A mathematical approach to Wick rotations

by

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Thesis submitted in fulfilment of
the requirements for the degree of
PHILOSOPHIAE DOCTOR
(PhD)

University
of Stavanger

Faculty of Science and Technology
Department of Mathematics and Physics

2019

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www.uis.no

ISBN: Click to enter ISBN.
ISSN: Click to enter ISSN.
PhD: Thesis UiS No. Click to enter PhD No.

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ABSTRACT. In this thesis we define Wick-rotations mathematically using pseudo-Riemannian geometry, and relate Wick-rotations to real geometric invariant theory (GIT). We discover some new results concerning the existence of Wick-rotations (of various signatures). For instance we show that a Wick-rotation of a pseudo-Riemannian space (at a fixed point p) to a Riemannian space forces the space to be *Riemann purely electric* (RPE). We also define *compatibility* among representations and relate them to real GIT and Wick-rotations. The polynomial curvature invariants of pseudo-Riemannian spaces are also considered and related to Wick-rotations.

Wick-rotations of a special class of pseudo-Riemannian manifolds (M, g) are also studied; namely Lie groups G equipped with left-invariant metrics. We prove some new results concerning the existence of real slices (of Lie algebras) of certain signatures of a holomorphic inner product space $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ (on a complex Lie algebra). The definition of a Cartan involution for a semisimple Lie algebra is defined for a general Lie algebra equipped with a pseudo-inner product: $(\mathfrak{g}, g)$, and the theorems of Cartan (concerning Cartan involutions) are generalised and proved. For instance we prove that a pseudo-Riemannian Lie group (G, g) can be Wick-rotated to a Riemannian Lie group $(\tilde{G}, \tilde{g})$ if and only if there exist a Cartan involution of the Lie algebra $\mathfrak{g}$.

Acknowledgements

I would like to thank the University of Stavanger (UiS), and my main advisor Sigbjørn Hervik for guiding me through the Ph.D project; thanks for always being available for whatever questions I had, and for letting me pursue the directions I wanted. Also a thanks to my second advisor Boris Kruglikov for questions I had during a stay at the University of Tromsø.

Part 1. Introduction

Wick-rotations arise from physics, named after *Gian Carlo Wick*, and is a mathematical trick of reducing a problem in Minkowski space to a problem of Euclidean space. This is done by transforming the Minkowski metric (which is a Lorentzian metric) to a Euclidean metric, and often the problem reduced to the Euclidean case will be easier to solve. Thus Wick-rotations are for instance of interest in fields like quantum mechanics, statistical mechanics and Euclidean gravity and thereon. One can ask of the limitations of such Wick-rotations, for when one can preform such a trick, and for a general spacetime one can ask the question if such a Wick-rotation to a Euclidean space is even possible.

In this thesis we consider a mathematical approach to Wick-rotations in the framework of pseudo-Riemannian geometry. A motivation for our approach also comes from the theory of Lie groups, i.e the example of a complex semisimple Lie group $G^{\mathbb{C}}$ equipped with its Killing form $-\kappa(\cdot,\cdot)$. The space $(G^{\mathbb{C}},-\kappa)$ is a holomorphic Riemannian space, and the real forms $G\subset G^{\mathbb{C}}$ give rise to pseudo-Riemannian spaces with real-valued metrics at the identity restricted from the complex Killing form. The real metrics are then the real Killing forms on the Lie algebra $\mathfrak{g}\subset\mathfrak{g}^{\mathbb{C}}$. Moreover a compact real form $U\subset G^{\mathbb{C}}$ gives rise to a Riemannian space in this way.

$$\begin{tikzcd}
(G^{\mathbb{C}},-\kappa) & (U,-\kappa_{|\mathfrak{u}}) \arrow[l, "Riemannian"'] \\
(G,-\kappa_{|\mathfrak{g}}) \arrow[u, "Pseudo"'] &
\end{tikzcd}$$

Thus we consider a general pseudo-Riemannian space $(\mathcal{M},g)$ of some signature (p,q), and consider the question: *When does there exist a Riemannian manifold $(\tilde{\mathcal{M}},\tilde{g})$ Wick-rotated to $(\mathcal{M},g)$?*

More generally we consider an arbitrary pseudo-Riemannian space $(\tilde{\mathcal{M}},\tilde{g})$ (not necessarily Riemannian) of some (possibly different) signature $(\tilde{p},\tilde{q})$ and ask the question of when it can be Wick-rotated to $(\mathcal{M},g)$. An interesting special case to consider are pseudo-Riemannian Lie groups (G,g), i.e Lie groups equipped with left-invariant metrics (not necessarily semisimple groups).

We are thus interested in finding necessary or sufficient conditions that enables a Wick-rotation to different signatures, with a special interest in the case of Wick-rotating to a Riemannian space.

A mathematical motivation for studying Wick-rotations in this thesis also comes from the classification of pseudo-Riemannian spaces, and that of the *polynomial curvature invariants* of pseudo-Riemannian spaces. At a point $p\in\mathcal{M}$, these invariants are special polynomial invariants of an action of a pseudo-orthogonal group $O(p,q)$ on a tensor space $\mathcal{V}$, restricted to the Riemann tensor R and its covariant derivatives $\nabla^k R$ up to some kth order viewed as vectors inside $\mathcal{V}$. The polynomial curvature invariants are related to the *Cartan equivalence principle* which relates the curvature tensors to the metric.

For example suppose we are given two pseudo-Riemannian spaces $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$, and we impose that they have the same polynomial curvature invariants, then we may ask: *How are* $(\mathcal{M}, g)$ *and* $(\tilde{\mathcal{M}}, \tilde{g})$ *related?* and *Under what conditions are the metrics Wick-rotated?*

The parts of the thesis is structured as follows.

(2) We provide some known concepts and mathematical tools used in the later parts. This part contains no original results.
(3) We define Wick-rotations by considering pseudo-Riemannian manifolds as real slices of a holomorphic Riemannian manifold. From a frame bundle viewpoint Wick-rotations between different pseudo-Riemannian spaces can then be studied through their structure groups which are real forms of the corresponding complexified Lie group (different real forms $O(p,q)$ of the complex Lie group $O(n,\mathbb{C})$). In this way, we can use real GIT (geometric invariant theory) to derive several new results regarding the existence, and non-existence, of such Wick-rotations. As an explicit example, we Wick rotate a known G_2-holonomy manifold to a pseudo-Riemannian manifold with split-G_2 holonomy.
(4) We show that a metric of arbitrary dimension and signature which allows for a standard Wick-rotation to a Riemannian metric necessarily has a purely electric Riemann and Weyl tensor.
(5) Motivated by Wick-rotations of pseudo-Riemannian manifolds, we study real geometric invariant theory (GIT) and compatible representations. We extend some of the results from earlier works [20, 21], in particular, we give some sufficient as well as necessary conditions for when pseudo-Riemannian manifolds are Wick-rotatable to other signatures. For arbitrary signatures, we consider a Wick-rotatable pseudo-Riemannian manifold with closed $O(p,q)$-orbits, and thus generalise the existence condition found in [21]. Using these existence conditions we also derive an invariance theorem for Wick-rotations of arbitrary signatures.
(6) We study Wick-rotations of left-invariant metrics on Lie groups, using results from real GIT ([20], [19]). An invariant for Wick-rotation of Lie groups is given, and we describe when a pseudo-Riemannian Lie group (a Lie group with a left-invariant metric) can be Wick-rotated to a Riemannian Lie group. We define a Cartan involution of a general Lie algebra, and prove a general version of É. Cartan's result, namely the existence and conjugacy of Cartan involutions.

(7) Continuing with the ideas of Part 6, we consider a holomorphic inner product space on a complex Lie algebra: $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$. We prove some new results regarding the existence of a compact real form (of Lie algebras) $\mathfrak{u} \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$, i.e $\mathfrak{u}$ is a real form with Euclidean signature: $g^{\mathbb{C}}(\mathfrak{u}, \mathfrak{u}) > 0$.

(8) The bibliography of the thesis.

Part 2. Preliminaries

1. Cartan involutions of linear Lie groups

In this section we introduce some known concepts and tools that we use throughout this thesis. The Lie algebras we speak of in this section, shall be defined over $\mathbb{R}$ or $\mathbb{C}$. We shall distinguish by saying real (complex), the same goes for Lie groups.

We recall that any Lie algebra $\mathfrak{g}$ has a Killing form: $\kappa(\cdot,\cdot)$, defined by:

$$\kappa(x,y) := tr\Big(ad(x)\circ ad(y)\Big),\ \ x,y\in\mathfrak{g},$$

where $\mathfrak{g}\xrightarrow{ad(x)}\mathfrak{g}\in End(\mathfrak{g})$, is defined by $ad(x)(y) := [x,y],\ \ x,y\in\mathfrak{g}$. The Killing form κ is easily seen to be *associative* or *invariant*, i.e it satisfies $\kappa([x,y],z) = \kappa(x,[y,z])$ for all $x,y,z\in\mathfrak{g}$.

Definition 1.1. A Lie algebra is said to be *semisimple* if $\kappa(\cdot,\cdot)$ is non-degenerate. A Lie group G with Lie algebra $\mathfrak{g}$ is said to be *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple.

Definition 1.2. A Lie algebra is said to be *simple* if it is non-abelian and does not have any non-trivial ideals.

A Lie algebra can be shown to be semisimple if and only if it does not have any non-trivial abelian ideals, thus solvable Lie algebras do not belong to this class. It is also a standard result that any semisimple Lie algebra can written into a direct sum of simple ideals [16].

Definition 1.3. A *Cartan involution* of a semisimple Lie algebra is an involution of Lie algebras: $\mathfrak{g}\xrightarrow{\theta}\mathfrak{g}$, such that $-\kappa(\cdot,\theta(\cdot))$ is positive definite.

The eigenspace decomposition w.r.t a Cartan involution is often called a *Cartan decomposition*. As an example consider the semisimple real Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, then the map $x\mapsto -x^t$, is a Cartan involution, and thus a Cartan decomposition is given by:

$$\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{so}(2)\oplus\{x\in\mathfrak{sl}_2(\mathbb{R})|x^t = x\}.$$

By a result of Cartan, every semisimple Lie algebra has a Cartan involution, and moreover any two Cartan involutions are conjugate by an inner automorphism (see for example [16]):

Theorem 1.4. *Any real semisimple Lie algebra has a Cartan involution which is unique up to conjugation.*

By Ado's theorem every Lie algebra is linear, thus for a general Lie algebra one can define the notion of a Cartan involution:

Definition 1.5. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a real Lie subalgebra. Then a *Cartan involution* of $\mathfrak{g}$ is an involution of Lie algebras that restricts from an involution $\mathfrak{gl}(V) \xrightarrow{\theta} \mathfrak{gl}(V)$ such that $\theta_{|_{\mathfrak{sl}(V)}}$ is a Cartan involution of $\mathfrak{sl}(V)$.

It is a fact that any Cartan involution of $\mathfrak{sl}(V)$ has the form $x \mapsto -x^*$, where $\langle -, - \rangle$ is some inner product on V with $\langle x^*(v_1), v_2 \rangle = \langle v_1, x(v_2) \rangle$ for all $v_1, v_2 \in V$. Thus the Cartan involutions of $\mathfrak{gl}(V)$ also have this form.

One shall note that the definition of a Cartan involution of a semisimple real Lie algebra coincides with the previous definition. Indeed if $\mathfrak{g} \subset \mathfrak{gl}(V)$ is semisimple and θ is a Cartan involution in the sense of Definition 1.3, then $\mathfrak{g} \subset \mathfrak{sl}(V)$ and by using Mostow's Theorem ([33], Thm 6) one can find a Cartan involution of $\mathfrak{sl}(V)$ extending θ. Conversely if θ is a Cartan involution of $\mathfrak{gl}(V)$ leaving $\mathfrak{g}$ invariant, then it follows that θ is a Cartan involution of $\mathfrak{g}$ in the sense of Definition 1.3 ([16], Ch. IX, Lem. 2.2).

Definition 1.6. A Lie algebra $\mathfrak{g}$ is said to be *reductive* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$ where $[\mathfrak{g}, \mathfrak{g}]$ is either trivial or a semisimple Lie algebra.

For example $\mathfrak{gl}_n(\mathbb{R})$ is reductive with $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{sl}_n(\mathbb{R}) \oplus \langle I_n \rangle$, and an example of a Cartan involution is the map $x \mapsto -x^t$.

More generally a Lie subalgebra $\mathfrak{g} \subset \mathfrak{h}$ is said to be reductive in $\mathfrak{h}$ if the representation of $\mathfrak{g}$:

$$x \cdot y := [x, y] \in \mathfrak{h}, \;\; x \in \mathfrak{g}, \;\; y \in \mathfrak{h},$$

is completely reducible.

Clearly the real Lie algebras for which a Cartan involution exist belong to the class of reductive Lie algebras. Moreover if θ is a Cartan involution of $\mathfrak{g}$, and $\mathfrak{z}(\mathfrak{g}) = V_+ \oplus V_-$ is the eigenspace decomposition w.r.t θ, then:

$$-\kappa(\cdot, \theta(\cdot)) + B(\cdot, \theta(\cdot)) > 0$$

is positive definite, where κ is the Killing form of $[\mathfrak{g}, \mathfrak{g}]$ and B is a pseudo-inner product (i.e a non-degenerate symmetric bilinear form) on $\mathfrak{z}(\mathfrak{g})$ of signature (p, q) for $p := Dim(V_+)$ and $q := Dim(V_-)$.

Recall that a *quadratic Lie algebra* $\mathfrak{g}$ is a Lie algebra equipped with an invariant pseudo-inner product $g(\cdot, \cdot)$, i.e g is a symmetric non-degenerate bilinear form on $\mathfrak{g}$ satisfying: $g([x, y], z) = g(x, [y, z]), \forall x, y, z \in \mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is said to be *compact* if there exist such a g which is also an inner product, i.e g is also positive definite. For example the class of Lie algebras: $\mathfrak{g}$, with a Cartan involution $\theta = 1$, are compact; by noting the above pseudo-inner product: $-\kappa(-,-) + B(-,-)$.

For linear Lie groups one defines:

Definition 1.7. Let $G \subset GL(V)$ be a real linear Lie group, then an involution $G \xrightarrow{\Theta} G$ of Lie groups is said to be a *Cartan involution* if Θ restricts from a Cartan involution of $GL(V)$ (i.e the differential is a Cartan involution of $\mathfrak{gl}(V)$).

If θ is a Cartan involution of a semisimple Lie algebra of a Lie group G (not necessarily linear) then there exists a unique involution $G \xrightarrow{\Theta} G$ with differential θ.

Thus for a general semisimple Lie group one have the notion of a Cartan involution (see for example [12]):

Theorem 1.8. *Let G be a semisimple real Lie group, and θ a Cartan involution of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then there exist a unique involution of Lie groups: $G \xrightarrow{\Theta} G$ satisfying:*

1. *The differential of Θ at $1 \in G$ is θ, i.e $d\Theta = \theta$.*
2. *The fix points of Θ denoted by $K \subset G$ has Lie algebra $\mathfrak{k}$ and $k \in K$ are those elements of G satisfying $[Ad(k), \theta] = 0$.*
3. *K is compact if and only if $Z(G_0)$ is finite and G has finitely many connected components (fcc).*
4. *There is a diffeomorphism $K \times \mathfrak{p} \to G$ given by $(k, x) \mapsto ke^x$.*

For example if we consider the semisimple linear group: $SL_2(\mathbb{R})$, then the map $g \mapsto (g^{-1})^t$, is a Cartan involution, with fix-points $K = SO(2)$. Another example is $SL_2(\mathbb{C})$ viewed as a real Lie group, then the map $g \mapsto (g^{-1})^\dagger$ is a Cartan involution with fix-points $K = SU(2)$.

There are many distinct definitions of a *reductive Lie group*, however some authors define a *real reductive Lie group* to be a linear Lie group $G \subset GL(V)$ together with a Cartan involution Θ (see [29]). For example a *real reductive algebraic group* $G \subset GL(V)$, is a real algebraic group such that $\mathfrak{g}$ is reductive in $\mathfrak{gl}(V)$, and it belongs to this class of real reductive Lie groups (see [35]).

Let $G \subset GL(V)$ be such a real linear Lie group with a Cartan involution Θ, and $\theta := d\Theta$ be the Cartan involution of its Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denotes the Cartan decomposition w.r.t θ (i.e the eigenspace decomposition), and $K \subset G$ is the fix points of Θ, then we can globally write $G = Ke^{\mathfrak{p}}$, where K is maximally compact and has Lie algebra $\mathfrak{k}$. Now any Cartan involution of $GL(V)$ has the form $A \mapsto (A^{-1})^*$ where

$$\langle A^*(v_1), v_2 \rangle = \langle v_1, A(v_2) \rangle,$$

for some inner product on V. Thus $K \subset O(V, \langle \cdot, \cdot \rangle)$, and $e^{\mathfrak{p}}$ consists of symmetric operators w.r.t $\langle \cdot, \cdot \rangle$. Therefore such groups G are also *self-adjoint*, i.e $G^* = G$.

Definition 1.9. Let $\mathfrak{g}^{\mathbb{C}}$ denote a complex Lie algebra, then a real Lie subalgebra $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is said to be a *real form* if there exist a conjugation map σ (i.e σ is a real involution satisfying $\sigma(ix) = -ix$) with fix points $\mathfrak{g}$.

Thus as a real Lie algebra we may write: $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. All the real forms up to isomorphism of a complex semisimple Lie algebra are in bijection with the conjugacy classes of involutions of the complex Lie algebra (see [12]):

$$\{[\mathfrak{g}] | \mathfrak{g} \textit{ a real form}\} \to \{[\theta] | \theta \textit{ an involution}\}.$$

The map is well-defined by sending a real form $\mathfrak{g}$ to a complexified Cartan involution $\theta^{\mathbb{C}}$. Thus from the complex Killing form perspective: $\kappa(\cdot,\cdot)$, the existence of an involution θ of $\mathfrak{g}^{\mathbb{C}}$ gives rise to a real form $\mathfrak{g}$ which is real-valued on $\kappa(\cdot,\cdot)$, i.e $\kappa(\mathfrak{g},\mathfrak{g}) \in \mathbb{R}$. The restriction is in fact just the real Killing form of $\mathfrak{g}$.

For example $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}(2)$ are all the real forms (up to isomorphism) of $\mathfrak{sl}_2(\mathbb{C})$; by noting the conjugation maps: $x \mapsto \bar{x}$, and $x \mapsto -x^{\dagger}$.

Any complex semisimple Lie algebra has a special real form which is unique up to isomorphism, namely a real form $\mathfrak{u}$ for which the Cartan involution is just $\theta = 1$, also called a *compact real form*. Thus the restriction of the complex Killing form: $-\kappa(\cdot,\cdot)$ to $\mathfrak{u}$ is of Euclidean signature.

For the following result, see for example [16]:

Theorem 1.10. *Any complex semisimple Lie algebra has a compact real form. Moreover up to isomorphism a compact real form is unique.*

If $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ is a compact real form then the conjugation map τ of $\mathfrak{u}$ is a Cartan involution of $\mathfrak{g}^{\mathbb{C}}$ viewed as a real Lie algebra. Thus a Cartan decomposition is given by: $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$. Moreover the real Killing form of $\mathfrak{g}^{\mathbb{C}}$ is just $2Re(\kappa(\cdot,\cdot))$, where κ is the complex Killing form and Re is the real part.

One shall also note that if $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is a semisimple real form, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, then $\mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$.

A simple real Lie algebra $\mathfrak{g}$ fall into one of the following two classes [40]; either its complexification $\mathfrak{g}^{\mathbb{C}}$ is simple in which case all of its real forms are simple, or $\mathfrak{g}$ has a complex structure $\mathfrak{g} \xrightarrow{J} \mathfrak{g}$, in which case $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$ where $\mathfrak{g}$ is viewed as the complex Lie algebra constructed by J. Recall that a complex structure is an endomorphism J on $\mathfrak{g}$ satisfying $J^2 = -1$ and $[J(x), y] = J([x, y])$ for all $x, y \in \mathfrak{g}$. For example viewing the complex simple Lie algebra: $\mathfrak{sl}_2(\mathbb{C})$, as a real Lie algebra denoted: $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$, then the latter real simple Lie algebra has a complex structure and is a real form of $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

Definition 1.11. Let G be a real Lie group, then the *universal complexification group* of G is a pair $(G^{\mathbb{C}}, \eta)$, where $G^{\mathbb{C}}$ is a complex Lie group, and $G \xrightarrow{\eta} G^{\mathbb{C}}$ is a real Lie homomorphism, satisfying the universal property. This means that for any real Lie homomorphism $G \xrightarrow{\psi} H^{\mathbb{C}}$ into a complex Lie group $H^{\mathbb{C}}$ there is a unique Lie homomorphism $G^{\mathbb{C}} \xrightarrow{l} H^{\mathbb{C}}$, such that the following diagram commutes:

$$\begin{tikzcd} G^{\mathbb{C}} \arrow[r, "l"] & H^{\mathbb{C}} \\ G \arrow[u, "\eta"] \arrow[r, "\psi"] & H^{\mathbb{C}} \arrow[u, "1"'] \end{tikzcd} \tag{1}$$

For example consider the complex semisimple linear Lie group $SL_2(\mathbb{C})$, then it is the universal complexification group of $SU(2)$ and of the universal covering group $\widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$. Note that $\widetilde{SL_2(\mathbb{R})}$ is not linear. Another example is if we view $SL_2(\mathbb{C})$ as a real Lie group of dimension 6, and use that $SL_2(\mathbb{C})$ is simply connected, then the universal complexification group is just the product: $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$.

Definition 1.12. ([31]) A *linearly complex reductive Lie group* $G^{\mathbb{C}}$ is a complex Lie group that is the universal complexification of a compact real Lie group G.

If $(G^{\mathbb{C}}, \eta)$ is the universal complexification of a compact group U (i.e $G^{\mathbb{C}}$ is linearly complex reductive) then η is injective with closed image, thus by identifying $U \cong \eta(U) \subset G^{\mathbb{C}}$, then U is a compact real form (see defn below), and moreover there is a diffeomorphism [31]:

$$U \times \mathfrak{u} \to G^{\mathbb{C}}, \; (u, x) \mapsto ue^{ix}.$$

One shall note that the linearly complex reductive Lie groups are precisely the complex reductive algebraic groups, for more details on these groups see for example [34, 31]. For example a 1-dimensional complex tori $\mathbb{C}^{\times}$ is a linearly complex reductive Lie group which is the universal complexification of the circle S^1.

There are many distinct definitions in the literature of a *real form* of a complex Lie group, however we shall occasionally use this one [12]:

Definition 1.13. A real Lie subgroup G of a complex Lie group $G^{\mathbb{C}}$ is said to be a *real form* if $\mathfrak{g}$ is a real form of the Lie algebra of $G^{\mathbb{C}}$, and moreover as a group product we have $G^{\mathbb{C}} = GG^{\mathbb{C}}_0$ where $G^{\mathbb{C}}_0$ is the identity component.

Note when $G^{\mathbb{C}}$ is connected then for a real Lie group $G \subset G^{\mathbb{C}}$ to be a real form is just the condition that the Lie algebra $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$.

Definition 1.14. A real form $U \subset G^{\mathbb{C}}$ shall be called a *compact real form* if U is compact.

For example, if a complex Lie group (with finitely many components (fcc)) have a compact real form then it must belong to the class of linearly complex reductive Lie groups [31]. All the semisimple complex Lie groups belong to this class.

2. Geometric invariant theory (GIT)

Our fields $\mathbb{K}$ of interest in this section are $\mathbb{C} \supset \mathbb{R}$. We begin by defining some preliminary concepts (see for example [48]). An *affine variety* or an *algebraic set* X shall mean a subset $X \subset \mathbb{K}^n$ of the form:

$$X = \{x \in \mathbb{K}^n | (\forall f \in I)(f(x) = 0)\} := Z_{\mathbb{K}}(I),$$

for some ideal $I \subset \mathbb{K}[X_1, \ldots, X_n]$. Set $\mathcal{I}_{\mathbb{K}}(X)$ for the set of all the polynomials $f \in \mathbb{K}[X_1, \ldots, X_n]$ such that $f(X) = 0$.

Definition 2.1. An algebraic set $X \subset \mathbb{C}^n$ is said to be *defined over* $\mathbb{R}$ if $\mathcal{I}_{\mathbb{C}}(X)$ is generated by real polynomials in $\mathbb{R}[X_1, \ldots, X_n]$.

Thus in such a case the real points $X(\mathbb{R}) := X \cap \mathbb{R}^n$ is a real algebraic subset of $\mathbb{R}^n$, such that the Zariski-closure of $X(\mathbb{R})$ in $\mathbb{C}^n$ is X. On the other hand if $X \subset \mathbb{R}^n$ is algebraic then the Zariski-closure of X in $\mathbb{C}^n$, denoted $X^{\mathbb{C}}$ (often called a complexification of X) is defined over $\mathbb{R}$, and if $\mathcal{I}_{\mathbb{R}}(X) = \langle f_1, \ldots, f_k \rangle$ then also $\mathcal{I}_{\mathbb{C}}(X^{\mathbb{C}}) = \langle f_1, \ldots, f_k \rangle$.

Definition 2.2. Let X, Y be complex algebraic sets both defined over $\mathbb{R}$, and $X \xrightarrow{F} Y$ be a morphism. Then F is said to be defined over $\mathbb{R}$ if F restricts to a real morphism: $X(\mathbb{R}) \xrightarrow{F_{|X(\mathbb{R})}} Y(\mathbb{R})$.

One shall also note that the real polynomial algebra $\mathbb{R}[X(\mathbb{R})]$ is a real form of $\mathbb{C}[X]$, by noting the conjugation map:

$$\mathbb{C}[X] \ni f \mapsto \bar{f} \in \mathbb{C}[X].$$

For a real algebraic group $G \subset GL(E)$ the Zariski-closure in $GL(E^{\mathbb{C}})$ is a complex algebraic group $G^{\mathbb{C}}$, and is often called an *algebraic complexification* of G. One shall note that $G \subset G^{\mathbb{C}}$ is a real form.

If $G^{\mathbb{C}}$ is a complex algebraic group defined over $\mathbb{R}$ and $G^{\mathbb{C}}$ acts on a complex affine variety X, also defined over $\mathbb{R}$, then the action is said to be defined over $\mathbb{R}$ if the map $G^{\mathbb{C}} \times X \to X$ is defined over $\mathbb{R}$. Thus in such a case there is a real action of $G(\mathbb{R})$ on $X(\mathbb{R})$, and if $\mathbb{C}[X]^{G^{\mathbb{C}}}$ denotes the $G^{\mathbb{C}}$-invariant polynomials of the action, then the conjugation map above leaves the algebra invariant, and it is straightforward to show that the corresponding real form is precisely the real polynomial invariants: $\mathbb{R}[X(\mathbb{R})]^{G(\mathbb{R})}$.

Geometric invariant theory (GIT) in algebraic geometry (first developed by *D. Mumford*) is concerned with the problem when an algebraic group G acts on a variety X, and the question of when there exist a quotient $\left(\frac{X}{G}, X \to \frac{X}{G}\right)$ where $\frac{X}{G}$ is a variety and $X \to \frac{X}{G}$ a morphism satisfying certain geometrical properties. One special case to consider is when a complex algebraic group $G^{\mathbb{C}}$ acts rationally on a complex vector space $V^{\mathbb{C}} \cong \mathbb{C}^k$. We shall recall some results over $\mathbb{C}$.

Let $G^{\mathbb{C}}$ be a linearly complex reductive Lie group (for example a semi-simple complex Lie group), and $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ be any representation of Lie groups. Then $\rho^{\mathbb{C}}$ is also a rational representation w.r.t its unique algebraic group structure. Let $\mathbb{C}[V^{\mathbb{C}}]$ denote the coordinate ring of polynomials of $V^{\mathbb{C}}$, and $\mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}}$, denote the subalgebra of $G^{\mathbb{C}}$-invariant polynomials, i.e $f \in \mathbb{C}[V^{\mathbb{C}}]$ satisfying

$$f(g \cdot v) = f(v), \ \forall g \in G^{\mathbb{C}}, \ v \in V^{\mathbb{C}}.$$

Then the following theorem hold [49]:

Theorem 2.3 (Hilbert)**.** *The polynomial ring of invariants:* $\mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}}$*, is a finitely generated* $\mathbb{C}$*-algebra.*

Let $\mathcal{I} := \{f_1, \dots, f_N\}$ be a finite generating set for $\mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}}$, and define a map:

$$V^{\mathbb{C}} \xrightarrow{p} \mathbb{C}^N, \quad v \mapsto (f_1(v), \dots, f_N(v)).$$

Let Y denote its closed image in $\mathbb{C}^N$, then Y is an affine variety with coordinate ring: $\mathbb{C}[Y] \cong \mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}}$. The map p is a *good categorial quotient*, i.e the following hold:

Theorem 2.4 ([49]). *$V^{\mathbb{C}} \xrightarrow{p} Y$ is a good categorial quotient:*

(1) p is surjective.
(2) If $W_1 \subset V^{\mathbb{C}} \supset W_2$ are closed and $G^{\mathbb{C}}$-invariant, and $W_1 \cap W_2 = \emptyset$ then $p(W_1) \cap p(W_2) = \emptyset$.
(3) If $W \subset V^{\mathbb{C}}$ is closed and $G^{\mathbb{C}}$-invariant, then $p(W) \subset Y$ is also closed.
(4) p is a $G^{\mathbb{C}}$-invariant morphism, i.e $p(gv) = p(v)$ for all $g \in G^{\mathbb{C}}$ and $v \in V^{\mathbb{C}}$.
(5) For any open subset $U \subset Y$ there is an isomorphism: $\mathbb{C}[U] \xrightarrow{p^*} \mathbb{C}[p^{-1}(U)]^{G^{\mathbb{C}}}$, i.e p is a categorial quotient.

The pair $(Y, V^{\mathbb{C}} \xrightarrow{p} Y)$ is often simply called a *GIT quotient* and Y is denoted by $Y := \frac{V^{\mathbb{C}}}{\overline{\overline{G^{\mathbb{C}}}}}$. One shall note by case (2) that if $y \in Y$, then there is a unique closed orbit in $p^{-1}(y)$ i.e $G^{\mathbb{C}}v \subset p^{-1}(y)$. Thus one can think of Y as the set of all closed orbits of the $G^{\mathbb{C}}$-action, by the bijection: $y \mapsto G^{\mathbb{C}}v$. Note that $\frac{X}{\overline{\overline{G}}}$ is not necessarily the orbit space of the action, i.e p does not necessarily satisfy:

$$p(v_1) = p(v_2) \Leftrightarrow Gv_1 = Gv_2, \ \forall v_1, v_2 \in V^{\mathbb{C}}.$$

Indeed this criterion would require that all the orbits of the action of G are closed, however if this is true then such an action is said to be closed, and locally the action is always closed; in fact there always exist a G-invariant open subset $U \subset V^{\mathbb{C}}$ such that the restriction: $U \xrightarrow{p_{|U}} p(U)$, gives rise to a GIT quotient: $\left(U, p_{|_U}\right)$, that is also an orbit space ([49]).

We now recall some results of GIT over $\mathbb{R}$ used in this thesis from [7, 35, 39]. Let $G^{\mathbb{C}} \subset GL(E^{\mathbb{C}})$ be a linearly complex reductive Lie group defined over $\mathbb{R}$, and denote the real points: $G^{\mathbb{C}}(\mathbb{R}) := G^{\mathbb{C}} \cap GL(E)$, which is an algebraic real form. Now consider any closed Lie subgroup $G \subset G^{\mathbb{C}}(\mathbb{R})$ containing the identity component $G^{\mathbb{C}}(\mathbb{R})_0$ such that the Zariski-closure in $GL(E^{\mathbb{C}})$ is $G^{\mathbb{C}}$. Let $G^{\mathbb{C}}$ act rationally on a complex vector space $V^{\mathbb{C}}$, and assume the action is defined over $\mathbb{R}$. Thus the action restricts to a Lie group action of G on V. These are the assumptions of *Richardson* and *Slowody* [7]. A class of real Lie groups G obtained in this way are for instance the class of real linear semisimple Lie groups which are fcc [39].

For such type of Lie groups G, we have the following theorem:

Theorem 2.5 ([7]). *Let $G \subset GL(E)$ be as above. Then the following statements hold:*

(1) There exist a global Cartan involution of $GL(E)$ leaving G invariant.

(2) *If* $\mathfrak{gl}(E) \xrightarrow{\theta} \mathfrak{gl}(E)$ *is a Cartan involution leaving* $\mathfrak{g}$ *invariant, then* $\Theta(G) \subset G$*, where* Θ *is the global Cartan involution of* $GL(E)$ *with differential* θ*.*
(3) *All Cartan involutions of* G *are conjugate by an inner automorphism of* G*.*
(4) *Let* $G \xrightarrow{\rho_V^G} GL(V)$ *be a real representation. Then given any global Cartan involution* Θ *of* G*, then there exist a global Cartan involution* Θ' *of* $GL(V)$ *such that:* $\rho_V^G(\Theta(g)) = \Theta'(\rho_V^G(g)), \forall g \in G$*.*

In this thesis we are mainly interested in the pseudo-orthogonal groups $O(p,q)$, which are naturally real forms of $O(p+q,\mathbb{C})$. Therefore throughout this section we can take as examples $G := O(p,q)$ and $G^{\mathbb{C}} := O(p+q,\mathbb{C})$. These groups are formally defined as follows:

Let $g(-,-)$ be a real valued non-degenerate symmetric bilinear form (also called a *pseudo-inner product*) on a real finite dimensional vector space V. Then the isometry group of $g(-,-)$, denoted $O(V,g)$ is a real Lie group, and if (p,q) denotes the signature of $g(-,-)$, then we define $O(p,q) := O(V,g)$. This group is semisimple for $p+q \geq 3$, and in general it is a real reductive algebraic group. If θ denotes the involution of V such that $g(\cdot,\theta(\cdot))$ is positive definite, then the involution:

$$O(p,q) \ni f \mapsto \theta f \theta,$$

is a Cartan involution of $O(p,q)$. Thus

$$K := \{f \in O(p,q) | [\theta, f] = 0\} \subset O(p,q),$$

is a maximally compact subgroup and one can show that K is isomorphic to the product of Lie groups: $O(p) \times O(q)$.

By complexifying g to $g^{\mathbb{C}}$, then $(V^{\mathbb{C}}, g^{\mathbb{C}})$ becomes a holomorphic inner product space such that the isometries: $O(p+q,\mathbb{C})$, is a complex Lie group. By complexifying the real group $O(p,q)$ by the map:

$$f \mapsto f^{\mathbb{C}}, \ f \in O(p,q),$$

then $O(p,q)$ becomes embedded as a real form of $O(p+q,\mathbb{C})$. In fact $O(p+q,\mathbb{C})$ is the universal complexfication group of $O(p,q)$, and it is a linearly complex reductive Lie group, which is semisimple for $p+q \geq 3$. Moreover since $O(p+q,\mathbb{C}) \subset GL(V^{\mathbb{C}})$ and $O(p,q) \subset GL(V)$, then naturally the Zariski-closure of $O(p,q)$ in $GL(V^{\mathbb{C}})$ is also precisely $O(p+q,\mathbb{C})$.

If $W \subset V^{\mathbb{C}}$ is a real form which is real-valued and positive definite on $g^{\mathbb{C}}$, say $\tilde{g} := g^{\mathbb{C}}_{|W}$ then the isometries of $(W,\tilde{g})$ denoted $O(p+q)$ is a compact real form of $O(p+q,\mathbb{C})$.

Suppose $G \xrightarrow{\rho_V^G} GL(V)$ is any real representation, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, is a Cartan decomposition with global decomposition: $G = Ke^{\mathfrak{p}}$, where K has Lie algebra $\mathfrak{k}$. Then one can choose a K-invariant inner product $\langle \cdot,\cdot \rangle$ on V such that $d\rho_V^G(\mathfrak{p})$ consists of symmetric operators w.r.t $\langle \cdot,\cdot \rangle$.

One defines w.r.t ρ_V^G and $\langle \cdot,\cdot \rangle$:

Definition 2.6. A vector $v \in V$ is a *minimal vector* if $||g \cdot v|| \geq ||v||$ for all $g \in G$, where $||v||^2 := \langle v, v \rangle$.

We denote $\mathcal{M}(G, V) \subset V$ for the set of minimal vectors. As an example consider the adjoint representation $G \xrightarrow{Ad} GL(\mathfrak{g})$, with $G := O(p, q)$ $(p + q \geq 3)$. Then choosing a Cartan involution θ of $\mathfrak{g} := \mathfrak{o}(p, q)$ we may take $||v||^2 := -\kappa(v, \theta(v))$ as our inner product. Moreover it is straight forward to show that the minimal vectors are precisely those vectors $v \in \mathfrak{g}$ satisfying $[v, \theta(v)] = 0$. Thus if $\mathfrak{o}(p, q) = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition w.r.t θ, then $\mathfrak{k} \cup \mathfrak{p} \subset \mathcal{M}(G, \mathfrak{g})$.

The following theorem relates the closure (w.r.t the classical topology) of a real orbit to the set $\mathcal{M}(G, V)$:

Theorem 2.7 (Richardson, Slodowy, [7])**.** *The following statements hold:*

(1) A real orbit Gv is closed if and only if $Gv \cap \mathcal{M}(G, V) \neq \emptyset$.
(2) If v is a minimal vector then $Gv \cap \mathcal{M}(G, V) = Kv$.
(3) If Gv is not closed then there exist $p \in \mathfrak{p}$ such that $e^{tp} \cdot v \to \alpha \in V$ exist as $t \to \infty$, and $G\alpha \subset V$ is closed. Moreover $G\alpha \subset \overline{Gv}$ is the unique closed orbit in the closure.
(4) A vector $v \in V$ is minimal if and only if $\Big(\forall x \in \mathfrak{p}\Big)\Big(\langle x \cdot v, v \rangle = 0\Big)$, where $x \cdot v$ is the differential action $d\rho_V^G(x)(v)$.

One shall note that the above theorem is also proved for a class of reductive Lie groups [29] that extends the class of groups G in [7].

Let $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ denote (by assumption) the complexification of ρ i.e the following diagram commutes:

$$\begin{array}{ccc} G^{\mathbb{C}} & \xrightarrow{\rho^{\mathbb{C}}} & GL(V^{\mathbb{C}}) \\ {\scriptstyle i}\big\uparrow & & {\scriptstyle i}\big\uparrow \\ G & \xrightarrow{\rho} & GL(V) \end{array} \tag{2}$$

Let U be a compact real form of $G^{\mathbb{C}}$ such that the Lie algebra is $\mathfrak{k} \oplus i\mathfrak{p}$, then one can choose (see [7]) a U-invariant Hermitian inner product $H(\cdot, \cdot)$ on $V^{\mathbb{C}}$ which is compatible with V, i.e

$$H(V, V) \in \mathbb{R},$$

such that:

$$\mathcal{M}(G, V) \subset \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}).$$

Parts (1) and (2) of the above theorem is known as the *Kempf-Ness Theorem* for actions $\rho^{\mathbb{C}}$. One shall also note that if $G = U$ is a compact real form, then $\mathcal{M}(U, V) = V$, since $\mathfrak{p} = 0$ in this case.

The following theorem connects the real and complex case [7, 35, 39]:

Theorem 2.8. *The following statements are true:*

(1) If $v \in V$ then a real orbit $Gv \subset V$ is closed (w.r.t the classical topology) if and only if $G^{\mathbb{C}}v \subset V^{\mathbb{C}}$ is closed.
(2) If $v \in V$ then $G^{\mathbb{C}}v \cap V = \cup_{j=1}^{N} Gv_j$ for some natural number $N \geq 1$.

In the case where $G = U$ is a compact real form of $G^{\mathbb{C}}$ then any real orbit $Gv \subset V$ is closed and the following special case of the previous theorem holds:

Theorem 2.9 ([7]). *If $G = U$ is a compact real form of $G^{\mathbb{C}}$ then for any $v \in V$ we have $G^{\mathbb{C}}v \cap V = Gv$.*

Part 3. Wick-rotations and real GIT

The following part is precisely the published paper in the Journal of Geometry and Physics:

Wick-rotations and real GIT, C. Helleland, S. Hervik, *J. Geom. Phys. 123 (2018) 343-361*, https://doi.org/10.1016/j.geomphys.2017.09.009.

Abstract. We define Wick-rotations by considering pseudo-Riemannian manifolds as real slices of a holomorphic Riemannian manifold. From a frame bundle viewpoint Wick-rotations between different pseudo-Riemannian spaces can then be studied through their structure groups which are real forms of the corresponding complexified Lie group (different real forms $O(p,q)$ of the complex Lie group $O(n,\mathbb{C})$). In this way, we can use real GIT (geometric invariant theory) to derive several new results regarding the existence, and non-existence, of such Wick-rotations. As an explicit example, we Wick rotate a known G_2-holonomy manifold to a pseudo-Riemannian manifold with split-G_2 holonomy.

1. Introduction

In this paper we will study so-called Wick-rotations which were first used in physics as a mathematical trick relating Minkowski space to flat Euclidean space. Here we will generalise the concept of Wick-rotations to more general pseudo-Riemannian geometries by considering the complexification to a holomorphic Riemannian manifold. The real pseudo-Riemannian manifolds will now manifest themselves as real slices of the complex holomorphic geometry. Utilizing this description we define Wick-rotations of pseudo-Riemannian manifolds as well as the stronger concept of a *standard* Wick rotation.

There are previous works considering complex geometry and Wick-rotations in different contexts [1, 2, 25, 4, 5, 6]. Here, we will define the Wick-rotations based on observations made in [25] which is related to the definition of *Wick-related spaces* given in [6]. In fact, we adopt this definition but define the stronger concepts of *Wick-rotations* and *standard Wick-rotations*. This enables us to connect the study of Wick-rotations to real GIT [35, 7] which recently has seen its appearence in the classification of pseudo-Riemannian geometries [8, 9]. Using old, as well as some new, results from real GIT we give several results regarding the possibily of Wick-rotating pseudo-Riemannian spaces to different signatures (see also [21]).

In this paper we will reserve the notion of *Riemannian space* to the case when the metric is positive definite (of signature $(++..+)$). The Lorentzian case is the case of signature $(-++..+)$. Note also the existence of the "anti-isometry" which switches the sign of the metric: $g \mapsto -g$. This anti-isometry induces the group isomorphism $O(p,q) \to O(q,p)$ and hence our results are independent under this map.

2. Mathematical Preliminaries

Let E be a complex vector space. By considering only scalar multiplications by $\mathbb{R} \subset \mathbb{C}$, we can define a real vector space $E_{\mathbb{R}}$ whose points are identical to E. Multiplication with i in E defines an automorphism $J : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}$ satisfying:

$$J \circ J = -\mathrm{Id}. \tag{3}$$

An endomorphism J satisfying eq.(3) is called a *complex structure* of $E_{\mathbb{R}}$. This correspondence between the complex vector space E and the real vector space $E_{\mathbb{R}}$ equipped with the complex structure J defines an isomorphism of these categories.

If we consider a complex vector subspace $W \subset E$, then its corresponding real vector space $W_{\mathbb{R}}$ is a vector subspace of $E_{\mathbb{R}}$ being invariant under J. On the other hand, if W is a vector subspace of $E_{\mathbb{R}}$ being invariant under J then we say that W is a *complex subspace*.

The linear map J can also be extended to the complexification $E^{\mathbb{C}} := E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The complexification $E^{\mathbb{C}}$ can then be split into the sum of the eigenspaces:

$$E^{1,0} = \{u \in E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \big| Ju = iu\}$$

and

$$E^{0,1} = \{u \in E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \big| Ju = -iu\}.$$

2.1. **Real form of a complex vector space.** A complex vector space E may be the complexification of a real vector space W:

$$E = W^{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C},$$

in which case we will call W a *real form* of E.

Assume now that W is a vector space over $\mathbb{R}$. Then W is naturally a real form of the complexification: $W^{\mathbb{C}}$, indeed the field extension $\mathbb{R} \hookrightarrow \mathbb{C}$ induces the inclusion $W \hookrightarrow W^{\mathbb{C}}, \quad w \mapsto w \otimes 1$. Furthermore, complex conjugation in $\mathbb{C}$ gives rise to an anti-linear involution ρ of $W^{\mathbb{C}}$:

$$\rho(w \otimes z) = w \otimes \bar{z}.$$

The fixed-point set of ρ is W. Such a map is called a *conjugation map* of $W^{\mathbb{C}}$ associated to W.

Some special examples of conjugation maps can be easily found among semi-simple complex Lie algebras. For example $\mathfrak{sl}(2, \mathbb{C})$ has real forms $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$ associated to the conjugation maps $X \mapsto -X^{\dagger}$ and $X \mapsto \bar{X}$ respectively.

Definition 2.1. Let E be a complex vector space with a complex structure $J : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}$. Then real linear subspace $W \subset E_{\mathbb{R}}$ is called totally real if $W \cap J(W) = 0$.

In particular we see that if W is a maximal totally real subspace, then W is a real form of E. We note that if W is totally real implies that the composition

$$W^{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow E^{\mathbb{C}} \longrightarrow E^{1,0} \cong E,$$

where the second map is the projection onto $E^{1,0}$, is injective. This is important to us in the following as this implies that there is a connection between the complexification of real pseudo-Riemannian geometry and holomorphic Riemannian geometry. We note also that if W is a real form of E then the complex dimension of E and the real dimension of W are equal.

2.2. **Real slices.**

Definition 2.2. A *holomorphic inner product* is a complex vector space E equipped with a non-degenerate complex bilinear form g.

For a holomorphic inner product space E we can always choose an orthonormal basis. By doing so we can identify E with $\mathbb{C}^n$ and the holomorphic inner product can be written as

$$g_0(X,Y) = X_1Y_1 + ... + X_nY_n, \tag{4}$$

where $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$.

Using this orthonormal basis it is also convenient to consider the group of transformations leaving the holomorphic inner product invariant. Consider a complex-linear map $A : E \longrightarrow E$. Using an orthonormal basis, we can represent the map by a complex matrix $\mathsf{A} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. Requiring that $g_0(A(X), A(Y)) = g_0(X,Y)$, for all X, Y, implies that $\mathsf{A}^t\mathsf{A} = 1$. Consequently, the matrix A must be a complex orthogonal matrix; i.e., $\mathsf{A} \in O(n, \mathbb{C})$.

Definition 2.3. Given a holomorphic inner product space (E, g). Then if $W \subset E$ is a real linear subspace for which $g\big|_W$ is non-degenerate and real valued, i.e., $g(X,Y) \in \mathbb{R}$, $\forall X, Y \in W$, we will call W a *real slice.*

Some standard examples of real slices can be found by considering the holomorphic inner product space $(\mathbb{C}^n, g_0)$ with standard basis $\{e_1, ..., e_n\}$. The real subspace:

$$W = \mathbb{R}^n_p := \mathrm{span}\{ie_1, ..., ie_p, e_{p+1}, ..., e_n\}, \tag{5}$$

is a real slice for any $0 \leq p \leq n$. The restriction of g_0 to W in this case is the standard pseudo-Euclidean metric with signature $(p, n-p)$. Using the standard coordinates $z_k = x_k + iy_k$ for $\mathbb{C}^n$, we see that the restriction $x_1 = ... = x_p = y_{p+1} = ... = y_n = 0$ gives us the real slice $\mathbb{R}^n_p$.

Let us assume that W and $\widetilde{W}$ are real slices of $(\mathbb{C}^n, g_0)$. Consider the real slice W with real non-degenerate bilinear form h. By choosing a pseudo-orthonormal basis, we can write:

$$h(X,Y) = -X_1Y_1 - ... - X_pY_p + X_{p+1}Y_{p+1} + ... + X_nY_n, \tag{6}$$

where $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ (real) for some p. This space has complexification $W^{\mathbb{C}}$ by allowing X and Y to be complex n-tuples. By restricting $g_0(-,-)$ to $W^{\mathbb{C}}$, we see that $W^{\mathbb{C}}$ is $\mathbb{C}^n$ in an orthonormal frame. Doing the same for $\widetilde{W}$ we note that $(\widetilde{W})^{\mathbb{C}}$ is also $\mathbb{C}^n$ in (possibly another) orthonormal frame. However, since orthonormal frames are related by the action of the group $O(n,\mathbb{C})$, the real slices W and $\widetilde{W}$ are related via the action of the group $O(n,\mathbb{C})$ on $\mathbb{C}^n$. Indeed, since any n-dimensional complex holomorphic inner product space (E,g) can be identified with $(\mathbb{C}^n, g_0)$, any two real slices of E are related through the action of $O(n,\mathbb{C})$ on E.

Definition 2.4. Let $W \subset (E,g)$ be a real slice. We say an involution $W \xrightarrow{\theta} W$, is a *Cartan involution* of W, if $g_\theta(\cdot,\cdot) := g\big|_W(\cdot,\theta(\cdot))$, is an inner product on W.

Of course Cartan involutions always exist as linear maps, and the definition generalises the notion of a Cartan involution of a semi-simple Lie algebra. Indeed special examples can be found within semi-simple real forms $\mathfrak{g}$ of a complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$, which are real slices w.r.t the holomorphic Killing form: $-\kappa(-,-)$, on the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. So a Cartan involution: $\mathfrak{g} \xrightarrow{\theta} \mathfrak{g}$, will in this case in fact be an involution of Lie algebras, and will be unique up to conjugation by inner automorphisms of $\mathfrak{g}$. Explicit examples of real slices are the pseudo-orthogonal real Lie algebras $\mathfrak{o}(p,q)$ of $\mathfrak{o}(n,\mathbb{C})$ with signatures $\Big(\binom{p}{2} + \binom{q}{2}, 2pq\Big)$.

It is also important to note the following (see e.g., [6]):

Proposition 2.5. *The real slices of a holomorphic inner product space are totally real subspaces.*

2.3. **Compatible real forms.** Associated to any real form W of a complex vector space $E \cong W^{\mathbb{C}}$ we know that there is a conjugation map $E \xrightarrow{\sigma} E$ with fix points W. The space E may have another real form $\widetilde{W}$, also with a conjugation map $\tilde{\sigma}$ which fixes pointwise $\widetilde{W}$. So we have the notion of *compatibility* among two real forms in the following definition:

Definition 2.6. The two real forms W and $\widetilde{W}$ of E are said to be compatible if their conjugation maps commute, i.e $[\sigma, \tilde{\sigma}] = 0$.

For two compatible real forms W and $\widetilde{W}$ of E we may write:

$$W = (W \cap \widetilde{W}) \oplus (W \cap i\widetilde{W}) \ \ \textit{and} \ \ \widetilde{W} = (W \cap \widetilde{W}) \oplus (\widetilde{W} \cap iW).$$

In the case of Lie algebras the real forms will have conjugation maps which are also real Lie homomorphisms. As an example consider the real forms $\mathfrak{o}(p,q)$ and $\mathfrak{o}(\tilde{p},\tilde{q})$ embedded into $\mathfrak{o}(n,\mathbb{C})$ with $n = p + q = \tilde{p} + \tilde{q}$, with corresponding conjugation maps:

$$X \mapsto -I_{p,q}\bar{X}I_{p,q}, \quad X \mapsto -I_{\tilde{p},\tilde{q}}\bar{X}I_{\tilde{p},\tilde{q}},$$

where $I_{p,q} := (a_{ij})$ is the $n \times n$ diagonal matrix with entries: $a_{ii} = 1$ for $1 \leq i \leq p$, and $a_{ii} = -1$ for $p+1 \leq i \leq n$. It is easy to see that $\mathfrak{o}(p,q)$ is compatible with $\mathfrak{o}(\tilde{p},\tilde{q})$, and also observe that a Cartan involution for both real forms may be chosen to be $X \mapsto \bar{X}$, i.e the Cartan involutions also commute, and we may choose the compact real form:

$$\mathfrak{o}(n) = \{X \in \mathfrak{o}(n,\mathbb{C}) | X = \bar{X}\},$$

which will be compatible with both $\mathfrak{o}(p,q)$ and $\mathfrak{o}(\tilde{p},\tilde{q})$. This means that if

$$\mathfrak{o}(p,q) = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{o}(\tilde{p},\tilde{q}) = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}},$$

denotes the Cartan decompositions then we have:

$$\mathfrak{o}(n) = \mathfrak{k} \oplus i\mathfrak{p} = \tilde{\mathfrak{k}} \oplus i\tilde{\mathfrak{p}}.$$

We shall refer to such a triple: $\Big(\mathfrak{o}(p,q), \mathfrak{o}(\tilde{p},\tilde{q}), \mathfrak{o}(n)\Big)$, as a *compatible triple* of real forms. We define this not only for Lie algebras, but also for general real slices of the same dimension:

Definition 2.7. Let W and $\widetilde{W}$ be real slices of (E,g). Assume they are both real forms of $W^{\mathbb{C}} \subset (E,g)$. Let V be another real slice of E, and a real form of $W^{\mathbb{C}}$, with Euclidean signature. Suppose $W, \widetilde{W}$ and V are pairwise compatible, then a triple: $(W, \widetilde{W}, V)$, will be called a *compatible triple*.

Note that a compatible triple $(W, \widetilde{W}, V)$, implies that we may choose Cartan involutions of $W, \widetilde{W}$ and V which commute, this is by construction. We shall often refer to V as a *compact real slice* of E. In the case of $W \cap \tilde{W} = 0$, we note that $W = i\tilde{W}$, so this corresponds to an anti-isometry, i.e., changing the metric from: $g \mapsto -g$, a standard example is the compatible triple: $\Big(i\mathbb{R}\oplus\mathbb{R}, \mathbb{R}\oplus i\mathbb{R}, \mathbb{R}^2\Big)$, in $(\mathbb{C}^2, g)$, with $g(-,-)$ the standard holomorphic inner product. However there exist compatible triples not of this form, i.e., with $W \cap \tilde{W} \neq 0$, and to find such examples, it is sufficient to look at compatible triple of Lie algebras. In fact, we may say something stronger in the case of a compatible triple of semi-simple Lie algebras.

Indeed we now show that if we have a compatible triple of semi-simple Lie algebras $(\mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{u})$ with $\mathfrak{u}$ compact like in the example above, then the compact/non-compact parts of the Cartan decompositions of the real forms must intersect. We denote $\mathfrak{k}$ (respectively $\tilde{\mathfrak{k}}$) for the compact part, and $\mathfrak{p}$ (respectively $\tilde{\mathfrak{p}}$) for the non-compact part. This is clear if $\mathfrak{g} = \tilde{\mathfrak{g}}$, so assume they are not equal nor isomorphic.

Proposition 2.8. *Assume $\mathfrak{g} \ncong \tilde{\mathfrak{g}}$. We have $\mathfrak{k} \cap \tilde{\mathfrak{k}} \neq 0$, and if none of the real forms are compact and they are both simple then also $\mathfrak{p} \cap \tilde{\mathfrak{p}} \neq 0$.*

Proof. We may assume none of the real forms are compact, because then the first part is trivial. Suppose that $\mathfrak{t} \cap \tilde{\mathfrak{t}} = 0$, then it is easy to check that $\mathfrak{t} \subset i\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{t}} \subset i\mathfrak{p}$. Indeed if $x \in \mathfrak{t}$ then because $\mathfrak{g} \cap \tilde{\mathfrak{g}} = \mathfrak{p} \cap \tilde{\mathfrak{p}}$ then $x = p + i(\tilde{t} + \tilde{p})$ for suitable $p, \tilde{p} \in \tilde{\mathfrak{p}}$ and $\tilde{t} \in \tilde{\mathfrak{t}}$. But then,

$$x - i\tilde{p} = p + i\tilde{t} \in \mathfrak{u} \cap i\mathfrak{u},$$

and consequently $x = i\tilde{p}$. The case $\tilde{\mathfrak{t}} \subset i\mathfrak{p}$ is similar. But then $[\mathfrak{t}, \mathfrak{t}] \subset \tilde{\mathfrak{t}} \cap \mathfrak{t}$, i.e must be zero, and similarly $[\tilde{\mathfrak{t}}, \tilde{\mathfrak{t}}] \subset \tilde{\mathfrak{t}} \cap \mathfrak{t}$. So we conclude that $\tilde{\mathfrak{t}}$ and $\mathfrak{t}$ must be abelian. Now the only simple Lie algebra with abelian compact part is $\mathfrak{sl}(2, \mathbb{R})$, i.e it follows that

$$\mathfrak{g} \cong \tilde{\mathfrak{g}} \cong \oplus_j^k \mathfrak{sl}(2, \mathbb{R}),$$

for a suitable k. But since we assume $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are non-isomorphic, then this is a contradiction. Now for the second statement suppose $\mathfrak{p} \cap \tilde{\mathfrak{p}} = 0$. Then one easily checks that $\tilde{\mathfrak{p}} \subset i\mathfrak{t}$, and it is a standard result that $[\tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}] = \tilde{\mathfrak{t}}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{t}$ using that the real forms are simple, and so therefore $\tilde{\mathfrak{t}} \subset [i\mathfrak{t}, i\mathfrak{t}] \subset \mathfrak{t}$. Of course we similarly must have $\mathfrak{p} \subset i\tilde{\mathfrak{t}}$, so we conclude that $\mathfrak{t} = \tilde{\mathfrak{t}}$. Hence $\mathfrak{g} \cap \tilde{\mathfrak{g}} = \mathfrak{t} = \tilde{\mathfrak{t}}$. However this will require $\tilde{\mathfrak{p}} \subset i\mathfrak{t} = i\tilde{\mathfrak{t}}$, proving that $\tilde{\mathfrak{p}} = 0$. Hence $\tilde{\mathfrak{g}}$ is compact, which contradicts our assumptions. The proposition is proved. □

3. Holomorphic Riemannian manifolds

3.1. **Complexification of real manifolds.** We will now consider the case where we have a real pseudo-Riemannian manifold. The aim is to consider analytic continuations of such via its complexification and it is thus necessary to assume that the manifold is analytic. We will also assume that the real dimension of the real manifold, and the complex dimension of the complex manifold are equal (unless stated otherwise).

Let us first start with a few definitions (see [6]).

Definition 3.1. Given a complex manifold M with complex Riemannian metric g. If a submanifold $N \subset M$ for any point $p \in N$ we have that T_pN is a real slice of (T_pM, g) (in the sense of Defn. 2.3), we will call N a real slice of (M, g).

This definition implies that the induced metric from M is real valued on N. N is therefore a pseudo-Riemannian manifold. This further implies that real slices are totally real manifolds.

We will also define the notion of Wick-related spaces, Wick-rotated spaces, as well as a standard Wick-rotation.

Definition 3.2 (Wick-related spaces)**.** Two pseudo-Riemannian manifolds P and Q are said to be *Wick-related* if there exists a holomorphic Riemannian manifold (M, g) such that P and Q are embedded as real slices of M.

Wick-related spaces was defined in [6]. However, we also find it useful to define:

Definition 3.3 (Wick-rotation)**.** If two Wick-related spaces intersect at a point p in M, then we will use the term *Wick-rotation*: the manifold P can be Wick-rotated to the manifold Q (with respect to the point p).

Definition 3.4 (Standard Wick-rotation)**.** Let the P and Q be Wick-related spaces having a common point p. Then if the tangent spaces T_pP and T_pQ are embedded: $T_pP, T_pQ \hookrightarrow (T_pP)^{\mathbb{C}} \cong (T_pQ)^{\mathbb{C}} \hookrightarrow T_pM$ such that they form a compatible triple, then we say that the spaces P and Q are related through a *standard Wick-rotation.*

We note in the case where P and Q are Wick-rotated by a standard Wick-rotation, and Q is a real slice of Euclidean signature (i.e., it is a Riemannian space), then the tangent spaces: T_pP and T_pQ, can be embedded into $(T_pP)^{\mathbb{C}} \cong (T_pQ)^{\mathbb{C}}$, such that they are compatible real forms. Also in the case where both real slices: P and Q, are Wick-rotated of the same signatures, then they are also Wick-rotated by a standard Wick-rotation. Indeed we can identify $T_pP \cong T_pQ$ (as symmetric non-degenerate bilinear spaces), and in this case the real slices will be compatible with each other (since they are equal as sets in $(T_pP)^{\mathbb{C}}$). Moreover there is a natural compact real slice: $W \subset (T_pP)^{\mathbb{C}}$, which is compatible with T_pP.

The following proposition is immediate by definition of a compatible triple:

Proposition 3.5. *Two Wick-rotated spaces P and Q by a standard Wick-rotation gives rise to Cartan involutions of T_pP and T_pQ which commute.* $\square$

These three definitions are of increasing speciality; Wick-related spaces need not intersect at a point p; nor is there a guarantee that Wick-rotated spaces have commuting Cartan involutions. This all depends on the way the real forms are imbedded into the complexification $O(n, \mathbb{C})$.

However, in physics, all examples of Wick-rotations (known to the authors) are standard Wick-rotations in the sense above.

3.2. **Complex differential geometry.** It is useful to review some of the results from complex differential geometry especially in the holomorphic setting.

A complex Riemannian manifold is a complex manifold M equipped with a symmetric, $\mathbb{C}$-bilinear, non-degenerate form g. A vector field is holomorphic if and only if it has holomorphic component functions with respect to any local complex coordinates. The holomorphic tangent bundle TM, can be constructed using the construction $E^{1,0}$ via the complexification of TM. Similarly, a tensor field T over the holomorphic tangent bundle is holomorphic if and only if the component functions $T^{\mu_1 \dots \mu_l}{}_{\nu_1 \dots \nu_k}$ are holomorphic with respect to any local holomorphic coordinates $\{z_1, ..., z_n\}$ on M. Note also that the sum or tensor multiplication of two holomorphic tensors are holomorphic, so is the contraction of a holomorphic tensor.

For any complex Riemannian manifold there is a unique Levi-Civita connection ∇ (just as in pseudo-Riemannian case) satisfying:

$$\text{(7)} \qquad 1. \quad [X,Y] = \nabla_X Y - \nabla_Y X \qquad \text{(torsion-free)},$$

$$\text{(8)} \qquad 2. \quad \nabla_X g = 0, \qquad \text{(metric compatible)}$$

for all vector fields X and Y.

For a holomorphic metric, the Levi-Civita connection ∇ is also holomorphic (as follows from the Koszul equations), so is the Lie bracket. This implies that the holomorphic Riemann curvature tensor,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{9}$$

is also holomorphic. Hence, for a holomorphic metric, the connection, and all the curvature tensors inherit this property: they are all holomorphic.

We remark that this implies that the standard equations for computing the connection coefficients, Riemann curvature tensors etc. which are known from the pseudo-Riemannian case can be used more or less unaltered for a holomorphic Riemannian manifold. Furthermore, this has profound consequences for us as the complexification of a real pseudo-Riemannian manifold is a holomorphic Riemannian manifold. Thus, given a real slice $N \subset M$ then the curvature tensors of N are uniquely extended to a neighbourhood of N in M.

We also remark that since contractions, tensor products preserve holomorphy, polynomial curvature scalars (as considered in [25]) are also holomorphic and is uniquely determined by knowing them on a real slice. Consequently, we get the following result: If a pseudo-Riemannian space N is obtained from a pseudo-Riemannian space P by Wick-rotation w.r.t. a point p, then their polynomial curvature invariants match at p.

Thus, we note the following important facts (M is the ambient holomorphic complex Riemannian manifold) :

(1) For two Wick-related spaces, all the Riemannian curvature tensors can be obtained from M by restricting to the real slices.
(2) If a curvature tensor is identically zero for a pseudo-Riemannian manifold, N, then it is identically zero in a neighbourhood of N in M.

3.3. **Real slices from a frame-bundle perspective.** In the frame-bundle formulation of differential geometry, the Riemannian case is a frame-bundle with an $O(n)$ structure group. In general, the pseudo-Riemannian case, has a $O(p,q)$ structure group. As we saw earlier, holomorphic Riemannian geometry has $O(n,\mathbb{C})$ structure group. The relation between the real slices (with structure group $O(p,q)$) and the holomorphic Riemannian case is related through the complexification of $O(p,q)^{\mathbb{C}} \cong O(n,\mathbb{C})$.

In the Wick-rotated case, at the intersection point the different real slices with structure groups $O(p,q)$ and $O(\tilde{p},\tilde{q})$ will both be embedded in $O(n,\mathbb{C})$. Indeed if P and Q are Wick-rotated at a point $p \in M$ of the same real dimension, say

n, then T_pP and T_pQ are real slices of T_pM, of say signatures (p,q) and $(\tilde{p},\tilde{q})$ respectively. Now since these are totally real spaces, we have natural embeddings of

$$(T_pP)^{\mathbb{C}} \hookrightarrow T_pM, \ \textit{and} \ (T_pQ)^{\mathbb{C}} \hookrightarrow T_pM.$$

We can restrict the metric g on T_pM to $(T_pP)^{\mathbb{C}}$ and $(T_pQ)^{\mathbb{C}}$ so they become holomorphic inner product subspaces of T_pM. In particular since we have an isomorphism: $(T_pQ)^{\mathbb{C}} \xrightarrow{\psi} (T_pP)^{\mathbb{C}}$ (as holomorphic inner product spaces), then we have natural embeddings:

$$T_pP, T_pQ \hookrightarrow (T_pP)^{\mathbb{C}} \subset T_pM,$$

as real slices of $(T_pP)^{\mathbb{C}}$, with their signatures: (p,q) and $(\tilde{p},\tilde{q})$, respectively. Moreover when restricting to $(T_pP)^{\mathbb{C}}$ on the holomorphic metric g on M, gives another holomorphic inner product, with structure group:

$$O(n,\mathbb{C}) := \{(T_pP)^{\mathbb{C}} \xrightarrow{f} (T_pP)^{\mathbb{C}} | g(f(-),f(-)) = g(-,-)\}.$$

The pseudo-orthogonal groups: $O(p,q)$ (structure group of P) and $O(\tilde{p},\tilde{q})$ (structure group of Q), will now be embedded as real forms via ψ into $O(n,\mathbb{C})$.

A tensor x over the point $p \in P \cap Q$ w.r.t P will therefore be considered as a vector $x \in V$ for some appropriate vector space, and similarly a tensor $\tilde{x}$ w.r.t to Q over the same point p will be in another real form $\tilde{V} \subset V^{\mathbb{C}}$. This could be, for example, the Riemann tensor or covariant derivatives of the Riemann tensor, restricted to the point. If the two spaces are Wick-rotated the orbits Gx, and $\tilde{G}\tilde{x}$ where $G := O(p,q)$ and $\tilde{G} := O(\tilde{p},\tilde{q})$, are embedded into the same complex orbit $G^{\mathbb{C}}x \cong G^{\mathbb{C}}\tilde{x}$, for $G^{\mathbb{C}} := O(n,\mathbb{C})$. Hence, we have the two embeddings:

$$\begin{tikzcd} O(n,\mathbb{C}) \cdot x \arrow[r, equal] & O(n,\mathbb{C}) \cdot \tilde{x} \\ O(p,q) \cdot x \arrow[u] & O(\tilde{p},\tilde{q}) \cdot \tilde{x} \arrow[u] \end{tikzcd}$$

A necessary condition for the existence of Wick-rotated x and $\tilde{x}$ is therefore the existence of a complex orbit in which both real orbits are embedded. In the case of a standard Wick-rotation we know that the tangent spaces T_pP and T_pQ form a compatible triple with a compact real slice, say, W (a real form of $(T_pP)^{\mathbb{C}}$ of Euclidean signature w.r.t g), when embedded into $(T_pP)^{\mathbb{C}} \subset T_pM$. So w.r.t W, there is a compact real form: $O(n)$, also embedded in $O(n,\mathbb{C})$ (as above). Denote now $\mathfrak{o}(p,q)$, $\mathfrak{o}(\tilde{p},\tilde{q})$ and $\mathfrak{o}(n)$, for the real forms (of Lie algebras) of $O(p,q), O(\tilde{p},\tilde{q})$ and $O(n)$ respectively, embedded into $\mathfrak{o}(n,\mathbb{C})$ (the Lie algebra of $O(n,\mathbb{C})$) w.r.t a standard Wick-rotation. Then we have the following observation:

Lemma 3.6. *The triple of real forms:* $\Big(\mathfrak{o}(p,q), \mathfrak{o}(\tilde{p},\tilde{q}), \mathfrak{o}(n)\Big)$, *embedded into* $\mathfrak{o}(n,\mathbb{C})$ *under a standard Wick-rotation is also a compatible triple of Lie algebras.*

Proof. Denote σ for the conjugation map of T_pP and $\tilde{\sigma}$ for the conjugation map of T_pQ, and let τ be the conjugation map of W. We note that the map:

$$\mathfrak{o}(n,\mathbb{C}) \to \mathfrak{o}(n,\mathbb{C}), \quad f \mapsto \sigma \circ \bar{f},$$

with $\bar{f}(v_1 + iv_2) := f(v_1) - if(v_2), \forall v_1, v_2 \in \mathfrak{o}(p,q)$, is a conjugation map with fixed points $\mathfrak{o}(p,q)$, this is easy to check. Similarly by replacing σ with $\tilde{\sigma}$ and τ we get conjugation maps associated to $\mathfrak{o}(\tilde{p},\tilde{q})$ and $\mathfrak{o}(n)$. Note that since the conjugation maps: $\sigma, \tilde{\sigma}$ and τ all commute, implies that the conjugation maps associated to $\mathfrak{o}(p,q), \mathfrak{o}(\tilde{p},\tilde{q})$ and $\mathfrak{o}(n)$ must also commute. The lemma is proved. □

4. Lie groups

An important class of examples can be found for Lie groups. Lie groups are analytic manifolds and can be equipped with left-invariant metrics of arbitrary signature (at least locally, ignoring the question of global geodesic completeness).

4.1. Complex Lie groups and their real forms.

Definition 4.1. A real Lie subgroup G of a complex Lie group $G^{\mathbb{C}}$ is said to be a *real form* if $\mathfrak{g}$ is a real form of the Lie algebra of $G^{\mathbb{C}}$, and moreover as a group product we have $G^{\mathbb{C}} = GG_0^{\mathbb{C}}$ where $G_0^{\mathbb{C}}$ is the identity component.

Given a real Lie group, G (which is an analytic manifold), we can complexify the Lie group using the Lie algebra, $\mathfrak{g}$. Since the identity component of the Lie group is determined by a neighbourhood of the identity, the exponential map – mapping the Lie algebra onto a neighbourhood of the identity – enables us to complexify the Lie group. The complexified Lie group $G^{\mathbb{C}}$ is then a Lie group of real dimension twice that of the real one: $\dim_{\mathbb{R}}(G^{\mathbb{C}}) = 2\dim_{\mathbb{R}}(G)$.

For two real forms $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ of a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, we can use the exponential map to create embeddings:

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d, "\exp"'] & \mathfrak{g}^{\mathbb{C}} \arrow[d, "\exp"'] & \tilde{\mathfrak{g}} \arrow[l] \arrow[d, "\exp"] \\ G \arrow[r] & G^{\mathbb{C}} & \tilde{G} \arrow[l] \end{tikzcd} \tag{10}$$

Hence, since the exponential map is a holomorphic map, equipping the group $G^{\mathbb{C}}$ a holomorphic metric, this can be considered as a holomorphic Riemannian manifold.

Consider therefore a complex Lie group $G^{\mathbb{C}}$. How can we find possible real forms of this Lie group? Henceforth, we will consider the semi-simple groups having semi-simple real forms. This case was completely classified by Cartan and the existence of real forms hinges on the existence of a *Cartan involution* θ which is a map $\theta : \mathfrak{g} \to \mathfrak{g}$, see Def. 2.4 and paragraph below. Such a Cartan involution is unique up to inner automorphisms.

Consider two real forms (of same dimension) of some complex Lie group $G^{\mathbb{C}}$. Then these necessarily have the unit element in common since the unit element is unique. Hence these real forms are necessarily Wick-rotated with respect to each other. Since they share the unit element, and their corresponding Lie algebras are tangent spaces over the unit element, it makes sense to compare their Cartan involutions. Assume the real forms have Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, with corresponding Cartan involutions θ and θ', respectively. These two Cartan involutions do not necessarily commute, i.e., $[\theta, \theta'] \neq 0$, but they would commute if the real forms are related through a standard Wick-rotation.

Important examples are:

(1) The real forms $O(p, q)$ of $O(n, \mathbb{C})$, where $n = p + q$.
(2) The real forms $SU(p, q)$, and $SL(n, \mathbb{R})$ of $SL(n, \mathbb{C})$.
(3) The real forms $Sp(p, q)$ of $Sp(n, \mathbb{C})$.
(4) The real forms G_2 (compact) and split-G_2 of $G_2^{\mathbb{C}}$.

Explicitly, the group $SU(2)$, which can be parameterised by the product of matrices

$$\begin{bmatrix} e^{ix_1} & 0 \\ 0 & e^{-ix_1} \end{bmatrix} \begin{bmatrix} \cosh(ix_2) & \sinh(ix_2) \\ \sinh(ix_2) & \cosh(ix_2) \end{bmatrix} \begin{bmatrix} \cos(x_3) & \sin(x_3) \\ -\sin(x_3) & \cos(x_3) \end{bmatrix} \tag{11}$$

By holomorphic extension, $x_k \mapsto z_k = x_k + iy_k$, and then restricting to the real section $(z_1, z_2, z_3) = (iy_1, iy_2, x_3)$ we obtain the following parameterisation of $SL(2, \mathbb{R})$:

$$\begin{bmatrix} e^{-y_1} & 0 \\ 0 & e^{y_1} \end{bmatrix} \begin{bmatrix} \cosh(y_2) & -\sinh(y_2) \\ -\sinh(y_2) & \cosh(y_2) \end{bmatrix} \begin{bmatrix} \cos(x_3) & \sin(x_3) \\ -\sin(x_3) & \cos(x_3) \end{bmatrix}. \tag{12}$$

In this example, the groups $SL(2, \mathbb{R})$ and $SU(2)$ are related through a standard Wick-rotation.

4.2. **Example: Split G_2-holonomy manifolds.** As an example, let us construct a pseudo-Riemannian split-G_2-holonomy manifold from a known Riemannian G_2-holonomy manifold [10].

Let us assume that the metric is of dimension 7 with $S^3 \times S^3$ hypersurfaces. Let us first see how we can use the Killing form to construct an Einstein metric. Since $S^3 \cong SU(2)$, we can equip these hypersurfaces with a left-invariant Riemannian metric proportional to the Killing form, κ, on each factor, i.e., using the left invariant frame, $h(X, Y) = -\lambda \kappa_{S^3 \times S^3}(X, Y)$, $\lambda > 0$. Since the Killing form is non-degenerate and negative-definite on a compact semi-simple group, the metric h is positive definite. We can now do a Wick-rotation of these Lie groups as explained above to another real form of $SU(2)^{\mathbb{C}} \cong SL(2, \mathbb{C})$. The other real form of this complex group is $SL(2, \mathbb{R})$. The Killing form will like-wise be Wick-rotated to the corresponding form of $SL(2, \mathbb{R})$. The Killing form of $SL(2, \mathbb{R})$ is also non-degenerate but of signature $(-++)$. The Wick-rotated form of h will therefore be:

$$\tilde{h}(X,Y) = -\lambda\kappa_{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}(X,Y),$$

which is also non-degenerate but of signature $(----++)$ or (4,2). This would be the standard bi-invariant Einstein metric on $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$.

With some amendments we can also provide with a Wick-rotation of the metrics in [10] to a 7-dimensional pseudo-Riemannian manifold of signature $(4,3)$. Let σ^i and Σ^i be a set of left-invariant one-forms on two copies of $SU(2)$ so that $\mathrm{d}\sigma^1 = -\sigma^2\wedge\sigma^3$ and $\mathrm{d}\Sigma^1 = -\Sigma^2\wedge\Sigma^3$ etc. (cyclic permutation). For simplicity, we write $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ for the column vectors of one-forms:

$$\boldsymbol{\sigma} = \begin{bmatrix}\sigma^1\\ \sigma^2\\ \sigma^3\end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix}\Sigma^1\\ \Sigma^2\\ \Sigma^3\end{bmatrix}.$$

Using the parameterisation eq.(11), the (real) left-invariant one-forms can be written:

$$\begin{aligned}\sigma^1 &= 2\left[\cos(2x_3)\mathrm{d}x_2 + \cos(2x_2)\sin(2x_3)\mathrm{d}x_1\right]\\ \sigma^2 &= 2\left[-\sin(2x_3)\mathrm{d}x_2 + \cos(2x_2)\cos(2x_3)\mathrm{d}x_1\right]\\ \sigma^3 &= 2\left[\mathrm{d}x_3 - \sin(2x_2)\mathrm{d}x_1\right],\end{aligned}$$

and similar for $\boldsymbol{\Sigma}$. A G_2-holonomy metric can now be written [10]:

$$ds^2 = \alpha^2\mathrm{d}r^2 + \beta^2\left(\boldsymbol{\sigma}-\boldsymbol{A}\right)^T\left(\boldsymbol{\sigma}-\boldsymbol{A}\right) + \gamma^2\boldsymbol{\Sigma}^T\boldsymbol{\Sigma}, \tag{13}$$

where $\alpha=\alpha(r)$, $\beta=\beta(r)$, $\gamma=\gamma(r)$, and the $\boldsymbol{A}$ is the connection one-form $\boldsymbol{A}=\frac{1}{2}\boldsymbol{\Sigma}$. When

$$\alpha^2 = (1-r^{-3})^{-1},\ \beta^2 = \frac{1}{9}r^2(1-r^{-3}),\ \gamma^2 = \frac{1}{12}r^2,$$

this metric is Ricci-flat and has G_2-holonomy.

A Wick-rotation can now be accomplished by Wick-rotating each copy of S^3 to $SL(2,\mathbb{R})$ given explicitly by eqs.(11-12). By aligning the Wick-rotation for each copy, so that $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ transform identically, then the metric above can be Wick-rotated into:

$$ds^2 = \alpha^2\mathrm{d}r^2 + \beta^2\left(\tilde{\boldsymbol{\sigma}}-\tilde{\boldsymbol{A}}\right)^T\boldsymbol{\eta}\left(\tilde{\boldsymbol{\sigma}}-\tilde{\boldsymbol{A}}\right) + \gamma^2\tilde{\boldsymbol{\Sigma}}^T\boldsymbol{\eta}\tilde{\boldsymbol{\Sigma}}, \tag{14}$$

where $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\Sigma}}$ are the corresponding real left-invariant one-forms on two copies of $SL(2,\mathbb{R})$, $\tilde{\boldsymbol{A}}=\frac{1}{2}\tilde{\boldsymbol{\Sigma}}$ and $\boldsymbol{\eta}=\mathrm{diag}(-1,-1,1)$. It is clearly essential here that the one-form $(\tilde{\boldsymbol{\sigma}}-\tilde{\boldsymbol{A}})$ is real on $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ but by aligning the Wick-rotations on each copy of $SU(2)$ this can be accomplished. The Wick-rotated metric is therefore Ricci-flat and of signature $(4,3)$. As the holonomy group is generated by the Riemann curvature tensor (and its covariant derivatives), the holonomy group of the Wick-rotated space would be another real form of the complexified $G_2^{\mathbb{C}}$ Lie group. Since the only other real form is the split-G_2, then the resulting space is of split-G_2 holonomy.

5. A standard Wick-rotation to a real Riemannian space

In Section 3.3 we saw a necessarily condition put on the orbits $O(p,q)x$ and $O(\tilde{p},\tilde{q})\tilde{x}$, for the existence of two Wick-rotated spaces P and Q of M. We now give a stronger necessary condition on the orbits in the case of a standard Wick rotation to a real Riemannian space. In the Riemannian case we have $\tilde{p} = n$ and $\tilde{q} = 0$, so one of the structure groups is a compact real form: $O(n)$, of $O(n,\mathbb{C})$.

We will use tools from real GIT applied to semi-simple groups to obtain a necessary condition on the orbits.

5.1. **Minimal vectors and closure of real semi-simple orbits.**
Throughout this section, our groups will always be semi-simple of finitely many components.

Let now G be a real semi-simple Lie group with finitely many components, and assume it is a real form of a complex Lie group $G^{\mathbb{C}}$. This immediately implies that our groups are all linear. Suppose $G \xrightarrow{\rho} GL(V)$ is a representation of G. We shall say that a complex representation $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ is a *complexified action* of ρ if the following diagram commutes:

$$\begin{tikzcd} G^{\mathbb{C}} \arrow[r, "\rho^{\mathbb{C}}"] & GL(V^{\mathbb{C}}) \\ G \arrow[u, "i"] \arrow[r, "\rho"] & GL(V) \arrow[u, "i"] \end{tikzcd} \tag{15}$$

Let now $\mathfrak{g} \xrightarrow{\theta} \mathfrak{g}$ be a Cartan involution of $\mathfrak{g}$ and $G \xrightarrow{\Theta} G$ be the corresponding unique Cartan involution of G which lifts θ, i.e $d\Theta = \theta$. Denote the Cartan decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and similarly let $K \subset G$ be the maximally compact subgroup with Lie algebra $\mathfrak{k}$, so we have Cartan decomposition of $G = Ke^{\mathfrak{p}}$. Let $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$, then $\mathfrak{u}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$ which is compatible with $\mathfrak{g}$. As a real Lie group denote the Cartan decomposition $G^{\mathbb{C}} = Ue^{i\mathfrak{u}}$, where U is a compact real form of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{u}$. Note that $K \subset U$, in this way.

We can choose a K-invariant inner product $\langle -,- \rangle$ on V w.r.t the action ρ, which is compatible with a U-invariant Hermitian inner product $\langle -,- \rangle_{\mathbb{C}}$ on $V^{\mathbb{C}}$ w.r.t $\rho^{\mathbb{C}}$. The inner product $\langle -,- \rangle$ can be chosen such that it has the two properties:

(1) $d\rho(\mathfrak{k})$ consists of skew-symmetric maps w.r.t $\langle -,- \rangle$.
(2) $d\rho(\mathfrak{p})$ consists of symmetric maps w.r.t $\langle -,- \rangle$.
In particular the real part of $\langle -,- \rangle_{\mathbb{C}}$ will have property (1) and (2) with respect to the Cartan decomposition: $(\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}} = \mathfrak{u} \oplus i\mathfrak{u}$.

For proofs of these facts we refer to [35] and [7].

By *Borel* and *Chandra* (see [35]) and also [7] we have the following theorem relating the closure of the complex orbit to that of the real orbit for real vectors $v \in V$:

Theorem 5.1 (BC)**.** *The following statements hold:*

(1) Given a real vector $v \in V$ then the real orbit Gv is closed in V w.r.t the classical topology on V if and only if $G^{\mathbb{C}}v$ is closed in $V^{\mathbb{C}}$.
(2) Given a real vector $v \in V$ then $G^{\mathbb{C}}v \cap V$ is a finite disjoint union of real G-orbits.

Two real orbits: Gv_1 and Gv_2 in the disjoint union $G^{\mathbb{C}}v \cap V$ are often said to be *conjugate*. Set $||-||^2 := \langle -,-\rangle$, for the norm on V.

Definition 5.2. A minimal vector $v \in V$ is one which satisfies

$$\Big(\forall g \in G\Big)\Big(||g \cdot v|| \geq ||v||\Big).$$

The set of minimal vectors will be denoted by $\mathcal{M}(G,V) \subset V$. Of course one observes the special case where G is a compact group then all vectors are minimal, i.e $\mathcal{M}(G,V) = V$. As an example consider $G := SL(2,\mathbb{R})$, and $G^{\mathbb{C}} := SL(2,\mathbb{C})$ with the representations being the adjoint actions. Take Cartan involution: $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R})$, given by $x \mapsto -x^t$. Then it is not difficult to show that if $x \in \mathfrak{so}(2)$ we have

$$G^{\mathbb{C}}x \cap \mathfrak{sl}(2,\mathbb{R}) = Gx \cup G(-x),$$

and if $x \in \mathfrak{p}$, then $G^{\mathbb{C}}x \cap \mathfrak{sl}(2,\mathbb{R}) = Gx$. This classifies what happens in all cases where the complex orbit: $G^{\mathbb{C}}x$, is closed and $x \in \mathfrak{sl}(2,\mathbb{R})$ (by the theorem below).

Now we also have the following theorem by *Richardson* and *Slodowy* in [7], which relates the closure of a real orbit to the existence of a minimal vector:

Theorem 5.3 (RS)**.** *The following statements hold:*

(1) A real orbit Gv is closed if and only if $Gv \cap \mathcal{M}(G,V) \neq \emptyset$.
(2) If v is a minimal vector then $Gv \cap \mathcal{M}(G,V) = Kv$.
(3) If Gv is not closed then there exist $p \in \mathfrak{p}$ and an $\alpha \in \overline{Gv}$ such that $e^{tp} \cdot v \to \alpha \in V$ exist as $t \to -\infty$, and $G\alpha$ is closed. Moreover $G\alpha$ is the unique closed orbit in the closure $\overline{Gv} \subset V$.
(4) A vector $v \in V$ is minimal if and only if $\Big(\forall x \in \mathfrak{p}\Big)\Big(\langle x \cdot v, v\rangle = 0\Big)$, where $x \cdot v$ is the action $d\rho(x)(v)$.

Parts (1), (2) and (4) of the theorem is known as the *Kempf-Ness Theorem*, for which it was first proved for linearly complex algebraic groups.

Remark 5.4. The above two theorems are proved in an algebraic geometric setting, for which the group is a real linearly reductive group. To see that one can apply these results to semi-simple Lie groups (not necessarily algebraic), we refer also to a *remark* in [39]. We recall the fact that any complex semi-simple Lie group is algebraic, and also any holomorphic representation of $G^{\mathbb{C}}$ is also rational. Theorem 5.3 in fact also hold for general reductive Lie groups, see [29].

5.2. **Compatible triples and intersection of real orbits.** Our aim in this section is to explore the connection between compatible triples of semi-simple Lie algebras, and the intersection of real orbits. We prove a theorem, stating that if one of the real forms, $\tilde{G}$ say, is compact, and compatible with another real form G of $G^{\mathbb{C}}$, then two real orbits can only belong to the same complex orbit if they intersect.

We continue with the notation of the previous subsection.

So we now apply the well-known theorems of the previous subsection to the situation of compatible triples, i.e suppose we have another real form of $G^{\mathbb{C}}$, say $\tilde{G}$, with Lie algebra $\tilde{\mathfrak{g}}$. We shall assume that the triple $\Big(\mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{u}\Big)$ is a compatible triple of real forms. So we can choose a Cartan involution of $\tilde{\mathfrak{g}}$ say $\tilde{\theta}$, which commutes with θ (the Cartan involution of $\mathfrak{g}$), i.e $[\theta, \tilde{\theta}] = 0$. Denote $\tilde{G} = \tilde{K}e^{\tilde{\mathfrak{p}}}$ for the Cartan decomposition of $\tilde{G}$, and similarly we denote the local Cartan decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$. So we have $K, \tilde{K} \subset U$. In fact by Proposition 2.8 we can assume that $K \cap \tilde{K} \neq 1$ have a non-trivial intersection.

Consider $\tilde{G} \xrightarrow{\tilde{\rho}} GL(\tilde{V})$ to be a representation of $\tilde{G}$ also with the same complexification $\rho^{\mathbb{C}}$ as ρ, with $\tilde{V} \subset V^{\mathbb{C}}$ another real form. We can put similarly a U-invariant Hermitian form on $V^{\mathbb{C}}$ (possibly different from the one compatible with V), which is compatible with $\tilde{V}$. So we have another commutative diagram, like the one in the previous subsection for G:

$$\begin{array}{ccc} G^{\mathbb{C}} & \xrightarrow{\rho^{\mathbb{C}}} & GL(V^{\mathbb{C}}) \\ {\scriptstyle i}\big\uparrow & & {\scriptstyle i}\big\uparrow \\ \tilde{G} & \xrightarrow{\tilde{\rho}} & GL(\tilde{V}) \end{array} \tag{16}$$

A good example to have in mind for such a situation is the compatible triple $\Big(\mathfrak{o}(p,q), \mathfrak{o}(\tilde{p},\tilde{q}), \mathfrak{o}(n)\Big)$ described in section 2.3. The action can be the adjoint action $\rho^{\mathbb{C}} := Ad$ of the complex orthogonal group $O(n, \mathbb{C})$, with real forms $O(p,q)$ and $O(\tilde{p},\tilde{q})$ with their adjoint actions: $\rho := Ad$ and $\tilde{\rho} := Ad$. Here the inner products associated to the adjoint representations can of course be

$$-\kappa(-, \theta(-)), \quad -\kappa(-, \tilde{\theta}(-)), \quad -\kappa(-, \tau(-)) \text{ (Hermitian)},$$

where τ is the conjugation map of $\mathfrak{o}(n)$. Moreover a minimal vector $x \in \mathfrak{o}(p,q)$ in this setting will just be a vector satisfying $[x, \theta(x)] = 0$.

We observe that the example of the adjoint action generalises. Indeed if V and $\tilde{V}$ are also compatible real forms of $V^{\mathbb{C}}$, then we may assume w.l.o.g, that the minimal vectors of our actions: $\mathcal{M}(G,V)$ of ρ and $\mathcal{M}(\tilde{G},\tilde{V})$ of $\tilde{\rho}$, are both contained in the minimal vectors of the complexified action: $\mathcal{M}(G^{\mathbb{C}},V^{\mathbb{C}})$ of $\rho^{\mathbb{C}}$. We refer to *Appendix* A, for a proof of this fact.

Now we prove our main theorem under the compatibility conditions of the Lie algebras and of the vector spaces the groups act on:

Theorem 5.5. *Suppose we have a compatible triple of semi-simple Lie algebras:* $\left(\mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{u}\right)$. *Assume also that V and $\tilde{V}$ are compatible real forms of $V^{\mathbb{C}}$. Let $v \in V$ and $\tilde{v} \in \tilde{V}$. Then the following statements hold:*

(1) Suppose $\tilde{v} \in G^{\mathbb{C}}v$. Then if $v \in \mathcal{M}(G,V)$ and $\tilde{v} \in \mathcal{M}(\tilde{G},\tilde{V})$ we have $Uv = U\tilde{v}$, i.e $Kv, \tilde{K}\tilde{v} \subset Uv$.

(2) If $\tilde{G} := U$ is the compact real form compatible with G, then

$$Gv, \tilde{G}\tilde{v} \subset G^{\mathbb{C}}v \Leftrightarrow Gv \cap \tilde{G}\tilde{v} \neq \emptyset.$$

Proof. For case (1) if v and $\tilde{v}$ are minimal vectors in V and $\tilde{V}$ respectively then they are also minimal vectors in $V^{\mathbb{C}}$. So $v, \tilde{v} \in G^{\mathbb{C}}v \cap \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}) = Uv$, by Theorem 5.3, and (1) follows. For case (2), we note that since $\tilde{G} := U$ is the compact real form, then $\tilde{G}\tilde{v}$ is closed, and so since $\tilde{G}\tilde{v} \subset G^{\mathbb{C}}v$ then $G^{\mathbb{C}}v$ is also closed and in particular so is the real orbit Gv, by Theorem 5.1. Hence we can choose a minimal vector $v_1 \in Gv$ which is also minimal in $G^{\mathbb{C}}v$, by Theorem 5.3, i.e $v_1 \in \tilde{G}\tilde{v}$, as $\tilde{G} := U$, and so proves (2). The theorem is proved. $\square$

Although the theorem guarantees intersection between orbits, there are cases where the orbits intersect in a unique vector. Indeed take $G := SL(2,\mathbb{R})$ and $\tilde{G} := SU(2)$, and let the action be the adjoint action. The Lie algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ are naturally compatible, w.r.t to the standard embedding into $\mathfrak{sl}(2,\mathbb{C})$. It is not difficult to show that whenever $Gx, \tilde{G}\tilde{x}$ belong to the same complex orbit: $G^{\mathbb{C}}x$, for $G^{\mathbb{C}} := SL(2,\mathbb{C})$, then $Gx \cap \tilde{G}\tilde{x} = \{x\}$.

Remark 5.6. We point out that case (2) of Theorem 5.5, does not require V and $\tilde{V}$ to be compatible real forms of $V^{\mathbb{C}}$.

5.3. **The real Riemannian case.** Using Theorem 5.5 we finally derive a necessary condition for the existence of a standard Wick-rotation to a real Riemannian space, following the notation in Section 3.3 we have:

Corollary 5.7. *Suppose P and Q are two Wick-rotated spaces of M by a standard Wick-rotation. Let Q be a real Riemannian space. Suppose $x \in V$ and $\tilde{x} \in \tilde{V}$ are* *two Wick-rotatable tensors. Then the real orbits $O(p,q)x$ and $O(n)\tilde{x}$ intersect, i.e $O(p,q)x \cap O(n)\tilde{x} \neq \emptyset$.*

Proof. By Section 3.3 , using Lemma 3.6 we can apply Theorem 5.5, and so the result follows. $\square$

For general Wick-rotated spaces P and Q by a standard Wick-rotation, we note that if the complex orbit: $O(n,\mathbb{C})x = O(n,\mathbb{C})\tilde{x}$, is closed, then a necessary condition is that for the maximally compact subgroups:

$$K := O(p) \times O(q) \subset O(p,q), \text{ and } \tilde{K} := O(\tilde{p}) \times O(\tilde{q}) \subset O(\tilde{p},\tilde{q}),$$

the orbits $K\alpha$ and $\tilde{K}\tilde{\alpha}$ (of minimal vectors) must both be embedded into the same compact orbit: $O(n)\alpha$, i.e we have the following diagram of embeddings:

$$\begin{tikzcd}
O(n,\mathbb{C}) \cdot x \arrow[r, equal] & O(n,\mathbb{C}) \cdot \tilde{x} \\
O(n) \cdot \alpha \arrow[u] \arrow[r, equal] & O(n) \cdot \tilde{\alpha} \arrow[u] \\
O(p) \times O(q) \cdot \alpha \arrow[u] & O(\tilde{p}) \times O(\tilde{q}) \cdot \tilde{\alpha} \arrow[u]
\end{tikzcd}$$

This is all by Theorem 5.5, noting that the tensor products: V and $\tilde{V}$, for which the groups act on, are compatible real forms of $V^{\mathbb{C}}$ under a standard Wick-rotation. In order to generalise the previous Corollary, we need to know more about how these K-orbits are embedded, and in the case of same signatures one needs to know how many real K-orbits there are in the compact orbit: $O(n)\alpha$, i.e when is $l = 1$, in the intersection:

$$O(n)\alpha \cap V = \cup_j^l K x_j?$$

There are examples where $l = 1$ and $l \neq 1$ for different $O(p,q)$, such examples can be found within the adjoint action, as we will see in the next section.

5.4. **The adjoint action of the Lorentz groups $O(n-1,1)$.** As an example of a semi-simple Lie group action using Section 5.1, we consider the adjoint action of the Lorentz group $O(n-1,1) \subset O(n,\mathbb{C})$ on its Lie algebra: $\mathfrak{o}(n-1,1)$. Here we view $O(n-1,1)$ embedded into $O(n,\mathbb{C}) := \{g \in GL(n,\mathbb{C}) | gg^t = I\}$ as a real form fixed by the conjugation map:

$$g \in O(n,\mathbb{C}) \mapsto I_{n-1,1}\bar{g}^{-1}I_{n-1,1} \in O(n,\mathbb{C}).$$

We prove that whenever the complex orbit $O(n,\mathbb{C})\boldsymbol{x}$ is closed for a real vector $\boldsymbol{x} \in \mathfrak{o}(n-1,1)$, then there is a unique real closed orbit in the complex orbit, i.e $O(n,\mathbb{C})\boldsymbol{x} \cap \mathfrak{o}(n-1,1) = O(n-1,1)\boldsymbol{x}$. We demonstrate that this is the only pseudo-orthogonal group $O(p,q) \subset O(n,\mathbb{C})$ with this property under the adjoint action, where we view $O(p,q)$ as the fix points of the conjugation map: $g \in O(n,\mathbb{C}) \mapsto I_{p,q}\bar{g}^{-1}I_{p,q}$.

Let $\mathfrak{o}(n-1,1) \xrightarrow{\theta} \mathfrak{o}(n-1,1)$, be the Cartan involution given by:

$$\boldsymbol{x} \mapsto Ad(I_{n-1,1})(\boldsymbol{x}) = \bar{\boldsymbol{x}}.$$

We use the standard norm on $\mathfrak{o}(n-1,1)$, given by $||\boldsymbol{x}||^2 := \lambda\kappa_\theta(\boldsymbol{x},\boldsymbol{x})$, where $\lambda > 0$, is chosen such that $||\boldsymbol{x}||^2 = tr(\boldsymbol{x}^2)$. Observe that here the global Cartan decomposition of $O(n-1,1)$ is given by: $O(n-1,1) = Ke^{\mathfrak{p}}$, where $K = O(n-1) \times O(1)$, with local Cartan decomposition: $\mathfrak{o}(n-1,1) = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{k}$ consisting of matrices in block form $\mathfrak{k} \cong \mathfrak{so}(n-1) \oplus \mathfrak{so}(1)$, and $\mathfrak{p}$, consists of matrices of the form:

$$\begin{pmatrix} 0_{(n-1)\times(n-1)} & iA_{(n-1)\times 1} \\ -iA^t_{(n-1)\times 1} & O_{1\times 1} \end{pmatrix}, \quad A \in \mathbb{R}^{n-1}.$$

We will borrow the following two standard results from linear algebra:

Lemma 5.8 (*Spectral theorem for skew-symmetric matrices*)**.** *If $\boldsymbol{x} \in \mathfrak{so}(n)$ then there exist an orthogonal matrix $g \in O(n)$, such that:*

$$g\boldsymbol{x}g^{-1} = \begin{bmatrix} \mathfrak{so}(2) & & & & & & & & \\ & \mathfrak{so}(2) & & & & & & & \\ & & \mathfrak{so}(2) & & & & & & \\ & & & \ddots & & & & & \\ & & & & \mathfrak{so}(2) & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{bmatrix},$$

where $\mathfrak{so}(2)$ is the 2×2 matrix of the form $\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$.

Corollary 5.9. *Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathfrak{so}(n)$ be skew-symmetric matrices, having identical characteristic polynomials. Then there exist $g \in O(n)$ such that $g\boldsymbol{x}_1 g^{-1} = \boldsymbol{x}_2$.*

By applying these results, we can prove the following lemma:

Lemma 5.10. *For any vector $\boldsymbol{x} \in \mathfrak{k} \cup \mathfrak{p}$ we have*

$$O(n,\mathbb{C})\boldsymbol{x} \cap \mathfrak{o}(n-1,1) = O(n-1,1)\boldsymbol{x}.$$

Proof. Suppose first $\boldsymbol{x} \in \mathfrak{o}(n-1,1)$ and $\tilde{\boldsymbol{x}} \in \mathfrak{o}(n-1,1)$ belong to the compact real part $\mathfrak{k} = \mathfrak{so}(n-1) \oplus \mathfrak{so}(1)$, and moreover lie in the same $O(n,\mathbb{C})$-orbit. Then we can remove the last row and column of the matrices $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$, and they will still have the same characteristic polynomial. Call these $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$, then they are in $\mathfrak{so}(n-1)$ and so must lie in the same $O(n-1)$-orbit by Corollary 5.9. Now by extending a matrix in $O(n-1)$ to a matrix in K (in the obvious way) then $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ must lie in the same $O(n-1,1)$-orbit.

Consider now the case where $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \mathfrak{p}$ are contained in the non-compact part. It is easy to see by calculating the characteristic polynomials that if $\boldsymbol{x}, \tilde{\boldsymbol{x}}$ lie in the same $O(n, \mathbb{C})$-orbit then they lie in the $(n-2)$-sphere:

$$S^{n-2} := \{\boldsymbol{x} \in \mathfrak{p} \Big| ||\boldsymbol{x}|| = 1\},$$

where the norm $||, ||$ is proportional to the Killing form: $\kappa(\cdot, \cdot)$ on $\mathfrak{o}(n-1, 1)$ restricted to $\mathfrak{p}$. Now we already know that

$$Ad(K_0)_{|\mathfrak{p}} := \{\mathfrak{p} \xrightarrow{Ad(k)} \mathfrak{p} | k \in K_0\},$$

where $K_0 = SO(n-1) \times SO(1) \cong SO(n-1)$, is contained in the space of isometries (as a closed matrix subgroup):

$$Ad(K_0)_{|\mathfrak{p}} \subset Isom(S^{n-2}) \cong O(n-1),$$

w.r.t the restricted norm metric. Now observe that $Ad(K_0)_{|\mathfrak{p}}$ has Lie algebra $\cong \mathfrak{o}(n-1, 1)$, and is connected. This follows because $\mathfrak{t} \xrightarrow{ad} \mathfrak{gl}(\mathfrak{p})$ is faithful, as $\mathfrak{o}(n-1, 1)$ is simple for all $n \geq 1$, hence the kernel of the restricted adjoint action: $K_0 \xrightarrow{Ad} GL(\mathfrak{p})$ is a discrete subgroup of K_0. So clearly:

$$Ad(K_0)_{|\mathfrak{p}} = Isom(S^{n-2})_0 \cong SO(n-1),$$

however $Isom(S^{n-2})_0$ acts transitively on S^{n-2}, and so proves that $\boldsymbol{x}, \tilde{\boldsymbol{x}}$ lie in the same K_0-orbit, and hence in the same $O(n-1, 1)-$orbit. This proves the lemma. $\square$

Using the previous lemma we can prove our claim, that $O(n, \mathbb{C})\boldsymbol{x}$ has a unique real Lorentz orbit: $O(n-1, 1)\boldsymbol{x}$, when it is closed, and $\boldsymbol{x} \in \mathfrak{o}(n-1, 1)$.

Theorem 5.11. *For any minimal vector $\boldsymbol{x} \in \mathfrak{o}(n-1, 1)$ we have*

$$O(n, \mathbb{C})\boldsymbol{x} \cap \mathfrak{o}(n-1, 1) = O(n-1, 1)\boldsymbol{x}.$$

Proof. Let $\boldsymbol{x}$ be a minimal vector of $\mathfrak{o}(n-1, 1)$, i.e we can write

$$\boldsymbol{x} := \begin{pmatrix} A & \\ & 0 \end{pmatrix} + \begin{pmatrix} 0_{(n-1)\times(n-1)} & ix \\ -ix^t & 0 \end{pmatrix},$$

where $A \in \mathfrak{so}(n-1)$ with $Ax = 0$. Suppose now that there is another minimal vector:

$$\tilde{\boldsymbol{x}} := \begin{pmatrix} \tilde{A} & \\ & 0 \end{pmatrix} + \begin{pmatrix} 0_{(n-1)\times(n-1)} & i\tilde{x} \\ -i\tilde{x}^t & 0 \end{pmatrix},$$

belonging to the same complex orbit as $\boldsymbol{x}$. Denote $\boldsymbol{x} = t + p$ and $\tilde{\boldsymbol{x}} = \tilde{t} + \tilde{p}$ for the components defined previously. We may assume $A, \tilde{A}, x, \tilde{x}$ are all non-zero. Let V_0 (respectively $\tilde{V_0}$) denote the kernel of the linear maps: $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, corresponding to the matrices A (respectively $\tilde{A}$). Note that $x \in V_0$ and $\tilde{x} \in \tilde{V_0}$. By the previous lemma, we know that there exist $k_1, k_2 \in K$, such that $k_1 \cdot t = \tilde{t}$ and $k_2 \cdot p = \tilde{p}$. This means that $k_1 V_0 = \tilde{V_0}$. Suppose $Dim(V_0) = 1$. Then $k_1 x = \lambda \tilde{x}$, for some $\lambda \in \mathbb{R}$. Now using the norm induced by the inner product on $\mathfrak{p}$: $\kappa_\theta(-,-)$, we can say that $||p|| = ||\tilde{p}||$, i.e x and $\tilde{x}$ have the same Euclidean norm in $\mathbb{R}^{n-1}$. This means that $\lambda = 1$, and hence $k_1 \cdot \boldsymbol{x} = \tilde{\boldsymbol{x}}$.

Now for the general case, assume $Dim(V_0) > 1$. We may assume A has the form $A := \begin{pmatrix} A_1 & \\ & 0_l \end{pmatrix}$, for some $A_1 \in \mathfrak{so}(n-l)$, where $l \geq 2$, and A_1 has kernel of dimension 1, as an operator: $\mathbb{R}^{n-l} \to \mathbb{R}^{n-l}$. Indeed we may assume that A_1 has the form:

$$\begin{bmatrix} \mathfrak{so}(2) & & & & & \\ & \mathfrak{so}(2) & & & & \\ & & \mathfrak{so}(2) & & & \\ & & & \ddots & & \\ & & & & \mathfrak{so}(2) & \\ & & & & & 0 \end{bmatrix},$$

by Lemma 5.8 and Corollary 5.9. Moreover if $x = (x_1, x_2, \dots, x_{n-1})^T$, then denote $x_1 := (x_1, x_2, \dots, x_{n-l})^T \in \mathbb{R}^{n-l}$. One can see that $A_1 x_1 = 0$. We can do this also to $\tilde{A}$, and denote similarly the $(n-l)\times(n-l)$ matrix by $\tilde{A}_1$, and the vector by $\tilde{x}_1$, as $k_1 t = \tilde{t}$. There exist $k \in O(n-l) \times O(1)$ such that $k \cdot A_1 = \tilde{A}_1$ by Corollary 5.9, i.e we may assume $k_1 = \begin{pmatrix} k & \\ & I_l \end{pmatrix}$. So applying the previous argument to this case, we have $kx_1 = \tilde{x}_1$, and hence $k_1 \cdot \boldsymbol{x} = \tilde{\boldsymbol{x}}$. This proves the theorem. $\square$

Now for general $O(p,q)$ with $p, q \neq 1$, then there is always a minimal vector $\boldsymbol{x} \in \mathfrak{o}(p,q)$ such that the closed orbit $O(n,\mathbb{C})\boldsymbol{x}$ has at least two disjoint real orbits: $O(p,q)\boldsymbol{x}_1$ and $O(p,q)\boldsymbol{x}_2$. To see this, note that if $p, q > 1$ then we can choose the block matrix of the form:

$$\boldsymbol{x} := \begin{bmatrix} \mathfrak{so}(2) & 0 \\ 0 & \mathfrak{so}(2) \end{bmatrix} \in \mathfrak{so}(p) \oplus \mathfrak{so}(q),$$

where the $\mathfrak{so}(2)$-blocks have different characteristic polynomials. Consider the same matrix, but with the blocks interchanged. Call this matrix $\boldsymbol{x}_1 \in \mathfrak{so}(p) \oplus \mathfrak{so}(q)$. Then these are in the same $O(n)$-orbit by Corollary 5.9, but can not be related in the same $O(p) \times O(q)$-orbit, hence $O(p,q)\boldsymbol{x}$ and $O(p,q)\boldsymbol{x}_1$ are two disjoint orbits in $O(n,\mathbb{C})\boldsymbol{x}$.

5.5. **Uniqueness of real orbits and the class of complex Lie groups.** An interesting question is: When is there a unique real orbit in the complex orbit, i.e., when does one have $G^{\mathbb{C}} v \cap V = Gv$, for $v \in V$? In this section we give a class of groups for which this holds. Recall that one such class is the compact groups, easily deduced from the results of *Richardson* and *Slodowy*, i.e., Theorem 5.3 in [7].

We prove that if G (an arbitrary Lie group) has the structure of a complex Lie group, and V has a complex structure $V \xrightarrow{J} V$, such that $G \xrightarrow{\rho} GL(V)$ is a complex representation w.r.t J, then we have uniqueness. We of course assume that $G \subset G^{\mathbb{C}}$ is a real form of some complex Lie group, and we have the following commutative diagram (as before):

$$
\begin{array}{ccc}
G^{\mathbb{C}} & \xrightarrow{\rho^{\mathbb{C}}} & GL(V^{\mathbb{C}}) \\
{\scriptstyle i}\Big\uparrow & & {\scriptstyle i}\Big\uparrow \\
G & \xrightarrow{\rho} & GL(V)
\end{array}
\tag{17}
$$

Theorem 5.12. *Let G be a complex Lie group, and as a real Lie group let it be a real form of some complex Lie group $G^{\mathbb{C}}$. Suppose V has a complex structure J and the representations are as in the commutative diagram (17), with ρ a complex representation w.r.t J. Then if $v \in V$, we have a unique real G-orbit in the complex orbit: $G^{\mathbb{C}}v$, i.e $G^{\mathbb{C}}v \cap V = Gv$.*

We shall prove this statement using the structure theory for Lie groups, as a reference for the results we use, we refer to [31].

Consider now the identity component G_0 of G, which is a real form of the identity component $G_0^{\mathbb{C}}$. Let $\widetilde{G_0}$ and $\widetilde{G_0^{\mathbb{C}}}$ be the universal covering groups.

We will now discuss some covering theory and universal complexification of Lie groups, for proofs we again refer to: [31], for example chapter 15.

Now if $\mathfrak{g}$ has a complex structure J then $\mathfrak{g}$ has the structure of a complex Lie algebra of dimension $\frac{1}{2}Dim(\mathfrak{g})$. So G_0 is a complex Lie group. In particular we note that $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{g} \oplus \bar{\mathfrak{g}}$, where $\bar{\mathfrak{g}}$ is the complex structure on $\mathfrak{g}$ obtained from $-J$. This isomorphism takes

$$x + iy \mapsto \Big(x + J(y), x - J(y)\Big), \ \ x, y \in \mathfrak{g},$$

and so therefore $\mathfrak{g}$ can be identified with the set $\{(x, x)|x \in \mathfrak{g}\}$ as a real form of $\mathfrak{g} \oplus \mathfrak{g}$. So the universal complexification group of the universal covering $\widetilde{G_0}$, is just the universal covering: $\widetilde{G_0^{\mathbb{C}}}$, and thus must be isomorphic (as Lie groups) to the product:

$$\widetilde{G_0^{\mathbb{C}}} \cong \widetilde{G}_0 \times \widetilde{G}_0.$$

Here the right component: $\widetilde{G}_0$, of the product is called the *opposite complex Lie group* of $\widetilde{G}_0$, since it has complex Lie algebra $\bar{\mathfrak{g}}$. The left and right components of the product are not necessarily isomorphic (as Lie groups) unless $\mathfrak{g}$ (as a complex Lie algebra), has the existence of a real form, like for instance if $\mathfrak{g}$ were reductive. The universal complexification map is simply the diagonal embedding:

$$g \mapsto (g, g), \ \ g \in \widetilde{G}_0.$$

Moreover w.r.t to this map $\widetilde{G}_0$ is a real form of $\widetilde{G_0^{\mathbb{C}}}$ identified as the image: $\{(g,g)|g \in \widetilde{G}_0\} \subset \widetilde{G_0^{\mathbb{C}}}$.

An example to have in mind is $G := O(3,1)$, with $G_0 \cong SO(3,\mathbb{C})$, and $\widetilde{G_0} \cong SL(2,\mathbb{C})$ with $\widetilde{G_0^{\mathbb{C}}} \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$.

We prove the theorem in steps:

Step 1. We extend the action to the universal covering groups.
Let $\widetilde{G}_0 \xrightarrow{p} G_0$ be the universal covering map of G_0. By the discussion above, we may assume w.l.o.g that we are dealing with the groups: $\widetilde{G_0^{\mathbb{C}}} := \widetilde{G}_0 \times \widetilde{G}_0$ and the real form: $\widetilde{G}_0 := \{(g,g)|g \in \widetilde{G}_0\} \subset \widetilde{G_0^{\mathbb{C}}}$. Set $\tilde{\mathfrak{g}}$ for the Lie algebra of $\widetilde{G}_0$, and let $\widetilde{G_0^{\mathbb{C}}} \xrightarrow{p_{\mathbb{C}}} G_0^{\mathbb{C}}$, be the unique lift of the Lie isomorphism:

$$\tilde{\mathfrak{g}}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$$

induced from the following commutative diagram of Lie algebras:

$$\begin{tikzcd}
\tilde{\mathfrak{g}} \arrow[r, "p_*"] \arrow[d, "i"'] & \mathfrak{g} \arrow[r, "i"] & \mathfrak{g}^{\mathbb{C}} \\
\tilde{\mathfrak{g}}^{\mathbb{C}} \arrow[urr, "\exists!"] & &
\end{tikzcd}$$

Explicitly this unique map is given by: $\tilde{x} + i\tilde{y} \mapsto p_*(\tilde{x}) + ip_*(\tilde{y}),\ \tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}}$.
The map $p_{\mathbb{C}}$ is a universal covering map of $G_0^{\mathbb{C}}$, simply because $G_0^{\mathbb{C}}$ and the cover is connected, so the map is surjective. In particular we can induce an action of the covering groups on V and $V^{\mathbb{C}}$ using the action: $G^{\mathbb{C}} \xrightarrow{\rho} GL(V^{\mathbb{C}})$ restricted to the identity components: $G_0, G_0^{\mathbb{C}}$. The following commutative diagram of Lie groups illustrates the induced action:

$$\begin{tikzcd}
\widetilde{G}_0 \arrow[r, "p"] \arrow[d, "i"'] & G_0 \arrow[r, "i"] & G_0^{\mathbb{C}} \arrow[d, "\rho"] \\
\widetilde{G_0^{\mathbb{C}}} \arrow[urr, "p_{\mathbb{C}}"] \arrow[rr, "\rho \circ p_{\mathbb{C}}"] & & GL(V^{\mathbb{C}})
\end{tikzcd}$$

Step 2. We extend an equivalent action of the covering groups to $V \oplus V \cong V^{\mathbb{C}}$.
Now since V is a complex vector space with a complex structure J, we have an isomorphism $V^{\mathbb{C}} \xrightarrow{\phi} V \oplus V$, given by

$$v_1 + iv_2 \mapsto \Big(v_1 + J(v_2), v_2 - J(v_2)\Big).$$

So we may extend our action to $V \oplus V$ using ϕ, given by the commutative diagram as follows:

$$\begin{tikzcd}
\widetilde{G_0^{\mathbb{C}}} \arrow[r, "\rho\circ p_{\mathbb{C}}"] & GL(V^{\mathbb{C}}) \arrow[r, "\phi"] & GL(V\oplus V) \\
\widetilde{G_0} \arrow[u, "i"] \arrow[r] & GL(V) \arrow[u, "i"] \arrow[r, "\phi"] & GL(V) \arrow[u, "i"]
\end{tikzcd}$$

Step 3. There is a unique real orbit in the complex orbit.

We claim that for $(g,h) \in \widetilde{G_0^{\mathbb{C}}}$, the action is given by the product action:

$$(g,h)\cdot(v_1,v_2) := \Big((g,g)\cdot v_1, (h,h)\cdot v_2\Big),$$

where $(g,h)\cdot v_j := (\rho\circ p_{\mathbb{C}})(g,h)(v_j)$. Indeed since $\widetilde{G_0^{\mathbb{C}}}$ is the universal complexification group, and the product action above is clearly holomorphic, since our action $G \to GL(V)$ is holomorphic, then it is enough to show that the action of the real form $\widetilde{G_0}$ on V is the product action. By definition we have:

$$(g,g)\cdot(v,v) := \phi\Big((g,g)\cdot(\phi^{-1}(v,v))\Big) = \phi((g,g)\cdot v) = \Big((g,g)\cdot v, (g,g)\cdot v\Big), \forall v \in V.$$

Now clearly we have a unique real orbit in the complex orbit, i.e:

$$\widetilde{G_0^{\mathbb{C}}}v \cap V = \widetilde{G_0}v, \forall v \in V.$$

So finally we derive our theorem:

Proof of theorem 5.12. We note that in order to prove the statement for the groups G and $G^{\mathbb{C}}$, then it is enough to prove the statement for the restricted action of the identity components. This is seen as follows. By definition of G being a real form of $G^{\mathbb{C}}$, (see Definition 4.1), this means that $G^{\mathbb{C}} = GG_0^{\mathbb{C}}$ as abstract groups. So if we have $v_1, v_2 \in V$ belonging to the same $G^{\mathbb{C}}$-orbit then write $gh\cdot v_1 = v_2$ for $g \in G$ and $h \in G_0^{\mathbb{C}}$, so clearly $g^{-1}v_2$ and v_1 belong to the same $G_0^{\mathbb{C}}$-orbit. Now if $G_0v_1 = G_0g^{-1}v_2$, then also $Gv_1 = Gv_2$. Finally to prove it for the identity components it is enough to prove the statement for the induced action of the universal covering groups: $\widetilde{G_0}$ and $\widetilde{G_0^{\mathbb{C}}}$, on V and $V^{\mathbb{C}}$ defined as above. But since the statement has already been proven for this case, then the theorem is proved. □

We can naturally apply the theorem to the adjoint action of any complex Lie group G. For example $O(3,1)$ has the structure of a complex Lie group, and is the only one among the pseudo-orthogonal Lie groups: $O(p,q)$. So we can apply the theorem to for instance the diagram:

$$
\begin{array}{ccc}
O(4,\mathbb{C}) & \xrightarrow{Ad} & GL(\mathfrak{o}(4,\mathbb{C})) \\
{\scriptstyle i}\big\uparrow & & {\scriptstyle i}\big\uparrow \\
O(3,1) & \xrightarrow{Ad} & GL(\mathfrak{o}(3,1))
\end{array}
\tag{18}
$$

6. Applications to the pseudo-Riemannian setting

For the Lorentzian spaces it is useful to use the boost-weight decomposition and the corresponding algebraic classification of tensors [14], see Appendix B. This turns out to give a very crisp result as to which spaces have non-closed orbits and hence cannot be Wick-rotated to a Riemannian space:

Theorem 6.1. *Given a Lorentzian manifold and assume that (any of) the curvature tensors is of proper type II, III, or N. Then it cannot be Wick-rotated to a real Riemannian space.*

Proof. Considering the real orbits Gx and $\tilde{G}\tilde{x}$, where $G = O(n)$ and $\tilde{G} = O(n-1,1)$ embedded into the same $G^{\mathbb{C}}x$. Then by Theorem 5.1, we have that the real orbits are (topologically) closed if and only if $G^{\mathbb{C}}x$ is closed. Since, $O(n)$ is compact, Gx is necessary compact and closed. Hence, $G^{\mathbb{C}}x$, is closed, implying $\tilde{G}\tilde{x}$ is closed. However, tensors of proper type II, III, and N, do not have closed orbits, see [9]. □

This result can be generalised to the pseudo-Riemannian case by classifying the types of tensors that give non-closed orbits. For the pseudo-Riemannian case, the Lie algebra $\mathfrak{g} = \mathfrak{o}(p,q)$ can be split into a positive eigenspace and negative eigenspace of the Cartan involution:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

If the orbit Gx is not closed, then by Theorem 5.3 there exists an $X \in \mathfrak{p}$ and a $v_0 \in \overline{Gx} \setminus Gx \subset V$ so that $\exp(tX)x \to v_0$ as $t \to -\infty$. Thus, this implies the existence of a v_0 on the boundary of Gx which is not in Gx. The orbit Gx is therefore not closed. Note also that the Lie algebra element $X \in \mathfrak{p}$ generates a one-parameter group $B_t := \{\exp(Xt) \ : \ t \in \mathbb{R}\} \subset G$ manifesting this limit.

We recall that a tensor T living on a pseudo-Riemannian manifold can be decomposed using the boost-weight decomposition with respect to a "null-frame" [27]. Let $k = \min(p,q)$ be the real rank of the group $O(p,q)$. Then, in terms of a orthonormal frame:

$$g(X,Y) = -X_1Y_1 - ... - X_kY_k + X_{k+1}Y_{k+1} + ... + X_nY_n. \tag{19}$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be the largest abelian subalgebra of $\mathfrak{p}$. All such will have dimension equal to k. For each $\lambda \in \mathfrak{a}^*$ (the dual of $\mathfrak{a}$) we define

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} : [y,x] = \lambda(y)x \text{ for } y \in \mathfrak{a}\}.$$

The λ is called the *restricted root* of $(\mathfrak{g}, \mathfrak{a})$ if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$ (see, e.g., [16]). Let Σ be the set of restricted roots. The Lie algebra $\mathfrak{g}$ can now be decomposed in the restricted root decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda.$$

Here, $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is the centraliser of $\mathfrak{a}$ in $\mathfrak{k}$.

Let now $\mathfrak{a}' \subset \mathfrak{a}$ be the set of regular elements:

$$\mathfrak{a}' = \{x \in \mathfrak{a} : \lambda(x) \neq 0, \forall \lambda \in \Sigma\}.$$

The set $\mathfrak{a}'$ is the complement of hyperplanes, and let $\mathfrak{a}^+$ be one connected component of $\mathfrak{a}'$ (this is called a Weyl chamber). We say that a root, $\lambda \in \Sigma$, is positive if it has only positive values on $\mathfrak{a}^+$, and *simple* if it cannot be written as a sum of positive roots. If $\{\alpha_1, ..., \alpha_k\}$ is the set of simple roots, then the set $\mathfrak{a}^+$ can be given by:

$$\mathfrak{a}^+ = \{x \in \mathfrak{a} : \alpha_1(x) > 0, ..., \alpha_k(x) > 0\}.$$

By normalising, we can find k linearly independent elements $\mathcal{X}^I \in \mathfrak{a} \subset \mathfrak{p}$, $I = 1, ..., k$, satisfying the following criteria:

(1) $[\mathcal{X}^I, \mathcal{X}^J] = 0$, and
(2) $\alpha_J(\mathcal{X}^I) = \delta^I_J$.

Since $\mathcal{X}^I \in \mathfrak{o}(p, q) \subset \mathrm{End}(T_pM)$, and is symmetric with respect to the inner product $g_\theta(-,-)$ defined in Defn. 2.4, the eigenvalues are real, and we can find simultaneous eigenvectors of T_pM. The corresponding eigenvector decomposition w.r.t. the set $\{\mathcal{X}^I\}$ is identical to the so-called *boost-weight decomposition* [14].

By letting the $\mathcal{X}^I$ act tensorally on $V = \bigoplus T_pM$, an eigenvector decomposition of V can also be achieved. Note that the metric g, as a symmetric tensor, is a zero-eigenvector of all $\mathcal{X}^I$ due to the fact that $\mathcal{X}^i \in \mathfrak{o}(p, q)$. Hence, the duality map, $\sharp : T_pM \to T_p^*M$ induced by the metric $v \overset{\sharp}{\mapsto} v^\sharp \equiv g(v, -)$ preserves the boost-weight decomposition. Thus an arbitrary tensor T can now be decomposed by the eigenvalues with respect to $\mathcal{X}^I$

$$T = \sum_{\mathbf{b} \in \Gamma} (T)_{\mathbf{b}},$$

where $\Gamma \subset \mathbb{Z}^k$ is a finite subset of $\mathbb{Z}^k$.

Lemma 6.2. *Let $x \in \mathfrak{p}$. Then there exists an $\tilde{x} \in Hx$, where H is the stabilizer of the Cartan involution, so that $\tilde{x} = \lambda_1\mathcal{X}^1 + ... + \lambda_k\mathcal{X}^k$, where $\mathcal{X}^I$ are the elements given above.*

Proof. The stabilizer of the Cartan involution, H, fulfills $H^{-1}\theta H = \theta$, and is the largest compact subgroup of $O(p, q)$. In particular, $O(p) \times O(q) \subset H$. In terms of the orthonormal frame, we can write $x \in \mathfrak{p}$ in block-form:

$$x = \begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix},$$

where A is a $p \times q$ matrix. The action of $O(p) \times O(q)$ on x induces the following action on A: $A \mapsto h^{-1}Ag$ where $(h, g) \in O(p) \times O(q)$. By the singular value decomposition, we can always find a $(h, g) \in O(p) \times O(q)$ so that $h^{-1}Ag$ is diagonal. Thus $h^{-1}Ag = \mathrm{diag}(\lambda_1, ..., \lambda_k)$ and $\tilde{x} = \lambda_1\mathcal{X}^1 + ... + \lambda_k\mathcal{X}^k$. □

Using the representative $\tilde{x}$ rather than x, we can now give a criteria for when a tensor does not have a closed orbit. Given a non-closed orbit Gx. Then there exists a $\tilde{x} = \lambda_1\mathcal{X}^1 + ... + \lambda_k\mathcal{X}^k \in \mathfrak{p}$ so that for $\mathbf{b} = (b_1, ..., b_k)$ we have when $t \to \infty$

$$B_t(T) = \exp(t\tilde{x})(T) = \sum_{\mathbf{b}\in\Gamma} \exp[t(b_1\lambda_1 + ... + b_k\lambda_k)(T)_{\mathbf{b}} \to v_0.$$

For this limit to exist we have either:

(1) $(T)_{\mathbf{b}} = 0$, or
(2) $b_1\lambda_1 + ... + b_k\lambda_k \leq 0$,

for all $\mathbf{b} \in \Gamma$. Tensors for which such a $\tilde{x} \in \mathfrak{p}$ exists are referred to as tensors possessing the S^G-property [17]. If v_0 is not it the G-orbit of T, then the orbit is not closed.

6.1. **Pseudo-Riemannian examples.**
4-dimensional Neutral examples: Walker metrics. The Walker metrics allow for an invariant null-plane and provide with examples of metrics that do not allow for a Wick-rotation. Walker [43] showed that the requirement of an invariant 2-dimensional null plane implies that the (Walker) metric can be written in the canonical form:

$$\mathrm{d}s^2 = 2\mathrm{d}u(\mathrm{d}v + A\mathrm{d}u + C\mathrm{d}U) + 2\mathrm{d}U(\mathrm{d}V + B\mathrm{d}U), \tag{20}$$

where A, B and C are functions that may depend on all of the coordinates. By introducing the null-coframe:

$$\{e^1, e^2, e^3, e^4\} = \{\mathrm{d}u, \mathrm{d}v + A\mathrm{d}u + C\mathrm{d}U, \mathrm{d}U, \mathrm{d}V + B\mathrm{d}U\} \tag{21}$$

We express the Riemann tensor in terms of this frame so that:

$$R = R_{ijkl}e^i \wedge e^i \otimes e^k \wedge e^l$$

Then define the boost-weight of a component, R_{ijkl}, as the a pair (b_1, b_2) where

$$(b_1, b_2) = (\#(2) - \#(1), \#(4) - \#(3)),$$

where $\#(n)$ means the number of indices equal to n. We note that the isometry $\phi: \{e^1, e^2, e^3, e^4\} \mapsto \{e^3, e^4, e^1, e^2\}$ interchanges the boost components: $(b_1, b_2) \mapsto (b_2, b_1)$.

For the Walker metrics, one can easily compute the Riemann tensor and one observes that $R_{ijkl} = 0$ if $b_1 + b_2 > 0$. Different functional forms of the functions

A, B, and C, gives various possibilities for the remaining components. Whether the orbits form closed orbits or not is summarised in the following table. Here we have assumed that the types are "proper", i.e., it is not possible to find other frames so that it is in a simpler category.

R_{ijkl}	$b_1+b_2>0$	$b_1+b_2<0$	$0<b_1=-b_2$	$b_1=-b_2<0$	Closed?
W1	0	$\neq 0$	Any	Any	No
W2	0	0	$\neq 0$	$\neq 0$	Yes
W3	0	0	0	$\neq 0$	No
W4	0	0	0	0	Yes

The generic Walker metric (type W1) is not closed and thus not possible to Wick-rotate to a Riemannian space. Examples of Walker metrics are given in [17]. As simple examples in each category (a, b, c and d are non-zero constants):

$$\begin{aligned}
&\mathrm{W1}: \quad && \mathrm{d}s_1^2 = 2\mathrm{d}u(\mathrm{d}v + V\mathrm{d}u) + 2\mathrm{d}U(\mathrm{d}V + av^4\mathrm{d}U)\\
&\mathrm{W2}: \quad && \mathrm{d}s_2^2 = 2\mathrm{d}u(\mathrm{d}v + (av^2+bV^2)\mathrm{d}u) + 2\mathrm{d}U(\mathrm{d}V + (cv^2+dV^2)\mathrm{d}U)\\
&\mathrm{W3}: \quad && \mathrm{d}s_3^2 = 2\mathrm{d}u(\mathrm{d}v + (av^2+bV^2)\mathrm{d}u) + 2\mathrm{d}U(\mathrm{d}V + cV^2\mathrm{d}U)\\
&\mathrm{W4}: \quad && \mathrm{d}s_4^2 = 2\mathrm{d}u(\mathrm{d}v + av^2\mathrm{d}u) + 2\mathrm{d}U(\mathrm{d}V + bV^2\mathrm{d}U) \qquad (22)
\end{aligned}$$

Of these, the W4 example metric can be Wick-rotated to a Riemannian space, while the W1 and W3 cannot (in general) due to the fact that they do not have closed orbits. However, both the W3 and W4 examples can be Wick-rotated to Lorentzian spaces.

Acknowledgements

This work was supported through the Research Council of Norway, Toppforsk grant no. 250367: *Pseudo-Riemannian Geometry and Polynomial Curvature Invariants: Classification, Characterisation and Applications.*

Appendix A. On compatible Hermitian inner products

We follow the notation in Subsection 5.2, and prove an extension of the statement *2.9* in [7]. Recall that a Hermitian inner product $\langle -,-\rangle$ on $V^{\mathbb{C}}$ is said to be *compatible* with V, if $\langle -,-\rangle \in \mathbb{R}$ on V.

Proposition A.1. *Assume we have a compatible triple:* $\left(\mathfrak{g},\tilde{\mathfrak{g}},\mathfrak{u}\right)$. *If* V *and* $\tilde{V}$ *are compatible real forms of* $V^{\mathbb{C}}$*, then there exist a* U*-invariant Hermitian inner product on* $V^{\mathbb{C}}$ *which is compatible with* V *and* $\tilde{V}$*.*

Proof. Let $G^{\mathbb{C}} \subset GL(n,\mathbb{C})$ for $n \geq 1$ minimal. Now since $n \leq Dim_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$, then we may as well assume that $G, \tilde{G} \subset G^{\mathbb{C}} \subset GL(\mathfrak{g}^{\mathbb{C}})$. In particular the conjugation maps σ and $\tilde{\sigma}$ of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ commute, and are naturally contained in $GL(\mathfrak{g}^{\mathbb{C}})$, by assumption. Denote $\mathfrak{g}^{\mathbb{C}} \xrightarrow{J} \mathfrak{g}^{\mathbb{C}}$ for the complex structure. We now follow the *proof of 2.9* in [7], and do a slight change.

Set $(G^{\mathbb{C}})^* \subset GL(\mathfrak{g}^{\mathbb{C}})$ for the closed subgroup generated by $G^{\mathbb{C}}, J, \sigma$ and $\tilde{\sigma}$. Similarly define K^* (respectively $\tilde{K}^*$) to be the compact subgroups of $(G^{\mathbb{C}})^*$ generated by K, J and σ (respectively $\tilde{K}, J$ and $\tilde{\sigma}$). Now since our Lie algebras form a compatible triple, then we know that $K, \tilde{K} \subset U$. So we also define U^* to be the compact subgroup of $(G^{\mathbb{C}})^*$ generated by U, J, σ and $\tilde{\sigma}$. Then clearly $U^* \cap G^{\mathbb{C}} = U$.

Now let σ_1 and $\tilde{\sigma}_1$ be the conjugation maps of V and $\tilde{V}$ respectively. Also let $V^{\mathbb{C}} \xrightarrow{\tilde{J}} V^{\mathbb{C}}$ be the complex structure. Now we can easily extend the real representation: $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$, to a real representation: $(G^{\mathbb{C}})^* \xrightarrow{(\rho^{\mathbb{C}})^*} GL(V^{\mathbb{C}})$, by simply defining:

$$(\rho^{\mathbb{C}})^* = \rho^{\mathbb{C}},\ \textit{on}\ G^{\mathbb{C}},\quad (\rho^{\mathbb{C}})^*(\sigma) = \sigma_1,\ (\rho^{\mathbb{C}})^*(\tilde{\sigma}) = \tilde{\sigma}_1,\ \textit{and}\ (\rho^{\mathbb{C}})^*(J) = \tilde{J}.$$

The map is well-defined since $[\sigma_1, \tilde{\sigma}_1] = 0$, as V and $\tilde{V}$ are assumed to be compatible. It now follows that there exist a U-invariant Hermitian form, which is compatible with V and $\tilde{V}$, as in the *proof of 2.9* in [7]. The proposition is proved. $\square$

An immediate corollary is the following:

Corollary A.2. *Assume we have a compatible triple:* $\left(\mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{u}\right)$*. Let* V *and* $\tilde{V}$ *be compatible real forms of* $V^{\mathbb{C}}$*. Then we can assume w.l.o.g that* $\mathcal{M}(G, V)$ *(the minimal vectors of* ρ*) and* $\mathcal{M}(\tilde{G}, \tilde{V})$ *(the minimal vectors of* $\tilde{\rho}$*) are both contained in the same set of minimal vectors of the complexified action* $\rho^{\mathbb{C}}$*, i.e., we have embeddings*

$$\begin{array}{ccc} \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}) & \xleftarrow{\ i\ } & \mathcal{M}(\tilde{G}, \tilde{V}) \\ \big\uparrow{\scriptstyle i} & & \\ \mathcal{M}(G, V) & & \end{array}$$

Appendix B. The boost-weight decomposition in the Lorentzian case

An algebraic classification of tensors T has been developed [14] which is based on the existence of certain normal forms of (23) through successive application of null rotations and spin-boost. In the special case where T is the Weyl tensor in four dimensions, this classification reduces to the well-known Petrov classification. However, the boost weight decomposition can be used in the classification of any tensor T in arbitrary dimensions.

Given a covariant tensor T with respect to a null frame, $\{\ell, n, m^i\}$, the effect of a boost $\ell \mapsto e^{\lambda}\ell$, $n \mapsto e^{-\lambda}n$ allows T to be decomposed according to its boost weight components

$$T = \sum_b (T)_b \tag{23}$$

where $(T)_b$ denotes the boost weight b components (with respect to the above-mentioned boost) of T. As an application, a Riemann tensor of type G has the following decomposition

$$R = (R)_{+2} + (R)_{+1} + (R)_0 + (R)_{-1} + (R)_{-2} \tag{24}$$

in every null frame. A Riemann tensor is algebraically special if there exists a frame in which certain boost weight components can be transformed to zero, these are summarized in Table 1.

In general we can define the following algebraic special cases:

Definition B.1. A tensor, T, is of

- type II if there exists a frame such that all the positive boost-weight components are zero, $(T)_{b>0} = 0$;
- type D if there exists a frame such that T has only boost-weight 0 components, $T = (T)_0$;
- type III if there exists a frame such T has only negative boost weights, $(T)_{b\geq 0} = 0$.

If the tensor is of neither of these cases the tensor is of type I/G, or more general [14].

This implies that a tensor, T, of type II has only non-positive boost weight components:

$$T = \sum_{b\leq 0} (T)_b, \qquad \text{(type II)}.$$

Riemann type	**Conditions**
G	—
I	$(R)_{+2} = 0$
II	$(R)_{+2} = (R)_{+1} = 0$
III	$(R)_{+2} = (R)_{+1} = (R)_0 = 0$
N	$(R)_{+2} = (R)_{+1} = (R)_0 = (R)_{-1} = 0$
D	$(R)_{+2} = (R)_{+1} = (R)_{-1} = (R)_{-2} = 0$
O	all vanish (Minkowski space)

Table 1. The relation between Riemann types and the vanishing of boost weight components. For example, $(R)_{+2}$ corresponds to the frame components R_{1i1j}.

Part 4. **A Wick-rotatable metric is purely electric**

The following part is precisely the published paper in the Journal of Geometry and Physics:

A Wick-rotatable metric is purely electric, C. Helleland, S. Hervik, *J. Geom. Phys. 123 (2018) 424-429*, https://doi.org/10.1016/j.geomphys.2017.09.015.

Abstract. We show that a metric of arbitrary dimension and signature which allows for a standard Wick-rotation to a Riemannian metric necessarily has a purely electric Riemann and Weyl tensor.

1. Introduction

In quantum theories a Wick-rotation is a mathematical trick to relate Minkowski space to Euclidean space by a complex analytic extension to imaginary time. This enables us to relate a quantum mechanical problem to a statistical mechanical one relating time to the inverse temperature. This trick is highly successful and is used in a wide area of physics, from statistical and quantum mechanics to Euclidean gravity and exact solutions.

In spite of its success, there is a question about its range of applicability. A question we can ask is: *Given a spacetime, does there exist a Wick-rotation to transform the metric to a Euclidean one?*

Here we will give a partial answer to this question and will give a necessary condition for a Wick-rotation (as defined below) to exist. However, before we prove our main theorem, we need to be a bit more precise with what we mean by a Wick-rotation. Consider a pseudo-Riemannian metric (of arbitrary dimension and signature). We need to allow for more general coordinate transformations than the real diffeomorphisms preserving the metric signature – namely to complex analytic continuations of the real metric [1, 2] .

Consider a point p and a neighbourhood, U, of p. Assume this nighbourhood is an analytic neighbourhood and that x^μ are coordinates on U so that $x^\mu \in \mathbb{R}^n$. We will adapt the coordinates to the point p so that p is at the origin of this coordinate system. Consider now the complexification of $x^\mu \mapsto x^\mu + iy^\mu = z^\mu \in \mathbb{C}^n$. This complexification enables us to consider the complex analytic neighbourhood $U^{\mathbb{C}}$ of p.

Furthermore, let $g^{\mathbb{C}}_{\mu\nu}$ be a complex bilinear form (a holomorphic metric) induced by the analytic extension of the metric:

$$g_{\mu\nu}(x^\rho)\mathrm{d}x^\mu\mathrm{d}x^\nu \mapsto g^{\mathbb{C}}_{\mu\nu}(z^\rho)\mathrm{d}z^\mu\mathrm{d}z^\nu.$$

Next, consider a real analytic submanifold containing p: $\bar{U} \subset U^{\mathbb{C}}$ with coordinates $\bar{x}^\mu \in \mathbb{R}^n$. The imbedding $\iota : \bar{U} \mapsto U^{\mathbb{C}}$ enables us to pull back the complexified metric $\boldsymbol{g}^{\mathbb{C}}$ onto $\bar{U}$:

$$\bar{\boldsymbol{g}} \equiv \iota^* \boldsymbol{g}^{\mathbb{C}}. \tag{25}$$

In terms of the coordinates $\bar{x}^\mu$: $\bar{\boldsymbol{g}} = \bar{g}_{\mu\nu}(\bar{x}^\rho)\mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu$. This bilinear form may or may not be real. However, *if the bilinear form $\bar{g}_{\mu\nu}(\bar{x}^\rho)\mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu$ is real (and non-degenerate) then we will call it an analytic extension of $g_{\mu\nu}(x^\rho)\mathrm{d}x^\mu \mathrm{d}x^\nu$ with respect to p,* or simply *a Wick-rotation* of the real metric $g_{\mu\nu}(x^\rho)\mathrm{d}x^\mu \mathrm{d}x^\nu$. This clearly generalises the concept of Wick-rotations from the standard Minkowskian setting to a more general setting [25].

In the following, let us call the Wick-rotation, in the sense above, for $\bar{\phi}$; i.e., $\bar{\phi} : U \to \bar{U}$. We note that this transformation is complex, and we can assume, since U is real analytic, that $\bar{\phi}$ is analytic.

The Wick-rotation in the sense above, leaves the point p stationary. It therefore induces a linear transformation, M, between the tangent spaces T_pU and $T_p\bar{U}$. The transformation M is complex and therefore may change the metric signature; consequently, even if the metric $\bar{g}_{\mu\nu}$ is real, it does not necessarily need to have the same signature of $g_{\mu\nu}$.

Consider now the curvature tensors, R and $\nabla^{(k)}R$ for $g_{\mu\nu}$, and $\bar{R}$ and $\bar{\nabla}^{(k)}\bar{R}$ for $\bar{g}_{\mu\nu}$. Since both metrics are real, their curvature tensors also have to be real. The analytic continuation, in the sense above, induces a linear transformation of the tangent spaces; consequently, this would relate the Riemann tensors R and $\bar{R}$ through a complex linear transformation. It is useful to introduce an orthonormal frame $\mathbf{e}_\mu$. The orthonormal frames $\mathbf{e}_\mu$ and $\bar{\mathbf{e}}_\mu$ are related through their complexified frame $\mathbf{e}^{\mathbb{C}}_\mu$. We can define a complex orthonormal frame requiring the inner product[1] $\langle \mathbf{e}^{\mathbb{C}}_\mu, \mathbf{e}^{\mathbb{C}}_\nu \rangle = \delta_{\mu\nu}$. This inner product is invariant under the complex orthogonal transformations, $O(n,\mathbb{C})$. The real frames $\mathbf{e}_\mu$ and $\bar{\mathbf{e}}_\mu$ are obtained by restricting the complex frame. As an example, consider the standard holomorphic inner product space $(\mathbb{C}^n, \boldsymbol{g}^{\mathbb{C}}_0)$ and $(\mathbf{e}^{\mathbb{C}}_1, ..., \mathbf{e}^{\mathbb{C}}_n)$ the standard basis. Then a real subspace is $V = \mathrm{span}_{\mathbb{R}}(i\mathbf{e}^{\mathbb{C}}_1, ..., i\mathbf{e}^{\mathbb{C}}_p, \mathbf{e}^{\mathbb{C}}_{p+1}, ..., \mathbf{e}^{\mathbb{C}}_n)$, and the corresponding metric (obtained from $\boldsymbol{g}^{\mathbb{C}}_0$ by restriction) is real. All such real subspaces V (of different signatures) are obtained from such identifications and hence different real subspaces V are related via the action of the complex orthogonal group $O(n,\mathbb{C})$ (for more details, see e.g. [6, 20]).

Hence, we consider the real vector spaces T_pU and $T_p\bar{U}$ as embedded in the complexified vector space $(T_pU)^{\mathbb{C}} \cong (T_p\bar{U})^{\mathbb{C}}$. The real frames are thus related though a restriction of a complex frame having an $O(n,\mathbb{C})$ structure group. If moreover the tangent spaces T_pU and $T_p\bar{U}$ are embedded:

$$T_pU, T_p\bar{U} \hookrightarrow (T_pU)^{\mathbb{C}} \cong (T_p\bar{U})^{\mathbb{C}},$$

[1] This is a not really a proper inner product since it is not positive definite, but rather a $\mathbb{C}$-bilinear non-degenerate form defining a *holomorphic inner product*.

such that they form a *compatible triple* [2], then we shall say that the real submanifolds: U and $\bar{U}$, are related through a *standard Wick-rotation*. A standard Wick-rotation allows us to choose commuting Cartan involutions of the real metrics.

Note the special case where $\bar{U}$ is Riemannian, then the condition of being Wick-rotated by a standard Wick-rotation, is just the condition that the conjugation maps of T_pU, and $T_p\bar{U}$ must commute when embedded into $(T_pU)^{\mathbb{C}}$, i.e T_pU and $T_p\bar{U}$ are compatible real forms.

We refer to the manuscript [20], for more details about standard Wick-rotations and the connection with real GIT, and the special case of $\bar{U}$ being Riemannian.

By using $\bar{\phi}$ we can relate the metrics $\boldsymbol{g} = \bar{\phi}^*\bar{\boldsymbol{g}}$. Since the map is analytic (albeit complex), the curvature tensors are also related via $\bar{\phi}$. If R and $\bar{R}$ are the Riemann curvature tensors for U and $\bar{U}$ respectively, then these are related, using an orthonormal frame, via an $O(n,\mathbb{C})$ transformation. Consider the components of the Riemann tensor as a vector in some $\mathbb{R}^N \subset \mathbb{C}^N$. If there exists a Wick-rotation of the metric at p, then the (real) Riemann curvature tensors of U and $\bar{U}$ must be real restrictions of vectors that lie in the same $O(n,\mathbb{C})$ orbit in $\mathbb{C}^N$.

Note: This definition of a Wick-rotation does not include the more general analytic continuations defined by Lozanovski [4]. In particular, we consider *one* particular metric (thus not a family of them) and we require that the point p is fixed and is therefore more of a complex rotation.

. In the following we will utilise the study of real orbits of semi-simple groups, see e.g. [7, 39]. In particular, the considerations made in [9] will be useful. For a more general introduction to the structure of Lie algebras including the Cartan involution, see, for example [16, 40].

2. The electric/magnetic parts of a tensor

Following [9], we can introduce the electric and magnetic parts of a tensor by considering the eigenvalue decomposition of the tensor under the Cartan involution θ of the real Lie algebras $\mathfrak{o}(p,q)$. This involution can be extended to all tensors, and to vectors $\mathbf{v} \in T_pM$ in particular. Considering an orthonormal frame, so that:

$$\boldsymbol{g}(\mathbf{e}_\mu, \mathbf{e}_\mu) = \begin{cases} -1, & 1 \leq \mu \leq p \\ +1 & p+1 \leq \mu \leq p+q = n, \end{cases}$$

[2] Let W and $\widetilde{W}$ be real slices of a holomorphic inner product space: (E,g). Assume they are both real forms of $W^{\mathbb{C}} \subset (E,g)$. Let V be another real slice of E, and a real form of $W^{\mathbb{C}}$, with Euclidean signature. Suppose $W, \widetilde{W}$ and V are pairwise compatible (i.e their conjugation maps commute pairwise), then a triple: $(W,\widetilde{W},V)$, will be called a *compatible triple*. **Examples**: $\Big(\mathbb{R}\oplus i\mathbb{R}, i\mathbb{R}\oplus\mathbb{R}, \mathbb{R}^2\Big)$ with $E := \mathbb{C}^2$, and $\Big(\mathfrak{o}(p,q), \mathfrak{o}(\tilde{p},\tilde{q}), \mathfrak{o}(n)\Big)$ with $E := \mathfrak{o}(n,\mathbb{C})$ and $g := \kappa(-,-)$ (the Killing form).

the $\theta : T_pM \to T_pM$, can be defined as the linear operator:

$$\theta(\mathbf{e}_\mu) = \begin{cases} -\mathbf{e}_\mu, & 1 \leq \mu \leq p \\ +\mathbf{e}_\mu & p+1 \leq \mu \leq p+q = n. \end{cases}$$

Clearly, this implies that the bilinear map:

$$\langle X, Y \rangle_\theta := \boldsymbol{g}(\theta(X), Y), \qquad X, Y \in T_pM$$

defines a positive definite inner-product on T_pM. This Cartan involution can be extended tensorially to arbitrary tensor products.

Given a Cartan involution θ, then since $\theta^2 = \mathrm{Id}$, its eigenvalues are ± 1 and any tensor T has an eigenvalue decomposition:

$$T = T_+ + T_-, \qquad \text{where } \theta(T_\pm) = \pm T_\pm.$$

A space is called *purely electric* (PE) if there exists a Cartan involution so that the Weyl tensor decomposes as $C = C_+$ [9]. Furthermore, a space is called *purely magnetic* (PM) if the Weyl tensor decomposes as $C = C_-$. If this property occurs also for the Riemann tensor, we call the space Riemann purely electric (RPE) or magnetic (RPM), respectively. Clearly, RPE implies PE.

3. The Riemann curvature operator

The Riemann curvature tensor can (pointwise) be seen as a bivector operator:

$$\mathrm{Riem} : \wedge^2\Omega_p(M) \to \wedge^2\Omega_p(M).$$

In a pseudo-Riemannian space of signature (p, q) the metric $\boldsymbol{g}$ will provide an isomorphism between the space of bivectors, $\wedge^2\Omega_p(M)$, and the Lie algebra $\mathfrak{g} = \mathfrak{o}(p, q)$. This can be seen as follows. The Lie algebra $\mathfrak{o}(p, q)$ is defined through the action of $O(p, q)$ on the tangent space T_pM: For any $G \in O(p, q)$, $G : T_pM \to T_pM$ so that $\boldsymbol{g}(G \cdot v, G \cdot u) = \boldsymbol{g}(v, u)$ for all $v, u \in T_pM$. Using the exponential map $\exp : \mathfrak{o}(p, q) \to O(p, q)$, we get the requirement that $\boldsymbol{g}(X(v), u) + \boldsymbol{g}(v, X(u)) = 0$ for any $X \in \mathfrak{o}(p, q)$. Consequently, X is antisymmetric with respect to the metric $\boldsymbol{g}$. In terms of the basis vectors, we can write $X = (X^\mu{}_\nu)$ and the antisymmetry condition implies that by raising an index we get $X^{\mu\nu} = -X^{\nu\mu}$ and can therefore be considered as a bivector. Since the dimensions match, the metric thus provides with an isomorphism between the Lie algebra $\mathfrak{o}(p, q)$ and the space of bivectors $\wedge^2\Omega_p(M)$ at a point[3].

Consequently, the Riemann curvature operator can also be viewed as an endomorphism of $V := \mathfrak{g}$ treated as a vector space. Consider therefore any $\mathsf{R} \in \mathrm{End}(V)$:

$$\mathsf{R} : V \to V.$$

[3]Indeed, this is a mere consequence of the fact that there is an $O(p, q)$-module isomorphism between T_pM and T_p^*M.

This endomorphism can be split in a symmetric and anti-symmetric part, $\mathsf{R} = \mathsf{S} + \mathsf{A}$, with respect to the metric induced by $\boldsymbol{g}$ (which we also will call $\boldsymbol{g}$ and is proportional to the Killing form κ on V)[4]:

$$\boldsymbol{g}(\mathsf{S}(x), y) = \boldsymbol{g}(\mathsf{S}(y), x), \quad \boldsymbol{g}(\mathsf{A}(x), y) = -\boldsymbol{g}(\mathsf{A}(y), x) \qquad \forall x, y \in \mathfrak{g}.$$

This metric is invariant under the Lie group action of $G = O(p,q)$:

$$\boldsymbol{g}(h \cdot x, h \cdot y) = \boldsymbol{g}(x, y),$$

where $h \cdot x$ is the natural Lie group action on the Lie algebra given by the adjoint: $h \cdot x := \mathrm{Ad}_h(x) = h^{-1}xh$.

Consider now a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$. Then we define the inner-product on $V = \mathfrak{g}$ as follows:

$$\langle x, y \rangle_\theta = \boldsymbol{g}(\theta(x), y),$$

which is just proportional to $\kappa_\theta(-,-) := -\kappa(-, \theta(-))$. We can now, similarly, split any $\mathsf{R} \in \mathrm{End}(V)$ in a symmetric and anti-symmetric part, $\mathsf{R} = \mathsf{R}_+ + \mathsf{R}_-$, with respect to the inner-product $\langle -, - \rangle_\theta$:

$$\langle \mathsf{R}_+(x), y \rangle_\theta = \langle \mathsf{R}_+(y), x \rangle_\theta\,, \quad \langle \mathsf{R}_-(x), y \rangle_\theta = -\langle \mathsf{R}_-(y), x \rangle_\theta\,, \qquad \forall x, y \in \mathfrak{g}.$$

We shall denote $V = \mathfrak{k} \oplus \mathfrak{p}$, for the Cartan decomposition w.r.t θ, where $\mathfrak{k}$ is the compact part and $\mathfrak{p}$ is the non-compact part.

Suppose now that the real submanifolds U and $\bar{U}$ are two Wick-rotatable spaces (of the same dimension) by a standard Wick-rotation at a fixed intersection point p, but with one of the real slices being Riemannian. So we can set $V := \mathfrak{o}(p,q)$ as before, and introduce (similarly as with V above), $\tilde{V} := \mathfrak{o}(n)$, a compact real form of $V^{\mathbb{C}} := \mathfrak{o}(n, \mathbb{C})$. These real forms V and $\tilde{V}$, will naturally be compatible when embedded into $V^{\mathbb{C}}$, w.r.t to a standard Wick-rotation, i.e it lets us fix a Cartan involution θ, such that $\mathfrak{k} = V \cap \tilde{V}$, and $\mathfrak{p} = V \cap i\tilde{V}$. Again we refer to the paper [20] for details.

The space of endomorphisms, $\mathrm{End}(V)$, is also a vector space with the group action given by conjugation:

$$(g \cdot X)(v) := gX(g^{-1}vg)g^{-1}, \quad X \in \mathrm{End(V)},\ v \in V,\ g \in G.$$

Call this action ρ. We can thus define $\mathcal{V} := \mathrm{End}(V)$, and extend the Cartan involution, θ, as well as $\boldsymbol{g}$ tensorially to $\mathcal{V}$. We define analogously an inner product on $\mathcal{V}$:

$$\langle\!\langle X, Y \rangle\!\rangle_\theta = \boldsymbol{g}(\theta(X), Y), \qquad X, Y \in \mathcal{V}.$$

The inner product can assume to have the following properties (see [7]) w.r.t the action ρ:

[4]That the metric induced by $\boldsymbol{g}$ is proportional to the Killing form can be seen either by explicit computation, or from considering κ as a even-ranked tensor over $V^* \otimes V^*$ which is invariant under the action of $O(p,q)$. By, e.g., section 5.3.2 in [41], this tensor is necessarily proportional to the metric tensor on V induced by $\boldsymbol{g}$.

(1) The inner product is K-invariant, where $K \cong O(p) \times O(q)$ is the maximally compact subgroup of G with Lie algebra $\mathfrak{t}$.
(2) $d\rho(\mathfrak{t}) : \mathcal{V} \to \mathcal{V}$ consists of skew-symmetric maps w.r.t $\langle\langle X, Y\rangle\rangle_\theta$.
(3) $d\rho(\mathfrak{p}) : \mathcal{V} \to \mathcal{V}$ consists of symmetric maps w.r.t $\langle\langle X, Y\rangle\rangle_\theta$.

With such an inner product, enables us to apply the results in [7], i.e we can make use of minimal vectors for determining the closure of real orbits.

Defining $\tilde{\mathcal{V}} := \mathrm{End}(\tilde{\mathrm{V}})$ similarly, we have $\mathcal{V}, \tilde{\mathcal{V}} \subset \mathcal{V}^{\mathbb{C}}$ where $\mathcal{V}^{\mathbb{C}} := \mathrm{End}(V^{\mathbb{C}})$. Now since V and $\tilde{V}$ are real forms of $V^{\mathbb{C}}$ then $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are real forms of $\mathcal{V}^{\mathbb{C}}$. This is seen in the following way. A map $\mathsf{R} \in \mathcal{V}$ can be extended to the complex linear map $\mathsf{R}^{\mathbb{C}} \in \mathcal{V}^{\mathbb{C}}$ by defining:

$$\mathsf{R}^{\mathbb{C}}(x+iy) := \mathsf{R}(x) + i\mathsf{R}(y), \quad x, y \in V.$$

So we view a map R as the complex linear map $\mathsf{R}^{\mathbb{C}}$. Thus regard $\tilde{\mathcal{V}}$ like this as well. We shall just write R instead of $\mathsf{R}^{\mathbb{C}}$.

We thus assume we have two endomorphisms (the Riemann curvature operators): $\mathsf{R} : V \to V$ (arbitrary pseudo-Riemannian), and $\tilde{\mathsf{R}} : \tilde{V} \to \tilde{V}$ (Riemannian). Now since we have the two real slices: U and $\bar{U}$, which are Wick-rotated at the point p, then necessarily $\mathsf{R} \in \mathcal{V}$ and $\tilde{\mathsf{R}} \in \tilde{\mathcal{V}}$ must be conjugated by an element $g \in G^{\mathbb{C}} := O(n, \mathbb{C})$.

Set now $G := O(p, q)$ (with Lie algebra $V := \mathfrak{o}(p, q)$) and $\tilde{G} := O(n)$ (with Lie algebra $\tilde{V} := \mathfrak{o}(n)$) for the real forms embedded into $G^{\mathbb{C}}$ (with Lie algebra $V^{\mathbb{C}} := \mathfrak{o}(n, \mathbb{C})$) w.r.t a standard Wick-rotation[5]. Now we have a commutative diagram of conjugation actions:

$$\begin{array}{ccc} G^{\mathbb{C}} & \xrightarrow{\rho^{\mathbb{C}}} & GL(\mathcal{V}^{\mathbb{C}}) \\ {\scriptstyle i}\big\uparrow & & {\scriptstyle i}\big\uparrow \\ G & \xrightarrow{\rho} & GL(\mathcal{V}) \end{array} \tag{26}$$

Where $\rho^{\mathbb{C}}$ is also the action given by conjugation, where $G^{\mathbb{C}}$ is viewed as a real Lie group, and $\mathcal{V}^{\mathbb{C}}$ is also viewed as a real vector space. We similarly have such a diagram for the the group $\tilde{G}$, where the conjugation action: $\tilde{\rho}$, on $\tilde{\mathcal{V}}$ also extends to $\rho^{\mathbb{C}}$.

Now our real Riemann curvature operators from U and $\bar{U}$: R and $\tilde{\mathsf{R}}$, will now lie in the same complex orbit, i.e $G^{\mathbb{C}} \cdot \mathsf{R} = G^{\mathbb{C}} \cdot \tilde{\mathsf{R}}$.

[5] G and $\tilde{G}$ are the structure groups of the real metrics restricting from the holomorphic metric, and thus consist of isometries: $T_pU \to T_pU$ and $T_p\bar{U} \to T_p\bar{U}$ of the real metrics respectively. These groups are naturally embedded into $O(n, \mathbb{C})$ as real forms, by complexification: $f \mapsto f^{\mathbb{C}}$.

So therefore in what follows, we will consider the real orbits, $G \cdot \mathsf{R}$, $\tilde{G} \cdot \tilde{\mathsf{R}}$ and its complexified orbit $G^{\mathbb{C}} \cdot \mathsf{R}$ defined by the conjugation action of the group on an endomorphism: $\mathsf{R} \in \mathcal{V}$ and $\tilde{\mathsf{R}} \in \tilde{\mathcal{V}}$, as follows [7, 39, 9]

$$\begin{aligned} G \cdot \mathsf{R} &:= \{h \cdot \mathsf{R} \mid h \in O(p,q)\} \subset \mathcal{V} \\ \tilde{G} \cdot \tilde{\mathsf{R}} &:= \{h \cdot \tilde{\mathsf{R}} \mid h \in O(n)\} \subset \tilde{\mathcal{V}} \\ G^{\mathbb{C}} \cdot \mathsf{R} &:= \{h \cdot \mathsf{R} \mid h \in O(n,\mathbb{C})\} \subset \mathcal{V}^{\mathbb{C}}. \end{aligned}$$

Theorem 3.1. *Suppose* $\mathsf{R} = \mathsf{S} + \mathsf{A} \in \mathcal{V}$ *where* S, A *are the symmetric/antisymmetric parts w.r.t* $\boldsymbol{g}$ *respectively. Assume that there exists a (real)* $\tilde{\mathsf{R}} \in G^{\mathbb{C}} \cdot \mathsf{R}$ *so that* $\tilde{\mathsf{R}} \in \tilde{\mathcal{V}}$ *i.e we assume:* $G^{\mathbb{C}} \cdot \mathsf{R} = G^{\mathbb{C}} \cdot \tilde{\mathsf{R}}$*. Then there exists a Cartan involution* θ' *of* V *such that* $\mathsf{R}_+ = \mathsf{S}$ *and* $\mathsf{R}_- = \mathsf{A}$*, where* $\mathsf{R}_+, \mathsf{R}_-$ *are the symmetric/antisymmetric parts w.r.t* $\langle -, - \rangle_{\theta'}$ *respectively.*

Proof. Consider the orbits $G \cdot \mathsf{R}$ and $\tilde{G} \cdot \tilde{\mathsf{R}}$. Since the group $\tilde{G}$ is compact, the orbit $\tilde{G} \cdot \tilde{\mathsf{R}}$ is necessarily closed in $\tilde{\mathcal{V}}$; consequently, $G \cdot \mathsf{R}$, is closed as well and possesses a minimal vector[6] [7]. Denote by $\mathcal{M}(G^{\mathbb{C}}, \mathcal{V}^{\mathbb{C}})$ the set of minimal vectors in $\mathcal{V}^{\mathbb{C}}$. Assume that $X \in G \cdot \mathsf{R} \subset \mathcal{V}$ is minimal, then X is also a minimal vector in the complex orbit: $G^{\mathbb{C}} \cdot \mathsf{R}$. However since G and $\tilde{G}$ are compatible real forms (i.e V and $\tilde{V}$ are compatible[7]), and $\tilde{G}$ is a compact real form of $G^{\mathbb{C}}$, then necessarily:

$$G^{\mathbb{C}} \cdot \mathsf{R} \cap \mathcal{M}(G^{\mathbb{C}}, \mathcal{V}^{\mathbb{C}}) = \tilde{G} \cdot \tilde{\mathsf{R}} \subset \tilde{\mathcal{V}},$$

so we deduce that $X \in G \cdot \mathsf{R} \cap \tilde{G} \cdot \tilde{\mathsf{R}} \subset \mathcal{V} \cap \tilde{\mathcal{V}}$.

Now we can choose $g \in G$ such that $g \cdot \mathsf{R} = X$, hence we can conjugate our fixed Cartan involution θ using g, and therefore work with R instead of X. Thus we may assume w.l.o.g that $X := \mathsf{R}$. Now R leaves invariant both V and $\tilde{V}$, in particular implying that:

$$\mathsf{R}(V \cap \tilde{V}) \subset V \cap \tilde{V} \quad \textit{and} \quad \mathsf{R}(V \cap i\tilde{V}) \subset V \cap i\tilde{V}.$$

However again by the compatibility of V and $\tilde{V}$ in $V^{\mathbb{C}}$, we know that $V \cap \tilde{V} = \mathfrak{k}$ and $V \cap i\tilde{V} = \mathfrak{p}$ are the compact/non-compact parts respectively w.r.t our fixed Cartan involution θ. So R and θ commute: $[\mathsf{R}, \theta] = 0$, which immediately implies that $\mathsf{R}_+ = \mathsf{S}$ and $\mathsf{R}_- = \mathsf{A}$ w.r.t θ as required. The theorem is proved.

□

In the case of the Riemann tensor, this is symmetric as a bivector operator with respect to the metric, so we have $\mathsf{R} = \mathsf{S}$, consequently, we get the immediate corollary:

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[6] A vector $X \in V$ is minimal if the norm function $||-|| := \sqrt{\langle -, - \rangle_\theta}$ along an orbit attains a minimum at X; i.e., $||X|| \leq ||h \cdot X||$, $\forall h \in G$.

[7] The conjugation maps of V and $\tilde{V}$ in $V^{\mathbb{C}}$ commute: $\sigma : V^{\mathbb{C}} \to V^{\mathbb{C}}$ and $\tilde{\sigma} : V^{\mathbb{C}} \to V^{\mathbb{C}}$, with $[\sigma, \tilde{\sigma}] = 0$.

Corollary 3.2. *A metric (of arbitrary dimension and signature) allowing for a standard Wick-rotation at a point p to a Riemannian metric, has a purely electric Riemann tensor, and is consequently purely electric, at p.*

We note that this result applies for a general classes of Wick-rotatable metrics. For example, by complexification of the Lie algebras, it is possible to include Wick-rotations between all of the spaces: de Sitter (dS), anti-de Sitter (AdS), the Riemannian sphere (S^n), and hyperbolic space, (H^n). These are all group quotients G/H of different groups G and H. This seems at first sight paradoxical since these have different signs of the curvature. Thus if $\mathsf{R} = g^{-1} \cdot \tilde{\mathsf{R}}$ as claimed in the proof, they would necessarily have the same Ricci scalar[8] . To understand this we first note that when we Wick-rotate to a Riemannian space we may risk to get either a positive definite metric, $\boldsymbol{g}(v,v) \geq 0$, or a negative definite metric, $\boldsymbol{g}(v,v) \leq 0$. The overall sign is conventional and we say that switching the sign using the "anti-isometry", $\boldsymbol{g} \mapsto -\boldsymbol{g}$ is a matter of convention. Note that this switch of the metric gives the same metric for the metric induced by $\boldsymbol{g}$ on the Lie algebra.

Consider the simple example of the complex holomorphic metric

$$\boldsymbol{g}_{\mathbb{C}} = \frac{1}{(1+z_1^2+...+z_n^2)^2}\left[dz_1^2+...+dz_n^2\right] \tag{27}$$

Locally, the two real slices $(z_1,...,z_n) = (x_1,...,x_n)$, and $(z_1,...,z_n) = (iy_1,...,iy_n)$, give a neighbourhood of S^n and H^n respectively. However, note that for hyperbolic space, the induced metric has the "wrong" sign (it is negative definite). Therefore, considering for example the Ricci tensor (by lowering indices appropriately), we get $R_{\mu\nu} = \lambda g_{\mu\nu}$, $\lambda > 0$, for both real slices, and the sign of the curvature is encaptured in whether the metric is positive or negative definite.

4. Discussion

Using techniques from real invariant theory we have considered a class of metrics allowing for a complex Wick-rotation to a Riemannian space. We have showed that these necessarily are rescricted, in particular, they are purely electric. The result is independent of dimension and signature and shows that if such a Wick rotation is allowable, then we necessarily restrict ourselves to classes of spaces where the "magnetic" degrees of freedom have to vanish (at the point p).

There are many examples of purely electric spaces (see [9, 4] and references therein). In particular, a purely electric Lorentzian spacetime is of type G, I_i, D or O [9]. Thus spacetimes not of these types provide with examples of spaces where such a Wick rotation is not allowed. Non-Wick-rotatable metrics include the

[8] The Riemann endomorphism has components related to the Riemann tensor in $T_pM \otimes T_p^*M \otimes (T_pM \otimes T_p^*M)^*$, i.e., $R^{\alpha}{}_{\beta\gamma}{}^{\delta}$. Thus the Ricci scalar is obtained by taking the double trace showing the Ricci scalar is the same after Wick-rotating.

classes of Kundt metrics [42] in Lorentzian geometry, and the Walker metrics [43] of more general signature. Also the metrics considered in [44] are in general non-Wick-rotatable metrics. Note that the plane-wave metrics are non-Wick-rotatable metrics.

These results have profound consequences for quantization frameworks where such Wick-rotation is used, since they give a clear restriction of the class of metrics that allows for such a Wick rotation. Clearly, also in the context of quantum gravity, the (real) gravitational degrees of freedom will be restricted by assuming the existence of such a Wick-rotation.

It is worth mentioning that there are quantization procedures which work in the Lorentzian signature all the way through, in particular, there is the algebraic approach to QFT on curved spacetime [45, 46]. For details on renormalization in Lorentzian signature (without Wick rotation), see e.g., [47].

Acknowledgements

This work was in part supported through the Research Council of Norway, Toppforsk grant no. 250367: *Pseudo-Riemannian Geometry and Polynomial Curvature Invariants: Classification, Characterisation and Applications.*

Part 5. Real GIT with applications to compatible representations and Wick-rotations

The following part is precisely the published paper in the Journal of Geometry and Physics:

C. Helleland, S. Hervik, *Real GIT with applications to compatible representations and Wick-rotations*, https://doi.org/10.1016/j.geomphys.2019.03.007.

Abstract. Motivated by Wick-rotations of pseudo-Riemannian manifolds, we study real geometric invariant theory (GIT) and compatible representations. We extend some of the results from earlier works [20, 21], in particular, we give some sufficient as well as necessary conditions for when pseudo-Riemannian manifolds are Wick-rotatable to other signatures. For arbitrary signatures, we consider a Wick-rotatable pseudo-Riemannian manifold with closed $O(p,q)$-orbits, and thus generalise the existence condition found in [21]. Using these existence conditions we also derive an invariance theorem for Wick-rotations of arbitrary signatures.

1. Introduction

Let (M, g) be a real analytic pseudo-Riemannian manifold. Here we will ask the question: When can such a manifold be Wick-rotated to a (different) pseudo-Riemannian manifold?

A partial answer to this question has already been given in the special case where (M, g) (of arbitrary signature) is Wick-rotated to a Riemannian space at a fixed point p, implying that (M, g) would have to be Riemann purely electric (RPE), see [21]. Standard examples of Wick-rotations can be found within Lie groups, indeed any two semi-simple real forms: $G \subset G^{\mathbb{C}} \supset \tilde{G}$, of a complex Lie group are Wick-rotated, where the Lie groups are equipped with their left-invariant Killing forms: $-\kappa(\cdot,\cdot)$ respectively. As explored in [20], the existence of a Wick-rotation at a fixed point p implies the existence of a Wick-rotation of the isometry groups of the pseudo-inner products on the tangent spaces at p: $O(p,q) \subset O(n,\mathbb{C}) \supset O(\tilde{p},\tilde{q})$ at the identity element. We continue this study by using results of real GIT applied to actions of these groups. The results are then applied to Wick-rotations, and we give partial answers to the question above in the case of arbitrary signatures (not necessarily Riemannian).

Another motivation behind studying such Wick-rotations are considering pseudo-Riemannian spaces having identical polynomial curvature invariants [22, 25, 27, 41]. Consider two pseudo-Riemannian manifolds (M, g) and $(\tilde{M}, \tilde{g})$. Assume that all of their polynomial curvature invariants are identical, what can we then say about the relation between the two spaces? Indeed, here we will address this question locally and we reach a partial classification of spaces with identical invariants. Indeed, again, the Wick-rotations play an important role in this classification.

Our paper is organised as follows. We begin by the study of real GIT, and apply the results to *compatible representations*, which are defined and purely motivated by the study of Wick-rotations in [20, 21]. Many of these results obtained are generalisations of previous results [7, 20, 21, 6]. These results are then applied to pseudo-Riemannian manifolds and holomorphic Riemannian manifolds. The main GIT results of our paper is Section 5, which we apply to the setting of Wick-rotations (Section 6).

In this paper we will reserve the notion of *Riemannian space* to the case when the metric is positive definite (of signature $(++..+)$) while a *Lorentzian space* has signature $(-++..+)$. Note also that the existence of the "anti-isometry" which switches the sign of the metric, $g \mapsto -g$ which induces the group isomorphism $O(p,q) \to O(q,p)$.

2. Mathematical Preliminaries

2.1. Real slices and compatibility.

Definition 2.1. A *holomorphic inner product space* is a complex vector space E equipped with a non-degenerate complex bilinear form g.

For a holomorphic inner product space E we can always choose an orthonormal basis. By doing so we can identify E with $\mathbb{C}^n$ and the holomorphic inner product can be written as

$$g_0(X,Y) = X_1Y_1 + ... + X_nY_n, \tag{28}$$

where $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$.

Using this orthonormal basis it is also convenient to consider the group of transformation leaving the holomorphic inner product invariant. Consider a complex-linear map $A : E \longrightarrow E$. Using an orthonormal basis, we can represent the map as a complex matrix $\mathsf{A} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. Requiring that $g_0(A(X), A(Y)) = g_0(X,Y)$, for all X, Y, implies that $\mathsf{A}^t\mathsf{A} = \mathsf{1}$. Consequently, the matrix A must be a complex orthogonal matrix; i.e., $\mathsf{A} \in O(n,\mathbb{C})$.

Definition 2.2. Given a holomorphic inner product space (E, g). Then if $W \subset E$ is a real linear subspace for which $g\big|_W$ is non-degenerate and real valued, i.e., $g(X,Y) \in \mathbb{R},\ \forall X, Y \in W$, we will call W a *real slice.*

A non-degenerate symmetric real bilinear form shall be called a pseudo-inner product.

We recall that a *conjugation map* σ of a complex vector space E, is a real linear isomorphism: $E \xrightarrow{\sigma} E$, which is anti-linear, i.e $\sigma(ix) = -i\sigma(x)$ for all $x \in E$. The fix points of such a map, defines what is called a *real form* of E. Thus for a complex Lie group G, an *anti-holomorphic involution* (or *real structure*): $G \xrightarrow{F} G$, is an involution of real Lie groups such that the differential at 1: $\mathfrak{g} \xrightarrow{dF} \mathfrak{g}$, is a conjugation map.

Let $W \subset (E, g)$ be a real slice of dimension: $Dim_{\mathbb{R}}(W) = Dim_{\mathbb{C}}(E)$ (i.e W is a real form of E). Denote (p, q) for the signature of the restricted pseudo-inner product: $g\big|_W(-,-)$. Let $O(p, q)$ denote the real Lie group consisting of isometries of the pseudo-inner product space: $\Big(W, g\big|_W(-,-)\Big)$, then $O(p, q)$ is a real form of $O(n, \mathbb{C})$ (the isometries of (E, g)), by noting the anti-holomorphic involution (real structure): $A \mapsto \sigma \circ A \circ \sigma$, where σ is the conjugation map of W in E.

Definition 2.3. Let $W \subset (E, g)$ be a real slice. We say an involution $W \xrightarrow{\theta} W$, is a *Cartan involution* of W, if $g_\theta(\cdot, \cdot) := g\big|_W(\cdot, \theta(\cdot))$, is an inner product on W.

We note that the definition generalises the notion of a Cartan involution of a semi-simple Lie algebra.

Definition 2.4. Two real forms V and $\widetilde{V}$ of E are said to be compatible if their conjugation maps commute, i.e $[\sigma, \tilde{\sigma}] = 0$.

Let $V, \tilde{V}$ and W be real slices of (E, g) (all of the same real dimension as $Dim_{\mathbb{C}}(E)$). Assume $g\big|_W(-,-)$ is an inner product, such a real slice is referred to as a *compact real slice.* If all of their conjugation maps are pairwise compatible, then we shall refer to the triple: $\Big(V, \tilde{V}, W\Big)$, as a *compatible triple.*

We shall say that $V \subset (E, g)$ is a real form, to mean that V is a real slice and $Dim_{\mathbb{R}}(V) = Dim_{\mathbb{C}}(E)$.

For Lie groups we define compatibility locally:

Definition 2.5. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ be two real Lie subgroups of a complex Lie group such that the real Lie algebras are real forms of $\mathfrak{g}^{\mathbb{C}}$. Then we say G and $\tilde{G}$ are *compatible* if the Lie algebras are compatible.

For example the abelian Lie groups: $S^1 \subset \mathbb{C}^\times \supset \mathbb{R}^\times$ are compatible w.r.t to the real structures: $z \mapsto \frac{1}{\overline{z}}$ and $z \mapsto \overline{z}$ respectively. This is also an example of a compatible triple: $\Big(\mathbb{R}^\times, S^1, S^1\Big)$, in the sense of the following definition:

Definition 2.6. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ and $U \subset G^{\mathbb{C}}$ be real Lie subgroups of a complex Lie group such that the real Lie algebras are real forms of $\mathfrak{g}^{\mathbb{C}}$. Moreover assume U is compact. Then we say $\Big(G, \tilde{G}, U\Big)$ is a *compatible triple* if the Lie algebras are pairwise compatible.

2.2. **A Wick-rotation implies a standard Wick-rotation.** We recall some definitions from [20], and prove the equivalence:

$$\exists \text{ A Wick-rotation} \Leftrightarrow \exists \text{ A standard Wick-rotation.}$$

Definition 2.7. Given a complex manifold $M^{\mathbb{C}}$ with complex Riemannian metric $g^{\mathbb{C}}$. If a submanifold $M \subset M^{\mathbb{C}}$ for any point $p \in M$ we have that T_pM is a real slice of $(T_pM^{\mathbb{C}}, g^{\mathbb{C}})$ (in the sense of Defn. 2.2), we will call M a real slice of $(M^{\mathbb{C}}, g^{\mathbb{C}})$.

This definition implies that the induced metric from $M^{\mathbb{C}}$ is real valued on M. M is therefore a pseudo-Riemannian manifold. This further implies that real slices are totally real manifolds.

Definition 2.8 (Wick-related spaces)**.** Two pseudo-Riemannian manifolds M and $\tilde{M}$ are said to be *Wick-related* if there exists a holomorphic Riemannian manifold $(M^{\mathbb{C}}, g^{\mathbb{C}})$ such that M and $\tilde{M}$ are embedded as real slices of $M^{\mathbb{C}}$.

Wick-related spaces were defined in [6]. However, we also find it useful to define:

Definition 2.9 (Wick-rotation)**.** If two Wick-related spaces (of the same real dimension) intersect at a point p in $M^{\mathbb{C}}$, then we will use the term *Wick-rotation*: the manifold M can be Wick-rotated to the manifold $\tilde{M}$ (with respect to the point p).

Remark 2.10. Throughout this paper, we shall always assume that $Dim_{\mathbb{R}}(M) = Dim_{\mathbb{R}}(\tilde{M}) = Dim_{\mathbb{C}}(M^{\mathbb{C}})$.

Definition 2.11 (Standard Wick-rotation)**.** Let the M and $\tilde{M}$ be Wick-related spaces (of the same dimension) having a common point p. Then if the tangent spaces T_pM and $T_p\tilde{M}$ are embedded:

$$T_pM, T_p\tilde{M} \hookrightarrow (T_pM)^{\mathbb{C}} \cong (T_p\tilde{M})^{\mathbb{C}} \hookrightarrow T_pM^{\mathbb{C}},$$

such that they form a compatible triple with a compact real slice $W \subset (T_pM)^{\mathbb{C}} \cong (T_p\tilde{M})^{\mathbb{C}}$, then we say that the spaces M and $\tilde{M}$ are related through a *standard Wick-rotation*.

It is useful to note that a standard Wick-rotation: $(M, g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g})$, at a common point p, induces a Wick-rotation of Lie groups at 1: $O(p,q) \subset O(n,\mathbb{C}) \supset O(\tilde{p},\tilde{q})$. This observation is for instance used in [21], and is seen as follows. Let $\{e_1, \ldots, e_p, \ldots e_n\}$ be a pseudo-orthonormal basis of $g(-,-)$, and θ denote the Cartan involution of g w.r.t this basis. Then $\{e_1, \ldots, e_p, ie_{p+1}, \ldots ie_n\} := \{y_1, \ldots y_n\}$ is an orthonormal basis of $g^{\mathbb{C}}(-,-)$. Thus define a holomorphic inner product $\mathbf{g}^{\mathbb{C}}$ on $End(T_pM^{\mathbb{C}})$ by:

$$\mathbf{g}^{\mathbb{C}}(f,h) := \sum_{1 \leq l \leq n} g^{\mathbb{C}}(f(y_l), h(y_l)).$$

It is easy to check that $End(T_pM) \subset \left(End(T_pM^{\mathbb{C}}), \mathbf{g}^{\mathbb{C}}\right) \supset End(T_p\tilde{M})$ are real forms, precisely because T_pM and $T_p\tilde{M}$ are compatible with the compact real slice: $W := \langle y_1, \dots, y_p, iy_{p+1}, \dots iy_n \rangle$. A natural choice of Cartan involution Θ of the induced pseudo-inner product $\mathbf{g}$ on $End(T_pM)$ is given by:

$$f \mapsto \theta f \theta, \ f \in End(T_pM).$$

Note that if we restrict to the pseudo-orthogonal Lie algebra $\mathfrak{o}(p,q) \subset End(T_pM)$, then Θ leaves invariant $\mathfrak{o}(p,q)$. Moreover if $p+q \geq 3$ then Θ is a Cartan involution of the semi-simple Lie algebra: $\mathfrak{o}(p,q)$. An easy calculation shows that $\mathbf{g}$ is invariant under the conjugation action of $O(p,q)$ on $End(T_pM)$:

$$h \cdot f := hfh^{-1}, \ h \in O(p,q), \ f \in End(T_pM).$$

Thus $\mathbf{g}$ induces a bi-invariant metric on $O(p,q)$. If $p+q \neq 4$ but $p+q \geq 3$, then the Lie algebra $\mathfrak{o}(n,\mathbb{C})$ is simple, thus $\mathbf{g}$ is proportional to the Killing form. If $p+q=4$, then because $\mathfrak{o}(4)$ is simple, and Θ is a Cartan involution of $\mathfrak{o}(p,q)$ and of $\mathbf{g}$, then it follows that $\mathbf{g}$ is again proportional to the Killing form.

Finally we note that the setup above is really just a tensor action by viewing $f^{\mathbb{C}} \in End(T_pM^{\mathbb{C}})$ as a tensor in the tensor product $v^{\mathbb{C}} \in T_pM^{\mathbb{C}} \otimes T_pM^{\mathbb{C}}$ w.r.t to a $O(n,\mathbb{C})$-module isomorphism $f^{\mathbb{C}} \mapsto v^{\mathbb{C}}$. Indeed let $f^{ij} \in End(T_pM^{\mathbb{C}})$ be defined by the matrix $(f^{ij})_{ij} = 1$, and otherwise zero, w.r.t the basis: $Y := \{y_1, \dots, y_n\}$ defined above. Then $\{f^{ij}\}_{ij}$ running over all $1 \leq i,j \leq n$ form a basis for $End(T_pM^{\mathbb{C}})$. We define an isomorphism:

$$End(T_pM^{\mathbb{C}}) \xrightarrow{\phi^{\mathbb{C}}} T_pM^{\mathbb{C}} \otimes T_pM^{\mathbb{C}}, \quad f^{ij} \mapsto y_i \otimes y_j.$$

An easy calculation shows that $\phi^{\mathbb{C}}(gfg^{-1}) = g \cdot \phi^{\mathbb{C}}(f)$, where g acts on tensors by $g \cdot (v_1 \otimes v_2) := g(v_1) \otimes g(v_2)$, i.e $\phi^{\mathbb{C}}$ is an isomorphism of $O(n,\mathbb{C})$-modules. An easy calculation shows that $\phi^{\mathbb{C}}$ maps $End(T_pM) \mapsto T_pM \otimes T_pM$ by noting that

$$\{f^{ij} | 1 \leq j \leq p\} \cup \{if^{ij} | p+1 \leq j \leq n\}$$

is a basis for $End(T_pM)$. Trivially it maps $End(W) \mapsto W \otimes W$. It remains to show that it also maps $End(T_p\tilde{M}) \mapsto T_p\tilde{M} \otimes T_p\tilde{M}$. To see this we note that since $T_p\tilde{M}$ is compatible with W, then we may choose a pseudo-orthogonal basis: $\{\tilde{e}_1, \dots, \tilde{e}_{\tilde{p}}, \dots, \tilde{e}_n\}$ of $\tilde{g}$, and define analogously a map:

$$\tilde{\phi}^{\mathbb{C}} : \tilde{f}^{ij} \mapsto \tilde{y}_i \otimes \tilde{y}_j,$$

w.r.t the real basis: $\{\tilde{y}_1, \dots, \tilde{y}_n\} := \{\tilde{e}_1, \dots, \tilde{e}_{\tilde{p}}, i\tilde{e}_{\tilde{p}+1}, \dots, i\tilde{e}_n\}$ of W. Thus let $g \in O(n)$ be the map sending $y_j \mapsto \tilde{y}_j$, then $\tilde{f}^{ij} = gf^{ij}g^{-1}$, i.e

$$\phi^{\mathbb{C}}(\tilde{f}^{ij}) = \phi^{\mathbb{C}}(gf^{ij}g^{-1}) = g \cdot \phi^{\mathbb{C}}(f^{ij}) := g(y_i) \otimes g(y_j) = \tilde{y}_i \otimes \tilde{y}_j = \tilde{\phi}^{\mathbb{C}}(\tilde{f}^{ij}).$$

Thus since $\tilde{\phi}^{\mathbb{C}}$ maps analogously $End(T_p\tilde{M})$ into $T_p\tilde{M} \otimes T_p\tilde{M}$, then so does $\phi^{\mathbb{C}}$. Therefore we conclude that the map $\phi^{\mathbb{C}}$ also induce an isomorphism of $O(p,q)$, $O(\tilde{p},\tilde{q})$ and $O(n)$ modules respectively.

We explore the induced isometry action of $O(n,\mathbb{C})$ on a more general tensor product space in Section 6.

The motivation behind the definition of a standard Wick-rotation comes from the following lemma together with results from real GIT.

Lemma 2.12 ([20], Lemma 3.6)**.** *The triple of real forms:* $\Big(\mathfrak{o}(p,q),\mathfrak{o}(\tilde{p},\tilde{q}),\mathfrak{o}(n)\Big)$*, embedded into* $\mathfrak{o}(n,\mathbb{C})$ *under a standard Wick-rotation is a compatible triple of Lie algebras.*

We begin by observing that the definition of a Wick-rotation is in fact equivalent to the definition of a standard Wick-rotation, i.e we may always find such an embedding of the tangent spaces, we only need to use the following lemma:

Lemma 2.13. *Let* $\Big(\mathbb{C}^n,\langle -,-\rangle\Big)$ *be the standard holomorphic inner product space, i.e* $\langle Z_1,Z_2\rangle := \sum_{i=1}^n z_i^1 z_i^2$ *for any* $Z_1 := (z_1^1,\dots,z_1^n)\in\mathbb{C}^n \ni Z_2 := (z_2^1,\dots,z_2^n)$*. Then there exist a compatible triple:* $\Big(\mathbb{R}^n(p,q),\mathbb{R}^n(\tilde{p},\tilde{q}),\mathbb{R}^n(n,0)\Big)$ *of any signatures* $p+q=\tilde{p}+\tilde{q}=n+0=n$*.*

Proof. For a signature $p+q=n$, there is a conjugation map $\mathbb{C}^n\to\mathbb{C}^n$, defined by $Z\mapsto I_{p,q}\bar{Z}$ where $I_{p,q}$ is the diagonal matrix with diagonal entries: $(+1,\dots,+1,-1,\dots,-1)$ ($+1$ p-times, -1 q-times). It gives rise to a real slice $\mathbb{R}(p,q)\subset\mathbb{C}^N$, so because $[I_{p,q},I_{\tilde{p},\tilde{q}}]=0$ we have a compatible triple:

$$\Big(\mathbb{R}^n(p,q),\mathbb{R}^n(\tilde{p},\tilde{q}),\mathbb{R}^n(n,0)\Big).$$

The lemma is proved. □

Corollary 2.14. *If* M *and* $\tilde{M}$ *are Wick-rotated at* $p\in M\cap\tilde{M}$*, then they are also Wick-rotated by a standard Wick-rotation.*

Proof. Let M and $\tilde{M}$ be Wick-rotated at $p\in M\cap\tilde{M}$. By Lemma 2.13 and since $(T_pM)^{\mathbb{C}}\cong\mathbb{C}^n$ as holomorphic inner product spaces, then we can also find a real slice V of $(T_pM)^{\mathbb{C}}$ with signature $(\tilde{p},\tilde{q})$, such that T_pM and V form a compatible triple with a compact real slice W. Thus we can extend a real isomorphism: $V\xrightarrow{\psi}T_p\tilde{M}$, to an isomorphism $T_pM^{\mathbb{C}}\to T_pM^{\mathbb{C}}$, such that $\Big(\psi^{-1}(T_p\tilde{M}),T_pM,W\Big)$ form a compatible triple. This proves that M and $\tilde{M}$ are Wick-rotated by a standard Wick-rotation. The corollary is proved. □

Thus the results from [20], [21] hold for Wick-rotated spaces, and we shall therefore always assume a Wick-rotation instead of a standard Wick-rotation.

2.3. **Real GIT for semi-simple groups.** *Convention*: For a Lie group G which has finitely many connected components we say G is fcc.

Let G be a real semi-simple linear group which is fcc, and $G\xrightarrow{\rho_V^G}GL(V)$, be a real representation. Denote: $G=Ke^{\mathfrak{p}}$, to be the Cartan decomposition w.r.t a global Cartan involution: $G\xrightarrow{\Theta}G$, where $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ w.r.t $d\Theta := \theta$. Let $\langle -,-\rangle$ be a K-invariant inner product on V such that $d\rho_V^G(\mathfrak{p})$ consists of symmetric operators w.r.t $\langle -,-\rangle$. A vector $v \in V$ is said to be a *minimal vector* if:

$$(\forall g \in G)(||g \cdot v|| \geq ||v||),$$

the set of minimal vectors shall be denoted by $\mathcal{M}(G,V) \subset V$.

The following theorem by *Richardson* and *Slodowy* in [7], which relates the closure of a real orbit to the existence of a minimal vector, is worth mentioning:

Theorem 2.15 (RS). *The following statements hold:*

(1) A real orbit Gv is closed if and only if $Gv \cap \mathcal{M}(G,V) \neq \emptyset$.
(2) If v is a minimal vector then $Gv \cap \mathcal{M}(G,V) = Kv$.
(3) If Gv is not closed then there exist $p \in \mathfrak{p}$ such that $e^{tp} \cdot v \to \alpha \in V$ exist as $t \to \infty$, and $G\alpha \subset V$ is closed. Moreover $G\alpha \subset \overline{Gv}$ is the unique closed orbit in the closure.
(4) A vector $v \in V$ is minimal if and only if $\Big(\forall x \in \mathfrak{p}\Big)\Big(\langle x \cdot v, v\rangle = 0\Big)$, where $x \cdot v$ is the differential action $d\rho_V^G(x)(v)$.

Parts (1), (2) and (4) of the theorem is known as the *Kempf-Ness Theorem*, for which it was first proved for linearly complex reductive groups. One shall also remark that Theorem 2.15 also holds for a more general class of real reductive Lie groups which includes the class of semi-simple linear groups which are fcc ([29]).

We also recall:

Definition 2.16. Let $G^{\mathbb{C}}$ be a complex Lie group. A closed real Lie subgroup $G \subset (G^{\mathbb{C}})_{\mathbb{R}}$ is said to be a *real form*, if $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is a real form, and $G^{\mathbb{C}} = G \cdot G_0^{\mathbb{C}}$ (abstract group product). If $U \subset G^{\mathbb{C}}$ is a real form which is compact, then we shall say it is a *compact real form*.

Note that G is fcc if and only if $G^{\mathbb{C}}$ is fcc, and moreover if $G^{\mathbb{C}}$ is fcc, and U a compact real form, then U must be a maximally compact subgroup of $G^{\mathbb{C}}$.

For a real form $G \subset G^{\mathbb{C}}$, a complex action $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ is a *complexified action* of a real action: $G \xrightarrow{\rho_V^G} GL(V)$, if $\rho^{\mathbb{C}}(G)(V) = \rho_V^G(G)(V)$. Let G be semi-simple and the notation as above, then if τ denotes the conjugation map of the compact real form: $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$, then τ restricted to $\mathfrak{g}$ is precisely θ. If $(G^{\mathbb{C}})_{\mathbb{R}} = Ue^{i\mathfrak{u}}$ is the corresponding Cartan decomposition w.r.t τ, then it is possible to choose a U-invariant Hermitian inner product: $H(-,-)$ on $V^{\mathbb{C}}$ which is compatible with V, note that $K \subset U$, and we have that:

$$\mathcal{M}(G,V) \subset \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}).$$

Let $G \subset GL(V)$ (V a real vector space) be a semi-simple linear Lie group which is fcc, and $G^{\mathbb{C}} \subset GL(V^{\mathbb{C}})$ be the Zariski-closure of G. We recall the following known result:

Theorem 2.17 ([39], Lemma 2.2 + Remark p.3)**.** *If $v \in V$, then $Gv \subset V$ is closed if and only if $G^{\mathbb{C}}v \subset V^{\mathbb{C}}$ is closed. Also $G^{\mathbb{C}}v \cap V$ is a finite disjoint union of real orbits: $Gv_j \subset V$.*

If $U \subset G^{\mathbb{C}} \supset G$ are compatible real forms with U a compact real form, together with real representations: ρ_V^G and ρ_W^U which have the same complexification, and V, W are compatible real forms of $V^{\mathbb{C}}$ then the following result hold:

Theorem 2.18 ([20])**.** *Assume the assumptions above. Then there exist $v \in V$ and $w \in W$ such that $Uw \subset G^{\mathbb{C}}v \supset Gv$ if and only if $Gw \cap Gv \neq \emptyset$.*

We end the section with an example. Consider the notation of the example in the previous section (paragraph after Defn 2.11), i.e the conjugation action:

$$O(n, \mathbb{C}) \to GL\Big(End(T_pM^{\mathbb{C}})\Big), \quad g \cdot f := gfg^{-1}, \quad n \geq 3.$$

Put the $O(n)$-invariant Hermitian inner product: $H := \mathbf{g}^{\mathbb{C}}(\cdot, \mathcal{T}(\cdot))$ on $End(T_pM^{\mathbb{C}})$, where $\mathcal{T}$ is the conjugation map: $\mathcal{T} := f \mapsto \tau f \tau$ of $End(W) \subset End(T_pM^{\mathbb{C}})$, and τ is the conjugation map of the compact real slice $W \subset T_pM^{\mathbb{C}}$. It is not difficult to see that $f \in End(T_pM^{\mathbb{C}})$ is a minimal vector if and only if

$$\mathbf{g}^{\mathbb{C}}(x, [f_+, f_-]) = 0, \forall x \in i\mathfrak{o}(n),$$

where $f = f_+ + f_-$ is the eigenspace decomposition w.r.t to $\mathcal{T}$. Thus the closed orbits: $O(n, \mathbb{C}) \cdot f$, are precisely those which intersect $\mathcal{M}(O(n, \mathbb{C}), End(T_pM^{\mathbb{C}}))$. If we moreover restrict our vector space to the Lie algebra: $\mathfrak{o}(n, \mathbb{C})$, then the action is just the adjoint action, and thus the minimal vectors are precisely those $f \in \mathfrak{o}(n, \mathbb{C})$ satisfying $[f_+, f_-] = 0$.

2.4. **Real GIT for linearly real reductive groups.** In this subsection we shall extend Theorem 2.17 to real forms: $G \subset G^{\mathbb{C}}$, which are linearly real reductive.

Remark 2.19. Note that in the definition of a real form, although $G^{\mathbb{C}} \subset GL(V^{\mathbb{C}})$ for some real vector space V, then G is not necessarily contained in $G^{\mathbb{C}} \cap GL(V)$. For example: $SU(2) \subset SL_2(\mathbb{C}) \subset GL_2(\mathbb{C})$, but $SU(2)$ is not contained in $SL_2(\mathbb{C}) \cap GL_2(\mathbb{R}) = SL_2(\mathbb{R})$. However since $SL_2(\mathbb{C})$ is the universal complexification group of $SU(2)$, then we may find a real vector space V such that $SU(2) \subset GL(V)$, and $SL_2(\mathbb{C}) \subset GL(V^{\mathbb{C}})$, but this is not part of our assumptions in the definition of a real form.

Definition 2.20 ([31])**.** A *linearly complex reductive Lie group* $G^{\mathbb{C}}$ is a complex Lie group containing a compact subgroup U such that $G^{\mathbb{C}}$ is the universal complexification group of U.

In fact the complex Lie groups $G^{\mathbb{C}}$ which are fcc and have a compact real form are precisely the linearly complex reductive groups. Thus this class of groups are all self-adjoint by ([32], Lemma 5.1), and so the class of groups lends itself to Theorem 2.15 by ([29]). All such groups $G^{\mathbb{C}}$ are algebraic (canonically), and so are fcc ([7], 8.3). One should also note that a complex Lie group $G^{\mathbb{C}}$ which is fcc and has a reductive Lie algebra is linearly complex reductive if and only if $Z(G_0^{\mathbb{C}})_0 \cong (\mathbb{C}^\times)^k$ (a complexified tori), see for example ([31], Chapter 15).

Definition 2.21. A real linear group G shall be called *linearly real reductive* if G is fcc and G_0 is linearly real reductive in the sense of ([31], Definition 16.1.4), i.e $Z(G_0)$ is compact and $\mathfrak{g}$ is reductive.

Thus G is also a real reductive Lie group in the sense of ([29]), i.e there is a faithful representation with closed image: $G \subset GL(V)$, together with a global Cartan involution of $GL(V)$ leaving G invariant.

All semi-simple complex Lie groups are linearly complex reductive, and all real semi-simple linear groups which are fcc are linearly real reductive. One shall also note that the class of linearly real reductive Lie groups G are precisely the Lie groups (fcc) which are completely reducible (i.e every representation is completely reducible).

In contrary to semi-simple real forms, not all real forms of a linearly complex reductive group are linearly real reductive. Indeed take $G^{\mathbb{C}} := \mathbb{C}^\times$, then it is linearly complex reductive, with a compact real form $U \cong S^1$. But $G := \mathbb{R}^\times$ is also a real form, however it is not linearly real reductive, but it's nevertheless a real reductive Lie group in the sense of ([29]). We also see that if G is linearly real reductive and is a real form of some complex group $G^{\mathbb{C}}$, then $Z(G_0^{\mathbb{C}})_0$ has compact real form: $Z(G_0)_0$, which must be a torus, and thus $G^{\mathbb{C}}$ is linearly complex reductive.

The following extends a property of semi-simple groups:

Lemma 2.22. *Suppose G is linearly real reductive. Then the image of G under any real representation $\rho_V^G(G) \subset GL(V)$ is closed.*

Proof. Since G is fcc then we may assume w.l.o.g that G is connected. Now the Lie algebra $\mathfrak{g}$ is reductive thus $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$. Now $Z(G) \subset G$ is compact and has Lie algebra: $\mathfrak{z}(\mathfrak{g})$. Let $G' \subset G$ be the unique connected Lie subgroup of G with Lie algebra $\mathfrak{g}'$. Then G' is semi-simple and connected, and since G is connected then it is generated by $\langle G', Z(G)\rangle$. Also since $Z(G)$ is compact then the image $H'' := \rho_V^G(Z(G))$ is compact, and by ([31], Corollary 14.5.7), the image $H' := \rho_V^G(G') \subset GL(V)$ is closed. The image $H := \rho_V^G(G)$ is generated by H' and H''. Recall that the topology of $GL(V) \subset End(V)$ is a metric subspace with an induced norm metric: $d(-,-)$ on $End(V)$, satisfying $d(gh,0) \leq d(g,0)d(h,0)$ for all $g,h \in End(V)$. Now suppose $(y_n) \subset H$ is any convergent sequence in $GL(V)$. Then clearly $y_n = a_n b_n$ for sequences $(a_n) \subset H'$ and $(b_n) \subset H''$. Thus since H'' is compact then (b_n) is a bounded sequence, and so we may choose a subsequence $(b_{m(k)})$ converging to $\beta \in H''$. It follows that $(a_{m(k)})$ must converge as well using the norm metric, thus it converges for some $\alpha \in H'$. But then $\lim_{n\to\infty}(y_n) = \lim_{k\to\infty}(y_{m(k)}) = \lim_{k\to\infty}(a_{m(k)}b_{m(k)}) = \alpha\beta \in H$. This shows that $\rho_V^G(G) \subset GL(V)$ is closed as required. The lemma is proved. $\square$

We now extend Theorem 2.17 to the case where our real form is linearly real reductive:

Proposition 2.23. *Let $G \subset G^{\mathbb{C}}$ be a real form which is of type real linearly real reductive. Assume $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ is a complexified Lie group action of a real Lie group action: $G \xrightarrow{\rho_V^G} GL(V)$. Then Theorem 2.17 holds.*

Proof. Now since $G^{\mathbb{C}}$ is algebraic, and $\rho^{\mathbb{C}}$ is a rational representation w.r.t the algebraic structure ([34], Theorem 5.11), then the image $H^{\mathbb{C}} := \rho^{\mathbb{C}}(G^{\mathbb{C}})$ is a complex algebraic subgroup of $GL(V^{\mathbb{C}})$. The group $H^{\mathbb{C}}$ is fcc since $G^{\mathbb{C}}$ is fcc, and is a linearly complex reductive group, since if $U \subset G^{\mathbb{C}}$ is a compact real form, then $\rho^{\mathbb{C}}(U)$ is a compact real form of $H^{\mathbb{C}}$. Now since H is assumed to be fcc then $H := \rho_V^G(G) \subset GL(V)$ is a real closed subgroup of $H^{\mathbb{C}}$ by Lemma 2.22. Now if Q is the Zariski-closure of H in $H^{\mathbb{C}}$, then $H_0^{\mathbb{C}} \subset Q^0$ where Q^0 is the Zariski-connected component of Q. Also since $H^{\mathbb{C}} = H \cdot H_0^{\mathbb{C}} \subset H \cdot Q^0 \subset Q$, then we have $H^{\mathbb{C}} = Q$, so H is Zariski-dense in $H^{\mathbb{C}}$, and in particular $H^{\mathbb{C}}$ is defined over $\mathbb{R}$. Thus denote the real algebraic subgroup: $H^{\mathbb{C}}(\mathbb{R}) := H^{\mathbb{C}} \cap GL(V) \subset H^{\mathbb{C}}$ then it is a real form under the anti-holomorphic involution: $X \mapsto \overline{X}$. Also $H^{\mathbb{C}}(\mathbb{R})_0 \subset H \subset H^{\mathbb{C}}(\mathbb{R}) \subset H^{\mathbb{C}}$, because $H^{\mathbb{C}}(\mathbb{R})$ and H have the same Lie algebras, and moreover note that $H \subset H^{\mathbb{C}}(\mathbb{R})$ is closed. Thus if we consider the identity representation: $H^{\mathbb{C}} \to GL(V^{\mathbb{C}})$, then we have exactly the assumptions in [7], and we can mimic the proof of ([39], Lemma 2.2). But given $v \in V$ then $H^{\mathbb{C}}v := G^{\mathbb{C}}v$ and $Hv := Gv$ so the proposition follows. $\square$

We also make a note of the following theorem, which is well-known for semi-simple linear Lie groups which are fcc, and also holds for reductive algebraic groups in the context of rational representations ([35]). The theorem also applies to the class of linearly real reductive groups:

Theorem 2.24. *Let $G \subset GL(E)$ be a linearly real reductive Lie group. Then the following statements hold:*

(1) There exist a global Cartan involution of $GL(E)$ leaving G invariant.
(2) If $\mathfrak{gl}(E) \xrightarrow{\theta} \mathfrak{gl}(E)$ is a Cartan involution leaving $\mathfrak{g}$ invariant, then $\Theta(G) \subset G$, where Θ is the global Cartan involution of $GL(E)$ with differential θ.
(3) All Cartan involutions of G are conjugate by an inner automorphism of G.

(4) Let $G \xrightarrow{\rho_V^G} GL(V)$ be a real representation. Then given any global Cartan involution Θ of G, then there exist a global Cartan involution Θ' of $GL(V)$ such that: $\rho_V^G(\Theta(g)) = \Theta'(\rho_V^G(g)), \forall g \in G$.

Proof. Since the center: $\mathfrak{z}(\mathfrak{g})$, of $\mathfrak{g}$ is algebraic because $Z(G_0)$ is compact, then $\mathfrak{g}$ is also algebraic since it is a reductive Lie algebra, thus we can mimic the proof of ([39], Remark p.3). Therefore by the results of ([7]) cases (1), (2) and (3) follows. Case (4). Since $Z(G_0)$ is compact, then $\rho_V^G(Z(G_0)) \subset GL(V)$ is an algebraic subgroup with Lie algebra $d\rho_V^G(\mathfrak{z}(\mathfrak{g}))$, and so the image $d\rho_V^G(\mathfrak{g})$ is an algebraic reductive subalgebra in $\mathfrak{gl}(V)$. Thus following the steps in the proof of ([35], Proposition 13.5), case (4) follows. The theorem is proved. □

Corollary 2.25. *Let $G \subset G^{\mathbb{C}} \supset U$ be two compatible real forms where G is linearly real reductive, and U is a compact real form. Suppose $G^{\mathbb{C}} \subset GL(V^{\mathbb{C}})$, then there exist a U-invariant Hermitian form on $V^{\mathbb{C}}$ such that $G^{\mathbb{C}}$ and G are both self-adjoint.*

Proof. Let $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ be the Lie algebra of U, i.e it is a compact real form of $\mathfrak{g}^{\mathbb{C}}$. By ([32]) the group $G^{\mathbb{C}}$ is self-adjoint w.r.t a Hermitian inner product $H(-,-)$ on $V^{\mathbb{C}}$. Let $\mathfrak{gl}(V^{\mathbb{C}}) \xrightarrow{\tau} \mathfrak{gl}(V^{\mathbb{C}})$ be the conjugation map of $\mathfrak{gl}(V^{\mathbb{C}})$ with fix points: $\mathfrak{u}(n)$, leaving $\mathfrak{g}^{\mathbb{C}}$ invariant w.r.t $H(-,-)$. Now since $\mathfrak{g}$ is compatible with $\mathfrak{u}$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{k} \subset \mathfrak{u}$ and $\mathfrak{p} \subset i\mathfrak{u}$. Thus τ also leaves invariant $\mathfrak{g}$. Now by identifying the real groups: $\Big(GL(V^{\mathbb{C}})\Big)_{\mathbb{R}} \cong GL\Big((V^{\mathbb{C}})_{\mathbb{R}}\Big)$, then τ induces a Cartan involution of $\mathfrak{gl}\Big((V^{\mathbb{C}})_{\mathbb{R}}\Big)$ w.r.t the real part of $H(-,-)$, leaving the copy of $\mathfrak{g} \hookrightarrow \mathfrak{gl}\Big((V^{\mathbb{C}})_{\mathbb{R}}\Big)$ invariant. Thus the corresponding global Cartan involution of $GL\Big((V^{\mathbb{C}})_{\mathbb{R}}\Big)$ leaves the copy $G \hookrightarrow GL\Big((V^{\mathbb{C}})_{\mathbb{R}}\Big)$ invariant by (2) of Theorem 2.24, and so the global conjugation map of $GL(V^{\mathbb{C}})$ with differential τ must also leave the original copy of G invariant. The corollary is proved. □

Remark 2.26. Let $V \subset (V^{\mathbb{C}}, g^{\mathbb{C}})$ be a real form of a holomorphic inner product space, and consider the linear isometry groups: $G := O(p,q) \subset G^{\mathbb{C}} := O(n,\mathbb{C})$. Then as a Lie group $G^{\mathbb{C}}$ is linearly complex reductive for all $n \geq 1$, and for $n > 2$ the real form $O(p,q)$ is semi-simple, while for $n = 1$ the group G is finite thus is linearly real reductive. For $n = 2$ then G is not linearly real reductive, but is the real points of $O(2,\mathbb{C})$, i.e is a reductive algebraic group, thus the group satisfies the assumptions of the setup in [7]. Therefore all the results obtained here in this section, can also be applied to a real form: $O(p,q) \subset O(n,\mathbb{C})$ for all $p + q = n$.

In regards to Wick-rotations we are mainly interested in the real forms: $O(p,q) \subset O(n,\mathbb{C}) \supset O(\tilde{p},\tilde{q})$, for $p + q = \tilde{p} + \tilde{q} = n$.

3. Balanced representations

Throughout sections 3, 4 and 5, when considering a complex Lie group $G^{\mathbb{C}}$ it shall always be of type linearly complex reductive. Moreover a real form $G \subset G^{\mathbb{C}}$ shall always be assumed to be either linearly real reductive or in the case where $G^{\mathbb{C}}$ is defined over $\mathbb{R}$, the real points: $G = G_{\mathbb{R}}$. The groups to have in mind are $O(p,q) \subset O(n,\mathbb{C})$.

Definition 3.1 ([7], *Section* 5.2)**.** Let $G \xrightarrow{\rho_V^G} GL(V)$ be a real representation, then ρ_V^G is said to be *balanced representation* if there exist an involution $V \xrightarrow{\theta} V$, and a global Cartan involution: $G \xrightarrow{\Theta} G$ such that:

$$\Big(\forall g \in G\Big)\Big(\rho_V^G(\Theta(g)) = \theta \circ \rho_V^G(g) \circ \theta\Big).$$

For example in the case of the adjoint action of a semi-simple Lie group G, then the involutions balancing the action are precisely: $\pm\theta$, where $\mathfrak{g} \xrightarrow{\theta} \mathfrak{g}$ is a Cartan involution of the Lie algebra: $\mathfrak{g}$. Note also that any real representation: $U \to GL(W)$ of a compact Lie group U is balanced, since the global Cartan involution of U is 1_U and thus 1_W is an involution balancing the action.

It is also worth noting that if our group G has the property that a global Cartan involution of G: $\Theta = Ad(k)$ for some $k \in K$ of order 2, then all representations are naturally balanced, since one may take $V \xrightarrow{\rho_V^G(k)} V$ as a natural choice of involution balancing a representation ρ_V^G. This is the case for example with the pseudo-orthogonal groups: $O(p,q)$. The group $SL_2(\mathbb{R})$ does not have this property for instance.

It is not difficult to see that an involution θ balancing a representation gives rise to a G-invariant symmetric non-degenerate bilinear form: $\langle -,-\rangle$ on V such that θ is a Cartan involution, i.e $\langle v, \theta(v)\rangle > 0$ for all $v \neq 0$, (see for example [7], *Section* 5.2). Note that $-\theta$ is also an involution balancing the action, and θ can not be conjugate to $-\theta$ by the action of G.

Definition 3.2. Let $G \xrightarrow{\rho_V^G} GL(V)$ be a balanced real representation and θ an involution balancing ρ_V^G. Let $\langle -,-\rangle$ be a (G,θ)-invariant symmetric non-degenerate bilinear form on V. Then any Cartan involution θ' of $\langle -,-\rangle$ is said to be an *inner Cartan involution* of ρ_V^G if it is conjugate by the action of G to θ.

In the case of the adjoint action for semi-simple groups, then fixing the Killing form: $-\kappa(-,-)$ on $\mathfrak{g}$, the inner Cartan involutions are precisely the Cartan involutions contained in $Aut(\mathfrak{g})$.

Remark 3.3. Hereon whenever we consider a balanced representation ρ_V^G we shall always fix a G-invariant non-degenerate symmetric bilinear form: $\langle -, - \rangle$ on V, and speak of the inner Cartan involutions of $\langle -, - \rangle$.

We shall also consider complex representations, and therefore define analogously:

Definition 3.4. Suppose $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ is a complex representation. We say $\rho^{\mathbb{C}}$ is balanced if $(\rho^{\mathbb{C}})_{\mathbb{R}}$ (the real representation) is balanced w.r.t a conjugation map: $V^{\mathbb{C}} \xrightarrow{\tau} V^{\mathbb{C}}$.

Note that the definition is a generalisation of the adjoint action of semi-simple Lie groups to general actions. We also extend Definition 3.2 to balanced complex actions $\rho^{\mathbb{C}}$, i.e if τ balances $\rho^{\mathbb{C}}$ then any real involution: $\rho^{\mathbb{C}}(g)\tau\rho^{\mathbb{C}}(g^{-1})$ for some $g \in G^{\mathbb{C}}$ shall be called an inner Cartan involution of $V^{\mathbb{C}}$. One observes that given a τ which balances a complex action, then we may choose a $G^{\mathbb{C}}$-invariant Hermitian form $H(-,-)$ on $V^{\mathbb{C}}$.

In the case where $V^{\mathbb{C}}$ is an irreducible $\mathfrak{g}^{\mathbb{C}}$-module there are restrictions on the involutions balancing the representation:

Proposition 3.5. *Let $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ be a balanced complex representation. Assume that $V^{\mathbb{C}}$ is an irreducible $\mathfrak{g}^{\mathbb{C}}$-module. Then any two real involutions: $V^{\mathbb{C}} \xrightarrow{\tau, \tilde{\tau}} V^{\mathbb{C}}$ balancing $\rho^{\mathbb{C}}$ are conjugate by the action of $G_0^{\mathbb{C}}$ up to scaling of ± 1.*

Proof. Assume τ and $\tilde{\tau}$ are two conjugation maps which balances $\rho^{\mathbb{C}}$, so there exist global Cartan involutions: $\Theta, \tilde{\Theta}$ of $G^{\mathbb{C}}$, such that:

$$\rho(\Theta(g)) = \tau \circ \rho(g) \circ \tau, \quad \rho(\tilde{\Theta}(g)) = \tilde{\tau} \circ \rho(g) \circ \tilde{\tau}, \forall g \in G^{\mathbb{C}}.$$

Now since Θ and $\tilde{\Theta}$ are conjugate by an inner automorphism of $G^{\mathbb{C}}$, then it is not difficult to see that there exist $h \in G_0^{\mathbb{C}}$, such that

$$\rho(h)\tau\rho(h^{-1}) \circ \tilde{\tau} \circ \rho(g) = \rho(g) \circ \rho(h)\tau\rho(h^{-1}) \circ \tilde{\tau}, \forall g \in G^{\mathbb{C}}.$$

Thus $f := \rho(h)\tau\rho(h^{-1}) \circ \tilde{\tau}$ is a complex linear map which is a $G^{\mathbb{C}}$-module isomorphism, using the exponential map this is also a $\mathfrak{g}^{\mathbb{C}}$-module isomorphism on Lie algebra level, i.e for the differential action: $\mathfrak{g}^{\mathbb{C}} \xrightarrow{d\rho^{\mathbb{C}}} \mathfrak{gl}(V^{\mathbb{C}})$. Now since $V^{\mathbb{C}}$ is irreducible, then by Schur's lemma we must have that $f = \lambda 1_{V^{\mathbb{C}}}$, for some $\lambda \in \mathbb{C}$. Thus $\lambda^2 = 1$ since $\lambda^2 1_{V^{\mathbb{C}}} = (\lambda\tilde{\tau})^2 = (\rho(h)\tau(\rho(h))^{-1})^2 = 1_{V^{\mathbb{C}}}$, and so the proposition is proved. □

Remark 3.6. Note that Proposition 3.5 fails in the case of the trivial representation, indeed any conjugation map σ of $V^{\mathbb{C}} \neq 0$ will balance the trivial representation. So if σ is a conjugation map of $V^{\mathbb{C}}$ then σ and $i\sigma$ are not conjugate by the action of $G^{\mathbb{C}}$ up to ± 1. In general it even fails for a non-trivial reducible representation as well. Indeed let Ad be the adjoint action, then it is non-trivial, and τ be a conjugation map of a compact real form, then τ will balance Ad. Consider the representation $0_{V^{\mathbb{C}}} \oplus Ad$ for $V^{\mathbb{C}}$ any non-zero complex vector space. Then this is a non-trivial representation, and for example if σ is any conjugation map of $V^{\mathbb{C}}$ then the two involutions: $\sigma \oplus \tau$ and $i\sigma \oplus \tau$ both balance this representation, however they cannot be conjugated by the action of $G^{\mathbb{C}}$ up to ± 1.

A complex action is balanced in the following sense:

Proposition 3.7. *Suppose $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ is a complex representation. Then there is a compact real form: $U \subset G^{\mathbb{C}}$, and a real form $W \subset V^{\mathbb{C}}$, such that $\rho^{\mathbb{C}}(U)(W) \subset W$ if and only if $\rho^{\mathbb{C}}$ is balanced.*

Proof. Suppose a compact real form: $U \subset G^{\mathbb{C}}$ restricts to an action on a real form: $W \subset V^{\mathbb{C}}$. Denote Θ for the corresponding Cartan involution of $(G^{\mathbb{C}})_{\mathbb{R}}$ with fix points U. Then clearly:

$$\tau(\rho^{\mathbb{C}}(u)(w_1 + iw_2)) = \rho^{\mathbb{C}}(u)(w_1 - iw_2) = \rho^{\mathbb{C}}(u)(\tau(w_1 + iw_2)).$$

Also if $g := e^{ix}$ for $x \in \mathfrak{u}$ (the Lie algebra of U), then:

$$\rho^{\mathbb{C}}(e^{ix})(w) = \sum_{n:=2k} \frac{1}{n!}(d\rho^{\mathbb{C}})^n(ix)(w) + \sum_{n:=2k+1} \frac{1}{n!}(d\rho^{\mathbb{C}})^n(ix)(w) = w_1' + iw_2',$$

for $w_1', w_2' \in W$. Thus,

$$\rho^{\mathbb{C}}(e^{-ix})(w) = \sum_{n:=2k} \frac{1}{n!}(d\rho^{\mathbb{C}})^n(ix)(w) - \sum_{n:=2k+1} \frac{1}{n!}(d\rho^{\mathbb{C}})^n(ix)(w) = w_1' - iw_2',$$

and so $\tau(\rho^{\mathbb{C}}(e^{-ix})(w)) = \rho^{\mathbb{C}}(e^{ix})(w)$. So since $(V^{\mathbb{C}})_{\mathbb{R}} = W \oplus iW$, then it follows that $\rho^{\mathbb{C}}$ is balanced w.r.t τ. Conversely this is clear, since if the action is balanced then one has the equation:

$$\rho^{\mathbb{C}}(\Theta(g)) = \tau \circ \rho^{\mathbb{C}}(g) \circ \tau,$$

where Θ is some Cartan involution of $(G^{\mathbb{C}})_{\mathbb{R}}$, and τ is some conjugation map in $V^{\mathbb{C}}$. Denote U for the compact real form of $G^{\mathbb{C}}$ which is the fix points of Θ, and W for the real form of $V^{\mathbb{C}}$, which is the fix points of τ, then clearly $\rho^{\mathbb{C}}(U)(W) \subset W$ as required. The proposition is proved. $\square$

In other words a complex action is balanced if and only if it is a complexified action of a real action of a compact real form. An example of a complex action which is not a complexification of any real action of a compact real form, is the faithful action of $G^{\mathbb{C}} := SL_2(\mathbb{C})$ on $V^{\mathbb{C}} := \mathbb{C}^2$ by $X \cdot v := Xv$. Indeed it is enough to show it for the compact real form: $U := SU(2) \subset G^{\mathbb{C}}$ (as all compact real forms are isomorphic). If this was the case, then the restricted action of $SU(2)$ on a real form $W \subset V^{\mathbb{C}}$ would also be faithful locally, and thus we could embed $\mathfrak{su}(2) \hookrightarrow \mathfrak{gl}(2, \mathbb{R})$. However all semi-simple Lie subalgebras of $\mathfrak{gl}(2, \mathbb{R})$ are contained in $\mathfrak{sl}_2(\mathbb{R})$, and hence we would obtain: $\mathfrak{su}(2) \cong \mathfrak{sl}_2(\mathbb{R})$, which is false. It is however a complexified action of the real form: $G := SL_2(\mathbb{R}) \subset G^{\mathbb{C}}$ acting on $V := \mathbb{R}^2 \subset V^{\mathbb{C}}$, which is also non-balanced, indeed if it were balanced then $\mathfrak{sl}_2(\mathbb{R}) \cong d\rho_V^G(\mathfrak{g}) \subset \mathfrak{o}(p,q)$ is a Lie subalgebra for some $p+q=2$, this is impossible, as $\mathfrak{o}(2)$ and $\mathfrak{o}(1,1)$ are both abelian.

Note that this example can be generalised to the faithful action of $SL_n(\mathbb{C})$ acting on $V^{\mathbb{C}} := \mathbb{C}^n$ for any $n \geq 2$.

Recall that two representations: $G_1 \xrightarrow{\rho_{V_1}^{G_1}} GL(V_1)$, and $G_2 \xrightarrow{\rho_{V_2}^{G_2}} GL(V_2)$ are said to be *isomorphic* if there are Lie group isomorphisms: $G_1 \xrightarrow{\psi_1} G_2$ and $GL(V_1) \xrightarrow{\psi_2} GL(V_2)$, such that: $\rho_{V_1}^{G_1} = \psi_2 \circ \rho_{V_2}^{G_2} \circ \psi_1$. We write $\rho_{V_1}^{G_1} \cong \rho_{V_2}^{G_2}$.

Corollary 3.8. *Let $G^{\mathbb{C}} \xrightarrow{\Psi} GL(V^{\mathbb{C}})$ be a complex representation. Assume $V^{\mathbb{C}}$ is an irreducible $\mathfrak{g}^{\mathbb{C}}$-module. Let $U \subset G^{\mathbb{C}} \supset \tilde{U}$ be compact real forms, and $W \subset V^{\mathbb{C}} \supset \tilde{W}$ be real forms. Suppose $U \xrightarrow{\rho_W^U} GL(W)$ and $\tilde{U} \xrightarrow{\rho_{\tilde{W}}^{\tilde{U}}} GL(\tilde{W})$ are two real representations with $\Psi = (\rho_W^U)^{\mathbb{C}} = (\rho_{\tilde{W}}^{\tilde{U}})^{\mathbb{C}}$. Then $\rho_W^U \cong \rho_{\tilde{W}}^{\tilde{U}}$.*

Proof. Two real representations: ρ_W^U and $\rho_{\tilde{W}}^{\tilde{U}}$, with complexification Ψ, give rise to two balanced Cartan involutions: τ_W and $\tau_{\tilde{W}}$, namely the conjugation maps with fix points W and $\tilde{W}$ respectively, by Proposition 3.7. Now following the proof of Proposition 3.5, we know that there exist $g \in (G^{\mathbb{C}})_{\mathbb{R}}$ such that $\Psi(g) \circ \tau_W \circ \Psi(g^{-1}) = \lambda \tau_{\tilde{W}}$ $(\lambda = \pm 1)$, with $Ad(g)(U) := gUg^{-1} = \tilde{U}$. We note that if $\lambda = -1$, then $\Psi(g^{-1})(i\tilde{W}) = W$, and if $\lambda = 1$, then $\Psi(g^{-1})(\tilde{W}) = W$. However since the action $U \xrightarrow{\rho_{iW}^U} GL(iW)$ given by: $u \cdot iw := i\rho_W^U(w)$ is isomorphic to ρ_W^U, then we can assume w.l.o.g that $\lambda = 1$. Thus we have isomorphisms: $U \xrightarrow{Ad(g)} \tilde{U}$, and $GL(W) \xrightarrow{Ad(\Psi(g))} GL(\tilde{W})$, where $Ad(\Psi(g))(f) := \Psi(g) f \Psi(g^{-1})$. One easily checks that:

$$\rho_W^U = Ad(\Psi(g)) \circ \rho_{\tilde{W}}^{\tilde{U}} \circ Ad(g),$$

and thus proves the corollary.

$\square$

Let $O(p,q) \subset GL(V)$, be defined as the isometry group of some non-degenerate symmetric bilinear form: $\langle -, - \rangle$, of signature $p+q = Dim(V)$. Then for ρ_V^G to be balanced is just a stronger version of Theorem 2.24 (case 4), i.e we may choose Θ' to be a Cartan involution of an $O(p,q)$ group:

Proposition 3.9. *Let $G \xrightarrow{\rho_V^G} GL(V)$ be a real representation. Then ρ_V^G is balanced if and only if there exist a pseudo-orthogonal group $O(p,q) \subset GL(V)$ and a Cartan involution of $O(p,q)$ leaving $\rho_V^G(G)$ invariant.*

Proof. Suppose $V \xrightarrow{\theta} V$ is an involution balancing ρ_V^G w.r.t Θ of G, let $\langle -,- \rangle$ be a $\Big(G, \theta\Big)$-invariant symmetric non-degenerate bilinear form of some signature $p+q=n$. Denote $O(p,q) \subset GL(V)$, for the isometry group of $\langle -,- \rangle$, then $\rho_V^G(G) \subset O(p,q) \subset GL(V)$. Now $g \mapsto \theta \circ g \circ \theta$, is a global Cartan involution of $O(p,q)$, thus

$$\rho_V^G(g) \mapsto \theta \circ \rho_V^G(g) \circ \theta = \rho_V^G(\Theta(g)) \in \rho_V^G(G),$$

for the fixed global Cartan involution Θ of G. Conversely suppose there exist a pseudo-orthogonal group $O(p,q) \subset GL(V)$ and a global Cartan involution Θ' of $O(p,q)$ leaving $\rho_V^G(G)$ invariant. Note that $p, q \neq 1$. Let $\langle -,- \rangle$ be the symmetric non-degenerate bilinear form of signature $p+q=n$ associated to $O(p,q)$. Then let θ be any Cartan involution of $\langle -,- \rangle$ w.r.t Θ', i.e it balances the isometry action of $O(p,q)$ on V. Now let Θ be a global Cartan involution of G, and let Θ_1 be a global Cartan involution of $GL(V)$ extending Θ', by Theorem 2.24. Also there exist a global Cartan involution Θ_2 of $GL(V)$, such that $\Theta_2(\rho_V^G(g)) = \rho_V^G(\Theta(g))$, again by Theorem 2.24. Thus since Θ_1 and Θ_2 are conjugated in $GL(V)$, then $\Theta_2 = Ad(g) \circ \Theta_1 \circ Ad(g^{-1})$ for some $g \in GL(V)$, hence $Ad(g)(\theta) := \theta'$ is an involution that will satisfy:

$$\theta' \circ \rho_V^G(g) \circ \theta' = \rho_V^G(\Theta(g)), \forall g \in G,$$

and so ρ_V^G is balanced as required. $\square$

4. Compatible representations

Definition 4.1. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ be real forms, and $G \xrightarrow{\rho_V^G} GL(V)$ and $\tilde{G} \xrightarrow{\rho_{\tilde{V}}^{\tilde{G}}} GL(\tilde{V})$ be real representations of Lie groups. Suppose $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ is a complexified action of both ρ_V^G and $\rho_{\tilde{V}}^{\tilde{G}}$. Then we say that ρ_V^G is *compatible* with $\rho_{\tilde{V}}^{\tilde{G}}$, if the following two criterions are fulfilled:

(1) G and $\tilde{G}$ are compatible real forms of $G^{\mathbb{C}}$.
(2) V and $\tilde{V}$ are compatible real forms of $V^{\mathbb{C}}$.

Note that a real representation $G \to GL(V)$ with a complexification is always compatible with itself, and moreover if $U \subset G^{\mathbb{C}}$ is a compact real form, then a real Lie group action: $U \to GL(W)$, can always be complexified to a complex action: $G^{\mathbb{C}} \to GL(W^{\mathbb{C}})$, simply because $G^{\mathbb{C}}$ is the universal complexification group of U.

Definition 4.2. Let $\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}$ and ρ_W^U be pairwise compatible representations, where $U \subset G^{\mathbb{C}}$, is a compact real form. Then the triple: $\Big(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\Big)$ is said to be a *compatible triple*.

Remark 4.3. When considering a compatible triple: $\Big(\mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{u}\Big)$ of Lie algebras, there is a natural good choice of Cartan involutions, indeed the conjugation map τ of $\mathfrak{u}$, restricts to Cartan involutions: $\theta := \tau_{|_{\mathfrak{g}}}$ and $\tilde{\theta} := \tau_{|_{\tilde{\mathfrak{g}}}}$. In this way the global Cartan involutions of our groups $G = Ke^{\mathfrak{p}}$ and $\tilde{G} = \tilde{K}e^{\tilde{\mathfrak{p}}}$ are such that $K \subset U \supset \tilde{K}$, where $G^{\mathbb{C}} = Ue^{i\mathfrak{u}}$ is the global Cartan involution of $G^{\mathbb{C}}$, where U has Lie algebra $\mathfrak{u}$, see Corollary 2.25.

From ([20], Proposition A.2), a compatible pair: $\left(\rho^G_V, \rho^{\tilde{G}}_{\tilde{V}}\right)$ was considered. We now extend this result for compatible triples. We recall that an Hermitian inner product $H(-,-)$ on $V^{\mathbb{C}}$ which is real on a real subspace $V' \subset V^{\mathbb{C}}$ is said to be *compatible* with V'.

Lemma 4.4. *Suppose* $\left(\rho^G_V, \rho^{\tilde{G}}_{\tilde{V}}, \rho^U_W\right)$ *is a compatible triple. Then there exist a* U*-invariant Hermitian inner product* $H(-,-)$ *on* $V^{\mathbb{C}}$ *which is compatible with* $V, \tilde{V}$ *and* W*.*

Proof. Since U is compact then so is $\rho^{\mathbb{C}}(U) \subset (GL(V^{\mathbb{C}}))_{\mathbb{R}} \cong GL((V^{\mathbb{C}})_{\mathbb{R}}$. Set $E := (V^{\mathbb{C}})_{\mathbb{R}}$ for the real vector space with complex structure J. Then the complex structure on E: $E \xrightarrow{J} E$ is an element of $GL(E)$, and so are all the conjugation maps: $\sigma_V, \tilde{\sigma}_{\tilde{V}}$ and τ_W. Define the subgroup $U^* := \langle \rho^{\mathbb{C}}(U), J, \sigma_V, \tilde{\sigma}_{\tilde{V}}, \tau_W \rangle \subset GL(E)$ then U^* is a compact subgroup of $GL(E)$ since $\rho^{\mathbb{C}}(U) \subset U^*$ is closed and the quotient group $\frac{U^*}{\rho^{\mathbb{C}}(U)}$ is finite, using that $\left(V, \tilde{V}, W\right)$ is a compatible triple. Now by the compatibility conditions on the Lie algebras we have that: $K \subset U \supset \tilde{K}$, thus $\rho^{\mathbb{C}}(K) \subset \rho^{\mathbb{C}}(U) \supset \rho^{\mathbb{C}}(\tilde{K})$. The inclusion ϕ: $U^* \hookrightarrow GL(E)$, is a real representation of a compact Lie group. So there exist a U^*-invariant inner product $\langle -,- \rangle$ on E. Since $\langle -,- \rangle$ is J-invariant then it is easy to see that there exist a unique Hermitian inner product $H(-,-)$ on $V^{\mathbb{C}}$ with real part $\langle -,- \rangle$ on E. It is easy to check that $H(-,-)$ is U-invariant and therefore: $d\rho^{\mathbb{C}}(i\mathfrak{u})$ consists of Hermitian operators on $H(-,-)$. Also $H(-,-)$ is clearly $\left(V, \tilde{V}, W\right)$-compatible by construction. The lemma is proved.

□

We thus also have an extended version of ([20], Corollary A.2), concerning minimal vectors, which is essentially ([7], Lemma 8.1) applied to each real representation:

Corollary 4.5. *Suppose* $\left(\rho^G_V, \rho^{\tilde{G}}_{\tilde{V}}, \rho^U_W\right)$ *is a compatible triple. Then there is a* $\left(V, \tilde{V}, W\right)$*-compatible* U*-invariant Hermitian inner product* $H(-,-)$ *on* $V^{\mathbb{C}}$ *such that:*

$$\mathcal{M}(U, W) \cup \mathcal{M}(\tilde{G}, \tilde{V}) \cup \mathcal{M}(G, V) \subset \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}).$$

Note that $\mathcal{M}(U, W) = W$, since U is a compact real form. Now it follows from Proposition 3.7, that a compatible triple must be a balanced triple, i.e every real representation in the triple must be balanced:

Corollary 4.6. *Let* $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$ *be a compatible triple. Then there exist an involution:* τ*, balancing* $\rho^{\mathbb{C}}$*, such that* $\tau(V) \subset V$*,* $\tau(\tilde{V}) \subset \tilde{V}$ *and* $\tau_W = 1_W$*. Thus* ρ_V^G *and* $\rho_{\tilde{V}}^{\tilde{G}}$ *must also be balanced, with involutions:* $\theta := \tau_V$ *and* $\tilde{\theta} := \tau_{\tilde{V}}$ *respectively.*

Proof. By Proposition 3.7, the conjugation map τ with fix points: W, which balance $\rho^{\mathbb{C}}$. Now since W is pairwise compatible with V and $\tilde{V}$, then obviously τ leaves V and $\tilde{V}$ invariant. Thus since the global Cartan involution of $(G^{\mathbb{C}})_{\mathbb{R}}$ w.r.t the compact real form U restricts to global Cartan involutions of G and $\tilde{G}$ respectively, then obviously $\theta := \tau_V$ and $\tilde{\theta} := \tau_{\tilde{V}}$ balance ρ_V^G and $\rho_{\tilde{V}}^{\tilde{G}}$ respectively. The corollary follows. $\square$

Remark 4.7. We note in the proof of Lemma 4.4, that the U-invariant Hermitian inner product on $V^{\mathbb{C}}$ may be chosen to be invariant under τ from Corollary 4.6.

We have the following criterion for a vector to be a minimal vector w.r.t a balanced Cartan involution: θ:

Lemma 4.8 ([7], Lemma 5.1.1)**.** *Let* $G \to GL(V)$ *be a balanced real representation, and* θ *be an inner Cartan involution. Let* $v = v_+ + v_- \in V$ *be the Cartan decomposition, then* $v \in \mathcal{M}(G, V)$ *if and only if* $\langle x \cdot v_+, v_- \rangle = 0$ *for all* $x \in \mathfrak{p}$*, where* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is the Cartan decomposition of* $\mathfrak{g}$ *for which* θ *is balanced.*

In particular we see that if $V = V_+ \oplus V_-$ w.r.t θ, then $V_+ \cup V_- \subseteq \mathcal{M}(G, V)$. There are cases where $V_+ \cup V_- = \mathcal{M}(G, V)$, for example the adjoint action of $SL_2(\mathbb{R})$ on $\mathfrak{sl}_2(\mathbb{R})$ or the matrix action of $O(p, q)$ on $\mathbb{R}^n$ with $n = p + q$.

Now for a compatible triple: $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$, where $V^{\mathbb{C}} = W \oplus iW$ w.r.t τ from Corollary 4.6, and $H(-, \tau(-))$ is a U-invariant Hermitian inner product compatible with $V, \tilde{V}$ and W, then we can characterise the minimal vectors as follows:

(1) $\mathcal{M}(G, V) = \{v \in V | H(x \cdot v_+, v_-) = 0, \forall x \in \mathfrak{p} \subset i\mathfrak{u}\}$.
(2) $\mathcal{M}(\tilde{G}, \tilde{V}) = \{\tilde{v} \in \tilde{V} | H(x \cdot \tilde{v}_+, \tilde{v}_-) = 0, \forall x \in \tilde{\mathfrak{p}} \subset i\mathfrak{u}\}$.
(3) $\mathcal{M}(U, W) = W$.
(4) $\mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}) = \{v \in V^{\mathbb{C}} | H(x \cdot w_1, iw_2) = 0, \forall x \in i\mathfrak{u}\}$.

5. Compatible real orbits

Definition 5.1. Let $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}\right)$ be a compatible pair. Suppose $v \in V$ and $\tilde{v} \in \tilde{V}$ are such that $\tilde{v} \in G^{\mathbb{C}} v$, then we shall say that Gv is *compatible* with $\tilde{G}\tilde{v}$.

We shall write $Gv \sim \tilde{G}\tilde{v}$ for two compatible real orbits. One notes that if U is compact, then by ([7]): $Uv_1 \sim Uv_2$ if and only if $Uv_1 = Uv_2$, this is however not true for general groups, see for example the adjoint action of $SL_2(\mathbb{R})$ on $\mathfrak{sl}_2(\mathbb{R})$.

Theorem 5.2. *Let $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$ be a compatible triple. Suppose $v \in V$ and $\tilde{v} \in \tilde{V}$ are such that $\tilde{G}\tilde{v} \sim Gv$. Assume $G^{\mathbb{C}}v \subset V^{\mathbb{C}}$ is closed. Then there exist inner Cartan involutions θ and $\tilde{\theta}$ of V and $\tilde{V}$ respectively, such that if $v = v_+ + v_-$ and $\tilde{v} = \tilde{v}_+ + \tilde{v}_-$ are the Cartan decompositions, then:*

$$Gv_+ \sim \tilde{G}\tilde{v}_+, \text{ and } Gv_- \sim \tilde{G}\tilde{v}_-.$$

Proof. Since $G^{\mathbb{C}}v \subset V^{\mathbb{C}}$, is closed, then so are the real orbits: $Gv \subset V$, and $\tilde{G}\tilde{v} \subset \tilde{V}$ by Proposition 2.23, thus we can choose minimal vectors $X \in Gv$ and $\tilde{X} \in \tilde{G}\tilde{v}$. Now since X and $\tilde{X}$ are also minimal vectors in $G^{\mathbb{C}}v$, then $\tilde{X} \in U \cdot X$ (by Corollary 4.5). So X and $\tilde{X}$ have components which lie in the same $G^{\mathbb{C}}$-orbit, this follows since the U-action preserves the W-components and iW-components. But there exist $g \in G$ and $\tilde{g} \in \tilde{G}$, such that $g \cdot v = X$ and $\tilde{g} \cdot \tilde{v} = \tilde{X}$. So by conjugating our fixed inner Cartan involution of ρ_V^G by the action of g, and similarly for $\rho_{\tilde{V}}^{\tilde{G}}$ by the action of $\tilde{g}$ we obtain the result. The theorem is proved. $\square$

Following the proof of the theorem, then an interesting corollary is the following:

Corollary 5.3. *Let $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$ be a compatible triple. Suppose $v \in V$ and $\tilde{v} \in \tilde{V}$ are such that: $\tilde{G}\tilde{v} \sim Gv$. Then $Gv \cap V_+ \neq \emptyset$ (respectively $Gv \cap V_- \neq \emptyset$) if and only if $\tilde{G}\tilde{v} \cap \tilde{V}_+ \neq \emptyset$ (respectively $\tilde{G}\tilde{v} \cap \tilde{V}_- \neq \emptyset$).*

Proof. If $v_+ \in Gv \cap V_+$ then as $V_+ \subset \mathcal{M}(G, V)$, the real orbit: $Gv \subset V$, must be closed. Thus $\tilde{G}\tilde{v} \subset \tilde{V}$ must also be closed, and so we may choose a minimal vector $\beta \in \tilde{G}\tilde{v}$. But since $v_+ \in W$ and $Uv_+ \subset W$, because U acts on W, then by Lemma 4.5, we have $\beta \in Uv_+ \subset W$, thus $\beta \in \tilde{V} \cap W = \tilde{V}_+$. The other case is identical, since $U \cdot iW \subset iW$. The corollary is proved. $\square$

Thus by letting $\tilde{G} := U$ and $\tilde{V} := W$ and $\rho_{\tilde{V}}^{\tilde{G}} := \rho_W^U$ then: $\left(\rho_V^G, \rho_W^U, \rho_W^U\right)$ is a compatible triple and we get a new version of ([20], Theorem 5.5 (case 2)), in view of inner Cartan involutions of the action:

Theorem 5.4. *Let (ρ_V^G, ρ_W^U) be a compatible pair, then the following two statements hold:*

(1) Let $v \in V$, then the following statements are equivalent:
- *A There exist $w \in W$ such that $Uw \sim Gv$.*
- *B There exist an inner Cartan involution $V \xrightarrow{\theta} V$ such that $\theta(v) = v$.*
- *C There exist $w \in W$ such that $Uw \cap Gv \neq \emptyset$.*

(2) Let $v \in V$, then the following statements are equivalent:
- *A There exist $iw \in iW$ such that $U \cdot iw \sim Gv$.*
- *B There exist an inner Cartan involution $V \xrightarrow{\theta} V$ such that $\theta(v) = -v$.*
- *C There exist $iw \in iW$ such that $U \cdot iw \cap Gv \neq \emptyset$.*

Proof. We prove case (1) as case (2) is identical. ($A \Rightarrow B$). Let $v \in V$ and write $v = v_+ + v_-$ w.r.t our inner Cartan involution: θ. If there exist $w \in W$ such that $Gv \sim Uw$, then by Theorem 5.2, it follows that $Gv_- \sim Uw_- = \{0\}$, since the inner Cartan involution of ρ_W^U is just the identity, and thus there exist $g \in G$ such that $g \cdot v \in Uw$, i.e $\theta(g \cdot v) = g \cdot v$. Therefore by conjugating θ by the action of g, we get a new inner Cartan involution θ', such that $\theta'(v) = v$. ($B \Rightarrow C$). Now if $\theta'(v) = v$ for some inner Cartan involution, then since θ' is conjugated to θ by definition, then it follows that there exist $g \in G$ such that $\theta(g \cdot v) = g \cdot v$, i.e $g \cdot v \in V_+ \subset W$, and thus $Gv \cap U \cdot (g \cdot v) \neq \emptyset$, but $Uw = U \cdot (g \cdot v)$. ($C \Rightarrow A$). Finally if $v' \in Gv \cap Uw$ then clearly $Gv \sim Uv'$ for $v' \in W$. Thus the equivalences are established, and so the theorem is proved. $\square$

Observe that the equivalence $A \Leftrightarrow C$ of case (1) is precisely Theorem 2.18. Now combining Corollary 5.3 with Theorem 5.4 we get the following invariance result of compatible real orbits:

Corollary 5.5. *Let $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$ be a compatible triple. Suppose $v \in V$ and $\tilde{v} \in \tilde{V}$ are such that $Gv \sim \tilde{G}\tilde{v}$. Then there exist an inner Cartan involution $V \xrightarrow{\theta} V$ such that $\theta(v) = v$ (respectively $\theta(v) = -v$) if and only if there exist an inner Cartan involution $\tilde{V} \xrightarrow{\tilde{\theta}} \tilde{V}$ such that $\tilde{\theta}(\tilde{v}) = \tilde{v}$ (respectively $\tilde{\theta}(\tilde{v}) = -\tilde{v}$).*

Proof. It is enough to consider the case where $\theta(v) = v$. If θ is an inner Cartan involution of V such that $\theta(v) = v$, then by Theorem 5.4 case (1), $Gv \cap Uw \neq \emptyset$ for some $w \in W$. Thus the minimal vectors of $G^{\mathbb{C}}v$ is just $Uw \subset W$. In particular $\tilde{G}\tilde{v}$ must be closed as well, and thus $\tilde{G}\tilde{v} \cap Uw \neq \emptyset$, so we can choose an inner Cartan involution $\tilde{\theta}$ of $\rho_{\tilde{V}}^{\tilde{G}}$ such that $\tilde{\theta}(\tilde{v}) = \tilde{v}$. The converse is identical, and so the corollary is proved. $\square$

Corollary 5.6. *Let (ρ_V^G, ρ_W^U) be a compatible pair. Let $v_1 \in V$, and $G^{\mathbb{C}}v_1 \cap V = Gv_1 \cup Gv_2 \cup \cdots \cup Gv_k$ for some natural number $k \geq 1$. Then there exist an inner Cartan involution θ_j of ρ_V^G for some $1 \leq j \leq k$ such that $\theta_j(v_j) = v_j$ (respectively $\theta_j(v_j) = -v_j$) if and only if there exist inner Cartan involutions: θ_i for all $1 \leq i \leq k$ such that $\theta_i(v_i) = v_i$ (respectively $\theta_i(v_i) = -v_i$).* $\square$

For non-closed orbits we can also apply Theorem 5.2 to their boundaries:

Corollary 5.7. *Suppose we have compatible triple: $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$. Let $v \in V$ and $\tilde{v} \in \tilde{V}$. Assume $G^{\mathbb{C}}v$ is not closed, and $Gv \sim \tilde{G}\tilde{v}$. Let $p \in \mathfrak{p}$ and $\tilde{p} \in \tilde{\mathfrak{p}}$ be such that the limits exist: $e^{tp} \cdot v \to \alpha \in \overline{Gv} - Gv$ and $e^{t\tilde{p}} \cdot \tilde{v} \to \tilde{\alpha} \in \overline{\tilde{G}\tilde{v}} - \tilde{G}\tilde{v}$ where $G\alpha$ and $\tilde{G}\tilde{\alpha}$ are closed (Theorem 2.15). Then there exist inner Cartan involutions θ and $\tilde{\theta}$ of V and $\tilde{V}$ respectively, such that if $\alpha = \alpha_+ + \alpha_-$ and $\tilde{\alpha} = \tilde{\alpha}_+ + \tilde{\alpha}_-$ are the Cartan decompositions, then:*

$$G\alpha_+ \sim \tilde{G}\tilde{\alpha}_+, \text{ and } G\alpha_- \sim \tilde{G}\tilde{\alpha}_-.$$

Proof. Since there is a unique closed $G^{\mathbb{C}}$-orbit in the closure $\overline{G^{\mathbb{C}}v}$ (Theorem 2.15), and $Gv \sim \tilde{G}\tilde{v}$, then the real orbits in the closures must be compatible, i.e $G\alpha \sim \tilde{G}\tilde{\alpha}$. Thus we may apply Theorem 5.2, and the corollary follows. □

We end this section with an example illustrating the falsehood of Theorem 5.4 in the case where both groups are non-compact:

Example 5.8. ***[Not all compatible real orbits need to intersect]****. Let $G := SL_2(\mathbb{R}) \subset G^{\mathbb{C}} := SL_2(\mathbb{C}) \supset U := SU(2)$ be the standard matrix representations, and consider the adjoint actions of these groups on their Lie algebras respectively. It is easy to see that G is compatible with U. We can find $v' \neq v \in \mathfrak{g} \cap \mathfrak{u}$ such that $Gv \subset G^{\mathbb{C}}v \supset Gv'$ but $Gv \neq Gv'$. Consider the induced product action of the semi-simple groups:*

$$H := G \times G \subset G^{\mathbb{C}} \times G^{\mathbb{C}} \supset G \times U := \tilde{H},$$

acting on $\mathfrak{h} := \mathfrak{g} \times \mathfrak{g}$ and $\tilde{\mathfrak{h}} := \mathfrak{g} \times \mathfrak{u}$ respectively. Then $\left(H, \tilde{H}, U \times U\right)$ is a compatible triple, and $\left(\mathfrak{h}, \tilde{\mathfrak{h}}, \mathfrak{u} \times \mathfrak{u}\right)$ is also a compatible triple. Thus we have the setup of compatible representations. Now we note that:

$$H \cdot (v, v) \subset H^{\mathbb{C}}(v, v) \supset \tilde{H} \cdot (v', v'),$$

however if there exist $(v_1, v_2) \in H \cdot (v, v) \cap \tilde{H} \cdot (v', v')$, then there exist $g \in G$ such that $g \cdot v = v'$ which is impossible. Hence $H \cdot (v, v) \sim \tilde{H} \cdot (v', v')$ are compatible real orbits, but cannot intersect.

6. Applications to Wick-rotations of arbitrary signatures

6.1. The isometry action of $O(n, \mathbb{C})$ on tensors.

In this subsection we consider Wick-rotations and recall the setup from [20]. We use the isometry action of the complex orthogonal group: $O(n, \mathbb{C})$ on a tensor product space, induced from the isometry action of the holomorphic metric, and apply the results of Section 5 to obtain necessary conditions for the existence of a Wick-rotation at a common fix point p. We begin by observing that we indeed have the setup of compatible representations (see Section 4).

Suppose now that $(M, g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g})$ are Wick-rotated at $p \in M \cap \tilde{M}$, and consider now the complex isometry action $\rho^{\mathbb{C}}$ of $O(n, \mathbb{C})$ on $T_pM^{\mathbb{C}}$:

$$g \cdot v := g(v), \;\; g \in O(n, \mathbb{C}), \;\; v \in T_pM^{\mathbb{C}}.$$

Now by using an isomorphism: $T_pM^{\mathbb{C}} \xrightarrow{\psi} T_pM^{\mathbb{C}}$, as in Corollary 2.14, then we have a compatible triple: $\left(T_pM, \psi^{-1}(T_p\tilde{M}), W\right)$, and we know by Lemma 2.12, that the corresponding pseudo-orthogonal groups: $\left(O(p,q), O(\tilde{p},\tilde{q}), O(n)\right)$ also form a compatible triple (by definition). Thus the corresponding real isometry actions of our pseudo-inner products: $g(-,-)$, $\tilde{g}(-,-)$ and $g^{\mathbb{C}}_{|W}(-,-)$, are restrictions of $\rho^{\mathbb{C}}$. Denote them by $\rho^{O(p,q)}_{T_pM}$, $\rho^{O(\tilde{p},\tilde{q})}_{\psi^{-1}(T_p\tilde{M})}$ and $\rho^{O(n)}_{W}$ respectively, then they form a compatible triple: $\left(\rho^{O(p,q)}_{T_pM}, \rho^{O(\tilde{p},\tilde{q})}_{\psi^{-1}(T_p\tilde{M})}, \rho^{O(n)}_{W}\right)$, in the sense of Definition 4.2.

The map ψ and the isometry action $\rho^{\mathbb{C}}$ naturally extends tensorially to complexified tensors: $v^{\mathbb{C}} \in \mathcal{V}^{\mathbb{C}} := \left(\bigotimes_{i=1}^k T_pM^{\mathbb{C}}\right) \otimes \left(\bigotimes_{i=1}^m (T_pM^{\mathbb{C}})^*\right)$ at a point p. Denote Ψ for the extension map of ψ to tensors $\mathcal{V}^{\mathbb{C}}$, i.e $\Psi(-) := \psi \cdot (-)$. Then it is easy to check that the triple: $\left(\mathcal{V}, \Psi^{-1}(\tilde{\mathcal{V}}), \mathcal{W}\right)$ also form a compatible triple, where we define:

$$\mathcal{V} := \left(\bigotimes_{i=1}^{k} T_pM\right) \bigotimes \left(\bigotimes_{i=1}^{m} (T_pM)^*\right), \quad \tilde{\mathcal{V}} := \left(\bigotimes_{i=1}^{k} T_p\tilde{M}\right) \bigotimes \left(\bigotimes_{i=1}^{m} (T_p\tilde{M})^*\right),$$

and $\mathcal{W} := \left(\bigotimes_{i=1}^k W\right) \otimes \left(\bigotimes_{i=1}^m W^*\right)$.

Thus the real isometry tensor actions also naturally form a compatible triple: $\left(\rho^{O(p,q)}_{\mathcal{V}}, \rho^{O(\tilde{p},\tilde{q})}_{\Psi^{-1}(\tilde{\mathcal{V}})}, \rho^{O(n)}_{\mathcal{W}}\right)$.

Let $\{e_1, \ldots, e_p, \ldots, e_n\}$ be a pseudo-orthonormal basis of the metric g, and θ the Cartan involution w.r.t this basis. Then

$$\{y_1, \ldots, y_n\} := \{e_1, \ldots, e_p, ie_{p+1}, \ldots, ie_n\},$$

is an orthonormal basis of $g^{\mathbb{C}}$. Note that the span of $\{y_1, \ldots, y_n\}$ is precisely the compact real slice W, and moreover the conjugation map τ of W in $T_pM^{\mathbb{C}}$ restricts to θ. We can extend the holomorphic metric $g^{\mathbb{C}}$ (at p) to a holomorphic inner product $\mathbf{g}^{\mathbb{C}}$ on $\mathcal{V}^{\mathbb{C}}$ by defining:

$$\mathbf{g}^{\mathbb{C}}\left(\otimes_{i=1}^k v_i^{\mathbb{C}} \otimes_{j=1}^m w_j^{\mathbb{C}^*}, \otimes_{s=1}^k \tilde{v}_s^{\mathbb{C}} \otimes_{t=1}^m \tilde{w}_t^{\mathbb{C}^*}\right) := \sum_{1\leq i,s\leq n} g^{\mathbb{C}}(v_i^{\mathbb{C}}, \tilde{v}_s^{\mathbb{C}}) + \sum_{1\leq j,t\leq n} g^{\mathbb{C}}(w_j^{\mathbb{C}}, \tilde{w}_t^{\mathbb{C}}),$$

using the isomorphism:

$$T_pM^{\mathbb{C}} \xrightarrow{v^{\mathbb{C}^*}} (T_pM^{\mathbb{C}})^*, \; v \mapsto g^{\mathbb{C}}(v, -).$$

We see that $\mathcal{V} \subset (\mathcal{V}^{\mathbb{C}}, \mathbf{g}^{\mathbb{C}}) \supset \Psi^{-1}(\tilde{\mathcal{V}})$ are real forms (i.e real slices). Denote $\mathbf{g}$ for the induced pseudo-inner product on $\mathcal{V}$. The Cartan involution θ of g extends in the obvious way to a Cartan involution Θ of $\mathbf{g}$, by

$$\otimes_{i=1}^k v_i \otimes_{j=1}^m v_j^* \mapsto \otimes_{i=1}^k \theta(v_i) \otimes_{j=1}^m v_j^* \circ \theta,$$

which is just the action of θ on tensors, i.e $\Theta = \rho^{O(p,q)}_{\mathcal{V}}(\theta)(v)$. Now the inner Cartan involutions of the action (w.r.t $\mathbf{g}$) are just those conjugate to Θ by definition (see definition in Section 3). This means that the inner Cartan involutions are precisely those which are extensions from a Cartan involution of the metric g.

Moreover because T_pM and $\psi^{-1}(T_p\tilde{M})$ are both compatible with W, then we also have that $\mathcal{V}$ and $\Psi^{-1}(\tilde{\mathcal{V}})$ are compatible with the $O(n)$-invariant Hermitian inner product: $\mathbf{g}^{\mathbb{C}}(\cdot,\mathcal{T}(\cdot))$, where $\mathcal{T}$ is the conjugation map of $\mathcal{W}\subset\mathcal{V}^{\mathbb{C}}$ defined by the action: $\mathcal{T}(v^{\mathbb{C}}):=\tau\cdot v^{\mathbb{C}}$. Thus the isometry actions lend themselves to the results of Section 5.

Remark 6.1. The isometry tensor product action and everything defined in this section extends in the natural way to finite sums of the form:

$$\bigoplus_{k,m}\Big(\Big(\bigotimes_{i=1}^{k}T_pM^{\mathbb{C}}\Big)\bigotimes\Big(\bigotimes_{i=1}^{m}(T_pM^{\mathbb{C}})^*\Big)\Big).$$

Thus from heron and to the end of this paper we assume the isometry tensor action on such sums and thus replace: $\mathcal{V}^{\mathbb{C}}$ with this sum.

Definition 6.2. Let M and $\tilde{M}$ be two Wick-rotatable real slices at $p\in M\cap\tilde{M}$. Then two tensors $v\in\mathcal{V}$ and $\tilde{v}\in\tilde{\mathcal{V}}$ are said to be *Wick-rotatable* at p, if they lie in the same $O(n,\mathbb{C})$-orbit, i.e

$$O(n,\mathbb{C})\cdot v=O(n,\mathbb{C})\cdot\tilde{v}.$$

Note that if v and $\tilde{v}$ are two Wick-rotatable tensors, then using the map Ψ above, we see that $O(p,q)v\sim O(\tilde{p},\tilde{q})\Psi^{-1}(\tilde{v})$ are two compatible real orbits (see Definition 5.1).

The most obvious example of two Wick-rotatable tensors, are of course the real metrics themselves: $g\in T^2(T_pM)$ and $\tilde{g}\in T^2(T_p\tilde{M})$ at the common point p, simply because they are restrictions of the holomorphic metric at p. Thus from the metrics it follows that the real Levi-Civita connections: $\nabla\in T^2(T_pM)$ and $\tilde{\nabla}\in T^2(T_p\tilde{M})$ must also be restrictions of the holomorphic Levi-Civita connection: $\nabla^{\mathbb{C}}$, on the tangent spaces at p. Thus furthermore the real Riemann tensors: R and $\tilde{R}$ restricted to the tangent spaces at p are also restrictions of the holomorphic Riemann tensor. As seen in [21], one can for instance view them as vectors: $R\in End(\mathfrak{o}(p,q))$ and $\tilde{R}\in End(\mathfrak{o}(\tilde{p},\tilde{q}))$. From the Riemann tensors it also follows that the real Ricci curvatures: $ric_g\in T^2(T_pM)$ and $ric_{\tilde{g}}\in T^2(T_p\tilde{M})$ and the real Ricci operators: $Ric_g\in End(T_pM)$ and $Ric_{\tilde{g}}\in End(T_p\tilde{M})$ must also be Wick-rotatable respectively.

6.2. **Purely electric/magnetic spaces.** Let (M,g) be a pseudo-Riemannian space of signature (p,q), and let $p\in M$ be a point, and $\theta\in O(p,q)$ be a Cartan involution of $g_p(-,-)$. Consider the isometry tensor action of $O(p,q)$ on $\mathcal{V}$ from the previous section:

$$O(p,q)\xrightarrow{\rho_{\mathcal{V}}^{O(p,q)}}GL(\mathcal{V}).$$

Then θ naturally extends to an involution $\Theta := \rho_{\mathcal{V}}^{O(p,q)}(\theta)$ on $\mathcal{V}$, and the metric naturally induces a pseudo-inner product: $\mathbf{g}(-,-)$ on $\mathcal{V}$ such that Θ is a Cartan involution.

Let now $R \in \mathcal{V}$ be the Riemann tensor of M at p for $\mathcal{V}$ some tensor product. If there exist a Cartan involution Θ such that $\Theta(R) = R$ (respectively $\Theta(R) = -R$), then the space (M, g) at p is called *Riemann purely electric* (RPE) (respectively *Riemann purely magnetic* (RPM)). If there is such a Θ for the Weyl tensor at p, then (M, g) at p is called *purely electric* (PE) (respectively *purely magnetic* (PM)).

6.3. **Invariance theorem for Wick-rotation at a point p.** We now follow the notation of Section 6.1 for the isometry action on tensor products, and apply the results of Section 5 to these actions. For the results in this section and the next we can for instance consider the Wick-rotatable tensors mentioned in the last paragraph after Defn 6.2. Recall the result given in [21], where a Wick-rotation of a Riemannian real slice and an arbitrary pseudo-Riemannian real slice was considered. There the following result was proven:

Theorem 6.3 ([21])**.** *Let (M, g) and $(\tilde{M}, \tilde{g})$ be Wick-rotated at $p \in M \cap \tilde{M}$. Assume $(\tilde{M}, \tilde{g})$ is Riemannian. Then the pseudo-Riemannian space (M, g) is Riemann purely electric (RPE) at p.*

We note in the case where $(\tilde{M}, \tilde{g})$ is Riemannian, then the complex orbit: $O(n, \mathbb{C})v \subset \mathcal{V}^{\mathbb{C}}$, for two Wick-rotatable tensors is always closed. Moreover any Cartan involution for a Riemannian space is just the identity ($\theta = 1$), and thus when extended to tensors, this is just the identity as well ($\Theta = 1$).

Thus for arbitrary signatures the following result is a generalisation:

Theorem 6.4. *Let (M, g) and $(\tilde{M}, \tilde{g})$ be Wick-rotated at $p \in M \cap \tilde{M}$ of arbitrary signatures. Suppose $v \in \mathcal{V}$ and $\tilde{v} \in \tilde{\mathcal{V}}$ are two Wick-rotated tensors at p. Assume $O(n, \mathbb{C})v$ is closed. Then there exist Cartan involutions θ and $\tilde{\theta}$ of $g(-,-)$ and $\tilde{g}(-,-)$ at p respectively, such that if $v = v_+ + v_-$ and $\tilde{v} = \tilde{v}_+ + \tilde{v}_-$ are the Cartan decompositions w.r.t the extended Cartan involutions on $\mathcal{V}$ and $\tilde{\mathcal{V}}$, then v_+ and $\tilde{v}_+$ are Wick-rotated at p and so are v_- and $\tilde{v}_-$ at p.*

Proof. We apply Theorem 5.2 to the compatible triple:
$\left(\rho_{\mathcal{V}}^{O(p,q)}, \rho_{\Psi^{-1}(\tilde{\mathcal{V}})}^{O(\tilde{p},\tilde{q})}, \rho_{\mathcal{W}}^{O(n)}\right)$, as defined in Section 6.1, together with the compatible real orbits: $O(p, q)v \sim O(\tilde{p}, \tilde{q})\Psi^{-1}(\tilde{v})$. The results then follow to $\tilde{v}$, as $\Psi(-) = g \cdot -$ for some $g \in O(n, \mathbb{C})$. □

Thus from the theorem there are Cartan involutions such that the components must be Wick-rotated also at p. Note that Theorem 5.4 (case 1), is precisely Theorem 6.3. Recall now the definition given in Section 6.2, then we have the following invariance result for Wick-rotation at a point which also extends Theorem 6.3:

Corollary 6.5. ***[Invariance of Wick-rotation]****.* *Let (M,g) and $(\tilde{M},\tilde{g})$ be Wick-rotated at $p\in M\cap\tilde{M}$ of arbitrary signatures. Then M is (PE), (RPE), (PM) or (RPM) if and only if $\tilde{M}$ is (PE), (RPE), (PM) or (RPM) respectively.*

Proof. This is precisely Corollary 5.5 with $v=R$ (respectively $v=W$) and $\tilde{v}:=\tilde{R}$ (respectively $\tilde{v}=\tilde{W}$) being the Riemann tensors at p (respectively the Weyl tensors at p), applied to the compatible triple:
$\left(\rho_{\mathcal{V}}^{O(p,q)},\rho_{\Psi^{-1}(\tilde{\mathcal{V}})}^{O(\tilde{p},\tilde{q})},\rho_{\mathcal{W}}^{O(n)}\right)$, as defined in Section 6.1. □

One may conjecture that Theorem 6.4 also hold for non-closed orbits: $O(n,\mathbb{C})v$, so this is a natural follow-up question to ask. However for a non-closed orbit: $O(n,\mathbb{C})v$, we do have the following result on the boundaries of the orbits:

Corollary 6.6. *Let (M,g) and $(\tilde{M},\tilde{g})$ be Wick-rotated at $p\in M\cap\tilde{M}$ of arbitrary signatures. Let $v\in\mathcal{V}$ and $\tilde{v}\in\tilde{\mathcal{V}}$. Assume $O(n,\mathbb{C})v$ is not closed, and that v is Wick-rotatable to $\tilde{v}$ at p. Let $x\in\mathfrak{p}$ and $\tilde{x}\in\tilde{\mathfrak{p}}$ be such that the limits exist: $e^{tx}\cdot v\to\alpha\in\overline{O(p,q)v}-O(p,q)v$ and $e^{t\tilde{x}}\cdot\tilde{v}\to\tilde{\alpha}\in\overline{O(\tilde{p},\tilde{q})\tilde{v}}-O(\tilde{p},\tilde{q})\tilde{v}$ where $O(p,q)\alpha$ and $O(\tilde{p},\tilde{q})\tilde{\alpha}$ are closed (Theorem 2.15). Then there exist extended Cartan involutions Θ and $\tilde{\Theta}$ of $\mathcal{V}$ and $\tilde{\mathcal{V}}$ respectively, such that if $\alpha=\alpha_++\alpha_-$ and $\tilde{\alpha}=\tilde{\alpha}_++\tilde{\alpha}_-$ are the Cartan decompositions, then α_+ is Wick-rotated at p to $\tilde{\alpha}_+$ and α_- is Wick-rotated at p to $\tilde{\alpha}_-$.*

Proof. We apply Corollary 5.7 to the compatible triple:
$\left(\rho_{\mathcal{V}}^{O(p,q)},\rho_{\Psi^{-1}(\tilde{\mathcal{V}})}^{O(\tilde{p},\tilde{q})},\rho_{\mathcal{W}}^{O(n)}\right)$, as defined in Section 6.1, and to the compatible real orbits: $O(p,q)v\sim O(\tilde{p},\tilde{q})\Psi^{-1}(\tilde{v})$. □

6.4. **A note on Wick-rotations of the same signatures.** Suppose M and $\tilde{M}$ are Wick-rotated at a common point p and have the same signatures. Then if $v\in\mathcal{V}$ and $\tilde{v}\in\tilde{\mathcal{V}}$ are Wick-rotated at p, we may choose Ψ (in Definition 6.2) such that $\Psi^{-1}(\tilde{v})\in\mathcal{V}$. Thus we have the following:

Proposition 6.7. *Suppose (M,g) and $(\tilde{M},\tilde{g})$ are Wick-rotated at $p\in M\cap\tilde{M}$ and have the same signatures $p+q=n$. Let $v\in\mathcal{V}$ and $\tilde{v}\in\tilde{\mathcal{V}}$ be two Wick-rotatable tensors at p. Assume there is a unique real orbit in the complex orbit $O(n,\mathbb{C})v$, i.e $O(n,\mathbb{C})v\cap\mathcal{V}=O(p,q)v$. Then there is a homeomorphism:*

$$\mathcal{V}\supset O(g_p(-,-),T_pM)\cdot v\cong O(\tilde{g}_p(-,-),T_p\tilde{M})\cdot\tilde{v}\subset\tilde{\mathcal{V}}.$$

Proof. Since $v\in\mathcal{V}$ and $\tilde{v}\in\tilde{\mathcal{V}}$ are Wick-rotatable tensors at p, then $\Psi^{-1}(\tilde{v})\in O(n,\mathbb{C})v$, where we may choose ψ to be an isomorphism: $T_pM\cong T_p\tilde{M}$. Thus $\Psi^{-1}(\tilde{v})\in\mathcal{V}$, and so

$$O(\tilde{g}_p(-,-),T_p\tilde{M})\cdot\tilde{v}\cong O(p,q)\cdot\Psi^{-1}(\tilde{v})=O(p,q)v:=O(g_p(-,-),T_pM)\cdot v.$$

□

Thus for two Wick-rotated Riemannian real slices at a common point we have:

Corollary 6.8. *Suppose (M,g) and $(\tilde{M},\tilde{g})$ are Wick-rotated Riemannian slices at $p\in M\cap\tilde{M}$. Let $v\in\mathcal{V}$ and $\tilde{v}\in\tilde{\mathcal{V}}$ be two Wick-rotatable tensors at p. Then there is a diffeomorphism of embedded submanifolds:*

$$\mathcal{V}\supset O(g_p(-,-),T_pM))\cdot v\cong O(\tilde{g}_p(-,-),T_p\tilde{M})\cdot\tilde{v}\subset\tilde{\mathcal{V}}.$$

Proof. By ([7], Proposition 8.3.1), we can apply Proposition 6.7, and the result follows. □

6.5. **Wick-rotatable metrics.** Here we will consider two pseudo-Riemannian metrics (M,g) and $(\tilde{M},\tilde{g})$ of possibly different signature and give sufficient conditions when such are Wick-rotated.

Proposition 6.9. *Assume that $v\in V$ and $\tilde{v}\in\tilde{V}$ have closed orbits Gv and $\tilde{G}\tilde{v}$, and that their polynomial invariants are identical. Then they are Wick-rotated in the sense that there is a $G^{\mathbb{C}}\supset G,\tilde{G}$ so that*

$$Gv\subset G^{\mathbb{C}}v\supset\tilde{G}\tilde{v}$$

Proof. Since v and $\tilde{v}$ have identical invariants and their corresponding orbits are closed, then due to Thm. 2.17, then the corresponding complex orbits are closed too. Then, since the invariants separate the complex orbits, the complex orbits are identical and the result follows. □

Let now $\mathcal{V}^{(k)}$ be the vector space associated with the components of tensors, ref. Section 6.1, so that $\bigoplus_{i=0}^{k}\nabla^{(i)}\text{Riem}\in\mathcal{V}^{(k)}$, where $\nabla^{(i)}\text{Riem}$ indicates the ith covariant derivative of the Riemann curvature tensor. Then:

Theorem 6.10. *Let $v^k\in\mathcal{V}^{(k)}$ and $\tilde{v}^k\in\tilde{\mathcal{V}}^{(k)}$ be the curvature tensors of (M,g) and $(\tilde{M},\tilde{g})$, respectively. Assume that there exists points $q\in M$ and $\tilde{q}\in\tilde{M}$ so that the corresponding orbits Gv^k and $\tilde{G}\tilde{v}^k$ are closed and their invariants are identical for all k. Then the metrics are Wick-rotated w.r.t a common point $q=\tilde{q}$.*

Proof. By the above proposition, Gv^k and $\tilde{G}\tilde{v}^k$ are Wick-rotated for a $q\in M$ and $\tilde{q}\in\tilde{M}$, for all k. Since the metrics are real analytic, there exists neighbourhoods $U\subset M$ and $\tilde{U}\subset\tilde{M}$, of q and $\tilde{q}$ respectively, which can be embedded into a complex neighbourhood $U^{\mathbb{C}}$ so that $q=\tilde{q}\in U^{\mathbb{C}}$. The real analytic structure can now be extended to an analytic structure on $U^{\mathbb{C}}$ and the complexified orbit $G^{\mathbb{C}}v^k$ at $q=\tilde{q}$ give rise to complex curvature tensors. These can now be (maximally) analytically extended to an analytic metric $g^{\mathbb{C}}$ on a neighbourhood in $U^{\mathbb{C}}$ (for simplicity, call this neighbourhood $U^{\mathbb{C}}$). These real analytic structures $(U,g|_U)$ and $(\tilde{U},\tilde{g}|_{\tilde{U}})$ thus are Wick-rotated, both being restrictions of the complex holomorphic $(U^{\mathbb{C}},g^{\mathbb{C}})$. Since Wick-rotation is a local criterion, the theorem now follows. □

This implies that for closed orbits, the metrics are necessarily Wick-rotated as long as their invariants are identical. If the orbits are not closed, we have a result which is point-wise. By evaluating the curvature tensors at a point, we can use the following result.

Theorem 6.11. *Assume that $v \in V$ and $\tilde{v} \in \tilde{V}$ have identical invariants. Then there exist $p \in \mathfrak{p}$ and $\tilde{p} \in \tilde{\mathfrak{p}}$ so that $v_0 := \lim_{t\to\infty} e^{tp} \cdot v$ and $\tilde{v}_0 := \lim_{t\to\infty} e^{t\tilde{p}} \cdot \tilde{v}$ are Wick-rotated.*

Proof. This follows from Theorem 2.15, and the fact that a point in the orbit, $x \in Gv$ and any point its closure $x_0 \in \overline{Gv}$ have identical invariants. Since there is a unique closed orbit in the closure $\overline{Gv}$, the result follows. □

Note that we say that a metric (M, g) is characterised by its invariants is exactly when v has a closed orbit. This implies that for two metrics being characterised by its invariants which have identical invariants are related by Wick rotations.

6.6. **Universal metrics.** A pseudo-Riemannian metric is called *universal* if all conserved symmetric rank-2 tensors constructed from the metric, the Riemann tensor and its covariant derivatives are multiples of the metric. Hence, universal metrics are metrics which obey $T_{\mu\nu} = \lambda g_{\mu\nu}$, for all symmetric conserved tensors $T_{\mu\nu}$ constructed from the metric and the curvature tensors (recall that conserved implies $\nabla^\mu T_{\mu\nu} = 0$) [36]. We note that this constuction can be lifted holomorphically to the holomorphic Riemannian manifold and thus implies $T^{\mathbb{C}}_{\mu\nu} = \lambda g^{\mathbb{C}}_{\mu\nu}$. Thus, universality is *preserved under Wick rotation.* This straight-forwardly leads to:

Proposition 6.12. *Assume that (M, g) and $(\tilde{M}, \tilde{g})$ are two Wick-rotated pseudo-Riemannian manifolds. Then (M, g) is universal if and only if $(\tilde{M}, \tilde{g})$ is universal.*

In the Riemannian case, all such metrics are classified. Indeed, all Riemannian universal spaces are locally homogeneous space where the isotropy group acts irreducibly on the tangent space [37]. In other signatures this is no longer true as there are universal examples of both Kundt and Walker type which are not locally homogeneous [36, 38]. It is, however, interesting to study those that are Wick-rotatable to the Riemannian case and relate these to the irreducibly-acting isotropy group.

As an example, consider the following four-dimensional Riemannian metric,

$$g = g_{S^2} \oplus g_{S^2}, \tag{29}$$

where g_{S^2} is the unit metric on the sphere. This has an isotropy group $O(2) \times O(2) \times \mathbb{Z}_2$, where the $\mathbb{Z}_2$ interchanges the two spheres. Each of the two spheres can be Wick-rotated to other two-dimensional spaces of constant curvature:

$$g_{S^2} \longmapsto g_{dS}\ (-+), \qquad -g_{AdS}\ (+-), \qquad -g_{H^2}\ (--),$$

where $(A)dS$ is (anti-)de Sitter space, and H^2 is the unit hyperbolic space. These can now be combined in various ways to get various Wick-related spaces being universal. For example,

$$g_N = g_{S^2} \oplus (-g_{H^2})$$

is a universal metric of neutral signature, and

$$g_L = g_{S^2} \oplus g_{dS}$$

is a universal metric of Lorentzian signature. We note that the interchange symmetry $\mathbb{Z}_2$ of the Riemannian metric is not necessarily an isotropy of the Wick-related metrics. Indeed, in both examples above, there exist vectors $X \in T_pM$ so that $g_\bullet(X,X) > 0$, while $g_\bullet(A(X), A(X)) < 0$, where $A(X)$ is the action of the non-trivial element of $\mathbb{Z}_2$ on X. Thus, the $\mathbb{Z}_2$ action cannot be an isotropy of $g_\bullet$. On the other hand, the symmetry $\mathbb{Z}_2$ preserves the signature and maps metrics onto other Wick-related metrics.

6.7. **On the set of tensors with identical invariants.** Let (M, g) be a pseudo-Riemannian manifold and denote $v^{(l)} := \nabla^{(l)}\mathrm{Riem}$ for the lth covariant derivative of the Riemann tensor at a fixed point $p \in M$. Define $V^{(l)}$ to be a tensor product space for which $v^{(l)} \in V^{(l)}$. Set $\mathcal{V}^{(k)} := \bigoplus_{l=0}^{k} V^{(l)}$, then it contains all the covariant derivatives $v^{(l)}$, up to order k of the Riemann tensor at the fixed point p. The isometry group $O(p,q)$ of the pseudo-inner product $g(-,-)$ (at p) acts on $\mathcal{V}^{(k)}$ by the tensor product action (as defined in Section 6.1). Consider the algebra of polynomial invariants $\mathbb{R}[\mathcal{V}^{(k)}]^{O(p,q)}$ of the action. Let I be the polynomial invariants restricted to the set of all the covariant derivatives of the Riemann tensor. Then I is defined to be the set of *polynomial curvature invariants*. Moreover let I_k denote the polynomial invariants $\mathbb{R}[\mathcal{V}^{(k)}]^{O(p,q)}$ restricted to the set of all the $v^{(l)}$ up to kth order. Moreover, $I = I_k$ is finitely generated [41], which means that we can find a finite number of generators for I: $I = \langle f_1, f_2, \dots, f_N \rangle$. Set $\mathcal{V} := \mathcal{V}^{(k)}$ then the set of invariants I defines a polynomial function:

$$\mathfrak{I} : \ \mathcal{V} \longrightarrow \mathbb{R}^N, \quad v \mapsto (f_1(v), \dots, f_N(v)).$$

We recall that the space (M, g) is said to be VSI if $I = \{0\}$, and is said to be VSI_k if $I_k = 0$.

Let $x_p \in \mathcal{V}$, and consider the set $S := \mathfrak{I}^{-1}(\mathfrak{I}(x_p)) \subset \mathcal{V}$, which is the set of all tensors in $\mathcal{V}$ having identical invariants as x_p.

Let S_i be the connected components of S so that $S = \cup_{i=1}^{n} S_i$, and $S_i \cap S_j = \emptyset$, $i \neq j$. Then, regarding the topology of the set S:

Proposition 6.13. *Let S be as above. Then:*

(1) If $\mathfrak{I}(S) = 0$ (VSI), then S is connected, and $\{0\} \subset S$ is the unique closed orbit in S.

(2) If S consists of $n \geq 1$ connected components, S_i, then there exist unique closed orbits $Gx_i \subset S_i$, for each $i = 1, ..., n$. These closed orbits are necessarily Wick-rotated.

Proof. First we note that since the function $\mathfrak{I}$ is polynomial, the set S is closed in $\mathcal{V}$. Recall also that there is a finite number of closed orbits in S, hence, also in each component S_i. Furthermore, each non-closed orbit has a unique closed orbit on its boundary.

Let S_i be one of the connected components and assume that A_1 and A_2 are two disjoint closed orbits in S_i; $A_1, A_2 \subset S$. Let

$$U_I := \bigcup \left\{ Gx \subset S_i : A_I \cap \overline{Gx} \neq \emptyset \right\}, \quad I = 1, 2;$$

i.e., the union of sets having A_I as part of their closure. Consider the intersection $V = \overline{U_1} \cap \overline{U_2}$. There are now two possibilities:

$V \neq \emptyset$: Since the intersection of two closed sets are closed, V is nonempty and closed. Moreover, there are no orbits Gx in V which are closed since orbits in U_I have a unique closed orbit on their boundary (namely A_I). Choose therefore a non-closed orbit Gx in V. Then $\overline{Gx}$ contains a (unique) closed orbit, but since this is necessarily in V, this leads to a contradiction.

$V = \emptyset$: Then U_1 and U_2 are disconnected. Define $W := S_i \setminus U_1 \cup U_2$ which is necessarily nonempty. Using the same argument as above, there needs to be a non-closed orbit in W with a closed orbit $A_3 \subset \overline{W}$ in its closure. Note that A_3 cannot be A_1 or A_2, because then the non-closed orbit in W should have been in U_I (hence, not in W). We can now do the same as above and define

$$U_3 := \bigcup \left\{ Gx \subset S_i : A_3 \cap \overline{Gx} \neq \emptyset \right\}.$$

Then this again implies that there is another closed orbit A_4, etc. This must terminate since there is a finite number of closed orbits in S_i. Hence, this leads to a contradiction and the closed orbit in S_i is thus unique.

The first part of the proposition now follows since $\{0\}$ is obviously the only closed orbit in S for which $\mathfrak{I}(S) = 0$. □

So in the sense of curvature tensors with identical invariants, each component S_i is characterised by its unique closed orbit. Of course, this closed orbit could be S_i itself (which it would be in the Riemannian case), but in the pseudo-Riemannian case more complicated structures of S_i are possible. We should also recall that this is the structure at a point $p \in M$. To study the structure in a neighbourhood of M is a considerably more difficult task.

Consider now a Wick-rotation at p: $(M, g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g})$. Let $v \in \mathcal{V}$ and $\tilde{v} \in \tilde{\mathcal{V}}$ be the curvature tensors of covariant derivatives of the Riemann tensors (respectively) of lth order, i.e $\tilde{v} \in O(n, \mathbb{C})v$ are Wick-rotatable. Consider the function $\mathfrak{I}$ defined for (M,g) as above. Then we can also consider (in exactly the same way as for $\mathcal{V}$ above) a function:

$$\tilde{\mathfrak{I}}:\ \tilde{\mathcal{V}} \longrightarrow \mathbb{R}^{\tilde{N}},$$

for $\tilde{\mathcal{V}}$, and $\tilde{I}=\tilde{I}_{\tilde{k}}$ is generated by a finite set of generators of $\mathbb{R}[\tilde{\mathcal{V}}]^{O(\tilde{p},\tilde{q})}$ (restricted to the covariant derivatives of the Riemann tensor up to some $\tilde{k}$th order) w.r.t the action of $O(\tilde{p},\tilde{q})$ on $\tilde{\mathcal{V}}$. Let $1\leq l\leq max\{k,\tilde{k}\}$ where k is as above. Define analogously:

$$\tilde{S}:=\tilde{\mathfrak{I}}^{-1}(\tilde{\mathfrak{I}}(\tilde{v}))=\cup_{j=1}^{m}\tilde{S}_j,$$

where $\tilde{S}_j$ are the connected components. Set $G:=O(p,q)$, $\tilde{G}:=O(\tilde{p},\tilde{q})$ and $G^{\mathbb{C}}:=O(n,\mathbb{C})$, and denote by the previous proposition $\tilde{G}\tilde{v}_j$ (respectively Gv_i) for the unique closed orbits in each component $\tilde{S}_j$(respectively S_i).

By Section 6.1 and the notation there, we can choose $\Psi(-):=g\cdot -$ for some $g\in G^{\mathbb{C}}$ such that $Gv\sim g(\tilde{G})\cdot(g\cdot\tilde{v})$ are compatible real orbits, where $\rho_{\tilde{\mathcal{V}}}^{\tilde{G}}\cong\rho_{g(\tilde{\mathcal{V}})}^{g(\tilde{G})}$ as real representations via g. Now the map Ψ is a morphism of affine complex varieties, and therefore the algebra of polynomial invariants: $\mathbb{R}[\tilde{\mathcal{V}}]^{\tilde{G}}\cong\mathbb{R}[g(\tilde{\mathcal{V}})]^{g(\tilde{G})}$ of the actions are related precisely via the action of g. Thus the set $\tilde{S}$ is mapped to $g\cdot\tilde{S}$, and $\tilde{I}$ is mapped to $g\cdot\tilde{I}$ and so on.

We have the following result:

Corollary 6.14. *Let S and $\tilde{S}$ be defined as above, and $\tilde{v}\in O(n,\mathbb{C})v$ be as above. Then*

(1) For all $1\leq i\leq n$ and $1\leq j\leq m$ the tensors: v_i and $\tilde{v}_j$ are Wick-rotatable tensors.

(2) $\mathfrak{I}(S)=0\Leftrightarrow\tilde{\mathfrak{I}}(\tilde{S})=0$. In particular if $\mathfrak{I}(S)=0$ then $S\cap\tilde{S}\neq\emptyset$.

(3) If $(\tilde{M},\tilde{g})$ is Riemannian, (i.e $\tilde{G}:=O(n)$ is compact), then there exist a $g\in O(n,\mathbb{C})$ such that $S_i\cap g\cdot\tilde{S}\neq\emptyset$ for all $1\leq i\leq n$. Thus for all $1\leq i\leq n$ there exist Cartan involutions θ_i of the metric $g(-,-)$ such that $\theta_i\cdot v_i=v_i$.

Proof. For all cases it is enough to assume $Gv\sim\tilde{G}\tilde{v}$ are compatible (see the paragraph before the statement). For case (1), suppose first that $Gv\subset V$ is closed, thus so is $\tilde{G}\tilde{v}\subset\tilde{\mathcal{V}}$. Hence since $v\in S_j$ for some j, then $Gv\subset S_j$ and is the unique closed orbit in S_j. Similarly $\tilde{G}\tilde{v}\subset\tilde{S}_i$ is the unique closed orbit for some i. So because $Gv\sim Gv_j$ for all j and $\tilde{G}\tilde{v}\sim\tilde{G}\tilde{v}_i$ for all i,j by the previous proposition, then also $Gv_j\sim\tilde{G}\tilde{v}_i$ for all i,j, and the closed case follows. Suppose now that Gv is not closed. Then let $Gx\subset\overline{Gv}$ and $\tilde{G}\tilde{x}\subset\overline{\tilde{G}\tilde{v}}$ be the unique closed orbits in the closures. Now since x and v (respectively $\tilde{x}$ and $\tilde{v}$) have the same invariants then there are i,j such that $Gx\subset S_i$ and $\tilde{G}\tilde{x}\subset\tilde{S}_j$. But since $Gv\sim\tilde{G}\tilde{v}$, then also $Gx\sim\tilde{G}\tilde{x}$ by uniqueness of closed orbits in the closure: $\overline{G^{\mathbb{C}}v}$, and so the statement follows.

For case (2), if $\mathcal{J}(S) = 0$, then $0 \in S$, and thus if $Gv_j \subset S_j$ is closed and $\tilde{G}\tilde{v}_i \subset \tilde{S}_i$ is closed, then by the proof of (1): $\tilde{G}\tilde{v}_i \sim Gv_j \sim G \cdot 0 = \{0\}$, proving that $\mathcal{J}(\tilde{v}_i) = 0$, and thus $\mathcal{J}(\tilde{S}) = 0$. The converse is symmetric so identical. The second statement follows since $0 \in S \cap \tilde{S}$.

For case (3), since $\tilde{S} = \tilde{G}\tilde{v}$, and $Gv_j \sim \tilde{G}\tilde{v}$ for all j by following the proof of (1), then $Gv_j \cap \tilde{G}v \neq \emptyset$, i.e it follows that $Gv_j \cap \tilde{S} \neq \emptyset$, and so $S_j \cap \tilde{S} \neq \emptyset$. Now by Theorem 5.4, the last part of the statement follows. □

Acknowledgements

This work was supported through the Research Council of Norway, Toppforsk grant no. 250367: *Pseudo-Riemannian Geometry and Polynomial Curvature Invariants: Classification, Characterisation and Applications.*

Part 6. Wick-rotations of pseudo-Riemannian Lie groups

The following part is an independent paper submitted to the Journal of Geometry and Physics.

Abstract. We study Wick-rotations of left-invariant metrics on Lie groups, using results from real GIT ([20], [19]). An invariant for Wick-rotation of Lie groups is given, and we describe when a pseudo-Riemannian Lie group (a Lie group with a left-invariant metric) can be Wick-rotated to a Riemannian Lie group. We define a Cartan involution of a general Lie algebra, and prove a general version of É. Cartan's result, namely the existence and conjugacy of Cartan involutions.

1. Introduction

This paper is motivated first of all by the study of Wick-rotations of pseudo-Riemannian manifolds defined in [20]. Given a pseudo-Riemannian manifold (M, g) of signature (p, q), then it is interesting know whether it can be Wick-rotated to another space $(\tilde{M}, \tilde{g})$ (w.r.t a fixed point $p \in M \cap \tilde{M}$) of signature $\tilde{p}+\tilde{q} = p+q$. In ([21], [20], [19]) the isometry action of the pseudo-orthogonal group $O(p, q)$ acting on tensors restricted to p is explored. For instance it is proved that if $\tilde{p} = 0$ (i.e $\tilde{g}$ is Riemannian) then there is a Cartan involution of the metric $\theta \in O(p, q)$ (at p) which fixes the Riemann tensor R under the isometry action, i.e $\theta \cdot R = R$. Thus (M, g) is *Riemann purely electric* (*RPE*) at p. More generally it is proved that for a space to be *purely electric* (respectively *purely magnetic*) or (RPE) (respectively *Riemann purely magnetic*) is preserved under a Wick-rotation at a common fixed point p.

A particular subclass of Wick-rotations which is of interest in its own right and deserves to be explored, is the class of *Lie groups* G equipped with left-invariant metrics, so called *pseudo-Riemannian Lie groups*. If we look at a semi-simple complex Lie group $G^{\mathbb{C}}$ equipped with the left-invariant Killing form: $-\kappa$, then there are natural examples of Wick-rotations to find at the identity point, simply because there exist real forms. Moreover by the theory of semi-simple Lie groups, one may always Wick-rotate a real form $(G, -\kappa) \subset (G^{\mathbb{C}}, -\kappa)$ to a Riemannian Lie group, simply because of the existence of a Cartan involution of the Lie algebra $\mathfrak{g}$. Thus motivated by this example, then for a general pseudo-Riemannian Lie group (G, g), an interesting question one may ask:

Given a pseudo-Riemannian Lie group (G, g)*, when can it be Wick-rotated to a Riemannian Lie group* $(\tilde{G}, \tilde{g})$*?*

Suppose (G, g) is Wick-rotated to a Riemannian Lie group $(\tilde{G}, \tilde{g})$, then in view of the results given in ([21], [20], [19]), then the so called Wick-rotatable tensors restricted to $\mathfrak{g}$ must be fixed by the isometry action (induced from the metric) of some (linear) Cartan involution $\theta \in O(p, q)$ of the metric. This could for instance be the Riemann tensor R (as mentioned above), and is related to the fact that R

can be embedded into the same complex orbit as $\tilde{R}$ (the Riemann tensor of $(\tilde{G}, \tilde{g})$ restricted to $\tilde{\mathfrak{g}}$), i.e

$$O(p+q, \mathbb{C}) \cdot R \ni \tilde{R}.$$

However some tensors for a left-invariant metric (for instance the Levi-Civita connection, the Riemann tensor and so on..) are very interlinked with the Lie bracket of the Lie algebra $\mathfrak{g}$. Moreover in the semi-simple case (equipped with the left-invariant Killing form) such tensors are naturally fixed by the Cartan involutions of the Lie algebra: $\theta \in Aut(\mathfrak{g})$. For example the Levi-Civita connection is given by: $\nabla_x y = \frac{1}{2}[x, y]$, thus naturally $\theta \cdot \nabla_x y = \nabla_x y$.

The author of this paper therefore pondered about the existence of a Cartan involution: $\theta \in Aut(\mathfrak{g})$, for a general left-invariant metric (on a general Lie group G) which can be Wick-rotated to a Riemannian Lie group $\tilde{G}$.

We prove an invariant for Wick-rotations of Lie groups, and give a complete answer to the question above, where we show that the answer is precisely related to the existence of a Cartan involution of the Lie algebra. Our main result of this paper is Theorem 3.1:

Theorem A. *Suppose (G, g) is a pseudo-Riemannian Lie group that can be Wick-rotated to another Lie group $(\tilde{G}, \tilde{g})$. Then there exist a Cartan involution of $\mathfrak{g}$ if and only if there exist a Cartan involution of $\tilde{\mathfrak{g}}$.*

We begin this paper by defining every notion we shall use throughout, and recall the definitions of Wick-rotations in [20]. Some new definitions are also given, in particular we define a *Wick-rotation of a Lie group*, and a *Cartan involution* of a general Lie algebra. We also state the results we use from [19], which makes the proofs easier to follow.

Remark 1.1. In this paper a Riemannian space shall always denote the signature: $(+, +, \cdots, +)$, and a Lorentzian space shall denote the signature: $(+, +, \cdots, +, -)$ and so on. The anti-isometry map $g \mapsto -g$ induces an isomorphism $O(p, q) \cong O(q, p)$. If we change signature via this anti-isometry map, then our results in this paper will be related precisely via this map as well. Moreover using a right-invariant metric instead of a left-invariant metric does not change the results of this paper.

Conventions: Throughout this paper κ shall denote the Killing form of a Lie algebra. A product of vector spaces $V \times V$ shall often be denoted by just V^2. A complex Lie group shall always be denoted by the symbol: $G^{\mathbb{C}}$.

2. Preliminaries

2.1. **Real forms and left-invariant metrics.** In this paper a real Lie group G shall be said to be an *immersive real form* of a complex Lie group $G^{\mathbb{C}}$, if there is a real immersion $G \to G^{\mathbb{C}}$ (of Lie groups) where $G^{\mathbb{C}}$ is viewed as a real Lie group, such that $\mathfrak{g}$ is embedded as a real form of $\mathfrak{g}^{\mathbb{C}}$ (the Lie algebra of $G^{\mathbb{C}}$). If the immersion is also injective then we shall call G a *virtual real form*. A virtual real form G which is also an embedding (i.e the image of G is closed in $G^{\mathbb{C}}$), we shall say that the real form is an *embedded real form*. An embedded real form which also satisfies: $G^{\mathbb{C}} = G \cdot G_0^{\mathbb{C}}$ (abstract group product) shall be said to be a *real form*.

Note that a connected embedded real form is also a real form. All these specialised "complexifications" divide the Lie groups into different classes. For instance if G is a connected semi-simple Lie group, then it is a fact that G is a virtual real form if and only if G is linear.

One shall note that given any connected real Lie group G, then we can complexify the Lie algebra via an inclusion $i\colon \mathfrak{g} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$. We can find a complex connected Lie group: $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Thus using the exponential maps of these groups, we can find a smooth map (of real Lie groups): $G \to G^{\mathbb{C}}$ with differential i, thus G is an immersive real form of $G^{\mathbb{C}}$.

We shall abuse notation and write $G \subset G^{\mathbb{C}}$ for an immersive real form.

Example 2.1. *Consider the complex orthogonal group: $O(4,\mathbb{C})$, then the map: $g \mapsto I_{3,1}\overline{g}I_{3,1}$, is a conjugation map (i.e the differential is a conjugation map), where $(I_{3,1})_{ii} = +1$ for $1 \le i \le 2$, $(I_{3,1})_{33} = -1$ and zero otherwise. The fix points of this map is just $O(1,3)$, which is an example of a real form of $O(4,\mathbb{C})$. Consider the universal covering group $G := \widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$, then it is a fact that G is not a virtual real form of any complex Lie group. However G is an immersive real form of $SL_2(\mathbb{C})$.*

Let G be a real Lie group, then a left-invariant metric g on G is a pseudo-Riemannian metric satisfying:

$$g_{gh}(L_{gh^*}(x_h), L_{gh^*}(y_h)) = g_h(x_h, y_h), \forall g, h \in G,\ \forall x_h, y_h \in T_hG,$$

where L_{g^*} is the push-forward of the translation map: $G \xrightarrow{L_g} G\colon h \mapsto gh$. Instead of writing $g_e(-,-)$ for the metric at the identity point, we simply write just $g(-,-)$. A bi-invariant metric g on a real Lie group G is a left-invariant metric which is also right-invariant i.e L_g above is replaced with $R_g : h \mapsto hg$.

On a real vector space V a symmetric non-degenerate bilinear form g shall be referred to as a *pseudo-inner product*, and an inner product in the case of positive definite. A pair (V, g) shall be referred to as a pseudo-inner product space (respectively inner product space). If we have a Lie algebra $\mathfrak{g}$ with a pseudo-inner product g which satisfies:

$$g([x,y],z) = g(x,[y,z]), \quad x,y,z \in \mathfrak{g},$$

then g shall be called *invariant*. Such a pair: $(\mathfrak{g}, g)$ is called a *quadratic Lie algebra*. For example the pair: $\left(\mathfrak{sl}_2(\mathbb{R}), -\kappa\right)$ is a quadratic Lie algebra, however the 3-dimensional Heisenberg Lie algebra: $\mathfrak{h}_3(\mathbb{R})$, is never a quadratic Lie algebra. We recall that an ideal $\mathfrak{I} \lhd \mathfrak{g}$ is called *non-degenerate* if $\mathfrak{g} = \mathfrak{I} \oplus \mathfrak{I}^{\perp}$ w.r.t the invariant form g. In the case that $\mathfrak{g}$ is a reductive Lie algebra, then all ideals are in fact non-degenerate.

A *holomorphic inner product* $g^{\mathbb{C}}$ on a complex vector space $V^{\mathbb{C}}$ shall be a symmetric non-degenerate complex bilinear form. The definitions of left-invariance and so on above are analogous in the case of a complex Lie group equipped with a holomorphic metric.

Definition 2.2. A real Lie group G equipped with a left-invariant metric g, denoted (G, g) shall be called a *pseudo-Riemannian Lie group*. If g is also a Riemannian metric then the pair (G, g) shall be called a *Riemannian Lie group*. A complex Lie group $G^{\mathbb{C}}$ equipped with a left-invariant holomorphic metric, shall be called a *holomorphic Riemannian Lie group* (or a *complex Riemannian Lie group*).

Definition 2.3. Let (G, g_1) and (H, g_2) be two pseudo-Riemannian Lie groups. Then G is said to be isometric to H if there exist a Lie group isomorphism: $G \xrightarrow{F} H$, such that $F_* : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of pseudo-inner product spaces: $(\mathfrak{g}, g_1) \cong (\mathfrak{h}, g_2)$. The spaces are said to be locally isometric if there exist a local homomorphism $G \supset U \xrightarrow{F} V \subset H$ such that F_* is an isomorphism of pseudo-inner product spaces: $(\mathfrak{g}, g_1) \cong (\mathfrak{h}, g_2)$.

The left-invariant metrics on a real Lie group G are in bijections with the pseudo-inner products on the Lie algebra $\mathfrak{g}$. So we shall always work with a pseudo-inner product g on the Lie algebra and induce a left-invariant metric on the Lie group by:

$$g_h(x_h, y_h) := g\Big(L_{h^{-1}_*}(x_h), L_{h^{-1}_*}(y_h)\Big), \quad x_h, y_h \in T_hG.$$

We note that for a compact Lie group G, we can always complexify it to a complex Lie group: $G^{\mathbb{C}}$, such that $G \subset G^{\mathbb{C}}$ is a real form, by using the universal complexification group. In particular starting from a compact Lie group with a left-invariant metric we naturally have a candidate for a holomorphic Riemannian Lie group such that $G \subset G^{\mathbb{C}}$ is a real form. Recall that the universal complexification group of a real Lie group G, is a pair: $(G^{\mathbb{C}}, \eta)$, where η is a real Lie homomorphism: $G \to G^{\mathbb{C}}$, satisfying the universal property (see for instance [31]). For example the pseudo-orthogonal groups: $O(p, q)$ has universal complexification group $O(p + q, \mathbb{C})$.

2.2. **Wick-rotations of pseudo-Riemannian manifolds.** We recall some of the definitions of Wick-rotations given in [20], and define a Wick-rotation of a pseudo-Riemannian Lie group.

Definition 2.4. Given a holomorphic inner product space $(E, g^{\mathbb{C}})$. Then if $V \subset E$ is a real linear subspace for which $g := g^{\mathbb{C}}\big|_V$ is non-degenerate and real valued, i.e., $g(X, Y) \in \mathbb{R}$, $\forall X, Y \in V$, we will call V a *real slice*.

Remark 2.5. In this paper we always assume $V \subset (E, g^{\mathbb{C}})$ has the same real dimension as the complex dimension of E. Thus V is also a real form of E, i.e there is a conjugation map $E \xrightarrow{\sigma} E$ with fix points V. We shall simply refer to $V \subset (E, g^{\mathbb{C}})$ as a real form in such a case, to mean both a real slice and a real form.

Thus in the definition $(V, g := g^{\mathbb{C}}\big|_V)$ is a pseudo-inner product space, and if (p, q) denotes the signature of g, then the isometry group $O(p, q)$ of (V, g) is a real Lie group and is a real form of $O(p+q, \mathbb{C})$ (the isometries of $(E, g^{\mathbb{C}})$). Indeed if σ is the conjugation map of V in E then note the involution F of real Lie groups:

$$g \mapsto \sigma g \sigma, \ g \in O(p+q, \mathbb{C}).$$

The differential of this map is a conjugation map, and $O(p, q)$ is the fix points of F, i.e is a real form. Such a map F is often called a *real structure*.

Definition 2.6. Given a complex (holomorphic) manifold $M^{\mathbb{C}}$ with complex (holomorphic) Riemannian metric $g^{\mathbb{C}}$. If a submanifold $M \subset M^{\mathbb{C}}$ for any point $p \in M$ we have that T_pM is a real slice of $(T_pM^{\mathbb{C}}, g^{\mathbb{C}})$ (in the sense of Defn. 2.4), we will call M a real slice of $(M^{\mathbb{C}}, g^{\mathbb{C}})$.

This definition implies that the induced metric from $M^{\mathbb{C}}$ is real valued on M. M is therefore a pseudo-Riemannian manifold.

Definition 2.7 (Wick-related spaces)**.** Two pseudo-Riemannian manifolds M and $\tilde{M}$ are said to be *Wick-related* if there exists a holomorphic Riemannian manifold $(M^{\mathbb{C}}, g^{\mathbb{C}})$ such that M and $\tilde{M}$ are embedded as real slices of $M^{\mathbb{C}}$.

Definition 2.8 (Wick-rotation)**.** If two Wick-related spaces (of the same real dimension) intersect at a point p in $M^{\mathbb{C}}$, then we will use the term *Wick-rotation*: the manifold M can be Wick-rotated to the manifold $\tilde{M}$ (with respect to the point p).

We now define a Wick-rotation of a pseudo-Riemannian Lie group:

Definition 2.9 (Wick-rotation of a pseudo-Riemannian Lie group)**.** Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ be two immersive real forms which are Wick-related in $(G^{\mathbb{C}}, g^{\mathbb{C}})$ for $g^{\mathbb{C}}$ a left-invariant holomorphic metric. Then we shall say that the pseudo-Riemannian Lie group (G, g) is Wick-rotated to $(\tilde{G}, \tilde{g})$.

Thus from the definition: $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ is a real slice of Lie groups, and shall write (p, q) for the signature of g. If there is another real slice $(\tilde{G}, \tilde{g}) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ of Lie groups, then we shall refer to the signature of $\tilde{g}$ as $(\tilde{p}, \tilde{q})$. We shall often just say a Wick-rotations of Lie groups. Note that two Lie groups which are Wick-related are also Wick-rotated at the identity point $p := 1$.

The definition implies that two Wick-rotatable metrics on real Lie groups are left-invariant themselves, and also note that a Wick-rotation of Lie groups induces in the obvious way a Wick-rotation of the identity components. Moreover the property of bi-invariance for connected groups is an invariant:

Proposition 2.10. *Suppose (G, g) is Wick-rotatable to $(\tilde{G}, \tilde{g})$ and they are both connected. Then $g(-,-)$ is bi-invariant if and only if $\tilde{g}(-,-)$ is bi-invariant.*

Proof. The proofs given in ([24], Lemma 7.1 and 7.2) also hold for pseudo-Riemannian left-invariant metrics, with $\mathfrak{o}(n)$ replaced with $\mathfrak{o}(p,q)$. Moreover if the metric $g(-,-)$ is bi-invariant, then because $ad(\mathfrak{g}) \subset \mathfrak{o}(p,q) \subset \mathfrak{o}(n,\mathbb{C})$, and $ad(\mathfrak{g})^{\mathbb{C}} = ad(\mathfrak{g}^{\mathbb{C}})$ it follows that the holomorphic metric must also be bi-invariant, thus also $\tilde{g}(-,-)$. The converse is identical. □

Note that the property of being connected or simply connected are not necessarily preserved under a Wick-rotation. However under a Wick-rotation of real forms, then being connected is conserved.

Example 2.11. *Let $SL_2(\mathbb{R}) \subset SL_2(\mathbb{C}) \supset SU(2)$ be the natural inclusions. Then they are real forms, and Wick-rotated w.r.t to the holomorphic Killing form κ on $\mathfrak{sl}_2(\mathbb{C})$. Note that $(SL_2(\mathbb{R}), \kappa)$ is Lorentzian and $(SU(2), \kappa)$ has signature: $(-,-,-)$.*

We also define:

Definition 2.12. Let $V \subset (E, g^{\mathbb{C}})$ be a real slice. We say an involution $V \xrightarrow{\theta} V \in O(p,q)$, is a *Cartan involution* of $g := g^{\mathbb{C}}\big|_V$, if $g_\theta(\cdot,\cdot) := g^{\mathbb{C}}\big|_V(\cdot,\theta(\cdot))$, is an inner product on V. If $\theta = 1$ then V is said to be a *compact real slice*, or in the case that V is also a real form, then V shall be said to be a *compact real form*.

Note the resemblance (in the definition) with a compact real form of a complex semi-simple Lie algebra and its Killing form. In the case of Lie algebras: $(\mathfrak{g}, g) \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$, then a Cartan involution θ of g is not necessarily a homomorphism of Lie algebras, since we do not know it they exist. We do not even know if there exist a compact real form which is also a Lie subalgebra of $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$. But we know if $\mathfrak{g}$ is semi-simple, and $g^{\mathbb{C}} = -\kappa$, then there exist a Cartan involution θ which is also homomorphism of the Lie algebra.

But more generally we shall define:

Definition 2.13. Let $\mathfrak{g} \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a real form. A Cartan involution θ of $\mathfrak{g}$ is a Cartan involution of $g := g^{\mathbb{C}}_{|_{\mathfrak{g}}}(-,-)$ which is also a homomorphism of Lie algebras.

Thus a Cartan involution of g is only a linear Cartan involution of the pseudo-inner product g, but a Cartan involution of $\mathfrak{g}$ is a Cartan involution of g which is also a homomorphism of Lie algebras. Currently at this point we only know that Cartan involutions of $\mathfrak{g}$ exist when $\mathfrak{g}$ is abelian or $\mathfrak{g}$ is semi-simple equipped with the Killing form: $-\kappa$. One shall note that there are examples where they do not exist, indeed by changing the sign to: κ, then it is straight forward to show that there are no Cartan involutions of $\mathfrak{g}$.

Definition 2.14. Two real forms V and $\widetilde{V}$ of E are said to be compatible if their conjugation maps commute, i.e $[\sigma, \tilde{\sigma}] = 0$.

Often we shall refer to a pair $(V, \tilde{V})$ as a compatible pair, to mean that the spaces are compatible.

We recall from [20], that if $(E, g^{\mathbb{C}})$ is a holomorphic inner product space, and $V, \tilde{V}$ and W are real forms such that W is a compact real form (i.e of Euclidean signature), then if they are pairwise compatible, the triple: $\left(V, \tilde{V}, W\right)$, is said to be a *compatible triple*. Note that Example 2.11 is an example of a compatible triple:

$$\left(V := \mathfrak{sl}_2(\mathbb{R}), \tilde{V} := \mathfrak{su}(2), W := \mathfrak{su}(2)\right).$$

We shall call the eigenspace decomposition of a Cartan involution: θ, for the Cartan decomposition. By the uniqueness of a signature associated to a pseudo-inner product g then all Cartan involutions are conjugate in $O(p, q)$. In fact given two Cartan involutions: θ_j $(j = 1, 2)$ then $g \mapsto \theta_j g \theta_j$ is a global Cartan involution of $O(p, q)$. Thus if $g\theta_1 g^{-1} = \theta_2$ for some $g \in O(p, q)$, then writing $g = k_2 e^x$, where k_2 commutes with θ_2 and $x \in \mathfrak{o}(p, q)$, we obtain $\theta_1 = e^x \theta_2 e^{-x}$, and therefore θ_1, θ_2 are conjugate by an element $g \in O(p, q)_0$.

Suppose now we have a Wick-rotation of two real Lie groups: $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$. Let $\theta \in O(p, q)$ be a Cartan involution of the metric g, and let W denote the corresponding unique compact real form associated with θ, i.e $W := V_+ \oplus iV_-$, where $\mathfrak{g} = V_+ \oplus V_-$ is the Cartan decomposition. Then by [19] it is possible to find a real form $\tilde{V} \subset \mathfrak{g}^{\mathbb{C}}$ (as vector spaces) and a linear isomorphism: $\tilde{V} \xrightarrow{\phi} \tilde{\mathfrak{g}}$ such that $\phi^{\mathbb{C}} \in O(n, \mathbb{C})$, and $(\mathfrak{g}, \tilde{V}, W)$ is a compatible triple. So consider the triple: $\left(\mathfrak{o}(p, q), \mathfrak{o}(\tilde{p}, \tilde{q}), \mathfrak{o}(n)\right)$, of Lie algebras of the isometry groups associated with the compatible triple $(\mathfrak{g}, \tilde{V}, W)$.

Then the following straightforward result is important to note:

Lemma 2.15 ([20], Lemma 3.6)**.** *The triple of real forms:* $\left(\mathfrak{o}(p, q), \mathfrak{o}(\tilde{p}, \tilde{q}), \mathfrak{o}(n)\right)$, *embedded into* $\mathfrak{o}(n, \mathbb{C})$ *is a compatible triple of Lie algebras.*

Thus we note that up to an isometry $g \in O(n, \mathbb{C})$ we may assume our two Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ (viewed as a vector space) form a compatible triple with a compact real form $W \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$.

2.3. **Real GIT on compatible representations.** In this section we recall some definitions and results of [19] that we shall use.

Definition 2.16. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ be two real Lie subgroups of a complex Lie group such that the real Lie algebras are real forms of $\mathfrak{g}^{\mathbb{C}}$. Then we say G and $\tilde{G}$ are *compatible* if the Lie algebras are compatible.

Definition 2.17. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ and $U \subset G^{\mathbb{C}}$ be real Lie subgroups of a complex Lie group such that the real Lie algebras are real forms of $\mathfrak{g}^{\mathbb{C}}$. Moreover assume U is compact. Then we say $\left(G, \tilde{G}, U\right)$ is a *compatible triple* if the Lie algebras are pairwise compatible.

If we use Lemma 2.15, in the context of Wick-rotations (see the previous section), then the triple of isometry groups: $\left(O(p,q), O(\tilde{p},\tilde{q}), O(n)\right)$ form a compatible triple when the pseudo-inner product spaces they are isometries of, form a compatible triple.

From now on when considering a real form: $G \subset G^{\mathbb{C}}$, then $G^{\mathbb{C}}$ shall be of type linearly complex reductive, and G should either be linearly real reductive, or in the case where $G^{\mathbb{C}} \subset GL(V^{\mathbb{C}})$ is defined over $\mathbb{R}$, the real points: $G := GL(V) \cap G^{\mathbb{C}}$. This is the assumptions in the paper [19]. Thus we may for instance use the pseudo-orthogonal group $O(p,q) \subset O(n,\mathbb{C})$ defined as the isometry group of some pseudo-inner product space: $(V,g) \subset (V^{\mathbb{C}}, g^{\mathbb{C}})$. A compact real form of $G^{\mathbb{C}}$ shall always be denoted by U.

Definition 2.18 ([7])**.** Let $G \xrightarrow{\rho_V^G} GL(V)$ be a real representation, then ρ_V^G is said to be *balanced representation* if there exist an involution $V \xrightarrow{\theta} V$, and a global Cartan involution: $G \xrightarrow{\Theta} G$ such that:

$$\left(\forall g \in G\right)\left(\rho_V^G(\Theta(g)) = \theta \circ \rho_V^G(g) \circ \theta\right).$$

Thus if we have an involution θ of V balancing our action, then w.r.t the global Cartan involution Θ of G with Cartan decomposition: $G = Ke^{\mathfrak{p}}$, there exist a pseudo-inner product $g(-,-)$ on V such that θ is a Cartan involution of $g(-,-)$, and the inner product $g_\theta(-,-) := g(-,\theta(-))$ is K-invariant. Let $\mathcal{M}(G,V)$ denote the minimal vectors of our action, i.e those $v \in V$ satisfying: $||g \cdot v||^2 \geq ||v||$ for all $g \in G$, where $||v||^2 := g_\theta(v,v)$. Then if $V = V_+ \oplus V_-$ is the Cartan decomposition, we naturally have $V_+ \cup V_- \subset \mathcal{M}(G,V)$. The Cartan involutions of $g(-,-)$ which are conjugate by the action of G to θ are defined as the *inner Cartan involutions* of $g(-,-)$.

A complex action: $\rho^{\mathbb{C}}$ of $G^{\mathbb{C}}$ acting on $V^{\mathbb{C}}$ is said to be a complexified action of a real action ρ_V^G if $\rho^{\mathbb{C}}(g)(v) = \rho(g)(v)$ for all $g \in G$ and $v \in V$.

Definition 2.19. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ be real forms, and $G \xrightarrow{\rho_V^G} GL(V)$ and $\tilde{G} \xrightarrow{\rho_{\tilde{V}}^{\tilde{G}}} GL(\tilde{V})$ be real representations of Lie groups. Suppose $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$

is a complexified action of both ρ_V^G and $\rho_{\tilde{V}}^{\tilde{G}}$. Then we say that ρ_V^G is *compatible* with $\rho_{\tilde{V}}^{\tilde{G}}$, if the following two criterions are fulfilled:

(1) G and $\tilde{G}$ are compatible real forms of $G^{\mathbb{C}}$.
(2) V and $\tilde{V}$ are compatible real forms of $V^{\mathbb{C}}$.

Definition 2.20. Let $\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}$ and ρ_W^U be pairwise compatible representations, where $U \subset G^{\mathbb{C}}$, is a compact real form. Then the triple: $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$ is said to be a *compatible triple*.

If we have such a compatible triple, then all the real actions in the triple are balanced, and we can choose pseudo-inner products $g(-,-)$ and $\tilde{g}(-,-)$ on V and $\tilde{V}$ respectively, in such a way that they restrict from the same Hermitian form on $V^{\mathbb{C}}$. Moreover if τ denotes the conjugation map of W in $V^{\mathbb{C}}$ then it restricts to Cartan involutions: θ (of g) and $\tilde{\theta}$ (of $\tilde{g}$). The Cartan involutions also balance the real actions respectively. In particular the inner products g_θ and $\tilde{g}_{\tilde{\theta}}$ both restrict from the U-invariant Hermitian inner product $H(-,\tau(-))$. The minimal vectors satisfy:

$$\mathcal{M}(G,V) \subset \mathcal{M}(G^{\mathbb{C}},V^{\mathbb{C}}) \supset \mathcal{M}(\tilde{G},\tilde{V}), \quad W \subset \mathcal{M}(G^{\mathbb{C}},V^{\mathbb{C}}).$$

Denote the Cartan decompositions by $V = V_+ \oplus V_-$ and $\tilde{V} = \tilde{V}_+ \oplus \tilde{V}_-$ respectively.

Definition 2.21. Let $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}\right)$ be a compatible pair. Suppose $v \in V$ and $\tilde{v} \in \tilde{V}$ are such that $\tilde{v} \in G^{\mathbb{C}}v$, then we shall say that Gv is *compatible* with $\tilde{G}\tilde{v}$. We write $Gv \sim \tilde{G}\tilde{v}$.

It is important to note the following result:

Theorem 2.22 ([19])**.** *Let $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$ be a compatible triple. Suppose $v \in V$ and $\tilde{v} \in \tilde{V}$ are such that: $\tilde{G}\tilde{v} \sim Gv$. Then $Gv \cap V_+ \neq \emptyset$ (respectively $Gv \cap V_- \neq \emptyset$) if and only if $\tilde{G}\tilde{v} \cap \tilde{V}_+ \neq \emptyset$ (respectively $\tilde{G}\tilde{v} \cap \tilde{V}_- \neq \emptyset$).*

Observe that if there exist $v_+ \in Gv$, then $\theta(v_+) = v_+$, i.e if $g \in G$ is such that $g \cdot v = v_+$, then there is an inner Cartan involution θ' of $g(-,-)$ such that $\theta'(v) = v$ using g.

We shall also state the following important result:

Theorem 2.23 ([19])**.** *Let (ρ_V^G, ρ_W^U) be a compatible pair. Let $v \in V$, then the following statements are equivalent:*

A There exist $w \in W$ such that $Uw \sim Gv$.
B There exist an inner Cartan involution $V \xrightarrow{\theta} V$ such that $\theta(v) = v$.
C There exist $w \in W$ such that $Uw \cap Gv \neq \emptyset$.

In fact if there is a $w \in W$ and $v \in V$ such that $Uw \sim Gv$ then:

$$\emptyset \neq Uw \cap Gv = Gv \cap \mathcal{M}(G,V) = Kv,$$

where $K = U \cap G$.

A worked out example of compatible representations is given in the next section in the context of Wick-rotations of Lie groups.

2.4. **The isometry action on bilinear forms into the Lie algebra.** In this section we shall consider the action that we are going to use to prove our main result of this paper. We shall explain in detail that under a Wick-rotation, the isometry groups of the pseudo-inner product spaces induces compatible representations (see Defn. Section 2.3).

Suppose we have a Wick-rotation of pseudo-Riemannian Lie groups: $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$. As we have seen we can choose a map $g \in O(n, \mathbb{C})$ such that we obtain a compatible triple: $(\mathfrak{g}, \tilde{V}, W)$, with $\tilde{V} := g(\tilde{\mathfrak{g}})$. We shall denote $\tilde{g}$ also for the pseudo-inner product on $\tilde{V}$ restricted from $g^{\mathbb{C}}$. We can choose a pseudo-orthonormal basis: $\{e_1, \ldots, e_p, \ldots, e_n\}$ (of g) and similarly $\{\tilde{e}_1, \ldots, \tilde{e}_{\tilde{p}}, \ldots, \tilde{e}_n\}$ (of $\tilde{g}$), such that W is the real span of both the sets: $Y := \{e_1, \ldots, e_p, ie_{p+1} \ldots ie_n\}$ and $\tilde{Y} := \{\tilde{e}_1, \ldots, \tilde{e}_{\tilde{p}}, i\tilde{e}_{\tilde{p}+1}, \ldots, i\tilde{e}_n\}$. Denote the corresponding Cartan involutions by θ (of g) and $\tilde{\theta}$ (of $\tilde{g}$). Note that Y and $\tilde{Y}$ are both an orthonormal basis of $g^{\mathbb{C}}$.

Consider the complex isometry action of $O(n, \mathbb{C})$ on $\mathfrak{g}^{\mathbb{C}}$ by $g \cdot x := g(x)$. This action restricts to the real isometry actions of $O(p, q)$ on $\mathfrak{g}$ and $O(\tilde{p}, \tilde{q})$ on V respectively. Let $\mathcal{V}$ and $\tilde{\mathcal{V}}$ denote the real vector spaces of bilinear forms: $\mathfrak{g}^2 \to \mathfrak{g}$ (respectively $\tilde{V}^2 \to \tilde{V}$). Thus $\mathcal{V} \subset \mathcal{V}^{\mathbb{C}} \supset \tilde{\mathcal{V}}$ are real forms, where $\mathcal{V}^{\mathbb{C}}$ is the complex vector space of complex bilinear forms: $(\mathfrak{g}^{\mathbb{C}})^2 \to \mathfrak{g}^{\mathbb{C}}$. The complex isometry action naturally extends to a complex action of $O(n, \mathbb{C})$ on $b \in \mathcal{V}^{\mathbb{C}}$, by

$$(g \cdot b)(x, y) := g\Big(b(g^{-1}(x), g^{-1}(y))\Big), \ x, y \in \mathfrak{g}^{\mathbb{C}}, \ g \in O(n, \mathbb{C}).$$

Note that the action again restricts to action of the real isometry groups on $\mathcal{V}$ and $\tilde{\mathcal{V}}$ respectively. Denote the real actions by ρ and $\tilde{\rho}$ respectively. The Cartan involution $\theta \in O(p, q)$ (respectively $\tilde{\theta} \in O(\tilde{p}, \tilde{q})$) naturally extends to an involution of $\mathcal{V}$ (respectively $\tilde{\mathcal{V}}$), by the action: $\rho(\theta)$ (respectively $\tilde{\rho}(\tilde{\theta})$). The holomorphic inner product $g^{\mathbb{C}}$ extends naturally to a holomorphic inner product: $\mathbf{g}^{\mathbb{C}}$, by defining:

$$\mathbf{g}^{\mathbb{C}}(b_1, b_2) := \sum_{j}^{n} g^{\mathbb{C}}\Big(b_1(y_j, y_j), b_2(y_j, y_j)\Big).$$

Observe that if we change basis w.r.t to $\tilde{Y}$ instead then we obtain the same holomorphic inner product. Indeed this follows since we can find $g \in O(n, \mathbb{C})$ sending $Y \mapsto \tilde{Y}$. It is easy to check that $\mathcal{V} \subset (\mathcal{V}^{\mathbb{C}}, \mathbf{g}^{\mathbb{C}}) \supset \tilde{\mathcal{V}}$ are real slices. Similarly if we define $\mathcal{W}$ to be all bilinear forms: $W^2 \to W$, then by construction $\mathcal{W}$ is a compact real form of $(\mathcal{V}^{\mathbb{C}}, \mathbf{g}^{\mathbb{C}})$. Observe that the three real forms form a compatible triple in $\mathcal{V}^{\mathbb{C}}$. Therefore the actions form a compatible triple (see Section 2.3). There is a natural choice of $O(n)$-invariant Hermitian inner product on $\mathcal{V}^{\mathbb{C}}$, namely: $H := \mathbf{g}^{\mathbb{C}}(\cdot, \mathcal{T}(\cdot))$, where $\mathcal{T}$ is the conjugation map of $\mathcal{W}$. This Hermitian inner product restricts to inner products on $\mathcal{V}, \tilde{\mathcal{V}}$ and $\mathcal{W}$. Observe that the inner Cartan involutions of ρ (respectively $\tilde{\rho}$) are those conjugate to $\rho(\theta)$ (respectively $\tilde{\rho}(\tilde{\theta})$).

2.5. **Wick-rotatable tensors of pseudo-Riemannian manifolds.** For a Wick-rotation of Lie groups it is worth noting that the action in the previous section is just an example of a tensor action of $O(n,\mathbb{C})$ on a general tensor space of finite form:

$$\mathcal{V}^{\mathbb{C}} := \bigoplus_{k,m} \Big(\Big(\bigotimes_{i=1}^{k} \mathfrak{g}^{\mathbb{C}} \Big) \bigotimes \Big(\bigotimes_{i=1}^{m} (\mathfrak{g}^{\mathbb{C}})^* \Big) \Big),$$

induced from the isometry action of the holomorphic metric $g^{\mathbb{C}}$. Analogously we define:

$$\mathcal{V} := \bigoplus_{k,m} \Big(\Big(\bigotimes_{i=1}^{k} \mathfrak{g} \Big) \bigotimes \Big(\bigotimes_{i=1}^{m} \mathfrak{g}^* \Big) \Big), \quad \tilde{\mathcal{V}} := \bigoplus_{k,m} \Big(\Big(\bigotimes_{i=1}^{k} \tilde{\mathfrak{g}} \Big) \bigotimes \Big(\bigotimes_{i=1}^{m} \tilde{\mathfrak{g}}^* \Big) \Big).$$

The real isometry groups: $O(p,q)$ (respectively $O(\tilde{p},\tilde{q})$) restrict to acting on $\mathcal{V}$ (respectively $\tilde{\mathcal{V}}$).

More generally for a Wick-rotation of pseudo-Riemannian manifolds:

$$(M,g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g}),$$

at a common point $p \in M \cap \tilde{M}$, then by replacing $\mathfrak{g}$ with T_pM (respectively $\tilde{\mathfrak{g}}$ with $T_p\tilde{M}$), and $\mathfrak{g}^{\mathbb{C}}$ with $T_pM^{\mathbb{C}}$, we obtain the induced tensor action on real forms: $\mathcal{V} \subset \mathcal{V}^{\mathbb{C}} \supset \tilde{\mathcal{V}}$.

One shall note that the metrics, Cartan involutions all extend naturally to these spaces via the tangent spaces. Moreover if $g \in O(n,\mathbb{C})$ is such that T_pM and $g(T_p\tilde{M})$ form a compatible triple with a compact real form $W \subset T_pM^{\mathbb{C}}$, then naturally also $\mathcal{V}$ and $g \cdot \tilde{\mathcal{V}}$ form a compatible triple with

$$\mathcal{W} := \bigoplus_{k,m} \Big(\Big(\bigotimes_{i=1}^{k} W \Big) \bigotimes \Big(\bigotimes_{i=1}^{m} W^* \Big) \Big).$$

For example the induced action of $O(n,\mathbb{C})$ on $End(T_pM^{\mathbb{C}})$ given by conjugation: $g \cdot f := gfg^{-1}$ is just the tensor action: $g \cdot (v_1 \otimes v_2) := g(v_1) \otimes g(v_2)$, for an $O(n,\mathbb{C})$-module isomorphism: $End(T_pM^{\mathbb{C}}) \cong T_pM^{\mathbb{C}} \otimes T_pM^{\mathbb{C}}$. For a more detailed explanation of this example, and on the tensor action in general we refer to [19].

Consider the action in the previous section for instance, then one should observe that the complex Lie bracket $v := [-,-]$ of $\mathfrak{g}^{\mathbb{C}}$ is a vector in $\mathcal{V}$, but also there is a $g \in O(n,\mathbb{C})$ such that $\tilde{v} := g \cdot v \in g \cdot \tilde{\mathcal{V}}$, i.e v and $\tilde{v}$ lie in the same complex orbit: $O(n,\mathbb{C})v \ni \tilde{v}$, in such a way that $O(p,q)v \sim O(\tilde{p},\tilde{q})\tilde{v}$ are compatible real orbits.

Thus it useful to define for general tensors $v \in \mathcal{V}$ and $\tilde{v} \in \tilde{\mathcal{V}}$:

Definition 2.24 ([19])**.** Let (M, g) and $(\tilde{M}, \tilde{g})$ be two Wick-rotatable pseudo-Riemannian manifolds at a common point p. Then two tensors $v \in \mathcal{V}$ and $\tilde{v} \in \tilde{\mathcal{V}}$ are said to be *Wick-rotatable* at p, if they lie in the same $O(n, \mathbb{C})$-orbit, i.e

$$O(n, \mathbb{C})v = O(n, \mathbb{C})\tilde{v}.$$

One should note the subset of Wick-rotatable tensors consisting of those in the intersection: $v \in \mathcal{V} \cap \tilde{\mathcal{V}}$. Then there is a map $g \in O(n, \mathbb{C})$, such that v and $g \cdot v \in g \cdot \tilde{\mathcal{V}}$ lie in the same complex orbit such that $O(p, q)v \sim O(\tilde{p}, \tilde{q})\tilde{v}$ are compatible real orbits. More generally if v and $\tilde{v}$ are Wick-rotatable i.e by definition $O(n, \mathbb{C})v = O(n, \mathbb{C})\tilde{v}$, then also $O(n, \mathbb{C})v = O(n, \mathbb{C})g \cdot \tilde{v}$. The main point is to be able to embed the vectors into the same complex orbit, such that we may apply the results of Section 2.3.

Let (M, g) be a pseudo-Riemannian manifold of signature (p, q), and $\theta \in O(p, q)$ be a Cartan involution of $g(-, -)$. Consider the isometry tensor action of $O(p, q)$ on $\mathcal{V}$ as above:

$$O(p, q) \xrightarrow{\rho_{\mathcal{V}}^{O(p,q)}} GL(\mathcal{V}).$$

Then θ naturally extends to an involution $\Theta := \rho_{\mathcal{V}}^{O(p,q)}(\theta)$ on $\mathcal{V}$, and the metric naturally induces a pseudo-inner product: $\mathbf{g}(-, -)$ on $\mathcal{V}$ such that Θ is a Cartan involution. Let now $R \in \mathcal{V}$ be the Riemann tensor of M at p for $\mathcal{V}$ some tensor space. For example R may be considered as a multilinear form into T_pM: $T_pM^3 \to T_pM$, where the action is given by:

$$(g \cdot R)(x, y, z) := g\Big(R(g^{-1}(x), g^{-1}(y), g^{-1}(z))\Big), \ x, y, z \in T_pM, \ g \in O(p, q).$$

Another approach is to consider R as a map in $End(\mathfrak{o}(p, q)) \subset End\Big(End(T_pM)\Big)$ at the point p, where the action is given by:

$$(g \cdot R)(X) := gR(g^{-1}Xg)g^{-1}, \ X \in \mathfrak{o}(p, q), \ g \in O(p, q).$$

The Riemann tensor at p is viewed in this way for instance in [21]. One may show that these two actions are equivalent up to an $O(p, q)$-module isomorphism, by identifying the spaces with the tensor space: $T_pM \otimes T_pM \otimes T_pM \otimes T_pM$.

We also recall the following definition:

Definition 2.25. If there exist a Cartan involution Θ such that $\Theta(R) = R$ (respectively $\Theta(R) = -R$), then the space (M, g) at p is called *Riemann purely electric* (RPE) (respectively *Riemann purely magnetic* (RPM)). If there is such a Θ for the Weyl tensor at p, then (M, g) at p is called *purely electric* (PE) (respectively *purely magnetic* (PM)).

Any Riemannian space (M, g) is RPE at any point $p \in M$, since the identity map $\theta := 1_{T_pM}$ is a Cartan involution of the metric g at any point, thus the Cartan involution extended to tensors: $\mathcal{V}$ is also the identity map, i.e $\Theta(R) = R$.

The Levi-Civita connection ∇ of a real slice of a holomorphic Riemannian manifold $(M,g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}})$ at $p \in M$, restricts from the complex Levi-Civita connection: $\nabla^{\mathbb{C}}$ at p of the complex manifold $M^{\mathbb{C}}$. Thus the real Riemann curvature tensor R (of M) at p restricts from the complex Riemann curvature tensor $R^{\mathbb{C}}$ of $M^{\mathbb{C}}$ (at p). Moreover if ric_g denotes the real Ricci curvature: $T_pM^2 \to \mathbb{R}$, defined by:

$$ric_g(x,y) := Tr\Big(z \mapsto R(z,y)(x)\Big),$$

then using a real basis of T_pM also for $T_pM^{\mathbb{C}}$ we see that restricting the complex Ricci curvature: $ric_{g^{\mathbb{C}}}$ on $M^{\mathbb{C}}$ to T_pM we get ric_g. Similarly the real Ricci operator:

$$Ric_g \in End(T_pM), \quad g_p(Ric_g(x), y) = ric_g(x,y),$$

restricts form the complex Ricci curvature operator of $M^{\mathbb{C}}$ (at p).

This means that in terms of Wick-rotations of pseudo-Riemannian manifolds at a common point p: $(M,g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g})$, we see that the pairs of tensors:

$$(\nabla, \tilde{\nabla}),\ (R, \tilde{R}),\ (ric_g, ric_{\tilde{g}}),\ (Ric_g, Ric_{\tilde{g}}),$$

are examples of Wick-rotatable tensors (at p) in the intersection $\mathcal{V} \cap \tilde{\mathcal{V}}$. The induced isometry action of $O(n,\mathbb{C})$ on these tensors (induced from the isometry action of the metric) can be naturally seen as the actions:

$$(g\cdot\nabla)(x,y) := g(\nabla_{g^{-1}x} g^{-1}y), \quad (g\cdot R)(x,y,z) := g\Big(R(g^{-1}x, g^{-1}y, g^{-1}z)\Big)$$

and

$$(g\cdot ric_g)(x,y) := ric_g(gx, gy), \quad (g\cdot Ric_g)(x) := (g\circ Ric_g \circ g^{-1})(x).$$

An immediate result is the following:

Theorem 2.26. *Let $(M,g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g})$ be a Wick-rotation at a common point $p \in M \cap \tilde{M}$. Assume $(\tilde{M}, \tilde{g})$ is a Riemannian space. Then the following statements hold:*

(1) There exist a Cartan involution θ of g such that $\nabla_{\theta(x)}\theta(y) = \theta(\nabla_x y)$ for all $x, y \in T_pM$.

(2) There exist a Cartan involution θ of g such that $ric_g\Big(\theta(x), \theta(y)\Big) = ric_g(x,y)$ for all $x, y \in T_pM$.

(3) There exist a Cartan involution θ of g such that $[\theta, Ric_g] = 0$.

(4) There exist a Cartan involution θ of g such that $R\Big(\theta(x), \theta(y)\Big)\Big(\theta(z)\Big) = \theta\Big(R(x,y)(z)\Big)$ for all $x, y, z \in T_pM$. Thus (M,g) is (RPE) at p.

Proof. It is enough to spell out the proof for the first case, as the other cases are identical. Let $v := \nabla \in \mathcal{V}$ and $\tilde{v} := \tilde{\nabla} \in \tilde{\mathcal{V}}$, and consider the isometry tensor action as above. The vectors v and $\tilde{v}$ are Wick-rotatable, thus up to a map $g \in O(n,\mathbb{C})$ we can assume the real actions are compatible, and that v and $\tilde{v}$ lie in the same complex orbit, such that the real orbits: $O(p,q)v \sim O(\tilde{p},\tilde{q})$ are compatible. The result now follows from Theorem 2.23, since $O(\tilde{p},\tilde{q}) = O(n)$ is a compact real form of $O(n,\mathbb{C})$. □

One shall note that Case (4) of the theorem is proved in [21]. We shall strengthen Theorem 2.26 for Wick-rotations of pseudo-Riemannian Lie groups in the last section of the paper, by proving that a Cartan involution of g may be chosen to be a homomorphism of Lie algebras.

3. An invariant of Wick-rotation of Lie groups

In this section we shall prove the main theorem of the paper, which is an invariance result based on the existence of a Cartan involution of the Lie algebras (Defn. 2.13).

Let $(G,g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G},\tilde{g})$ be a Wick-rotation of Lie groups. Consider the action in Section 2.4 and following the notation there, then by our preparations, the main result is now easy deducible:

Theorem 3.1. *Suppose (G,g) is a pseudo-Riemannian Lie group that can be Wick-rotated to another Lie group $(\tilde{G},\tilde{g})$. Then there exist a Cartan involution of $\mathfrak{g}$ if and only if there exist a Cartan involution of $\tilde{\mathfrak{g}}$.*

Proof. Consider the group action and the notation as in Section 2.4. Thus if $v := [-,-]$ is the Lie bracket of $\mathfrak{g}^{\mathbb{C}}$ then $v \in \mathcal{V}$ and restricts to the Lie bracket of $\tilde{\mathfrak{g}}$. We can choose $g \in O(n,\mathbb{C})$ such that $g \cdot v \in \tilde{\mathcal{V}}$, i.e v and $\tilde{v} := g \cdot v$ lie in the same complex orbit, thus $O(p,q)v \sim O(\tilde{p},\tilde{q})\tilde{v}$ are compatible real orbits. Suppose θ is a Cartan involution of $\mathfrak{g}$, and denote $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$ (respectively $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_+ \oplus \tilde{\mathcal{V}}_-$) for the Cartan decomposition w.r.t to $\rho(\theta)$ (respectively $\tilde{\rho}(\tilde{\theta})$). Then the action of θ on v fixes v, i.e $\rho(\theta)(v) := \theta \cdot v = v$, thus $v \in \mathcal{V}_+$. Hence the real orbit: $O(p,q)v$ intersects $\mathcal{V}_+$. But then by Theorem 2.22, it follows that there exist also $\tilde{v}' \in \tilde{\mathcal{V}}_+ \cap O(\tilde{p},\tilde{q})\tilde{v}$. Therefore choose $h \in O(\tilde{p},\tilde{q})$ such that $h \cdot \tilde{v} = \tilde{v}'$. By conjugating $\tilde{\rho}(\tilde{\theta})$ by h we obtain a Cartan involution $\tilde{\theta}'$ of $\tilde{g}$ such that $\tilde{\theta}' \cdot \tilde{v} = \tilde{v}$. Finally since $\tilde{V} := g(\tilde{\mathfrak{g}})$ for some $g \in O(n,\mathbb{C})$ then the Cartan involution $g^{-1}\tilde{\theta}' g$ fixes v, i.e is a Cartan involution of $\tilde{g}$ and a homomorphism of Lie algebras. The converse is symmetric. The theorem is proved. □

We find it useful to define for future exploration:

Definition 3.2. A property of a pseudo-Riemannian Lie group (G,g) is said to be *Wick-rotatable* if it is an invariant under a Wick rotation of Lie groups.

Corollary 3.3. *The existence of a Cartan involution of $\mathfrak{g}$ is Wick-rotatable.*

Other Wick-rotatable properties include: being semi-simple, abelian, nilpotent, solvable, reductive. Note that being simple, is not Wick-rotatable, indeed as an example consider the Lie group $O(1,3)$ with the left-invariant metric being the

Killing form. Then $\mathfrak{o}(1,3)$ is simple, but we may Wick-rotate $O(1,3)$ to $O(2,2)$ which is semi-simple but not simple, as $\mathfrak{o}(2,2) \cong \mathfrak{sl}_2(\mathbb{R})^2$.

We can now answer the question for when an arbitrary left-invariant metric can be Wick-rotated to a Riemannian left-invariant metric. One should compare the result with semi-simple Lie groups equipped with the left-invariant Killing form: $g := -\kappa$.

Corollary 3.4. *Suppose $(G,g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ is a real slice of Lie groups. Then (G,g) can be Wick-rotated to a Riemannian Lie group $(\tilde{G}, \tilde{g})$ if and only if there exist a Cartan involution of $\mathfrak{g}$.*

Proof. $(\Rightarrow)$. The identity map $\tilde{\mathfrak{g}} \xrightarrow{1} \tilde{\mathfrak{g}}$ is a Cartan involution of $\tilde{\mathfrak{g}}$. Thus by Theorem 3.1 the direction follows. Conversely suppose θ is a Cartan involution of $\mathfrak{g}$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, for the Cartan decomposition. Then is is not difficult to show that $\tilde{\mathfrak{g}} := \mathfrak{k} \oplus i\mathfrak{p}$ is a Lie algebra and is a real form of $\mathfrak{g}^{\mathbb{C}}$. Moreover the complex metric $g^{\mathbb{C}}(-,-)$ restricts to an inner product on $\tilde{\mathfrak{g}}$ by construction. Thus if we let $\tilde{G}$ be the unique connected Lie subgroup of $G^{\mathbb{C}}$ (the real Lie group) with Lie algebra $\tilde{\mathfrak{g}}$, then the corollary follows. $\square$

In view of Remark 1.1 with the signature change $g \mapsto -g$, if (G,g) can be Wick-rotated to a signature $(-,-,\cdots,-)$, then $(G,-g)$ can be Wick-rotated to a Riemannian space, thus there would exist a Cartan involution of $\mathfrak{g}$ w.r.t $-g$. We note in the Corollary that w.r.t the existing Cartan involution, then the Wick-rotated Riemannian Lie group may be chosen to be a virtual real form. Moreover note that since a Wick-rotation is a local condition then on Lie algebra level we have proved:

Corollary 3.5. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, where $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra. Let $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ be a real form which is a real slice. Assume there exist a compact real form $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ which is also a real Lie subalgebra. Let σ be the conjugation map of $\mathfrak{g}$. Then there exist an automorphism $\phi \in Aut(\mathfrak{g}^{\mathbb{C}}) \cap O(n,\mathbb{C})$ such that: $\sigma(\phi(\mathfrak{u})) \subset \phi(\mathfrak{u})$.*

Note in the corollary that if τ denotes the conjugation map of the compact real form $\phi(\mathfrak{u}) \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$, then the map $\theta^{\mathbb{C}} := \sigma\tau$ restricts to a Cartan involution θ of $\mathfrak{g}$.

Thus we have proved a general version of É. Cartan's result: ([16], Thm 7.1). Note also that the proof given there for the semi-simple case w.r.t to the Killing form is not valid for a general pair: $(\mathfrak{g}, g)$ as above, indeed following the notation of the proof, it is not obvious that $N := \sigma\tau \in O(n,\mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$.

One shall note that it may be the case that a pseudo-Riemannian Lie group (G,g) can be Wick-rotated to more than one Riemannian Lie group, in such a case we have the following (again one should compare this to semi-simple compact real forms w.r.t $-\kappa$):

Proposition 3.6. *Suppose there exist two Riemannian Wick-rotatable Lie groups: (G, g) and $(\tilde{G}, \tilde{g})$. Then (G_0, g) and $(\tilde{G}_0, \tilde{g})$ are locally isometric Lie groups. In particular if moreover G and $\tilde{G}$ are both simply connected then G and $\tilde{G}$ are isometric Lie groups.*

Proof. Choose a map $g \in O(n, \mathbb{C})$ mapping $\mathfrak{g} \mapsto \tilde{\mathfrak{g}}$. Consider the action and notation of Section 2.4. Using the map g the Lie bracket $v := [-,-]$ of $\mathfrak{g}^{\mathbb{C}}$ lies in $\mathcal{V}$, and also $\tilde{v} := g^{-1} \cdot v \in \mathcal{V}$. Thus $O(n, \mathbb{C})v \ni \tilde{v}$. But since $O(n)$ (the isometries of $(\mathfrak{g}, g)$) is compact, then we may choose $h \in O(n)$ such that the vectors v and $\tilde{v}$ lie in the same $O(n)$-orbit, i.e $h \cdot v = \tilde{v}$. Or in other words:

$$gh \cdot v = v.$$

Now since h maps $\mathfrak{g}$ to $\mathfrak{g}$ by definition and gh fixes v, i.e fixes the complex Lie bracket. Then $gh \in O(n, \mathbb{C})$ is an automorphism of complex Lie algebras, and it maps $\mathfrak{g} \mapsto \tilde{\mathfrak{g}}$. Therefore since the metrics are left-invariant we can conclude that (G_0, g) and $(\tilde{G}_0, \tilde{g})$ are locally isometric Lie groups as required. Finally if G and $\tilde{G}$ are both simply connected then since any local isometry is also an isometry, it follows that $(G, g) \cong (\tilde{G}, \tilde{g})$ are isometric Lie groups. The proposition is proved. □

Thus as a corollary for compact real forms:

Corollary 3.7. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, where $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra. Let $\mathfrak{u}_1 \subset \mathfrak{g}^{\mathbb{C}} \supset \mathfrak{u}_2$ be two real Lie subalgebras which are compact real forms. Then there exist a linear isomorphism: $\mathfrak{u}_1 \xrightarrow{\phi} \mathfrak{u}_2$, such that $\phi \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$.*

In the case of a complex semi-simple Lie group: $(G^{\mathbb{C}}, -\kappa)$, equipped with the left-invariant Killing form, then any compact real form: $\mathfrak{u} \subset \mathfrak{g}$, gives rise to a real form: $U \subset G^{\mathbb{C}}$ (thus is by definition a Riemannian real slice of Lie groups). It follows by the theory of semi-simple Lie groups that any two compact real forms of $G^{\mathbb{C}}$ are isomorphic Lie groups, and thus also isometric Lie groups.

Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice of Lie groups. Recall again the action of Section 2.4, and consider the Lie bracket $[-,-]$ of $\mathfrak{g}^{\mathbb{C}}$. Thus $[-,-] \in \mathcal{V}$ (the bilinear forms $\mathfrak{g}^2 \to \mathfrak{g}$.) Suppose as usual that the signature of g is (p, q). Then from real GIT there are a finite number of real $O(p, q)$-orbits in the complex orbit: $O(n, \mathbb{C}) \cdot [-,-]$, i.e

$$O(n, \mathbb{C}) \cdot [-,-] \cap \mathcal{V} = \cup_{i=1}^{m} O(p, q) v_i,$$

for some $m \geq 1$. We shall put an equivalence relation on the real slices of Lie groups of $(G^{\mathbb{C}}, g^{\mathbb{C}})$ by the relation of local isometry. Let $[(G, g)]$ denote an equivalence class.

We can thus generalise Proposition 3.6 in the following sense:

Theorem 3.8. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice of Lie groups, and (p, q) be the signature of g. Let $O(n, \mathbb{C}) \cdot [-, -] \cap \mathcal{V} = \cup_{i=1}^{m} O(p, q)v_i$. Then there are exactly m equivalence classes (up to a local isometry) of real slices of Lie groups in $(G^{\mathbb{C}}, g^{\mathbb{C}})$ with signature (p, q). In particular a Wick-rotation of two Lie groups (of the same signature) are locally isometric if and only if $m = 1$.*

Proof. Suppose $(\tilde{G}, \tilde{g})$ is Wick-rotated to (G, g) of the same signature. Let $h \in O(n, \mathbb{C})$ be such that $h(\mathfrak{g}) = \tilde{\mathfrak{g}}$, then $\tilde{v} := h^{-1} \cdot [-, -] \in \mathcal{V}$ is in the same complex orbit as $[-, -]$. Thus we have a mapping of an equivalence class:

$$[(\tilde{G}, \tilde{g})] \mapsto O(p, q)\tilde{v}.$$

The map does not depend on the choice of h, since if $h_1 \in O(n, \mathbb{C})$ also maps $h_1(\mathfrak{g}) = \tilde{\mathfrak{g}}$, then $h^{-1}h_1 \in O(p, q)$, and $h^{-1}h_1 \cdot \tilde{v} = h^{-1} \cdot [-, -]$. The map is well-defined. Indeed let (G_1, g_1) map to $O(p, q)v_1 := O(p, q) \cdot (h_1^{-1} \cdot [-, -])$ for some $h_1 \in O(n, \mathbb{C})$ with $h_1(\mathfrak{g}) = \mathfrak{g}_1$. Assume (G_1, g_1) is locally isometric to $(\tilde{G}, \tilde{g})$. Then there exist $g \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$ such that $g(\mathfrak{g}_1) = \tilde{\mathfrak{g}}$, therefore:

$$g_1 := h^{-1}gh_1 \in O(p, q), \quad g_1 \cdot v_1 = h^{-1}g \cdot [-, -] = h^{-1} \cdot [-, -] = \tilde{v},$$

using that g fixes the Lie bracket.

To see that the map is injective, then suppose $[(G_j, g_j)]$ maps to the same orbit for $j = 1, 2$. Then by definition: $[(G_j, g_j)] \mapsto O(p, q) \cdot (h_j^{-1} \cdot [-, -])$ for maps $h_j \in O(n, \mathbb{C})$ with $h_j(\mathfrak{g}) = \mathfrak{g}_j$. Thus since the orbits are the same, then choose $g \in O(p, q)$ such that $g \cdot (h_1^{-1} \cdot [-, -]) = h_2^{-1} \cdot [-, -]$, i.e $h_2gh_1^{-1} \cdot [-, -] = [-, -]$ so that $h_2gh_1^{-1} \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$. Note that $h_2gh_1^{-1}$ maps $\mathfrak{g}_1 \mapsto \mathfrak{g}_2$. It follows that $[(G_1, g_1)] = [(G_2, g_2)]$ as required.

It remains to show that the map is surjective. Indeed if $v_j \in \mathcal{V}$ is among the $v_1, \ldots, v_m$, then there exist $h \in O(n, \mathbb{C})$ such that $h \cdot v_j = [-, -]$. If $V_1 \subset \mathfrak{g}^{\mathbb{C}}$ denotes the real form (of vector spaces) $h(\mathfrak{g})$, then:

$$[V_1, V_1] = h\Big(v(h^{-1}(V_1), h^{-1}(V_1))\Big) \subset h(v(\mathfrak{g}, \mathfrak{g})) \subset h(\mathfrak{g}) := V_1.$$

Therefore V_1 is a real form of Lie algebras, thus redefine $V_1 := \mathfrak{g}_1$. Let G_1 be the virtual Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_1$, then G_1 is a real slice of Lie groups of signature (p, q). Thus $[(G_1, g_1)] \mapsto O(p, q)v_j$, which proves that the map is surjective. The theorem is proved. □

There are classes of Lie algebras with $m = 1$, for instance the trivial case of abelian Lie algebras. However in general $m \neq 1$. Indeed even a semi-simple Lie algebra is not determined by the signature of its Killing form: $-\kappa$. As an example consider the semi-simple real forms $\mathfrak{o}(1, 4) \subset (\mathfrak{o}(5, \mathbb{C}), -\kappa) \supset \mathfrak{o}(2, 3)$. Then the signatures are $(6, 4)$ and $(4, 6)$ respectively. Thus $\mathfrak{o}(1, 4) \oplus \mathfrak{o}(2, 3)$ is a real form of $(\mathfrak{o}(5, \mathbb{C})^2, -\kappa)$ of signature $(10, 10)$. But also if $\mathfrak{o}(5, \mathbb{C})_{\mathbb{R}}$ denotes the real Lie algebra of $\mathfrak{o}(5, \mathbb{C})$, then it is also a real form of $\mathfrak{o}(5, \mathbb{C})^2$ which is simple, also of signature $(10,10)$, thus

$$\mathfrak{o}(5,\mathbb{C})_{\mathbb{R}} \ncong \mathfrak{o}(1,4) \oplus \mathfrak{o}(2,3),$$

and so $m \geq 2$ in this example.

We now give two examples, one where a Lie group is Wick-rotatable to a Riemannian Lie group, and the other where a Lie group is not Wick-rotatable to a Riemannian Lie group.

Example 3.9. *Let $H_3(\mathbb{R}) \subset H_3(\mathbb{C})$ be the 3-dimensional real and complex Heisenberg groups. The Lie algebra of $H_3(\mathbb{R})$ denoted: $\mathfrak{h}_3(\mathbb{R})$, is the set of strictly upper triangular 3×3 matrices. A basis of the Lie algebra is given by $\{e_1, e_2, e_3\}$ with,*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

We may identify $\{e_j\}_j$ with the standard basis of $\mathbb{R}^3$. Let $g(-,-)$ be the standard Lorentzian pseudo-inner product on $\mathbb{R}^3$, i.e of signature $(+,+,-)$. Thus $(H_3(\mathbb{R}), -g)$ is a real slice (of Lie groups) of $(H_3(\mathbb{C}), -g^{\mathbb{C}})$. Note that $g(-,-)$ is not bi-invariant, since $g([e_1,e_2],e_3) = -1 \neq g(e_1,[e_2,e_3]) = 0$. Define the linear map: $\theta \in End(\mathfrak{h}_3(\mathbb{R}))$ by:

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \mapsto -\lambda_1 e_1 - \lambda_2 e_2 + \lambda_3 e_3.$$

Then it is easy to show that this is an involution of Lie algebras, and moreover θ is a Cartan involution of $\mathfrak{h}_3(\mathbb{R})$ w.r.t $-g(-,-)$, thus by Corollary 3.4 it follows that $H_3(\mathbb{R})$ can be Wick-rotated to a Riemannian Lie group $\tilde{G}$. Note that $\tilde{G}$ is the real form of $H_3(\mathbb{C})$ consisting of matrices of the form: $\begin{bmatrix} 1 & ix & iy \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$ for $x, y, z \in \mathbb{R}$.

Example 3.10. *Consider the real form: $G := SL_2(\mathbb{R})^2 \subset G^{\mathbb{C}} := SL_2(\mathbb{C})^2$. Then we can equip G with a left-invariant metric $g(-,-)$ of signature $(3,3)$, by equipping one copy with $-\kappa$ and the other copy with κ. The real forms up to isomorphism of $\mathfrak{sl}_2(\mathbb{C})^2$ are:*

$$\mathfrak{sl}_2(\mathbb{R})^2, \ \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{su}(2), \ \mathfrak{su}(2)^2, \ \mathfrak{o}(1,3).$$

Let $\tilde{\mathfrak{g}}$ be one of these real forms (except the last one), then we may Wick-rotate G to the corresponding real forms $\tilde{G}$ of $SL_2(\mathbb{C})^2$ of signature either: $(3,3)$ or $(1,5)$. In the case of Wick-rotating to $(SU(2)^2, \tilde{g})$ we get a signature of $(3,3)$. Now note that if G can be Wick-rotated to a signature: $(0,6)$ or Riemannian: $(6,0)$, then we can find (by Corollary 3.4) a Cartan involution of $\mathfrak{su}(2)^2$ w.r.t $-\tilde{g}$ or $+\tilde{g}$ respectively:

$$\mathfrak{su}(2)^2 \xrightarrow{\theta} \mathfrak{su}(2)^2.$$

Suppose the Cartan involution is w.r.t $\tilde{g}$. Then if $\mathfrak{su}(2)^2 = \mathfrak{k} \oplus \mathfrak{p}$, is the Cartan decomposition w.r.t θ, we have $\mathfrak{g}_1 := \mathfrak{k} \oplus i\mathfrak{p} \cong \mathfrak{o}(1,3)$. Indeed $\theta^{\mathbb{C}}$ is a Cartan *involution of $\mathfrak{g}_1$ (w.r.t $-\kappa$), thus $-\kappa$ has signature $(3,3)$, hence it must be the case that $\mathfrak{g}_1 \cong \mathfrak{o}(1,3)$. We recall that $\mathfrak{t} \cong \mathfrak{su}(2)$, and that the Killing form of $\mathfrak{t}$ is just:*

$$(X,Y) \mapsto 4Tr(XY).$$

Now we note that $\mathfrak{t}^{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$ is simple and naturally $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{sl}_2(\mathbb{C})^2$ simply because $\mathfrak{t} \subset \mathfrak{su}(2)^2$ is contained in a real form of $\mathfrak{sl}_2(\mathbb{C})^2$. Thus when restricting $\tilde{g}$ on $\mathfrak{t}$ we must get $\tilde{g}(X,Y) = -\lambda\kappa_{\mathfrak{t}}(X,Y) = -4\lambda Tr(XY)$ for some $\lambda > 0$. Now if $X := (x,y) \in \mathfrak{t}$, then:

$$\begin{aligned}\tilde{g}(X,X) &:= -\kappa(x,x) + \kappa(y,y) = -4Tr(x^2) + 4Tr(y^2) \\ &= -4\lambda Tr(X^2) = -4\lambda(Tr(x^2) + Tr(y^2)).\end{aligned}$$

Thus we conclude that $y = 0$, and therefore:

$$\mathfrak{t} = \{(x,0)|x \in \mathfrak{su}(2)\} \subset \mathfrak{su}(2)^2, \quad \mathfrak{p} = \{(0,y)|y \in \mathfrak{su}(2)\} \subset \mathfrak{su}(2)^2.$$

This is impossible since then $\theta = 1 \oplus -1$ which is not a Lie homomorphism. The argument for the signature case: $(0,6)$, is identical with the change: $\tilde{g} \mapsto -\tilde{g}$. We conclude that (G,g) can not be Wick-rotated to a Riemannian Lie group nor of signature $(0,6)$.

One shall note that Proposition 3.6 does not hold for a general non-Riemannian signature. Indeed consider the previous example then $SL_2(\mathbb{R})^2$ has signature $(3,3)$ and can be Wick-rotated to $SU(2)^2$ also of signature $(3,3)$, but they are not locally isometric (since their Lie algebras are non-isomorphic). Thus $m \geq 2$ in the previous theorem.

We end this section by considering a result on semi-simple Lie groups.

Proposition 3.11. *Let $(G,g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice, and G be semi-simple. Then (G,g) can be Wick-rotated to a Riemannian compact Lie group if and only if there exist a Cartan involution θ of $\mathfrak{g}$ (w.r.t g) which is also a Cartan involution of $\mathfrak{g}$ (w.r.t $-\kappa$).*

Proof. $(\Rightarrow)$. If (G,g) is Wick-rotated to a Riemannian Lie group, then by Corollary 3.4, we can choose a Cartan involution θ of $\mathfrak{g}$. Denote $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition. Then following the proof of Corollary 3.4, then we can find a Riemannian Lie group $\tilde{G}$ with Lie algebra: $\tilde{\mathfrak{g}} := \mathfrak{k} \oplus i\mathfrak{p}$, which is Wick-rotated to G in $(G^{\mathbb{C}}, g^{\mathbb{C}})$. By Proposition 3.6, $\tilde{\mathfrak{g}}$ is compact, since we can Wick-rotate G to a Riemannian compact Lie group (by assumption). But since θ is a Cartan involution of $\mathfrak{g}$ (w.r.t $-\kappa$) if and only if $\tilde{\mathfrak{g}}$ is compact, then the direction is proved. $(\Leftarrow)$. Suppose θ is a Cartan involution of $\mathfrak{g}$ w.r.t $g(-,-)$ and $-\kappa$ simultaneously. Thus if $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$ associated with θ, then there exist a compact real form $U \subset G^{\mathbb{C}}$ with Lie algebra $\mathfrak{u}$. The proposition follows. $\square$

Thus since we may lift a local Cartan involution: $\mathfrak{g} \xrightarrow{\theta} \mathfrak{g}$, to a global Cartan involution: $G \xrightarrow{\Theta} G$, then in view of the previous proposition, there is a Θ which is an isometry of (G, g), i.e $\Theta \in Isom(G)$. Observe also that if there exist a real slice of Lie groups of $G^{\mathbb{C}}$ which is compact Riemannian, then the possible signatures (p, q) w.r.t $g^{\mathbb{C}}$ is a subset of the possible signatures of $-\kappa$ (of $\mathfrak{g}^{\mathbb{C}}$).

It is tempting to think that if (G, g) can be Wick-rotated to a compact semi-simple Riemannian Lie group, then it would be locally isometric to $(G, -\lambda\kappa)$ ($\lambda > 0$). However this is false, indeed consider $G := SL_2(\mathbb{R})^2$ equipped with the metric $g := -\kappa \oplus -\kappa$. We can Wick-rotate to the compact Riemannian Lie group: $SU(2)^2$. Consider the real form $\cong \mathfrak{o}(1, 3)$ identified with the set:

$$\{(x, x) | x \in \mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}\} \subset \mathfrak{sl}_2(\mathbb{C})^2.$$

If (G, g) is locally isometric to $\lambda\kappa$ for some $\lambda \in \mathbb{R}$, then we can Wick-rotate to a $\tilde{G} \subset G^{\mathbb{C}}$ with Lie algebra $\mathfrak{o}(1, 3)$. However $\mathfrak{o}(1, 3)$ on $g^{\mathbb{C}}$ is not a real slice. We thus conclude that (G, g) is not locally isometric to $(G, \lambda\kappa)$ for any $\lambda \in \mathbb{R}$.

Remark 3.12. One shall observe that if (G, g) and $(\tilde{G}, \tilde{g})$ are pseudo-Riemannian spaces, where G and $\tilde{G}$ are Lie groups, but the metrics are not assumed to be left-invariant, then the proof of Theorem 3.1 is still valid. The direction ($\Rightarrow$) of Corollary 3.4 is also valid, however the direction ($\Leftarrow$) does not necessarily hold.

4. Conjugacy of Cartan involutions

Given a pseudo-Riemannian Lie group (G, g), with two Cartan involutions θ_j ($j = 1, 2$) of $\mathfrak{g}$, one may wonder if they are conjugate in $Aut(\mathfrak{g})$. This is in fact true as we will show here, and we note again the resemblance with semi-simple Lie groups G and Cartan involutions of $\mathfrak{g}$ (w.r.t $-\kappa$).

Theorem 4.1. *Suppose $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ is a pseudo-Riemannian Lie group. Assume there exist two Cartan involutions: θ_1, θ_2 of $\mathfrak{g}$. Then θ_1 is conjugate to θ_2 in $Aut(\mathfrak{g})_0 \cap O(p, q)_0$.*

Proof. Write $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ for the Cartan decompositions w.r.t θ_1 and θ_2 respectively. Denote also: $\mathfrak{k}_j \oplus i\mathfrak{p}_j := \mathfrak{u}_j$ ($j = 1, 2$) for the real forms of $\mathfrak{g}^{\mathbb{C}}$. There exist Wick-rotations of G to connected virtual real forms: $U_j \subset G^{\mathbb{C}}$ with Lie algebras $\mathfrak{u}_j$ which are Riemannian (by Corollary 3.4). If σ denotes the conjugation map w.r.t $\mathfrak{g}$, and τ_j denotes the conjugation map of $\mathfrak{u}_j$, then we have $\theta_j^{\mathbb{C}} = \sigma\tau_j$. Now since θ_1 is conjugate to θ_2 in $O(p, q)_0$, i.e there is a $\phi \in O(p, q)_0$ such that

$$\phi\theta_1\phi^{-1} = \theta_2,$$

as linear maps, then $g := \phi^{\mathbb{C}}$ sends $\mathfrak{u}_1 \mapsto \mathfrak{u}_2$. Consider the action in Section 2.4 and the notation there. If $v := [-, -]$ is the complex Lie bracket of $\mathfrak{g}^{\mathbb{C}}$, then $v \in \mathcal{V}$ and $w := g^{-1} \cdot v \in \mathcal{W}$ (i.e is a bilinear form $\mathfrak{u}_1^2 \to \mathfrak{u}_1$) lie in the same complex orbit. Note that $(\mathfrak{g}, \mathfrak{u}_1)$ is a compatible pair. Now w is a minimal vector since it belongs to $\mathcal{W}$, i.e:

$$w \in O(n)_0 w \cap O(p, q)_0 v = K_0 v,$$

where

$$K := \{g \in O(p,q) | g\theta_1 = \theta_1 g\} \subset O(p,q),$$

is the maximal compact subgroup associated with the fixed global Cartan involution of $O(p,q)$: $g \mapsto \theta_1 g \theta_1$. Thus there is an element $k_0 \in K_0 \subset O(p,q)_0$ such that $k_0 v = w$, in other words: $gk_0 \cdot v = v$. Hence $gk_0 \in O(p,q)_0 \cap Aut(\mathfrak{g})$, and it follows that: $[\sigma, gk_0] = 0$, i.e

$$\theta_2 = gk_0 \circ \theta_1 \circ k_0^{-1} g^{-1}.$$

The corollary is proved. □

Thus on Lie algebras we get the following nice corollary:

Corollary 4.2. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, where $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra. Let $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ be a real form which is a real slice. Then any two Cartan involutions of $\mathfrak{g}$ are conjugate in $Aut(\mathfrak{g})_0 \cap O(p,q)_0$.*

Note that the corollary is a generalised version (for general pseudo-inner product spaces) of É. Cartan's result: ([16], Thm 7.2). Let us give an example (of a non-compact real form) where there is a unique Cartan involution of the Lie algebra:

Example 4.3. *If we consider again the real Heisenberg group: $(H_3(\mathbb{R}), -g)$ and follow Example 3.9, then calculating the derivation algebra of $\mathfrak{h}_3(\mathbb{R})$ w.r.t to the basis $\{e_j\}$, then the matrices have the form: $\begin{bmatrix} a & c & 0 \\ e & b & 0 \\ f & l & a+b \end{bmatrix}$, for $a, b, c, e, l, f \in \mathbb{R}$. Thus $Dim\Big(\mathfrak{Der}(\mathfrak{h}_3(\mathbb{R}))\Big) = 6$. Now if such a derivation D belongs to $\mathfrak{o}(1,2)$, then an easy calculation shows that $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$, i.e the Lie algebra of $Aut(\mathfrak{h}_3(\mathbb{R})) \cap O(1,2)$ has dimension 1. Now in view of the previous theorem if θ_1 is another Cartan involution of $\mathfrak{h}_3(\mathbb{R})$, there would exist an element $A \in Aut(\mathfrak{h}_3(\mathbb{R}))_0 \cap O(1,2)_0$ such that $A\theta A^{-1} = \theta_1$, i.e $\theta_1 = \theta$ since A is diagonal w.r.t our basis. We conclude that there exist a unique Cartan involution of $\mathfrak{h}_3(\mathbb{R})$, namely θ.*

Recall that for a real semi-simple Lie algebra $\mathfrak{g}$ equipped with the Killing form: $-\kappa$. Then it is proved in Helgason ([16]) that given any involution $\tilde{\theta}$ of $\mathfrak{g}$ there exist a Cartan involution of $\mathfrak{g}$ commuting with $\tilde{\theta}$. We can also prove a generalised version of this result for a general pseudo inner product space: $(\mathfrak{g}, g)$, by mimicking the proof given for semi-simple Lie algebras in [16] together with Corollary 3.5.

Corollary 4.4. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, where $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra. Let $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ be a real form which is a real slice. Suppose there exist a compact real form of $\mathfrak{g}^{\mathbb{C}}$ which is also a real Lie subalgebra. Let $\tilde{\theta} \in Aut(g) \cap O(p,q)$ be an involution of $\mathfrak{g}$. Then there exist a Cartan involution θ of $\mathfrak{g}$ that commutes with $\tilde{\theta}$, i.e $[\tilde{\theta}, \theta] = 0$.*

Proof. Let θ' be a Cartan involution of $\mathfrak{g}$ by Corollary 3.5. By mimicking the proof of ([16], Thm 7.1) in view of Exercise (4, Ch.3, [16]), we apply the proof given there to the inner product $g_{\theta'}(-,-) := g(-,\theta'(-))$, together with the symmetric operator: $N := \tilde{\theta}\theta'$. Thus there exist a $\psi \in O(p,q) \cap Aut(\mathfrak{g})$ such that $[\psi\theta'\psi^{-1}, \tilde{\theta}] = 0$, therefore let $\theta := \psi\theta'\psi^{-1}$. ☐

5. Wick-rotating a Lorentzian signature

If we assume our left-invariant metric on our Lie group G is Lorentzian or of signature $(+,-,\cdots,-)$, then being able to Wick-rotate to a Riemannian space puts some constraints on the structure of the Lie algebra (in view of Corollary 3.4). Now since a Wick-rotation is a local condition, it would be interesting to know what type of Lie algebra allows for a Wick-rotation to a Riemannian Lie group.

We recall by the fundamental *Levi-Malcev theorem* that our Lie algebra $\mathfrak{g}$ can be written as a semi-direct sum $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{h}$, where $\mathfrak{h}$ is the radical of $\mathfrak{g}$ and $\mathfrak{s} \subset \mathfrak{g}$ is either trivial or a semi-simple subalgebra of $\mathfrak{g}$ called the Levi-factor.

It is clear that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{s}^{\mathbb{C}} \ltimes \mathfrak{h}^{\mathbb{C}}$, and if $\tilde{\mathfrak{g}}$ is another real form of $\mathfrak{g}^{\mathbb{C}}$, then writing a Levi-decomposition: $\tilde{\mathfrak{g}} = \tilde{\mathfrak{s}} \ltimes \tilde{\mathfrak{h}}$, then $\tilde{\mathfrak{h}}$ is a real form of $\mathfrak{h}^{\mathbb{C}}$. To see that $\tilde{\mathfrak{s}}$ is a real form of $\mathfrak{s}^{\mathbb{C}}$, we note that there exist a $k \geq 1$ such that

$$\mathfrak{s}^{\mathbb{C}} = [\mathfrak{g}^{\mathbb{C}(k)}, \mathfrak{g}^{\mathbb{C}(k)}] \supset [\tilde{\mathfrak{g}}^{(k)}, \tilde{\mathfrak{g}}^{(k)}] = \tilde{\mathfrak{s}}.$$

In view of the existence of an involution of Lorentzian decomposition we can say the following:

Proposition 5.1. *Let $(G,g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice of Lie groups. Then the following statements hold:*

1. *Suppose $g(-,-)$ has Lorentzian signature. If (G,g) can be Wick-rotated to a Riemannian Lie group $(\tilde{G},\tilde{g})$ then $\mathfrak{s} = 0$ or $\mathfrak{h} \neq 0$. Moreover if $\tilde{\mathfrak{s}}$ is a Levi-factor of $\tilde{\mathfrak{g}}$, then $\tilde{\mathfrak{s}} \cong \mathfrak{s}$.*
2. *Suppose $g(-,-)$ has signature $(+,-,\cdots,-)$. If (G,g) can be Wick-rotated to a Riemannian Lie group, then either $\mathfrak{s} = 0$ or $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$.*

Proof. For case (1), assume $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{h}$ for $\mathfrak{s} \neq 0$, and choose a Cartan involution θ of $\mathfrak{g}$. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition. Then θ leaves $\mathfrak{s}$ invariant: $\theta(\mathfrak{s}) \subset \mathfrak{s}$. Indeed note that since $\mathfrak{h}$ is solvable, then there exist $k \geq 1$ such that the k^{th}-derived algebra satisfies: $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{s}$, thus it follows that θ must leave $\mathfrak{s}$ invariant, and hence we can write,

$$\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{k}) \oplus (\mathfrak{s} \cap \mathfrak{p}).$$

We claim that $\mathfrak{s} \cap \mathfrak{p} = 0$, indeed suppose not, i.e $\mathfrak{p} \subset \mathfrak{s}$ thus $[\mathfrak{s}, \mathfrak{p}] \subset \mathfrak{p}$ so $\mathfrak{p}$ is an abelian non-trivial ideal of $\mathfrak{s}$, contradicting the semi-simplicity of $\mathfrak{s}$. Thus θ fixes $\mathfrak{s}$ point wise. Moreover $\mathfrak{p} \lhd \mathfrak{g}$ is an abelian ideal, and so therefore $\mathfrak{p} \subset \mathfrak{h}$, i.e $\mathfrak{h} \neq 0$. Finally since $g(-,-)$ restricted to $\mathfrak{s}$ and $\tilde{g}(-,-)$ restricted to $\tilde{\mathfrak{s}}$ is positive definite, then $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ give rise to a Wick-rotation of two Riemannian Lie groups, thus by Proposition 3.6 it follows that $\mathfrak{s} \cong \tilde{\mathfrak{s}}$, and case (1) is proved. For case (2) suppose $\mathfrak{g}$ is non-solvable (i.e $\mathfrak{s} \neq 0$), then again w.r.t θ we see that

$$\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{k}) \oplus (\mathfrak{s} \cap \mathfrak{p}),$$

where $\mathfrak{s} \cap \mathfrak{k} \neq 0$, since if not then $\mathfrak{s} \subset \mathfrak{p}$, i.e $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$, which is a contradiction. Now since $\theta^{\mathbb{C}}$ is a Cartan involution of a real form $\tilde{\mathfrak{g}} \subset \mathfrak{s}^{\mathbb{C}}$, then $-\kappa$ on $\tilde{\mathfrak{g}}$ must also have the signature $(+, -, \cdots, -)$, this follows since $\mathfrak{t}^{\mathbb{C}}$ is 1-dimensional. Now finally if $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ is a Cartan decomposition, then $\tilde{\mathfrak{k}}$ is abelian and 1-dimensional, thus it follows that $\tilde{\mathfrak{g}} \cong \mathfrak{sl}_2(\mathbb{R})$ see for example (Prop. 13.1.10, [31]). We conclude that $\mathfrak{s}^{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$, and hence also $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$. The proposition is proved. □

Thus restricting to the class of semi-simple Lie algebras it is impossible to Wick-rotate a Lorentzian metric to a Riemannian metric. However even for a nilpotent Lie algebra the converse of (1) is not necessarily true, indeed consider the nilpotent Lie algebra $\mathfrak{h}_3(\mathbb{R})$ of 3×3 strictly upper triangular matrices. Then if θ is an involution with $Dim(\mathfrak{p}) = 1$, we must be able to find a basis $\{x_1, x_2, x_3\}$ such that

$$[x_1, x_2] = C_{12}^1 x_1 + C_{12}^2 x_2, \quad [x_1, x_3] = C_{13}^3 x_3, \quad [x_2, x_3] = C_{23}^3 x_3.$$

But since $[\mathfrak{h}_3(\mathbb{R}), \mathfrak{h}_3(\mathbb{R})]$ has dimension 1, then it follows that $C_{12}^1 = 0 = C_{12}^2$. Moreover since $\mathfrak{h}_3(\mathbb{R})$ is nilpotent of class 2, then we conclude also that $C_{12}^3 = 0 = C_{23}^3$, i.e $\mathfrak{h}_3(\mathbb{R})$ would have to be abelian, thus it is not possible to Wick-rotate a Lorentzian metric on $H_3(\mathbb{R})$ to a Riemannian metric.

In view of case (2), there are examples of metrics (of signature $(+, -, \cdots, -)$) that are Wick-rotatable to a Riemannian metric within: $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{h}_3(\mathbb{R})$ and even $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{h}_3(\mathbb{R})$.

If we impose the condition that the metric is bi-invariant, i.e $(\mathfrak{g}, g)$ is a quadratic Lie algebra, then we have the following equivalence result:

Corollary 5.2. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice of Lie groups. Suppose g is bi-invariant. Then the following statements hold:*

(1) *Suppose $g(-,-)$ has Lorentzian signature. Then (G, g) can be Wick-rotated to a Riemannian Lie group $(\tilde{G}, \tilde{g})$ if and only if $\mathfrak{g}$ is abelian or $\mathfrak{g}$ is a direct sum of $\mathfrak{h} \neq 0$ and $\mathfrak{s}$ is compact semi-simple. Moreover $\tilde{\mathfrak{g}} \cong \mathfrak{g}$.*
(2) *Suppose $g(-,-)$ has signature $(+, -, \cdots, -)$. Then (G, g) can be Wick-rotated to a Riemannian Lie group, if and only if either $\mathfrak{g}$ is abelian or $\mathfrak{g}$ is a direct sum of $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{h}$ abelian.*

Proof. Case (1). Since g is bi-invariant then $\tilde{\mathfrak{g}}$ must be reductive, because $\tilde{g}$ is bi-invariant and a Riemannian metric. Thus the complexification is also reductive, i.e so are the real forms, thus $\mathfrak{g}$ is reductive. This means that either $\mathfrak{s} = 0$ or $\mathfrak{s}$ is semi-simple, and $\mathfrak{h}$ is abelian. So if $\mathfrak{g}$ is non-abelian then $\mathfrak{s}$ is semi-simple. Now by the proof of the previous proposition case (1), then given a Cartan involution of $\mathfrak{g}$ we must have that θ fixes point wise $\mathfrak{s}$. This means that g restricted to $\mathfrak{s}$ is an inner product. If $\mathfrak{s}^{\mathbb{C}}$ is simple, then $g_{|\mathfrak{s}}$ must be proportional to the Killing form: $\lambda\kappa$ ($\lambda \in \mathbb{R}$). Now since g is positive definite on $\mathfrak{s}$ then $\lambda > 0$ i.e $\mathfrak{s}$ is compact. If $\mathfrak{s}^{\mathbb{C}}$ is not simple then on each simple ideal, $g^{\mathbb{C}}$ is proportional to the Killing form. There are two cases to consider, either $\mathfrak{s}$ is simple (in which case $\mathfrak{s}$ has a complex structure) or each simple ideal $\mathfrak{J}$ of $\mathfrak{s}$ has a simple complexification $\mathfrak{J}^{\mathbb{C}} \lhd \mathfrak{s}^{\mathbb{C}}$ or has a complex structure. See for instance (Thm 6.94, [40]). Assume $\mathfrak{s}$ is simple, then $\mathfrak{s}^{\mathbb{C}} \cong \mathfrak{s} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a complex Lie algebra. Thus $g^{\mathbb{C}}$ restricted to $\mathfrak{s}$ is proportional to the complex Killing form on $\mathfrak{s}$, say $\lambda\kappa$. Thus viewing $\mathfrak{s}$ as a real Lie algebra, we get that $g^{\mathbb{C}}$ restricts to something proportional to the real part: $\lambda\frac{1}{2}Re(\kappa) = g$, which is positive definite by assumption. Therefore $\lambda \in \mathbb{R}$. But the real Killing form of $\mathfrak{s}$ is precisely $2Re(\kappa)$, so we conclude that either the Killing form is positive definite or negative definite, this is impossible. The argument for the other case is a combination of the previous two arguments. We conclude that $\mathfrak{s}$ is semi-simple compact. Now finally it follows that $\mathfrak{g} \cong \tilde{\mathfrak{g}}$ by the previous proposition and that $\mathfrak{h} \cong \tilde{\mathfrak{h}}$ (since they are abelian of the same dimension).

Conversely if $\mathfrak{g}$ is abelian then the statement is trivial, therefore assume $\mathfrak{s}$ is compact semi-simple. Then $\mathfrak{s} := [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$ forms an orthogonal sum w.r.t g. Thus g restricted to $\mathfrak{s}$ must be positive definite, indeed restricting g on a compact simple ideal (which is a non-degenerate ideal) $\mathfrak{J} \lhd \mathfrak{s}$ we get something proportional to the Killing form on $\mathfrak{J}$: $\lambda\kappa$. Thus if $\lambda > 0$ then this would contradict g having Lorentzian signature. Therefore $\lambda < 0$. Hence g on $\mathfrak{h}$ must have Lorentzian signature, and so we can easily find a Cartan involution $\theta_{\mathfrak{h}}$ of $\mathfrak{h}$ such that $1_{\mathfrak{s}} \oplus \theta_{\mathfrak{h}}$ is a Cartan involution of $\mathfrak{g}$, now use Corollary 3.4.

Case (2). Again since $\mathfrak{g}$ must be reductive, then by the previous proposition case (2), if $\mathfrak{g}$ is not abelian then $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{h}$ is abelian. Conversely let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$. If $\mathfrak{s} = 0$, then the statements is obviously true. Suppose therefore that $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$. Note that $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$ is an orthogonal direct sum w.r.t g, i.e $[\mathfrak{g}, \mathfrak{g}]^{\perp} = \mathfrak{z}(\mathfrak{g})$. Thus g restricted to $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$ forms a quadratic Lie algebra, but since $\mathfrak{sl}_2(\mathbb{C})$ is simple, then g must be proportional to the Killing form: $\lambda\kappa$ ($\lambda \in \mathbb{R}$). Note that $\lambda < 0$ since otherwise g would not be able to have signature: $(+, -, - \dots, -)$. Also note that g restricted to $\mathfrak{h}$ must be of signature: $(-, -, \dots, -)$. Thus choose any Cartan involution $\theta_{\mathfrak{s}}$ of $\mathfrak{s}$, and the Cartan involution $\theta_{\mathfrak{h}}$ of $\mathfrak{h}$ of the form: $\theta_{\mathfrak{h}}(x) := -x$. Then $\theta_{\mathfrak{s}} \oplus \theta_{\mathfrak{h}}$ is a Cartan involution of $\mathfrak{g}$, and the statement follows by Corollary 3.4. □

Thus a solvable Lie group (G, g) with a bi-invariant (non-Riemannian) metric is not Wick-rotatable to a Riemannian Lie group.

6. A remark on Wick-rotatable tensors of Lie groups

Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is Riemannian. By Corollary 3.4 there exist a Cartan involution θ of $\mathfrak{g}$. Recall the section on Wick-rotatable tensors. We prove in this section that if $\tilde{v} = v \in \mathcal{V} \cap \tilde{\mathcal{V}}$

are two tensors on the Lie algebras, then they are Wick-rotatable with respect to an embedding $\phi^{-1} \in H^{\mathbb{C}}$ into the same $H^{\mathbb{C}}$-orbit for

$$H^{\mathbb{C}} := Aut(\mathfrak{g}^{\mathbb{C}}) \cap O(n, \mathbb{C}) \subset O(n, \mathbb{C}),$$

such that $(\mathfrak{g}, \phi^{-1}(\tilde{\mathfrak{g}}))$ is a compatible pair (i.e also a compatible triple). Denote $H := Aut(\mathfrak{g}) \cap O(p, q)$. Note that $H \subset H^{\mathbb{C}}$ is a real form. Indeed the real structure of $O(n, \mathbb{C})$ fixing $O(p, q)$ given by $A \mapsto \sigma A \sigma$ where σ is the conjugation map w.r.t $\mathfrak{g}$, leaves $H^{\mathbb{C}}$ invariant, and thus fixes H. Note also that a global Cartan involution Θ: $A \mapsto \theta A \theta$ of $O(p, q)$ where θ is a Cartan involution of $\mathfrak{g}$, also leave H invariant. Thus Θ is a global Cartan involution of H. The arguments above are analogous for the real from: $\tilde{H} := O(\tilde{p}, \tilde{q}) \cap Aut(\tilde{\mathfrak{g}})$.

Lemma 6.1. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is Riemannian. Then any two tensors $v = \tilde{v} \in \mathcal{V} \cap \tilde{\mathcal{V}}$ can be embedded into the same $H^{\mathbb{C}}$-orbit for some $\phi^{-1} \in H^{\mathbb{C}}$ such that $\Big(\mathfrak{g}, \phi^{-1}(\tilde{\mathfrak{g}})\Big)$ is a compatible pair.*

Proof. By Corollary 3.4, we can choose a Cartan involution θ of $\mathfrak{g}$. Moreover by Proposition 3.6 there is an isomorphism of Lie algebras: $\phi \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$ sending $\mathfrak{u} \mapsto \tilde{\mathfrak{g}}$, where $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ w.r.t the Cartan decomposition of θ. Thus $\phi^{-1}(\tilde{\mathfrak{g}}) = \mathfrak{u}$ and $\mathfrak{g}$ are compatible. Let now $\tilde{v} = v \in \mathcal{V} \cap \tilde{\mathcal{V}}$ be two Wick-rotatable tensors, using the isometry tensor action of ϕ^{-1} on v, then $\phi^{-1} \cdot v$ and v lie in the same $H^{\mathbb{C}}$-orbit as required. $\square$

Theorem 6.2. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is Riemannian. Let $\tilde{v} = v \in \mathcal{V} \cap \tilde{\mathcal{V}}$ be two tensors (i.e they are also Wick-rotatable). There exist a Cartan involution θ of $\mathfrak{g}$ such that $\theta \cdot v = v$.*

Proof. Consider the real forms: $H \subset H^{\mathbb{C}} \supset \tilde{H}$ as above. Then one simply note that $H^{\mathbb{C}}$ is naturally algebraic, and moreover is a linearly complex reductive Lie group, simply because $\tilde{H}$ is a compact real form. Now since $O(n, \mathbb{C})$ and $Aut(\mathfrak{g}^{\mathbb{C}})$ are naturally algebraic groups defined over $\mathbb{R}$, then so is $H^{\mathbb{C}}$. Moreover H and $\tilde{H}$ are the real points of $H^{\mathbb{C}}$ (respectively). Thus the groups are naturally among the class of groups considered in Section 2.3. The theorem follows by Lemma 6.1 and Theorem 2.23. $\square$

Thus we can restate a stronger version of Theorem 2.26 for Lie groups:

Theorem 6.3. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is a Riemannian Lie group. Then the following statements hold:*

1. *There exist a Cartan involution θ of $\mathfrak{g}$ such that $\nabla_{\theta(x)}\theta(y) = \theta(\nabla_x y)$ for all $x, y \in \mathfrak{g}$.*
2. *There exist a Cartan involution θ of $\mathfrak{g}$ such that $ric_g\Big(\theta(x), \theta(y)\Big) = ric_g(x, y)$ for all $x, y \in \mathfrak{g}$.*

(3) *There exist a Cartan involution θ of $\mathfrak{g}$ such that $[\theta, Ric_g] = 0$.*
(4) *There exist a Cartan involution θ of $\mathfrak{g}$ such that $R\Big(\theta(x), \theta(y)\Big)\Big(\theta(z)\Big) = \theta\Big(R(x,y)(z)\Big)$ for all $x, y, z \in \mathfrak{g}$.*

Any of the properties 1-4 of the previous theorem, are (using also Theorem 3.1) Wick-rotatable. Thus we state as a stronger result for Lie groups (compare with [21]):

Corollary 6.4. *Let (G, g) be a pseudo-Riemannian Lie group. Then the property of being Riemann purely electric (RPE) at 1 w.r.t to a Cartan involution θ of $\mathfrak{g}$ is Wick-rotatable.*

Proof. Follows by Theorem 3.1 and Theorem 6.3. □

We end this section by also noting the following result:

Corollary 6.5. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is Riemannian. Then*

$$\Big(\forall x \in \mathfrak{g} \cap \tilde{\mathfrak{g}}\Big)\Big(\exists \theta \in Aut(\mathfrak{g})\Big)\Big(\theta(x) = x\Big),$$

where θ is a Cartan involution of $\mathfrak{g}$.

Proof. Consider the isometry action of $O(n, \mathbb{C})$ (restricted to $H^{\mathbb{C}}$ defined above) on the complex Lie algebra: $\mathfrak{g}^{\mathbb{C}}$, i.e

$$g \cdot x := g(x), \ g \in O(n, \mathbb{C}), \ x \in \mathfrak{g}^{\mathbb{C}}.$$

Let $x = \tilde{x} \in \mathfrak{g} \cap \tilde{\mathfrak{g}}$. Then $x = \tilde{x}$ are two Wick-rotatable tensors, thus w.r.t a choice of $g \in H^{\mathbb{C}}$ we can assume that the real actions (of H and $\tilde{H}$) are compatible. Moreover x and $\tilde{x}$ lie in the same complex orbit, such that $O(p, q)x \sim O(n)\tilde{x}$ are compatible real orbits. We can now finish the proof by applying Theorem 6.2. □

7. Wick-rotating an algebraic soliton

A pseudo-Riemannian Lie group (G, g), such that the Ricci operator $Ric_g \in \mathfrak{gl}(\mathfrak{g})$ has the form:

$$Ric_g = \lambda \cdot 1_{\mathfrak{g}} + D,$$

where $\lambda \in \mathbb{R}$ and $D \in \mathfrak{der}(\mathfrak{g})$ (a derivation) is called an *algebraic soliton* (defined in [26]). If D can be taken to be $D = 0$, then (G, g) is said to be Einstein, and moreover if the Lie algebra is also nilpotent (resp. solvable), then an algebraic soliton (G, g), is said to be a *Ricci nilsoliton* (resp. *solsoliton*). For a discussion of Riemannian Ricci nilsolitons we refer to for example [23]. However we shall only be interested in Wick-rotating such a geometry.

We shall prove a result regarding the existence of a Wick-rotation of an algebraic soliton to a Riemannian Lie group, by using the results of the previous section.

Lemma 7.1. *The property of being an algebraic soliton is Wick-rotatable.*

Proof. Let (G, g) be Wick-rotatable to $(\tilde{G}, \tilde{g})$. Suppose (G, g) is an algebraic soliton. The Ricci operator: $\mathfrak{g} \xrightarrow{Ric_g} \mathfrak{g}$ on G is also a restriction of the Ricci operator on $G^{\mathbb{C}}$. So if $Ric_g = \lambda \cdot 1_{\mathfrak{g}} + D$ for some $\lambda \in \mathbb{R}$ and $D \in \mathfrak{der}(\mathfrak{g})$, then also

$$Ric_{g^{\mathbb{C}}} = (\lambda \cdot 1_{\mathfrak{g}})^{\mathbb{C}} + D^{\mathbb{C}} = \lambda 1_{\mathfrak{g}^{\mathbb{C}}} + D^{\mathbb{C}}.$$

Note that $D^{\mathbb{C}}$ is a derivation of $\mathfrak{g}^{\mathbb{C}}$, thus when restricting to $Ric_{\tilde{g}}$ we see that $D^{\mathbb{C}}$ must leave $\tilde{\mathfrak{g}}$ invariant and is thus a derivation $\tilde{D}$ of $\tilde{\mathfrak{g}}$ as required. The lemma follows. □

Note from the lemma that $D \in End(\mathfrak{g})$ and $\tilde{D} \in End(\tilde{\mathfrak{g}})$ are Wick-rotatable tensors, under the isometry action:

$$g \cdot f := gfg^{-1}, \; g \in O(n, \mathbb{C}), \; f \in End(\mathfrak{g}^{\mathbb{C}}).$$

Corollary 7.2. *The property of being a Ricci nilsoliton (resp. solsoliton) is Wick-rotatable.*

Corollary 7.3. *The property of being Einstein is Wick-rotatable.*

Applying the previous section, we get the following necessary condition for when an algebraic soliton can be Wick-rotated to a Riemannian algebraic soliton:

Theorem 7.4. *Suppose (G, g) is an algebraic soliton, with $Ric_g = \lambda \cdot 1_{\mathfrak{g}} + D$, which can be Wick-rotated to a Riemannian algebraic soliton: $(\tilde{G}, \tilde{g})$ with $Ric_{\tilde{g}} = \lambda \cdot 1_{\tilde{g}} + \tilde{D}$. Then there exist a Cartan involution θ of $\mathfrak{g}$ such that $[\theta, D] = 0$.*

Proof. The derivations: $D^{\mathbb{C}} = \tilde{D}^{\mathbb{C}} \in \mathcal{V} \cap \tilde{\mathcal{V}}$ are Wick-rotatable (see the proof of Lemma 7.1). Thus w.r.t a choice of map $g \in H^{\mathbb{C}}$ we can assume w.l.o.g that D and $\tilde{D}$ lie in the same complex orbit under the conjugation action: $H^{\mathbb{C}} \cdot D \ni \tilde{D}$. By Theorem 6.2 there is a Cartan involution θ of $\mathfrak{g}$ such that $\theta \cdot D := \theta D \theta = D$ or in other words $[\theta, D] = 0$. The theorem is proved. □

Example 7.5. *We follow Example 3.9. Thus consider the real 3-dimensional Heisenberg group $(H_3(\mathbb{R}), -g)$, then this is a Ricci nilsoliton of signature $(+, -, -)$. Indeed one can calculate with respect to the basis $\{e_1, e_2, e_3\}$, that the Ricci operator can be written uniquely as:*

$$Ric_{-g} = -\frac{3}{2} \cdot I_3 + D,$$

where $D(e_1) = e_1$, $D(e_2) = e_2$, $D(e_3) = 2e_3$. Note that the Cartan involution θ commutes with D, i.e $[\theta, D] = 0$. Thus when restricting to the Wick-rotated Riemannian Ricci nilsoliton $(\tilde{G}, \tilde{g})$ with Lie algebra: $\tilde{\mathfrak{g}} := \langle ie_1, ie_2, e_3 \rangle \subset \mathfrak{h}_3(\mathbb{C})$, we get the corresponding Ricci operator expressed as:

$$Ric_{\tilde{g}} = -\frac{3}{2} \cdot I_3 + \tilde{D},$$

with $\tilde{D}(ie_1) = ie_1,\ \tilde{D}(ie_2) = ie_2,\ \tilde{D}(e_3) = 2e_3.$

Part 7. Holomorphic inner product spaces on Lie algebras

We continue with the notation from the previous part, and prove some results for a holomorphic inner product space $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ resembling complex semisimple Lie algebras equipped with their Killing form. We also prove when there exist a compact real form (of Lie algebras) in the case where $g^{\mathbb{C}}$ is invariant, i.e $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ is a quadratic Lie algebra.

1. Real slices up to isomorphism

Suppose $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ is a holomorphic inner product space on a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Let $O(n, \mathbb{C})$ be the isometry group. We recall that a compact real form $\mathfrak{u} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$, is a real form of Lie algebras which is also a real slice of Euclidean signature. If $\phi \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$ is an involution such that when $\mathfrak{g}^{\mathbb{C}} = V_+ \oplus V_-$, is the eigenspace decomposition, then $\mathfrak{g} := V_+ \oplus iV_-$ is a real slice, i.e $g^{\mathbb{C}}(\mathfrak{g}, \mathfrak{g}) \in \mathbb{R}$, then we shall write $\phi \in \mathcal{O}$ for such a map. We can put an equivalence relation on maps $\mathcal{O}$ by conjugacy in $O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$. We recall that if $\mathfrak{g} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ is a real form, then a Cartan involution θ of $\mathfrak{g}$ is an involution of Lie algebras such that $g^{\mathbb{C}}_{|\mathfrak{g}}(\cdot, \theta(\cdot)) > 0$.

The following theorem should be compared with a similar result of semisimple Lie algebras equipped with their Killing form (see [12], Thm 1.3).

Theorem 1.1. *If $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ has a compact real form, then there is a bijection between isomorphism classes of real forms $\mathfrak{g} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ and conjugacy classes of $\mathcal{O}$.*

Proof. Let $\mathfrak{g} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ be a real form, then we can choose a Cartan involution of $\mathfrak{g}$ say θ by (Part 6, Corollary 3.5). Define the map $[\mathfrak{g}] \mapsto [\theta^{\mathbb{C}}]$. The map is well-defined (Part 6, Theorem 4.1) since any two Cartan involutions are conjugate in $O(p, q) \cap Aut(\mathfrak{g})$, where (p, q) is the signature of the induced pseudo-inner product from $g^{\mathbb{C}}$. To see that the map is surjective, let $\phi \in \mathcal{O}$, and set $\mathfrak{g} := V_+ \oplus iV_-$ for the real form $\mathfrak{g} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$. Then ϕ restricted to $\mathfrak{g}$ is an involution, and if σ denotes its conjugation map then $[\sigma, \phi] = 0$. But we may choose a Cartan involution θ of $\mathfrak{g}$ such that $[\theta, \phi] = 0$ by (Part 6, Corollary 4.4). Thus $\theta^{\mathbb{C}} = \sigma\tau$ for τ a conjugation map of a compact real form $\mathfrak{u} \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$. Thus

$$\sigma\tau\phi = \phi\sigma\tau,$$

therefore

$$\sigma\tau\phi = \sigma\phi\tau,$$

or in other words by canceling σ we obtain $[\tau, \phi] = 0$ so that ϕ is in fact a Cartan involution of $\mathfrak{g}$, and hence $[\mathfrak{g}] \mapsto [\phi]$.

Suppose now that $\mathfrak{g}_j$ are two real forms for $j = 1, 2$ such that the images are the same: $[\theta_1^{\mathbb{C}}] = [\theta_2^{\mathbb{C}}]$. Then if σ_j denotes the conjugation maps, and $\mathfrak{u}_j$ are the compact real forms compatible with $\mathfrak{g}_j$, then $\theta_j^{\mathbb{C}} = \sigma_j \tau_j$. But since the maps are conjugate in $O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$, say by ϕ, thus $\phi\theta_1^{\mathbb{C}}\phi^{-1} = \theta_2^{\mathbb{C}}$ then it is easy to see that: $\phi(\mathfrak{u}_1) = \mathfrak{u}_2$. Thus

$$\theta_2^{\mathbb{C}} = \phi\theta_1^{\mathbb{C}}\phi^{-1} = \phi\sigma_1\tau_1\phi^{-1} = \phi\sigma_1\phi^{-1}\phi\tau_1\phi^{-1} = \phi\sigma_1\phi^{-1}\tau_2,$$

thus cancelling τ_2 we obtain: $\phi\sigma_1\phi^{-1} = \sigma_2$, which proves that $[\mathfrak{g}_1] = [\mathfrak{g}_2]$, and hence the map is injective. The theorem is proved. $\square$

2. On the existence of a compact real form

Let $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ be a holomorphic inner product space, and let $W \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ be a real form of Euclidean signature, i.e a compact real form. We are interested in the question when W is also a compact real form in the sense of Lie algebras, i.e when W is also a compact real form of Lie algebras. Consider the isometry action of $G^{\mathbb{C}} := O(n, \mathbb{C})$ on the complex vector space $V^{\mathbb{C}}$ of bilinear forms into the Lie algebra, i.e the action is given by:

$$(g \cdot v)(x, y) := g\Big(v(g^{-1}(x), g^{-1}(y))\Big).$$

Note that the Lie bracket $[-,-] \in V^{\mathbb{C}}$.

Our action is clearly balanced (see Part 6, Section 2.4), since if τ is the conjugation map w.r.t W, then the map $v \mapsto \tau \cdot v$ balances the complex action. Consider the eigenspace decomposition w.r.t the action of τ for the Lie bracket: $[-,-]$, i.e $[-,-] = \beta_+ + \beta_-$. Then $\beta_+, \beta_- \in \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}})$ are minimal vectors, and in fact the components define Lie brackets on $\mathfrak{g}^{\mathbb{C}}$:

Lemma 2.1. *β_+ and β_- define Lie brackets on the complex vector space: $\mathfrak{g}^{\mathbb{C}}$.*

Proof. Let us begin by showing that $\beta_+(x, y) = -\beta_+(y, x)$ and $\beta_-(x, y) = -\beta_-(y, x)$ for all $x, y \in \mathfrak{g}^{\mathbb{C}}$. If $x, y \in \mathfrak{g}^{\mathbb{C}}$, then since $[x, y] + [y, x] = 0$ we have: $\beta_+(y, x) + \beta_-(y, x) + \beta_+(x, y) + \beta_-(x, y) = 0$, but by the action of τ, we also have:

$$0 = \beta_+(y, x) - \beta_-(y, x) + \beta_+(x, y) - \beta_-(x, y).$$

Combining the two equations we get the desired property. By the Jacobi-identity of $[-,-]$ we get that

$$\beta_+(x, \beta_+(y, z) + \beta_-(y, z)) + \beta_-(x, \beta_+(y, z) + \beta_-(y, z)) + \beta_+(y, \beta_+(z, x) + \beta_-(z, x))$$
$$+\beta_-(y, \beta_+(z, x) + \beta_-(z, x)) + \beta_+(z, \beta_+(x, y) + \beta_-(x, y)) + \beta_-(z, \beta_+(x, y) + \beta_-(x, y)) = 0.$$

Call this equation $(*)$. By the action of τ on both sides of $(*)$ gives the reduced equation:

$\beta_+(x,\beta_+(y,z)+\beta_-(y,z))+\beta_+(y,\beta_+(z,x)+\beta_-(z,x))+\beta_+(z,\beta_+(x,y)+\beta_-(x,y))=0.$

Finally substituting x,y,z with $\tau(x),\tau(y),\tau(z)$ we get

$\tau\Big(\beta_+(x,\beta_+(y,z)-\beta_-(y,z))+\beta_+(y,\beta_+(z,x)-\beta_-(z,x))+\beta_+(z,\beta_+(x,y)-\beta_-(x,y))\Big)=0,$

in other words:

$$\begin{aligned}\beta_+(x,\beta_+(y,z))+\beta_+(y,\beta_+(z,x))+\beta_+(z,\beta_+(x,y))=0,\\ \beta_+(x,\beta_-(y,z))+\beta_+(y,\beta_-(z,x))+\beta_+(z,\beta_-(x,y))=0.\end{aligned}$$

Putting these two equation into the first equation $(*)$ above gives also:

$$\begin{aligned}\beta_-(x,\beta_+(y,z))+\beta_-(y,\beta_+(z,x))+\beta_-(z,\beta_+(x,y))&\\ +\beta_-(x,\beta_-(y,z))+\beta_-(y,\beta_-(z,x))+\beta_-(z,\beta_-(x,y))&=0.\end{aligned}$$

Now finally substituting x,y,z again with $\tau(x),\tau(y),\tau(z)$ gives the equations:

$$\begin{aligned}\beta_-(x,\beta_+(y,z))+\beta_-(y,\beta_+(z,x))+\beta_-(z,\beta_+(x,y))=0\\ \beta_-(x,\beta_-(y,z))+\beta_-(y,\beta_-(z,x))+\beta_-(z,\beta_-(x,y))=0,\end{aligned}$$

which proves the lemma. □

Corollary 2.2. *The complex orbit $O(n,\mathbb{C})\cdot[-,-]$ is closed.*

Proof. Given an orthonormal basis $\{e_j\}_j$ of W, then we can equip $V^{\mathbb{C}}$ with the Hermitian inner product: H, given by

$$H(\alpha,\beta):=\sum_l g^{\mathbb{C}}(\alpha(e_l,e_l),(\tau\cdot\beta)(e_l,e_l)),$$

such that the minimal vectors are precisely those vectors $\beta=\beta_++\beta_-$ satisfying

$$H(X\cdot\beta_+,\beta_-)=0,\ \forall X\in\mathfrak{o}(n).$$

Thus by the previous lemma $[-,-]$ is a minimal vector, and so the complex orbit: $O(n,\mathbb{C})\cdot[-,-]$, is closed as required. □

We define $\mathfrak{g}^{\mathbb{C}}_+(W)$ and $\mathfrak{g}^{\mathbb{C}}_-(W)$ for these complex Lie algebras equipped with the Lie brackets β_+ and β_- respectively. Note that if $\mathfrak{g}^{\mathbb{C}}$ is abelian then $\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}^{\mathbb{C}}_+(W)=\mathfrak{g}^{\mathbb{C}}_-(W)$. Denote κ_+ and κ_- for the Killing forms respectively, then we have:

Proposition 2.3. *If $\kappa=0$ then $\kappa_++\kappa_-=0$.*

Proof. By definition,

$$\begin{aligned}0=\kappa(x,y)&=tr([x,[y,-]])\\ &=tr(\beta_+(x,\beta_+(y,-)+\beta_-(y,-)))+tr(\beta_-(x,\beta_+(y,-)+\beta_-(y,-)))\\ &=\kappa_+(x,y)+\kappa_-(x,y)+tr(\beta_+(x,\beta_-(y,-)))+tr(\beta_-(x,\beta_+(y,-))).\end{aligned}$$

Now since $\tau \cdot \beta_+ = \beta_+$ and $-\tau \cdot \beta_- = \beta_-$ then τ leaves the Killing forms κ_+ and κ_- invariant. Thus it follows that

$$tr(\beta_+(x, \beta_-(y, -))) + tr(\beta_-(x, \beta_+(y, -))) = 0,$$

on $(x, y) \in W \times iW$ or $(x, y) \in iW \times W$. In other words for such pairs (x, y):

$$0 = \kappa_+(x, y) + \kappa_-(x, y),$$

but then the equation also holds on $x, y \in W \oplus iW = \mathfrak{g}^{\mathbb{C}}$. The proposition is proved. $\square$

Clearly we have:

Proposition 2.4. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, then the following are equivalent:*

(1) $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ has a compact real form of Lie algebras.
(2) $\mathfrak{g}^{\mathbb{C}}_-(W)$ is abelian for some compact real form W.
(3) $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right) \cong \left(\mathfrak{g}^{\mathbb{C}}_+(W), g^{\mathbb{C}}\right)$ for some compact real form W.

Proof. (1) $\Rightarrow$ (2). Suppose such a compact real form $\mathfrak{u}$ of Lie algebras exist, then the action of the conjugation map τ w.r.t $\mathfrak{u}$ clearly fixes $[-, -]$, so that $\beta_- = 0$ or equivalently $\mathfrak{g}^{\mathbb{C}}_-(\mathfrak{u})$ is abelian. (2) $\Rightarrow$ (3). If $\mathfrak{g}^{\mathbb{C}}_-(W)$ is abelian for some compact real form W, then w.r.t the conjugation map τ of W we have: $[-, -] = \beta_+(-, -)$, thus trivially: $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right) \cong \left(\mathfrak{g}^{\mathbb{C}}_+(W), g^{\mathbb{C}}\right)$. (3) $\Rightarrow$ (1). Suppose $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right) \cong \left(\mathfrak{g}^{\mathbb{C}}_+(W), g^{\mathbb{C}}\right)$ for some compact real form W. Let $\mathfrak{g}^{\mathbb{C}} \xrightarrow{\phi} \mathfrak{g}^{\mathbb{C}}_+(W)$ be such an isomorphism, then $\phi^{-1}(W)$ is a compact real form of Lie algebras of $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$. The proposition is proved. $\square$

Note that W is a compact real form of Lie algebras in $\left(\mathfrak{g}^{\mathbb{C}}_+(W), g^{\mathbb{C}}\right)$, since $\tau \cdot \beta_+ = \beta_+$, and iW is a compact real form of Lie algebras of $\left(\mathfrak{g}^{\mathbb{C}}_-(W), -g^{\mathbb{C}}\right)$, since $-\tau \cdot \beta_- = \beta_-$. Thus the pairs lend themselves to Theorem 1.1.

Proposition 2.5. *If $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ is quadratic, then so are the pairs: $\left(\mathfrak{g}^{\mathbb{C}}_+(W), g^{\mathbb{C}}\right)$ and $\left(\mathfrak{g}^{\mathbb{C}}_-(W), -g^{\mathbb{C}}\right)$.*

Proof. If $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ is quadratic then $g^{\mathbb{C}}([x, y], z) = g^{\mathbb{C}}(x, [y, z])$ for all $x, y, z \in \mathfrak{g}^{\mathbb{C}}$, thus we obtain:

$$g^{\mathbb{C}}(\beta_+(x, y), z) + g^{\mathbb{C}}(x, \beta_+(y, z)) + g^{\mathbb{C}}(\beta_-(x, y), z) + g^{\mathbb{C}}(x, \beta_-(y, z)) = 0.$$

Now since $g^{\mathbb{C}}(\tau(x), \tau(y)) = \overline{g^{\mathbb{C}}(x, y)}$ for all $x, y \in \mathfrak{g}^{\mathbb{C}}$ then substituting x, y, z in the equation with $\tau(x), \tau(y), \tau(z)$ we get the equation:

$$\overline{g^{\mathbb{C}}(\beta_+(x, y), z) + g^{\mathbb{C}}(x, \beta_+(y, z)) - g^{\mathbb{C}}(\beta_-(x, y), z) - g^{\mathbb{C}}(x, \beta_-(y, z))} = 0.$$

Now taking $x, y, z \in W$ then $g^{\mathbb{C}}(\beta_+(x,y),z) \in \mathbb{R}$ and $g^{\mathbb{C}}(x,\beta_+(y,z)) \in \mathbb{R}$ are real numbers. Thus by combining the two equations proves the result for vectors in W and hence also for $x, y, z \in \mathfrak{g}^{\mathbb{C}}$, using that $\mathfrak{g}^{\mathbb{C}} = W \oplus iW$. ☐

In such a case where $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ is a quadratic Lie algebra then $\mathfrak{g}_+^{\mathbb{C}}(W)$ and $\mathfrak{g}_-^{\mathbb{C}}(W)$ must be reductive. As an example consider the quadratic Lie algebra: $\left(\mathfrak{sl}_2(\mathbb{C})^2, -\kappa \oplus \kappa\right)$. Then for the compact real form $W := \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$ one can easily calculate that $\mathfrak{sl}_2(\mathbb{C})^2_+(W) \cong \mathfrak{sl}_2(\mathbb{C})^2_-(W) \cong \mathfrak{sl}_2(\mathbb{C})^2$. However $\left(\mathfrak{sl}_2(\mathbb{C})^2, -\kappa \oplus \kappa\right)$ does not have a compact real form of Lie algebras.

Proposition 2.6. *If $\mathfrak{g}^{\mathbb{C}}$ is abelian (resp.nilpotent, resp.solvable) then so are $\mathfrak{g}_+^{\mathbb{C}}(W)$ and $\mathfrak{g}_-^{\mathbb{C}}(W)$ respectively.*

Proof. Set $\mathfrak{g} := \mathfrak{g}^{\mathbb{C}}$. *The abelian case.* If $\mathfrak{g}$ is abelian then $\beta_+(x,y) + \beta_-(x,y) = 0$ for all $x, y \in \mathfrak{g}$, thus by the action of τ we also get $\beta_+(x,y) - \beta_-(x,y) = 0$, hence $\beta_+(x,y) = 0 = \beta_-(x,y)$ for all $x, y \in \mathfrak{g}$.

The nilpotent case. Let $\mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k]$ for $\mathfrak{g}^0 = \mathfrak{g}$. Similarly define $\beta_+^{k+1} := \beta_+(\mathfrak{g}, \beta_+^k)$ for $\beta_+^0 = \mathfrak{g}$, and $\beta_-^{k+1} := \beta_-(\mathfrak{g}, \beta_-^k)$ with $\beta_-^0 = \mathfrak{g}$. Suppose $\mathfrak{g}^k = 0$ for some $k \geq 1$. Then by definition:

$$0 = \mathfrak{g}^k = \beta_+(\mathfrak{g}, \mathfrak{g}^{k-1}) + \beta_-(\mathfrak{g}, \mathfrak{g}^{k-1}).$$

Thus also

$$0 = \mathfrak{g}^k = \beta_+(\tau(\mathfrak{g}), \tau(\mathfrak{g})^{k-1}) + \beta_-(\tau(\mathfrak{g}), \tau(\mathfrak{g})^{k-1}),$$

i.e

$$\beta_+(\mathfrak{g}, \mathfrak{g}^{k-1}) = 0 = \beta_-(\mathfrak{g}, \mathfrak{g}^{k-1}).$$

Call this last equation: $\beta_+(\mathfrak{g}, \mathfrak{g}^{k-1}) = 0$ by $(*)$. Using τ we also have,

$$\beta_+(\tau(\mathfrak{g}), \tau(\mathfrak{g})^{k-1}) = \tau(\beta_+(\mathfrak{g}, \beta_+(\mathfrak{g}, \mathfrak{g}^{k-2}) - \beta_-(\mathfrak{g}, \mathfrak{g}^{k-2}))) = 0,$$

thus $\beta_+(\mathfrak{g}, \beta_+(\mathfrak{g}, \mathfrak{g}^{k-2}) - \beta_-(\mathfrak{g}, \mathfrak{g}^{k-2})) = 0$. Finally combining this last equation with $(*)$ we obtain $\beta_+(\mathfrak{g}, \beta_+(\mathfrak{g}, \mathfrak{g}^{k-2})) = 0$. Thus continuing in a similar fashion with this equation using τ we eventually obtain (in a total of k-steps):

$$\beta_+^k = 0.$$

Redefining the equation $(*)$ to be $\beta_-(\mathfrak{g}, \mathfrak{g}^{k-1}) = 0$ instead, we similarly also obtain that $\beta_-^k = 0$.

The solvable case. It is enough to show that $\kappa_+(x_1, \beta_+(x_2, x_3)) = 0 = \kappa_-(x_1, \beta_-(x_2, x_3))$ for all $x_1, x_2, x_3 \in \mathfrak{g}$. But by setting $x := x_1$ and $y := [x_2, x_3]$ into the proof of Proposition 2.3, then

$$\kappa_+(x_1, \beta_+(x_2, x_3)) + \kappa_+(x_1, \beta_-(x_2, x_3)) \\ + \kappa_-(x_1, \beta_+(x_2, x_3)) + \kappa_-(x_1, \beta_-(x_2, x_3)) = 0.$$

Thus by substituting x_1, x_2, x_3 by $\tau(x_1), \tau(x_2), \tau(x_3)$ we obtain the equations,

$$\kappa_+(x_1, \beta_+(x_2, x_3)) + \kappa_-(x_1, \beta_+(x_2, x_3)) = 0$$
$$\kappa_+(x_1, \beta_+(x_2, x_3)) + \kappa_-(x_1, \beta_-(x_2, x_3)) = 0.$$

Finally by taking $x_1, x_2, x_3 \in W$ gives $\kappa_-(x_1, \beta_-(x_2, x_3)) \notin \mathbb{R}$ and $\kappa_+(x_1, \beta_+(x_2, x_3)) \in \mathbb{R}$ since $W \subset \mathfrak{g}^{\mathbb{C}}_+(W)$ and $iW \subset \mathfrak{g}^{\mathbb{C}}_-(W)$ are real forms of Lie algebras, i.e we must have:

$$\kappa_+(x_1, \beta_+(x_2, x_3)) = 0 = \kappa_-(x_1, \beta_-(x_2, x_3)),\ \forall x_1, x_2, x_3 \in \mathfrak{g},$$

as required. The proposition is proved.

□

In the case of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$, the existence of a compact real form of $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ is determined as follows.

Proposition 2.7. *Let $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ be quadratic and $\mathfrak{g}^{\mathbb{C}}$ semisimple where $\mathfrak{g}^{\mathbb{C}} = \oplus_l^N \mathfrak{g}_l$ is its decomposition into simple ideals. Then there exist a compact real form of Lie algebras if and only if $g^{\mathbb{C}}_{|\mathfrak{g}_l} = \lambda_l \kappa_l$, where $\lambda_l < 0$ for all $1 \le l \le N$, and κ_l denotes the Killing form on each simple ideal $\mathfrak{g}_l$.*

Proof. Denote g_l for the restriction of $g^{\mathbb{C}}$ to a simple ideal $\mathfrak{g}_l$. It is clear that g_l is a symmetric non-degenerate invariant bilinear form on $\mathfrak{g}_l$, and it is a standard result that g_l is proportional to the Killing form, i.e $g_l = \lambda_l \kappa_l$ for some $\lambda_l \in \mathbb{C}$. Suppose first the case where $N = 1$ (i.e $\mathfrak{g}^{\mathbb{C}}$ is simple) then $g^{\mathbb{C}} = \lambda\kappa$. So if $\mathfrak{u}$ is a compact real form of $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$, then $\kappa(\mathfrak{u}, \mathfrak{u}) \in \mathbb{R}$, thus $\lambda \in \mathbb{R}$. Now $\lambda < 0$ since if not then $-1_{\mathfrak{u}}$ is a Cartan involution of $\mathfrak{u}$ w.r.t its Killing form. This is impossible. Conversely if $g^{\mathbb{C}}$ is proportional to the Killing form and $\lambda < 0$ then any compact real form $\mathfrak{u}$ of $\left(\mathfrak{g}^{\mathbb{C}}, -\kappa\right)$ is also a compact real form of $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$.

Suppose now the general case that $N > 1$, thus $\mathfrak{g}^{\mathbb{C}}$ is semisimple but not simple. Let $\mathfrak{u} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ be a compact real form of Lie algebras. There are two cases to consider, either $\mathfrak{u}$ is simple (in which case it has a complex structure such that $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{u} \oplus \mathfrak{u}$), or $\mathfrak{u} = \oplus_l^k \mathfrak{u}_l$, where $\mathfrak{u}_l^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is a simple ideal (i.e $\mathfrak{u}_l^{\mathbb{C}} = \mathfrak{g}_l$) or equal to $\mathfrak{u}_l \oplus \mathfrak{u}_l$. See for instance (Thm 6.94, [40]). By the simple case above it is enough to prove the case where $\mathfrak{u}$ is simple with a complex structure (i.e $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{u} \oplus \mathfrak{u}$), and show that such a compact real form does not exist. We note that $\mathfrak{u}$ can be identified as the set $\{(x, x) | x \in \mathfrak{u}\}$ where $\mathfrak{g}^{\mathbb{C}}$ is identified with $\mathfrak{u} \oplus \mathfrak{u}$. Now on each copy of the simple ideal $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$ then $g^{\mathbb{C}}$ is proportional to the Killing form, thus $g^{\mathbb{C}} = \lambda_1 \kappa \oplus \lambda_2 \kappa$. But $g^{\mathbb{C}}$ must be real on $\{(x, x) | x \in \mathfrak{u}\}$ which is impossible. The proposition is proved. □

Thus if $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ is quadratic and $\mathfrak{g}^{\mathbb{C}}$ is semisimple then the compact real forms of Lie algebras (if they exist) are just compact real forms of $(\mathfrak{g}^{\mathbb{C}}, -\kappa)$. For quadratic

Lie algebras we can completely classify the problem for when there exist a compact real form:

Theorem 2.8. *Let* $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ *be quadratic. Then there exist a compact real form of* $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ *if and only if* $\mathfrak{g}^{\mathbb{C}}$ *is reductive and either* $\mathfrak{g}^{\mathbb{C}}$ *is abelian or* $g^{\mathbb{C}}$ *is real and positive definite on any compact real form* $\mathfrak{u}$ *of* $\Big([\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}], -\kappa\Big)$.

Proof. $(\Rightarrow)$. Suppose $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ have a compact real form of Lie algebras $\mathfrak{u}$. Then since $\mathfrak{u}$ is reductive then so is $\mathfrak{g}^{\mathbb{C}}$, thus if $\mathfrak{g}^{\mathbb{C}}$ is not abelian then the derived Lie subalgebra: $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$ must be semisimple. Now restricting $g^{\mathbb{C}}$ to $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$ and the center: $\mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$ we also get quadratic Lie algebras. Moreover $[\mathfrak{u}, \mathfrak{u}]$ is a compact real form of $\Big([\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}], g^{\mathbb{C}}\Big)$, so by the previous proposition the direction follows. $(\Leftarrow)$. Assume $\mathfrak{g}^{\mathbb{C}}$ is reductive, and if $\mathfrak{g}^{\mathbb{C}}$ is abelian the result is clearly true, so assume $\mathfrak{u}$ is a compact real form of $\Big([\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}], -\kappa\Big)$ for which $g^{\mathbb{C}}$ is real and positive definite. Then choosing any compact real form W of $(\mathfrak{z}(\mathfrak{g}^{\mathbb{C}}), g^{\mathbb{C}})$ then $\mathfrak{u} \oplus W$ is a compact real form of $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$. The theorem is proved. $\square$

Thus we obtain in view of the definitions given in Part 6:

Corollary 2.9. *Let* $(G^{\mathbb{C}}, g^{\mathbb{C}})$ *be a holomorphic Riemannian Lie group equipped with a bi-invariant holomorphic metric. Then there exist a Riemannian real slice of Lie groups* $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ *if and only if* $\mathfrak{g}^{\mathbb{C}}$ *is reductive and either* $\mathfrak{g}^{\mathbb{C}}$ *is abelian or* $g^{\mathbb{C}}$ *is real and positive definite on any compact real form:* $\mathfrak{u} \subset \Big([\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}], -\kappa\Big)$.

Part 8. The bibliography

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