TALES FROM WONDERLAND

Doctoral thesis by

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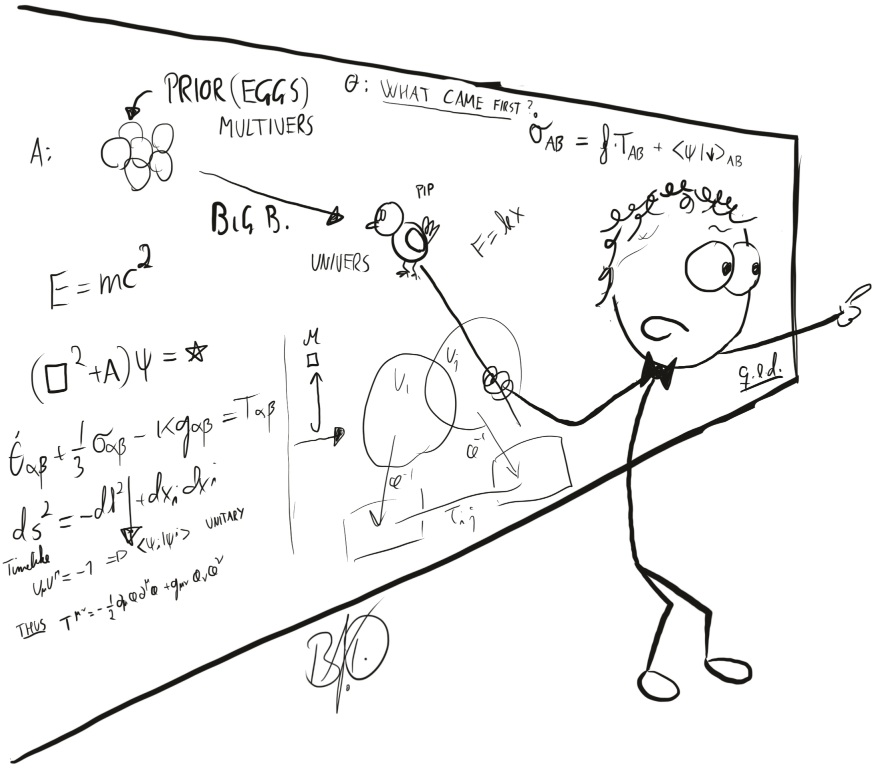
To my father

who entrusted me with the burdensome

torch of understanding.

Lo and behold!



"... AND THUS THE UNIVERSE IS CLEARLY A CHICKEN."

Preface

This dissertation compiles five papers, written in the period 2016–2019, during a 3-year PhD program at the University of Stavanger in Norway, under the supervission of Professor Sigbjørn Hervik, and with Professor Anders Tranberg as co-supervisor. The papers are preceded by five introductory chapters. These chapters aim at providing a sufficient basis for understanding the content of the papers. It is, however, generally assumed that the reader will be a physicist or a mathematician, or someone who otherwise has found the required time to build the mathematical backbone needed.

Since the author has a physicists' background himself, he has however generously provided an appendix entitled *A physicist's guide to mathematical jargon*. Hopefully this will make the current dissertation a bit more of a page-turner for some of you.

B. D. N.

Acknowledgments

First of all, I would like to thank my supervisor, Prof. Sigbjørn Hervik, for entrusting me with the opportunity to undertake this project, and for drawing on his seemingly infinite knowledge whenever I would be bold enough to knock on his door. I also appreciate his efforts in giving me opportunities to teach and his good sense of humor. Let me also express my gratitude to Dr. Mikjel Thorsrud for his patience, for cheerful hours and for fruitful collaboration. It is my hope that the future holds continued collaboration with you both.

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Matthew, Marco, Pavel and all the rest of my colleagues in Stavanger: thank you for useful discussions and for making my stay in Siddis-town more of an enjoyable experience. In the same manner I thank Dr. Clarkson for hosting me at Queen Mary and the rest of the group for useful input and merry occasions in the SCR lounge.

I will forever be indebted to Prof. Iver Brevik at NTNU (and also to Prof. Jens O. Andersen!) without whose encouragement I would never have pursued a Phd degree within the field of cosmology[a].

Fredrik. This thesis would no doubt have had a much 'interesting' look if it had not been for your persistent help on LaTeX and layout issues, and your notorious eye for detail. Thank you for providing an excellent

[a] I should however perhaps have listened to him when I asked him if he could recommend going into theoretical physics for a living. His answer was prompt and clear: 'No!´

template!

Mum and dad: It is high time that I express my gratitude for the fact that you taught me to pursue knowledge and to value truth. Mikael – our (occasionally heated) discussions and creative mind-wanderings are treasured; I look forward to closer collaboration in the future. John – our late nights at Jørpeland will always be a fond memory. Your company and our many conversations have been most enjoyable.

Finally, and above all, I thank my wife Beate for faithfully standing by my side. Without your support in love, patience and encouragement, this work would have made little or no sense to me.

Ben David Normann

List of papers

The papers contained in this dissertation are listed beneath, and henceforth referred to as **Papers I-V**.

I Normann B.D., Hervik S., Ricciardone A. and Thorsrud M.
Bianchi cosmologies with p-form gauge fields,
Class. Quantum Grav. **35** 095004 (2018).
DOI:10.1088/1361-6382/aab3a7. ArXiv:1712.08752v2 [gr-qc].

II Normann B.D., Hervik S., *Approaching Wonderland*,
Class. Quantum Grav. **37** 085002 (2020).
DOI: 10.1088/1361-6382/ab719b. ArXiv:1909.11962v2 [gr-qc].

III Normann B.D., Hervik S., *Collins in Wonderland*,
Class. Quantum Grav. (Accepted).
ArXiv:1910.12083v2 [gr-qc].

IV Thorsrud M. and Normann B.D. and Pereira T. *Extended FLRW Models: dynamical cancellation of cosmological anisotropies*
Class. Quantum Grav. **37** 065015 (2020).
DOI: 10.1088/1361-6382/ab6f7f. ArXiv:1911.05793v2 [gr-qc].

V Normann B.D. and Clarkson C. *Recursion relations for gravitational lensing*
General Relativity and Gravitation **52** (2020) 3.
DOI: 10.1007/s10714-020-02677-z. arXiv:1904.04471v2 [gr-qc].

Contents

INTRODUCTION

> “We are now approaching lunar sunrise, and for all the people back on Earth, the crew of Apollo 8 has a message that we would like to send to you:
>
> “In the beginning God created the heaven and the earth. And the earth was without form, and void; and darkness was upon the face of the deep. And the Spirit of God moved upon the face of the waters. And God said, Let there be light: and there was light. And God saw the light, that it was good: and God divided the light from the darkness.”
>
> William Anders

Chapter 1
Background

1.1 History

It is interesting that the crew on board the Apollo 8 mission chose to read from a perhaps 2 000–3 000 years old document—the account of creation in the Hebrew Bible—when they reported back to the distant Earth from lunar sunrise.

Understanding one's origin seems to be a quest we have set out for, for as long as we have existed. Our understanding of the cosmos has, however, differed greatly over the ages. We may model this difference as a projection $\mathcal{P}(t)$ from a Platonic space of possibilities Ω_{p} to a representation-space E_p [1];

$$\mathcal{P}(t) : \Omega_p \to E_p. \tag{1.1}$$

The nature of Ω_p will now entail the issues of *ontology*; what possibilities actually exist, whereas *epistemology* – what we know about that which exists – is contained in $E_p(t)$. Ancient texts were written in times with altogether different ideas about ontology from our own. Where many a modern mind will take *material* existence to be primary, the ancient thinker would be oriented toward *functional* existence [2]. As a result, *naming* was important, as names were thought to convey truth about that being named. For instance, the Babylonian creation myth *Enuma Elish* opens in the following way.

"*At a time when even the glories above had yet to be named, And unuttered was the word for the world which lay beneath...*"(Enuma Elish, 1 000–1 700 BCE) [3].

The differences between (for instance) this creation account and our modern, Western account of origins seem infinite, and hence $\mathcal{P}(t)$ demonstrably varies with time. Following G.F.R. Ellis [1, Chapter 1] we shall categorize the epics of ancient times as part of the fuller field of *cosmologia*, since they deal with (the origin of) matters such as *function* and *purpose* and the like. The questions of modern scientific endeavour—such as "What is mass-energy?" or "What is space-time?"—were considered derivative issues and drew little or no interest. Even Aristotle would base his cosmological treatise "On the Heavens" [4] on philosophical deduction. In this doctoral dissertation, however, we shall consider only the far lesser field of *cosmology*, the study of which we define as follows.

> Definition 1 (The study of cosmology). The study of the physical Universe at large, as projected onto a scientific method[a] through mathematics.
>
> [a] We leave aside the issues connected with deciding *what* method.

Having thus cleared the waters, we see that the modern field of cosmology is but a part of the fuller field of cosmologia. Taking such a view, it becomes understandable how advocates of modern science would convey their message to the Earthlings through words from Genesis 1. The message they spoke simply found no basis within science: it was a message of cosmologia.

As a matter of curiosity a few words on etymology is in place. The word "cosmology" derives from Greek $\kappa\acute{o}\sigma\mu o\varsigma$ and $\lambda o\gamma\acute{\iota}\alpha$, which translates into English as something like "the study of the ordained[b]." In Greek writings from Homer and down $\kappa\acute{o}\sigma\mu o\varsigma$ seems to originally have been related to the (aesthetic) *ordering* of something[c]. In the Greek translation of the Hebrew Bible (Septuagint) it is used for the arrangement of the stars ("the heavenly hosts"). Actually Pythagoras might have been the first to use the word for *the world*, although he possibly referred only to *the heavens* [5].

[b] As opposed to chaos.

[c] Also in modern language a phantom of this connection is evident: The etymology of *cosmetics* goes back to the Greek word $\kappa\acute{o}\sigma\mu o\varsigma$.

This doctoral dissertation is a mathematical endeavour embarking upon questions relating to cosmology as defined above. The reader interested in questions of more philosophical character is highly encouraged to consider the book *The Philosophy of Cosmology* [1] and also Ellis' paper [6] and further references therein.

1.2 Principles of physical cosmology

As an observatory, planet Earth is thought to be situated in a galaxy that is typical among other typical ones in a typical cluster among clusters in a typical supercluster among superclusters [7, Chapter 2]. The anthropocentric view of preceding generations[a] has been exchanged with the more recent Copernican Principle[b], which could be taken to express the opposite standing point.

> Principle 1 (The Copernican Principle). Planet Earth is not a privileged place of (cosmological) observation.

If our observations are arch-typical for observations performed anywhere else in the Universe, then it follows that the Universe must look essentially the same everywhere. Philosophically, this leads to a version of the cosmological principle [9, Chapter 2] which we shall refer to as the weak version.

> Principle 2 (Weak Cosmological Principle). The Universe presents the same aspects from every point.

Observations from the vantage point of Earth suggest that the Universe is (quite) isotropic [10, 11], meaning that it looks the same in every spatial direction. By the Copernican Principle, the Universe must therefore be everywhere isotropic. We may therefore also make a stronger version of the cosmological principle.

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[a] The roman emperor Marcus Aurelius nevertheless calls the Earth a mere "...point in space..." in his work [8, book 4 Sec. III], originally written in Greek around AD 170–180.

[b] Not so strange at all, considering the shift in attention from cosmologia toward cosmology.

> Principle 3 (Strong Cosmological Principle). The Universe is isotropic around every point.

Hence, there is an implication from the strong to the weak, but not the converse. Both versions, however, imply that the Universe is homogeneous[a]. Clearly, this is not true at every scale[b]. These principles should therefore be understood as expressing the idea that *there exists an averaging scale* at which the Universe is homogeneous.

Both versions of the cosmological principle amounts to put severe restrictions (laws) on the different universes realised, E_p, out of all theoretically possible ones Ω_p. In the language of the previous section, such principles are projections $\mathcal{P}_{\mathrm{cosm.pr.}}$ between the two categories. Naturally they do not come without philosophical quandary. The mathematical study undertaken in this thesis, could be taken to address one of them:

How did the Universe become the way it is?

1.3 Isotropy: chance or necessity?

Strategy. The strategy underpinning our approach, is that of turning mainstream astronomy on its head. Instead of asking what the Universe looks like, given 'theory + data', and what principle one may derive such Universe from, one works the other way around, beginning instead with disregarding the principle in question. In our case we discard of the Strong Cosmological Principle and ask: *How likely is an everywhere isotropic Universe*, if we do not take it as an a-priori postulate?

Assumptions. This question will be sought answered through assumptions concerning the geometry and matter content of the Universe. These aspects will be further discussed in the two following chapters, after which

[a] Homogeneity (in cosmology) refers to sameness-of-observation as observed from different points in space. This is distinct from isotropy, which refers to sameness-of-observation as one varies the direction of the telescope.

[b] None of us would insist on seeing the same in all directions in everyday life!

we state our assumptions in a final chapter, where the project and the results are summarized.

Outcome. Applying the kind of approach laid out above to any principle will reveal if the principle is a result of *chance* or of *necessity*. Necessity occurs if the principle turns out to be a consequence of the theory. In this case it is no longer a principle. Chance, however, is made relevant if the principle does *not* follow from the theory. In that case the following should occur.

(i) Reexamination of the data to see if hasty conclusions have been drawn.

(ii) If the outcome of (i) is negative, one must proceed with acceptance[a] in the hope that there is yet to be found a theory in which the principle is indeed a necessity and not a result of mere chance.

In our particular case, the specific outcome will be to pin down the equilibrium points of the dynamical systems constructed for the development of the different invariant sets of initial conditions of the universes considered. As we shall see, the equilibrium points that are stable into the future are given relevant roles as (possible) future asymptotic states of the whole invariant set of initial conditions under consideration.

The bottom line. If isotropy[b] correctly describes the Universe at some scale, then one may hope for the correct theory of gravity to be capable of uniquely (or at least in a probabilistic manner) describing how that came to be.

> **Summary (Philosophical justification):** This project intends to contribute toward the longstanding investigation of how likely the observed Universe is [12]. More specifically, we will investigate these issues for a certain type of matter content in a Universe where we assume General Relativity (GR) to be the correct theory of gravity.

[a] And consider consulting a philosopher.
[b] Or any other characteristic, for that sake!

Chapter 2

Homogeneous, anisotropic cosmologies

This chapter builds extensively on chapters 6 and 15 in *Einstein's General Theory of Relativity* [13], chapter 1 in *Dynamical Systems in Cosmology* [14] and chapters 2 and 3 in *Lecture Notes* [15].

As mentioned in the previous chapter we will rely on the Copernican principle and hence we are interested in cosmologies with (three-dimensional) homogeneous spatial sections. Furthermore, we restrict attention to a four-dimensional manifold with three-dimensional (spatial) orbits of homogeneity. In the following sections we introduce (briefly) the mathematical machinery necessary in order to be precise about what we mean by this, and point toward more substantial literature on the different topics introduced. We shall assume that the reader is somewhat familiar with basic concepts of differential geometry and dynamical-systems theory. For instance, the concepts of a *manifold* and of a *p-form* are concidered known. Since the signing author has a physics' background himself, he grants a generous portion of sympathy for whom such concepts may be unfamiliar. Consequently, Appendix A is provided as a look-up tool for (a few) central definitions.

In the rest of this dissertation we assume a torsion-free, Lorentzian manifold with a metric of signature $(-,+,+,+)$. Furthermore, numerical indices $\{0,1,2,3\}$ are used to index the orthonormal frame (to be introduced) and the letters $\{t,x,y,z\}$ are used to index the coordinate basis. Greek indices are taken to run over all four space-time components, whereas Latin indices $\{a,b,c,\cdots,m,n\}$[a] run over spatial components only. We use $\mathbf{e}_\mu$ to refer to a general basis vector, and $\boldsymbol{\omega}^\mu$ to refer to a general

[a] We will stick to these letters to avoid confusion: The letters $\{t,x,y,z\}$ *always* refer to particular components of the coordinate basis.

basis one-form. Throughout, we shall assume a space-time foliation such that $\mathbf{e}_0 = \partial_0 = \partial_t$ is orthonormal to spatial hypersurfaces, and (the velocity of) our congruence of observers $\mathbf{u}$ will be aligned along $\mathbf{e}_t$. Throughout, $\mathcal{M}$ denotes a manifold, and we use units such that $\mathrm{c} = 8\pi\mathrm{G} = 1$, where c refers to the speed of light and G is the gravitational constant. Also, δ^μ_ν are the components of the identity-matrix (Diag$(1,1,1,1)$) and $\varepsilon_{\alpha\beta\gamma\delta}$ is the totally antisymmetric symbol specified by $\varepsilon_{0123} = 1$ and $(\dot{})$ denotes differentiation with respect to proper time[a], except in Chapter 4 and Paper V, where it denotes differentiation along the null curve. Finally, take the following definition of a cosmology.

> Definition 2 (A cosmology). In this treatise we take a cosmology to consist of the triple $(\mathcal{M}, \boldsymbol{g}, \mathbf{u})$, where $\mathcal{M}$ is the four-dimensional space-time manifold, $\boldsymbol{g}$ is the space-time metric and $\mathbf{u}$ is the time-like (velocity-field of the) congruence of fundamental observers.

2.1 Homogeneity

In order to define what we mean by a homogeneous space, we shall make use of the mathematical concept of an isometry; a mapping that preserves the metric. Take $\phi = \phi(p, t)$ to be a one-parameter group of diffeomorphisms. Then the following definition.

> Definition 3 (Isometry). We say that ϕ is an isometry if
>
> $$\phi^* \mathbf{g} = \mathbf{g} \tag{2.1}$$
>
> where $\mathbf{g}$ is the metric tensor.

One may use a one-parameter group of diffeomorphisms to formalise the idea of comparing the metric at different locations on a manifold $\mathcal{M}$. If the difference between the metric at two points p and q differ at most by

[a] Proper time coincides with *coordinate-time* whenever the observers are comoving, which will be the case in this dissertation.

a coordinate transformation, then they are the same. To be precise, we let $q = \phi(p,t)$, and note that the pull-back $\phi^*(p,t)$ now induces a way to compare the metric $\hat{\mathbf{g}}$ at q with the metric $\mathbf{g}$ at p. More specifically, let $\{x^\mu\}$ be coordinates at p and $\{y^\mu = \phi^*(x)\}$ coordinates at $q = \phi(p,t)$. Then

$$\hat{\mathbf{g}} = \phi^*(p,t)\mathbf{g} = \frac{\partial y^\alpha}{\partial x^\nu}\frac{\partial y^\beta}{\partial x^\mu}\, g_{\alpha\beta}\, \mathbf{dx}^\mu \otimes \mathbf{dx}^\nu. \tag{2.2}$$

Furthermore, if $\hat{\mathbf{g}} = \mathbf{g}$, then we say that $\phi^*(p,t)$ is an isometry. The transformation has preserved the metric.

Definition 4 (Isometry group). The isometry group $\mathrm{Isom}(\mathcal{M})$ is defined such that

$$\mathrm{Isom}(\mathcal{M}) \equiv \{\phi : \mathcal{M} \mapsto \mathcal{M} \,|\, \phi\, \text{isometry}\}\,. \tag{2.3}$$

From the two above definitions we can now formalise the notion of a homogeneous space.

Definition 5 (Homogeneous (/transitive) space). If, for each pair of points p, $q \in \mathcal{M}$ there is a $\phi \in \mathrm{Isom}(\mathcal{M})$ such that $\phi(p) = q$, then we say that $\mathcal{M}$ is a **homogeneous space**.

Homogeneity is therefore a measure of how similar a manifold looks as we move from point to point. To the one-parameter group of diffeomorphisms is attached a notion of an underlying vector field. This vector field is at every point p tangent to the orbit of p. A vector field $\boldsymbol{\xi}$ is said to be Killing if

$$\pounds_{\boldsymbol{\xi}}\, \mathbf{g} = 0 \tag{2.4}$$

where $\pounds$ is the Lie derivative, and $\mathbf{g}$ is the metric tensor. Hence, for a homogeneous manifold, there must exist Killing-vectors generating isometries connecting any two points on the manifold. The Killing-vector fields become important as we seek to understand a related but different concept: isotropy.

2.2 Isotropy

For a manifold $\mathcal{M}$ of dimension n to be homogeneous, the number k of Killing-vectors $\{\boldsymbol{\xi}_\alpha\}$ must be equal to or larger than n. Hence we require

$$k \geq n \qquad \text{homogeneity requirement.} \tag{2.5}$$

By such it becomes possible to generate n independent translations at a point $p \in \mathcal{M}$. Since the Killing-vectors at a point p live in the tangent-space $T_p\mathcal{M}$, and since $\dim(T_p\mathcal{M}) = n$, it follows that in the case where $k > n$, not all Killing-vectors can be linearly independent. We denote the difference as

$$d = k - n. \tag{2.6}$$

Thus the number d is a measure of how many transformations are left that will leave the metric invariant upon having subtracted the n translations following from the homogeneity requirement on an n-dimensional manifold. d is a measure on what we call isotropy. As an example, consider the maximally symmetric three-spaces, $n = 3$, where $k = n(n+1)/2 = 6$. From (2.6) we find $d = 3$. The remaining transformations are the three rotations. To formalise the concept of isotropy, we use Lie Groups and Lie algebras.

2.2.1 Connection to Lie groups and Lie algebras

It is instructive to note that the isometries of a manifold $\mathcal{M}$ form a Lie group. A Lie group is a group that is also a manifold.

Definition 6 (Lie group). A Lie group G is a topological group that has the following properties:

1) G is a manifold.
2) The group multiplication $m : G \times G \longmapsto G$ is smooth.
3) Inversion $i : G \longmapsto G$ is smooth.

Note also the following definition of a Lie algebra.

Definition 7 (Lie algebra). A real (or complex) Lie algebra, $\mathfrak{g}$, is a (finite-dimensional) vector space equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \longmapsto \mathfrak{g}$ which satisfies the following properties:

1) $[\mathbf{X}, \mathbf{X}] = 0\,, \quad \forall \quad \mathbf{X} \in \mathfrak{g}$

2) The **Jacobi identity**:

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0\,, \quad \forall \quad \mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{g}. \tag{2.7}$$

From 1) and 2) it may be inferred that the bilinear map is skew-symmetric: $[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}]$. A Lie algebra is thus a vector space, and a Lie group is a group manifold. The following theorem reveals the connection between them.

Theorem 2.1 (The Lie algebra of a Lie group). *Let G be a Lie group. Then the tangent space of G at the identity element, T_eG, is a Lie algebra; i.e.,*

$$\mathfrak{g} = T_eG. \tag{2.8}$$

The Killing-vectors form a Lie algebra, and, as discussed above, they are also generators of isometries. In fact it turns out that the Lie algebra of the Killing-vectors is isomorphic to the Lie algebra of Isom($\mathcal{M}$). Furthermore, let the structure constants of the Killing-vectors be $D^\lambda_{\mu\nu}$, and the structure constants of the Lie algebra at the identity element of the Lie group be $C^\lambda_{\mu\nu}$, such that

$$[\boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\mu] = D^\lambda_{\mu\nu}\boldsymbol{\xi}_\lambda \qquad \text{and} \qquad [\mathbf{e}_\mu, \mathbf{e}_\nu] = C^\lambda_{\mu\nu}\mathbf{e}_\lambda, \tag{2.9}$$

where $\{\mathbf{e}_\mu\}$ is a basis for the Lie algebra of the isometry group. If these vector fields coincide at one point, then they coincide everywhere [15, Sec. 3.3] and it may be shown that

$$D^\lambda_{\mu\nu} \;=\; -C^\lambda_{\mu\nu}. \tag{2.10}$$

Understanding the Lie algebra of the Killing-vector fields, or equivalently, the Lie algebra of the isometry group, will therefore determine the (algebraic) properties of the isometry group. It will not, however, determine the action of the group. The concept of isotropy is now formally defined through the isotropy subgroup (stabiliser) of the isometry group.

Definition 8 (Isotropy subgroup). Take a point $p \in \mathcal{M}$. Then the isotropy subgroup of p is

$$\mathcal{I}_p(\mathcal{M}) = \{\phi \in \text{Isom}(\mathcal{M}) \,|\, \phi(p) = p\}. \tag{2.11}$$

In the case where not all the Killing-vectors at a point p are linearly independent, they will necessarily span a tangent space of dimension $s < r$, where r is the dimension of the isometry group. We call this difference

$$d = r - s \qquad \text{(measure of isotropy)} \tag{2.12}$$

and say that this is the *dimension* of the isotropy subgroup. This provides a more formal definition of the d used in (2.6). The Killing-vector fields that vanish at p, form a subgroup of dimension d that leaves the point p fixed. Taking all this together, we now have tools to classify both the isotropic and homogeneous properties of a space:

- ⋆ The dimension d of the isotropy subgroup of the manifold $(M, \mathbf{g})$ determines the **isotropic properties** of the manifold.
- ⋆ The dimension s of the orbit of the isometry group (i.e. $\dim(\text{span}(\boldsymbol{\xi}_\alpha))$) at a point p determines the **homogeneity properties** of the manifold.

We are interested in four-dimensional space-time. The case $s = 4$ must then correspond to static universe models, since (in that case) no change may occur as we move along time. Since we shall allow for expanding models, however, we shall require instead that the dimension of the orbit of a point $p \in \mathcal{M}$ under the isometry group equals the dimension of the spatial sections. Hence we require $s = 3$. Cosmologies with even less symmetries ($s < 3$) are also not considered in this work. Note for instance that $s = 0$ is the fully inomogeneous case. Having specified s, we must also specify the dimension d of the isotropy subgroup.

- ⋆ $d = 3$: *Isotropic.* This necessarily corresponds to $r = 6$. These models are maximally symmetric and correspond to the so-called Friedmann-Lemaître-Robertson-Walker models. We refer to a model of this class as a FLRW model.
- ⋆ $d = 1$: *Locally rotationally symmetric* (LRS). In this case we must have $r = 4$. Note the two further possibilities:
  - ◇ G_4 has a subgroup G_3 that acts *simply transitively* on the three-dimensional orbits. We obtain the **LRS Bianchi models**.
  - ◇ G_4 is *multiply transitive* (it does not have a subgroup that acts simply transitively). We obtain the **Kantowski-Sachs** models.
- ⋆ $d = 0$: *Anisotropic.* **The Bianchi models** (G_3).

Observe that the more symmetric cases ($d = 3$ and $d = 1$) have G_3 subgroups that, with one exception, act simply transitively. We may therefore restrict our study to $d = 0$, where $d = 3$ and $d = 1$ will show up as special cases. The exception is, as mentioned, the Kantowski–Sachs model, which we will not consider in this thesis, except briefly in **Paper IV**, where we look for shear-free solutions. As a summary: in this work we consider

$$\boxed{s = 3 \qquad \text{and} \qquad d = 0.} \tag{2.13}$$

We saw that the algebraic properties of the Lie group can be understood in terms of the corresponding Lie algebra. To understand the three-dimensional, fully anisotropic cases, we shall therefore (in a later section) classify the three-dimensional, real Lie-algebras. This is what gives rise to the so-called Bianchi models.

2.3 Left-invariant basis

In this section, we consider the left action as defined by the elements of a Lie group. The reader is referred to [15] for details and proofs. Let a and g be elements of the Lie group G. Then the left action is a mapping

$$L_a : G \to G \quad , \quad L_a(g) = ag. \tag{2.14}$$

This induces a mapping between tangent and co-tangent spaces in the following way.

$$L_{*a} : T_g G \to T_{ag} G \tag{2.15}$$

$$L_a^* : T_{ag}^* G \to T_g^* G \tag{2.16}$$

The idea is to use this induced map to construct a left-invariant vector field (and correspondingly a left-invariant one-form basis). In order to do so, we note the following theorem.

Theorem 2.2. *An n-dimensional Lie-group G has n left-invariant vector fields being linearly independent at every point.*

Hence we may compute the members of the tangent space at the identity element, and then left-translate these vectors across the Lie group. By such we obtain a left-invariant basis for the vectors over the whole manifold. The members of the vector-space $T_e G$ at the identity element e form a Lie algebra, and in our case, this Lie-algebra must be of one of the Bianchi types. By such, we may find left-invariant basis-fields for each Bianchi type, which in turn allows for constructing a left-invariant metric (for each type). Let $\{\boldsymbol{\omega}^\mu\}$ be a field of one-forms constituting a basis for the co-tangent bundle of G. Then we define

$$\boldsymbol{\omega}^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu. \tag{2.17}$$

By application of the induced pull-back one may now show that these are left-invariant one-forms. Furthermore, upon taking the exterior derivative, one finds

$$\mathbf{d}\boldsymbol{\omega}^{\lambda} = -\frac{1}{2}C^{\lambda}_{\mu\nu}\boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu}. \tag{2.18}$$

Since the Bianchi types may be classified according to their structure coefficients, this equation shows that one may construct a left-invariant metric corresponding to any of the Bianchi types. In particular, having found one-forms that fulfill Eq. (2.18), the line-element is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\boldsymbol{\omega}^{\mu} \otimes \boldsymbol{\omega}^{\nu}, \tag{2.19}$$

where $g_{\mu\nu}$ are the components of the metric tensor. Similarly, one could have defined a right-action, showing that the right-invariant vectors correspond to the Killing-vectors $\{\boldsymbol{\xi}_{\mu}\}$. The left-invariant frame is then the frame that is invariant under the action of the Killing-vectors.

2.4 Orthonormal-frame formalism

In this thesis, we adopt the orthonormal-frame approach. The reason is two-fold:

1. The Einstein equations will reduce to a set of first-order differential equations.
2. The physical meaning of variables is less disguised.

Take the basis vectors to be $\{\mathbf{e}_{\mu}\}$, and the corresponding basis one-forms to be $\{\boldsymbol{\omega}^{\nu}\}$. Then the orthonormal frame is defined through

$$\mathbf{e}_{\mu} \cdot \mathbf{e}_{\nu} = \mathbf{g}(\mathbf{e}_{\mu}, \mathbf{e}_{\nu}) \; = \; \eta_{\mu\nu} \qquad \text{and} \qquad \omega^{\mu}(\mathbf{e}_{\nu}) = \delta^{\mu}_{\nu}. \tag{2.20}$$

Here $\eta_{\mu\nu} =$ diag$(-1, 1, 1, 1)$ is the Minkowski metric. Let ∇ denote the Koszul connection. The components of the directional derivative of a basis vector $\mathbf{e}_{\mu}$ are now

$$\nabla_{\nu}\mathbf{e}_{\mu} = \Gamma^{\lambda}{}_{\mu\nu}\,\mathbf{e}_{\lambda}, \tag{2.21}$$

where ${\Gamma^\lambda}_{\mu\nu}$ are the connection coefficients. The exterior derivative $\mathbf{d}$ of a basis vector is

$$\mathbf{d}\mathbf{e}_\mu = \mathbf{e}_\nu \otimes {\Omega^\nu}_\mu = \mathbf{e}_\nu \otimes {\Gamma^\nu}_{\mu\alpha}\boldsymbol{\omega}^\alpha, \tag{2.22}$$

Hence, the *connection one-forms* ${\Omega^\nu}_\mu$ are defined to give the νth component of the change of basis vector $\mathbf{e}_\mu$. In an orthonormal frame such changes are necessarily reduced to rotations (since the frame must remain orthonormal), and we consequently refer to the connection one-forms as *rotation one-forms*. Calculating the exterior derivative of the components $g_{\mu\nu}$ of the metric tensor, one finds $\mathbf{d}g_{\mu\nu} = \Omega_{\mu\nu} + \Omega_{\nu\mu}$. In an orthonormal frame this must imply

$$\Omega_{\mu\nu} = -\Omega_{\nu\mu}. \tag{2.23}$$

where $\Omega_{\mu\nu} = g_{\mu\lambda}\Omega^\lambda_\nu$. This relation greatly simplifies calculations.

2.4.1 Frame rotations

In this thesis we employ the orthonormal frame to study the Bianchi models of Solvable type. Their isometry group admit a two-dimensional Abelian subgroup G_2. Throughout this dissertation we choose to align our frame such that $\{\mathbf{e}_2, \mathbf{e}_3\}$ span the Lie algebra that generates this subgroup. Hence $\mathbf{e}_1$ is orthonormal to it. To every point on the space-time manifold there is attached an orthonormal frame (a vierbein). The rotation Ω^α of the frame around the axis aligned with $\mathbf{e}_\alpha$ is defined as

$$\Omega^\alpha \equiv -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}u_\beta\mathbf{e}_\gamma \cdot \dot{\mathbf{e}}_\delta. \tag{2.24}$$

Here $\varepsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric symbol fixed by $\varepsilon_{0123} = 1$. In the following we show that for a congruence of time-like observers $\mathbf{u} = \mathbf{e}_0$ these correspond to three rotations of the spatial frame. Indeed: putting $\beta = 0$ in the above equation, it becomes clear that there are only spatial rotations Ω^i. Using the properties of the antisymmetric symbol, and the fact that $u_0 = -1$, we find

$$\Omega^1 = -\mathbf{e}_2 \cdot \dot{\mathbf{e}}_3 = \mathbf{e}_3 \cdot \dot{\mathbf{e}}_2, \tag{2.25}$$

where we have arbitrarily chosen $i = 1$ for clarity. Since the length of the basis vectors in the orthonormal, spatial triad are $|\mathbf{e}_i| = 1$, the frame may only change its orientation. Hence, since the frame is rigid, the only possibility is a rotation around the $\mathbf{e}_1$-axis. Furthermore, since $i = 1$ was an arbitrary choice, we may generally conclude that Ω^i does give the rotation of the frame around axis i, as expected. To see this even more explicitly, consider the rotation of a frame $(\mathbf{e}_2, \mathbf{e}_3)$ relative to a frame $(\tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$ of gyroscopes[a]. Then the change of basis vectors must be given by the time-derivative of the rotation matrix in two dimensions. Differentiating, and pulling out the common factor $\dot{\phi}_1$ one finds

$$\begin{bmatrix} \dot{\mathbf{e}}_2 \\ \dot{\mathbf{e}}_3 \end{bmatrix} = -\dot{\phi}_1 \begin{bmatrix} \sin\phi_1 & -\cos\phi_1 \\ \cos\phi_1 & \sin\phi_1 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{e}}_2 \\ \tilde{\mathbf{e}}_3 \end{bmatrix}. \tag{2.26}$$

From this it is straight forward to confirm that the magnitude of the change of $\mathbf{e}_2$ when rotated around another axis $\mathbf{e}_1$ is given by

$$|\dot{\mathbf{e}}_2| = \dot{\phi}_1 \qquad \text{and hence} \qquad \dot{\mathbf{e}}_2 = \dot{\phi}_1 \mathbf{e}_3. \tag{2.27}$$

Since the frame is rigid, we similarly find $\dot{\mathbf{e}}_3 = -\dot{\phi}_1 \mathbf{e}_2$. In this thesis, $\mathbf{e}_1$ is as mentioned chosen orthonormal to the orbits of G_2, and rotations around this axis will be a major focus. We therefore simply define $\phi \equiv \phi_1$. Inserting (2.27) into (2.25) and using that $\mathbf{e}_j \cdot \mathbf{e}_j = 1$ in an orthonormal basis we therefore find

$$\Omega_1 = \dot{\phi}. \tag{2.28}$$

The frame rotations $(\Omega_1, \Omega_2, \Omega_3)$ may be seen as **gauge freedom**. As explained in the following section, aligning $\mathbf{e}_1$ orthogonal to the orbits of G_2 is a gauge choice: one must specify the frame rotations Ω_2 and Ω_3. Note that in our case ϕ is constant on the orbit of G_3.

[a] a Fermi-propagated frame.

2.5 The Bianchi models in an orthonormal frame

As explained earlier, the Bianchi types correspond to distinct three-dimensional Lie algebras. A certain Bianchi type may therefore be studied through the structure coefficients corresponding to its Lie algebra. We denote these as ${\gamma^\lambda}_{\mu\nu}$, and follow standard procedure by invoking the Behr decomposition. The spatial parts of the structure coefficients are then decomposed into a symmetric matrix n^{ab} and a 1-index object a_a. We have

$$ {\gamma^c}_{ab} = \varepsilon_{abm} n^{mc} + a_a {\delta_b}^c - a_b {\delta_a}^c. \tag{2.29} $$

Furthermore, the congruence of observers is in our case hypersurface orthogonal[a]. This implies that the motion is geodesic ($\dot{u}_a = 0$) and that the congruence is irrotational ($\omega_{\mu\nu} = 0$). Using that $\nabla(\mathbf{u}\cdot\mathbf{e}_i) = 0$, where $\theta_{\mu\nu}$ is the expansion-tensor, one finds upon some straight forward algebra that $\theta_{\mu\nu} = {\Gamma^0}_{\mu\nu}$. Since we use an orthonormal frame, we may use equation (2.23); $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$. It allows for expressing ${\Gamma^\alpha}_{\mu\nu}$ in terms of the structure coefficients ${\gamma^\alpha}_{\mu\nu}$. The result is that the mixed structure coefficients become

$$ {\gamma^a}_{0b} = -{\sigma_b}^a - H{\delta_b}^a - {\varepsilon^a}_{bm}\Omega^m. \tag{2.29} $$

Here H is the Hubble-Lemaître parameter and σ^{ab} is the shear-tensor. Refer to Appendix B for a general decomposition of the four-velocity field, and to [13, Chpt. 15] for more on the Bianchi models in an orthonormal frame. The remaining structure coefficients vanish; ${\gamma^0}_{0a} = \dot{u}_a = 0$ and ${\gamma^0}_{ab} = -2{\varepsilon_{ab}}^m\omega_m = 0$.

2.5.1 The Jacobi identity

The Jacobi identity must be fulfilled for all vectors. Taking the Jacobi identity for the triple $(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$ implies that the vector $\mathbf{a}$ lies in the kernel of the matrix n^{ij};

$$ n^{ij} a_j = 0. \tag{2.30} $$

[a]where the hypersurface is defined by the orbits of the isometry group.

The Jacobi identity for the triple $(\mathbf{u}, \mathbf{e}_a, \mathbf{e}_b)$ provides evolution equations for the structure coefficients. In particular, for comoving observers $\mathbf{u} = \partial_t$, we find

$$\dot{a}_i = -\frac{1}{3}\theta a_i - \sigma_{ij}a^j + \varepsilon_{ijk}a^j \Omega^k \,, \tag{2.31}$$

$$\dot{n}_{ab} = -\frac{1}{3}\theta n_{ab} + 2n^k_{(a}\varepsilon_{b)kl}\Omega^l + 2n_{k(a}\sigma^k_{b)}. \tag{2.32}$$

The different invariant sets of the system of evolution and constraint equations obtained through the Jacobi identity give rise to the different Bianchi-models I–IX. Without loss of generality [14, chapters 1.5 and 1.6], a choice is made such that $\mathbf{e}_1$ points in the direction of the vector $\mathbf{a}$, leaving the remaining frame vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ defined up to a rotation. We shall adopt the choice

$$\mathbf{a} = (a, 0, 0) \qquad\qquad \text{1+1+2 decomposition.} \tag{2.33}$$

As a consequence, the equations for $\dot{a}_2$ and $\dot{a}_3$ immediately imply

$$\Omega_A = \varepsilon_{AB}\sigma^{1B}, \tag{2.34}$$

and also

$$n^{1i} = 0. \tag{2.35}$$

In (2.34) capital letters run over $\{2, 3\}$ and ε_{AB} is the totally antisymmetric symbol with $\varepsilon_{23} = 1$. Note that Eq. (2.33) carries no information for models where $\mathbf{a} = 0$ (the so-called Class-A models; to be introduced). The gauge-choice (2.34) may still be made, however, in all class-A models that admit a G_2 subgroup. By such, it becomes possible to make this choice for all types except VIII and IX, which do not admit a G_2 subgroup of isometries. In this dissertation, we only consider the Bianchi models of Solvable type[a]. By the above equation two of the frame rotations are specified. There remains in this way only one rotational gauge freedom: rotation of the

[a] As shown in **Paper 1**, the types VIII and IX do not admit for isotropy-breaking degrees of freedom for the matter type we intend to study; the p-form with $p = \{1, 3\}$. These types are consequently less interesting.

frame around the $\mathbf{e}_1$-axis.

2.6 Bianchi classification (the solvable types)

In the following we give the specifications for each individual Bianchi model of Solvable type in terms of the sign of the eigenvalues, which remain invariant under time evolution, as shown explicitly in for instance [16]. It is possible to use the remaining gauge freedom of Ω_1 to diagonalize n^{ij}. By such, we would obtain simpler expressions for the different Bianchi types. In the following, however, we choose to keep the gauge freedom in the equations since that is the route taken in the papers contained in this work. Decomposing n^{ij} according to

$$n_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n_+ + \sqrt{3}n_- & \sqrt{3}n_\times \\ 0 & \sqrt{3}n_\times & n_+ - \sqrt{3}n_- \end{pmatrix} \tag{2.36}$$

we find the three eigenvalues

$$n_1 = 0 \quad , \quad n_2 = n_+ + \sqrt{3}\sqrt{n_-^2 + n_\times^2} \quad , \quad n_3 = n_+ - \sqrt{3}\sqrt{n_-^2 + n_\times^2}. \tag{2.37}$$

As mentioned, one may show from the evolution equations that the signs of n_2 and n_3 are preserved in time (cf. [14] and [16]). One may also show that a^2 evolves proportionally to $n_2 n_3$. Id est: $\mathrm{d}\left(a^2/(n_2 n_3)\right)/\mathrm{d}t = 0$. It follows that

$$a^2 = h(n_+^2 - 3n_-^2 - 3n_\times^2) \tag{2.38}$$

for some constant h. This constant is the so-called group-parameter of Bianchi types VI_h and VII_h. All Bianchi models with $a = 0$ are so-called **class-A** models, and the rest, where $a \neq 0$ are **class-B** models. In the following we give the specifications of all invariant Bianchi-sets of Solvable type, henceforth denoted $\mathcal{B}(i)$. In accord with the notion used so far, we let $C^i{}_{jk}$ denote the structure constants of the Lie algebra that generates the group of isometries for each of these Bianchi types.

The left-invariant one-forms listed for each type fulfill (2.18).

- ⋆ $\mathcal{B}$(I): Class A. n^{ij} has three zero-eigenvalues. The specifications are

$$a = n_+ = n_- = n_\times = 0. \tag{2.39}$$

Hence G_3 is Abelian and $C^i{}_{jk} = 0$. A set of left-invariant one-forms is $\{\mathbf{dx} \quad , \quad \mathbf{dy} \quad , \quad \mathbf{dz}\}$.

- ⋆ $\mathcal{B}$(II): Class A. n^{ij} has two zero-eigenvalues. Furthermore,

$$a = 0 \quad \text{and} \quad n_+^2 - 3(n_-^2 + n_\times^2) = 0. \tag{2.40}$$

The non-zero commutator is $C^3{}_{12} = -1$ and a set of left-invariant one-forms is $\{\mathbf{dx} \quad , \quad \mathbf{dy} \quad , \quad \mathbf{dz} - x\mathbf{dy}\}$.

- ⋆ $\mathcal{B}$(III): Class B. n^{ij} has one zero-eigenvalue. The two others are of opposite signs. Type III is decomposable into the Lie algebras of dimension 1 and 2. It may also be seen as the special case $h = -1$ of type VI_h. Its specification is

$$a^2 = 3(n_-^2 + n_\times^2) - n_+^2. \tag{2.41}$$

The non-zero structure constant is $C^3{}_{13} = 1$ and a set of left-invariant one-forms is $\{\mathbf{dx} \quad , \quad \mathbf{dy} \quad , \quad e^{-x}\mathbf{dz}\}$.

- ⋆ $\mathcal{B}$(IV): Class B. n^{ij} has two zero-eigenvalues. Furthermore,

$$a \neq 0 \quad \text{and} \quad n_+^2 - 3(n_-^2 + n_\times^2) = 0. \tag{2.42}$$

The non-zero structure constants are $C^3{}_{13} = C^3{}_{12} = C^2{}_{12} = 1$. A set of left-invariant one-forms is
$\{\mathbf{dx} \quad , \quad e^{-x}\mathbf{dy} \quad , \quad e^{-x}(\mathbf{dz} - x\mathbf{dy})\}$.

- ⋆ $\mathcal{B}$(V): Class B. n^{ij} has three zero-eigenvalues. The specifications are

$$a \neq 0 \quad \text{and} \quad n_\times = n_- = n_+ = 0. \tag{2.43}$$

The non-zero structure constants are $C^3{}_{13} = 1$ and $C^2{}_{12} = 1$. A set of

left-invariant one-forms is
$\{\mathbf{dx} \quad , \quad e^{-x}\mathbf{dy} \quad , \quad e^{-x}\mathbf{dz}\}$.

- ⋆ $\mathcal{B}(\mathrm{VI}_0)$: Class A. n^{ij} has one zero-eigenvalue. The specifications are

$$a = 0 \quad \text{and} \quad n_+^2 - 3(n_-^2 + n_\times^2) < 0. \tag{2.44}$$

Non-vanishing structure constants are $C_{12}^2 = -1$ and $C_{13}^3 = 1$. A set of left-invariant one-forms is
$\{\mathbf{dx} \quad , \quad e^{x}\mathbf{dy} \quad , \quad e^{-x}\mathbf{dz}\}$.

- ⋆ $\mathcal{B}(\mathrm{VI}_h)$: Class B. n^{ij} has one zero-eigenvalue. This is a one-parameter family of invariant sets. The specifications are

$$a^2 = h(n_+^2 - 3(n_-^2 + n_\times^2)) \quad \text{and} \quad h < 0. \tag{2.45}$$

Non-vanishing structure constants are $C_{12}^2 = p$ and $C_{13}^3 = 1$. A set of left-invariant one-forms is
$\{\mathbf{dx} \quad , \quad e^{-px}\mathbf{dy} \quad , \quad e^{-x}\mathbf{dz}\}$.

- ⋆ $\mathcal{B}(\mathrm{VII}_0)$: Class A. n^{ij} has one zero-eigenvalue. The specifications are

$$a = 0 \quad \text{and} \quad n_+^2 - 3(n_-^2 + n_\times^2) > 0. \tag{2.46}$$

The non-vanishing structure constants are ${C^2}_{13} = -1$and ${C^3}_{12} = 1$. A set of left-invariant one-forms is
$\{\mathbf{dx} \quad , \quad (\sin x\mathbf{dz} - \cos x\mathbf{dy}) \quad , \quad (\cos x\mathbf{dz} + \sin x\mathbf{dy})\}$.

- ⋆ $\mathcal{B}(\mathrm{VII}_h)$: Class B. n^{ij} has one zero-eigenvalue. This is a one-parameter family of invariant sets, and the specifications are

$$a^2 = h(n_+^2 - 3(n_-^2 + n_\times^2)) \quad \text{and} \quad h > 0. \tag{2.47}$$

The non-vanishing structure constants are ${C^2}_{12} = {C^3}_{13} = q$, ${C^2}_{13} = -1$and ${C^3}_{12} = 1$. A set of left-invariant one-forms is
$\{\mathbf{dx} \quad , \quad e^{-qx}(\sin x\mathbf{dz} - \cos x\mathbf{dy}) \quad , \quad e^{-qx}(\cos x\mathbf{dz} + \sin x\mathbf{dy})\}$.

2.7 Dynamical-systems approach

The reader is referred to [17] for a thorough introduction to dynamical systems in general. For instance, center-manifold theory—which has been used in this thesis—is explained in Sec. 2.12 therein. We also refer to [14, Chapter 4] for an introduction to dynamical systems theory as applied to cosmology.

2.7.1 Expansion normalization

Since we are interested in self-similar cosmological models, we 'factor out' the overall isotropic expansion encoded in the Hubble–Lemaître parameter H. This is done by constructing expansion-normalized variables, as follows. Define a dimensionless time-parameter τ according to

$$l = \mathrm{e}^{\tau} \tag{2.48}$$

where l is the overall scale-factor of the isotropic expansion. Id est; let

$$H = \dot{l}/l. \tag{2.49}$$

Taking the two above equations together gives

$$\frac{1}{H} = \frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{2.50}$$

where t is the proper time of comoving observers[a] and H is the Hubble–Lemaître parameter. Meanwhile $(\dot{\ })$ denotes derivation with respect to proper time (as before), $(\,'\,)$ denotes, henceforth, derivation with respect to dynamical time τ. In the following we give the expansion-normalized quantities. Let σ denote shear and let n and a be the decomposition of the structure coefficients as before. With the conventions given in Appendix C[b]

[a] or, equivalently, the coordinate time.
[b] Note that our conventions slightly differ from that used by [14, 18, 19].

we now find the following normalizations.

$$\begin{aligned}
&\Sigma_+ = \frac{\sigma_+}{H}, \quad \Sigma_- = \frac{\sigma_-}{H}, \quad \Sigma_\times = \frac{\sigma_\times}{H}, \quad \Sigma_2 = \frac{\sigma_2}{H}, \quad \Sigma_3 = \frac{\sigma_3}{H},\\
&N_+ = \frac{n_+}{H}, \quad N_- = \frac{n_-}{H}, \quad N_\times = \frac{n_\times}{H}, \quad N_2 = \frac{n_2}{H}, \quad N_3 = \frac{n_3}{H},\\
&A = \frac{a}{H}, \quad \Sigma^2 = \frac{\sigma_{ab}\sigma^{ab}}{6H^2}.
\end{aligned}$$

Take next a general anisotropic matter-sector with energy-momentum tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\, h_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \pi_{\mu\nu}. \tag{2.51}$$

Here $\pi_{\mu\nu}$ is the anisotropic stress, q_ν the heat-flow, ρ the energy density seen by an observer with four-velocity $\mathbf{u}$ and p the isotropic pressure (refer to Appendix D for details. The normalized variables are now

$$\begin{aligned}
&\Pi_+ = \frac{\pi_+}{H^2}, \quad \Pi_- = \frac{\pi_-}{H^2}, \quad \Pi_\times = \frac{\pi_\times}{H^2}, \quad \Pi_2 = \frac{\pi_2}{H^2}d,\\
&\Pi_3 = \frac{\sigma_3}{H^2}, \quad P = \frac{p}{3H^2}, \quad \Omega = \frac{\rho}{3H^2}, \quad \Xi_i = \frac{q_i}{3H^2}.
\end{aligned}$$

By the above normalization the equations of motion become an autonomous system of differential equations and all equilibrium points will represent self-similar cosmologies (to be defined). The resulting dynamical system will be on the form

$$\mathbf{X}' = F(\mathbf{X}), \qquad \mathcal{C}_i(\mathbf{X}) = 0, \tag{2.52}$$

where $\mathbf{X}$ is the n-dimensional state space vector of the system, $\mathcal{C}_i(\mathbf{X}) = 0$ is the set of constraints, F is an n-dimensional vector function. The local stability of the self-similar cosmological solutions represented by equilibrium points, $\mathbf{X}_0$ (where $F(\mathbf{X}_0) = 0$), may now be computed by looking at displacements from such points to linear order:

$$(\delta\mathbf{X})' = J\,(\delta\mathbf{X}). \tag{2.53}$$

Here J is the Jacobian matrix of the system. The eigenvalues l are given by the equation

$$\det(J - I\,l) = 0, \tag{2.54}$$

where I is the Identity matrix. Finally; the equation for the Hubble expansion now elegantly decouples from the rest of the system, and may be shown to be

$$H' = -(1+q)H, \tag{2.55}$$

where q is the so-called deceleration-parameter, generally defined as $q = -\ddot{l}/(\dot{l})^2$.

2.7.2 Gauge freedom: scalars and spin-n quantities

Following [20] the gauge freedom is left in the equations[a] introducing the (expansion-normalized) local angular velocity R_a of a Fermi-propagated axis, with respect to the triad $\{\mathbf{e}_a\}$, with components

$$R_1 \equiv \frac{\Omega_1}{H} = \phi' \quad \text{and} \quad \mathbf{R}_c \equiv R_2 + iR_3 \equiv \frac{\Omega_2}{H} + i\frac{\Omega_3}{H}. \tag{2.56}$$

Recall that $\mathbf{R}_c$ is already fixed according to Eq. (2.34). There remains in this way only one rotational gauge freedom (rotation of the frame around the $\mathbf{e}_1$-axis). This is

$$\begin{bmatrix} \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{e}}_2 \\ \tilde{\mathbf{e}}_3 \end{bmatrix}. \tag{2.57}$$

The complex variable $\mathbf{R}_c$ is introduced in order to simplify the equations when the gauge symmetry is still not fixed. This is in accordance with [20][b] and becomes a particularly useful tool in constructing gauge independent

[a] This differs from the general treatment in [14], where the dynamical system is instead built from gauge-independent quantities only.

[b] Note that there are some small conventional discrepancies in the current notation compared to that of [20].

quantities). Define next the complex quantities

$$\begin{aligned} \mathbf{N}_\Delta &= N_- + iN_\times \ , & \mathbf{\Phi_1} &= \Xi_2 + i\Xi_3 \ , \\ \mathbf{\Sigma}_\Delta &= \Sigma_- + i\Sigma_\times \ , & \mathbf{\Pi}_1 &= \Pi_2 + i\Pi_3 \ , \\ \mathbf{\Sigma}_1 &= \Sigma_2 + i\Sigma_3 \ , & \mathbf{\Pi}_\Delta &= \Pi_- + i\Pi_\times . \end{aligned} \tag{2.58}$$

Some of the quantities introduced so far are independent under transformations over the remaining gauge freedom, (2.57), whereas others change. To distinguish these quantities from each other, note the following two definitions.

Definition 9 (Scalar). Any quantity invariant under the transformation (2.57) is said to be a scalar.

Definition 10 (Spin-n object). Any quantity $\mathbf{X}$ transforming such that

$$\mathbf{X} \to \exp{(in\phi)}\mathbf{X}$$

under the transformation (2.57) is said to be a spin-n object.

The above variables may now be classified as scalars or spin-n objects by looking at how they transform under the gauge transformation(2.57). For the geometric and shear-variables we find

$$\{A, N_+, \Sigma_+, \mathbf{\Sigma}_1, \mathbf{\Sigma}_\Delta, \mathbf{N}_\Delta\} \to \{A, N_+, \Sigma_+, \mathrm{e}^{i\phi}\mathbf{\Sigma}_1, \mathrm{e}^{2i\phi}\mathbf{\Sigma}_\Delta, \mathrm{e}^{2i\phi}\mathbf{N}_\Delta\} \,. \tag{2.58}$$

The rest we shall return to in the next chapter, where we discuss the matter sector more intently. Observe that the complex conjugates of the spin-n objects transform in a similar manner. In particular $\exp(ix)^* = \exp(-ix))$[c]. This makes it very easy to construct all sorts of physical variables (gauge-independent quantities; hence scalars) from the spin-n objects.

[c]By such for instance $\mathbf{\Sigma}_\Delta\mathbf{\Sigma}_\Delta^*$ becomes a scalar quantity, since the exponentials cancel out.

2.7.3 Equilibrium sets

As anticipated, in the dynamical-systems approach a relevant role is given to the equilibrium points, the stable of which function as asymptotic states of the system (at least locally). In order to formally define an equilibrium point in a gauge-independent manner, consider the definition of a scalar given in Def. 9. Then, a gauge independent definition of an equilibrium point is

> Definition 11 (Equilibrium point). An equilibrium point P is a set on which all scalars are constants on P as functions of τ.

The equilibrium points that we obtain through our dynamical systems, with the expansion normalization described in this section, will generally represent self-similar cosmological models. A definition of self-similar is as follows.

> Definition 12 (Homothety and Self-similar space-time). A self-similar space-time is a space-time possessing a proper homothety. A vector field $\mathbf{H}$ is said to be a (proper) homothety if
>
> $$\mathcal{L}_{\mathbf{H}}\mathbf{g} = k\,\mathbf{g}, \tag{2.59}$$
>
> where k is a (non-zero) constant.

The stable equilibrium points will, as mentioned, correspond (at least locally) to future asymptotic states of the system, and are therefore of special interest. Furthermore, one may note that the Einstein equations will be part of our dynamical systems, since we shall assume General Relativity as the theory of gravity. Henceforth, finding the metric that corresponds to an equilibrium point will provide exact solutions to the Einstein equations.

2.7.4 Gauge choices

An important part of the analysis will consist in choosing a certain gauge in which to study the equilibrium points. We have employed quite a few different choices in the papers contained in this thesis. The gauge choice must be invariant under time evolution. That is to say: One must choose the remaining gauge freedom (the tuple $(R_1, \phi_0 = \phi(t = 0))$ such that this choice remains invariant under the evolution of the dynamical system. Oftentimes, the choosing of a gauge consists in being able to set a variable X_i of the dynamical system to zero. Take $X_i = N_\times$ as an example. First, it must be possible to choose an initial orientation ϕ_0 such that $N_\times(\phi_0) = 0$. Second, one must find a choice for R_1 such that this remains true for *all* subsequent times. In our case this specifies the N_- - gauge. In the following we list gauge choices employed in our works.

- *Diagonal shear frame:* In this case the tetrad is aligned with the shear eigenvectors, so that the shear tensor takes a diagonal form.
- *Vector aligned frame:* Aligning the frame with the 3-vector part of the matter sector becomes an important option in the class-A types, since the geometrical vector **a** in that case is zero.
- *F-gauge ($R_1 = 0$):* This is in some sense a quite physical gauge[a]: R_1 specifies the angular velocity compared to a Fermi-Walker propagated frame, so equating this to zero means that one plane is following the frame of gyroscopes[b]. Note that there is still a $U(1)$-gauge freedom left: the initial configuration (ϕ_0) of the frame around the axis orthonormal to the plane spanned by the G_2 subgroup.
- *N_--gauge:* Use the gauge freedom to diagonalize N_{ab}. This gauge may be implemented by choosing

$$R_1 = \sqrt{3}\alpha\Sigma_\times \qquad \text{and} \qquad N_+ = \sqrt{3}\alpha N_-.$$

[a]Remember that $R_1 \equiv \phi'$.

[b]To fully align with the gyroscope frame one must additionally have $\Omega_2 = \Omega_3 = 0$.

It can be shown that this allows for choosing $N_\times = 0$ for all times if one chooses the inital orientation

$$\phi_0 = -\frac{1}{2}\tan\left(\frac{\tilde{\Sigma}_\times}{\tilde{\Sigma}_-}\right)$$

where $\tilde{\Sigma}_\times$, $\tilde{\Sigma}_-$ are variables referring to the frame following gyroscopes.

⋆ *Σ_3-gauge ($R_1 = \sqrt{3}\Sigma_\times$):*In this gauge $\mathbf{\Sigma}_1$ is imposed to be purely real, so $\Re\{\mathbf{\Sigma}_1\} = 0$. This gauge choice becomes natural in the analysis of type I in a G_2 frame aligned such that $\mathbf{V}_c = 0$.

2.7.5 Monotonic functions

Monotonic functions are of great value in dynamical-systems theory. Though valuable, the problem with monotonic functions is that they are, generally speaking, hard to find. In this section we shall make no attempt at giving a general procedure on how to obtain such functions for a given dynamical system. Instead, we shall simply make an observation regarding a special case. Take a dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{2.60}$$

where $\mathbf{x} \in \mathbb{R}^n$ and f is a function from $\mathbb{R}^n \to \mathbb{R}$. Let the system of equations be arranged such that the first k of these equations are on the form

$$\dot{x}_i = (b_i + a_{ij}g(x^j))x^i. \tag{2.61}$$

for some $k \in (0, n]$ and a function $g(x^j)$ where $j \in (0, n]$. Here b_i and a_{ij} are constants. Construct from the k first variables $\{x^i\}$ the function

$$Z = \prod_{j=1}^{k} x_j^{c_j}. \tag{2.62}$$

Calculating $\dot{Z}$ by inserting from (2.61), will now give an algebraic system of equations for the coefficients c_j that must be fulfilled in order for Z to be monotonic. In particular: if it is possible to choose the coefficients such that

$$c^i a_{ij}(g^j) = 0, \tag{2.63}$$

it follows that

$$\dot{Z} = (c^i b_i) Z, \tag{2.64}$$

and hence Z is a monotonic function. This was used in constructing some of the monotonic functions for $\mathcal{B}$(II). The methods of [21] were applied to more general cases.

2.8 Obtaining the line-element

In the following we describe an eloquent way to obtain the line-elements of the equilibrium points found in the orthonormal-frame formalism. Since we are looking at the Bianchi models, we know that the metric must be isotropic and on the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + h_{ij}(t) W^i W^j, \tag{2.65}$$

where, for each Bianchi type, the $\{W^i\}$ are left-invariant one-forms that fulfill the Lie algebra associated with that particular Bianchi type (see Section 2.6). Next, taking (2.49),(2.50) and (2.55) together, one may show that in the case of a constant deceleration-parameter

$$H(t) = \frac{1}{(1+q)t} \qquad \rightarrow \qquad l(t) = t^{1/(1+q)}. \tag{2.66}$$

The deceleration parameter is a scalar and hence it must by definition be constant on the equilibrium points. The equilibrium-point solutions that we seek are self-similar, and the coefficients are powers of t [22]. Consequently, it may be shown that the line-element Hence we may write on the form

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2a}\Big[& t^{4p}\left(W^1 + bt^cW^2 + rt^sW^3\right)^2 \\ & + t^{-2p+2w}\left(W^2 + ut^vW^3\right)^2 + t^{-2p-2w}(W^3)^2\Big]. \end{aligned} \tag{2.67}$$

The factor of t^{2a} is the overall isotropic scale-factor, which we previously called $l(t)$. By Eq. (2.66) we find the relation

$$a = \frac{1}{1+q}. \tag{2.68}$$

In our calculations we have used an orthonormal frame. To compare with the solutions thus obtained as equilibrium points, we need to relate the above metric to the orthonormal frame. To this end we find an orthonormal basis:

$$\omega^0 = \mathrm{d}t, \tag{2.69}$$

$$\omega^1 = t^{a+2p}(W^1 + bt^cW^2 + rt^sW^3), \tag{2.70}$$

$$\omega^2 = t^{a-p+w}(W^2 + ut^vW^3), \tag{2.71}$$

$$\omega^3 = t^{a-p-w}W^3. \tag{2.72}$$

In this basis the metric $\mathbf{g} = g_{\mu\nu}\omega^\mu \otimes \omega^\nu$ simplifies to $\mathbf{g} = \eta_{\mu\nu}\omega^\mu \otimes \omega^\nu$, where $\eta_{\mu\nu}$ is the Minkowski metric[a]. In an orthonormal frame with a time-parameter t orthonormal to a spatial hypersurface Σ_t, the extrinsic curvature $\mathbf{K}$ of the spatial sections Σ_t is given by

$$\mathbf{K} = \frac{1}{2}\pounds_{\partial_t}\mathbf{h} = \boldsymbol{\theta} \tag{2.73}$$

where $\mathbf{h}$ is the metric of the hypersurface and $\boldsymbol{\theta}$ is the expansion tensor. In an orthonormal frame we must have

$$\mathbf{h} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3. \tag{2.74}$$

[a] In particular: $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) = \eta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta(\mathbf{e}_\mu, \mathbf{e}_\nu) = g_{\alpha\beta}\,\omega^\alpha(\mathbf{e}_\mu)\omega^\beta(\mathbf{e}_\nu) = \eta_{\mu\nu}$.

Also note that in the hyper-surface orthogonal case the expansion-tensor is

$$\boldsymbol{\theta} = H\left(\delta_{ij} + \Sigma_{ij}\right)\omega^i \otimes \omega^j, \tag{2.75}$$

where Σ_{ij} is the expansion-normalized shear. Note next the rule

$$\frac{1}{2}\pounds_{\partial_t}\left(\omega^i \otimes \omega^i\right) = \partial_t(\omega^i) \otimes \omega^i. \tag{2.76}$$

Applying this rule to Eq. (2.69) one finds

$$\begin{aligned}
&\frac{1}{2}\pounds_{\partial_t}(\omega^1 \otimes \omega^1) = \\
&(1+q)H\left[(a+2p)\omega^1 + bc\,t^{3p-w+c}\omega^2 + (rs\,t^{3p+w+s} - bcu\,t^{3p+c+v-a})\omega^3\right] \otimes \omega^1, \\
&\frac{1}{2}\pounds_{\partial_t}(\omega^2 \otimes \omega^2) = (1+q)H\left[(a-p+w)\omega^2 + uv\,t^{2w+v}\omega^3\right] \otimes \omega^2, \\
&\frac{1}{2}\pounds_{\partial_t}(\omega^3 \otimes \omega^3) = (1+q)H(a-p-w)\omega^3 \otimes \omega^3.
\end{aligned} \tag{2.77}$$

Taking all the above together, and using that the tensors $\mathbf{h}$ and $\boldsymbol{\theta}$ are symmetric we thus obtain a set of scalar equations by comparing term by term in (2.73). These are as follows.

$$\boxed{\begin{aligned}
\omega^1 \otimes \omega^1 :&\quad (1+q)(a+2p) = (1-2\Sigma_+), && (2.78)\\
\omega^2 \otimes \omega^2 :&\quad (1+q)(a-p+w) = (1+\Sigma_+ + \sqrt{3}\Sigma_-), && (2.79)\\
\omega^3 \otimes \omega^3 :&\quad (1+q)(a-p-w) = (1+\Sigma_+ - \sqrt{3}\Sigma_-), && (2.80)\\
\omega^2 \otimes \omega^1 :&\quad (1+q)bc \cdot t^{3p-w+c} = 2\sqrt{3}\Sigma_2, && (2.81)\\
\omega^3 \otimes \omega^1 :&\quad (1+q)(rs \cdot t^s - bcu \cdot t^{v+c}) \cdot t^{3p+w} = 2\sqrt{3}\Sigma_3, && (2.82)\\
\omega^3 \otimes \omega^2 :&\quad (1+q)uv \cdot t^{2w+v} = 2\sqrt{3}\Sigma_\times. && (2.83)
\end{aligned}}$$

From these equations, the coefficients may be calculated, and the line-element thus found. In the following we provide an explicit example.

2.8.1 Wonderland in type VI_h

In this case the algebra is (cf. Sec. 2.6) such that

$$C^2_{12} = -kf \quad \text{and} \quad C^3_{13} = -k. \tag{2.84}$$

The rest of the structure coefficients vanish. Also

$$C^j_{ij} = 2A_i \quad \to \quad C^2_{12} + C^3_{13} = 2A. \tag{2.85}$$

Inserting from (2.84) we find (expansion normalized) that

$$A = -\frac{k}{2}(1+f). \tag{2.86}$$

A left-invariant basis for the type VI_h algebra is

$$W^1 = \mathrm{d}x, \tag{2.87}$$

$$W^2 = e^{-kfx}\mathrm{d}y, \tag{2.88}$$

$$W^3 = e^{-kx}\mathrm{d}z. \tag{2.89}$$

The corresponding general, self-similar metric incorporating this geometry is therefore

$$\begin{aligned} \mathrm{d}s^2 = &-\mathrm{d}t^2 + t^{2a+4p}\left(\mathrm{d}x + bt^c e^{-kfx}\mathrm{d}y + rt^s e^{-kx}\mathrm{d}z\right)^2 \\ &+ t^{2a-2p+2w}\left(e^{-kfx}\mathrm{d}y + ut^v e^{-kx}\mathrm{d}z\right)^2 \\ &+ t^{2a-2p-2w}e^{-2kx}\mathrm{d}z^2. \end{aligned} \tag{2.90}$$

The shear-specifications for Wonderland in $\mathcal{B}(\mathrm{VI}_h)$ are:

$$\Sigma_+ = \frac{1}{2} - \frac{4}{3}\gamma \quad , \quad \Sigma_- = -\kappa\nu_3 \quad , \quad \Sigma_\times = \kappa\nu_2 \quad \text{and} \quad \Sigma_2 = \Sigma_3 = 0 \tag{2.91}$$

Here $(\nu_2, \nu_3) = (N_-, N_\times)$, and $-1 < \kappa \leq 0$ Since $\Sigma_2 = \Sigma_3 = 0$, we set $b = c = r = s = 0$. The remaining system is

$$(1+q)(a+2p) = (1-2\Sigma_+) \tag{2.92}$$
$$(1+q)(a-p+w) = (1+\Sigma_+ + \sqrt{3}\Sigma_-) \tag{2.93}$$
$$(1+q)(a-p-w) = (1+\Sigma_+ - \sqrt{3}\Sigma_-) \tag{2.94}$$
$$v = -2w \tag{2.95}$$
$$(1+q)uv = 2\sqrt{3}\Sigma_\times. \tag{2.96}$$

This results in the line-element

$$\begin{aligned} \mathrm{d}s^2 = &-\mathrm{d}t^2 + t^2\mathrm{d}x^2 + t^{\frac{2-\gamma}{\gamma}-\Gamma}\left(e^{-kfx}\mathrm{d}y + \frac{\nu_2}{\nu_3}t^{\Gamma}e^{-kx}\mathrm{d}z\right)^2 \\ &+ t^{\frac{2-\gamma}{\gamma}-\Gamma}e^{-2kx}\mathrm{d}z^2, \end{aligned} \tag{2.97}$$

where $\Gamma = \Gamma(\nu_3, \gamma)$ is actually a function defined such that $\Gamma(\nu_3, \gamma) = 4\kappa\nu_3/(\sqrt{3}\gamma)$. Also, $-1 \leq f < 1$. In Wonderland, $A = -\frac{3}{4}\kappa(2-\gamma)$. Using this with (2.86) one finds $k(1+f) = \frac{3}{2}(2-\gamma)\kappa$.

Summary (Geometry): We intend to study the Bianchi models of solvable type in the orthonormal-frame approach. This will permit for a dynamical-systems approach, and by expansion-normalization the equilibrium points of the dynamical system will generally represent self-similar cosmological models. We are particularly interested in those equilibrium points that are stable into the future.

Chapter 3

The matter sector

We consider a matter sector made up of a perfect fluid with barotropic equation of state and anisotropic matter. They are all non-interacting, meaning that

$$(T_i)^{\mu\nu}{}_{;\nu} = 0 \tag{3.1}$$

for each matter component i. In this chapter we describe the anisotropic matter in more detail, starting from a p-form action. But before that, we will have a look at what has already been investigated.

3.1 Existing contributions

The modern field of cosmology has been up and running for about 100 years by now, and the strategy described in the previous section is inherited from others [23, 24]. In particular, spatially homogeneous, anisotropic cosmologies have been studied systematically for many decades. As we have seen, such a cosmology are either of a Bianchi type [25], or it is Kantowski–Sachs [26].

The Bianchi models are well studied [27–29], and dynamical systems theory has been applied to cosmology for a long time [24, 30]. The early development eventually culminated in the book *Dynamical Systems in Cosmology* edited by Wainwright and Ellis [14]. In this book and other early works (e.g. [31]) perfect fluids with a barotropic equation of state are investigated. Such fluids give simple, physically relevant general-relativistic models, despite the fact that they most certainly represent approximations to reality. The question of whether or not bulk-viscosity should be added into the equations have been discussed for some time by now. Consider for instance my own works [32,33] and references therein. Such models do not, however, contain anisotropic stress. One way to include anisotropic stress, is to include shear viscosity [34]. Others have considered generalizations such as tilt, diffusion and vorticity [18,20,21,35–47].

Others have considered electromagnetic fields. The electromagnetic fieldstrength may from a mathematical point of view be described as a so-called 2-form. As such, electromagnetism may be seen as a particular realization of the more general p-form. This is described next.

3.2 The general p-form action

It is necessary to find a source that will sustain anisotropies, if one intends to investigate the possibilities for anisotropic hairs. The p-form is a natural candidate for describing a general anisotropic matter sector. For a recapitulation on what a p-form is, please refer to Appendix A.

For a Lorentzian manifold of dimension n, the canonical volume form η is given by the relation $\eta = \star 1$, and hence any Lorentz scalar (function) f defines a volume form $F = \star f$. Since the volume form is a top-form, integrating it will again give a scalar. A functional S may therefore be constructed in a coordinate-invariant manner as $S = \int \star f$. The question is now which volume form F we take to define our theory. In constructing a gauge theory, there is a natural choice. In particular, take $F = -\frac{1}{2}\mathcal{P} \wedge \star\mathcal{P}$, where $\mathcal{P}$ is a p-form constructed by the exterior derivative of a $(p-1)$-form $\mathcal{K}$. The action now reads

$$S = -\frac{1}{2}\int \mathcal{P} \wedge \star\mathcal{P}\,. \tag{3.2}$$

The Bianchi identity and the equations of motion may now be given in the language of exterior calculus by the following two equations.

$$\mathbf{d}\mathcal{P} = 0 \qquad \rightarrow \qquad \nabla_{[\alpha_0}\mathcal{P}_{\alpha_1\cdots\alpha_p]} = 0 \qquad \text{Bianchi Identity.} \tag{3.3}$$

$$\mathbf{d}\star\mathcal{P} = 0 \qquad \rightarrow \qquad \nabla_{\alpha_1}\mathcal{P}^{\alpha_1\cdots\alpha_p} = 0 \qquad \text{Equations of motion.} \tag{3.4}$$

The latter equation implies a sourcefree field, as dictated by the action. Note that the theories derived from the general p-form action (3.2) respect the following properties: (i) gauge invariance of the Lagrangian density $\mathcal{L}$[b], such that $\mathcal{L} \to \mathcal{L}$ under $\mathcal{K} \to \mathcal{K} + \mathbf{d}\mathcal{U}$, where $\mathcal{U}$ is a $(p-2)$ -form; (ii) only up to second order derivatives in equations of motion; (iii) Lagrangian is up to second order in field strength $\mathcal{P}$; (iv) constructed by exterior derivatives of a p-form and (v) minimally coupled to gravity.

Homogeneity. On the fieldstrength level, our theory is required (spatially) homogeneous. We must therefore have

$$\mathcal{P}(t, \mathbf{x}) \Rightarrow \mathcal{P}(t).$$

Note, however, that since we choose to build the p-form field from an underlying gauge field, we have

$$\mathcal{P}(t) = \mathbf{d}\mathcal{K}(t, \mathbf{x}). \tag{3.5}$$

Id est; the underlying gauge field $\mathcal{K}(t, \mathbf{x})$ may vary both with space and time. Hence, our study is richer than [48] where the gauge field is a function of time only.

3.2.1 Back to the literature

So what about the literature on the p-form action? Source-free electromagnetism ($p = 2$) has received more or less full attention in the context of Bianchi models [49–53]. Recently, a general study of coupled p-form actions in a cosmological context was undertaken [54, 55], though not generally in anisotropic backgrounds. In [56] inflationary scenarios realized by p-forms with arbitrary potentials were investigated, also in axisymmetric, anisotropic cosmologies. An inflationary scenario with a 3-form gauge field was also studied in [57].

A number of studies have looked at 3-form fields (typically with 4-form

[b] Where the Lagrangian density is defined such that $S = \int \sqrt{-g}\, \mathrm{d}^4x\, \mathcal{L}$

fieldstrength) to mimic dark energy, or as a way to couple dark energy to dark matter [58–60]. Dynamical-systems approaches have been used in some of these, but the interest typically lies in the gauge potential, and isotropy is therefore assumed. This is also true for most of the seemingly infinite amount of literature on inflation, with a few exceptions, such as [54, 61, 62]. A short notice regarding 3-form inflation was also given here [63].

Since observations seem to be quite compatible with the Universe being isotropic, the criteria for having shear-free anisotropic cosmologies have also drawn interest [64–67]. Shear-free cosmologies realised by a canonical massless 2-form field were studied in [68] (3-form fieldstrength). Generalizing the matter sources used in these works, one may seek to facilitate shear-free evolution by a general p-form action. It was recently shown [69] that the Bianchi type III is the only class of such cosmological models in which a shear-free solution (properly defined therein) with a lower-bounded Hamiltonian exists.

In ending this section we draw attention to the review (e)book by Coley [70], where actions both from scalar-field theories, scalar-tensor theories and string theory are considered. In particular, the fat list of references therein comprises a treasure for the interested reader.

From the above, we conclude that no systematic study of a p-form fieldstrength where $p \in \{1, 3\}$ exists in the literature on anisotropic cosmology. In this dissertation we therefore conduct a systematic analysis of a p-form fieldstrength with $p \in \{1, 3\}$. Observe next that since we are in four-dimensional spacetime, the Hodge-dual of a three-form is a one-form, and the Hodge-dual of a one-form is a three-form. The equations (3.3) and (3.4) must therefore (collectively) take precisely the same form for $p = 1$ as for $p = 3$. This is (more generally) discussed in for instance [71, Sec. 7.8], and may also be seen in the action (3.2) (up to a prefactor). We therefore and henceforth introduce a piece of new notation; the j-form field, defined as follows.

Definition 13 (j-form field). With a 'j-form field' we mean a matter-sector deriving from (3.2) with $p = j$, where $j \in \{1, 3\}$. The corresponding fieldstrength is a j-form. With the j-form field we associate a one-form $\mathcal{J}$, which we decompose such that

$$\mathcal{J}_\alpha = -w\, u_\alpha + v_\alpha\,, \tag{3.6}$$

where the 4-velocity u_α is time-like ($u_\alpha u^\alpha < 0$), whereas v_α is defined to be orthogonal to u_α and therefore space-like ($v_\alpha v^\alpha > 0$). Note the following.

- ⋆ If $j = 1$, then $\mathcal{J}$ denotes the j-form, which is a one-form.
- ⋆ If $j = 3$, then $\mathcal{J}$ denotes the Hodge dual of the j-form, which is a one-form.

Note that the difference between the case $j = 1$ and $j = 3$ will only be important when discussing the underlying gauge field.

3.3 The j-form field

Generally, for the p-form action (3.2), the energy-momentum tensor $T_{\alpha\beta}$ is given by

$$\begin{aligned} T_{\alpha\beta} &\equiv -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta g^{\alpha\beta}} \\ &= \frac{1}{p!}\left[p\,\mathcal{P}_\alpha{}^{\mu_2\cdots\mu_p}\mathcal{P}_{\beta\mu_2\cdots\mu_p} - \frac{1}{2}g_{\alpha\beta}\mathcal{P}^{\mu_1\cdots\mu_p}\mathcal{P}_{\mu_1\cdots\mu_p}\right], \end{aligned} \tag{3.7}$$

where $\mathcal{L} = -\frac{1}{2p!}\mathcal{P}^{\mu_1\cdots\mu_p}\mathcal{P}_{\mu_1\cdots\mu_p}$ is the Lagrangian density following from the same action. Because of the invariance of the equations (3.3) and (3.4) previously noted, it follows that for a j-form the energy-momentum tensor becomes

$$T^{\mathrm{f}}_{\mu\nu} = \mathcal{J}_\mu\mathcal{J}_\nu - \frac{1}{2}g_{\mu\nu}\,\mathcal{J}_\alpha\mathcal{J}^\alpha\,. \tag{3.8}$$

Here and later 'f' stands for 'form-fluid.'

3.3.1 Connection to scalar-field theories

Take now $j = 1$, such that $\mathcal{J}$ becomes a 1-form constructed from an underlying gauge-potential ϕ. Inserting $\mathcal{J}(t) = \partial_\mu \phi(t, \mathbf{x})\, \omega^\mu$ into (3.8) we may also write

$$\mathcal{L}_\mathrm{f} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi \quad \to \quad T^\mathrm{f}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\gamma \phi \partial^\gamma \phi, \tag{3.9}$$

where we have suppressed the arguments of ϕ for brevity. In terms of the gauge-potential ϕ, the Equations (3.3) and (3.4) now take the form

$$\mathbf{dd}\phi = 0 \quad \to \quad (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\phi = 0, \tag{3.10}$$

$$\mathbf{d}{\star}\mathbf{d}\phi = 0 \quad \to \quad \Box \phi = 0 \qquad \text{Klein–Gordon Eq.} \tag{3.11}$$

where we have defined $\Box = \nabla_\mu \nabla^\mu$. Equation (3.10) is the Bianchi identity. The latter equation, Equation (3.11), we recognize as the Klein–Gordon equation for a massless scalar field.

Our study of a j-form may be viewed as the study of a massless, *inhomogeneous* scalar field with a homogeneous gradient.

3.3.2 Equation of state and energy-momentum tensor components

In Definition 13 we mentioned that we decompose such that $\mathcal{J}_\alpha = -w\, u_\alpha + v_\alpha$. Hence, the j-form field has four independent components. We will, however, rather use the four quantities

$$\{w, v_1, v_2 + iv_3, v_2 - iv_3\}$$

as independent components (more precisely their expansion-normalized counterparts), since this complex form permits for a more compact notation. In real variables, however, the six energy-momentum tensors in the standard irreducible decomposition by the velocity field components u^α and the spatial metric components h_{ij} (refer to Appendix D for details) are as follows.

$$\rho_{\mathrm{f}} = \frac{1}{2}(w^2 + v^2)\,, \tag{3.12}$$

$$p_{\mathrm{f}} = \frac{1}{2}(w^2 - \frac{1}{3}v^2)\,, \tag{3.13}$$

$$q^i = -wv^i\,, \tag{3.14}$$

$$\pi_{ij} = \left(v_i v_j - \frac{1}{3}v^2 h_{ij}\right). \tag{3.15}$$

We omit the subscript ‘f’ on the off-diagonal elements q^i and π_{ij}, since we shall include no other matter sourcing with such components. The equation-of-state parameter ξ for the j-form fluid is defined by

$$p_{\mathrm{f}} = (\xi - 1)\rho_{\mathrm{f}}. \tag{3.16}$$

Hence, one finds the relation

$$\xi = \frac{w^2 - v^2/3}{w^2 + v^2} + 1 \qquad \rightarrow \qquad \frac{2}{3} \leq \xi \leq 2\,. \tag{3.17}$$

The range of ξ follows directly from requiring that $\mathcal{J}_\alpha \in \mathbb{R}$. Note that (3.17) is a dynamical equation of state, since the components of $\mathcal{J}$ in general change with time. The lower bound ($\xi = 2/3$) is found for $w = 0$ and the upper bound ($\xi = 2$) is found for $v = 0$. Note also that $w = v$ gives $\xi = 4/3$, as in the case of electromagnetic radiation.

Expansion-normalization. With the normalization introduced in the previous chapter, Sec 2.7.1, the expansion-normalized j-form components become

$$\Theta = \frac{w}{\sqrt{6}H} \quad , \quad V_i = \frac{v_i}{\sqrt{6}H}. \tag{3.18}$$

Analogously with the complex entities defined in (2.58) of the previous chapter, we now define the complex quantity

$$\mathbf{V}_c = V_2 + iV_3, \tag{3.19}$$

which transforms as a spin-1 quantity under the transformation (2.24). The four independent components of $\mathcal{J}$ are now $\{\Theta, V_1, \mathbf{V}_{\mathrm{c}}, \mathbf{V}_{\mathrm{c}}^*\}$.

3.3.3 1+1+2 decomposition

As mentioned in the previous chapter, the Bianchi types analyzed in this dissertation (I-VII$_h$) admit an Abelian G_2 subgroup and this allows for a 1+1+2 split of the four-dimensional space-time. This translates into a $1+1+2$ decomposition of the Jacobi identity (as we saw), the Einstein Field Equations and the Bianchi identities. Furthermore, we chose a *group-invariant orbit-aligned* frame, i.e. an orthonormal frame which is invariant under the action of G_2 [14]. In this way the complete set of *independent* basic variables reduces to

$$\{H, \sigma_{AB}, \sigma_{1A}, \Omega_1, n_{AB}, a\} \quad \text{and} \quad \{q_a, \pi_{AB}, \pi_{1A}, \rho_{\mathrm{f}}, \rho_{\mathrm{pf}}\}, \tag{3.20}$$

where the capital letter indices A, B run over 2 and 3 which are taken to be the two Killing-vector fields chosen tangential to the group orbits of the G_2 subgroup, all as before. Note that σ_{11} and π_{11} may be derived from the trace-free property of these tensors and the isotropic pressures from equations of state. The energy densities, ρ_{pf} and ρ_{f}, refer to the 'perfect fluid' and to the 'form-fluid,' respectively. Note that by the expansion-normalized quantities constructed in the previous chapter, we now find that under the gauge transformation (2.24), the matter variables transform in the following manner.

$$\{\Xi_1, \Pi_+, \mathbf{\Pi}_1, \mathbf{\Pi}_\Delta, \mathbf{\Phi_1}\} \to \{\Xi_1, \Pi_+, \mathrm{e}^{2i\phi}\mathbf{\Pi}_\Delta, \mathrm{e}^{i\phi}\mathbf{\Pi}_1, \mathrm{e}^{i\phi}\mathbf{\Phi_1}\}, \tag{3.21}$$

$$\{\Omega_{\mathrm{f}}, \Omega_{\mathrm{pf}}, \Theta, V_1, \mathbf{V}_c\} \quad \to \{\Omega_{\mathrm{f}}, \Omega_{\mathrm{pf}}, \Theta, V_1, \mathrm{e}^{i\phi}\mathbf{V}_c\}. \tag{3.22}$$

3.4 Motivation for studying the j-form field

Our motivation has largely been mathematical. Knowing that actions built from p-forms with $p \in \{2,4\}$[c] have already been covered by the literature on anisotropic cosmology, there is a sense of completeness with now also having investigated the remaining scenarios; $p = \{1,3\}$. Especially so, since it is clear that one cannot know the results before they are obtained.

Furthermore, it seems possible to draw motivation to study j-form fields from physical aspects as well. First of all, standard cosmology invokes inflation, cold dark matter (CDM) and a cosmological constant [10] to account for observations. The line of work contained in this thesis may be seen as another attempt at closing in on a better understanding of the 'dark side' of the Universe.

The aim of this dissertation is not a detailed study of observations, or to make claims toward such ends, and we suffice it to let the reader know about papers like [68, 72–77], where observations are discussed in the context of anisotropy. Also, for the interested reader, we mention that Section 4 in **Paper III**, discusses more intently the possible physical motivation for studying the j-form field.

Summary (The matter content): In this dissertation we shall study the so far unstudied cases of a general p-form action. Id est; we assume that the matter content consists of a j-form, where $j \in \{1,3\}$. Additionally, we add a non-tilted, non-phantom, barotropic perfect fluid $(0 \leq \gamma \leq 2)$. Denoting the energy densities by Ω_i we write

$$\Omega = \Omega_{j\mathrm{f}} + \Omega_{\mathrm{pf}}. \tag{3.23}$$

We will also investigate the effects of adding a cosmological constant Ω_Λ on top of Ω.

[c] Where, as mentioned, electromagnetism is studied through $p = 2$ and the cosmological constant through $p = 4$. These matters are more thoroughly discussed in **Paper I**.

Chapter 4

Probing the Night Sky

It seems as if Nature couldn't care less about my theories. No matter how fine they are: if they do not correctly describe her, she does not reveal herself through them. As a matter of fact; any *theory* that seeks to describe (parts of) Nature is judged by *the observation* of her. It becomes all the more important, therefore, to develop good probes on the night sky, such as to facilitate an as accurate judgement of theories as possible.

Today we collect information from the night sky in many different ways. The electromagnetic spectrum is read from the radio-wave frequency to the x-ray frequency, and gravitational-wave signals are caught on tape and analysed. However, non of these observations would be of any particular value without proper theoretical framework in which to interpret them. Theoretical development is therefor important. One such theoretical development came with the understanding that even light is deflected by gravity's 'pull'[a]. This effect is nowadays called *gravitational lensing* **(GL)**.

4.1 A brief history of GL

Isaac Newton himself added a couple of lines at the end of his 1704 work on optics, about light also being influenced by gravity in the same way as matter [78]. Following a century later, scientists such as John Michell [b] [80] and Pierre-Simon Laplace [81] would build on Newton's theory of gravity and conclude that light should be affected by gravity. Actually, Newton's

[a] Actually, gravity does not exert forces on the objects on which it acts. Rather, the notion of straight lines—geodesic motion— is affected. Hence 'pull' and not pull.

[b] This creative chap also speculated that there might be objets so massive that even light cannot escape their gravitational attraction. Today we call such objects *black holes*. An overview is provided in [79].

theory misses the correct lensing equation for a point mass by a factor two, only. This discrepancy between general relativity **(GR)** and Newton's theory of gravitation, however, was sufficient to discriminate between the two theories under a solar eclipse back in 1919 [82], favouring Einstein's theory over Newton's. A solid framework in which to study GL-effects was hence provided: GR. The contest between the two theories also displayed the power of the tool, leading to optimism concerning its capacities. Since then, the field of GL has been vibrant with contributions.

Einstein's calculations, however, led to pessimism about observing stars lensed by our sun [83]. It is fortunate, therefore, that Zwicky [84, 85] studied lensing effects by galaxies outside our own, successfully showing how such observation could determine the mass of distant galaxies. Later, further theoretical advancement came with the Norwegian astrophysicist Sjur Refsdal, who in the 1960s [86, 87] did pioneering work in using GL as a probe on cosmic parameters. By measuring the time delay between a light-ray travelling one way against a ray taking a different route around a lens (galaxy), the Hubble–Lemâitre constant was successfully calculated.

The modern field of GL is typically divided into two regimes [88, p. 21]: **weak lensing** ($\kappa < 1$) and **strong lensing** ($\kappa > 1$). Here κ is the dimensionless surface-mass density (or convergence) of the lens, and will be more properly discussed later on. Observation of strong lensing is rare. Observation of weak lensing is not rare, but subtle as it is a weak effect and requires statistical treatment. A typical application is to measure the distortion of background galaxies lensed by a (foreground) cluster of galaxies. This effect was first detected by Tyson et al. in 1990 [89] in a pioneering work on **cluster lens-mass reconstruction**. Since there is a map between surface-mass density and lensing, one early recognised GL as a good probe on the distribution of dark matter. Already in 1989, microlensing[c] was used to explore the nature of dark matter [90]. Since then the effect has been used to detect black holes, exoplanets and much more.

[c]Microlensing typically refers to the lensing of galactic or extra-galactic sources by lenses inside our galaxy.

As late as in 2018, the study of microlensing effects in images magnified through galaxy-cluster lensing effects, led to the discovery of Icarus, the most distant star observed so far [91, 92]. The field of strong lensing was recently reviewed by Anna Barnacka, who concludes that it provides unique physical information about the central structure of active galaxies [93]. Since GL-effects such as magnification, arcs and rings [94, 95] allow for deep-galaxy observation, one may conclude that improved accuracy of GL-measurements means an improved probe on the Universe of the distant past.

4.2 GL as a probe on dark matter

The first use of GL was, as mentioned, to pin down the correct theory of gravity. The dispute concerning whether or not GR really is the correct theory of gravity has not remained silent, however. Today we know that if the standard ΛCDM-cosmology[a] is correct, then we need a whole lot more matter in galaxies than the luminous part. The Planck survey suggests a ratio of luminous to non-luminous of about 1/5 [10, 96]. Furthermore, estimates suggest that in a galaxy cluster, about 80% of the mass is due to non-luminous sources [97, p. 300]. According to the same source (p. 327) observations of colliding galaxy clusters cannot properly be accounted for without the inclusion of collision-less dark matter—even if GR is modified on large scales. Within the paradigm of ΛCDM-cosmology, one has therefore set out to search for the nature of this so-called dark matter.

To this end it is important to map the gravitational potential of the Universe. A large share of today's cosmology is devoted to finding out, and this is where lensing comes in as an excellent tool [98, 99].

[a] Standard cosmology is named after its two main constituents: The cosmological constant (Λ) and cold dark matter (CDM).

4.3 Lens-mass reconstruction

One of the most prospering outputs of studying GL, has been the ability to do lens-mass reconstruction. Understanding how light-rays are being deflected by a lens, is the same as understanding the gravitational potential of the lens; its magnitude and distribution. This, in turn, translates into a map of dark matter in the lens. Early in the 1990s, **Kaiser and Squires** did pioneering work to this end; finding expressions for the mass in terms of the shear field measured in galaxy clusters [100,101]. A key improvement from previous attempts with weak lensing (e.g. [102, 103]) was that this approach was model-independent. Later Schneider and Seitz [104–106] developed the method further, using only local data to reconstruct the lens mass from the shear field. These early works have paved the way for an array of papers describing the implementation of Kaiser and Squires' work to obtain real data.

4.4 The two approaches to GL

In this dissertation, **The roulette formalism** will be used. This is a weak-lensing approach that aims at describing also strong lensing effects. Of course any divide between weak and strong lensing is in some sense artificial and imposed by the formalism. Since both weak and strong lensing will typically occur together, one should seek a formalism in which both regimes may be accounted for. Roughly speaking, one may say that there are two approaches to gravitational lensing.

- ⋆ *The lensing equation.* This approach starts with the gravitational field outside a point mass, and looks at how a light-ray is deflected. More complicated matter profiles of the lens is obtained by integrating over the lens-plane (to be defined). For strong lensing, this will typically be the way to go.
- ⋆ *The geodesic deviation equation.* Starting from the equation of geodesic

deviation, one calculates how light-rays scatter. This approach is more often used for weak lensing.

In order to create a unified treatment of the two regimes, it is natural to assume one of the above established approaches, seeking to extend it to include both regimes in an appropriate manner. In [107] they start by the lensing equation, thus creating a 'strong-lensing formalism for weak lensing'. The Roulette-approach is taking the opposite approach. By starting from the geodesic deviation equation, one seeks to implement strong gravitational lensing through the inclusion of higher-order terms. In the following we briefly describe the two approaches. Take in the following $\boldsymbol{\eta}$ and $\boldsymbol{R_L}$ to be two-dimensional vectors and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\vartheta}$ to be (three) two-dimensional vectors of angular coordinates. The following definitions are now used.

- ⋆ *Optical axis.* A straight line OA from the observer to the source-plane.
- ⋆ *Source-plane*: The plane orthogonal to OA at the source, which is at a distance χ_{S} along OA and spanned by $\boldsymbol{\eta}$.
- ⋆ *Lens-plane*: The plane orthogonal to OA at the lens, which is at a distance χ_{D} along OA and spanned by $\boldsymbol{R_L}$. Note that $\chi_{\mathrm{S}} - \chi_{\mathrm{D}} \equiv \chi_{\mathrm{DS}}$.
- ⋆ *Deflection-angle*: Let C be the path of the light-ray from the source to the lens-plane, and L the path from the lens-plane to the observer. Then $\boldsymbol{\alpha}(\boldsymbol{R_L})$ is the angle between these two lines, such that $\boldsymbol{\alpha}(\boldsymbol{R_L}) = 0$ in the case of no lensing.

Refer to Figure 4.1 for a more intuitive picture. Note also that we will assume the flat-sky approximation, meaning that the angular distance between the lensed objects observed on the celestial sphere are so small that we may assume flatness. Hence the name lens-*plane*.

4.4.1 Approach 1: The lensing equation

To derive the lensing equation, one starts by considering the deflection of a light-ray by a point-mass lens. Using the fact that for small angles

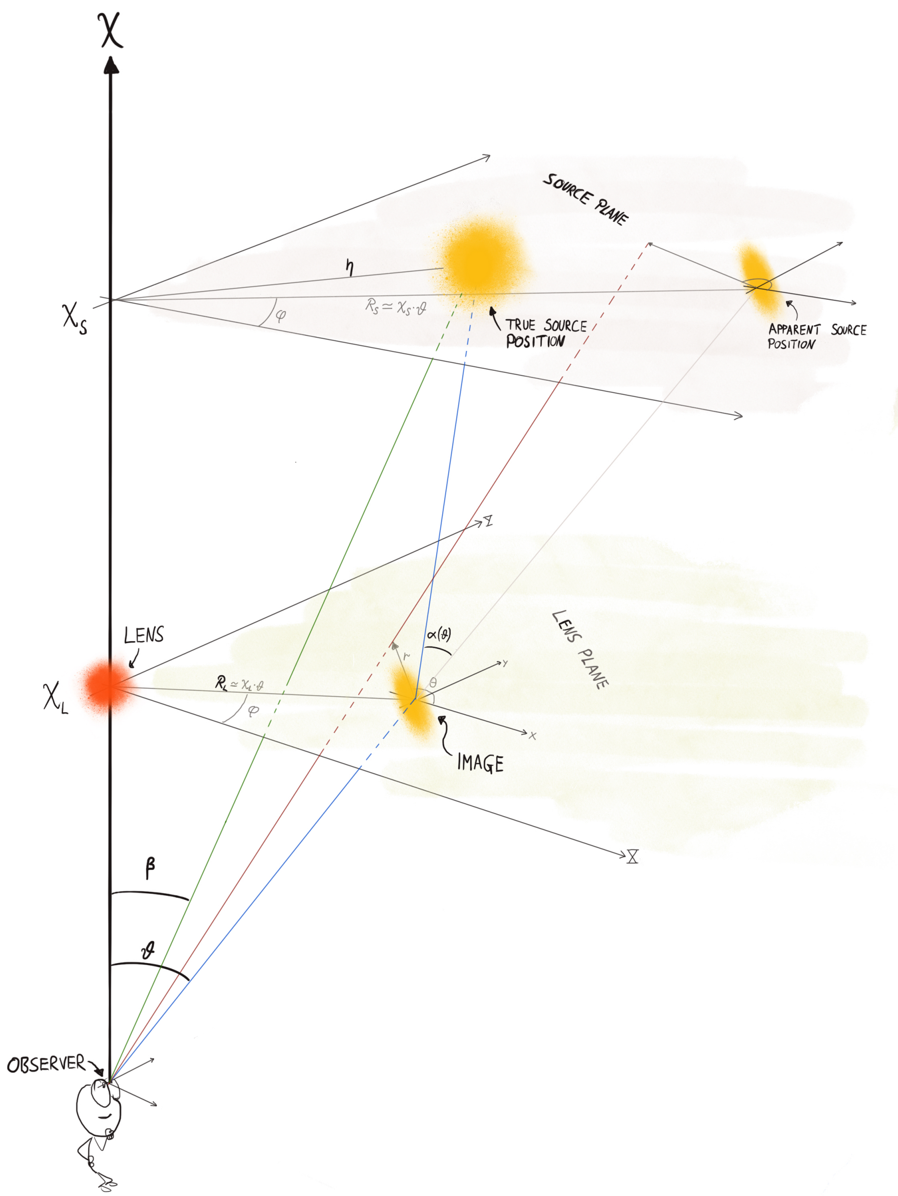

Figure 4.1: The figure shows the notation used for the Roulette formalism in the thin-lens approximation. The bold symbols are those defined and used in context of the lensing equation.

$\sin\alpha \approx \tan\alpha \approx \alpha$, one may now use Figure 4.1 to show geometrically that the lensing equation takes the following simple form,

$$\boldsymbol{\beta} = \boldsymbol{\vartheta} - \boldsymbol{\alpha}(\boldsymbol{\vartheta}), \tag{4.1}$$

where we have used angular coordinates such that $\boldsymbol{\eta} = \chi_{\mathrm{S}}\boldsymbol{\beta}$, $\boldsymbol{R_L} = \chi_{\mathrm{L}}\boldsymbol{\vartheta}$. The lensing-equation now maps any point in the source-plane to point(s) in the lens-plane by the condition that the points in the lens-plane are those seen by the observer. Considering next a geometrically *thin lens*, the incoming light-ray may still be approximated as a straight line $\boldsymbol{s} = s\hat{\boldsymbol{s}}$. The surface-mass density Σ[b] of the plane perpendicular to $\boldsymbol{s}$ may in this case be shown to be

$$\Sigma(\boldsymbol{R_L}) = \int \mathrm{d}s\, \rho(\boldsymbol{R_L}, s), \tag{4.2}$$

where $\boldsymbol{R_L}$ is a vector of coordinates (approximately) perpendicular to $\boldsymbol{s}$. From this, one defines the already mentioned dimensionless surface-mass density κ—also called *convergence*—as

$$\kappa(\boldsymbol{\vartheta}) = \frac{\Sigma(\chi_{\mathrm{D}}\boldsymbol{\vartheta})}{\Sigma_{\mathrm{cr}}} \qquad \text{where} \qquad \Sigma_{\mathrm{cr}} = \frac{\mathrm{c}^2}{4\pi \mathrm{G}}\frac{\chi_{\mathrm{S}}}{\chi_{\mathrm{D}}\chi_{\mathrm{DS}}}. \tag{4.3}$$

As always, c is here the speed of light and G is the gravitational constant. Σ_{cr} is the so-called critical surface mass density, which depends on the relative distances between source, lens and observer. If $\Sigma \geq \Sigma_{\mathrm{cr}}$, (which is the same as $\kappa > 1$), multiple images may occur. Hence we see that in the strong-lensing regime ($\kappa > 1$) the lensing-map may map one image to many lensed images. The kink-like behaviour of the light-ray is a good approximation, and the formalism described above is frequently used. On the other hand, in the weak-lensing regime ($\kappa < 1$), the lens is bigger than the so-called Einstein-radius, and rings or multiple images do therefore not occur.

Next, one defines the so-called *amplification-matrix* $\boldsymbol{A}$ according to

[b] **Caution!** Not to be confused with the Σ used to describe the *shear* of the congruence of observers elsewhere in this thesis. However, these are standard notations and the context will provide the correct interpretation.

$\mathbf{d}\boldsymbol{\beta} = \boldsymbol{A}(\boldsymbol{\vartheta})\, \boldsymbol{d\vartheta}$. Define now the *lensing potential* ψ according to

$$\tilde{\psi}(\boldsymbol{\vartheta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{\vartheta'} \tilde{\kappa}(\boldsymbol{\vartheta'}) \log |\boldsymbol{\vartheta' - \vartheta}| \,, \tag{4.4}$$

where $\tilde{\nabla} = (\partial_{\vartheta_1}, \partial_{\vartheta_2})$ is given in angular coordinates. One may show that this implies the relations

$$\boldsymbol{\alpha} = \tilde{\nabla}\psi \qquad \text{and} \qquad \kappa = \frac{1}{2}\tilde{\nabla}^2\psi. \tag{4.5}$$

The amplification matrix $\boldsymbol{A}$ now takes the form

$$\boldsymbol{A}(\boldsymbol{\vartheta}) = \begin{pmatrix} 1 - \tilde{\kappa} + \tilde{\gamma}_1 & \tilde{\gamma}_2 \\ \tilde{\gamma}_2 & 1 - \tilde{\kappa} - \tilde{\gamma}_1 \end{pmatrix} \tag{4.6}$$

with

$$\tilde{\kappa}(\vartheta) = \frac{1}{2}\tilde{\nabla}^2\tilde{\psi}(\vartheta), \tag{4.7}$$

$$\tilde{\gamma}_1 = \frac{1}{2}(\tilde{\psi}_{,22} - \tilde{\psi}_{,11}), \tag{4.8}$$

$$\tilde{\gamma}_2 = -\tilde{\psi}_{,12}. \tag{4.9}$$

Here $_{,i}$ refers to derivative w.r.t. ϑ_i, where $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2)$. Kaiser & Squires described a method for inverting the measured shear caused by the lens to obtain the surface-mass density of a lens [101]. Moreover, they obtained an inversion formula, yielding an expression for the surface mass density. Defining $\tilde{\boldsymbol{\gamma}} = \tilde{\gamma}_1 + i\tilde{\gamma}_2$ one finds

$$\tilde{\kappa}(\vartheta) = \frac{1}{\pi} \int_{\mathbb{R}^2} \mathrm{d}^2\vartheta' \Re \left[D(\boldsymbol{\vartheta} - \boldsymbol{\vartheta'}) \tilde{\gamma}(\boldsymbol{\vartheta'}) \right], \qquad \text{(Kaiser \& Squires' inversion formula)} \tag{4.10}$$

where $D = (\vartheta_1^2 - \vartheta_2^2 + 2i\vartheta_1\vartheta_2) / |\boldsymbol{\vartheta}|^4$. As mentioned, this formula gives the convergence (surface mass density) in terms of the shear-field generated by the lens. Kaiser & Squires' results are very well described by Schneider, who extended their work in two papers [104, 105] and finds a way to use only local data to reconstruct the lens-mass from the shear-field, and also points out the need for a generalized Kaiser & Squires' inversion procedure, to account for stronger lensing effects.

4.4.2 Approach 2: Starting from the geodesic-deviation equation

Weak lensing is, as the name suggests, a relatively weak effect, and the treatment becomes theoretically more involved. The starting-point is often the 1st order geodesic deviation equation (GDE)

$$\ddot{\xi}^a + R^a_{kbk}\xi^b = O(\xi, \dot{\xi})^2 \qquad \text{1st order GDE.} \tag{4.11}$$

Here $\boldsymbol{\xi} = \xi^a \mathbf{e}_a$ is the deviation vector, and $R^a{}_{kbk}$ are Riemann-tensor components where k as an index denotes the projection of that index in the direction of k, which is the direction of the tangent-vector of the light-ray. $(\dot{\ })$ denotes derivative along the null curve. Weak GL is traditionally assumed to only have two principle effects; magnification and shear, although higher-order effects, such as flexion and second flexion are sometimes included [108–110], and arise from including higher-order terms in the GDE. To second order in $(\xi, \ddot{\xi})$ one finds the so-called Bazanski equation [111–113].

4.4.3 The Roulette formalism

In 2015 Clarkson showed that integrating the second order GDE resulted in a Hessian map for GL. In two subsequent papers [114, 115], Clarkson set out to solve the GDE equation to arbitrary order, keeping only the maximum number of leading screen-space derivatives. This essentially amounts in the replacement

$$R^a_{k\xi k} \quad \to \quad \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\xi^b \nabla_b\right)^n R^a_{kbk} \tag{4.12}$$

in the above equation (4.11). By this procedure the GDE now takes the form

$$\ddot{\boldsymbol{\xi}} - \boldsymbol{\mathcal{R}}\boldsymbol{\xi} = \boldsymbol{F}, \tag{4.13}$$

where $\mathcal{R}$ is the optical tidal matrix,

$$\mathcal{R}_{AB} = -C_{AcBd}k^c k^d - \frac{1}{2}\delta_{AB}R_{cd}k^c k^d \tag{4.14}$$

where $C^{ab}{}_{cd}$ is the Weyl tensor and capital-letter indices refer to screen-space coordinates and R_{ab} is the Ricci tensor. Moreover, $\boldsymbol{F}$ is a matrix containing leading-order screen-space derivatives[c], such that

$$F^A_{(m)} = \xi^{A_1}_{(1)}\xi^{A_2}_{(1)}\cdots\xi^{A_m}_{(1)}\nabla_{A_1}\nabla_{A_2}\cdots\nabla_{A_{m-1}}\mathcal{R}^A{}_{A_m}. \tag{4.15}$$

By application of the same solution strategy as in [116], the general integral solution is

$$\boldsymbol{\xi} = \int_{\lambda_0}^{\lambda} \mathrm{d}\lambda' \left[\boldsymbol{\mathcal{K}}(\lambda) - \boldsymbol{\mathcal{J}}(\lambda)\boldsymbol{\mathcal{J}}^{-1}(\lambda')\boldsymbol{\mathcal{K}}(\lambda')\right]\boldsymbol{\mathcal{K}}^T(\lambda')\boldsymbol{F}(\lambda'), \tag{4.16}$$

where $\boldsymbol{\mathcal{J}}(\lambda)$ is the Jacobi-map, $\boldsymbol{\mathcal{K}}(\lambda)$ is the reciprocal Jacobi-map and λ is an affine parameter on the past light-cone of the light-ray. The technical details are nasty, and far beyond the scope of this introduction, and so we shall refer the reader to the sources for derivations. The important outcome is that the mth order map between the image and the source may now be shown to be given by

$$\xi^A_{(m)} = \mathcal{M}^A_{B_1\cdots B_m}\zeta^{B_1}\cdots\zeta^{B_m}, \tag{4.17}$$

where the matrix $\mathcal{M}$ may be decomposed into symmetric, trace-free tensors. By such it may be shown to fulfill the relation

$$\begin{aligned}\mathcal{M}_{AB_1\cdots B_m}\hat{\zeta}^{B_1}\cdots\hat{\zeta}^{B_m} = \sum_{s=0}^{m+1}\frac{[1-(-1)^{m+s}]}{4}\times \\ \left(\left[\left[C^+\alpha^m_s + \bar{\beta}^m_s\right]\boldsymbol{R}_- + \left[C^+\beta^m_s - \bar{\alpha}^m_s\right]\boldsymbol{R}_/\right]\boldsymbol{p}_{s-1}\right. \\ \left. + \left[\left[C^-\alpha^m_s - \bar{\beta}^m_s\right]\boldsymbol{I} + \left[C^-\beta^m_s + \bar{\alpha}^m_s\right]\boldsymbol{\varepsilon}\right]\boldsymbol{p}_{s+1}\right),\end{aligned} \tag{4.18}$$

[c] we here use capital letters $A, B, \cdots$ to denote screen-space components.

where $\hat{\zeta} = \zeta / r$ is the radial unit vector in the lens plane. Here we have defined

$$C^{\pm} = 1 \pm \frac{s}{m+1}. \tag{4.19}$$

Also, $\boldsymbol{I}$ is the 2×2 identity matrix, and the rest are the Pauli spin matrices:

$$\boldsymbol{\varepsilon}_{-} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \boldsymbol{R}_{-} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \boldsymbol{R}_{/} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.20}$$

The coefficients α_s^m and β_s^m (and $\hat{\alpha}_s^m$ and $\hat{\beta}_s^m$) are the so-called even (and odd) Roulette-amplitudes. These amplitudes each encode independent distortions. Actually, we may obtain expressions for each one of them by considering distortions of the unit circle. Consider a distortion induced by a vector

$$\boldsymbol{p}_{(s)} = \cos s\theta \mathbf{e}_x + \sin s\theta \mathbf{e}_y. \tag{4.21}$$

Multiplying this on the right by the row-vector $\boldsymbol{p}_{(n)}^{\mathrm{T}}$, and integrating around the unit circle, one finds

$$\frac{1}{\pi} \int \mathrm{d}\theta \, \boldsymbol{p}_{(m)} \boldsymbol{p}_{(n)}^{\mathrm{T}} = \begin{cases} \boldsymbol{I} \delta_{mn} & \text{for} \quad m, n > 0, \\ \boldsymbol{R}_{-} \delta_{|m|n} & \text{for} \quad m < 0, n > 0, \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \delta_{0n} & \text{for} \quad m = 0. \end{cases} \tag{4.22}$$

Next, applying this to solve the integral $\frac{1}{\pi} \int \mathrm{d}\theta \, \hat{\boldsymbol{\xi}}_{(m)} \boldsymbol{p}_{(n)}^{\mathrm{T}}$ one finds by (4.18) the following: For $s > 0$ the roulette amplitudes become

$$\alpha_s^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \hat{\xi}_{(m)}^A \left[p_A^{(s+1)} + R_{AB}^{-} p_{(s-1)}^B \right], \tag{4.23}$$

$$\beta_s^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \hat{\xi}_{(m)}^A \left[\varepsilon_{AB} p_{(s+1)}^B + R_{AB}^{/} p_{(s-1)}^B \right], \tag{4.24}$$

$$\hat{\alpha}_s^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \hat{\xi}_{(m)}^A \left[-C^{+} \varepsilon_{AB} p_{(s+1)}^B + C^{-} R_{AB}^{/} p_{(s-1)}^B \right], \tag{4.25}$$

$$\hat{\beta}_s^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \hat{\xi}_{(m)}^A \left[-C^{+} p_A^{(s+1)} + C^{-} R_{AB}^{-} p_{(s-1)}^B \right]. \tag{4.26}$$

For $s = 0$ one similarly finds

$$\alpha_0^m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \hat{\xi}^A_{(m)} p_A^{(1)}, \tag{4.27}$$

$$\beta_0^m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \varepsilon_{AB} \hat{\xi}^A_{(m)} p^B_{(1)}, \tag{4.28}$$

whereas the odd modes vanish. For a given order m of the map $\mathcal{M}$, these amplitudes contain the spin-s contributions to the distortion. By such, one has for any order m, decomposed the s distortions of an image into independent contributions. This formalism therefore entails not only the distortions named convergence and shear (recall that we discussed these effects from the lensing-equation point-of-view), but also higher order effects [a] to arbitrary order in leading screen-space derivatives. Consequently, one may view this approach as a *a weak-lensing approach to strong lensing*.

The weak-field approximation

Weak lensing should not be confused with the *weak-field approximation*, which merely states that the Newtonian potential ϕ_{N} is small in a linearisation around a Minkowski background ($\phi_{\mathrm{N}}/c^2 \ll 1$). Linearising around Minkowski space we write the perturbations with respect to Poisson gauge as

$$\mathrm{d}s^2 = -(1+2\varPhi)\mathrm{d}\eta^2 + (1-2\varPsi)\gamma_{ij}\mathrm{d}x^i x^j, \tag{4.29}$$

and define a lensing potential

$$\psi = \int_0^{\chi} \mathrm{d}\chi' \left(\frac{\chi-\chi'}{\chi\chi'}\right)(\varPhi+\varPsi). \tag{4.30}$$

We refer the reader to the discussion around eq. 102 in [115] for further details, and suffice it here to give the results, as follows: For a general *thin lens* in the *weak-field* and *flat-sky* approximation the (non-vanishing)

[a] The two first of which are typically named flexion and second flexion.

Roulette amplitudes may be shown to take the form

$$\alpha_s^m = -2^{-\delta_{0s}}\chi^{m+1}\sum_{k=0}^{m}\binom{m}{k}\left(\mathcal{C}_s^{m(k)}\partial_{\mathrm{X}} + \mathcal{C}_s^{m(k+1)}\partial_{\mathrm{Y}}\right)\partial_{\mathrm{X}}^{m-k}\partial_{\mathrm{Y}}^{k}\psi, \tag{4.31}$$

$$\beta_s^m = -\chi^{m+1}\sum_{k=0}^{m}\binom{m}{k}\left(\mathcal{S}_s^{m(k)}\partial_{\mathrm{X}} + \mathcal{S}_s^{m(k+1)}\partial_{\mathrm{Y}}\right)\partial_{\mathrm{X}}^{m-k}\partial_{\mathrm{Y}}^{k}\psi, \tag{4.32}$$

where χ is the distance from the observer to the lens. X, Y are coordinates in the lens-plane and $\psi = \psi(X, Y)$ is the lensing potential given in (4.30). Also the spin s is restricted such that $0 \leq s \leq m + 1$ and the roulette amplitudes α_s^m , β_s^m may be non-zero only if $m + s$ is odd. Finally;

$$\mathcal{C}_s^{m(k)} = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \sin^k\theta \cos^{m-k+1}\theta \cos s\theta, \tag{4.33}$$

$$\mathcal{S}_s^{m(k)} = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \sin^k\theta \cos^{m-k+1}\theta \sin s\theta. \tag{4.34}$$

The above expressions for the roulette-amplitudes α_s^m, β_s^m constitute the starting point of our **Paper V**. One may note that allthough the above formulas makes it possible in principle to calculate the amplitudes, the complexity of the expressions makes it a somewhat time-consuming business.

Why use the Roulette formalism?

There are primarily three reasons as to why we intend to use the Roulette formalism:

First; one of the clear advantages with using the Roulette formalism, is the ability to include higher-order terms and by such constucting a theory that encompasses both weak and strong lensing. Since cluster lenses will contain both strongly and weakly lensed images, this is highly beneficial. Today, the number of massive single lensed galaxies lensed by a distant galaxy is not so large (< 1000), but surveys like the Dark Energy Survey (DES), SLACS, and in particular upcoming surveys like the LSST [39], Wide-Field Infrared Survey Telescope and ESA's Euclide will change the picture dramatically. Among strong lensing effects alone, hundreds of thousands are expected to be observed [117]. An equally dramatic shift in the way we build models is no doubt needed (refer to [118], Conclusions) in order to capacitate analysis of all the data.

Second, the approach should in principle be applicable also to the exact GDE derived by Vines through an exponential map approach via bitensors [119]. This is highly beneficial, since solutions of the GDE to high order in $\boldsymbol{\xi}$ and its derivatives have not been developed beyond the aforementioned literature, in particular Clarkson's works.

Third, it should be noted that the roulette formalism is **parameter-free**. Hence we need no a-priori model for the lens.

Summary (GL and Roulette theory): Theoretical understanding of GL is important for many reasons, one of which is its promising future as a means to mapping the dark-matter distribution of the night sky. The Roulette-formalism for gravitational lensing is a scheme for integrating the non-linear geodesic-deviation equation to arbitrary order, by such extending the much praised Kaiser-Squires relations from weak to stronger lensing regimes. By such, weak and strong lensing may be treated within the same formalism.

Chapter 5

Summary

5.1 Research question and assumptions

Knowing that the Universe is quite isotropic, one may, as already discussed in Chapter 1, wonder how likely such a Universe is. We take the following research question.

> Question 1. Is the asymptotic future of a cosmology filled with a perfect fluid alongside j-form matter isotropic?

Take a cosmological model $(\mathcal{M}, \mathbf{g}, \mathbf{u})$. Then the research question stated above will be addressed under the following assumptions.

> Assumption 5.1 (Philosophy). We assume the weak cosmological principle (WCP). Id est; we assume that the manifold $\mathcal{M}$ of the model is homogeneous on spatial sections, meanwhile simultaneously allowing for anisotropies in the metric $\mathbf{g}$.

> Assumption 5.2 (Matter). We take as matter content a perfect fluid with barotropic, non-phantom[a] equation of state with which the fundamental observer $\mathbf{u}$ will be aligned and a j-form fluid. We also investigate the effects of adding a cosmological constant.
>
> [a]With 'barotropic' we mean that the fluid is a function of pressure only. 'Non-phantom' means that the equation-of-state parameter γ is not allowed to be negative.

> Assumption 5.3 (Theory). General Relativity is assumed to be the correct theory of gravity.

Finally, in repetition: the ruthless swiftness of time has forced us to leave the Kantowski-Sachs model out and concentrate on the Bianchi models.

5.2 Method

The Question 1 will be addressed within the boundaries set by the Assumptions 5.1-5.3 above, which specify what sort of models we intend to study. The following steps will be crucial in the actual analysis.

1) Write down the system of equations in the orthonormal frame (ONF). This ensures first-order differential equations. Furthermore, since homogeneity is required, no spatial derivatives are allowed for, and we are hence left with a system of ordinary differential equations (ODE).
2) The ODE will be studied as a dynamical system. Since each Bianchi model is an invariant set[a] we study each of these sets separately.
3) The next crucial step is to expansion-normalize (EN). The isotropic part of the expansion decouples from the system of differential equations. The equilibrium points of the expansion-normalized system of expanding cosmologies now represent self-similar cosmological models[b].
4) The next step in the analysis is to work out the stability of the equilibrium points. The stable (/unstable) equilibrium points are given relevant roles as candidates[c] for future (/past) asymptotic states of the invariant sets. Where found possible, the local analysis is supplemented by a global analysis.

The above points 1)-4) are summarized in Figure 5.1.

[a] Invariant sets: In our case these are interesting because a Universe model that belongs to an invariant set of state-space will remain in that set.
[b] Except if $q = -1$, for which $H' = 0$.
[c] I say 'candidates' since one must be able to exclude closed orbits in order to understand the global behaviour in the set. A local analysis alone does not exclude closed orbits.

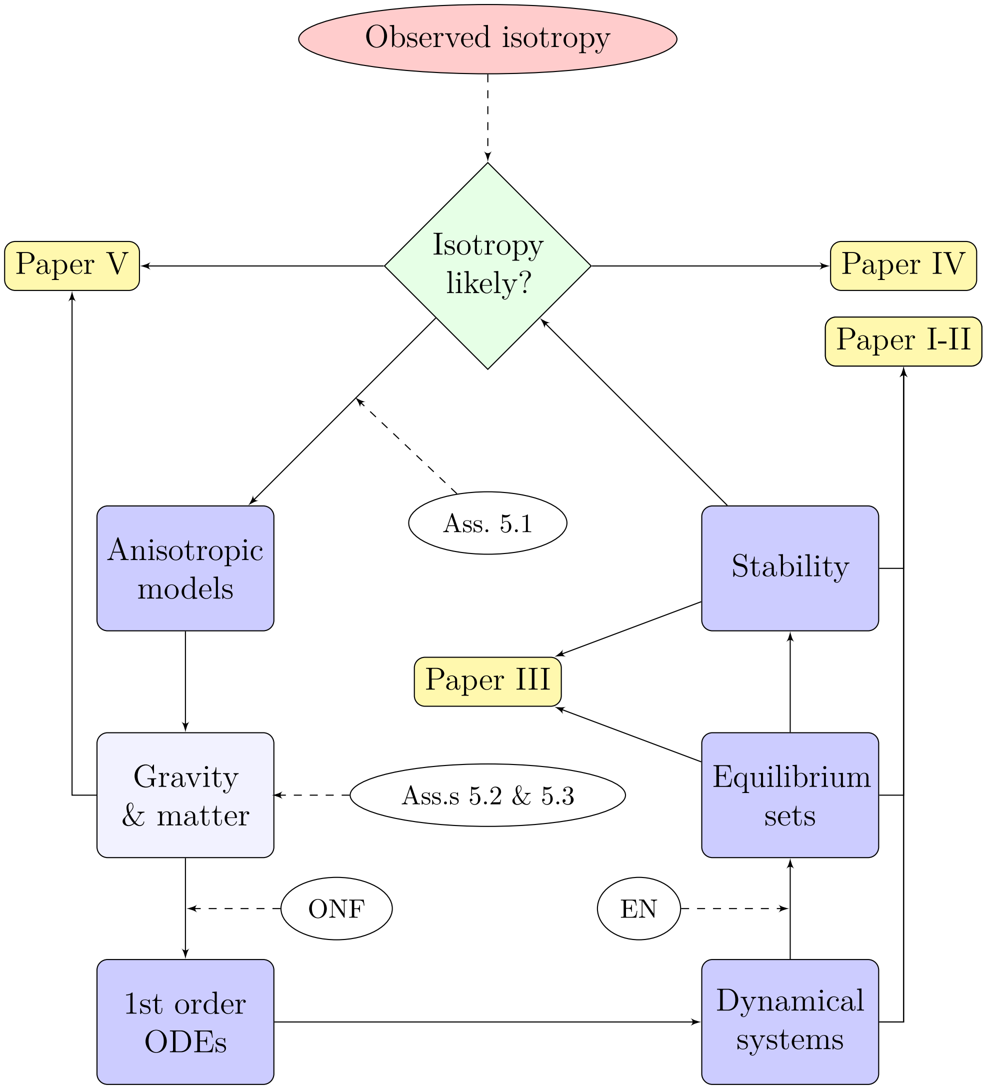

Figure 5.1: This schematics provides an overview of the work-flow entailed by the approach taken in this project. Observations (ellipse) lead to a research question (tilted square). To answer the question the project is broken down into various steps (non-tilted rectangles) that answer the question for each model tested (white-blue rectangle). It is also indicated where the assumptions Ass. 5.1-5.3 come in and where essential methods like the orthonormal frame (ONF) and expansion normalisation (EN) apply.

5.3 Breakdown into papers

This dissertation consists, as mentioned, of five papers. In this section I give a brief summary of each of the papers, alongside comments displaying my own contributions.

- ⋆ **Paper I [120]:** In this paper (i) the system of equations for a j-form matter content alongside a prefect fluid (and a cosmological constant) is written down, (ii) the allowed components of the j-form field in the Bianchi models is determined, (iii) past and future attractors of the dynamical systems $\mathcal{B}(\mathrm{I})$ and $\mathcal{B}(\mathrm{V})$ are found and (iv) global results are established.

  My contribution: With the exception of monotonic functions and parts of the written analysis, I have done most of the work.

- ⋆ **Paper II [121]:** In this paper, past and future attractors of the dynamical systems $\mathcal{B}(\mathrm{II})$, $\mathcal{B}(\mathrm{IV})$, $\mathcal{B}(\mathrm{VII}_0)$ and $\mathcal{B}(\mathrm{VII}_h)$ are obtained and some global results are established.

  My contribution: With the exception of the monotonic function used in the analysis of $\mathcal{B}(\mathrm{VII}_0)$ and proofreading I have done all the work.

- ⋆ **Paper III [122]:** In this paper, past and future attractors of the dynamical systems $\mathcal{B}(\mathrm{VI}_0)$ and $\mathcal{B}(\mathrm{VI}_h)$ are obtained. and some global results are established.

  My contribution: With the exception of proofreading and discussions I have done all the work.

- ⋆ **Paper IV [123]:** In this paper a unique shear-free cosmological model (of Bianchi type III) is discussed. The propagation of light is shown to be isotropic and the dynamical evolution of this model is shown to be the same as that of an FLRW model.

  My contribution: In this paper I have contributed in writing, making figures and discussions of the results. I also performed an extensive bit of local stability analysis of $\mathcal{B}(\mathrm{III})$ relevant for the project.

The final paper honors observations as the true judge of scientific correctness. The paper concerns gravitational lensing, which might be used as a probe on the distribution of (dark) matter in the Universe. Hence, an increased understanding of gravitational lensing might literally shed observational light on potential anisotropies in the (dark) matter distribution.

- **Paper V [124]:** In this paper we prove recursion relations that apply in the weak-field, thin-lens regime of the Roulette-formalism for gravitational lensing. These relations relate lower-order Roulette-amplitudes to higher-order ones, and serve to make computations exceedingly more economic.

  My contribution: Everything except initialising the project and proof-reading.

5.4 Main results

This section provides a brief overview of the main results obtained in this dissertation, whereupon we compare with relevant literature. In the following we shall for brevity use $\mathcal{B}_{\mathrm{S}}$ to refer to the union of all the invariant sets corresponding to the solvable Bianchi types *except* the type $\mathrm{VI}^{*}_{-1/9}$ and VI_{-1}. Id est;

$$\mathcal{B}_{\mathrm{S}} = \mathcal{B}(\mathrm{I}) \cup \mathcal{B}(\mathrm{II}) \cup \mathcal{B}(\mathrm{IV}) \cup \mathcal{B}(\mathrm{V}) \cup \mathcal{B}(\mathrm{VI}_0) \cup \mathcal{B}(\mathrm{VI}_{\tilde{h}}) \cup \mathcal{B}(\mathrm{VII}_0) \cup \mathcal{B}(\mathrm{VII}_h) \tag{5.1}$$

where $\tilde{h} = \{h < 0 |\, h \neq -1 \cup -1/9\}$. Also, we shall in most of this chapter refer to the same type of matter sector. We therefore define

$$\Omega \equiv \Omega_{j\mathrm{f}} + \Omega_{\mathrm{pf}}, \tag{5.2}$$

in accordance with (the summary of) the previous chapter. Recall that $\Omega_{j\mathrm{f}}$ is the energy-density of the j-form and Ω_{pf} is the energy-density of the perfect fluid with barotropic equation of state and equation-of-state-paramter $0 \leq \gamma \leq 2$.

Remark The results obtained for the various invariant sets have varied, and far from all the sets have resulted in global conclusions. From a dynamical-systems point of view, this distinction is crucial, since monotonic functions exclude the existence of closed orbits. Global solutions were mainly obtained for $\mathcal{B}(\mathrm{I})$, $\mathcal{B}(\mathrm{II})$ $\mathcal{B}(\mathrm{V})$, $\mathcal{B}(\mathrm{VI}_0)$ and $\mathcal{B}(\mathrm{VII}_0)$, and we refer the reader to the various papers for a detailed discussion.

5.4.1 No-hair theorems

In **Paper I** the equations for GR with Ω alongside a cosmological constant Ω_Λ were written down in an orthonormal frame. From these equations, certain conditions under which all universes of Solvable Bianchi type must and will isotropize irrespective of initial conditions were obtained, and formulated in two so-called no-hair theorems. The conditions concern the inclusion of a cosmological constant, or the restriction of the γ-range, and the isotropic state that asymptotically will be reached is so-called flat de Sitter, defined in the following.

> Definition 14 (Flat de Sitter universe). A flat de Sitter Universe is a universe which is maximally symmetric with flat spatial sections ($\Sigma^2 = A^2 = |\mathbf{N}_\Delta|^2 = 0$) and for which $q = -1$.

The fact that $q = -1$ implies both that $H' = 0$ in a flat de Sitter universe and that a de Sitter solution may be reached in the solvable Bianchi types if and only if $\Omega_\Lambda = 1$ or if $\Omega_{\mathrm{pf}} = 1$, $\gamma = 0$. With this definition in mind, we summarise next the content of the two theorems (Theorems 6.1 & 6.2) in Paper I in the follow conclusion.

> *Conclusion 1 (Necessity): Except for a set of measure zero, any space-time of Solvable Bianchi type and with a matter sector Ω will asymptotically isotropize under the following circumstances.*
>
> - ⋆ *If a cosmological constant Ω_Λ is added. In this case a de flat Sitter universe is reached asymptotically.*
> - ⋆ *If the equation-of-state parameter is restricted to $0 \leq \gamma < 2/3$. In this case a so-called quasi de Sitter universe with deceleration parameter $q = \frac{3}{2}\gamma - 1 < 0$ is reached asymptotically.*

Discussion

The Conclusion 1 is not surprising, and agrees with Wald's so-called cosmic no-hair theorem [125], and extends results previously established for perfect fluids (cf. [14, Thrm. 8.2]) to include also the j-form. Such extensions are found also elsewhere. Consider for instance the case of a tilted perfect fluid, as discussed in e.g. [35,36].

5.4.2 Anisotropic hairs

For the matter content Ω in $\mathcal{B}_\text{S}$ with $\gamma > 2/3$ the results change dramatically. It is a generic result (modulo sets of measure zero) that the future attractors are anisotropic. The anisotropic attractors *Plane Waves* (PW), *Wonderland*, *Edge* (E) and *Rope* (R) are summarised in the following. Figure 5.2 gives an overview of where these sets are stable. Please note in the following that we adopt notation such that

$$(N_+, N_-, N_\times, \Sigma_+, \Sigma_-, \Sigma_\times, \Sigma_3) \to (\nu_1, \nu_2, \nu_3, \beta_1, \beta_2, \beta_3, \beta_4), \tag{5.3}$$

alongside

$$\nu^2 = \sqrt{\nu_2^2 + \nu_3^2}. \tag{5.4}$$

This notation is in accordance with [121,122], and is introduced to abbreviate notation. For further details, refer to Section 5 in **Paper II**.

Wonderland, $\mathrm{W}(\kappa, \nu_1, \nu^2)$

Wonderland exists for $2/3 < \gamma < 2$ and is a fabric of equilibrum points with subsets in all the invariant sets belonging to $\mathcal{B}_\text{S}$. Moreover, it is an attractor on all of its existence in these sets. According to the most general specification of Wonderland, the non-vanishing variables are as follows.

$$\beta_1 = \frac{1}{4}(2-3\gamma) \quad , \quad \beta_2 = -\kappa\nu_3 \quad , \quad \beta_3 = \kappa\nu_2 \quad , \quad \nu_1\nu^2 = 0 \tag{5.5}$$

$$A = -\kappa(1+\beta_1) \quad , \quad V_1^2 = -\beta_1(1+\beta_1) - \nu^2 \quad , \quad \Theta = \kappa V_1. \tag{5.5}$$

Wonderland is hence a so-called *isolated*[d] set of equilibrium points. In particular

$$\Omega_\text{pf} = \frac{3}{4}(2-\gamma)(1-\kappa^2) \qquad \text{and} \qquad q = -1 + \frac{3}{2}\gamma. \tag{5.6}$$

The family $W(\kappa, \nu_1, \nu^2)$ may be divided into several subsets that belong to different invariant sets. They are as follows.

[d] Used to denote the fact that $\Omega_\text{pf} > 0$.

- $\star$ $\mathcal{B}(I) \supset \mathcal{P}_W \equiv \lim_{\kappa,\nu_1,\nu\to 0} W(\kappa,\nu_1,\nu^2)$.
- $\star$ $\mathcal{B}(V) \supset \mathcal{P}_{W(\kappa)} \equiv \lim_{\nu_1,\nu\to 0} W(\kappa,\nu_1,\nu^2)$.
- $\star$ $\mathcal{B}(\mathrm{VII}_h) \supset \mathcal{P}_{W(\kappa,\nu_1)} \equiv \lim_{\nu\to 0} W(\kappa,\nu_1,\nu^2)$.
- $\star$ $\mathcal{B}(\mathrm{VII}_0) \supset \mathcal{P}_{W(\nu_1)} \equiv \lim_{\kappa,\nu\to 0} W(\kappa,\nu_1,\nu^2)$.
- $\star$ $\mathcal{B}(\mathrm{VI}_h) \supset \mathcal{P}_{W(\kappa,\nu^2)} \equiv \lim_{\nu_1\to 0} W(\kappa,\nu_1,\nu^2)$.
- $\star$ $\mathcal{B}(\mathrm{VI}_0) \supset \mathcal{P}_{W(\nu^2)} \equiv \lim_{\kappa,\nu_1\to 0} W(\kappa,\nu_1,\nu^2)$.

Please refer to Sec. I in **Paper II** for a more nuanced categorization.

Plane Waves, PW(β_1,ν_1,ν^2)

The Plane-Wave fabric exists for $0 \le \gamma \le 2$ and is a three-parameter family of equilibrium points, denoted PW(β_1,ν_1,ν^2), that stretches across several invariant sets. It is specified as follows.

$$-1 < \beta_1 < 0 \quad , \quad \beta_2 = \nu_3 \quad , \quad \beta_3 = -\nu_2, \tag{5.7}$$

$$A = 1+\beta_1 \quad , \quad V_1^2 = -\beta_1(1+\beta_1) - \nu^2 \quad , \quad \Theta = -V_1. \tag{5.8}$$

This implies that

$$\Omega_{\mathrm{pf}} = 0 \qquad \text{and} \qquad q = -2\beta_1. \tag{5.9}$$

The family $PW(\beta_1,\nu_1,\nu^2)$ may be divided into subsets that belong to different Bianchi invariant sets. In particular,

- $\star$ $\mathcal{B}(\mathrm{VII}_h) \supset \mathcal{P}_{PW(\beta_1,\nu_1)} \equiv \lim_{\nu\to 0} PW(\beta_1,\nu_1,\nu^2)$.
- $\star$ $\mathcal{B}(\mathrm{VI}_h) \supset \mathcal{P}_{PW(\beta_1,\nu^2)} \equiv \lim_{\nu_1\to 0} PW(\beta_1,\nu_1,\nu^2)$.
- $\star$ $\mathcal{B}(\mathrm{V}) \supset \mathcal{P}_{PW(\beta_1)} \equiv \lim_{\nu_1,\nu\to 0} PW(\beta_1,\nu_1,\nu^2)$.
- $\star$ $\mathcal{B}(\mathrm{V}) \supset \mathcal{P}_M \equiv \lim_{\beta_1,\nu_1,\nu\to 0} PW(\beta_1,\nu_1,\nu^2)$.

M is here the Milne exact vacuum solution, and we have $M \subset PW(\beta_1) \subset PW(\beta_1,\nu_1,\nu^2)$.

Rotating-vector solutions

Two special types of solutions (The Rope and The Edge) of Bianchi type I were found. Since two of the frame-rotations are specified by $\Omega_A = \varepsilon_{AB}\sigma^{1B}$ (by Eq. (2.34)), one may observe that the orthonormal frame rotates for σ^{1B} different from zero. If one now finds that the vector $\mathbf{V}$[e] stays fixed along an axis different from the axis around which the frame rotates, this is properly interpreted as a rotating vector. We have discovered equilibrium sets of type I where there are rotating vectors in the above explained sense. In the Σ_3-gauge 2.7.4 and with $\mathbf{V}$ aligned along $\mathbf{e}_1$, the specification of these are as follows (as thoroughly discussed in **Paper I**).

The Rope (R) exists for $6/5 < \gamma < 4/3$ and is stable on all of its existence. The non-vanishing variables takes the following values.

$$\beta_1 = \frac{1}{4}(2-3\gamma) \quad , \quad \beta_2 = \frac{\sqrt{3}}{4}(6-5\gamma) \quad , \tag{5.10}$$

$$\beta_4 = \pm\frac{1}{2}\sqrt{\frac{15}{2}(2-\gamma)\left(\gamma-\frac{6}{5}\right)} \quad , \quad V_1 = \frac{3}{2}\sqrt{\frac{3}{2}(2-\gamma)\left(\gamma-\frac{10}{9}\right)}. \tag{5.11}$$

This gives

$$\Omega_{\mathrm{pf}} = 6-\frac{9}{2}\gamma \qquad , \qquad q = -1+\frac{3}{2}\gamma. \tag{5.12}$$

The Edge (E) exists for $0 \leq \gamma \leq 2$ and takes over the stability from Rope at $\gamma = 4/3$ and on. The non-vanishing variables takes the following values.

$$\beta_1 = -\frac{1}{2} \quad , \quad \beta_2 = -\frac{1}{2\sqrt{3}} \quad , \quad \beta_4 = \pm\frac{1}{\sqrt{6}} \quad , \quad V_1 = \frac{1}{\sqrt{2}}. \tag{5.13}$$

This gives

$$\Omega_{\mathrm{pf}} = 0 \qquad \text{and} \qquad q = 1. \tag{5.14}$$

[e]Here meant to refer to the 3-vector degree of freedom in the j-form $\mathcal{J} = (\Theta, \mathbf{V})$. Hence $\mathbf{V} = (V_1, V_2, V_3)$.

Conclusion 2 (Vector rotation): With the matter sector Ω there exist type I self-similar cosmologies that at every point in space possess a rotating vector-field. Moreover, the sets $\mathcal{B}(I)$ and $\mathcal{B}(II)$ are future asymptotic to these cosmologies for $6/5 < \gamma < 2$.

The above conclusion is based on the results of **Paper I & II**. In Figure 5.2 the anisotropic attractors of the various invariant sets are summarised.

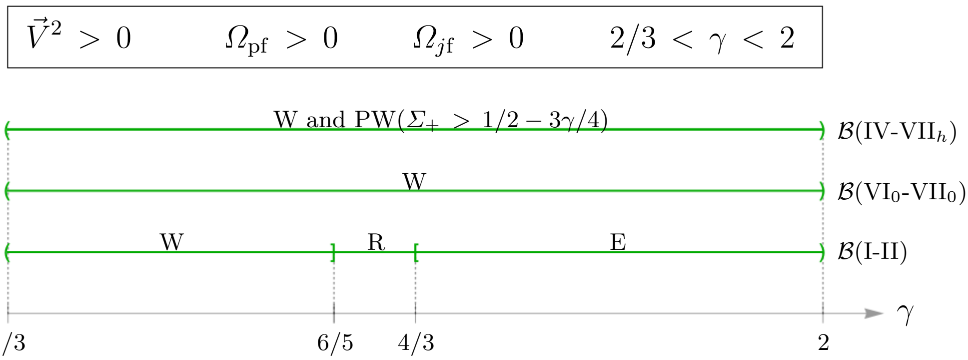

Figure 5.2: The figure summarises the stable equilibrium sets $2/3 < \gamma < 2$) in the studied models for the specifications given in the box above. Note that $\mathcal{B}$(IV-VII$_h$) is short for $\mathcal{B}$(IV)$\cup\mathcal{B}$(V)$\cup\mathcal{B}$(VI$_{\tilde{h}}$)$\cup\mathcal{B}$(VII$_h$).

Conclusion 3 (Chance): In the case where $\gamma > 2/3$ and except for a set of measure zero, isotropization will not occur for universes in $\mathcal{B}_{\rm S}$ with Ω where $\mathbf{V} \neq 0$. Moreover, anisotropic, self-similar attractors exist in all the invariant Bianchi sets belonging to $\mathcal{B}_{\rm S}$.

Discussion

As an illustration, this means that for a dust-filled ($\gamma = 1$) universe, a lasting isotropic state, like that observed around us today, is an infinitely unlikely asymptotic future in the universes belonging to $\mathcal{B}_{\rm S}$ and with a j-form. As an important side-mark, however, intermediate FLRW-like phases is far more likely. An example is provided in **Paper III**.

The results obtained here are not surprising. Also with only a non-tilted perfect fluid present, the stability picture changes for $\gamma > 2/3$. [a] [14]. The biggest difference is perhaps found for the sets $\mathcal{B}(\mathrm{I})$ and $\mathcal{B}(\mathrm{II})$. In $\mathcal{B}(\mathrm{I})$ one finds in the case of a non-tilted perfect fluid that the flat FLRW is the future attractor, whereas in $\mathcal{B}(\mathrm{II})$ it is the Collins-Stewart solution that takes over the stability for $\gamma > 2/3$. Furthermore, the Collins solution is in this case the attractor solution for $\Omega_{\mathrm{pf}} > 0$ in $\mathcal{B}(\mathrm{IV})$, $\mathcal{B}(\mathrm{VI}_0)$, $\mathcal{B}(\mathrm{VI}_{\tilde{h}})$ as well as in $\mathcal{B}(\mathrm{VII}_h)$. For $\Omega_{\mathrm{pf}} = 0$ and for $h > -(3\gamma-2)/(2-\gamma)$, however, Plane Waves take over the stability[b]. This is in perfect agreement with our analysis, which additionally includes a j-form, and where Wonderland takes over the role of the Collins solution. These matters are further discussed in **Paper III**, where it is also shown that the Wonderland-fabric reduces to precisely this solution in the limit $\Theta = V_1 = 0$.

Note that with only a perfect fluid present, and also (typically) with a homogeneous, magnetic field [49], self-similarity is broken in $\mathcal{B}(\mathrm{VII}_0)$. In the presence of the j-form, however, this self-similarity breaking is regularized, providing instead self-similar asymptotics, as further discussed in **Paper II** [121].

As mentioned, electromagnetic fields together with a non-tilted perfect fluid have also been investigated in the foregoing literature. Electro-vacuum Plane Waves are reported to be the future attractor for all expanding Universes of Bianchi-class B [53], also here under the condition $\Sigma_+ > 1/2 - 3\gamma/4$. Moreover, all models in $\mathcal{B}(\mathrm{I})$ isotropize into the future for $0 < \gamma \leq 4/3$ [51]. This is different from the j-form case, where we see from Figure 5.2 that isotropization is prevented for $\gamma > 2/3$ in $\mathcal{B}(\mathrm{I})$. Isotropization from $\gamma > 2/3$ occurs also for electromagnetic fields in $\mathcal{B}(\mathrm{II})$ [52] and $\mathcal{B}(\mathrm{VI}_0)$ [50]. The behaviour is, not surprisingly, less complicated in our case, where the most involved stability bifurcation tree is found in $\mathcal{B}(\mathrm{I})$. We have

$$\mathrm{FLRW} \xrightarrow{\gamma=2/3} \mathrm{W} \xrightarrow{\gamma=6/5} \mathrm{R} \xrightarrow{\gamma=4/3} \mathrm{E}.$$

[a] As for why the stability changes at certain γ-values, consider the physical intuition given in [36, Sec. 1].

[b] More precisely; the part of Plane Waves where $\Sigma_+ > -\frac{3}{4}\left(\gamma - \frac{2}{3}\right)$

In the magnetic case, these diagrams are more involved, as summarised in [52, Fig. 7].

Anisotropic, shear-free attractor

In **Paper IV** we study a corner of the Bianchi models that has not received so much attention; namely universes with underlying anisotropic Bianchi geometry that nevertheless expand isotropically. As pointed out in [64] such models require an isotropy-breaking matter field. By such, the matter isotropy may cancel out the geometrical anisotropy. In $\mathcal{B}$(III) there exists such a shear-free attractor with Ω, where $\Omega_{\mathrm{f}} > 0$. This solution was recently [69] proven to be the unique shear-free solution with anisotropic spatial curvature within spatially homogeneous orthogonal models that contain, in addition to a perfect fluid, a free canonical p-form gauge field. The shear-free model may be mapped onto a reference FLRW-model, and thus one cannot conclude from isotropic expansion that the background geometry is isotropic. The shear-free attractor in $\mathcal{B}$(III) is globally stable in the LRS subspace of $\mathcal{B}$(III), and locally stable with respect to *all* homogeneous perturbations.

> *Conclusion 4 (Anisotropic and shear-free): It is possible for a cosmology with Ω in an anisotorpic background to have an asymptotic shear-free yet anisotropic future, dynamically equivalent to that of an FLRW cosmology.*

5.4.3 Gravitational lensing

In **Paper V** we found recursion relations for the roulette amplitudes of the theory developed by Clarkson in [114]. In the ordinary theory, the Kaiser & Squires' relations [100] provides an efficient way to draw information from the shear field of a cluster lens. Our work generalises these relations to stronger lensing. Id est; higher order modes of distortion by the lens are calculated. In particular, by constructing the complex amplitude $\gamma_s^m = \alpha_s^m + i\beta_s^m$, we have proven that for the α_s^ms and β_s^ms given by (4.31) and (4.32), respectively, the following relations hold.

Recursion relations
$$\begin{cases} \gamma_{s+1}^{m+1} = D_+^+ \gamma_s^m = (C_+^+)_{s+1}^{m+1} \partial_c \gamma_s^m, \\ \gamma_{s-1}^{m+1} = D_-^+ \gamma_s^m = (C_-^+)_{s-1}^{m+1} \partial_c^* \gamma_s^m. \end{cases} \tag{5.14}$$

Here $\Box = \partial_c \partial_c^*$ and $\partial_c = \partial_\mathrm{x} + i\partial_\mathrm{y}$, where x, y are coordinates in the lens-plane cf. Fig. 4.1. The mth order coefficients of spin s coefficients are such that

$$(C_+^+)_s^m = 2^{\delta_{0(s-1)}} \frac{m+1}{m+1+s}\chi \qquad \text{and} \qquad (C_-^+)_s^m = 2^{-\delta_{0s}} \frac{m+1}{m+1-s}\chi, \tag{5.15}$$

The $+$ and $-$ signs are hence there to indicate whether we add or subtract to the number m (upper index) and s (lower index). Moreover, this result allowed us to rewrite the horrendous formulas (4.31) and (4.32) (which involved the integrals (4.34) and (4.33)) to the neat little inch

$$\gamma_s^m = \mathit{\Gamma}_s^m \Box^{a^-} \partial_c^s \psi. \tag{5.16}$$

Here $a^- = (m+1-s)/2$ and $\mathit{\Gamma}_s^m$ are numerical coefficients given by

$$\mathit{\Gamma}_s^m = \begin{cases} -(2^{-\delta_{0s}}) \frac{\chi^{m+1}}{2^m} \binom{m+1}{a^-} & m+s \quad \text{odd}, \\ 0 & \text{else}. \end{cases} \tag{5.16}$$

Sums and integrals are thus dismissed from the expressions. Finally; the recursion-relations makes it possible to express the derivatives of the lensing potential in terms of the roulette-modes, as follows.

$$\begin{aligned}
\psi_{\mathrm{x}} &= -\frac{1}{\chi}\alpha_1^0 \quad , \quad \psi_{\mathrm{y}} = -\frac{1}{\chi}\beta_1^0 \quad , \quad \psi_{\mathrm{xx}} = -\frac{1}{\chi^2}\left(\alpha_0^1 + \alpha_2^1\right) \quad , \\
\psi_{\mathrm{yy}} &= -\frac{1}{\chi^2}\left(\alpha_0^1 - \alpha_2^1\right) \quad , \quad \psi_{\mathrm{xy}} = -\frac{1}{\chi^2}\beta_2^1 \quad , \\
\psi_{3\mathrm{x}} &= -\frac{1}{\chi^3}\left(\alpha_1^2 + \alpha_3^2\right) \quad , \quad \psi_{3\mathrm{y}} = -\frac{1}{\chi^3}\left(\beta_1^2 - \beta_3^2\right) \quad , \\
\psi_{\mathrm{xyy}} &= -\frac{1}{3\chi^3}\left(\alpha_1^2 - 3\alpha_3^2\right) \quad , \quad \psi_{\mathrm{yxx}} = -\frac{1}{3\chi^3}\left(\beta_1^2 + 3\beta_3^2\right) \quad , \\
&\text{etc. .}
\end{aligned} \tag{5.17}$$

For brevity we used, in the above, notation such that $\partial_{\mathrm{x}}^n \partial_{\mathrm{y}}^m \psi \equiv \psi_{n\mathrm{x}m\mathrm{y}}$, where n, m are positive integers.

> *Conclusion 5: The Kaiser & Squires inversion procedure may, through the Roulette formalism, be extended to include higher-order terms, by such treating stronger lensing effects than those captured by convergence and shear.*

Paper I

Bianchi cosmologies with p-form gauge fields

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“ Why, sometimes I've believed as many as six impossible things before breakfast.

Lewis Carroll, *Through the looking-glass*

Appendix A

A physicist's guide to mathematical jargon

In this appendix the concepts most crucially underpinning this study are discussed and sometimes defined, all for reference purposes. Most of the following material is borrowed from [13, 15].

A.1 Exterior calculus

Take in the following an n-dimensional space. A p-form $\mathcal{P}$ is a totally antisymmetric tensor whose components have 'lower' indices. Using the wedge-product[c] $\wedge$ we write

$$\mathcal{P} = \frac{1}{p!}\mathcal{P}_{\mu_1\cdots\mu_p}\boldsymbol{\omega}^{\mu_1} \wedge \cdots \wedge \boldsymbol{\omega}^{\mu_p} \,. \tag{A.1}$$

In an n-dimensional space, one must have $p \leq n$. In such a space, the n-form is for this reason referred to as a top-form. Since top-forms can only have one component[d] all top-forms must be proportional. The volume-form is a top-form defined in the following manner.

$$\eta = \frac{1}{n!}\sqrt{|g|}\varepsilon_{\mu_1\cdots\mu_n}\boldsymbol{\omega}^{\mu_1} \wedge \cdots \wedge \boldsymbol{\omega}^{\mu_n}. \tag{A.2}$$

Here $|g|$ is the absolute value of the determinant of the metric tensor $\mathbf{g}$ and $\varepsilon_{\mu_1\cdots\mu_n}$ is the standard antisymmetric symbol of rank n. The Hodge dual $\star\mathcal{P}$ of the p-form $\mathcal{P}$ is now an $(n-p)$-form constructed from the

[c] The wedge-product signals that only totally antisymmetric combinations are counted.
[d] Namely the one corresponding to the basis-element $\boldsymbol{\omega}^1 \wedge \cdots \wedge \boldsymbol{\omega}^n$.

contraction of the volume-form with the p-form. That is,

$$\begin{aligned}\star\mathcal{P} &= \frac{1}{p!(n-p)!}\eta_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{n-p}}\mathcal{P}^{\mu_1\cdots\mu_p}\boldsymbol{\omega}^{\nu_1}\wedge\cdots\wedge\boldsymbol{\omega}^{\nu_{n-p}} \\ &= \frac{1}{(n-p)!} * \mathcal{P}_{\mu_1\cdots\mu_{n-p}}\boldsymbol{\omega}^{\mu_1}\wedge\cdots\wedge\boldsymbol{\omega}^{\mu_{n-p}}.\end{aligned} \tag{A.3}$$

The exterior derivative $\mathbf{d}$ is a mapping from the space of $(p-1)$-forms to the space of p-forms. Taking ∇ to represent the covariant derivative, we have for a $(p-1)$-form $\mathcal{K}$ that

$$\mathbf{d}\mathcal{K} = \frac{1}{(p-1)!}\nabla_{\mu_1}\mathcal{K}_{\mu_2\cdots\mu_p}\boldsymbol{\omega}^{\mu_1}\wedge\cdots\wedge\boldsymbol{\omega}^{\mu_p}\,. \tag{A.4}$$

Since both $\mathbf{d}\mathcal{K}$ and $\mathcal{P}$ are p-forms, we could have the relation $\mathcal{P}=\mathbf{d}\mathcal{K}$ If this relation is fulfilled we say that $\mathcal{P}$ is **exact**. Note that for p-forms, the exterior derivative has the property that

$$\mathbf{d}^2\mathcal{P} = 0. \tag{A.5}$$

This, however, is not generally true for *vectorial* p-forms. For more on this, and how to apply exterior calculus to general relativity, consult [13, Chpt. 6]. A p-form that fulfills the relation $\mathbf{d}\mathcal{P}=0$ is said to be **closed**. Hence, by (A.5), all exact p-forms are closed. The converse is however not always true and hence *Poincaré's lemma* comes in handy.

Lemma A.1 (Poincaré). *For any star-shaped*[a] *open set U there will, for any closed p-form $\mathcal{P}$, exist a $(p-1)$-form $\mathcal{K}$ such that $\mathcal{P}=\mathbf{d}\mathcal{K}$.*

[a]With star-shaped is here meant any region that is homomorphic to a region in a Euclidean space that has a point that can be connected to any other point in the region by a straight line.

A.2 A manifold $\mathcal{M}$ and its tangent- and cotangent spaces

Definition 15 (Manifold). A manifold $\mathcal{M}$ is a space satisfying the following properties:

1) There exists a family of open neighbourhoods U_i together with continuous one-to-one mappings $f_i : U_i \longmapsto \mathbb{R}^n$ with a continuous inverse for a number n.
2) The family of open neighbourhoods cover the whole of $\mathcal{M}$. We write

$$\mathcal{M} = \bigcup_i U_i. \tag{A.6}$$

One may want to attach the notion of vectors to points on the manifold. For instance, if a river flow on the surface of the Earth, its flow may be described by a vector field that uniquely assigns a vector to every point on the manifold (the surface of the Earth). These vectors will, however, not live on the manifold, but in the *tangent space* $T_p\mathcal{M}$ of every point p in the manifold $\mathcal{M}$. Consider all possible curves in $\mathcal{M}$ through p,and take the union of the tangent vectors of all these curves at a point p. The span of these vectors is a basis for $T_p\mathcal{M}$.

In a similar fashion, the co-tangent space $T_p^*\mathcal{M}$ is the space where the one-forms ($p = 1$ in (A.1)) live. Take $\{\mathbf{e}_\mu\}$ to be a basis for $T_p\mathcal{M}$. Then we take the one-forms $\{\boldsymbol{\omega}^\nu\}$ to be a basis for $T_p^*\mathcal{M}$, related to $\{\mathbf{e}_\mu\}$ by the relation

$$\boldsymbol{\omega}^\nu(\mathbf{e}_\mu) = \delta_\mu^\nu. \tag{A.7}$$

A.3 The orbit of a point p

Definition 16 (Diffeomorphism). Given two manifolds $\mathcal{M}$ and $\mathcal{N}$ a differentiable map $f : \mathcal{M} \mapsto \mathcal{N}$ is called a diffeomorphism if it is a bijection[a] and if also its inverse $f^{-1} : \mathcal{N} \mapsto \mathcal{M}$ is differentiable.

[a]There is a one-to-one correspondence between every element in $\mathcal{M}$ and $\mathcal{N}$.

Consider next a vector-field $\mathbf{v}$. The path $\phi(p,t)$ of a point p is such that

$$\frac{\partial \phi}{\partial t} = \mathbf{v}_{\phi(p,t)} \quad , \quad \phi(p,0) = p. \tag{A.8}$$

I.e; the vector field at a certain point $\phi(p,t)$ along the path gives the tangent of the path of p at that point[a]. Let us denote the path $\phi(p,t) \equiv \phi_t(p)$ for a fixed t. Then, for any t, $\phi_t(p)$ is a group element acting on p. As such, the path represents a one-parameter group of diffeomorphisms written in a coordinate-independent manner. Such diffeomorphisms depend only on the underlying vector-field $\mathbf{v}$, and are very useful in physics.

Definition 17 (Orbits of a group). A vector field $\mathbf{v}$ generates a one-parameter group of transformations (and vice versa). The *orbits* of this group are the integral curves of $\mathbf{v}$.

Definition 18 (Orbit of a point). The orbit of a point p in a manifold $\mathcal{M}$ under the group G is the set of all points into which p is mapped when all elements of G act on p.

A.4 Push-forward and pull-back

Having a smooth map $F : \mathcal{M} \to \mathcal{N}$ between two manifolds $\mathcal{M}$ and $\mathcal{N}$, one may induce a way to relate vectors in $T_p\mathcal{M}$ to vectors in $T_{f(p)}\mathcal{N}$, and similarely one-forms on $T^*_p\mathcal{M}$ to one-forms on $T^*_{f(p)}\mathcal{N}$.

Definition 19 (Pull-back and push forward). For all differentiable functions F one may define the push-forward F_* for a vector $\mathbf{v}$ and the pull-back F^* for a one-form $\boldsymbol{\alpha}$. Let $F : \mathcal{M} \mapsto \mathcal{N}$, and introduce a local

[a]The intuition must be a leaf on water. The leaf just follows the current (i.e. the vector field) wherever it goes.

set of coordinates $\{x^\mu\}$ on $\mathcal{M}$ such that the function F is expressed as $y^\mu = F^\mu(x)$ and thus becomes a local set of coordinates on $\mathcal{N}$. Then

$$\begin{aligned} F_*\mathbf{v} &= v^\beta \frac{\partial y^\mu}{\partial x^\beta}\frac{\partial}{\partial y^\mu} \\ F^*\boldsymbol{\alpha} &= \frac{\partial y^\mu}{\partial x^\beta}\alpha_\mu \mathbf{dx}^\beta \end{aligned} \tag{A.9}$$

where the latter equation follows from the requirement that $(F^*\boldsymbol{\alpha})(\mathbf{v}) = \boldsymbol{\alpha}(F_*\mathbf{v})$. Note that for a function f we have $(F^*f)(x) = f(y(x))$, and for $\boldsymbol{\alpha} = \mathbf{dy}^\nu$ the latter of the above formulae becomes the chain rule. Thus, in essence, we can think of this as coordinate transformations. The name 'pull-back' comes from the fact that the tensor defined on $\mathcal{N}$ is 'pulled back' to a tensor on $\mathcal{M}$. Similarely for 'push forward'.

A.5 Lie transport and killing-vectors

Consider next the concept of change of a tensor along a vector field. The Lie derivative $\pounds$ of a covariant[b] tensor $\mathbf{T}$ with respect to a vector-field $\mathbf{X}$ is defined as

$$\pounds_\mathbf{X}\mathbf{T} = \lim_{t\to 0}\frac{1}{t}\left(\phi_t^*\mathbf{T}_{\phi_t(x)} - \mathbf{T}_x\right). \tag{A.10}$$

Similarely, the Lie derivative of a contravariant tensor[c] $\mathbf{Y}$ with respect to the vector-field $\mathbf{X}$ is:

$$\pounds_\mathbf{X}\mathbf{Y} = \lim_{t\to 0}\frac{1}{t}\left(\phi_{-t*}\mathbf{Y}_{\phi_t(x)} - \mathbf{Y}_x\right) \tag{A.11}$$

We thus find that the Lie derivative relates to transport of tensors along vector fields. Since we push forward the vector at one point to a vector at a later point, it should not come as a surprise that the Lie derivative can be related to *parallel transport*, and thus the covariant derivative of a vector. Specifically, for a vector $\mathbf{Y}$ we have

$$\pounds_\mathbf{X}\mathbf{Y} = \left(X^\nu Y^\mu{}_{;\nu} - Y^\nu X^\mu{}_{;\nu}\right) = [\mathbf{X},\mathbf{Y}] = -\pounds_\mathbf{Y}\mathbf{X}. \tag{A.12}$$

[b]Lower indices on the components.
[c]Upper indices on the components.

Note also that

$$\pounds_{\mathbf{e}_\mu}\mathbf{Y} = \mathbf{Y}^{\alpha}{}_{,\mu}\mathbf{e}_\alpha. \tag{A.13}$$

The Lie derivative thus expresses the change of a vector as one moves along the integral curve of an underlying vector field.

> Definition 20 (Definition of Lie transport:). We say that a tensor $\mathbf{T}$ is Lie transported along a curve whose tangent vector field is $\mathbf{U}$ iff
>
> $$\pounds_{\mathbf{U}}\mathbf{T} = 0. \tag{A.14}$$

Now let $\mathbf{g}$ be the metric tensor.

> Definition 21 (Killing-vector field:). $\boldsymbol{\xi}$ is said to be a Killing vector-field if
>
> $$\pounds_{\boldsymbol{\xi}}\mathbf{g} = 0, \tag{A.15}$$
>
> where $\mathbf{g}$ is the metric tensor.

Appendix B

Relativistic decomposition of a velocity field

Take $\mathbf{u}$ to denote the four-velocity field and take the four-acceleration to be

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tilde{\tau}}, \tag{B.1}$$

where $\tilde{\tau}$ is the proper time. Denoting with ; the covariant derivative, and using $(\dot{})$ to denote differentiation with respect to proper time we have

$$\dot{u}_\alpha = a_\alpha = u_{\alpha;\mu}u^\mu. \tag{B.2}$$

The projection-operator $h_{\mu\nu}$ projects tensors onto the plane of simultaneity orthogonal to the four-velocity $\mathbf{u}$. The covariant derivative of the four-velocty may now be written

$$u_{\alpha;\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \dot{u}_\alpha u_\beta, \tag{B.3}$$

where θ is the expansion scalar, $\sigma_{\alpha\beta}$ is the shear tensor and $\omega_{\alpha\beta}$ is the vorticity tensor. These are defined such that

$$\theta = u^\mu_{;\mu}, \tag{B.4}$$

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)} - \frac{1}{3}u^\mu_{;\mu}h_{\alpha\beta} + \dot{u}_{(\alpha}u_{\beta)}, \tag{B.5}$$

$$\omega_{\alpha\beta} = u_{[\alpha;\beta]} + \dot{u}_{[\alpha}u_{\beta]}. \tag{B.6}$$

Here square brackets in the indices denote antisymmetric combination and ordinary brackets denote symmetric combination. Freely moving particles move along geodesics; curved space-time's answer to straight lines. Hence, in the case of no external forces we have

$$\mathbf{a} = 0. \tag{B.7}$$

In this thesis we assume that the fundamental observers move freely. Moreover, we assume co-motion, such that

$$\mathbf{u} = \partial_t. \tag{B.8}$$

Taking this together with (B.7) and using that $\theta = 3H$, where H is the Hubble parameter, one finally has

$$\theta = 3H, \tag{B.9}$$

$$u_{(\alpha;\beta)} = \sigma_{\alpha\beta} + H h_{\alpha\beta}, \tag{B.10}$$

$$\omega_{\alpha\beta} = 0. \tag{B.11}$$

Note also that the *expansion tensor* is

$$\theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}. \tag{B.12}$$

Appendix C

Conventions

The notation used in the paper is such that

$$x_{ab} = \begin{pmatrix} -2x_+ & \sqrt{3}x_2 & \sqrt{3}x_3 \\ \sqrt{3}x_2 & x_+ + \sqrt{3}x_- & \sqrt{3}x_\times \\ \sqrt{3}x_3 & \sqrt{3}x_\times & x_+ - \sqrt{3}x_- \end{pmatrix} \tag{C.1}$$

where x_{ab} is one of the traceless matrices π_{ab} or σ_{ab} (their normalized equivalents Π_{ab} and Σ_{ab} have the same structure). For the considered Bianchi type I-VII$_h$ models n_{ab} can always be written on the form

$$n_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n_+ + \sqrt{3}n_- & \sqrt{3}n_\times \\ 0 & \sqrt{3}n_\times & n_+ - \sqrt{3}n_- \end{pmatrix}. \tag{C.2}$$

Appendix D

Standard irreducible decomposition

Let $T^{\mu\nu}$ be the components of a rank (2,0) tensor and let $h_{\mu\nu}$ be the projection onto the hypersurfaces orthogonal to the 4-velocity u^{μ}. That is to say; we decompose the full metric $g_{\mu\nu}$ according to

$$g_{\mu\nu} = h_{\mu\nu} - u_{\mu}u_{\nu}. \tag{D.1}$$

Since u_{μ} is time-like, $h_{\mu\nu}$ will always represent spatial sections. As usual, we define the lower components according to

$$T_{\mu\nu} = T^{\alpha\beta} g_{\alpha\mu} g_{\beta\nu}. \tag{D.2}$$

Then, by (D.1), we find

$$T_{\mu\nu} = T^{\alpha\beta}(h_{\alpha\mu} - u_{\alpha}u_{\mu})(h_{\beta\nu} - u_{\beta}u_{\nu}). \tag{D.3}$$

Expanding the brackets we obtain

$$T_{\mu\nu} = h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} T_{\alpha\beta} + u_{\mu}u_{\nu}u^{\alpha}u^{\beta}T_{\alpha\beta} - u_{\mu}h^{\beta}{}_{\nu}u^{\alpha}T_{\alpha\beta} - u_{\nu}u^{\beta}h^{\alpha}{}_{\mu}T_{\alpha\beta}. \tag{D.4}$$

Projecting onto $u^{\mu}u^{\nu}$: The energy density ρ is the scalar quantity we observe in the comoving frame. It must be

$$\rho \equiv u^{\alpha}u^{\beta}T_{\alpha\beta}. \tag{D.5}$$

Projecting one index onto u^{μ} and one onto $h_{\alpha\mu}$: To get the energy flow q_{ν} we project one 'leg' on each side. We find

$$q_{\nu} \equiv -h_{\nu}{}^{\alpha}u^{\beta}T_{\alpha\beta} \tag{D.6}$$

The spatial part: The part of $T^{\mu\nu}$ projected onto spatial sections is

$$h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} T_{\alpha\beta}. \tag{D.7}$$

These will therefore give the purely space-like components of $T_{\mu\nu}$. More specifically, the isotropic pressure p is now given as the trace, whereas the rest, $\pi_{\mu\nu}$, represents shear. Hence

$$p = \frac{1}{3} h^{\mu\nu} T_{\mu\nu} \tag{D.8}$$

and

$$\pi_{\mu\nu} \equiv h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} T_{\alpha\beta} - p\, h_{\mu\nu}. \tag{D.9}$$

With these definitions we may rewrite $T_{\mu\nu}$.

Symmetric $T^{\mu\nu}$: Any symmetric tensor $T^{\mu\nu}$ may now be decomposed such that

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + p\, h_{\mu\nu} + 2q_{(\mu} u_{\nu)} + \pi_{\mu\nu}. \tag{D.10}$$

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