bar{\psi}\gamma_5\psi)\gamma_5\right)\psi \\ &= -\frac{1}{2}(\bar{\epsilon}\psi)(\bar{\psi}\psi) - \frac{1}{2}(\bar{\epsilon}\gamma_5\psi)(\bar{\psi}\gamma_5\psi) = 0.\end{aligned}$$

Thus, also the terms $(\bar{\epsilon}\psi)(\bar{\psi}\psi)$ and $(\bar{\epsilon}\gamma_5\psi)(\bar{\psi}\gamma_5\psi)$ vanish from $\delta_\epsilon S_{\text{WZ}}$. The remaining terms are

$$\begin{aligned}\delta_\epsilon S_{\text{WZ}} = \int &-\bar{\epsilon}\psi\lambda\left(2\lambda(\phi^2+\tilde{\phi}^2)\phi + m(3\phi^2+\tilde{\phi}^2)\right) - \bar{\epsilon}\gamma_5\psi\lambda\left(2\lambda(\phi^2+\tilde{\phi}^2)\tilde{\phi} + 2m\phi\tilde{\phi}\right) \\ &+\bar{\epsilon}\lambda\Big(2\lambda(\phi+\tilde{\phi}\gamma_5)(\phi^2+\tilde{\phi}^2) + 2\not{\partial}(\phi-\tilde{\phi}\gamma_5)(\phi-\tilde{\phi}\gamma_5) \\ &\quad + 2m(\phi+\tilde{\phi}\gamma_5)(\phi-\tilde{\phi}\gamma_5) + (\phi+\tilde{\phi}\gamma_5)^2\not{\partial} + (\phi+\tilde{\phi}\gamma_5)^2 m\Big)\psi\end{aligned}$$

The first term in each of the brackets cancel each other. The mass terms in the final bracket can be rewritten

$$\begin{aligned}2m(\phi+\tilde{\phi}\gamma_5)(\phi-\tilde{\phi}\gamma_5)\psi + (\phi+\tilde{\phi}\gamma_5)^2 m\psi &= 2m(\phi^2+\tilde{\phi}^2)\psi + (\phi^2+2\phi\tilde{\phi}-\tilde{\phi}^2)m\psi \\ &= m(3\phi^2+\tilde{\phi}^\psi) + 2m\phi\tilde{\phi}\psi,\end{aligned}$$

so all the mass terms cancel each other. We are left with

$$\begin{aligned}&2\not{\partial}(\phi-\tilde{\phi}\gamma_5)(\phi-\tilde{\phi}\gamma_5)\psi + (\phi+\tilde{\phi}\gamma_5)^2\not{\partial}\psi \\ &= 2\not{\partial}(\phi-\tilde{\phi}\gamma_5)(\phi-\tilde{\phi}\gamma_5)\psi - 2\partial_\mu(\phi+\tilde{\phi}\gamma_5)(\phi+\tilde{\phi}\gamma_5)\gamma^\mu\psi \\ &= 2\not{\partial}(\phi-\tilde{\phi}\gamma_5)(\phi-\tilde{\phi}\gamma_5)\psi - 2\partial_\mu\gamma^\mu(\phi-\tilde{\phi}\gamma_5)(\phi-\tilde{\phi}\gamma_5)\psi = 0.\end{aligned}$$

Thus, $\delta S_{\mathrm{WZ}} = 0$.

From subsection 6.3 we know that a massive Lagrangian cannot be invariant under conformal transformations. Let us now consider the massless Wess-Zumino model

$$\mathscr{L}_{\mathrm{mWZ}} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\tilde{\phi})^2 - \bar{\psi}\partial\!\!\!/\psi - \lambda\left(\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi + \frac{1}{2}\lambda(\phi^2 + \tilde{\phi}^2)^2\right). \tag{8.26}$$

The varying action is

$$\begin{aligned}\delta S_{\mathrm{mWZ}} = \int \delta\phi &\left[\Box\phi - \lambda\bar{\psi}\psi - 2\lambda^2(\phi^3 + \phi\tilde{\phi}^2)\right] \\ &+ \delta\tilde{\phi}\left[\Box\tilde{\phi} + \lambda\bar{\psi}\gamma_5\psi - 2\lambda^2(\tilde{\phi}^3 + \phi^2\tilde{\phi})\right] - \delta\bar{\psi}\left[\partial\!\!\!/ + 2\lambda(\phi - \tilde{\phi}\gamma_5)\right]\psi.\end{aligned} \tag{8.27}$$

We should check that this is invariant under infinitesimal conformal transformations. We only need to check either using the dilations or the special conformal transformations, due to $[P_\mu, K_\nu] = 2\eta_{\mu\nu}D - 2M_{\mu\nu}$. We check the dilations, since these are the simplest ones,

$$\delta_c\phi = c(x^\mu\partial_\mu + 1)\phi, \qquad \delta_c\tilde{\phi} = c(x^\mu\partial_\mu + 1)\tilde{\phi}, \qquad \delta_c\psi = c(x^\mu\partial_\mu + \frac{3}{2})\psi.$$

The transformation for $\bar{\psi}$ is simply $\delta_c\bar{\psi} = c(x^\mu\partial_\mu + \frac{3}{2})\bar{\psi}$. For the same reasons as above, the only remaining terms to be checked are

$$\begin{aligned}\delta S_{\mathrm{r}} = -\lambda \int &\delta\phi(\bar{\psi}\psi + 2\lambda\phi\tilde{\phi}^2) + \delta\tilde{\phi}(-\bar{\psi}\gamma_5\psi + 2\lambda\tilde{\phi}\phi^2) \\ &+ 2\delta\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi.\end{aligned} \tag{8.28}$$

Then, acting on the action with the infinitesimal dilation transformations yield

$$\begin{aligned}\delta_c S_{\mathrm{r}} = -\lambda \int &c(x^\mu\partial_\mu + 1)\left(\phi(\bar{\psi}\psi + 2\lambda\phi\tilde{\phi}^2) + \tilde{\phi}(-\bar{\psi}\gamma_5\psi + 2\lambda\tilde{\phi}\phi^2\right) \\ &+ 2c(x^\mu\partial_\mu + \frac{3}{2})\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi.\end{aligned}$$

The terms linear in λ are

$$\begin{aligned}\delta_c S_{\mathrm{r}} &= \lambda \int -(x^\mu\partial_\mu + 1)\phi\bar{\psi}\psi + (x^\mu\partial_\mu + 1)\tilde{\phi}\bar{\psi}\gamma_5\psi - 2(x^\mu\partial_\mu + \frac{3}{2})\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi \\ &= \lambda \int -\bar{\psi}(x^\mu\partial_\mu + 1)(\phi - \tilde{\phi}\gamma_5)\psi - 2(x^\mu\partial_\mu + \frac{3}{2})\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi \\ &= \lambda \int x^\mu\partial_\mu\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi + \bar{\psi}(\phi - \tilde{\phi}\gamma_5)x^\mu\partial_\mu\psi + (\delta^\mu_\mu - 1)(\phi - \tilde{\phi}\gamma_5)\bar{\psi}\psi \\ &\qquad - 2(x^\mu\partial_\mu + \frac{3}{2})\bar{\psi}(\phi - \tilde{\phi}\gamma_5)\psi \\ &= 0.\end{aligned}$$

By a similar computation, the terms linear in λ^2 vanish. Hence, the massless Wess-Zumino model is invariant under the conformal algebra.

Doing the same calculations as in subsection 8.1, we find the supersymmetry transformations generated by S to be

$$\begin{aligned}
\delta_\zeta \phi &= \bar{\zeta} x^\mu \gamma_\mu \psi, \\
\delta_\zeta \tilde{\phi} &= \bar{\zeta} x^\mu \gamma_\mu \gamma_5 \psi, \\
\delta_\zeta \psi &= -(\not{\partial} - \lambda(\phi + \tilde{\phi}\gamma_5))(\phi + \tilde{\phi}\gamma_5) x^\mu \gamma_\mu \zeta - 2(\phi - \tilde{\phi}\gamma_5)\zeta.
\end{aligned} \tag{8.29}$$

We find $\delta_\zeta \bar{\psi}$. The only additional term to the one we found in subsection 8.1 is $\lambda(\phi + \tilde{\phi}\gamma_5)^2 x^\mu \gamma_\mu \zeta = \lambda(\phi^2 + 2\phi\tilde{\phi}\gamma_5 - \tilde{\phi}^2) x^\mu \gamma_\mu \zeta$. This becomes

$$\begin{aligned}
\overline{\lambda(\phi^2 + 2\phi\tilde{\phi}\gamma_5 - \tilde{\phi}^2) x^\mu \gamma_\mu \zeta} &= \lambda\phi^2 x^\mu \overline{\gamma_\mu \zeta} + 2\lambda\phi\tilde{\phi} x^\mu \overline{\gamma_5 \gamma_\mu \zeta} - \lambda\tilde{\phi}^2 x^\mu \overline{\gamma_\mu \zeta} \\
&= \lambda\phi^2 x^\mu (\gamma_\mu \zeta)^t C + 2\lambda\phi\tilde{\phi} x^\mu (\gamma_5 \gamma_\mu \zeta)^t C - \lambda\tilde{\phi}^2 x^\mu (\gamma_\mu \zeta)^t C \\
&= -\lambda\phi^2 x^\mu \zeta^t \gamma_\mu^t C^t - 2\lambda\phi\tilde{\phi} x^\mu \zeta^t \gamma_\mu^t \gamma_5^t C^t + \lambda\tilde{\phi}^2 x^\mu \zeta^t \gamma_\mu^t C^t \\
&= -\lambda\phi^2 x^\mu \zeta^t (C\gamma_\mu)^t - 2\lambda\phi\tilde{\phi} x^\mu \zeta^t (C\gamma_5\gamma_\mu)^t \\
&\quad + \lambda\tilde{\phi}^2 x^\mu \zeta^t (C\gamma_\mu)^t \\
&= -\lambda\phi^2 x^\mu \zeta^t C\gamma_\mu + 2\lambda\phi\tilde{\phi} x^\mu \zeta^t C\gamma_5\gamma_\mu + \lambda\tilde{\phi}^2 x^\mu \zeta^t C\gamma_\mu \\
&= -\bar{\zeta}\lambda\phi^2 x^\mu \gamma_\mu - 2\bar{\zeta}\lambda\phi\tilde{\phi} x^\mu \gamma_\mu \gamma_5 + \bar{\zeta}\lambda\tilde{\phi}^2 x^\mu \gamma_\mu \\
&= -\bar{\zeta}\lambda x^\mu \gamma_\mu (\phi^2 + 2\phi\tilde{\phi}\gamma_5 - \tilde{\phi}^2) \\
&= -\bar{\zeta}\lambda x^\mu \gamma_\mu (\phi + \tilde{\phi}\gamma_5)^2.
\end{aligned}$$

Then

$$\delta_\zeta \bar{\psi} = -\bar{\zeta} x^\mu \gamma_\mu (\phi + \tilde{\phi}\gamma_5)(\overleftarrow{\not{\partial}} + \lambda(\phi + \tilde{\phi}\gamma_5)) - 2\bar{\zeta}(\phi - \tilde{\phi}\gamma_5).$$

Let us now see if $\mathscr{L}_{\mathrm{mWZ}}$ is invariant under these transformations:

$$\begin{aligned}
\delta_\zeta S_{\mathrm{mWZ}} = \int & \bar{\zeta} x^\mu \gamma_\mu \psi \left[\Box\phi - \lambda\bar{\psi}\psi - 2\lambda^2(\phi^3 + \phi\tilde{\phi}^2)\right] + \bar{\zeta} x^\mu \gamma_\mu \gamma_5 \psi \left[\Box\tilde{\phi} + \lambda\bar{\psi}\gamma_5\psi - 2\lambda^2(\tilde{\phi}^3 + \phi^2\tilde{\phi})\right] \\
& + \left[\bar{\zeta} x^\mu \gamma_\mu (\phi + \tilde{\phi}\gamma_5)(\overleftarrow{\not{\partial}} + \lambda(\phi + \tilde{\phi}\gamma_5)) + 2\bar{\zeta}(\phi - \tilde{\phi}\gamma_5)\right] \left[\not{\partial} + 2\lambda(\phi - \tilde{\phi}\gamma_5)\right] \psi.
\end{aligned}$$

We know from subsection 8.1 that terms not including any factors of λ cancel each other. By a trivial extension of how we found that the terms $\bar{\epsilon}\psi\bar{\psi}\psi$ and $\bar{\epsilon}\gamma_5\psi\bar{\psi}\gamma_5\psi$ vanish in the massive case, also $\bar{\zeta}\gamma_\mu\psi\bar{\psi}\psi$ and $\bar{\zeta}\gamma_\mu\gamma_5\psi\bar{\psi}\gamma_5\psi$ vanish. Hence, we are left with

$$\begin{aligned}
\delta_\zeta S_{\mathrm{mWZ}} = \lambda \int & \bar{\zeta} x^\mu \gamma_\mu \psi \left[-2\lambda(\phi^2 + \tilde{\phi}^2)\phi\right] + \bar{\zeta} x^\mu \gamma_\mu \psi \left[-2\lambda(\tilde{\phi}^2 + \phi^2)\tilde{\phi}\gamma_5\right] \\
& + \bar{\zeta}\Big[x^\mu \gamma_\mu \Big(2(\phi - \tilde{\phi}\gamma_5)\not{\partial}(\phi - \tilde{\phi}\gamma_5) + (\phi + \tilde{\phi}\gamma_5)^2 \not{\partial} + 2\lambda(\phi + \tilde{\phi}\gamma_5)^2(\phi - \tilde{\phi}\gamma_5)\Big) \\
& \qquad + 4(\phi - \tilde{\phi}\gamma_5)^2\Big]\psi.
\end{aligned}$$

The terms linear in λ^2 cancel. Let us now consider the terms linear in λ.

$$
\begin{aligned}
&\int 2x^\mu\gamma_\mu(\phi-\tilde{\phi}\gamma_5)\partial\!\!\!/(\phi-\tilde{\phi}\gamma_5)\psi + x^\mu\gamma_\mu(\phi+\tilde{\phi}\gamma_5)^2\gamma^\nu\partial_\nu\psi + 4(\phi-\tilde{\phi}\gamma_5)^2\psi \\
&\quad = \int 2x^\mu\gamma_\mu(\phi-\tilde{\phi}\gamma_5)\partial\!\!\!/(\phi-\tilde{\phi}\gamma_5)\psi - \delta^\mu_\nu\gamma_\mu(\phi+\tilde{\phi}\gamma_5)^2\gamma^\nu\psi \\
&\qquad\qquad - x^\mu\gamma_\mu\partial_\nu(\phi+\tilde{\phi}\gamma_5)^2\gamma^\nu\psi + 4(\phi-\tilde{\phi}\gamma_5)^2\psi \\
&\quad = \int 2x^\mu\gamma_\mu(\phi-\tilde{\phi}\gamma_5)\partial\!\!\!/(\phi-\tilde{\phi}\gamma_5)\psi - \gamma_\mu\gamma^\mu(\phi-\tilde{\phi}\gamma_5)^2\psi \\
&\qquad\qquad - 2x^\mu\gamma_\mu(\phi-\tilde{\phi}\gamma_5)\gamma^\nu\partial_\nu(\phi-\tilde{\phi}\gamma_5)\psi + 4(\phi-\tilde{\phi}\gamma_5)^2\psi \\
&\quad = \int -4(\phi-\tilde{\phi}\gamma_5)^2\psi + 4(\phi-\tilde{\phi}\gamma_5)^2\psi = 0.
\end{aligned}
$$

Thus $\mathscr{L}_{\mathrm{mWZ}}$ is invariant under the infinitesimal transformations generated by S.

As we discovered in subsection 8.1 there is one more generator in the conformal superalgebra, namely R. We need to check that $\mathscr{L}_{\mathrm{mWZ}}$ is invariant under the infinitesimal transformations generated by R. We recall the transformations:

$$
\delta_e\phi = e\tilde{\phi}, \qquad \delta_e\tilde{\phi} = -e\phi, \qquad \delta_e\bar{\psi} = \frac{1}{2}e\gamma_5\bar{\psi}.
$$

Let us apply these transformations to δS_{mWZ}:

$$
\begin{aligned}
\delta_e S_{\mathrm{mWZ}} = \int & e\tilde{\phi}\left[\Box\phi - \lambda\bar{\psi}\psi - 2\lambda^2(\phi^3+\phi\tilde{\phi}^2)\right] - e\phi\left[\Box\tilde{\phi} + \lambda\bar{\psi}\gamma_5\psi - 2\lambda^2(\tilde{\phi}^3+\phi^2\tilde{\phi})\right] \\
& - \frac{1}{2}e\gamma_5\bar{\psi}\left[\partial\!\!\!/ + 2\lambda(\phi-\tilde{\phi}\gamma_5)\right]\psi.
\end{aligned}
$$

It is easy to see that all terms cancel. Thus, the Wess-Zumino model is invariant under the conformal superalgebra.

In the next section we will investigate *gauge theories*. The bosonic field in gauge theories are not scalar fields ϕ, which are spin-0. They are instead spin-1 gauge fields A_μ, which are vector fields.

9 Supersymmetric Gauge Theories

Here we will consider gauge theories. In a gauge theory, one can remove redundant degrees of freedom in the Lagrangian by gauge transformations. The gauge transformations are symmetries of the Lagrangian. Gauge theories are built on Lie groups. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\{T_i\}$ be a basis for $\mathfrak{g}$. Then $\mathfrak{g}$ satisfies $[T_i, T_j] = {f_{ij}}^k T_k$, where ${f_{ij}}^k$ are the structure constant, which are anti-symmetric. If $[T_i, T_j] = 0$, we have an Abelian gauge theory, which we will see an example of, namely electromagnetism.

9.1 Super-Electromagnetism

The non-supersymmetry parts are taken from [7], chapter 1.8.

Let us start by recalling the Maxwell equations representing the laws of electromagnetism. Let a bold faced letter represent three-vectors, where $\boldsymbol{E} = (E_x, E_y, E_z)$ is the electric field and $\boldsymbol{B} = (B_x, B_y, B_z)$ is the magnetic field. Let $\boldsymbol{V}$ be a three-vector, then $\nabla \cdot \boldsymbol{V}$ is the divergence of $\boldsymbol{V}$, defined as

$$\nabla \cdot \boldsymbol{V} := \partial_x V_x + \partial_y V_y + \partial_z V_z.$$

$\nabla \times \boldsymbol{V}$ is the curl of $\boldsymbol{V}$, defined as

$$\nabla \times \boldsymbol{V} := (\partial_y V_z - \partial_z V_y)\mathbf{i} - (\partial_x V_z - \partial_z V_x)\mathbf{j} + (\partial_x V_y - \partial_y V_x)\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the standard basis vectors. Furthermore, let ρ be the electric charge density, and $\boldsymbol{j}$ be the electric current density. The Maxwell equations are then

$$\nabla \cdot \boldsymbol{E} = \rho, \tag{9.1}$$
$$\partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\boldsymbol{j}, \tag{9.2}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{9.3}$$
$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0. \tag{9.4}$$

Let A_μ be the four-vector consisting of the electric potential ϕ and the magnetic potential $\boldsymbol{A}$, $A_\mu = (\phi, \boldsymbol{A})$. Then, $\boldsymbol{E}$ and $\boldsymbol{B}$ are expressed in terms of A_μ as

$$\boldsymbol{E} = \nabla\phi - \partial_t \boldsymbol{A}, \tag{9.5}$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{9.6}$$

Here $\nabla\phi = (\partial_x\phi, \partial_y\phi, \partial_z\phi)$ is the gradient of ϕ. Let us define the **electromagnetic field tensor** $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. This is invariant under a **gauge transformation**

$$A_\mu \to A_\mu + \partial_\mu \alpha, \tag{9.7}$$

where α is any function. This is obvious since derivatives commute. We immediately see that $F_{\mu\nu} = -F_{\nu\mu}$. Then there are only 6 independent elements in this matrix. They are F_{01}, F_{02}, F_{03}, F_{12}, F_{13} and F_{23}. Let us calculate them using (9.5) and (9.6). Let i, j, k run from 1 to 3, then

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = \partial_t A_i - \nabla\phi = -E_i,$$
$$F_{ij} = \partial_i A_j - \partial_j A_i = \partial_i A_j - \partial_j A_i = \epsilon_{ij}{}^k B_k,$$

where $\epsilon_{ij}{}^k = 1$. Then we have $F_{01} = -E_x$, $F_{02} = -E_y$, $F_{03} = -E_z$, $F_{12} = B_z$, $F_{13} = -B_y$ and $F^{23} = B_x$. Hence,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{9.8}$$

Raising the indices changes the sign on the $\boldsymbol{E}$-components:

$$\begin{aligned}F_{\mu\nu} &= \eta_{\mu\rho}F^{\rho\sigma}\eta_{\sigma\nu}\\ &= \begin{pmatrix}-1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & B_z & -B_y\\ E_y & -B_z & 0 & B_x\\ E_z & B_y & -B_x & 0\end{pmatrix}\begin{pmatrix}-1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}\\ &= \begin{pmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0\end{pmatrix}.\end{aligned} \tag{9.9}$$

We notice that

$$\begin{aligned}&\partial_0 F^{00} + \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} = \partial_x E_x + \partial_y E_y + p_z E_z = \nabla\cdot\boldsymbol{E},\\ &\partial_0 F^{10} + \partial_1 F^{11} + \partial_2 F^{12} + \partial_3 F^{13} = -\partial_t E_x + \partial_y B_z - p_z B_y = -\partial_t E_x + (\nabla\times\boldsymbol{B})_x,\\ &\partial_0 F^{20} + \partial_1 F^{21} + \partial_2 F^{22} + \partial_3 F^{23} = -\partial_t E_y - \partial_x B_z + p_z B_x = -\partial_t E_y + (\nabla\times\boldsymbol{B})_y,\\ &\partial_0 F^{30} + \partial_1 F^{31} + \partial_2 F^{32} + \partial_3 F^{33} = -\partial_t E_z + \partial_x B_y - p_y B_x = -\partial_t E_z + (\nabla\times\boldsymbol{B})_z.\end{aligned}$$

Thus, if we let $j^\mu := (\rho, \boldsymbol{j})$, we can write two of the Maxwell equations, (9.1) and (9.2), as one equation,

$$\partial_\nu F^{\mu\nu} = j^\mu. \tag{9.10}$$

We also notice that

$$\begin{aligned}\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} &= \partial_t B_z + \partial_x E_y - \partial_y E_x = \partial_t B_z + (\nabla\times\boldsymbol{E})_z,\\ \partial_0 F_{31} + \partial_3 F_{10} + \partial_1 F_{03} &= \partial_t B_y + \partial_z E_x - \partial_x E_z = \partial_t B_y + (\nabla\times\boldsymbol{E})_y,\\ \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} &= \partial_t B_x + \partial_y E_z - \partial_z E_y = \partial_t B_x + (\nabla\times\boldsymbol{E})_x,\\ \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= \partial_x B_x + \partial_y B_y + \partial_z B_z = \nabla\cdot\boldsymbol{B}.\end{aligned}$$

Thus, we can write the final two Maxwell equations, (9.3) and (9.4), as one equation,

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0. \tag{9.11}$$

This is a **Bianchi identity**, which can be written compactly as $\partial_{[\rho}F_{\mu\nu]} = 0$. If any of ρ, μ and ν are identical, the Bianchi identity follows trivially.

The Lagrangian of the electromagnetic fields is

$$\mathscr{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu. \tag{9.12}$$

The first term can be expanded as

$$\begin{aligned}F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\\ &= (\partial_\mu A_\nu)^2 - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + (\partial_\nu A_\mu)^2\\ &= (\partial_\mu A_\nu)^2 - \partial_\mu(A_\nu \partial^\nu A^\mu) + A_\nu \partial^\nu \partial_\mu A^\mu\\ &\quad - \partial_\nu(A_\mu \partial^\mu A^\nu) + A_\mu \partial^\mu \partial_\nu A^\nu + (\partial_\mu A_\nu)^2\\ &= 2(\partial_\mu A_\nu)^2 - \partial_\mu(A_\nu \partial^\nu A^\mu) + \partial^\nu(A_\nu \partial_\mu A^\mu) - \partial^\nu A_\nu \partial_\mu A^\mu\\ &\quad - \partial_\nu(A_\mu \partial^\mu A^\nu) + \partial^\mu(A_\mu \partial_\nu A^\nu) - \partial^\mu A_\mu \partial_\nu A^\nu\\ &= 2(\partial_\mu A_\nu)^2 - 2(\partial_\mu A^\mu)^2\\ &\quad - \partial_\mu(A_\nu \partial^\nu A^\mu) + \partial^\nu(A_\nu \partial_\mu A^\mu) - \partial_\nu(A_\mu \partial^\mu A^\nu) + \partial^\mu(A_\mu \partial_\nu A^\nu).\end{aligned}$$

All terms in the last line are total derivatives which vanish in the Lagrangian. Thus, the Lagrangian can be written

$$\mathscr{L}_{\text{EM}} = -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}(\partial_\mu A^\mu)^2 + A_\mu j^\mu. \tag{9.13}$$

We find the equations of motion:

$$\begin{aligned}\delta S &= \int \delta A_\mu \left(\frac{\partial \mathscr{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu)}\right) = \int \delta A_\mu (j^\mu + \partial_\nu \partial^\nu A^\mu - \partial^\mu(\partial_\nu A^\nu)) \\ &= \int \delta A_\mu (j^\mu + \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu)) = \int \delta A_\mu (j^\mu + \partial_\nu F^{\nu\mu}). \end{aligned} \tag{9.14}$$

We have used that

$$\partial_\nu \frac{\partial(\partial_\rho A^\rho)^2}{\partial(\partial_\nu A_\mu)} = \partial_\nu \left(2(\partial_\rho A^\rho)\frac{\partial(\partial_\sigma A_\lambda)}{\partial(\partial_\nu A_\mu)}\eta^{\lambda\sigma}\right) = \partial_\nu \left(2(\partial_\rho A^\rho)\delta^\nu_\sigma \delta^\mu_\lambda \eta^{\lambda\sigma}\right) = 2\partial^\mu(\partial_\rho A^\rho).$$

Thus, the equations of motion are $j^\mu + \partial_\nu F^{\nu\mu} = 0$. This can be rewritten as $\partial_\nu F^{\mu\nu} = j^\mu$, which is one of the Maxwell equations found above.

Let us now check if $\mathscr{L}_{\text{EM}}$ is invariant under the Poincaré algebra. We try applying the obvious transformations

$$\delta_a A_\mu = a^\nu \partial_\nu A_\mu, \qquad \delta_b A_\mu = b^{\rho\sigma}(x_\rho \partial_\sigma - x_\sigma \partial_\rho) A_\mu \tag{9.15}$$

to (9.14). Then,

$$\begin{aligned}\delta_a S &= \int \delta_a A_\mu (j^\mu + \partial_\nu F^{\nu\mu}) = \int a^\rho \partial_\rho A_\mu (j^\mu + \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu)) \\ &= -\int a^\rho(-\partial_\rho A_\mu j^\mu + (\partial_\nu \partial^\nu A_\mu \partial_\rho A^\mu - \partial_\nu \partial^\mu A_\mu \partial_\rho A^\nu)) \\ &= -\int a^\rho(-\partial_\rho A_\mu j^\mu + (\partial_\nu \partial^\nu A^\mu \partial_\rho A_\mu - \partial_\mu \partial^\nu A_\nu \partial_\rho A^\mu)) \\ &= -\int a^\rho(-\partial_\rho A_\mu j^\mu + \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu)\partial_\rho A_\mu) \\ &= -\int a^\rho \partial_\rho A_\mu (-j^\mu + \partial_\nu F^{\nu\mu}). \end{aligned}$$

This does not vanish unless we have $j^\mu = 0$. Let us choose $j^\mu = 0$ from now on.

$$\begin{aligned}\delta_b S &= \int \delta_b A_\mu \partial_\nu F^{\nu\mu} = \int b^{\rho\sigma}(x_\rho \partial_\sigma - x_\sigma \partial_\rho) A_\mu \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &= -\int b^{\rho\sigma}(\eta_{\nu\rho}\partial_\sigma + x_\rho \partial_\sigma \partial_\nu) A_\mu (\partial^\nu A^\mu - \partial^\mu A^\nu) - (\rho \leftrightarrow \sigma) \\ &= -\int b^{\rho\sigma}\Big(\partial_\sigma A_\mu(\partial_\rho A^\mu - \partial^\mu A_\rho) - \eta_{\sigma\rho} A_\mu (\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &\qquad\qquad - \partial_\nu A_\mu x_\rho \partial_\sigma (\partial^\nu A^\mu - \partial^\mu A^\nu)\Big) - (\rho \leftrightarrow \sigma). \end{aligned}$$

The $\partial_\sigma A_\mu \partial_\rho A^\mu$ term, and the terms including $\eta_{\sigma\rho}$ vanishes since we subtract with the same term where ρ and σ change places. Then

$$\begin{aligned}\delta_b S &= -\int b^{\rho\sigma}\Big(-\partial_\sigma A_\mu \partial^\mu A_\rho - \partial_\nu A_\mu x_\rho \partial_\sigma (\partial^\nu A^\mu - \partial^\mu A^\nu)\Big) - (\rho \leftrightarrow \sigma)\\ &= -\int b^{\rho\sigma}\Big(A_\rho \partial_\sigma \partial^\mu A_\mu + (\delta^\nu_\rho + x_\rho \partial^\nu)\partial_\nu A_\mu \partial_\sigma A^\mu \\ &\qquad - (\delta^\mu_\rho + x_\rho \partial^\mu)\partial_\nu A_\mu \partial_\sigma A^\nu\Big) - (\rho \leftrightarrow \sigma)\\ &= -\int b^{\rho\sigma}\Big(A_\rho \eta_{\sigma\mu}\partial_\nu \partial^\mu A^\nu + \partial_\rho A_\mu \partial_\sigma A^\mu + x_\rho \partial_\nu \partial^\nu A_\mu \partial_\sigma A^\mu - \partial_\nu A_\rho \partial_\sigma A^\nu \\ &\qquad - x_\rho \partial_\nu \partial^\mu A_\mu \partial_\sigma A^\nu\Big) - (\rho \leftrightarrow \sigma)\\ &= -\int b^{\rho\sigma}\Big(A_\rho \eta_{\sigma\mu}\partial_\nu \partial^\mu A^\nu + x_\rho \partial_\nu \partial^\nu A^\mu \partial_\sigma A_\mu + A_\rho \eta_{\sigma\mu}\partial_\nu \partial^\mu A^\nu \\ &\qquad - x_\rho \partial_\mu \partial^\nu A_\nu \partial_\sigma A^\mu\Big) - (\rho \leftrightarrow \sigma)\\ &= -\int b^{\rho\sigma}\Big(2A_\rho \eta_{\sigma\mu}\partial_\nu \partial^\mu A^\nu + x_\rho \partial_\sigma A_\mu \partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu)\Big) - (\rho \leftrightarrow \sigma).\end{aligned}$$

$\partial_\rho A_\mu \partial_\sigma A^\mu$ cancels due to the same reasons as above. Thus, acting with δ_b on (9.14) gives

$$\begin{aligned}\delta_b S &= \int b^{\rho\sigma}(x_\rho \partial_\sigma - x_\sigma \partial_\rho) A_\mu \partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu)\\ &= -\int b^{\rho\sigma}\Big((x_\rho \partial_\sigma - x_\sigma \partial_\rho) A_\mu \partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &\qquad + 2A_\rho \eta_{\sigma\mu}\partial_\nu \partial^\mu A^\nu - 2A_\sigma \eta_{\rho\mu}\partial_\nu \partial^\mu A^\nu\Big).\end{aligned}$$

Hence $\delta_b S$ does not vanish. We need to modify the $\delta_b A_\mu$ transformation for the electromagnetic Lagrangian density to be Lorentz invariant. Let us try

$$\delta_b A_\mu = b^{\rho\sigma}\left((x_\rho \partial_\sigma - x_\sigma \partial_\rho) A_\mu - A_\rho \eta_{\sigma\mu} + A_\sigma \eta_{\rho\mu}\right). \tag{9.16}$$

Then, applying this transformation to the varying action leads to

$$\begin{aligned}\delta_b S &= \int b^{\rho\sigma}\left((x_\rho \partial_\sigma - x_\sigma \partial_\rho) A^i_\mu - A^i_\rho \eta_{\sigma\mu} + A^i_\sigma \eta_{\rho\mu}\right) \partial_\nu (\partial^\nu A^\mu_i - \partial^\mu A^\nu_i)\\ &= -\int b^{\rho\sigma}\left((x_\rho \partial_\sigma - x_\sigma \partial_\rho) A^i_\mu - 2A^i_\rho \eta_{\sigma\mu} + 2A^i_\sigma \eta_{\rho\mu} + A^i_\rho \eta_{\sigma\mu} - A^i_\sigma \eta_{\rho\mu}\right)\\ &\qquad \times \partial_\nu (\partial^\nu A^\mu_i - \partial^\mu A^\nu_i)\\ &= -\int b^{\rho\sigma}\left((x_\rho \partial_\sigma - x_\sigma \partial_\rho) A^i_\mu - A^i_\rho \eta_{\sigma\mu} + A^i_\sigma \eta_{\rho\mu}\right) \partial_\nu (\partial^\nu A^\mu_i - \partial^\mu A^\nu_i) = 0.\end{aligned}$$

Thus, (9.12) is invariant under the following infinitesimal Poincaré transformations:

$$\begin{aligned}\delta_a A_\mu &= a^\nu \partial_\nu A_\mu,\\ \delta_b A_\mu &= b^{\rho\sigma}\left((x_\rho \partial_\sigma - x_\sigma \partial_\rho) A_\mu - A_\rho \eta_{\sigma\mu} + A_\sigma \eta_{\rho\mu}\right).\end{aligned}$$

Let us now introduce a fermionic field Ψ, where the Lagrangian takes the form

$$\mathscr{L} = \frac{1}{2}\bar{\Psi}\partial\!\!\!/\Psi. \tag{9.17}$$

In subsection 6.2 we found that the varying action is

$$\delta S = -\int \delta\bar{\Psi}\not{\partial}\Psi, \tag{9.18}$$

and we saw that the Lagrangian is invariant under the following infinitesimal Poincaré transformations

$$\delta_a \Psi = a^\mu \partial_\mu \Psi, \tag{9.19}$$
$$\delta_b \Psi = b^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu + \Sigma_{\mu\nu})\Psi. \tag{9.20}$$

It is also invariant under a *global* transformation

$$\Psi \to \mathrm{e}^{\alpha\gamma_5}\Psi, \tag{9.21}$$

where $\alpha \in \mathbb{R}$. The exponential of a matrix X is defined as

$$\mathrm{e}^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}. \tag{9.22}$$

Then,

$$\begin{aligned}\mathrm{e}^{\alpha\gamma_5} &= \mathbb{1} + \alpha\gamma_5 + \frac{1}{2}\alpha^2\gamma_5^2 + \frac{1}{6}\alpha^3\gamma_5^3 + \dots \\ &= \mathbb{1} + \alpha\gamma_5 - \frac{1}{2}\alpha^2\mathbb{1} - \frac{1}{6}\alpha^3\gamma_5 + \dots.\end{aligned}$$

That is, for even powers of γ_5, we get $\pm\mathbb{1}$, while for odd powers of γ_5 we get $\pm\gamma_5$. We recall that

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!}x^{2k}, \qquad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}. \tag{9.23}$$

Thus, we can identify

$$\mathrm{e}^{\alpha\gamma_5} = \cos(\alpha)\mathbb{1} + \sin(\alpha)\gamma_5. \tag{9.24}$$

We find how $\bar{\Psi}$ transforms:

$$\begin{aligned}\bar{\Psi} \to \overline{\mathrm{e}^{\alpha\gamma_5}\Psi} &= (\mathrm{e}^{\alpha\gamma_5}\Psi)^t C = -\Psi^t(\mathrm{e}^{\alpha\gamma_5})^t C^t = -\Psi^t(C\mathrm{e}^{\alpha\gamma_5})^t \\ &= -\Psi^t(C(\cos(\alpha)\mathbb{1} + \sin(\alpha)\gamma_5))^t = \Psi^t C(\cos(\alpha)\mathbb{1} + \sin(\alpha)\gamma_5) \\ &= \bar{\Psi}\mathrm{e}^{\alpha\gamma_5}.\end{aligned}$$

Hence, the Lagrangian density transforms as

$$\begin{aligned}\mathscr{L} = \frac{1}{2}\bar{\Psi}\not{\partial}\Psi \to \frac{1}{2}\bar{\Psi}\mathrm{e}^{\alpha\gamma_5}\gamma^\mu\partial_\mu(\mathrm{e}^{\alpha\gamma_5}\Psi) &= \frac{1}{2}\bar{\Psi}(\cos(\alpha)\mathbb{1} + \sin(\alpha)\gamma_5)\gamma^\mu\mathrm{e}^{\alpha\gamma_5}\partial_\mu\Psi \\ &= \frac{1}{2}\bar{\Psi}\gamma^\mu(\cos(\alpha)\mathbb{1} - \sin(\alpha)\gamma_5)\mathrm{e}^{\alpha\gamma_5}\partial_\mu\Psi \\ &= \frac{1}{2}\bar{\Psi}\gamma^\mu\mathrm{e}^{-\alpha\gamma_5}(\mathrm{e}^{\alpha\gamma_5}\partial_\mu\Psi) = \frac{1}{2}\bar{\Psi}\not{\partial}\Psi.\end{aligned}$$

Let us couple the fermionic Lagrangian with $\mathscr{L}_{\mathrm{EM}}$, to construct the Lagrangian of supersymmetric electromagnetism:

$$\mathscr{L}_{\mathrm{SEM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\not{\partial}\Psi. \tag{9.25}$$

The varying action is

$$\delta S_{\text{SEM}} = \int \delta A_\mu \partial_\nu F^{\nu\mu} - \delta\bar{\Psi}\partial\!\!\!/\Psi. \tag{9.26}$$

We have already shown that this is invariant under the Poincaré algebra. We should check that it is also invariant under supersymmetry. [10] gives us the following supersymmetry transformations:

$$\delta_\epsilon A_\mu = \bar{\epsilon}\gamma_\mu\Psi, \tag{9.27}$$

$$\delta_\epsilon \Psi = -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon. \tag{9.28}$$

We find the transformation on $\bar{\Psi}$:

$$\begin{aligned}\delta_\epsilon\bar{\Psi} &= -\overline{\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon} = -\frac{1}{2}F_{\mu\nu}(\gamma^{\mu\nu}\epsilon)^t C = \frac{1}{2}F_{\mu\nu}\epsilon^t(\gamma^{\mu\nu})^t C^t = \frac{1}{2}F_{\mu\nu}\epsilon^t(C\gamma^{\mu\nu})^t \\ &= \frac{1}{2}F_{\mu\nu}\epsilon^t C\gamma^{\mu\nu} = \frac{1}{2}\bar{\epsilon}F_{\mu\nu}\gamma^{\mu\nu}.\end{aligned}$$

Then, applying the infinitesimal supersymmetry transformation to (9.26), we get

$$\delta_\epsilon S_{\text{SEM}} = \int \bar{\epsilon}\gamma_\mu\Psi\partial_\nu F^{\nu\mu} - \frac{1}{2}\bar{\epsilon}F_{\mu\nu}\gamma^{\mu\nu}\partial\!\!\!/\Psi.$$

Let us look at the last term:

$$-\frac{1}{2}\int \bar{\epsilon}F_{\mu\nu}\gamma^{\mu\nu}\gamma^\rho\partial_\rho\Psi = \frac{1}{2}\int \bar{\epsilon}\partial_\rho F_{\mu\nu}(\gamma^{\mu\nu\rho} + \eta^{\rho\nu}\gamma_\mu - \eta^{\rho\mu}\gamma_\nu)\Psi. \tag{9.29}$$

The term including $\gamma^{\mu\nu\rho}$ can be rewritten

$$\begin{aligned}\partial_\rho F_{\mu\nu}\gamma^{\mu\nu\rho} &= \frac{1}{3}(\partial_\rho F_{\mu\nu}\gamma^{\mu\nu\rho} + \partial_\mu F_{\nu\rho}\gamma^{\nu\rho\mu} + \partial_\nu F_{\rho\mu}\gamma^{\rho\mu\nu}) \\ &= \frac{1}{3}(\partial_\rho F_{\mu\nu}\gamma^{\mu\nu\rho} + \partial_\mu F_{\nu\rho}\gamma^{\mu\nu\rho} + \partial_\nu F_{\rho\mu}\gamma^{\mu\nu\rho}) \\ &= \frac{1}{3}(\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu})\gamma^{\mu\nu\rho} = 0.\end{aligned}$$

This vanishes due to the Bianchi identity (9.11), which we derived above. The last two terms in (9.29) are

$$\frac{1}{2}\int \bar{\epsilon}\partial_\rho F_{\mu\nu}(\eta^{\rho\nu}\gamma_\mu - \eta^{\rho\mu}\gamma_\nu)\Psi = \frac{1}{2}\int \bar{\epsilon}(\partial_\rho F^{\mu\rho}\gamma_\mu - \partial_\rho F^{\rho\nu}\gamma_\nu)\Psi = -\int \bar{\epsilon}\partial_\nu F^{\nu\mu}\gamma_\mu\Psi,$$

which cancel the first term in $\delta_\epsilon S_{\text{SEM}}$. Hence, $\delta_\epsilon S_{\text{SEM}} = 0$, and $\mathscr{L}_{\text{SEM}}$ is invariant under the supersymmetry transformations.

Using the supersymmetry transformations

$$\delta_\epsilon A_\mu = \bar{\epsilon}\gamma_\mu\Psi, \qquad \delta_\epsilon\Psi = -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon, \tag{9.30}$$

and $\delta_\epsilon \varphi = \bar{\epsilon} Q \varphi = -\epsilon^a Q_a \varphi$ we find how the supercharge Q acts on the two fields. On A_μ it is easily seen that $Q_a A_\mu = -(\gamma_\mu)_a{}^b \Psi_b$. On Ψ it is slightly more complicated:

$$\delta_\epsilon \Psi^a = -\epsilon^b Q_b \Psi^a = -\frac{1}{2} F_{\mu\nu} (\gamma^{\mu\nu})^a{}_b \epsilon^b.$$

Lowering the a index with C_{ac} gives

$$-Q_b \Psi_c = -\frac{1}{2} F_{\mu\nu} (\gamma^{\mu\nu})^a{}_b C_{ac} = -\frac{1}{2} F_{\mu\nu} (\gamma^{\mu\nu})_{cb}.$$

Relabelling, and using $(\gamma^{\mu\nu})_{ab} = (\gamma^{\mu\nu})_{ba}$, we find

$$Q_a A_\mu = -(\gamma_\mu)_a{}^b \Psi_b, \qquad Q_a \Psi_b = \frac{1}{2} F_{\mu\nu} (\gamma^{\mu\nu})_{ab}.$$

We calculate the brackets $[Q_a, Q_b]\varphi$, beginning with $\varphi = A_\mu$:

$$\begin{aligned}
[Q_a, Q_b] A_\mu &= Q_a(-(\gamma_\mu)_b{}^c \Psi_c) + Q_b(-(\gamma_\mu)_a{}^c \Psi_c) = -(\gamma_\mu)_b{}^c \frac{1}{2} F_{\nu\rho} (\gamma^{\nu\rho})_{ac} - (\gamma_\mu)_a{}^c \frac{1}{2} F_{\nu\rho} (\gamma^{\nu\rho})_{bc} \\
&= -\frac{1}{2} F_{\nu\rho} ((\gamma_\mu)_b{}^c (\gamma^{\nu\rho})_{ca} + (\gamma_\mu)_a{}^c (\gamma^{\nu\rho})_{cb} = -\frac{1}{2} F_{\nu\rho} ((\gamma_\mu \gamma^{\nu\rho})_{ba} + (\gamma_\mu \gamma^{\nu\rho})_{ab}).
\end{aligned}$$

We use (5.9) to find

$$(\gamma_\mu \gamma^{\nu\rho})_{ba} = (\gamma_\mu{}^{\nu\rho} + \delta_\mu^\nu \gamma^\rho - \delta_\mu^\rho \gamma^\nu)_{ab} = -(\gamma_\mu{}^{\nu\rho} - \delta_\mu^\nu \gamma^\rho + \delta_\mu^\rho \gamma^\nu)_{ab}$$

Then,

$$\begin{aligned}
[Q_a, Q_b] A_\mu &= -\frac{1}{2} F_{\nu\rho} ((\gamma_\mu \gamma^{\nu\rho})_{ba} + (\gamma_\mu \gamma^{\nu\rho})_{ab}) \\
&= -\frac{1}{2} F_{\nu\rho} (-(\gamma_\mu{}^{\nu\rho} - \delta_\mu^\nu \gamma^\rho + \delta_\mu^\rho \gamma^\nu)_{ab} + (\gamma_\mu{}^{\nu\rho} + \delta_\mu^\nu \gamma^\rho - \delta_\mu^\rho \gamma^\nu)_{ab}) \\
&= -\frac{1}{2} F_{\nu\rho} (2\delta_\mu^\nu (\gamma^\rho)_{ab} - 2\delta_\mu^\rho (\gamma^\nu)_{ab}) = (\gamma^\nu)_{ab} F_{\nu\mu} - (\gamma^\rho)_{ab} F_{\mu\rho} \\
&= (\gamma^\nu)_{ab} F_{\nu\mu} + (\gamma^\nu)_{ab} F_{\nu\mu} = 2(\gamma^\nu)_{ab} F_{\nu\mu}.
\end{aligned}$$

By the definition of $F_{\nu\mu}$ we have

$$[Q_a, Q_b] A_\mu = 2(\gamma^\nu)_{ab} \partial_\nu A_\mu - 2(\gamma^\nu)_{ab} \partial_\mu A_\nu = 2(\gamma^\nu)_{ab} P_\nu A_\mu - 2(\gamma^\nu)_{ab} \partial_\mu A_\nu. \tag{9.31}$$

Recognising the last term as a gauge transformation, $\partial_\mu A_\nu = 0$, (9.31) agrees with the Poincaré superalgebra found in subsection 8.1.

In the calculation of the other bracket, $[Q_a, Q_b]\Psi_c$, we use (B.4):

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\Psi = \epsilon_1^a \epsilon_2^b [Q_a, Q_b]\Psi. \tag{9.32}$$

By using the supersymmetry transformations, we get

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\Psi &= \delta_{\epsilon_1}(-\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon_2) - (1 \leftrightarrow 2) \\
&= -\frac{1}{2}\gamma^{\mu\nu}\epsilon_2(\partial_\mu(\delta_{\epsilon_1}A_\nu) - \partial_\nu(\delta_{\epsilon_1}A_\mu)) - (1 \leftrightarrow 2) \\
&= -\frac{1}{2}\gamma^{\mu\nu}\epsilon_2(\partial_\mu(\bar{\epsilon}_1\gamma_\nu\Psi) - \partial_\nu(\bar{\epsilon}_1\gamma_\mu\Psi)) - (1 \leftrightarrow 2) \\
&= -\frac{1}{2}\gamma^{\mu\nu}\epsilon_2\bar{\epsilon}_1(\gamma_\nu\partial_\mu\Psi - \gamma_\mu\partial_\nu\Psi) - (1 \leftrightarrow 2) \\
&= \frac{1}{2}\gamma^{\mu\nu}(\epsilon_1\bar{\epsilon}_2 - \epsilon_2\bar{\epsilon}_1)(\gamma_\nu\partial_\mu\Psi - \gamma_\mu\partial_\nu\Psi) \\
&= \frac{1}{2}\gamma^{\mu\nu}\left(\frac{1}{2}(\bar{\epsilon}_1\gamma^\rho\epsilon_2)\gamma_\rho - \frac{1}{4}(\bar{\epsilon}_1\gamma^{\rho\sigma}\epsilon_2)\gamma_{\rho\sigma}\right)(\gamma_\nu\partial_\mu\Psi - \gamma_\mu\partial_\nu\Psi),
\end{aligned}
$$

where we have used the Fierz identity (A.32) in the last line. Furthermore, since $\gamma^{\mu\nu}\gamma_\mu\partial_\nu = \gamma^{\nu\mu}\gamma_\nu\partial_\mu = -\gamma^{\mu\nu}\gamma_\nu\partial_\mu$, we get

$$
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\Psi = \left(\frac{1}{2}(\bar{\epsilon}_1\gamma^\rho\epsilon_2)\gamma^{\mu\nu}\gamma_\rho - \frac{1}{4}(\bar{\epsilon}_1\gamma^{\rho\sigma}\epsilon_2)\gamma^{\mu\nu}\gamma_{\rho\sigma}\right)\gamma_\nu\partial_\mu\Psi.
$$

The γ-matrices in the first term gives

$$
\gamma^{\mu\nu}\gamma_\rho\gamma_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_\rho\gamma_\nu.
$$

Using (5.29) $\gamma^\nu\gamma_\mu\gamma_\nu = -2\gamma_\mu$, (5.30) $\gamma^\rho\gamma_{\mu\nu}\gamma_\rho = 0$ and $\gamma^\mu\gamma_\mu = \delta^\mu_\mu = 4\mathbb{1}$, this becomes

$$
\begin{aligned}
\gamma^{\mu\nu}\gamma_\rho\gamma_\nu = -\gamma^\mu\gamma_\rho - \frac{1}{2}\eta^{\mu\sigma}\gamma^\nu(\gamma_{\sigma\rho} + \eta_{\sigma\rho})\gamma_\nu &= -(\gamma^m{}u_\rho + \delta^\mu_\rho) - \frac{1}{2}\delta^\mu_\rho\gamma^\nu\gamma_\nu \\
= -(-\gamma_\rho{}^\mu + \delta^\mu_\rho) - 2\delta^\mu_\rho = (\gamma_\rho\gamma^\mu - \delta^\mu_\rho) - 3\delta^\mu_\rho &= \gamma_\rho\gamma^\mu - 4\delta^\mu_\rho.
\end{aligned}
$$

In the second term, the γ-matrices are

$$
\begin{aligned}
\gamma^{\mu\nu}\gamma_{\rho\sigma}\gamma_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_{\rho\sigma}\gamma_\nu &= -\frac{1}{2}\eta^{\mu\lambda}\gamma^\nu\gamma_\lambda\gamma_{\rho\sigma}\gamma_\nu \\
&= -\frac{1}{2}\eta^{\mu\lambda}\gamma^\nu\left(\gamma_{\lambda\rho\sigma} + \eta_{\lambda\rho}\gamma_\sigma - \eta_{\lambda\sigma}\gamma_\rho\right)\gamma_\nu \\
&= -\frac{1}{2}\eta^{\mu\lambda}\left(\gamma^\nu(\epsilon_{\lambda\rho\sigma\delta}\gamma^\delta\gamma_5\gamma_\nu - 2\eta_{\lambda\rho}\gamma_\sigma + 2\eta_{\lambda\sigma}\gamma_\rho\right) \\
&= -\frac{1}{2}\eta^{\mu\lambda}\left(-\gamma^\nu(\epsilon_{\lambda\rho\sigma\delta}\gamma^\delta\gamma_\nu\gamma_5 - 2\eta_{\lambda\rho}\gamma_\sigma + 2\eta_{\lambda\sigma}\gamma_\rho\right) \\
&= -\frac{1}{2}\eta^{\mu\lambda}\left(2\epsilon_{\lambda\rho\sigma\delta}\gamma^\delta\gamma_5 - 2\eta_{\lambda\rho}\gamma_\sigma + 2\eta_{\lambda\sigma}\gamma_\rho\right) \\
&= -\epsilon^\mu{}_{\rho\sigma\delta}\gamma^\delta\gamma_5 + \delta^\mu_\rho\gamma_\sigma - \delta^\mu_\sigma\gamma_\rho.
\end{aligned}
$$

Rewriting the first term

$$
\epsilon^\mu{}_{\rho\sigma\delta}\gamma^\delta\gamma_5 = \gamma^\mu{}_{\rho\sigma} = \gamma_{\rho\sigma}{}^\mu = \gamma_{\rho\sigma}\gamma^\mu - \delta^\mu_\sigma\gamma_\rho + \delta^\mu_\rho\gamma_\sigma.
$$

We are left with only $-\gamma_{\rho\sigma}\gamma^\mu$. Thus,

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\Psi &= -\left(\frac{1}{2}(\bar{\epsilon}_1\gamma^\rho\epsilon_2)(\gamma_\rho\gamma^\mu - 4\delta^\mu_\rho) - \frac{1}{4}(\bar{\epsilon}_1\gamma^{\rho\sigma}\epsilon_2)\gamma_{\rho\sigma}\gamma^\mu\right)\partial_\mu\Psi \\
&= -\left(\frac{1}{2}(\epsilon^a_1(\gamma^\rho)_{ab}\epsilon^b_2)(\gamma_\rho\not{\partial} - 4\partial_\rho) - \frac{1}{4}(\epsilon^a_1(\gamma^{\rho\sigma})_{ab}\epsilon^b_2)\gamma_{\rho\sigma}\not{\partial}\right)\Psi \qquad (9.33)
\end{aligned}
$$

Equating (9.32) and (9.33)

$$\epsilon_1^a\epsilon_2^b[Q_a,Q_b]\Psi = -\epsilon_1^a\epsilon_2^b\left(\frac{1}{2}(\gamma^\rho)_{ab}(\gamma_\rho\partial\!\!\!/ - 4\partial_\rho) - \frac{1}{4}(\gamma^{\rho\sigma})_{ab}\gamma_{\rho\sigma}\partial\!\!\!/\right)\Psi.$$

Thus,

$$[Q_a,Q_b]\Psi = 2(\gamma^\rho)_{ab}P_\rho\Psi - \frac{1}{2}(\gamma^\rho)_{ab}\gamma_\rho\partial\!\!\!/\Psi + \frac{1}{4}(\gamma^{\rho\sigma})_{ab}\gamma_{\rho\sigma}\partial\!\!\!/\Psi. \tag{9.34}$$

Applying the equations of motion $\partial\!\!\!/\Psi = 0$, (9.34) agrees with the Poincaré superalgebra on-shell.

We also check for conformal invariance. The fermionic Lagrangian is already know to be invariant under the conformal algebra, so we only need to check $\mathscr{L}_{\text{EM}}$. Checking invariance under the dialtions is enough. We try the transformation

$$\delta_c A_\mu = cx^\nu\partial_\nu A_\mu. \tag{9.35}$$

Then,

$$\begin{aligned}
\delta_c S_{\text{EM}} &= \int cx^\rho\partial_\rho A^i_\mu\partial_\nu F_i^{\nu\mu} = -\int c\left(\delta^\rho_\rho A^i_\mu\partial_\nu F_i^{\nu\mu} + x^\rho A^i_\mu\partial_\rho\partial_\nu F_i^{\nu\mu}\right)\\
&= -\int c\left(4A^i_\mu\partial_\nu F_i^{\nu\mu} - (\delta^\rho_\nu + x^\rho\partial_\nu)A^i_\mu\partial_\rho(\partial^\nu A_i^\mu - \partial^\mu A_i^\nu\right)\\
&= -\int c\left(3A^i_\mu\partial_\nu F_i^{\nu\mu} - x^\rho\partial_\nu A^i_\mu\partial_\rho(\partial^\nu A_i^\mu - \partial^\mu A_i^\nu\right)\\
&= -\int c\left(3A^i_\mu\partial_\nu F_i^{\nu\mu} + (\eta^{\rho\nu} + x^\rho\partial^\nu)\partial_\nu A^i_\mu\partial_\rho A_i^\mu - (\eta^{\rho\mu} + x^\rho\partial_\mu)\partial_\nu A^i_\mu\partial_\rho A_i^\nu\right)\\
&= -\int c\left(3A^i_\mu\partial_\nu F_i^{\nu\mu} + \partial_\nu A^i_\mu\partial^\nu A_i^\mu - \partial_\nu A^i_\mu\partial^\mu A_i^\nu + x^\rho\partial^\nu\partial_\nu A^i_\mu\partial_\rho A_i^\mu - x^\rho\partial_\mu\partial_\nu A^i_\mu\partial_\rho A_i^\nu\right)\\
&= -\int c\left(3A^i_\mu\partial_\nu F_i^{\nu\mu} + \partial_\nu A^i_\mu(\partial^\nu A_i^\mu - \partial^\mu A_i^\nu) + x^\rho\partial^\nu\partial_\nu A^i_\mu\partial_\rho A_i^\mu - x^\rho\partial_\nu\partial_\mu A^i_\nu\partial_\rho A_i^\mu\right)\\
&= -\int c\left(3A^i_\mu\partial_\nu F_i^{\nu\mu} + \partial_\nu A^i_\mu F_i^{\nu\mu} + x^\rho\partial_\rho A^i_\mu\partial_\nu(\partial^\nu A_i^\mu - \partial^\mu A_i^\nu)\right)\\
&= -\int c\left(3A^i_\mu\partial_\nu F_i^{\nu\mu} - A^i_\mu\partial_\nu F_i^{\nu\mu} + x^\rho\partial_\rho A^i_\mu\partial_\nu F_i^{\nu\mu}\right)\\
&= -\int c\left(x^\rho\partial_\rho + 2A^i_\mu\right)\partial_\nu F_i^{\nu\mu} \neq 0.
\end{aligned}$$

We see that if we instead choose the transformation

$$\delta_c A_\mu = cx^\nu\partial_\nu A_\mu + A_\mu, \tag{9.36}$$

$\delta_c S_{\text{EM}} = 0$. Hence, S_{EM} is invariant under the conformal algebra.

Let us now check for invariance under the conformal superalgebra. [10] gives the following transformations:

$$\delta_\zeta A_\mu = \bar{\zeta}x^\nu\gamma_\nu\gamma_\mu\Psi, \qquad \delta_\zeta\Psi = \frac{1}{2}x_\rho F_{\mu\nu}\gamma^{\mu\nu}\gamma^\rho\zeta. \tag{9.37}$$

We find the transformation on $\bar{\Psi}$:

$$\begin{aligned}\delta_\zeta \bar{\Psi} &= \frac{1}{2} x_\rho F_{\mu\nu} \overline{\gamma^{\mu\nu}\gamma^\rho \zeta} = \frac{1}{2} x_\rho F_{\mu\nu} (\gamma^{\mu\nu}\gamma^\rho\zeta)^t C = -\frac{1}{2} x_\rho F_{\mu\nu} \zeta^t (\gamma^\rho)^t (\gamma^{\mu\nu})^t C^t \\ &= -\frac{1}{2} x_\rho F_{\mu\nu} \zeta^t (\gamma^\rho)^t C \gamma^{\mu\nu} = \frac{1}{2} x_\rho F_{\mu\nu} \zeta^t C \gamma^\rho \gamma^{\mu\nu} = \frac{1}{2} \bar{\zeta} x_\rho F_{\mu\nu} \gamma^\rho \gamma^{\mu\nu}. \quad (9.38)\end{aligned}$$

Applying these supersymmetry transformation to δS_{SEM} gives

$$\begin{aligned}\delta_\zeta S_{\text{SEM}} &= \int \delta_\zeta A_\mu \partial_\nu F^{\nu\mu} - \delta_\zeta \bar{\Psi} \partial\!\!\!/ \Psi = \int \bar{\zeta} x^\rho \gamma_\rho \gamma_\mu \Psi \partial_\nu F^{\nu\mu} - \frac{1}{2} \bar{\zeta} x_\rho F_{\mu\nu} \gamma^\rho \gamma^{\mu\nu} \gamma^\sigma \partial_\sigma \Psi \\ &= \int \bar{\zeta} x^\rho \gamma_\rho \gamma_\mu \Psi \partial_\nu F^{\nu\mu} + \frac{1}{2} \bar{\zeta} \eta_{\sigma\rho} F_{\mu\nu} \gamma^\rho \gamma^{\mu\nu} \gamma^\sigma \Psi + \frac{1}{2} \bar{\zeta} x_\rho \partial_\sigma F_{\mu\nu} \gamma^\rho \gamma^{\mu\nu} \gamma^\sigma \Psi.\end{aligned}$$

The second term vanishes since $\gamma^\rho \gamma^{\mu\nu} \gamma_\rho = 0$. In the last term we can rewrite

$$\gamma^{\mu\nu}\gamma^\sigma = \gamma^{\mu\nu\sigma} + \eta^{\sigma\nu}\gamma^\mu - \eta^{\sigma\mu}\gamma^\nu.$$

The $\gamma^{\mu\nu\sigma}$ part vanish due to the Bianchi identity, while the $\eta^{\sigma\nu}\gamma^\mu - \eta^{\sigma\mu}\gamma^\nu$ part cancel the final term in $\delta_\zeta S_{\text{SEM}}$. Hence, $\delta_\zeta S_{\text{SEM}} = 0$. Before we can conclude that δS_{SEM} is invariant under the conformal superalgebra we need to check R-symmetry. [10] tells us how R acts on the fields:

$$RA_\mu = 0, \qquad R\Psi = \frac{1}{2}\gamma_5 \Psi. \quad (9.39)$$

Using $\delta_e \varphi = eR\varphi$, we find

$$\delta_e A_\mu = 0, \qquad \delta_e \Psi = \frac{1}{2} e \gamma_5 \Psi, \delta_e \bar{\Psi} = \frac{1}{2} e \bar{\Psi} \gamma_5 \quad (9.40)$$

Thus,

$$\begin{aligned}\delta_e S_{\text{SEM}} &= \int \delta_e A_\mu \partial_\nu F^{\nu\mu} - \delta_e \bar{\Psi} \partial\!\!\!/ \Psi = -\frac{1}{2} \int e \bar{\Psi} \gamma_5 \gamma^\mu \partial_\mu \Psi \\ &= \frac{1}{2} \int e \Psi^a (\gamma^\mu \gamma_5)_{ab} \partial_\mu \Psi^b = -\frac{1}{2} \int e \partial_\mu \Psi^a (\gamma^\mu \gamma_5)_{ab} \Psi^b \\ &= \frac{1}{2} \int e \Psi^b (\gamma^\mu \gamma_5)_{ab} \partial_\mu \Psi^a = -\frac{1}{2} \int e \Psi^b (\gamma^\mu \gamma_5)_{ba} \partial_\mu \Psi^a = 0.\end{aligned}$$

We have shown that S_{SEM} is invariant under the conformal superalgebra.

Part IV
Summary

The thesis started out with computing the Poincaré and conformal algebra from Killing vector fields. This was done in arbitrary dimensions. The Clifford algebra was also computed in arbitrary dimensions, but some of the important results from that section was restricted to four dimensions, among them the charge conjugation matrix.

We also computed several examples, kinetic- massive- and interaction Lagrangians, and check whether or not the Poincaré and conformal algebra were symmetries of these. We found that the massive Lagrangian was not invariant under the conformal algebra, and that in four dimensions, only interactions of fourth power could be symmetric under the conformal algebra. It should be pointed out that the last point only applies to terms like ϕ^4, we saw in the Wess-Zumino model section that also terms like $\phi^2\tilde{\phi}^2$ can be symmetric under the conformal algebra.

We have also seen how the algebra proposed by Haag, Łopuszański and Sohnius, superalgebras, can generate symmetries of Lagrangians. We saw that in the Wess-Zumino model, the Poincaré superalgebra closed on-shell, and so did the confomral superalgebra. In the gauge theory electromagnetism, we also needed a gauge transformation for the superalgebras to close.

Appendices

A Some Proofs

In this Appendix we give proofs of identities and other things we have had a need for.

A.1 Clifford Algebras

This section is taken from [9], and shows how Table 1 was made.

A real Clifford algebra over the field $\mathbb{R}$, $C\ell(p,q)$, is an associative algebra containing the unit, $\mathbb{1}$. It has generators Γ_i, such that

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\eta_{ij}\mathbb{1}, \tag{A.1}$$

where $\eta_{ij} = \text{Diag}(\underbrace{-1,\ldots,-1}_{p}, \underbrace{+1,\ldots,+1}_{q})$. We may rewrite (A.1) as

$$\Gamma_i{}^2 = \eta_{ii}\mathbb{1}, \tag{A.2}$$

$$\Gamma_i\Gamma_j = -\Gamma_j\Gamma_i, \tag{A.3}$$

where $i \neq j$. We begin with looking at the Clifford algebra in dimensions 0, 1, and 2.

Dimension 0 is trivial. We have both $p = q = 0$, thus only the identity is contained in the algebra, $C\ell(0,0) = \{\mathbb{1}\} \cong \mathbb{R}$.

Dimension 1 is more interesting. We first consider $p = 1$, $q = 0$. Since p is negative in the Minkowski metric we have the generator Γ satisfying $\Gamma^2 = -1$. Then Γ is complex, and $C\ell(1,0) = \{\mathbb{1}, \Gamma \mid \Gamma^2 = -1\} \cong \mathbb{C}$. The other possibility is $p = 0$, $q = 1$. Now we have $\Gamma^2 = 1$, so that Γ is real. $C\ell(0,1) = \{\mathbb{1}, \Gamma \mid \Gamma^2 = 1\} \cong \mathbb{R} \oplus \mathbb{R}$.

Now we look at dimension 2. We start with $p = 2$, $q = 0$. Now there are two generators, Γ_1 and Γ_2. They satisfy $\Gamma_1^2 = \Gamma_2^2 = -1$, $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. We may choose

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{A.4}$$

We also need to see whether these anticommute or not.

$$\Gamma_1\Gamma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -\Gamma_2\Gamma_1. \tag{A.5}$$

We may relate these generators with the quaternions, $\Gamma_1 \leftrightarrow i$, $\Gamma_2 \leftrightarrow j$ and $\Gamma_1\Gamma_2 \leftrightarrow k$. Thus, we have the isomorphism $C\ell(2,0) \cong \mathbb{H}$. Next we look at the case $p = q = 1$. Now the generators must satisfy $\Gamma_1{}^2 = -\Gamma_2{}^2 = 1$, $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. A possible choice of Γ-matrices is

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma_1\Gamma_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\Gamma_2\Gamma_1. \tag{A.6}$$

These generators generate the real (2×2)-matrices, thus $C\ell(1,1) \cong \mathrm{Mat}_2(\mathbb{R})$. Only the case $p = 0$, $q = 2$ remains. The generators satisfy ${\Gamma_1}^2 = {\Gamma_2}^2 = 1$, $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. One choice of generators is

$$\Gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_1\Gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\Gamma_2\Gamma_1. \tag{A.7}$$

Thus, $\mathbb{1}, \Gamma_1, \Gamma_2, \Gamma_1\Gamma_2$ are the generators of $C\ell(2,0)$. These matrices are all real, and together they generate the real (2×2)-matrices, $C\ell(0,2) \cong \mathrm{Mat}_2(\mathbb{R})$.

To find the rest of the Clifford algebra we will take use of further isomorphisms, shown in Lemma A.1.

Lemma A.1. *The following isomorphisms hold:*

$$C\ell(0,d) \otimes C\ell(2,0) \cong C\ell(d+2,0), \tag{A.8}$$
$$C\ell(d,0) \otimes C\ell(0,2) \cong C\ell(0,d+2), \tag{A.9}$$
$$C\ell(p,q) \otimes C\ell(1,1) \cong C\ell(p+1,q+1). \tag{A.10}$$

Proof. We prove (A.8) first. Let $\gamma_1, \dots, \gamma_d$ be the generators of $C\ell(0,d)$ and let σ_1, σ_2 be the generators of $C\ell(2,0)$. We have the following relations:

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\mathbb{1},$$
$$\sigma_i\sigma_j + \sigma_j\sigma_i = -2\delta_{ij}\mathbb{1}.$$

We define a new matrix as

$$\Gamma_i = \begin{cases} \gamma_i \otimes \sigma_1\sigma_2 & , 1 \leqslant i \leqslant d, \\ \mathbb{1} \otimes \sigma_{i-d} & , i = d+1, d+2. \end{cases} \tag{A.11}$$

We need to know if these Γ_i satisfy $\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\eta_{ij}\mathbb{1}$.

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i$$
$$= \begin{cases} (\gamma_i \otimes \sigma_1\sigma_2)(\gamma_j \otimes \sigma_1\sigma_2) + (\gamma_j \otimes \sigma_1\sigma_2)(\gamma_i \otimes \sigma_1\sigma_2) & , 1 \leqslant i, j \leqslant d, \quad ① \\ (\gamma_i \otimes \sigma_1\sigma_2)(\mathbb{1} \otimes \sigma_{j-d}) + (\mathbb{1} \otimes \sigma_{j-d})(\gamma_i \otimes \sigma_1\sigma_2) & , 1 \leqslant i \leqslant d, j = d+1, d+2, ② \\ (\mathbb{1} \otimes \sigma_{i-d})(\gamma_j \otimes \sigma_1\sigma_2) + (\gamma_j \otimes \sigma_1\sigma_2)(\mathbb{1} \otimes \sigma_{i-d}) & , i = d+1, d+2, 1 \leqslant j \leqslant d, ③ \\ (\mathbb{1} \otimes \sigma_{i-d})(\mathbb{1} \otimes \sigma_{j-d}) + (\mathbb{1} \otimes \sigma_{j-d})(\mathbb{1} \otimes \sigma_{i-d}) & , i, j = d+1, d+2. \quad ④ \end{cases}$$

Prior to the calculations of equations ① - ④, we make the following observations. Let us first recall the matrices generating $C\ell(2,0)$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_1\sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -\sigma_2\sigma_1. \tag{A.12}$$

Since $i = d+1, d+2$ in σ_{i-d}, we have $\sigma_{i-d} = \sigma_1, \sigma_2$. Now, we find $\sigma_1\sigma_2\sigma_{i-d}$ and $\sigma_{i-d}\sigma_1\sigma_2$.

$$\sigma_1\sigma_2\sigma_{i-d} = \begin{cases} \sigma_2 & , i = d+1, \\ -\sigma_1 & , i = d+2. \end{cases} \quad \sigma_{i-d}\sigma_1\sigma_2 = \begin{cases} -\sigma_2 & , i = d+1, \\ \sigma_1 & , i = d+2. \end{cases}$$

This means that $\sigma_1\sigma_2\sigma_{i-d} + \sigma_{i-d}\sigma_1\sigma_2 = 0$. It is also useful to note that $\sigma_1\sigma_2\sigma_1\sigma_2 = -\mathbb{1}$.

Equation ① gives

$$\begin{aligned}&(\gamma_i\gamma_j)\otimes(\sigma_1\sigma_2\sigma_1\sigma_2)+(\gamma_j\gamma_i)\otimes(\sigma_1\sigma_2\sigma_1\sigma_2)\\&\quad=(\gamma_i\gamma_j+\gamma_j\gamma_i)\otimes(\sigma_1\sigma_2\sigma_1\sigma_2)\\&\quad=2\delta_{ij}\mathbb{1}\otimes-\mathbb{1}\\&\quad=-2\delta_{ij}\mathbb{1}.\end{aligned}$$

② gives

$$\begin{aligned}&(\gamma_i\mathbb{1})\otimes(\sigma_1\sigma_2\sigma_{j-d})+(\mathbb{1}\gamma_i)\otimes(\sigma_{j-d}\sigma_1\sigma_2)\\&\quad=\gamma_i\otimes(\sigma_1\sigma_2\sigma_{j-d}+\sigma_{j-d}\sigma_1\sigma_2)\\&\quad=0.\end{aligned}$$

③ gives

$$\begin{aligned}&(\mathbb{1}\gamma_j)\otimes(\sigma_{i-d}\sigma_1\sigma_2)+(\gamma_j\mathbb{1})\otimes(\sigma_1\sigma_2\sigma_{i-d})\\&\quad=\gamma_j\otimes(\sigma_{i-d}\sigma_1\sigma_2+\sigma_1\sigma_2\sigma_{i-d})\\&\quad=0.\end{aligned}$$

④ gives

$$\sigma_{i-d}\sigma_{j-d}+\sigma_{j-d}\sigma_{i-d}=-2\delta_{ij}\mathbb{1}.$$

Thus, $\Gamma_i\Gamma_j+\Gamma_j\Gamma_i=-2\delta_{ij}\mathbb{1}$, and Γ_i generates the algebra $C\ell(d+2,0)$. The proof of (A.9) is similar, with only some changes of sign.

(A.10) mixes the two above. The procedure is similar, but more complicated. Let $\gamma_1,\dots,\gamma_p$ and $\tilde\gamma_1\dots\tilde\gamma_q$ generate $C\ell(p,q)$, and σ_1 and σ_2 generate $C\ell(1,1)$. These generators satisfy

$$\begin{aligned}\gamma_i\gamma_j+\gamma_j\gamma_i&=-2\delta_{ij}\mathbb{1}, & \tilde\gamma_i\tilde\gamma_j+\tilde\gamma_j\tilde\gamma_i&=2\delta_{ij}\mathbb{1}, & \gamma_i\tilde\gamma_j+\tilde\gamma_j\gamma_i&=0,\\ {\sigma_1}^2&=-1, & {\sigma_2}^2&=1, & \sigma_1\sigma_2+\sigma_2\sigma_1&=0.\end{aligned}$$

This time we define two matrices

$$\Gamma_i=\begin{cases}\gamma_i\otimes\sigma_1\sigma_2 & ,1\le i\le p,\\ \mathbb{1}\otimes\sigma_1 & ,i=p+1.\end{cases},\quad \tilde\Gamma_i=\begin{cases}\tilde\gamma_i\otimes\sigma_1\sigma_2 & ,1\le i\le q,\\ \mathbb{1}\otimes\sigma_2 & ,i=q+1.\end{cases}\tag{A.13}$$

We recall that the generators of $C\ell(1,1)$ are

$$\sigma_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_1\sigma_2=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=-\Gamma_2\Gamma_1.\tag{A.14}$$

This gives

$$\sigma_1\sigma_2\sigma_{i-d}=\begin{cases}\sigma_2 & ,i=d+1,\\ \sigma_1 & ,i=d+2.\end{cases}\quad \sigma_{i-d}\sigma_1\sigma_2=\begin{cases}-\sigma_2 & ,i=d+1,\\ -\sigma_1 & ,i=d+2.\end{cases}$$

We also find that $\sigma_1\sigma_2\sigma_1\sigma_2=\mathbb{1}$.

Looking at the two matrices in (A.13) separately they are almost identical to those seen in the first part of the proof. Therefore, only when the two matrices mix will be considered. One investigates in the same way as in the first part, and find that all are zero. Letting $\Gamma_1, \ldots, \Gamma_{p+q}$ denote all the generators, where $\tilde{\Gamma}_i$ are $\Gamma_{p+1}, \ldots, \Gamma_{p+q}$, the Γ-matrices satisfy

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\eta_{ij}\mathbb{1}. \tag{A.15}$$

□

We may, then, find that

$$\begin{aligned}
C\ell(0,0)\otimes C\ell(0,2) &\cong C\ell(0,2) \cong \mathrm{Mat}_2(\mathbb{R}),\\
C\ell(0,2)\otimes C\ell(2,0) &\cong C\ell(4,0) \cong \mathrm{Mat}_2(\mathbb{H}),\\
C\ell(4,0)\otimes C\ell(0,2) &\cong C\ell(0,6) \cong \mathrm{Mat}_4(\mathbb{H}),\\
C\ell(0,6)\otimes C\ell(2,0) &\cong C\ell(8,0) \cong \mathrm{Mat}_{16}(\mathbb{R}),\ldots
\end{aligned}$$

We see that when we reach $C\ell(8,0)$ the process may start over again, thus $C\ell(0,0)$, $C\ell(0,2)$, $C\ell(4,0)$ and $C\ell(0,6)$ has periodicity 8. We also notice that $C\ell(0,0)$ and $C\ell(0,2)$ are both real matrices, while $C\ell(4,0)$ and $C\ell(0.6)$ are both quaternionic matrices. Thus it seems that for $p-q=0,6 \pmod 8$, the matrices are real and of dimension $2^{d/2}$, where $d=p+q$, and that for $p-q=4,6$ (mod 8), the matrices are quaternionic and of dimension $2^{(d-2)/2}$.

$$\begin{aligned}
C\ell(0,0)\otimes C\ell(2,0) &\cong C\ell(2,0) \cong \mathbb{H},\\
C\ell(2,0)\otimes C\ell(0,2) &\cong C\ell(0,4) \cong \mathrm{Mat}_2(\mathbb{H}),\\
C\ell(0,4)\otimes C\ell(2,0) &\cong C\ell(6,0) \cong \mathrm{Mat}_8(\mathbb{R}),\\
C\ell(6,0)\otimes C\ell(0,2) &\cong C\ell(0,8) \cong \mathrm{Mat}_{16}(\mathbb{R}),\ldots
\end{aligned}$$

These are again periodic of periodicity 8. These satisfy the same conditions as above.

$$\begin{aligned}
C\ell(0,1)\otimes C\ell(2,0) &\cong C\ell(3,0) \cong \mathbb{H}\oplus\mathbb{H},\\
C\ell(3,0)\otimes C\ell(0,2) &\cong C\ell(0,5) \cong \mathrm{Mat}_2(\mathbb{H})\oplus\mathrm{Mat}_2(\mathbb{H}),\\
C\ell(0,5)\otimes C\ell(2,0) &\cong C\ell(7,0) \cong \mathrm{Mat}_8(\mathbb{R})\oplus\mathrm{Mat}_8(\mathbb{R}),\\
C\ell(7,0)\otimes C\ell(0,2) &\cong C\ell(0,9) \cong \mathrm{Mat}_{16}(\mathbb{R})\oplus\mathrm{Mat}_{16}(\mathbb{R}),\ldots
\end{aligned}$$

Also $C\ell(0,1)$, $C\ell(3,0)$, $C\ell(0,5)$ and $C\ell(7,0)$ are periodic of periodicity 8. For $p-q=7 \pmod 8$ they are a direct sum of real matrices of dimension $2^{(d-1)/2}$, and for $p-q=3 \pmod 8$, a direct sum of quaternionic matrices of dimension $2^{(d-3)/2}$.

$$\begin{aligned}
C\ell(1,0)\otimes C\ell(0,2) &\cong C\ell(0,3) \cong \mathrm{Mat}_2(\mathbb{C}),\\
C\ell(0,3)\otimes C\ell(2,0) &\cong C\ell(5,0) \cong \mathrm{Mat}_4(\mathbb{C}),\ldots
\end{aligned}$$

$C\ell(1,0)$ and $C\ell(0,3)$ are periodic of periodicity 4. We may, however, extend the periodicity to 8, so that it matches with the periods found above. Then, for $p-q=1,5 \pmod 8$, we have complex matrices of dimension $2^{(d-1)/2}$.

We have now found the Clifford algebra over the field $\mathbb{R}^{p,q}$ where either p, q or both are 0. By induction we can prove that the results found above is true for any choice of p and q. We first notice that the direct product of p $C\ell(1,1)$s is isomorphic to $C\ell(p,p) \cong \mathrm{Mat}_{2^p}(\mathbb{R})$. Then we have that $C\ell(p,q)\otimes C\ell(n,n) \cong$

$C\ell(p+n,q+n)$ for any $p,q,n \in \mathbb{R}$. We can write any number p as $q+a+8n$, $p,q,n \in \mathbb{R}$, $0 \leq a \leq 7$. Then we find that $C\ell(p,q) = C\ell(q+a+8n,q) \cong C\ell(a+8n,0) \otimes C\ell(q,q) \cong C\ell(a+8n,0) \otimes \mathrm{Mat}_{2^q}(\mathbb{R})$. This means that $C\ell(p,q)$ is just a direct product between $C\ell(a+8n,0)$, which we already know how looks like, and $\mathrm{Mat}_{2^q}(\mathbb{R})$, which raises the the dimension of the matrix from $C\ell(a+8n)$ by 2^q. Then Table 1 follows.

$p-q$ mod 8	$C\ell(p,q)$	N
0,6	$\mathrm{Mat}_N(\mathbb{R})$	$2^{d/2}$
2,4	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-2)/2}$
1,5	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$
3	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$
7	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$

Table 3: Classification of Clifford algebras.

A.2 Proof of (5.9)

The proof of (5.9)

$$\begin{aligned}\gamma_{\mu_1\mu_2\ldots\mu_n}\gamma_\nu = {} & \gamma_{\mu_1\mu_2\ldots\mu_n\nu} + \eta_{\nu\mu_n}\gamma_{\mu_1\mu_2\ldots\mu_{n-1}} - \eta_{\nu\mu_{n-1}}\gamma_{\mu_1\mu_2\ldots\widehat{\mu_{n-1}}\mu_n} \\ & + \ldots + (-1)^{n-1}\eta_{\nu\mu_1}\gamma_{\mu_2\mu_3\ldots\mu_n}.\end{aligned} \tag{A.16}$$

follows.

Proof.

$$\begin{aligned}
\gamma_{\mu_1\mu_2\ldots\mu_n}\gamma_\nu &= \frac{1}{n!}\sum_{\sigma\in S}\text{sign}(\sigma)\big(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}(\gamma_{\mu_{\sigma(n)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n)}}) \\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu\gamma_{\mu_{\sigma(n)}}\big) \\
&= \frac{1}{n!}\sum_{\sigma\in S}\text{sign}(\sigma)\big(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}(\gamma_{\mu_{\sigma(n)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n)}}) \\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}(\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n-1)}})\gamma_{\mu_{\sigma(n)}} \\
&\qquad +\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}\gamma_\nu\gamma_{\mu_{\sigma(n-1)}}\gamma_{\mu_{\sigma(n)}}\big) \\
&= \cdots \\
&= \frac{1}{n!}\sum_{\sigma\in S}\text{sign}(\sigma)\big(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}(\gamma_{\mu_{\sigma(n)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n)}}) \\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}(\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n-1)}})\gamma_{\mu_{\sigma(n)}} \\
&\qquad +\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}\gamma_\nu\gamma_{\mu_{\sigma(n-1)}}\gamma_{\mu_{\sigma(n)}}+\cdots \\
&\qquad +(-1)^{n-1}(\gamma_{\mu_{\sigma(1)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(1)}})\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}} \\
&\qquad +(-1)^n\gamma_\nu\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\big) \\
&= \frac{1}{n!}\sum_{\sigma\in S}\text{sign}(\sigma)\big(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}\eta_{\nu\mu_{\sigma(n)}} \\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}\eta_{\nu\mu_{\sigma(n-1)}}\gamma_{\mu_{\sigma(n)}}+\cdots \\
&\qquad +(-1)^{n-1}\eta_{\nu\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\big) \\
&\quad +\frac{1}{n!}\frac{1}{2}\sum_{\sigma\in S}\text{sign}(\sigma)\big(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}(\gamma_{\mu_{\sigma(n)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n)}}) \\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}(\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n-1)}})\gamma_{\mu_{\sigma(n)}}+\cdots \\
&\qquad +(-1)^{n-1}(\gamma_{\mu_{\sigma(1)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(1)}})\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}} \\
&\qquad +(-1)^n 2\gamma_\nu\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\big)\,.
\end{aligned} \tag{A.17}$$

In the first sum of the last equality in (A.17) there are n terms, and each term is permuted once, and it changes sign for each permutation. Therefore, that sum can be written

$$\begin{aligned}
&\frac{1}{n!}\sum_{\sigma\,\in S}\text{sign}(\sigma)\big(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}\eta_{\nu\mu_{\sigma(n)}} \\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}\eta_{\nu\mu_{\sigma(n-1)}}\gamma_{\mu_{\sigma(n)}}+\cdots \\
&\qquad +(-1)^{n-1}\eta_{\nu\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\big) \\
&= \frac{1}{(n-1)!}\sum_{\sigma\in S}\text{sign}(\sigma)\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}\eta_{\nu\gamma_{\mu_{\sigma(n)}}} \\
&= \eta_{\nu\mu_n}\gamma_{\mu_1\mu_2\ldots\mu_(n-1)}-\eta_{\nu\mu_{(n-1)}}\gamma_{\mu_1\mu_2\ldots\widehat{\mu_{(n-1)}}\mu_n}+\cdots+(-1)^{n-1}\eta_{\nu\mu_1}\gamma_{\mu_2\mu_3\ldots\mu_n}.
\end{aligned} \tag{A.18}$$

The other sum can be rewritten as

$$
\begin{aligned}
&\frac{1}{n!}\frac{1}{2}\sum_{\sigma\,\in S}\operatorname{sign}(\sigma)\big(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}(\gamma_{\mu_{\sigma(n)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n)}})\\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}(\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n-1)}})\gamma_{\mu_{\sigma(n)}}\\
&\qquad +\cdots+(-1)^{n-1}(\gamma_{\mu_{\sigma(1)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(1)}})\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\\
&\qquad +(-1)^n 2\gamma_\nu\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\big)\\
&=\frac{1}{n!}\frac{1}{2}\sum_{\sigma\in S}\operatorname{sign}(\sigma)\big(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\gamma_\nu+(-1)^n\gamma_\nu\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\big)\\
&=\frac{1}{n!}\frac{1}{2}\sum_{\sigma\in S}\operatorname{sign}(\sigma)\bigg(\frac{2}{n+1}\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\gamma_\nu\\
&\qquad +\frac{n-1}{n+1}\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}(\gamma_{\mu_{\sigma(n)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n)}})\\
&\qquad -\frac{n-1}{n+1}\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu\gamma_{\mu_{\sigma(n)}}\\
&\qquad +(-1)^n\frac{2}{n+1}\gamma_\nu\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\\
&\qquad +(-1)^n\frac{n-1}{n+1}(\gamma_\nu\gamma_{\mu_{\sigma(1)}}+\gamma_{\mu_{\sigma(1)}}\gamma_\nu)\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\\
&\qquad +(-1)^{n-1}\frac{n-1}{n+1}\gamma_{\mu_{\sigma(1)}}\gamma_\nu\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\bigg)\\
&=\frac{1}{n!}\frac{1}{2}\sum_{\sigma\in S}\operatorname{sign}(\sigma)\bigg(\frac{2}{n+1}(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\gamma_\nu\\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu\gamma_{\mu_{\sigma(n)}}+(-1)^n\gamma_\nu\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\\
&\qquad +(-1)^{n-1}\gamma_{\mu_{\sigma(1)}}\gamma_\nu\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}})\\
&\qquad -\frac{n-3}{n+1}(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}(\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu+\gamma_\nu\gamma_{\mu_{\sigma(n-1)}})\gamma_{\mu_{\sigma(n)}}\\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-2)}}\gamma_\nu\gamma_{\mu_{\sigma(n-1)}}\gamma_{\mu_{\sigma(n)}}\\
&\qquad -(-1)^{n-1}\gamma_{\mu_{\sigma(1)}}(\gamma_\nu\gamma_{\mu_{\sigma(2)}}+\gamma_{\mu_{\sigma(2)}}\gamma_\nu)\gamma_{\mu_{\sigma(3)}}\cdots\gamma_{\mu_{\sigma(n)}}\\
&\qquad +(-1)^{n-2}\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\gamma_\nu\gamma_{\mu_{\sigma(3)}}\cdots\gamma_{\mu_{\sigma(n)}})\bigg)\\
&=\cdots\\
&=\frac{1}{n!}\frac{1}{2}\sum_{\sigma\in S}\operatorname{sign}(\sigma)\frac{2}{n+1}(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\gamma_\nu\\
&\qquad -\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n-1)}}\gamma_\nu\gamma_{\mu_{\sigma(n)}}+\cdots+(-1)^n\gamma_\nu\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}})\\
&=\frac{1}{(n+1)!}\sum_{\sigma\in S}\operatorname{sign}(\sigma)\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}}\gamma_{\sigma(\nu)}\\
&=\gamma_{\mu_1\mu_2\ldots\mu_n\nu}.
\end{aligned}
\tag{A.19}
$$

If n is odd, there will eventually only remain two $\frac{2}{n+1}$ terms, while if n is even, there will be two $\frac{1}{n+1}$ terms, one positive and the other negative, since one is multiplied with $(-1)^{n-n/2} = (-1)^{n/2}$ and the other has to go through $n/2-1$ permutations, so it is multiplied with $(-1)^{n/2-1}$. Adding the two last equations, we get the desired result. □

A.3 Proof of Proposition 5.2

This is the proof of Proposition 5.2

Proposition A.2. Let $\gamma = \gamma_{1\dots p(p+1)\dots n}$ where $\eta_{\mu\nu} = \text{Diag}(\underbrace{-1,\dots,-1}_{p},\underbrace{+1,\dots,+1}_{q})$ and $n = p + q$. Then γ satisfies the following identities:

$$\gamma_\mu \gamma = (-1)^{n-1}\gamma\gamma_\mu, \tag{A.20}$$

$$\gamma^2 = (-1)^{n(n+1)/2-q}. \tag{A.21}$$

In Minkowski space, $\gamma = \gamma_5$ satisfies

$$\gamma_{\mu\nu}\gamma_5 = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}, \tag{A.22}$$

$$\gamma_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5, \tag{A.23}$$

$$\gamma_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}\gamma_5. \tag{A.24}$$

where $\epsilon^{0123} = -\epsilon_{0123} = 1$.

Proof. We first prove (A.20). Since $\mu \in \{1\dots n\}$ we can choose $\mu = i$. For all $i \in \{1\dots n\}$ we can show

$$\gamma_i\gamma_1\dots\gamma_i\dots\gamma_n = (-1)^{i-1}\gamma_1\dots\gamma_i^2\dots\gamma_n. \tag{A.25}$$

Similarly, we find

$$\gamma_1\dots\gamma_i\dots\gamma_n\gamma_i = (-1)^{n-i}\gamma_1\dots\gamma_i^2\dots\gamma_n. \tag{A.26}$$

Then $(-1)^{i-1}$(A.25) $= (-1)^{n-i}$(A.26). Multiplying both sides with $(-1)^{i-1}$ gives the desired result,

$$\begin{aligned}\gamma_i\gamma_1\dots\gamma_i\dots\gamma_n &= (-1)^{i-1}(-1)^{n-i}\gamma_1\dots\gamma_i\dots\gamma_n\gamma_i\\ &= (-1)^{n-1}\gamma_1\dots\gamma_i\dots\gamma_n\gamma_i.\end{aligned}$$

We continue with (A.21).

$$\begin{aligned}\gamma^2 &= \gamma_1\dots\gamma_p\gamma_{p+1}\dots\gamma_n\gamma_1\dots\gamma_p\gamma_{p+1}\dots\gamma_n\\ &= (-1)^{n-1}\eta_{11}\gamma_2\dots\gamma_p\gamma_{p+1}\dots\gamma_n\gamma_2\dots\gamma_p\gamma_{p+1}\dots\gamma_n\\ &= \cdots\\ &= (-1)^{n+(n-1)+\cdots+(n-p+1)}\gamma_{p+1}\dots\gamma_n\gamma_{p+1}\dots\gamma_n\\ &= (-1)^{n+(n-1)+\cdots+(n-p+1)}(-1)^{q-1}\eta_{qq}\gamma_{p+2}\dots\gamma_n\gamma_{p+2}\dots\gamma_n\\ &= (-1)^{n+(n-1)+\cdots+(n-p+1)}(-1)^{(q-1)+(q-2)+\cdots+1}\\ &= (-1)^{n(n+1)/2-q}.\end{aligned}$$

The last four identities are only proved for four-dimensional space time. 0 is used for the time-component, while 1, 2, 3 are used for the space-components.

First out is (A.22),

$$\begin{aligned}
\gamma_{\mu\nu}\gamma_5 =&\gamma_5\gamma_{\mu\nu} = \frac{1}{2}\gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \\
=& \left(\eta_{3[\mu}\gamma_{012} - \eta_{2[\mu}\gamma_{013} + \eta_{1[\mu}\gamma_{023} - \eta_{0[\mu}\gamma_{123}\right)\gamma_{\nu]} \\
=&\eta_{3[\mu}\left(\gamma_{012\nu]} + \eta_{\nu]2}\gamma_{01} - \eta_{\nu]1}\gamma_{02} + \eta_{\nu]0}\gamma_{12}\right) - \eta_{2[\mu}\left(\gamma_{013\nu]} + \eta_{\nu]3}\gamma_{01} - \eta_{\nu]1}\gamma_{03} + \eta_{\nu]0}\gamma_{13}\right) \\
&+ \eta_{1[\mu}\left(\gamma_{023\nu]} + \eta_{\nu]3}\gamma_{02} - \eta_{\nu]2}\gamma_{03} + \eta_{\nu]0}\gamma_{23}\right) - \eta_{0[\mu}\left(\gamma_{123\nu]} + \eta_{\nu]3}\gamma_{12} - \eta_{\nu]2}\gamma_{13} + \eta_{\nu]1}\gamma_{23}\right) \\
=&\gamma_{01}\left(\eta_{3[\mu}\eta_{\nu]2} - \eta_{2[\mu}\eta_{\nu]3}\right) - \gamma_{02}\left(\eta_{3[\mu}\eta_{\nu]1} - \eta_{1[\mu}\eta_{\nu]3}\right) + \gamma_{12}\left(\eta_{3[\mu}\eta_{\nu]0} - \eta_{0[\mu}\eta_{\nu]3}\right) \\
&+ \gamma_{03}\left(\eta_{2[\mu}\eta_{\nu]1} - \eta_{1[\mu}\eta_{\nu]2}\right) - \gamma_{13}\left(\eta_{2[\mu}\eta_{\nu]0} - \eta_{0[\mu}\eta_{\nu]2}\right) + \gamma_{23}\left(\eta_{1[\mu}\eta_{\nu]0} - \eta_{1[\mu}\eta_{\nu]0}\right) \\
&+ \eta_{3[\mu}\gamma_{012\nu]} - \eta_{2[\mu}\gamma_{013\nu]} + \eta_{1[\mu}\gamma_{023\nu]} - \eta_{0[\mu}\gamma_{123\nu]} \\
=&2\gamma^{\rho\sigma}(\eta_{0[\rho}\eta_{\sigma]1}\eta_{3[\mu}\eta_{\nu]2} - \eta_{0[\rho}\eta_{\sigma]2}\eta_{3[\mu}\eta_{\nu]1} + \eta_{1[\rho}\eta_{\sigma]2}\eta_{3[\mu}\eta_{\nu]0} \\
&- \eta_{0[\rho}\eta_{\sigma]3}\eta_{2[\mu}\eta_{\nu]1} + \eta_{1[\rho}\eta_{\sigma]3}\eta_{2[\mu}\eta_{\nu]0} - \eta_{2[\rho}\eta_{\sigma]3}\eta_{1[\mu}\eta_{\nu]0}) \\
=&-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}.
\end{aligned}$$

The last equality follows since

$$\begin{aligned}
\epsilon_{\mu\nu\rho\sigma} =& \eta_{0[\mu}\eta_{\nu]1}\eta_{2[\rho}\eta_{\sigma]3} - \eta_{0[\mu}\eta_{\nu]2}\eta_{1[\rho}\eta_{\sigma]3} + \eta_{1[\mu}\eta_{\nu]2}\eta_{0[\rho}\eta_{\sigma]3} \\
&- \eta_{3[\mu}\eta_{\nu]1}\eta_{2[\rho}\eta_{\sigma]0} + \eta_{3[\mu}\eta_{\nu]2}\eta_{1[\rho}\eta_{\sigma]0} - \eta_{3[\mu}\eta_{\nu]0}\eta_{1[\rho}\eta_{\sigma]2}
\end{aligned}$$

Another thing to be pointed out is how the $\eta_{3[\mu}\gamma_{012\nu]}$ terms vanish. The vanish unless $\mu = \nu$. Then all terms but the one in question vanishes on the right hand side. The left hand side vanishes as well since $\gamma_{\mu\mu} = 0$. Thus $\eta_{3[\mu}\gamma_{012\nu]} - \eta_{2[\mu}\gamma_{013\nu]} + \eta_{1[\mu}\gamma_{023\nu]} - \eta_{0[\mu}\gamma_{123\nu]} = 0$.

Next is (A.23). For this we may consider all possibilities separately, there are only $\binom{4}{3} = 4$ of them:

$$\begin{aligned}
\epsilon_{012\sigma}\gamma^\sigma\gamma_5 &= \epsilon_{0120}\gamma^0\gamma_5 + \epsilon_{0121}\gamma^1\gamma_5 + \epsilon_{0122}\gamma^2\gamma_5 + \epsilon_{0123}\gamma^3\gamma_5 \\
&= -\gamma_3\gamma_5 = \gamma_5\gamma_3 = \gamma_{012}\gamma_3\gamma_3 = \gamma_{012}(\gamma_{33} + \eta_{33}) = \gamma_{012},
\end{aligned}$$

$$\begin{aligned}
\epsilon_{013\sigma}\gamma^\sigma\gamma_5 &= \epsilon_{0130}\gamma^0\gamma_5 + \epsilon_{0131}\gamma^1\gamma_5 + \epsilon_{0132}\gamma^2\gamma_5 + \epsilon_{0133}\gamma^3\gamma_5 \\
&= \gamma_2\gamma_5 = -\gamma_5\gamma_2 = -\gamma_{0123}\gamma_2 = -\gamma_{01232} - \eta_{23}\gamma_{012} + \eta_{22}\gamma_{013} - \eta_{21}\gamma_{023} + \eta_{20}\gamma_{123} = \gamma_{013},
\end{aligned}$$

$$\begin{aligned}
\epsilon_{023\sigma}\gamma^\sigma\gamma_5 &= \epsilon_{0230}\gamma^0\gamma_5 + \epsilon_{0231}\gamma^1\gamma_5 + \epsilon_{0232}\gamma^2\gamma_5 + \epsilon_{0233}\gamma^3\gamma_5 \\
&= -\gamma_1\gamma_5 = \gamma_5\gamma_1 = \gamma_{0123}\gamma_1 = \gamma_{01231} + \eta_{13}\gamma_{012} - \eta_{12}\gamma_{013} + \eta_{11}\gamma_{023} - \eta_{10}\gamma_{123} = \gamma_{023},
\end{aligned}$$

$$\begin{aligned}
\epsilon_{123\sigma}\gamma^\sigma\gamma_5 &= \epsilon_{1230}\gamma^0\gamma_5 + \epsilon_{1231}\gamma^1\gamma_5 + \epsilon_{1232}\gamma^2\gamma_5 + \epsilon_{1233}\gamma^3\gamma_5 \\
&= -\gamma_0\gamma_5 = -\gamma_0\gamma_0\gamma_{123} = -(\gamma_{00} + \eta_{00})\gamma_{123} = \gamma_{123}.
\end{aligned}$$

Finally, (A.24). Here there is only one (non-zero) possibility:

$$-\epsilon_{0123}\gamma_{0123} = \epsilon^{0123}\gamma_{0123} = \gamma_{0123}.$$

□

A.4 Proof of (5.18)

The proof of (5.18)

$$\gamma^t_{\mu_1\mu_2\ldots\mu_n} = (-1)^{n(n+1)/2} C\gamma_{\mu_1\mu_2\ldots\mu_n} C^{-1} \tag{A.27}$$

follows. We first look at how one term changes,

$$\begin{aligned}\gamma^t_{\mu_1}\gamma^t_{\mu_2}\ldots\gamma^t_{\mu_n} &= (-C\gamma_{\mu_1}C^{-1})(-C\gamma_{\mu_2}C^{-1})\ldots(-C\gamma_{\mu_n}C^{-1}) \\ &= (-1)^n C\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_n}C^{-1}.\end{aligned} \tag{A.28}$$

Then, the collection of all of the terms become

$$\begin{aligned}\gamma^t_{\mu_1\mu_2\ldots\mu_n} &= \sum_{\sigma\in S} \operatorname{sign}(\sigma)(\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\ldots\gamma_{\mu_{\sigma(n)}})^t \\ &= \sum_{\sigma\in S} \operatorname{sign}(\sigma)\gamma^t_{\mu_{\sigma(n)}}\ldots\gamma^t_{\mu_{\sigma(2)}}\gamma^t_{\mu_{\sigma(1)}} \\ &= (-1)^n C\left(\sum_{\sigma\in S} \operatorname{sign}(\sigma)\gamma_{\mu_{\sigma(n)}}\ldots\gamma_{\mu_{\sigma(2)}}\gamma_{\mu_{\sigma(1)}}\right)C^{-1}.\end{aligned} \tag{A.29}$$

There are $\sum_{i=1}^n (n-i)$ permutations needed to go from $\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_n}$ to $\gamma_{\mu_n}\ldots\gamma_{\mu_2}\gamma_{\mu_1}$. Thus

$$\begin{aligned}\gamma^t_{\mu_1\mu_2\ldots\mu_n} &= (-1)^{n+\sum_{i=1}^n(n-i)}\gamma_{\mu_1\mu_2\ldots\mu_n} \\ &= (-1)^{\sum_{i=0}^n(n-i)}\gamma_{\mu_1\mu_2\ldots\mu_n} \\ &= (-1)^{n(n+1)/2}\gamma_{\mu_1\mu_2\ldots\mu_n}.\end{aligned} \tag{A.30}$$

A.5 Fierz identity

The Fierz identity for fermionic spinors ϵ_1 and ϵ_2 is

$$\epsilon_1\bar\epsilon_2 = -\frac{1}{4}(\bar\epsilon_2\epsilon_1)\mathbb{1} + \frac{1}{4}(\bar\epsilon_2\gamma_5\epsilon_1)\gamma_5 - \frac{1}{4}(\bar\epsilon_2\gamma^\mu\epsilon_1)\gamma_\mu + \frac{1}{4}(\bar\epsilon_2\gamma^\mu\gamma_5\epsilon_1)\gamma_\mu\gamma_5 + \frac{1}{8}(\bar\epsilon_2\gamma^{\mu\nu}\epsilon_1)\gamma_{\mu\nu}. \tag{A.31}$$

This is an important identity in supersymmetry computations. A special case we have a use for is $\epsilon_1\bar\epsilon_2 - \epsilon_2\bar\epsilon_1$:

$$\begin{aligned}\epsilon_1\bar\epsilon_2 - \epsilon_2\bar\epsilon_1 = &-\frac{1}{4}(\bar\epsilon_2\epsilon_1 - \bar\epsilon_1\epsilon_2)\mathbb{1} + \frac{1}{4}(\bar\epsilon_2\gamma_5\epsilon_1 - \bar\epsilon_1\gamma_5\epsilon_2)\gamma_5 - \frac{1}{4}(\bar\epsilon_2\gamma^\mu\epsilon_1 - \bar\epsilon_1\gamma^\mu\epsilon_2)\gamma_\mu \\ &+ \frac{1}{4}(\bar\epsilon_2\gamma^\mu\gamma_5\epsilon_1 - \bar\epsilon_1\gamma^\mu\gamma_5\epsilon_2)\gamma_\mu\gamma_5 + \frac{1}{8}(\bar\epsilon_2\gamma^{\mu\nu}\epsilon_1 - \bar\epsilon_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu}.\end{aligned}$$

Using $C_{ab} = -C_{ba}$, $(\gamma_5)_{ab} = -(\gamma_5)_{ba}$, $(\gamma_\mu)_{ab} = (\gamma_\mu)_{ba}$, $(\gamma_\mu\gamma_5)_{ab} = -(\gamma_\mu\gamma_5)_{ba}$ and $(\gamma_{\mu\nu})_{ab} = (\gamma_{\mu\nu})_{ba}$, we find

$$\epsilon_1\bar\epsilon_2 - \epsilon_2\bar\epsilon_1 = \frac{1}{2}(\bar\epsilon_1\gamma^\mu\epsilon_2)\gamma_\mu - \frac{1}{4}(\bar\epsilon_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu}. \tag{A.32}$$

B Computation of Algebras

In this section we compute the commutation relations describing the Poincaré- and conformal (super)algebra. At the end of the Appendix B.3 we also show that the conformal algebra on $\mathbb{R}^{p,q}$ is isomorphic to so$(p+1, q+1)$.

B.1 The Poincaré Algebra

The calculations leading to the Poincaré algebra (4.24), (4.25) and (4.26) follows. We recall that $P_\mu = \partial_\mu$ and $M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$. Then the commutation relations are

$$[P_\mu, P_\nu] = [\partial_\mu, \partial_\nu] = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0,$$

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= [x_\mu \partial_\nu - x_\nu \partial_\mu, \partial_\rho] = [x_\mu \partial_\nu, \partial_\rho] - [x_\nu \partial_\mu, \partial_\rho] \\
&= x_\mu \partial_\nu (\partial_\rho) - \partial_\rho (x_\mu \partial_\nu) - x_\nu \partial_\mu (\partial_\rho) + \partial_\rho (x_\nu \partial_\mu) \\
&= x_\mu \partial_\nu \partial_\rho - \eta_{\rho\mu} \partial_\nu - x_\mu \partial_\rho \partial_\nu - x_\nu \partial_\mu \partial_\rho + \eta_{\rho\nu} \partial_\mu + x_\nu \partial_\rho \partial_\mu \\
&= \eta_{\rho\nu} \partial_\mu - \eta_{\rho\mu} \partial_\nu = \eta_{\rho\nu} P_\mu - \eta_{\rho\mu} P_\nu,
\end{aligned}$$

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= [x_\mu \partial_\nu - x_\nu \partial_\mu, x_\rho \partial_\sigma - x_\sigma \partial_\rho] \\
&= [x_\mu \partial_\nu, x_\rho \partial_\sigma] - [x_\mu \partial_\nu, x_\sigma \partial_\rho] - [x_\nu \partial_\mu, x_\rho \partial_\sigma] + [x_\nu \partial_\mu, x_\sigma \partial_\rho] \\
&= (x_\mu \partial_\nu (x_\rho \partial_\sigma) - x_\rho \partial_\sigma (x_\mu \partial_\nu)) \\
&\qquad - (\rho \leftrightarrow \sigma) - (\mu \leftrightarrow \nu) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) \\
&= (x_\mu \eta_{\nu\rho} \partial_\sigma + x_\mu x_\rho \partial_\nu \partial_\sigma - x_\rho \eta_{\sigma\mu} \partial_\nu - x_\rho x_\mu \partial_\sigma \partial_\nu) \\
&\qquad - (\rho \leftrightarrow \sigma) - (\mu \leftrightarrow \nu) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) \\
&= (\eta_{\nu\rho} x_\mu \partial_\sigma - \eta_{\sigma\mu} x_\rho \partial_\nu) - (\eta_{\nu\sigma} x_\mu \partial_\rho - \eta_{\rho\mu} x_\sigma \partial_\nu) \\
&\qquad - (\eta_{\mu\rho} x_\nu \partial_\sigma - \eta_{\sigma\nu} x_\rho \partial_\mu) + (\eta_{\mu\sigma} x_\nu \partial_\rho - \eta_{\rho\nu} x_\sigma \partial_\mu) \\
&= \eta_{\rho\nu} (x_\mu \partial_\sigma - x_\sigma \partial_\mu) - \eta_{\sigma\mu} (x_\rho \partial_\nu - x_\nu \partial_\rho) \\
&\qquad - \eta_{\sigma\nu} (x_\mu \partial_\rho - x_\rho \partial_\mu) + \eta_{\rho\mu} (x_\sigma \partial_\nu - x_\nu \partial_\sigma) \\
&= \eta_{\rho\nu} M_{\mu\sigma} - \eta_{\sigma\mu} M_{\rho\nu} - \eta_{\sigma\nu} M_{\mu\rho} + \eta_{\rho\mu} M_{\sigma\nu}.
\end{aligned}$$

Thus, the commutation relations describing the Poincaré algebra is

$$[P_\mu, P_\nu] = 0, \tag{B.1}$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \tag{B.2}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\rho\nu} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\rho} M_{\sigma\nu}. \tag{B.3}$$

B.2 The Poincaré Superalgebra

We compute the Poincaré superalgebra. The generators act on the fields as:

$$\begin{aligned}
&P_\mu \phi = \partial_\mu \phi, \quad M_{\mu\nu} \phi = (x_\mu \partial_\nu - x_\nu \partial_\mu) \phi, \quad && Q_a \phi = \psi_a, \\
&P_\mu \tilde{\phi} = \partial_\mu \tilde{\phi}, \quad M_{\mu\nu} \tilde{\phi} = (x_\mu \partial_\nu - x_\nu \partial_\mu) \tilde{\phi}, \quad && Q_a \tilde{\phi} = (\gamma_5)_a{}^b \psi_b, \\
&P_\mu \psi = \partial_\mu \psi, \quad M_{\mu\nu} \psi = (x_\mu \partial_\nu - x_\nu \partial_\mu + \Sigma_{\mu\nu}) \psi, \quad && Q_a \psi_b = -(\gamma^\mu)_{ab} \partial_\mu \phi + (\gamma^\mu \gamma_5)_{ab} \partial_\mu \tilde{\phi}.
\end{aligned}$$

We find the brackets defining the Poincaré superalgebra.

$$\begin{aligned}
[P_\mu, Q_a] \phi &= P_\mu \psi_a - Q_a \partial_\mu \phi = \partial_\mu \psi_a - \partial_\mu \psi_a = 0, \\
[P_\mu, Q_a] \tilde{\phi} &= P_\mu \gamma_5 \psi_a - Q_a \partial_\mu \tilde{\phi} = \gamma_5 \partial_\mu \psi - \partial_\mu (\gamma_5 \psi) = 0, \\
[P_\mu, Q_a] \psi_b &= P_\mu (-(\gamma^\nu)_{ab} \partial_\nu \phi + (\gamma^\nu \gamma_5)_{ab} \partial_\nu \tilde{\phi}) - Q_a \partial_\mu \psi_b \\
&= -(\gamma^\nu)_{ab} \partial_\nu \partial_\mu \phi + (\gamma^\nu \gamma_5)_{ab} \partial_\nu \partial_\mu \tilde{\phi} - \partial_\mu (-(\gamma^\nu)_{ab} \partial_\nu \phi + (\gamma^\nu \gamma_5)_{ab} \partial_\nu \tilde{\phi}) = 0.
\end{aligned}$$

Hence, $[P_\mu, Q_a]$ vanishes.

$$[M_{\mu\nu}, Q_a]\phi = M_{\mu\nu}\psi_a - Q_a(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi = (\Sigma_{\mu\nu})_a{}^b\psi_b = (\Sigma_{\mu\nu})_a{}^b Q_b\phi$$
$$[M_{\mu\nu}, Q_a]\tilde{\phi} = M_{\mu\nu}\gamma_5\psi_a - Q_a(x_\mu\partial_\nu - x_\nu\partial_\mu)\tilde{\phi} = \gamma_5(\Sigma_{\mu\nu})_a{}^b\psi_b = (\Sigma_{\mu\nu})_a{}^b Q_b\tilde{\phi}$$
$$\begin{aligned}[M_{\mu\nu}, Q_a]\,\psi_b &= M_{\mu\nu}\left(-(\gamma^\rho)_{ab}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ab}\partial_\rho\tilde{\phi}\right)\\ &\quad - Q_a\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\psi_b + (\Sigma_{\mu\nu})_b{}^c\psi_c\right)\\ &= -(\gamma^\rho)_{ab}\partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi + (\gamma^\rho\gamma_5)_{ab}\partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu)\tilde{\phi}\\ &\quad - (x_\mu\partial_\nu - x_\nu\partial_\mu)\left(-(\gamma^\rho)_{ab}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ab}\partial_\rho\tilde{\phi}\right)\\ &\quad - (\Sigma_{\mu\nu})_b{}^c\left(-(\gamma^\rho)_{ac}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ac}\partial_\rho\tilde{\phi}\right)\\ &= -(\gamma^\rho)_{ab}(\eta_{\rho\mu}\partial_\nu - \eta_{\rho\nu}\partial_\mu)\phi + (\gamma^\rho\gamma_5)_{ab}(\eta_{\rho\mu}\partial_\nu - \eta_{\rho\nu}\partial_\mu)\tilde{\phi}\\ &\quad + (\Sigma_{\mu\nu})_b{}^c\left((\gamma^\rho)_{ca}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ca}\partial_\rho\tilde{\phi}\right).\end{aligned}$$

Let us consider only the last line:

$$\begin{aligned}(\Sigma_{\mu\nu})_b{}^c(\gamma^\rho)_{ca} &= \frac{1}{2}(\gamma_{\mu\nu}\gamma^\rho)_{ba} = \frac{1}{2}(\gamma_{\mu\nu}{}^\rho + \delta^\rho_\nu\gamma_\mu - \delta^\rho_\mu\gamma_\nu)_{ba}\\ &= -\frac{1}{2}(\gamma_{\mu\nu}{}^\rho - \delta^\rho_\nu\gamma_\mu + \delta^\rho_\mu\gamma_\nu)_{ab}\\ &= -\frac{1}{2}((\gamma_{\mu\nu}\gamma^\rho - \delta^\rho_\nu\gamma_\mu + \delta^\rho_\mu\gamma_\nu) - \delta^\rho_\nu\gamma_\mu + \delta^\rho_\mu\gamma_\nu)_{ab}\\ &= -(\Sigma_{\mu\nu}\gamma^\rho - \delta^\rho_\nu\gamma_\mu + \delta^\rho_\mu\gamma_\nu)_{ab}.\end{aligned}$$

In the same way,

$$(\Sigma_{\mu\nu})_b{}^c(\gamma^\rho\gamma_5)_{ca} = (\Sigma_{\mu\nu}\gamma^\rho\gamma_5 - \delta^\rho_\nu\gamma_\mu\gamma_5 + \delta^\rho_\mu\gamma_\nu\gamma_5)_{ab}.$$

Then,

$$\begin{aligned}[M_{\mu\nu}, Q_a]\,\psi_b &= -(\gamma^\rho)_{ab}(\eta_{\rho\mu}\partial_\nu - \eta_{\rho\nu}\partial_\mu)\phi + (\gamma^\rho\gamma_5)_{ab}(\eta_{\rho\mu}\partial_\nu - \eta_{\rho\nu}\partial_\mu)\tilde{\phi}\\ &\quad - (\Sigma_{\mu\nu}\gamma^\rho - \delta^\rho_\nu\gamma_\mu + \delta^\rho_\mu\gamma_\nu)_{ab}\partial_\rho\phi + (\Sigma_{\mu\nu}\gamma^\rho\gamma_5 - \delta^\rho_\nu\gamma_\mu\gamma_5 + \delta^\rho_\mu\gamma_\nu\gamma_5)_{ab}\partial_\rho\tilde{\phi}\\ &= -(\Sigma_{\mu\nu}\gamma^\rho)_{ab}\partial_\rho\phi + (\Sigma_{\mu\nu}\gamma^\rho\gamma_5)_{ab}\partial_\rho\tilde{\phi} = -(\Sigma_{\mu\nu})_a{}^c(\gamma^\rho_{cb}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{cb}\partial_\rho\tilde{\phi})\\ &= (\Sigma_{\mu\nu})_a{}^c Q_c\psi_b.\end{aligned}$$

Hence, $[M_{\mu\nu}, Q_a] = (\Sigma_{\mu\nu})_a{}^b Q_b$.

$$\begin{aligned}[Q_a, Q_b]\phi &= Q_a\psi_b + Q_b\psi_a\\ &= -(\gamma^\mu)_{ab}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{ab}\partial_\mu\tilde{\phi} - (\gamma^\mu)_{ba}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{ba}\partial_\mu\tilde{\phi} = -2(\gamma^\mu)_{ab}\partial_\mu\phi,\\ [Q_a, Q_b]\tilde{\phi} &= Q_a(\gamma_5)_b{}^c\psi_c + Q_b(\gamma_5)_a{}^c\psi_c\\ &= (\gamma_5)_b{}^c(-(\gamma^\mu)_{ac}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{ac}\partial_\mu\tilde{\phi}) + (\gamma_5)_a{}^c(-(\gamma^\mu)_{bc}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{bc}\partial_\mu\tilde{\phi})\\ &= (\gamma_5)_b{}^c(-(\gamma^\mu)_{ca}\partial_\mu\phi - (\gamma^\mu\gamma_5)_{ca}\partial_\mu\tilde{\phi}) + (\gamma_5)_a{}^c(-(\gamma^\mu)_{cb}\partial_\mu\phi - (\gamma^\mu\gamma_5)_{cb}\partial_\mu\tilde{\phi})\\ &= (\gamma^\mu\gamma_5)_{ba}\partial_\mu\phi + (\gamma^\mu\gamma_5^2)_{ba}\partial_\mu\tilde{\phi}) + (\gamma^\mu\gamma_5)_{ab}\partial_\mu\phi + (\gamma^\mu\gamma_5^2)_{ab}\partial_\mu\tilde{\phi} = -2(\gamma^\mu)_{ab}\partial_\mu\tilde{\phi},.\end{aligned}$$

We use another method to see $[Q_a, Q_b]\psi_c$. Since $\delta_\epsilon\varphi = \bar{\epsilon}Q\varphi$,

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\varphi &= [\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q]\varphi = (\bar{\epsilon}_1 Q\bar{\epsilon}_2 Q - \bar{\epsilon}_2 Q\bar{\epsilon}_1 Q)\varphi = (\epsilon_1^a Q_a\epsilon_2^b Q_b - \epsilon_2^b Q_b\epsilon_1^a Q_a)\varphi\\ &= \epsilon_1^a\epsilon_2^b(Q_aQ_b + Q_bQ_a)\varphi = \epsilon_1^a\epsilon_2^b[Q_a, Q_b]\varphi\end{aligned} \tag{B.4}$$

We will take use of the Fierz identity (A.32)

$$\epsilon_1\bar{\epsilon}_2 - \epsilon_2\bar{\epsilon}_1 = \frac{1}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu - \frac{1}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu} \tag{B.5}$$

Now, computing (B.4) by applying the supersymmetry transformations

$$\delta_\epsilon\phi = \bar{\epsilon}\psi, \qquad \delta_\epsilon\tilde{\phi} = \bar{\epsilon}\gamma_5\psi, \qquad \delta_\epsilon\psi = \partial\!\!\!/(\phi + \tilde{\phi}\gamma_5)\epsilon, \tag{B.6}$$

we get

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi &= \delta_{\epsilon_1}(\partial\!\!\!/(\phi + \tilde{\phi}\gamma_5)\epsilon_2) - (\epsilon_1 \leftrightarrow \epsilon_2) = \partial\!\!\!/(\bar{\epsilon}_1\psi + \bar{\epsilon}_1\gamma_5\psi\gamma_5)\epsilon_2 - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \gamma^\mu\epsilon_2\bar{\epsilon}_1\partial_\mu\psi + \gamma^\mu\gamma_5\epsilon_2\bar{\epsilon}_1\gamma_5\partial_\mu\psi - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \gamma^\mu((\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2) + \gamma_5(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2)\gamma_5)\partial_\mu\psi \\
&= -\gamma^\mu\Big\{ \left(\frac{1}{2}(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu - \frac{1}{4}(\bar{\epsilon}_1\gamma^{\nu\rho}\epsilon_2)\gamma_{\nu\rho}\right) \\
&\qquad + \gamma_5\left(\frac{1}{2}(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu - \frac{1}{4}(\bar{\epsilon}_1\gamma^{\nu\rho}\epsilon_2)\gamma_{\nu\rho}\right)\gamma_5\Big\}\partial_\mu\psi,
\end{aligned}$$

where we have applied the Fierz identity (B.5) in the last step. Let us compute this as two separate terms. Consider first $(\bar{\epsilon}_1\gamma^\mu\epsilon_2)$-terms:

$$\begin{aligned}
\gamma^\mu(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu + \gamma^\mu\gamma_5(\frac{1}{2}(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu\gamma_5 &= \gamma^\mu(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu - \gamma^\mu(\frac{1}{2}(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu\gamma_5^2 \\
&= 2\gamma^\mu(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu.
\end{aligned}$$

The $(\bar{\epsilon}_1\gamma^{\nu\rho}\epsilon_2)$-terms are

$$\begin{aligned}
\gamma^\mu(\bar{\epsilon}_1\gamma^{\nu\rho}\epsilon_2)\gamma_{\nu\rho} + \gamma^\mu\gamma_5(\bar{\epsilon}_1\gamma^{\nu\rho}\epsilon_2)\gamma_{\nu\rho}\gamma_5 &= \gamma^\mu(\bar{\epsilon}_1\gamma^{\nu\rho}\epsilon_2)\gamma_{\nu\rho} + \gamma^\mu(\bar{\epsilon}_1\gamma^{\nu\rho}\epsilon_2)\gamma_{\nu\rho}\gamma_5^2 \\
&= 0.
\end{aligned}$$

Thus, only the terms of the first kind contribute. Using the Clifford condition $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbb{1}$, we find

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi &= -\gamma^\mu(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu\partial_\mu\psi = -(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma^\mu\gamma_\nu\partial_\mu\psi \\
&= -(\bar{\epsilon}_1\gamma^\nu\epsilon_2)(2\delta^\mu_\nu - \gamma_\nu\gamma^\mu)\partial_\mu\psi = -2\epsilon_1^a\epsilon_2^b(\gamma^\mu)_{ab}\partial_\mu\psi + \epsilon_1^a\epsilon_2^b(\gamma^\mu)_{ab}\gamma_\mu\partial\!\!\!/\psi.
\end{aligned} \tag{B.7}$$

Then equating (B.4) and (B.7), we find

$$\epsilon_1^a\epsilon_2^b[Q_a, Q_b]\psi = -2\epsilon_1^a\epsilon_2^b(\gamma^\mu)_{ab}\partial_\mu\psi + \epsilon_1^a\epsilon_2^b(\gamma^\mu)_{ab}\gamma_\mu\partial\!\!\!/\psi.$$

Thus, we have, with $\partial_\mu = P_\mu$,

$$[Q_a, Q_b]\psi = -2(\gamma^\mu)_{ab}P_\mu\psi + \epsilon_1^a\epsilon_2^b(\gamma^\mu)_{ab}\gamma_\mu\partial\!\!\!/\psi.$$

Using the equations of motion for ψ, $\partial\!\!\!/\psi = 0$, we get

$$[Q_a, Q_b]\psi = -2(\gamma^\mu)_{ab}P_\mu\psi.$$

This is the same type as $[Q_a, Q_b]$ on ϕ and $\tilde{\phi}$. Hence, $[Q_a, Q_b] = -2(\gamma^\mu)_{ab}P_\mu$.

The brackets defining the **Poincaré superalgebra**, in addition to the brackets found in Appendix B.1, are

$$[P_\mu, Q_a] = 0, \tag{B.8}$$

$$[M_{\mu\nu}, Q_a] = -(\Sigma_{\mu\nu})_a{}^bQ_b, \tag{B.9}$$

$$[Q_a, Q_b] = 2(\gamma^\mu)_{ab}P_\mu. \tag{B.10}$$

B.3 The Conformal Algebra

The calculations leading to the conformal algebra (4.45) - (4.50) follows. We recall that $P_\mu = \partial_\mu$, $M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$, $D = x^\mu\partial_\mu$ and $K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu$, where $x^2 = x_\nu x^\nu$. Then,

$$[P_\mu, D] = [\partial_\mu, x^\nu\partial_\nu] = \partial_\mu(x^\nu\partial_\nu) - x^\nu\partial_\nu(\partial_\mu) = \delta^\nu_\mu\partial_\nu = \partial_\mu = P_\mu,$$

$$\begin{aligned}
[P_\mu, K_\nu] &= [\partial_\mu, 2x_\nu x^\rho\partial_\rho - x^2\partial_\nu] = 2[\partial_\mu, x_\nu x^\rho\partial_\rho] - [\partial_\mu, x^2\partial_\nu] \\
&= 2(\partial_\mu(x_\nu x^\rho\partial_\rho) - x_\nu x^\rho\partial_\rho\partial_\mu) - \partial_\mu(x^2\partial_\nu) + x^2\partial_\nu\partial_\mu \\
&= 2(\eta_{\mu\nu}x^\rho\partial_\rho + x_\nu\delta^\rho_\mu\partial_\rho + x_\nu x^\rho\partial_\mu\partial_\rho - x_\nu x^\rho\partial_\rho\partial_\mu) - 2x_\mu\partial_\nu \\
&\quad - x^2\partial_\mu\partial_\nu + x^2\partial_\nu\partial_\mu \\
&= 2\eta_{\mu\nu}x^\rho\partial_\rho + 2x_\nu\partial_\mu - 2x_\mu\partial_\nu = 2\eta_{\mu\nu}D + 2M_{\nu\mu},
\end{aligned}$$

$$[D, D] = [x^\mu\partial_\mu, x^\nu\partial_\nu] = 0,$$

$$\begin{aligned}
[D, M_{\mu\nu}] &= [x^\rho\partial_\rho, x_\mu\partial_\nu - x_\nu\partial_\mu] = [x^\rho\partial_\rho, x_\mu\partial_\nu] - [x^\rho\partial_\rho, x_\nu\partial_\mu] \\
&= x^\rho\partial_\rho(x_\mu\partial_\nu) - x_\mu\partial_\nu(x^\rho\partial_\rho) - (\mu \leftrightarrow \nu) \\
&= x^\rho\eta_{\rho\mu}\partial_\nu + x^\rho x_\mu\partial_\rho\partial_\nu - x_\mu\delta^\rho_\nu\partial_\rho - x_\mu x^\rho\partial_\nu\partial_\rho - (\mu \leftrightarrow \nu) \\
&= x_\mu\partial_\nu - x_\mu\partial_\nu - (\mu \leftrightarrow \nu) = 0,
\end{aligned}$$

$$\begin{aligned}
[D, K_\mu] &= [x^\rho\partial_\rho, 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu] = 2[x^\rho\partial_\rho, x_\mu x^\nu\partial_\nu] - [x^\rho\partial_\rho, x^2\partial_\mu] \\
&= 2(x^\rho\partial_\rho(x_\mu x^\nu\partial_\nu) - x_\mu x^\nu\partial_\nu(x^\rho\partial_\rho)) - x^\rho\partial_\rho(x^2\partial_\mu) + x^2\partial_\mu(x^\rho\partial^\rho) \\
&= 2(x^\rho(\eta_{\rho\mu}x^\nu + x_\mu\delta^\nu_\rho + x_\mu x^\nu\partial_\rho)\partial_\nu - x_\mu x^\nu(\delta^\rho_\nu + x^\rho\partial_\nu)\partial_\rho) \\
&\quad - x^\rho(2x_\rho + x^2\partial_\rho)\partial_\mu + x^2(\delta^\rho_\mu + x^\rho\partial_\mu)\partial_\rho \\
&= 2(x_\mu x^\nu\partial_\nu + x^\nu x_\mu\partial_\nu + x^\rho x_\mu x^\nu\partial_\rho\partial_\nu) - 2(x_\mu x^\rho\partial_\rho - x_\mu x^\nu x^\rho\partial_\nu\partial_\rho) \\
&\quad - 2x^2\partial_\mu - x^2x^\rho\partial_\rho\partial_\mu + x^2\partial_\mu + x^2x^\rho\partial_\rho\partial_\mu \\
&= 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu = K_\mu,
\end{aligned}$$

$$\begin{aligned}
[M_{\mu\nu}, K_\rho] &= [x_\mu\partial_\nu - x_\nu\partial_\mu, 2x_\rho x^\sigma\partial_\sigma - x^2\partial_\rho] \\
&= [x_\mu\partial_\nu, x_\rho x^\sigma\partial_\sigma] - [x_\mu\partial_\nu, x^2\partial_\rho] - (\mu \leftrightarrow \nu) \\
&= x_\mu\partial_\nu(x_\rho x^\sigma\partial_\sigma) - x_\rho x^\sigma\partial_\sigma(x_\mu\partial_\nu) - x_\mu\partial_\nu(x^2\partial_\rho) + x^2\partial_\rho(x_\mu\partial_\nu) - (\mu \leftrightarrow \nu) \\
&= x_\mu(\eta_{\nu\rho}x^\sigma + x_\rho\delta^\sigma_\nu + x_\rho x^\sigma\partial_\nu)\partial_\sigma - x_\rho x^\sigma(\eta_{\sigma\mu} + x_\mu\partial_\sigma)\partial_\nu \\
&\quad - x_\mu(2x_\nu + x^2\partial_\nu)\partial_\rho + x^2(\eta_{\rho\mu} + x_\mu\partial_\rho)\partial_\nu - (\mu \leftrightarrow \nu) \\
&= \eta_{\nu\rho}x_\mu x^\sigma\partial_\sigma + x_\mu x_\rho\partial_\nu + x_\mu x_\rho x^\sigma\partial_\sigma\partial_\nu - x_\rho x_\mu\partial_\nu - x_\rho x_\mu x^\sigma\partial_\sigma\partial_\nu \\
&\quad - 2x_\mu x_\nu\partial_\rho - x^2x_\mu\partial_\nu\partial_\rho + \eta_{\rho\mu}x^2\partial_\nu + x^2x_\mu\partial_\nu\partial_\rho - (\mu \leftrightarrow \nu) \\
&= \eta_{\nu\rho}x_\mu x^\sigma\partial_\sigma - 2x_\mu x_\nu\partial_\rho + \eta_{\rho\mu}x^2\partial_\nu - (\mu \leftrightarrow \nu) \\
&= \eta_{\nu\rho}x_\mu x^\sigma\partial_\sigma - 2x_\mu x_\nu\partial_\rho + \eta_{\rho\mu}x^2\partial_\nu - (\eta_{\mu\rho}x_\nu x^\sigma\partial_\sigma - 2x_\nu x_\mu\partial_\rho + \eta_{\rho\nu}x^2\partial_\mu) \\
&= \eta_{\rho\nu}(x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu) - \eta_{\rho\mu}(x_\nu x^\sigma\partial_\sigma - x^2\partial_\nu) = \eta_{\rho\nu}K_\mu - \eta_{\rho\mu}K_\nu,
\end{aligned}$$

$$\begin{aligned}
[K_\mu, K_\nu] &= [2x_\mu x^\rho \partial_\rho - x^2\partial_\mu, 2x_\nu x^\sigma \partial_\sigma - x^2\partial_\nu] \\
&= 4[x_\mu x^\rho \partial_\rho, x_\nu x^\sigma \partial_\sigma] - 2[x_\mu x^\rho \partial_\rho, x^2\partial_\nu] - 2[x^2\partial_\mu, x_\nu x^\sigma \partial_\sigma] + [x^2\partial_\mu, x^2\partial_\nu] \\
&= 4[x_\mu x^\rho \partial_\rho, x_\nu x^\sigma \partial_\sigma] - 2([x_\mu x^\rho \partial_\rho, x^2\partial_\nu] - (\mu \leftrightarrow \nu)) + [x^2\partial_\mu, x^2\partial_\nu] \\
&= 4(x_\mu x^\rho \partial_\rho(x_\nu x^\sigma \partial_\sigma) - x_\nu x^\sigma \partial_\sigma(x_\mu x^\rho \partial_\rho)) \\
&\quad - 2(x_\mu x^\rho \partial_\rho(x^2\partial_\nu) - x^2\partial_\nu(x_\mu x^\rho \partial_\rho) - (\mu \leftrightarrow \nu)) + x^2\partial_\mu(x^2\partial_\nu) - x^2\partial_\nu(x^2\partial_\mu) \\
&= 4(x_\mu x^\rho(\eta_{\rho\nu}x^\sigma + x_\nu\delta^\sigma_\rho + x_\nu x^\sigma\partial_\rho)\partial_\sigma - x_\nu x^\sigma(\eta_{\sigma\mu}x^\rho + x_\mu\delta^\rho_\sigma + x_\mu x^\rho\partial_\sigma)\partial_\rho \\
&\quad - 2(x_\mu x^\rho(2x_\rho + x^2\partial_\rho)\partial_\nu - x^2(\eta_{\nu\mu}x^\rho + x_\mu\delta^\rho_\nu + x_\mu x^\rho\partial_\nu)\partial_\rho - (\mu \leftrightarrow \nu)) \\
&\quad + x^2(2x_\mu\partial_\nu + x^2\partial_\mu\partial_\nu - 2x_\nu\partial_\mu - x^2\partial_\nu\partial_\mu) \\
&= 4(x_\mu x_\nu x^\sigma\partial_\sigma + x_\mu x_\nu x^\sigma\partial_\sigma - x_\nu x_\mu x^\rho\partial_\rho - x_\nu x_\mu x^\rho\partial_\rho) \\
&\quad - 2(2x^2 x_\mu\partial_\nu + x^2 x_\mu x^\rho\partial_\rho\partial_\nu - \eta_{\nu\mu}x^2x^\rho\partial_\rho - x^2x_\mu\partial_\nu - x^2x_\mu x^\rho\partial_\rho\partial_\nu - (\mu \leftrightarrow \nu)) \\
&\quad + x^2(2x_\mu\partial_\nu - 2x_\nu\partial_\mu) \\
&= -2x^2(x_\mu\partial_\nu - \eta_{\nu\mu}x^\rho\partial_\rho - (x_\nu\partial_\mu - \eta_{\mu\nu}x^\rho\partial_\rho) + 2x^2(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
&= -2x^2(x_\mu\partial_\nu - x_\nu\partial_\mu) + +2x^2(x_\mu\partial_\nu - x_\nu\partial_\mu) = 0.
\end{aligned}$$

We have found that the non-vanishing commutation relations describing the conformal algebra are given by

$$[P_\mu, D] = P_\mu, \tag{B.11}$$
$$[P_\mu, K_\nu] = 2\left(\eta_{\nu\mu}D + M_{\nu\mu}\right), \tag{B.12}$$
$$[D, K_\mu] = K_\mu, \tag{B.13}$$
$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu, \tag{B.14}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\rho\nu} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\rho}M_{\sigma\nu}, \tag{B.15}$$
$$[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu. \tag{B.16}$$

We also show that on $\mathbb{R}^{p,q}$ the conformal algebra is isomorphic to so$(p+1, q+1)$. This algebra is the Lorentz algebra of dimension $p+q+2$,

$$\mathrm{so}(p+1, q+1) := \{X \in \mathrm{Mat}_{p+q+2} \mid X\eta_{\mu\nu} = -\eta_{\mu\nu}X^t\}, \tag{B.17}$$

where ^{t} denotes the transpose. It has $\frac{(p+q+2)^2-(p+q+2)}{2} = \frac{(p+q+2)(p+q+1)}{2}$ generators. In subsection 4.3 we found that the conformal algebra has $\frac{(m+2)(m+1)}{2}$ generators, where $m = p+q$ is the space-time dimension, so the number of generators in each algebra is the same. Let us see why the conformal algebra is isomorphic to so$(p+1, q+1)$.

Proof. Let the generators of so$(p+1, q+1)$ be denoted $Z_{\mu\nu}$, with $\mu, \nu = 1, \ldots, p+q+2$, and $Z_{\mu\nu} = 0$ when $\mu = \nu$. The generators satisfies (4.26):

$$[Z_{\mu\nu}, Z_{\rho\sigma}] = \eta_{\nu\rho}Z_{\mu\sigma} - \eta_{\mu\sigma}Z_{\rho\nu} - \eta_{\nu\sigma}Z_{\mu\rho} + \eta_{\mu\rho}Z_{\sigma\nu}. \tag{B.18}$$

The signature of the Minkowski metric on $\mathbb{R}^{p,q}$ is $(\underbrace{-,\cdots,-}_{p+1},\underbrace{+,\cdots,+}_{q+1})$.

Let $i, j = 1, \ldots, p, p+2, \ldots p+q+1$. The generators, $Z_{\mu\nu}$, can be written as a set of four different generators,

$$Z_{\mu\nu} = \{Z_{ij}, Z_{iA}, Z_{iB}, Z_{AB}\}.$$

Here A has been used instead of $(p+1)$, and B instead of $(p+q+2)$. It should be mentioned that there is nothing special with the choice of A and B, only that one of them should be negative, and the other positive in the Minkowski signature. We rewrite the generators as $Q = Z_{AB}$, $S_i = Z_{iA} - Z_{iB}$, and $T_i = Z_{iA} + Z_{iB}$. The commutation relations are

$$[Z_{ij}, Q] = [Z_{ij}, Z_{AB}] = \eta_{jA} Z_{iB} - \eta_{iB} Z_{Aj} - \eta_{jB} Z_{iA} + \eta_{iA} Z_{Bj} = 0,$$

$$\begin{aligned}[Z_{ij}, S_k] &= [Z_{ij}, Z_{kA} - Z_{kB}] = [Z_{ij}, Z_{kA}] - [Z_{ij}, Z_{kB}] \\ &= \eta_{jk} Z_{iA} - \eta_{kA} Z_{ij} - \eta_{jA} Z_{ik} + \eta_{ik} Z_{Aj} - \eta_{jk} Z_{iB} + \eta_{iB} Z_{kj} + \eta_{jB} Z_{ik} - \eta_{ik} Z_{Bj} \\ &= \eta_{jk} Z_{iA} + \eta_{ik} Z_{Aj} - \eta_{jk} Z_{iB} - \eta_{ik} Z_{Bj} = \eta_{jk} (Z_{iA} - Z_{iB}) - \eta_{ik} (Z_{jA} - Z_{jB}) \\ &= \eta_{jk} S_i - \eta_{ik} S_j,\end{aligned}$$

$$\begin{aligned}[Z_{ij}, T_k] &= [Z_{ij}, Z_{kA} + Z_{kB}] = [Z_{ij}, Z_{kA}] + [Z_{ij}, Z_{kB}] \\ &= \eta_{jk} Z_{iA} - \eta_{iA} Z_{kj} - \eta_{jA} Z_{ik} + \eta_{ik} Z_{Aj} + \eta_{jk} Z_{iB} - \eta_{iB} Z_{kj} - \eta_{jB} Z_{ik} + \eta_{ik} Z_{Bj} \\ &= \eta_{jk} Z_{iA} + \eta_{ik} Z_{Aj} + \eta_{jk} Z_{iB} + \eta_{ik} Z_{Bj} = \eta_{jk} (Z_{iA} + Z_{iB}) - \eta_{ik} (Z_{jB} + Z_{jA}) \\ &= \eta_{jk} T_i - \eta_{ik} T_j,\end{aligned}$$

$$[Q, Q] = [Z_{AB}, Z_{AB}] = 0,$$

$$\begin{aligned}[Q, S_i] &= [Z_{AB}, Z_{iA} - Z_{iB}] = [Z_{AB}, Z_{iA}] - [Z_{AB}, Z_{iB}] \\ &= \eta_{Bi} Z_{AA} - \eta_{AA} Z_{iB} - \eta_{BA} Z_{Ai} + \eta_{Ai} Z_{AB} - \eta_{Bi} Z_{AB} + \eta_{AB} Z_{iB} + \eta_{BB} Z_{Ai} - \eta_{Ai} Z_{BB} \\ &= Z_{iB} + Z_{Ai} = Z_{iB} - Z_{iA} = -S_i,\end{aligned}$$

$$\begin{aligned}[Q, T_i] &= [Z_{AB}, Z_{iA} + Z_{iB}] = [Z_{AB}, Z_{iA}] + [Z_{AB}, Z_{iB}] \\ &= \eta_{Bi} Z_{AA} - \eta_{AA} Z_{iB} - \eta_{BA} Z_{Ai} + \eta_{Ai} Z_{AB} + \eta_{Bi} Z_{AB} - \eta_{AB} Z_{iB} - \eta_{BB} Z_{Ai} + \eta_{Ai} Z_{BB} \\ &= Z_{iB} - Z_{Ai} = Z_{iB} + Z_{iA} = T_i,\end{aligned}$$

$$\begin{aligned}[S_i, S_j] &= [Z_{iA} - Z_{iB}, Z_{jA} - Z_{jB}] = [Z_{iA}, Z_{jA}] - [Z_{iA}, Z_{jB}] - \underbrace{[Z_{iB}, Z_{jA}]}_{=-[Z_{jA}, Z_{iB}]} + [Z_{iB}, Z_{jB}] \\ &= \eta_{Aj} Z_{iA} - \eta_{iA} Z_{jA} - \eta_{AA} Z_{ij} + \eta_{ij} Z_{AA} - \eta_{Aj} Z_{iB} + \eta_{iB} Z_{jA} + \eta_{AB} Z_{ij} - \eta_{ij} Z_{BA} \\ &\quad + \eta_{Ai} Z_{jB} - \eta_{jB} Z_{iA} - \eta_{AB} Z_{ji} + \eta_{ji} Z_{BA} + \eta_{Bj} Z_{iB} - \eta_{iB} Z_{jB} - \eta_{BB} Z_{ij} + \eta_{ij} Z_{BB} \\ &= Z_{ij} + \eta_{ij} Z_{AA} - \eta_{ij} Z_{BA} + \eta_{ji} Z_{BA} - Z_{ij} + \eta_{ij} Z_{BB} = 0,\end{aligned}$$

$$\begin{aligned}[S_i, T_j] &= [Z_{iA} - Z_{iB}, Z_{jA} + Z_{jB}] = [Z_{iA}, Z_{jA}] + [Z_{iA}, Z_{jB}] - \underbrace{[Z_{iB}, Z_{jA}]}_{=-[Z_{jA}, Z_{iB}]} - [Z_{iB}, Z_{jB}] \\ &= Z_{ij} + \eta_{ij} Z_{BA} + \eta_{ji} Z_{BA} + Z_{ij} = 2(Z_{ij} + \eta_{ij} Z_{BA}) = 2(Z_{ij} - \eta_{ij} Q) \\ &= -2(\eta_{ij} Q + Z_{ji}),\end{aligned}$$

$$[T_i, T_j] = [Z_{iA} + Z_{iB}, Z_{jA} + Z_{jB}] = [Z_{iA}, Z_{jB}] + [Z_{iA}, Z_{jB}] + \underbrace{[Z_{iB}, Z_{jA}]}_{=-[Z_{jA}, Z_{iB}]} + [Z_{iB}, Z_{jB}]$$
$$= Z_{ij} + \eta_{ij} Z_{AA} + \eta_{ij} Z_{BA} - \eta_{ji} Z_{BA} - Z_{ij} + \eta_{ij} Z_{BB} = 0.$$

The commutation relations are

$$[Z_{\mu\nu}, Z_{\rho\sigma}] = \eta_{\nu\rho} Z_{\mu\sigma} - \eta_{\mu\sigma} Z_{\rho\nu} - \eta_{\nu\sigma} Z_{\mu\rho} + \eta_{\mu\rho} Z_{\sigma\nu}, \tag{B.19}$$
$$[Z_{ij}, S_k] = \eta_{jk} S_i - \eta_{ik} S_j, \tag{B.20}$$
$$[Z_{ij}, T_k] = \eta_{jk} T_i - \eta_{ik} T_j, \tag{B.21}$$
$$[Q, S_i] = -S_i, \tag{B.22}$$
$$[Q, T_i] = T_i, \tag{B.23}$$
$$[S_i, T_j] = -2(\eta_{ji} Q + Z_{ji}). \tag{B.24}$$

Above we found that the conformal algebra is given by

$$[P_\mu, D] = P_\mu, \tag{B.25}$$
$$[P_\mu, K_\nu] = 2\left(\eta_{\nu\mu} D + M_{\nu\mu}\right), \tag{B.26}$$
$$[D, K_\mu] = K_\mu, \tag{B.27}$$
$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \tag{B.28}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\rho\nu} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\rho} M_{\sigma\nu}, \tag{B.29}$$
$$[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu. \tag{B.30}$$

We see that the commutation relations (B.19) - (B.24) are equal to the commutation relations describing the conformal algebra, (B.25) - (B.30), except that (B.24) is equal to minus (B.26), and (B.22) is equal to minus (B.25). Then we can relate $Z_{\mu\nu} = \alpha M_{\mu\nu}$, $S_i = \beta P_\mu$, $Q = \gamma D$, and $T_i = \delta K_\mu$. Choosing $\alpha = \gamma = 1$, and either $\beta = 1$ and $\delta = -1$ or $\beta = -1$ and $\delta = 1$ gives an isomorphism. □

B.4 The Conformal Superalgebra

We compute the brackets describing the conformal superalgebra.

We recall the infinitesimal transformations for D from subsection 6.2:

$$\delta_c \phi = c(x^\mu \partial_\mu + 1)\phi, \qquad \delta_c \tilde{\phi} = c(x^\mu \partial_\mu + 1)\tilde{\phi}, \qquad \delta_c \psi = c(x^\mu \partial_\mu + \frac{3}{2})\psi.$$

We find how D acts on the field from $\delta_c \varphi = cD\varphi$:

$$D\phi = (x^\mu \partial_\mu + 1)\phi, \qquad D\tilde{\phi} = (x^\mu \partial_\mu + 1)\tilde{\phi}, \qquad D\psi(x^\mu \partial_\mu + \frac{3}{2})\psi.$$

We can now find $[D, Q_a]$:

$$
\begin{aligned}
[D, Q_a]\phi &= D\psi_a - Q_a(x^\mu\partial_\mu + 1)\phi = (x^\mu\partial_\mu + \frac{3}{2})\psi_a - (x^\mu\partial_\mu + 1)\psi_a \\
&= \frac{1}{2}\psi_a = \frac{1}{2}Q_a\phi, \\
[D, Q_a]\tilde{\phi} &= D\gamma_5\psi_a - Q_a(x^\mu\partial_\mu + 1)\tilde{\phi} = \gamma_5(x^\mu\partial_\mu + \frac{3}{2})\psi_a - (x^\mu\partial_\mu + 1)\gamma_5\psi_a \\
&= \frac{1}{2}\gamma_5\psi_a = \frac{1}{2}Q_a\tilde{\phi}, \\
[D, Q_a]\psi_b &= D(-(\gamma^\mu)_{ab}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{ab}\partial_\mu\tilde{\phi}) - Q_a(x^\nu\partial_\nu + \frac{3}{2})\psi_b \\
&= -(\gamma^\mu)_{ab}\partial_\mu((x^\nu\partial_\nu + 1)\phi) + (\gamma^\mu\gamma_5)_{ab}\partial_\mu((x^\nu\partial_\nu + 1)\tilde{\phi}) \\
&\quad - (x^\nu\partial_\nu + \frac{3}{2})\left(-(\gamma^\mu)_{ab}\partial_\mu\phi + (\gamma^\mu\gamma_5)_{ab}\partial_\mu\tilde{\phi}\right) \\
&= -(\gamma^\mu)_{ab}(\delta^\nu_\mu\partial_\nu + x^\nu\partial_\nu\partial_\mu + \partial_\mu)\phi + (\gamma^\mu\gamma_5)_{ab}(\delta^\nu_\mu\partial_\nu + x^\nu\partial_\nu\partial_\mu + \partial_\mu)\tilde{\phi} \\
&\quad + (\gamma^\mu)_{ab}x^\nu\partial_\nu\partial_\mu\phi - (\gamma^\mu\gamma_5)_{ab}x^\nu\partial_\nu\partial_\mu\tilde{\phi} - \frac{3}{2}Q_a\psi_b \\
&= -2(\gamma^\mu)_{ab}\partial_\mu\phi + 2(\gamma^\mu\gamma_5)_{ab}\partial_\mu\tilde{\phi} - \frac{3}{2}Q_a\psi_b \\
&= 2Q_a\psi_b - \frac{3}{2}Q_a\psi_b = \frac{1}{2}Q_a\psi_b.
\end{aligned}
$$

Hence, $[D, Q_a] = \frac{1}{2}Q_a$.

The infinitesimal transformation for K_μ are

$$
\delta_d\phi = d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)\phi, \qquad \delta_d\tilde{\phi} = d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)\tilde{\phi},
$$
$$
\delta_d\psi = d^\mu(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\psi.
$$

We find how K_μ acts on the field from $\delta_d\varphi = d^\mu K_\mu\varphi$:

$$
K_\mu\phi = (2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)\phi, \qquad K_\mu\tilde{\phi} = (2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)\tilde{\phi},
$$
$$
K_\mu\psi = (2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\psi.
$$

We can now find $[K_\mu, Q_a]$:

$$\begin{aligned}
[K_\mu, Q_a]\phi &= K_\mu\psi_a - Q_a(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)\phi \\
&= (2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\psi_a - (2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)\psi_a \\
&= (x_\mu - x^\nu\gamma_{\nu\mu})\psi_a, \\
[K_\mu, Q_a]\phi &= K_\mu\gamma_5\psi_a - Q_a(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)\phi \\
&= \gamma_5(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\psi_a - (2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)\gamma_5\psi_a \\
&= \gamma_5(x_\mu - x^\nu\gamma_{\nu\mu})\psi_a, \\
[K_\mu, Q_a]\psi_b &= K_\mu(-(\gamma^\rho)_{ab}\partial_\rho\phi + (\gamma^\rho\gamma_5)_{ab}\partial_\rho\tilde{\phi}) - Q_a(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 3x_\mu)\psi_b \\
&\quad - Q_a(-x^\nu(\gamma_{\nu\mu})_b{}^c)\psi_c \\
&= \partial_\rho(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 2x_\mu)(-\phi(\gamma^\rho)_{ab} + \tilde{\phi}(\gamma^\rho\gamma_5)_{ab}) \\
&\quad - (2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu + 3x_\mu)\partial_\rho(-\phi(\gamma^\rho)_{ab} + \tilde{\phi}(\gamma^\rho\gamma_5)_{ab}) \\
&\quad + x^\nu(\gamma_{\nu\mu})_b{}^c\partial_\rho(-\phi(\gamma^\rho)_{ac} + \tilde{\phi}(\gamma^\rho\gamma_5)_{ac}) \\
&= (2\eta_{\rho\mu}(x^\nu\partial_\nu + 1) + 2x_\mu\partial_\rho - 2x_\rho\partial_\mu)(-\phi(\gamma^\rho)_{ab} + \tilde{\phi}(\gamma^\rho\gamma_5)_{ab}) \\
&\quad - x_\mu\partial_\rho(-\phi(\gamma^\rho)_{ab} + \tilde{\phi}(\gamma^\rho\gamma_5)_{ab}) + x^\nu(\gamma_{\nu\mu})_b{}^c\partial_\rho(-\phi(\gamma^\rho)_{ca} + \tilde{\phi}(\gamma^\rho\gamma_5)_{ca}) \\
&= (2\eta_{\rho\mu}(x^\nu\partial_\nu + 1) + x_\mu\partial_\rho - 2x_\rho\partial_\mu)(-\phi(\gamma^\rho)_{ab} + \tilde{\phi}(\gamma^\rho\gamma_5)_{ab}) \\
&\quad + x^\nu\partial_\rho(-\phi(\gamma_{\nu\mu}\gamma^\rho)_{ba} - \tilde{\phi}(\gamma_{\nu\mu}\gamma^\rho\gamma_5)_{ba}).
\end{aligned}$$

Hence, $[K_\mu, Q_a]$ does not correspond to any of the previous generators. We therefore introduce a new fermionic operator S_a. It seems we must multiply this generator with $(\gamma_\mu)_a{}^b$, since we have μ-subscript and γ-matrices. Then we have

$$[K_\mu, Q_a] = (\gamma_\mu)_a{}^b S_b. \tag{B.31}$$

Let us find how S_a acts on the fields. We will use the following two identities:

$$\begin{aligned}
\gamma_\mu\gamma^\mu &= \gamma_\mu\eta^{\mu\nu}\gamma_\nu = \frac{1}{2}(\eta^{\mu\nu}\gamma_\mu\gamma_\nu + \eta^{\nu\mu}\gamma_\mu\gamma_\nu) = \frac{1}{2}\eta^{\mu\nu}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) \\
&= \eta^{\mu\nu}\eta_{\mu\nu}\mathbb{1} = d\mathbb{1},
\end{aligned} \tag{B.32}$$

and

$$\begin{aligned}
\gamma^\mu\gamma_{\nu\mu} &= \frac{1}{2}\gamma^\mu(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu) = \frac{1}{2}\gamma^\mu(\gamma_\nu\gamma_\mu + \gamma_\mu\gamma_\nu - 2\gamma_\mu\gamma_\nu) \\
&= \frac{1}{2}\gamma^\mu(2\eta_{\mu\nu}\mathbb{1} - 2\gamma_\mu\gamma_\nu) = (1-d)\gamma_\nu.
\end{aligned} \tag{B.33}$$

With $d = 4$ they are $\gamma_\mu\gamma^\mu = 4\mathbb{1}$ and $\gamma^\mu\gamma_{\nu\mu} = -3\gamma_\nu$. Then we have

$$\gamma_\mu S_a\phi = (x_\mu - x^\nu\gamma_{\nu\mu})\psi_a.$$

We multiply both sides with $\frac{1}{4}\gamma^\mu$ from left:

$$S_a\phi = \frac{1}{4}\gamma^\mu(x_\mu - x^\nu\gamma_{\nu\mu})\psi_a = \frac{1}{4}(x_\mu\gamma^\mu + 3x^\nu\gamma_\nu)\psi_a = x^\mu\gamma_\mu\psi_a.$$

The next one is similar:

$$S_a\tilde{\phi} = \frac{1}{4}\gamma^\mu\gamma_5(x_\mu - x^\nu\gamma_{\nu\mu})\psi_a = -\frac{1}{4}\gamma_5(x_\mu\gamma^\mu + 3x^\nu\gamma_\nu)\psi_a = x^\mu\gamma_\mu\gamma_5\psi_a.$$

We will do some extra work on $[K_\mu, Q_a]\psi_b$. We rewrite the last line

$$\begin{aligned}
&x^\nu\partial_\rho(-\phi(\gamma_{\nu\mu}\gamma^\rho)_{ba} - \tilde{\phi}(\gamma_{\nu\mu}\gamma^\rho\gamma_5)_{ba}) \\
&\quad = x^\nu\partial^\rho(-\phi(\gamma_{\nu\mu\rho} + \eta_{\rho\mu}\gamma_\nu - \eta_{\rho\nu}\gamma_\mu)_{ba} - \tilde{\phi}((\gamma_{\nu\mu\rho} + \eta_{\rho\mu}\gamma_\nu - \eta_{\rho\nu}\gamma_\mu)\gamma_5)_{ba}) \\
&\quad = x^\nu\partial^\rho(\phi(\gamma_{\nu\mu\rho} - \eta_{\rho\mu}\gamma_\nu + \eta_{\rho\nu}\gamma_\mu)_{ab} - \tilde{\phi}((\gamma_{\nu\mu\rho} - \eta_{\rho\mu}\gamma_\nu + \eta_{\rho\nu}\gamma_\mu)\gamma_5)_{ab}) \\
&\quad = x^\nu\partial_\nu(\phi(\gamma_\mu)_{ab} - \tilde{\phi}(\gamma_\mu\gamma_5)_{ab}) - x^\nu\partial_\mu(\phi(\gamma_\nu)_{ab} - \tilde{\phi}(\gamma_\nu\gamma_5)_{ab}) \\
&\qquad + x^\nu\partial^\rho(-\phi(\gamma_{\mu\nu\rho})_{ab} + \tilde{\phi}(\gamma_{\mu\nu\rho}\gamma_5)_{ab}).
\end{aligned}$$

We find $S_a\psi_b = \frac{1}{4}(\gamma^\mu)_a{}^c[K_\mu, Q_c]\psi_b$:

$$\begin{aligned}
\frac{1}{4}(\gamma^\mu)_a{}^c[K_\mu, Q_c]\psi_b = \frac{1}{4}\Big\{&(2\eta_{\rho\mu}(x^\nu\partial_\nu + 1) + x_\mu\partial_\rho - 2x_\rho\partial_\mu)(-\phi(\gamma^\mu\gamma^\rho)_{ab} + \tilde{\phi}(\gamma^\mu\gamma^\rho\gamma_5)_{ab}) \\
&+ x^\nu\partial_\nu(\phi(\gamma^\mu\gamma_\mu)_{ab} - \tilde{\phi}(\gamma^\mu\gamma_\mu\gamma_5)_{ab}) - x^\nu\partial_\mu(\phi(\gamma^\mu\gamma_\nu)_{ab} - \tilde{\phi}(\gamma^\mu\gamma_\nu\gamma_5)_{ab}) \\
&+ x^\nu\partial^\rho(-\phi(\gamma^\mu\gamma_{\mu\nu\rho})_{ab} + \tilde{\phi}(\gamma_{\gamma^\mu\mu\nu\rho}\gamma_5)_{ab})\Big\}.
\end{aligned}$$

Using Proposition 5.2 and

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\lambda\delta} = -2(\delta^\lambda_\rho\delta^\delta_\sigma - \delta^\delta_\rho\delta^\lambda_\sigma),$$

we notice that

$$\begin{aligned}
\gamma^\mu\gamma_{\mu\nu\rho} &= \gamma^\mu\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 = \epsilon_{\mu\nu\rho\sigma}(\gamma^{\mu\sigma} + \eta^{\mu\sigma})\gamma_5 = \frac{1}{2}(\epsilon_{\mu\nu\rho\sigma}(\gamma^{\mu\sigma} + \eta^{\mu\sigma}) + \epsilon_{\sigma\nu\rho\mu}(\gamma^{\sigma\mu} + \eta^{\sigma\mu}))\gamma_5 \\
&= \frac{1}{2}(\epsilon_{\mu\nu\rho\sigma}(\gamma^{\mu\sigma} + \eta^{\mu\sigma}) - \epsilon_{\mu\nu\rho\sigma}(-\gamma^{\mu\sigma} + \eta^{\mu\sigma}))\gamma_5 = \epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\sigma}\gamma_5 = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\sigma\lambda\delta}\gamma_{\lambda\delta} \\
&= -\frac{1}{2}\epsilon_{\mu\sigma\nu\rho}\epsilon^{\mu\sigma\lambda\delta}\gamma_{\lambda\delta} = -\frac{1}{2}(-2\delta^\lambda_\nu\delta^\delta_\rho - \delta^\delta_\nu\delta^\lambda_\rho)\gamma_{\lambda\delta} = \gamma_{\nu\rho} - \gamma_{\rho\nu} = 2\gamma_{\nu\rho}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{4}(\gamma^\mu)_a{}^c[K_\mu, Q_c]\psi_b] = \frac{1}{4}\Big\{&(2\eta_{\rho\mu}(x^\nu\partial_\nu + 1) + x_\mu\partial_\rho - 2x_\rho\partial_\mu)(-\phi(\gamma^{\mu\rho} + \eta^{\mu\rho})_{ab} + \tilde{\phi}((\gamma^{\mu\rho} + \eta^{\mu\rho})\gamma_5)_{ab}) \\
&+ 4x^\nu\partial_\nu(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) - x_\nu\partial_\mu(\phi(\gamma^{\mu\nu} + \eta^{\mu\nu})_{ab} - \tilde{\phi}((\gamma^{\mu\nu} + \eta^{\mu\nu})\gamma_5)_{ab}) \\
&+ 2x_\nu\partial_\rho(-\phi(\gamma^{\nu\rho})_{ab} + \tilde{\phi}(\gamma^{\nu\rho}\gamma_5)_{ab})\Big\}.
\end{aligned}$$

We again rewrite some of the terms:

$$\begin{aligned}
&2\eta_{\rho\mu}(x^\nu\partial_\nu + 1)(-\phi(\gamma^{\mu\rho} + \eta^{\mu\rho})_{ab} + \tilde{\phi}((\gamma^{\mu\rho} + \eta^{\mu\rho})\gamma_5)_{ab}) \\
&= 2(x^\nu\partial_\nu + 1)(-\phi(\gamma^\mu{}_\mu)_{ab} + \tilde{\phi}(\gamma^\mu{}_\mu\gamma_5)_{ab}) + 2(x^\nu\partial_\nu + 1)(-4\phi C_{ab} + 4\tilde{\phi}(\gamma_5)_{ab}) \\
&= 8(x^\nu\partial_\nu + 1)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}),
\end{aligned}$$

$$
\begin{aligned}
&(x_\mu\partial_\rho - 2x_\rho\partial_\mu)(-\phi(\gamma^{\mu\rho} + \eta^{\mu\rho})_{ab} + \tilde{\phi}((\gamma^{\mu\rho} + \eta^{\mu\rho})\gamma_5)_{ab}) \\
&= (x_\mu\partial_\rho - 2x_\rho\partial_\mu)(-\phi(\gamma^{\mu\rho})_{ab} + \tilde{\phi}(\gamma^{\mu\rho}\gamma_5)_{ab}) \\
&\qquad + (x_\mu\partial_\rho - 2x_\rho\partial_\mu)\eta^{\mu\rho}(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) \\
&= x_\mu\partial_\rho(-\phi(\gamma^{\mu\rho})_{ab} + \tilde{\phi}(\gamma^{\mu\rho}\gamma_5)_{ab}) - 2x_\rho\partial_\mu(-\phi(\gamma^{\mu\rho})_{ab} + \tilde{\phi}(\gamma^{\mu\rho}\gamma_5)_{ab}) \\
&\qquad + (x_\mu\partial^\mu - 2x_\rho\partial^\rho)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) \\
&= x_\mu\partial_\rho(-\phi(\gamma^{\mu\rho})_{ab} + \tilde{\phi}(\gamma^{\mu\rho}\gamma_5)_{ab}) - 2x_\mu\partial_\rho(-\phi(\gamma^{\rho\mu})_{ab} + \tilde{\phi}(\gamma^{\rho\mu}\gamma_5)_{ab}) \\
&\qquad - x^\mu\partial_\mu(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) \\
&= x_\mu\partial_\rho(-\phi(\gamma^{\mu\rho})_{ab} + \tilde{\phi}(\gamma^{\mu\rho}\gamma_5)_{ab}) + 2x_\mu\partial_\rho(-\phi(\gamma^{\mu\rho})_{ab} + \tilde{\phi}(\gamma^{\mu\rho}\gamma_5)_{ab}) \\
&\qquad - x^\mu\partial_\mu(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) \\
&= 3x_\mu\partial_\rho(-\phi(\gamma^{\mu\rho})_{ab} + \tilde{\phi}(\gamma^{\mu\rho}\gamma_5)_{ab}) - x^\mu\partial_\mu(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}),
\end{aligned}
$$

$$
\begin{aligned}
&x_\nu\partial_\mu(\phi(\gamma^{\mu\nu} + \eta^{\mu\nu})_{ab} - \tilde{\phi}((\gamma^{\mu\nu} + \eta^{\mu\nu})\gamma_5)_{ab}) \\
&= x_\mu\partial_\nu(\phi(\gamma^{\nu\mu})_{ab} - \tilde{\phi}(\gamma^{\nu\mu}\gamma_5)_{ab}) + x^\mu\partial_\mu(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) \\
&= x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}) - x^\mu\partial_\mu(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}).
\end{aligned}
$$

Then S_a acts on ψ_b as

$$
\begin{aligned}
S_a\psi_b = \frac{1}{4}\Big\{&8(x^\nu\partial_\nu + 1)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + 3x_\mu\partial_\rho(-\phi(\gamma^{\mu\rho})_{ab} + \tilde{\phi}(\gamma^{\mu\rho}\gamma_5)_{ab}) \\
&- x^\mu\partial_\mu(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + 4x^\nu\partial_\nu(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) \\
&- x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}) + x^\mu\partial_\mu(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) \\
&+ 2x_\nu\partial_\rho(-\phi(\gamma^{\nu\rho})_{ab} + \tilde{\phi}(\gamma^{\nu\rho}\gamma_5)_{ab})\Big\} \\
= (x^\mu&\partial_\mu + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}).
\end{aligned}
$$

Now we have found how the generator S acts on the fields. We summarise it below:

$$
\begin{aligned}
&S_a\phi = x^\mu\gamma_\mu\psi_a, \qquad S_a\tilde{\phi} = x^\mu\gamma_\mu\gamma_5\psi_a, \\
&S_a\psi_b = (x^\mu\partial_\mu + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}).
\end{aligned}
$$

We have to compute the brackets containing S, to see if the algebra is now closed:

$$\begin{aligned}
[P_\mu, S_a]\phi &= P_\mu x^\nu \gamma_\nu \psi_a - S_a \partial_\mu \phi = x^\nu \gamma_\nu \partial_\mu \psi_a - \partial_\mu (x^\nu \gamma_\nu \psi_a) = -\delta_\mu^\nu \gamma_\nu \psi_a \\
&= -\gamma_\mu \psi_a = -(\gamma_\mu)_a{}^b Q_b \phi, \\
[P_\mu, S_a]\tilde{\phi} &= P_\mu x^\nu \gamma_\nu \gamma_5 \psi_a - S_a \partial_\mu \tilde{\phi} = x^\nu \gamma_\nu \partial_\mu \gamma_5 \psi_a - \partial_\mu (x^\nu \gamma_\nu \gamma_5 \psi_a) = -\delta_\mu^\nu \gamma_\nu \gamma_5 \psi_a \\
&= -\gamma_\mu \gamma_5 \psi_a = -(\gamma_\mu)_a{}^b Q_b \tilde{\phi}, \\
[P_\mu, S_a]\psi_b &= P_\mu \Big((x^\nu \partial_\nu + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\nu \partial_\rho (-\phi (\gamma^{\nu\rho})_{ab} + \tilde{\phi}(\gamma^{\nu\rho}\gamma_5)_{ab}) \Big) \\
&\quad - S_a \partial_\mu \psi_b \\
&= (x^\nu \partial_\nu + 2)\partial_\mu (-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\nu \partial_\rho \partial_\mu (-\phi (\gamma^{\nu\rho})_{ab} + \tilde{\phi}(\gamma^{\nu\rho}\gamma_5)_{ab}) \\
&\quad - \partial_\mu \Big((x^\nu \partial_\nu + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\nu \partial_\rho (-\phi (\gamma^{\nu\rho})_{ab} + \tilde{\phi}(\gamma^{\nu\rho}\gamma_5)_{ab}) \Big) \\
&= \partial_\mu (-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) - \partial_\rho (-\phi (\gamma_\mu{}^\rho)_{ab} + \tilde{\phi}(\gamma_\mu{}^\rho \gamma_5)_{ab}) \\
&= \partial_\mu (-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) - \partial_\rho (-\phi (\gamma_\mu \gamma^\rho - \delta_\mu^\rho)_{ab} + \tilde{\phi}((\gamma_\mu \gamma^\rho - \delta_\mu^\rho)\gamma_5)_{ab}) \\
&= -(\gamma_\mu)_a{}^c \partial_\rho (-\phi (\gamma^\rho)_{cb} + \tilde{\phi}(\gamma^\rho \gamma_5)_{cb}) = -(\gamma_\mu)_a{}^c Q_c \psi_b.
\end{aligned}$$

Hence, $[P_\mu, S_a] = -(\gamma_\mu)_a{}^b Q_b$.

$$\begin{aligned}
[M_{\mu\nu}, S_a]\phi &= M_{\mu\nu} x^\rho (\gamma_\rho)_a{}^b \psi_b - S_a (x_\mu \partial_\nu - x_\nu \partial_\mu)\phi \\
&= x^\rho (\gamma_\rho)_a{}^b ((x_\mu \partial_\nu - x_\nu \partial_\mu)\psi_b + (\Sigma_{\mu\nu})_b{}^c \psi_c) - (x_\mu \partial_\nu - x_\nu \partial_\mu) x^\rho (\gamma_\rho)_a{}^b \psi_b \\
&= x^\rho (\gamma_\rho \Sigma_{\mu\nu})_a{}^b \psi_b - (x_\mu (\gamma_\nu)_a{}^b - x_\nu (\gamma_\mu)_a{}^b)\psi_b \\
&= \frac{1}{2} x^\rho (\gamma_\rho \gamma_{\mu\nu})_a{}^b \psi_b - (x_\mu (\gamma_\nu)_a{}^b - x_\nu (\gamma_\mu)_a{}^b)\psi_b \\
&= \frac{1}{2} x^\rho (\gamma_{\rho\mu\nu} + \eta_{\rho\mu}\gamma_\nu - \eta_{\rho\nu}\gamma_\mu)_a{}^b \psi_b - (x_\mu (\gamma_\nu)_a{}^b - x_\nu (\gamma_\mu)_a{}^b)\psi_b \\
&= \frac{1}{2} x^\rho (\gamma_{\mu\nu\rho} - \eta_{\rho\mu}\gamma_\nu + \eta_{\rho\nu}\gamma_\mu)_a{}^b \psi_b \\
&= \frac{1}{2} x^\rho (\gamma_{\mu\nu}\gamma_\rho - \eta_{\rho\nu}\gamma_\mu + \eta_{\rho\mu}\gamma_\nu - \eta_{\rho\mu}\gamma_\nu + \eta_{\rho\nu}\gamma_\mu)_a{}^b \psi_b \\
&= \frac{1}{2} x^\rho (\gamma_{\mu\nu}\gamma_\rho)_a{}^b \psi_b = \frac{1}{2} (\gamma_{\mu\nu})_a{}^b x^\rho (\gamma_\rho)_b{}^c \psi_c = (\Sigma_{\mu\nu})_a{}^b S_b \phi, \\
[M_{\mu\nu}, S_a]\tilde{\phi} &= M_{\mu\nu} x^\rho (\gamma_\rho \gamma_5)_a{}^b \psi_b - S_a (x_\mu \partial_\nu - x_\nu \partial_\mu)\tilde{\phi} \\
&= x^\rho (\gamma_\rho \gamma_5)_a{}^b ((x_\mu \partial_\nu - x_\nu \partial_\mu)\psi_b + (\Sigma_{\mu\nu})_b{}^c \psi_c) - (x_\mu \partial_\nu - x_\nu \partial_\mu) x^\rho (\gamma_\rho \gamma_5)_a{}^b \psi_b \\
&= \frac{1}{2} (\gamma_{\mu\nu})_a{}^b x^\rho (\gamma_\rho \gamma_5)_b{}^c \psi_c = (\Sigma_{\mu\nu})_a{}^b S_b \tilde{\phi},
\end{aligned}$$

$$
\begin{aligned}
[M_{\mu\nu}, S_a]\psi_b &= M_{\mu\nu}\Big[(x^\rho\partial_\rho + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\rho\partial_\sigma(-\phi(\gamma^{\rho\sigma})_{ab} + \tilde{\phi}(\gamma^{\rho\sigma}\gamma_5)_{ab})\Big] \\
&\quad - S_a((x_\mu\partial_\nu - x_\nu\partial_\mu)\psi_b + (\Sigma_{\mu\nu})_b{}^c\psi_c) \\
&= \Big[(x^\rho\partial_\rho + 2)(x_\mu\partial_\nu - x_\nu\partial_\mu)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) \\
&\quad + x_\rho\partial_\sigma(x_\mu\partial_\nu - x_\nu\partial_\mu)(-\phi(\gamma^{\rho\sigma})_{ab} + \tilde{\phi}(\gamma^{\rho\sigma}\gamma_5)_{ab})\Big] \\
&\quad - \Big((x_\mu\partial_\nu - x_\nu\partial_\mu)\Big[(x^\rho\partial_\rho + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\rho\partial_\sigma(-\phi(\gamma^{\rho\sigma})_{ab} \\
&\quad + \tilde{\phi}(\gamma^{\rho\sigma}\gamma_5)_{ab})\Big] + (\Sigma_{\mu\nu})_b{}^c\Big[(x^\rho\partial_\rho + 2)(-\phi C_{ac} + \tilde{\phi}(\gamma_5)_{ac}) \\
&\quad + x_\rho\partial_\sigma(-\phi(\gamma^{\rho\sigma})_{ac} + \tilde{\phi}(\gamma^{\rho\sigma}\gamma_5)_{ac})\Big]\Big) \\
&= \Big(x^\rho\eta_{\rho\mu}\partial_\nu(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\rho\eta_{\sigma\mu}\partial_\nu(-\phi(\gamma^{\rho\sigma})_{ab} + \tilde{\phi}(\gamma^{\rho\sigma}\gamma_5)_{ab}) \\
&\quad - x_\mu\delta^\rho_\nu\partial_\rho(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + \eta_{\nu\rho}\partial_\sigma(-\phi(\gamma^{\rho\sigma})_{ab} + \tilde{\phi}(\gamma^{\rho\sigma}\gamma_5)_{ab})\Big) - (\mu \leftrightarrow \nu) \\
&\quad - (\Sigma_{\mu\nu})_b{}^c\Big[(x^\rho\partial_\rho + 2)(-\phi C_{ac} + \tilde{\phi}(\gamma_5)_{ac}) + x_\rho\partial_\sigma(-\phi(\gamma^{\rho\sigma})_{ac} + \tilde{\phi}(\gamma^{\rho\sigma}\gamma_5)_{ac})\Big]
\end{aligned}
$$

The first term in the first and second line cancel each other. Furthermore, in the calculation of $[M_{\mu\nu}, Q_a]\psi_b$, we saw that $(\Sigma_{\mu\nu})_b{}^c(\gamma^\rho)_{ca} = -(\Sigma_{\mu\nu}\gamma^\rho - \delta^\rho_\nu\gamma_\mu + \delta^\rho_\mu\gamma_\nu)_{ab}$. Similarly, we have

$$
\begin{aligned}
(\Sigma_{\mu\nu})_b{}^c(\gamma^{\rho\sigma})_{ac} &= -(\Sigma_{\mu\nu}\gamma^{\rho\sigma} + \delta^\sigma_\nu\gamma_\mu{}^\rho - \delta^\sigma_\mu\gamma_\nu{}^\rho - \delta^\rho_\nu\gamma_\mu{}^\sigma + \delta^\rho_\mu\gamma_\nu{}^\sigma)_{ab}, \\
(\Sigma_{\mu\nu})_b{}^c(\gamma^{\rho\sigma}\gamma_5)_{ac} &= ((-\Sigma_{\mu\nu}\gamma^{\rho\sigma} + \delta^\sigma_\nu\gamma_\mu{}^\rho - \delta^\sigma_\mu\gamma_\nu{}^\rho - \delta^\rho_\nu\gamma_\mu{}^\sigma + \delta^\rho_\mu\gamma_\nu{}^\sigma)\gamma_5)_{ab}.
\end{aligned}
$$

The terms not containing $\Sigma_{\mu\nu}$ cancel with the remaining term in $[M_{\mu\nu}, S_a]\psi_b$, thus

$$[M_{\mu\nu}, S_a]\psi_b = (\Sigma_{\mu\nu})_a{}^c S_c\psi_b$$

Hence, $[M_{\mu\nu}, S_a] = (\Sigma_{\mu\nu})_a{}^b S_b$.

$$
\begin{aligned}
[D, S_a]\phi &= Dx^\mu\gamma_\mu\psi_a - S_a(x^\nu\partial_\nu + 1)\phi = x^\mu\gamma_\mu(x^\nu\partial_\nu + \frac{3}{2})\psi_a - (x^\nu\partial_\nu + 1)x^\mu\gamma_\mu\psi_a \\
&= \frac{1}{2}x^\mu\gamma_\mu\psi_a - x^\mu\gamma_\mu\psi_a = -\frac{1}{2}x^\mu\gamma_\mu\psi_a, \\
[D, S_a]\tilde{\phi} &= Dx^\mu\gamma_\mu\gamma_5\psi_a - S_a(x^\nu\partial_\nu + 1)\tilde{\phi} = x^\mu\gamma_\mu\gamma_5(x^\nu\partial_\nu + \frac{3}{2})\psi_a - (x^\nu\partial_\nu + 1)x^\mu\gamma_\mu\gamma_5\psi_a \\
&= -\frac{1}{2}x^\mu\gamma_\mu\gamma_5\psi_a,
\end{aligned}
$$

$$
\begin{aligned}
[D,S_a]\psi_b &= D\left((x^\mu\partial_\mu+2)(-\phi C_{ab}+\tilde\phi(\gamma_5)_{ab})+x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab}+\tilde\phi(\gamma^{\mu\nu}\gamma_5)_{ab})\right)\\
&\qquad - S_a(x^\rho\partial_\rho+\frac{3}{2})\psi_b\\
&= (x^\mu\partial_\mu+2)(x^\rho\partial_\rho+1)(-\phi C_{ab}+\tilde\phi(\gamma_5)_{ab})+x_\mu\partial_\nu(x^\rho\partial_\rho+1)(-\phi(\gamma^{\mu\nu})_{ab}+\tilde\phi(\gamma^{\mu\nu}\gamma_5)_{ab})\\
&\qquad -(x^\rho\partial_\rho+\frac{3}{2})\left((x^\mu\partial_\mu+2)(-\phi C_{ab}+\tilde\phi(\gamma_5)_{ab})+x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab}+\tilde\phi(\gamma^{\mu\nu}\gamma_5)_{ab})\right)\\
&= (x^\mu\partial_\mu+x^\mu\partial_\mu+2-x^\rho\partial_\rho-\frac{3}{2}x^\mu\partial_\mu-3)(-\phi C_{ab}+\tilde\phi(\gamma_5)_{ab})\\
&\qquad +(2x_\mu\partial_\nu-x_\mu\partial_\nu-\frac{3}{2}x_\mu\partial_\nu)(-\phi(\gamma^{\mu\nu})_{ab}+\tilde\phi(\gamma^{\mu\nu}\gamma_5)_{ab})\\
&= -(\frac{1}{2}x^\mu\partial_\mu+1)(-\phi C_{ab}+\tilde\phi(\gamma_5)_{ab})-\frac{1}{2}x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab}+\tilde\phi(\gamma^{\mu\nu}\gamma_5)_{ab}).
\end{aligned}
$$

Hence, $[D,S_a]=-\frac{1}{2}S_a$.

$$
\begin{aligned}
[K_\mu,S_a]\phi &= K_\mu x^\rho\gamma_\rho\psi_a - S_a(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu+2x_\mu)\phi\\
&= x^\rho\gamma_\rho(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu+3x_\mu-x^\nu\gamma_{\nu\mu})\psi_a-(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu+2x_\mu)x^\rho\gamma_\rho\psi_a\\
&= x^\rho\gamma_\rho(x_\mu-x^\nu\gamma_{\nu\mu})\psi_a-(2x_\mu x^\nu\gamma_\nu-x^2\gamma_\mu)\psi_a\\
&= -x_\mu x^\nu\gamma_\nu\psi_a+x^2\gamma_\mu\psi_a-\frac{1}{2}x^\rho x^\nu\gamma_\rho(\gamma_\nu\gamma_\mu-\gamma_\mu\gamma_\nu)\psi_a\\
&= -x_\mu x^\nu\gamma_\nu\psi_a+x^2\gamma_\mu\psi_a-\frac{1}{2}x^\rho x^\nu((\gamma_{\rho\nu}+\eta_{\rho\nu})\gamma_\mu-(\gamma_{\rho\mu}+\eta_{\rho\mu})\gamma_\nu)\psi_a\\
&= -x_\mu x^\nu\gamma_\nu\psi_a+x^2\gamma_\mu\psi_a\\
&\qquad -\frac{1}{2}x^\rho x^\nu(\gamma_{\rho\nu\mu}+\eta_{\mu\nu}\gamma_\rho-\eta_{\mu\rho}\gamma_\nu+\eta_{\rho\nu}\gamma_\mu-\gamma_{\rho\mu\nu}-\eta_{\nu\mu}\gamma_\rho+\eta_{\nu\rho}\gamma_\mu+\eta_{\rho\mu}\gamma_\nu)\psi_a\\
&= -x_\mu x^\nu\gamma_\nu\psi_a+x^2\gamma_\mu\psi_a-x^\rho x^\nu(\epsilon_{\rho\nu\mu\sigma}\gamma^\sigma\gamma_5-\eta_{\mu\rho}\gamma_\nu+\eta_{\rho\nu}\gamma_\mu)\psi_a\\
&= -x^\rho x^\nu\epsilon_{\rho\nu\mu\sigma}\gamma^\sigma\gamma_5\psi_a=-x^\nu x^\rho\epsilon_{\nu\rho\mu\sigma}\gamma^\sigma\gamma_5\psi_a=x^\nu x^\rho\epsilon_{\rho\nu\mu\sigma}\gamma^\sigma\gamma_5\psi_a=0,\\
[K_\mu,S_a]\tilde\phi &= K_\mu x^\rho\gamma_\rho\gamma_5\psi_a-S_a(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu+2x_\mu)\phi\\
&= x^\rho\gamma_\rho\gamma_5(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu+3x_\mu-x^\nu\gamma_{\nu\mu})\psi_a-(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu+2x_\mu)x^\rho\gamma_\rho\gamma_5\psi_a\\
&= 0,\\
[K_\mu,S_a]\psi_b &= K_\mu\left((x^\rho\partial_\rho+2)(-\phi C_{ab}+\tilde\phi(\gamma_5)_{ab})+x_\rho\partial_\sigma(-\phi(\gamma^{\rho\sigma})_{ab}+\tilde\phi(\gamma^{\rho\sigma}\gamma_5)_{ab})\right)\\
&\qquad -S_a(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu+3x_\mu-x^\nu\gamma_{\nu\mu})\psi_b=\cdots=0.
\end{aligned}
$$

Hence, $[K_\mu,S_a]=0$.

$$
\begin{aligned}
[S_a,S_b]\phi &= S_a x^\mu(\gamma_\mu)_b{}^c\psi_c+S_b x^\mu(\gamma_\mu)_a{}^c\psi_c\\
&= x^\mu\Big\{(x^\nu\partial_\nu+2)\left[-\phi\Big((\gamma_\mu)_b{}^cC_{ac}+(\gamma_\mu)_a{}^cC_{bc}\Big)+\tilde\phi\Big((\gamma_\mu)_b{}^c(\gamma_5)_{ac}+(\gamma_\mu)_a{}^c(\gamma_5)_{bc}\Big)\right]\\
&\qquad +x_\nu\partial_\rho\left[-\phi\Big((\gamma_\mu)_b{}^c(\gamma^{\nu\rho})_{ac}+(\gamma_\mu)_a{}^c(\gamma^{\nu\rho})_{bc}\Big)+\tilde\phi\Big((\gamma_\mu)_b{}^c(\gamma^{\nu\rho}\gamma_5)_{ac}+(\gamma_\mu)_a{}^c(\gamma^{\nu\rho}\gamma_5)_{bc}\Big)\right]\Big\}\\
&= x^\mu\Big\{(x^\nu\partial_\nu+2)\left[\phi\Big((\gamma_\mu)_b{}^cC_{ca}+(\gamma_\mu)_a{}^cC_{cb}\Big)-\tilde\phi\Big((\gamma_\mu)_b{}^c(\gamma_5)_{ca}+(\gamma_\mu)_a{}^c(\gamma_5)_{cb}\Big)\right]\\
&\qquad +x_\nu\partial_\rho\left[-\phi\Big((\gamma_\mu)_b{}^c(\gamma^{\nu\rho})_{ca}+(\gamma_\mu)_a{}^c(\gamma^{\nu\rho})_{cb}\Big)+\tilde\phi\Big((\gamma_\mu)_b{}^c(\gamma^{\nu\rho}\gamma_5)_{ca}+(\gamma_\mu)_a{}^c(\gamma^{\nu\rho}\gamma_5)_{cb}\Big)\right]\Big\}\\
&= x^\mu\Big\{(x^\nu\partial_\nu+2)\left[\phi\Big((\gamma_\mu)_{ba}+(\gamma_\mu)_{ab}\Big)-\tilde\phi\Big((\gamma_\mu\gamma_5)_{ba}+(\gamma_\mu\gamma_5)_{ab}\Big)\right]\\
&\qquad +x_\nu\partial_\rho\left[-\phi\Big((\gamma_\mu\gamma^{\nu\rho})_{ba}+(\gamma_\mu\gamma^{\nu\rho})_{ab}\Big)+\tilde\phi\Big((\gamma_\mu\gamma^{\nu\rho}\gamma_5)_{ba}+(\gamma_\mu\gamma^{\nu\rho}\gamma_5)_{ab}\Big)\right]\Big\}.
\end{aligned}
$$

We notice that all terms containing $\tilde{\phi}$ vanishes. The first one is simply due to $(\gamma_\mu\gamma_5)_{ab} = -(\gamma_\mu\gamma_5)_{ba}$. The other one is slightly more complicated. We use Proposition 5.2, so that $\gamma^{\nu\rho}\gamma_5 = -\frac{1}{2}\epsilon^{\nu\rho\sigma\lambda}\gamma_{\sigma\lambda}$. Then,

$$(\gamma_\mu\gamma^{\nu\rho}\gamma_5)_{ba} + (\gamma_\mu\gamma^{\nu\rho}\gamma_5)_{ab} = -\frac{1}{2}\epsilon^{\nu\rho\sigma\lambda}\left[(\gamma_\mu\gamma_{\sigma\lambda})_{ba} + (\gamma_\mu\gamma_{\sigma\lambda})_{ab}\right].$$

We can then, using (5.9), rewrite

$$\begin{aligned}\gamma_\mu\gamma_{\sigma\lambda} &= \frac{1}{2}\gamma_\mu(\gamma_\sigma\gamma_\lambda - \gamma_\lambda\gamma_\sigma) = \frac{1}{2}\left((\gamma_{\mu\sigma} + \eta_{\mu\sigma})\gamma_\lambda - (\gamma_{\mu\lambda} + \eta_{\mu\lambda})\gamma_\sigma\right)\\ &= \frac{1}{2}\left(\gamma_{\mu\sigma\lambda} + \eta_{\lambda\sigma}\gamma_\mu - \eta_{\lambda\mu}\gamma_\sigma + \eta_{\mu\sigma}\gamma_\lambda - \gamma_{\mu\lambda\sigma} - \eta_{\sigma\lambda}\gamma_\mu + \eta_{\sigma\mu}\gamma_\lambda - \eta_{\mu\lambda}\gamma_\sigma\right)\\ &= \gamma_{\mu\sigma\lambda} + \eta_{\mu\sigma}\gamma_\lambda - \eta_{\mu\lambda}\gamma_\sigma. \end{aligned} \tag{B.34}$$

Then, using $(\gamma_{\mu\nu\rho})_{ab} = -(\gamma_{\mu\nu\rho})_{ba}$ and $(\gamma_\mu)_{ab} = (\gamma_\mu)_{ba}$,

$$\begin{aligned}(\gamma_\mu\gamma^{\nu\rho}\gamma_5)_{ba} + (\gamma_\mu\gamma^{\nu\rho}\gamma_5)_{ab} &= -\frac{1}{2}\epsilon^{\nu\rho\sigma\lambda}\left[(\gamma_{\mu\sigma\lambda} + \eta_{\mu\sigma}\gamma_\lambda - \eta_{\mu\lambda}\gamma_\sigma)_{ba} + (\gamma_{\mu\sigma\lambda} + \eta_{\mu\sigma}\gamma_\lambda - \eta_{\mu\lambda}\gamma_\sigma)_{ab}\right]\\ &= -\frac{1}{2}\epsilon^{\nu\rho\sigma\lambda}\left[(-\gamma_{\mu\sigma\lambda} + \eta_{\mu\sigma}\gamma_\lambda - \eta_{\mu\lambda}\gamma_\sigma)_{ab} + (\gamma_{\mu\sigma\lambda} + \eta_{\mu\sigma}\gamma_\lambda - \eta_{\mu\lambda}\gamma_\sigma)_{ab}\right]\\ &= -\epsilon^{\nu\rho\sigma\lambda}(\eta_{\mu\sigma}\gamma_\lambda - \eta_{\mu\lambda}\gamma_\sigma)_{ab} = -\epsilon^{\nu\rho\sigma\lambda}\eta_{\mu\sigma}(\gamma_\lambda)_{ab} + \epsilon^{\nu\rho\sigma\lambda}\eta_{\mu\lambda}(\gamma_\sigma)_{ab}\\ &= -\epsilon^{\nu\rho\sigma\lambda}\eta_{\mu\sigma}(\gamma_\lambda)_{ab} + \epsilon^{\nu\rho\lambda\sigma}\eta_{\mu\sigma}(\gamma_\lambda)_{ab}\\ &= -\epsilon^{\nu\rho\sigma\lambda}\eta_{\mu\sigma}(\gamma_\lambda)_{ab} - \epsilon^{\nu\rho\sigma\lambda}\eta_{\mu\sigma}(\gamma_\lambda)_{ab} = -2\epsilon^{\nu\rho\sigma\lambda}\eta_{\mu\sigma}(\gamma_\lambda)_{ab}.\end{aligned}$$

Now, the whole term including $\tilde{\phi}$ is

$$-2x^\mu x_\nu\partial_\rho\epsilon^{\nu\rho\sigma\lambda}\eta_{\mu\sigma}(\gamma_\lambda)_{ab} = -2x_\mu x_\nu\partial_\rho\epsilon^{\nu\rho\mu\lambda}(\gamma_\lambda)_{ab}.$$

This vanishes since

$$2x_\mu x_\nu\epsilon^{\nu\rho\mu\lambda} = x_\mu x_\nu\epsilon^{\nu\rho\mu\lambda} + x_\nu x_\mu\epsilon^{\mu\rho\nu\lambda} = x_\mu x_\nu\epsilon^{\nu\rho\mu\lambda} - x_\nu x_\mu\epsilon^{\nu\rho\mu\lambda} = 0.$$

Thus, $[S_a, S_b]\phi$ consists of only terms involving ϕ:

$$[S_a, S_b]\phi = x^\mu\Big\{(x^\nu\partial_\nu + 2)\phi\Big((\gamma_\mu)_{ba} + (\gamma_\mu)_{ab}\Big) - x_\nu\partial_\rho\phi\Big((\gamma_\mu\gamma^{\nu\rho})_{ba} + (\gamma_\mu\gamma^{\nu\rho})_{ab}\Big)\Big\}.$$

We again use (B.34). However this time the Minkowski metric is the Kronecker delta

$$\begin{aligned}[S_a, S_b]\phi &= x^\mu\Big\{(x^\nu\partial_\nu + 2)\phi\Big((\gamma_\mu)_{ab} + (\gamma_\mu)_{ab}\Big)\\ &\qquad - x_\nu\partial_\rho\phi\Big((\gamma_\mu{}^{\nu\rho} + \delta^\nu_\mu\gamma^\rho - \delta^\rho_\mu\gamma^\nu)_{ba} + (\gamma_\mu{}^{\nu\rho} + \delta^\nu_\mu\gamma^\rho - \delta^\rho_\mu\gamma^\nu)_{ab}\Big)\Big\}\\ &= x^\mu\Big\{2(x^\nu\partial_\nu + 2)\phi(\gamma_\mu)_{ab}\\ &\qquad - x_\nu\partial_\rho\phi\Big((-\gamma_\mu{}^{\nu\rho} + \delta^\nu_\mu\gamma^\rho - \delta^\rho_\mu\gamma^\nu)_{ab} + (\gamma_\mu{}^{\nu\rho} + \delta^\nu_\mu\gamma^\rho - \delta^\rho_\mu\gamma^\nu)_{ab}\Big)\Big\}\\ &= x^\mu\Big\{2(x^\nu\partial_\nu + 2)\phi(\gamma_\mu)_{ab} - 2x_\nu\partial_\rho\phi(\delta^\nu_\mu\gamma^\rho - \delta^\rho_\mu\gamma^\nu)_{ab}\Big\}\\ &= 2x^\mu(x^\nu\partial_\nu + 2)\phi(\gamma_\mu)_{ab} - 2x^\nu x_\nu\partial_\rho\phi(\gamma^\rho)_{ab} + 2x^\mu x_\nu\partial_\mu(\gamma^\nu)_{ab}\\ &= 2x_\mu(x^\nu\partial_\nu + 2)\phi(\gamma^\mu)_{ab} - 2x^\nu x_\nu\partial_\mu\phi(\gamma^\mu)_{ab} + 2x^\nu x_\mu\partial_\nu(\gamma^\mu)_{ab}\\ &= 2(\gamma^\mu)_{ab}\left(x_\mu x^\nu\partial_\nu + 2x_\mu - x^2\partial_\mu + x_\mu x^\nu\partial_\nu\right)\phi = 2(\gamma^\mu)_{ab}K_\mu\phi.\end{aligned}$$

Similarly one finds $[S_a, S_b]\tilde{\phi} = 2(\gamma^\mu)_{ab}K_\mu\tilde{\phi}$. For $[S_a, S_b]\psi$ we use (B.4):

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\psi = \zeta^a\zeta^b[S_a, S_b]\psi.$$

Applying the supersymmetry transformations, we get

$$\begin{aligned}[\delta_{\zeta_1}, \delta_{\zeta_2}]\psi = x^\mu x^\nu\gamma^\rho\gamma_\mu(\zeta_1\bar{\zeta}_2 - \zeta_2\bar{\zeta}_1)\gamma_\nu\partial_\rho\psi + x^\mu x^\nu\gamma^\rho\gamma_5\gamma_\mu(\zeta_1\bar{\zeta}_2 - \zeta_2\bar{\zeta}_1)\gamma_\nu\gamma_5\partial_\rho\psi \\ +2x_\mu(\zeta_1\bar{\zeta}_2 - \zeta_2\bar{\zeta}_1)\gamma_\mu\psi + 2x_\mu\gamma_5(\zeta_1\bar{\zeta}_2 - \zeta_2\bar{\zeta}_1)\gamma_\mu\gamma_5\psi.\end{aligned}$$

Applying the Fierz identity (A.32), we get, where we have used the equations of motion $\partial\!\!\!/\psi = 0$,

$$[S_a, S_b]\psi = 2(\gamma^\mu)_{ab}K_\mu\psi.$$

Hence, $[S_a, S_b] = 2(\gamma^\mu)_{ab}K_\mu$.

$$\begin{aligned}[Q_a, S_b]\phi &= Q_a x^\mu(\gamma_\mu)_b{}^c\psi_c + S_b\psi_a \\
&= x^\mu(\gamma_\mu)_b{}^c\partial_\nu(-\phi(\gamma^\nu)_{ac} + \tilde{\phi}(\gamma^\nu\gamma_5)_{ac}) + (x^\mu\partial_\mu + 2)(-\phi C_{ba} + \tilde{\phi}(\gamma_5)_{ba}) \\
&\quad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ba} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ba}) \\
&= x^\mu(\gamma_\mu)_b{}^c\partial_\nu(-\phi(\gamma^\nu)_{ca} - \tilde{\phi}(\gamma^\nu\gamma_5)_{ca}) + (x^\mu\partial_\mu + 2)(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) \\
&\quad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}) \\
&= x_\mu\partial_\nu(-\phi(\gamma^\mu\gamma^\nu)_{ba} - \tilde{\phi}(\gamma^\mu\gamma^\nu\gamma_5)_{ba}) + (x^\mu\partial_\mu + 2)(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) \\
&\quad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}) \\
&= x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu} + \eta^{\mu\nu})_{ba} - \tilde{\phi}((\gamma^{\mu\nu} + \eta^{\mu\nu})\gamma_5)_{ba}) + (x^\mu\partial_\mu + 2)(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) \\
&\quad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}) \\
&= x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu} - \eta^{\mu\nu})_{ab} - \tilde{\phi}((\gamma^{\mu\nu} - \eta^{\mu\nu})\gamma_5)_{ab}) + (x^\mu\partial_\mu + 2)(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) \\
&\quad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}) \\
&= x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} - \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}) + x^\mu\partial_\mu(\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) \\
&\quad + (x^\mu\partial_\mu + 2)(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab}) \\
&= -2x_\mu\partial_\nu\phi(\gamma^{\mu\nu})_{ab} + 2(-\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) + 2x^\mu\partial_\mu\phi C_{ab} \\
&= -\left(x_\mu\partial_\nu(\gamma^{\mu\nu})_{ab} + x_\nu\partial_\mu(\gamma^{\nu\mu})_{ab}\right)\phi + 2(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) + 2x^\mu\partial_\mu\phi C_{ab} \\
&= -\left(x_\mu\partial_\nu(\gamma^{\mu\nu})_{ab} - x_\nu\partial_\mu(\gamma^{\mu\nu})_{ab}\right)\phi + 2(\phi C_{ab} - \tilde{\phi}(\gamma_5)_{ab}) + 2x^\mu\partial_\mu\phi C_{ab} \\
&= -(x_\mu\partial_\nu - x_\nu\partial_\mu)\gamma^{\mu\nu})_{ab}\phi + 2(x^\mu\partial_\mu + 1)\phi C_{ab} - 2\tilde{\phi}(\gamma_5)_{ab} \\
&= -(\gamma^{\mu\nu})_{ab}M_{\mu\nu}\phi + 2D\phi C_{ab} - 2\tilde{\phi}(\gamma_5)_{ab},\end{aligned}$$

$$
\begin{aligned}
[Q_a, S_b]\tilde{\phi} &= Q_a x^\mu (\gamma_\mu\gamma_5)_b{}^c \psi_c + S_b(\gamma_5)_a{}^c\psi_c \\
&= x^\mu(\gamma_\mu\gamma_5)_b{}^c \partial_\nu(-\phi(\gamma^\nu)_{ac} + \tilde{\phi}(\gamma^\nu\gamma_5)_{ac}) + (\gamma_5)_a{}^c \Big((x^\mu\partial_\mu + 2)(-\phi C_{bc} + \tilde{\phi}(\gamma_5)_{bc}) \\
&\qquad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{bc} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{bc})\Big) \\
&= x^\mu(\gamma_\mu\gamma_5)_b{}^c \partial_\nu(-\phi(\gamma^\nu)_{ca} - \tilde{\phi}(\gamma^\nu\gamma_5)_{ca}) + (\gamma_5)_a{}^c (x^\mu\partial_\mu + 2)(\phi C_{cb} - \tilde{\phi}(\gamma_5)_{cb}) \\
&\qquad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{cb} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{cb}) \\
&= x_\mu\partial_\nu(\phi(\gamma^\mu\gamma^\nu\gamma_5)_{ba} + \tilde{\phi}(\gamma^\mu\gamma^\nu\gamma_5^2)_{ba}) + (x^\mu\partial_\mu + 2)(\phi(\gamma_5)_{ab} + \tilde{\phi}C_{ab}) \\
&\qquad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu}\gamma_5)_{ab} - \tilde{\phi}(\gamma^{\mu\nu})_{ab}) \\
&= x_\mu\partial_\nu(\phi((\gamma^{\mu\nu} + \eta^{\mu\nu})\gamma_5)_{ba} - \tilde{\phi}(\gamma^{\mu\nu} + \eta^{\mu\nu})_{ba}) + (x^\mu\partial_\mu + 2)(\phi(\gamma_5)_{ab} + \tilde{\phi}C_{ab}) \\
&\qquad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu}\gamma_5)_{ab} - \tilde{\phi}(\gamma^{\mu\nu})_{ab}) \\
&= x_\mu\partial_\nu(\phi((\gamma^{\mu\nu} - \eta^{\mu\nu})\gamma_5)_{ab} - \tilde{\phi}(\gamma^{\mu\nu} - \eta^{\mu\nu})_{ab}) + (x^\mu\partial_\mu + 2)(\phi(\gamma_5)_{ab} + \tilde{\phi}C_{ab}) \\
&\qquad + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu}\gamma_5)_{ab} - \tilde{\phi}(\gamma^{\mu\nu})_{ab}) \\
&= x_\mu\partial_\nu(\phi(\gamma^{\mu\nu}\gamma_5)_{ab} - \tilde{\phi}(\gamma^{\mu\nu})_{ab}) + x^\mu\partial_\mu(-\phi(\gamma_5)_{ab} + \tilde{\phi}C_{ab}) \\
&\qquad + (x^\mu\partial_\mu + 2)(\phi(\gamma_5)_{ab} + \tilde{\phi}C_{ab}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu}\gamma_5)_{ab} - \tilde{\phi}(\gamma^{\mu\nu})_{ab}) \\
&= -2x_\mu\partial_\nu\tilde{\phi}(\gamma^{\mu\nu})_{ab}) + 2(\phi(\gamma_5)_{ab} + \tilde{\phi}C_{ab}) + 2x^\mu\partial_\mu\tilde{\phi}C_{ab} \\
&= -(x_\mu\partial_\nu - x_\nu\partial_\mu)\gamma^{\mu\nu})_{ab}\tilde{\phi} + 2(x^\mu\partial_\mu + 1)\tilde{\phi}C_{ab} + 2\phi(\gamma_5)_{ab} \\
&= -(\gamma^{\mu\nu})_{ab}M_{\mu\nu}\tilde{\phi} + 2D\tilde{\phi}C_{ab} + 2\phi(\gamma_5)_{ab}.
\end{aligned}
$$

For the last one, we use (B.4):

$$[\delta_\epsilon, \delta_\zeta]\psi = \epsilon^a\zeta^b[Q_a, S_b]\psi.$$

Applying the supersymmetry transformations gives

$$
\begin{aligned}
[\delta_\epsilon, \delta_\zeta]\psi = &-x^\mu\gamma^\nu\gamma_\mu\zeta\bar{\epsilon}\partial_\nu\psi - x^\mu\gamma^\nu\gamma_5\gamma_\mu\zeta\bar{\epsilon}\gamma_5\partial_\nu\psi \\
&- 2\zeta\bar{\epsilon}\psi + 2\gamma_5\zeta\bar{\epsilon}\gamma_5\psi \\
&- \gamma^\nu\zeta\epsilon\bar{\zeta}\gamma_\mu\partial_\nu(x^\mu\psi) - \gamma^\nu\gamma_5\zeta\epsilon\bar{\zeta}\gamma_\mu\gamma_5\partial_\nu(x^\mu\psi).
\end{aligned}
$$

Applying the Fierz identity (A.31), and using $\bar{\zeta}\epsilon = \bar{\epsilon}\zeta$, $\bar{\zeta}\gamma_5\epsilon = \bar{\epsilon}\gamma_5\zeta$, $\bar{\zeta}\gamma^\mu\epsilon = -\bar{\epsilon}\gamma^\mu\zeta$, $\bar{\zeta}\gamma^\mu\gamma_5\epsilon = \bar{\epsilon}\gamma^\mu\gamma_5\zeta$ and $\bar{\zeta}\gamma^{\mu\nu}\epsilon = -\bar{\epsilon}\gamma^{\mu\nu}\zeta$, we find that

$$[Q_a, S_b]\psi = -(\gamma^{\mu\nu})_{ab}M_{\mu\nu}\tilde{\phi} + 2D\tilde{\phi}C_{ab}\frac{1}{2}(\gamma_5)_{ab}\psi.$$

Hence, we need an extra bosonic generator, let us call it R, which acts on the fields as

$$R\phi = \tilde{\phi}, \qquad R\tilde{\phi} = -\phi, \qquad R\psi = \frac{1}{2}\gamma_5\psi. \tag{B.35}$$

Then we have $[Q_a, S_b] = 2DC_{ab} - 2R(\gamma_5)_{ab} - (\gamma^{\mu\nu})_{ab}M_{\mu\nu}$.

We should also find the additional brackets:

$$
\begin{aligned}
[P_\mu, R]\phi &= P_\mu\tilde{\phi} - R\partial_\mu\phi = \partial_\mu\tilde{\phi} - \partial_\mu\tilde{\phi} = 0, \\
[P_\mu, R]\tilde{\phi} &= -P_\mu\phi - R\partial_\mu\tilde{\phi} = -\partial_\mu\phi + \partial_\mu\phi = 0,
\end{aligned}
$$

$$[P_\mu, R]\psi = P_\mu \frac{1}{2}\gamma_5\psi - R\partial_\mu\psi = \frac{1}{2}\gamma_5\partial_\mu\psi - \partial_\mu\frac{1}{2}\gamma_5\psi = 0.$$

Hence, $[P_\mu, R] = 0$.

$$\begin{aligned}
[M_{\mu\nu}, R]\phi &= M_{\mu\nu}\tilde{\phi} - R(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi = (x_\mu\partial_\nu - x_\nu\partial_\mu)\tilde{\phi} - (x_\mu\partial_\nu - x_\nu\partial_\mu)\tilde{\phi} = 0,\\
[M_{\mu\nu}, R]\tilde{\phi} &= -M_{\mu\nu}\phi - R(x_\mu\partial_\nu - x_\nu\partial_\mu)\tilde{\phi} = -(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi + (x_\mu\partial_\nu - x_\nu\partial_\mu)\phi = 0,\\
[M_{\mu\nu}, R]\psi &= M_{\mu\nu}\frac{1}{2}\gamma_5\psi - R(x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\psi\\
&= \frac{1}{2}\gamma_5(x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\psi - (x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\frac{1}{2}\gamma_5\psi = 0.
\end{aligned}$$

Hence, $[M_{\mu\nu}, R] = 0$.

$$\begin{aligned}
[D, R]\phi &= D\tilde{\phi} - R(x^\mu\partial_\mu + 1)\phi = (x^\mu\partial_\mu + 1)\tilde{\phi} - (x^\mu\partial_\mu + 1)\tilde{\phi} = 0,\\
[D, R]\tilde{\phi} &= -D\phi - R(x^\mu\partial_\mu + 1)\tilde{\phi} = -(x^\mu\partial_\mu + 1)\phi + (x^\mu\partial_\mu + 1)\phi = 0,\\
[D, R]\psi &= D\frac{1}{2}\gamma_5\psi - R(x^\mu\partial_\mu + \frac{3}{2})\psi = \frac{1}{2}\gamma_5(x^\mu\partial_\mu + \frac{3}{2})\psi - (x^\mu\partial_\mu + \frac{3}{2})\psi\frac{1}{2}\gamma_5\psi = 0.
\end{aligned}$$

Hence, $[D, R]\phi = 0$.

$$\begin{aligned}
[K_\mu, R]\phi &= K_\mu\tilde{\phi} - R(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x_\mu)\phi\\
&= (2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x_\mu)\tilde{\phi} - (2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x_\mu)\tilde{\phi} = 0,\\
[K_\mu, R]\tilde{\phi} &= -K_\mu\phi - R(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x_\mu)\tilde{\phi}\\
&= -(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x_\mu)\phi + (2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x_\mu)\phi = 0,\\
[K_\mu, R]\psi &= K_\mu\frac{1}{2}\gamma_5\psi - R(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\psi\\
&= \frac{1}{2}\gamma_5(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\psi - (2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\frac{1}{2}\gamma_5\psi\\
&= \frac{1}{2}(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\gamma_5\psi - (2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 3x_\mu - x^\nu\gamma_{\nu\mu})\frac{1}{2}\gamma_5\psi\\
&= 0.
\end{aligned}$$

Hence, $[K_\mu, R] = 0$.

$$\begin{aligned}
[R, Q_a]\phi &= R\psi_a - Q_a\tilde{\phi} = \frac{1}{2}(\gamma_5)_a{}^b\psi_b - (\gamma_5)_a{}^b\psi_b = -\frac{1}{2}(\gamma_5)_a{}^b\psi_b = -\frac{1}{2}(\gamma_5)_a{}^bQ_b\phi,\\
[R, Q_a]\tilde{\phi} &= R(\gamma_5)_a{}^b\psi_b + Q_a\phi = \frac{1}{2}(\gamma_5)_a{}^b(\gamma_5)_b{}^c\psi_c + \psi_a = \frac{1}{2}(\gamma_5)_a{}^b(\gamma_5)_b{}^c\psi_c - (\gamma_5)_a{}^b(\gamma_5)_b{}^c\psi_c\\
&= -\frac{1}{2}(\gamma_5)_a{}^b(\gamma_5)_b{}^c\psi_c = -\frac{1}{2}(\gamma_5)_a{}^bQ_b\tilde{\phi},
\end{aligned}$$

$$
\begin{aligned}
[R, Q_a]\psi_b &= R\partial_\mu(-\phi(\gamma^\mu)_{ab} + \tilde{\phi}(\gamma^\mu\gamma_5)_{ab}) - Q_a\frac{1}{2}(\gamma_5)_b{}^c\psi_c \\
&= \partial_\mu(-\tilde{\phi}(\gamma^\mu)_{ab} - \phi(\gamma^\mu\gamma_5)_{ab}) - \frac{1}{2}(\gamma_5)_b{}^c\partial_\mu(-\phi(\gamma^\mu)_{ac} + \tilde{\phi}(\gamma^\mu\gamma_5)_{ac}) \\
&= \partial_\mu(-\tilde{\phi}(\gamma_5\gamma^\mu\gamma_5)_{ab} + \phi(\gamma_5\gamma^\mu)_{ab}) - \frac{1}{2}(\gamma_5)_b{}^c\partial_\mu(-\phi(\gamma^\mu)_{ca} + \tilde{\phi}(\gamma^\mu\gamma_5)_{ca}) \\
&= -(\gamma_5)_a{}^c\partial_\mu(-\phi(\gamma^\mu)_{cb} + \tilde{\phi}(\gamma^\mu\gamma_5)_{cb}) - \frac{1}{2}\partial_\mu(\phi(\gamma^\mu\gamma_5)_{ba} - \tilde{\phi}(\gamma^\mu)_{ba}) \\
&= -(\gamma_5)_a{}^cQ_c\psi_b - \frac{1}{2}\partial_\mu(-\phi(\gamma^\mu\gamma_5)_{ab} - \tilde{\phi}(\gamma^\mu)_{ab}) \\
&= -(\gamma_5)_a{}^cQ_c\psi_b - \frac{1}{2}\partial_\mu(\phi(\gamma_5\gamma^\mu)_{ab} - \tilde{\phi}(\gamma_5\gamma^\mu\gamma_5)_{ab}) \\
&= -(\gamma_5)_a{}^cQ_c\psi_b + \frac{1}{2}(\gamma_5)_a{}^c\partial_\mu(-\phi(\gamma^\mu)_{cb} + \tilde{\phi}(\gamma^\mu\gamma_5)_{cb}) = -(\gamma_5)_a{}^cQ_c\psi_b + \frac{1}{2}(\gamma_5)_a{}^cQ_c\psi_b \\
&= -\frac{1}{2}(\gamma_5)_a{}^cQ_c\psi_b.
\end{aligned}
$$

Hence, $[R, Q_a] = -\frac{1}{2}(\gamma_5)_a{}^bQ_b$.

$$
\begin{aligned}
[R, S_a]\phi &= Rx^\mu(\gamma_\mu)_a{}^b\psi_b - S_a\tilde{\phi} = x^\mu(\gamma_\mu)_a{}^b\frac{1}{2}(\gamma_5)_b{}^c\psi_c - x^\mu(\gamma_\mu\gamma_5)_a{}^b\psi_b = -\frac{1}{2}x^\mu(\gamma_\mu\gamma_5)_a{}^b\psi_b \\
&= \frac{1}{2}x^\mu(\gamma_5\gamma_\mu)_a{}^b\psi_b = \frac{1}{2}(\gamma_5)_a{}^bx^\mu(\gamma_\mu)_b{}^c\psi_c = \frac{1}{2}(\gamma_5)_a{}^bS_b\phi, \\
[R, S_a]\tilde{\phi} &= Rx^\mu(\gamma_\mu\gamma_5)_a{}^b\psi_b + S_a\phi = x^\mu(\gamma_\mu\gamma_5)_a{}^b\frac{1}{2}(\gamma_5)_b{}^c\psi_c + x^\mu(\gamma_\mu)_a{}^b\psi_b \\
&= -\frac{1}{2}x^\mu(\gamma_5\gamma_\mu\gamma_5)_a{}^b\psi_b + x^\mu(\gamma_5\gamma_\mu\gamma_5)_a{}^b\psi_b = \frac{1}{2}x^\mu(\gamma_5\gamma_\mu\gamma_5)_a{}^b\psi_b \\
&= \frac{1}{2}(\gamma_5)_a{}^bx^\mu(\gamma_\mu\gamma_5)_b{}^c\psi_c = \frac{1}{2}(\gamma_5)_a{}^bS_b\tilde{\phi},
\end{aligned}
$$

$$
\begin{aligned}
[R, S_a]\psi_b &= R\Big[(x^\mu\partial_\mu + 2)(-\phi C_{ab} + \tilde{\phi}(\gamma_5)_{ab}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ab})\Big] \\
&\quad - S_a \frac{1}{2}(\gamma_5)_b{}^c\psi_c \\
&= \Big[(x^\mu\partial_\mu + 2)(-\tilde{\phi} C_{ab} - \phi(\gamma_5)_{ab}) + x_\mu\partial_\nu(-\tilde{\phi}(\gamma^{\mu\nu})_{ab} - \phi(\gamma^{\mu\nu}\gamma_5)_{ab})\Big] \\
&\quad - \frac{1}{2}(\gamma_5)_b{}^c\Big[(x^\mu\partial_\mu + 2)(-\phi C_{ac} + \tilde{\phi}(\gamma_5)_{ac}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ac} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ac})\Big] \\
&= \Big[(x^\mu\partial_\mu + 2)(-\phi(\gamma_5)_{ab} + \tilde{\phi}(\gamma_5^2)_{ab}) + x_\mu\partial_\nu(-\phi(\gamma_5\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma_5\gamma^{\mu\nu}\gamma_5)_{ab})\Big] \\
&\quad - \frac{1}{2}(\gamma_5)_b{}^c\Big[(x^\mu\partial_\mu + 2)(\phi C_{ca} - \tilde{\phi}(\gamma_5)_{ca}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{ca} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{ca})\Big] \\
&= (\gamma_5)_a{}^c\Big[(x^\mu\partial_\mu + 2)(-\phi C_{cb} + \tilde{\phi}(\gamma_5)_{cb}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu})_{cb} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{cb})\Big] \\
&\quad - \frac{1}{2}\Big[(x^\mu\partial_\mu + 2)(\phi(\gamma_5)_{ba} + \tilde{\phi} C_{ba}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu}\gamma_5)_{ba} - \tilde{\phi}(\gamma^{\mu\nu})_{ba})\Big] \\
&= (\gamma_5)_a{}^c S_c\psi_b \\
&\quad - \frac{1}{2}\Big[(x^\mu\partial_\mu + 2)(-\phi(\gamma_5)_{ab} - \tilde{\phi} C_{ab}) + x_\mu\partial_\nu(-\phi(\gamma^{\mu\nu}\gamma_5)_{ab} - \tilde{\phi}(\gamma^{\mu\nu})_{ab})\Big] \\
&= (\gamma_5)_a{}^c S_c\psi_b \\
&\quad - \frac{1}{2}\Big[(x^\mu\partial_\mu + 2)(-\phi(\gamma_5)_{ab} + \tilde{\phi}(\gamma_5^2)_{ab}) + x_\mu\partial_\nu(-\phi(\gamma_5\gamma^{\mu\nu})_{ab} + \tilde{\phi}(\gamma_5\gamma^{\mu\nu}\gamma_5)_{ab})\Big] \\
&= (\gamma_5)_a{}^c S_c\psi_b \\
&\quad - \frac{1}{2}(\gamma_5)_a{}^c\Big[(x^\mu\partial_\mu + 2)(-\phi C_{cb} + \tilde{\phi}(\gamma_5)_{cb}) + x_\mu\partial_\nu(-\phi(gamma^{\mu\nu})_{cb} + \tilde{\phi}(\gamma^{\mu\nu}\gamma_5)_{cb})\Big] \\
&= (\gamma_5)_a{}^c S_c\psi_b - \frac{1}{2}(\gamma_5)_a{}^c S_c\psi_b = \frac{1}{2}(\gamma_5)_a{}^c S_c\psi_b.
\end{aligned}
$$

Hence, $[R, S_a] = \frac{1}{2}(\gamma_5)_a{}^b S_b$.

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