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Analyticity in Several Complex Variables

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Contents

Introduction

For any $n \geq 1$, the holomorphy or complex differentiability of a function in the domain of $\mathbb{C}^n$ implies its analyticity. This fact was discovered by Cauchy in the year $1830 - 1840$ and it helps us to explain the nice properties of holomorphic functions. Now, when we come towards the integral representation of holomorphic functions of several variables, it becomes complicated as compare to the situation of one variable and we will see that the simple integral formulas in terms of boundary values will exist only for $\mathbb{C}^n$ domains that are the product of $\mathbb{C}^1$ domains. As the result, the function theory for a ball in $\mathbb{C}^n$ is different from the function theory for a polydisc.

We will see that there are many similarities between complex analysis in several variables and one variable but there are also some important differences between holomorphic functions of a single variable and holomorphic functions of n variables, for $n \geq 2$.

For example in a single variable, for every domain $\Omega \subset \mathbb{C}$ there will be a holomorphic function in Ω which will be holomorphic in no larger domain. But the situation will be different in case of several variables. If there is a bounded domain $\Omega \subset \mathbb{C}^n$ with connected boundary $\partial\Omega$ where $n \geq 2$ then every function holomorphic on a neighborhood of $\partial\Omega$ can be extended to a function holomorphic on Ω. This result was introduced by Friedrich Hartogs[1] and it is known as Hartogs' phenomenon.

Another new and unexpected property of functions of several complex variables is described by Hartogs in his theorem on separate analyticity, which states that for a function of several complex variables, the separate analyticity of a function in each of the variables implies its joint analyticity.

[1]Friedrich Mortiz "Fritz" Hartogs (20 May 1874-18 August 1943) was a German mathematician. His basic work was in several complex variables.

The main object of the thesis is to obtain the two results of Hartogs, *Hartogs phenomenon* and *Hartogs Theorem on separate analyticity* which have been described above.

The thesis is based on reading, understanding, and presenting the corresponding results from [1]-[10].

The thesis comprises four Chapters. In Ch. 1, first, we define complex numbers and holomorphic functions in a single complex variable, then we show that holomorphic functions satisfy Cauchy- Riemann equations. We also describe the Cauchy integral representation of holomorphic functions and it's consequences. We show that holomorphic functions have power series representation and they are also called analytic functions. Then we define *inhomogeneous Cauchy Riemann equations* for a single complex variable. The Cauchy Green formula for a function of class C^1 is proved and then it is shown that Cauchy Green transform provides the solution of inhomogeneous Cauchy-Riemann equation for the case of a single complex variable. At the end of the chapter, a short introduction on Power series, Taylor series, Maclaurin series, and Laurent series is given, and it is proved that a function analytic in an annulus domain can be represented by Laurent series.

In Ch. 2, there are two sections. In the first section, we define harmonic functions in $\mathbb{R}^m$ and give some simple examples of harmonic functions. We describe the relation between harmonic functions and analytic functions in a single complex variable. Then we give some basic properties of harmonic functions and prove that harmonic functions satisfy the mean value property. Likewise, in the second section, we define subharmonic functions which are a class of harmonic functions. Then we give examples of subharmonic functions. In the end, Hartogs' lemma for subharmonic functions is proved which will have an application in the proof of Hartogs' theorem on separate analyticity.

In Ch. 3, there are six sections. In the first section, we define the $\mathbb{C}^n$ space and different types of domains in that space. In section 2, we define holomorphic functions in $\mathbb{C}^n$ in three different ways. Then in the third section, we show that a continuous and separately holomorphic function f in a polydisc can be represented by Cauchy integral formula and a multiple power series, which means that holomorphic functions are analytic. Then by Abel's lemma, we show that analytic functions are holomorphic and we show that a separately holomorphic and continuous function in a polydisc is jointly holomorphic in the disc. At the end of this section, in uniqueness theorem, we prove that if a function f is holomorphic in a domain $D \subset \mathbb{C}^n$ and it vanishes at some point in a nonempty subset of D then it will be 0 all over the D. In the fourth section, we prove the Schwarz lemma, and then we show that a separately holomorphic and bounded function in a polydisc will be continuous at each point of the disc with respect to all the variables. And then we see that a function which is continuous with respect to each variable separately in a polydisc D will be bounded in a smaller polydisc $W \subset D$. We also see that if a separately holomorphic function in a polydisc D is jointly holomorphic in a smaller polydisc $W \subset D$ then it will be jointly holomorphic in the disc D. Then finally it is proved that a separately holomorphic function in a domain D is jointly holomorphic in D which is *the fundamental theorem of Hartogs* or *Hartogs' theorem on separate analyticity*. In section five, we define complete Reinhardt domains and domains of convergence of the power series in $\mathbb{C}^n$ for $n \geq 2$ and prove that the domains of convergence of multiple power series are complete Reinhardt set. And it is also observed that the role of complete Reinhardt domains is the same for functions of several variables as of discs in case of a single complex variable. In the last section, we prove that a holomorphic function f on a connected multicircular domain D can be represented by a uniformly convergent multiple Laurent series in D.

In Ch. 4, first, we prove the original Hartogs' result which says that if a function f is analytic on a Hartogs' figure then it extends analytically on the whole unit bidisc. In the second section, we prove another Hartogs' result which states that if a function is holomorphic in a spherical shell then it extends holomorphically on the whole unit ball. In the third section,we define *the* $\overline{\partial}$*- problem*. Then by using the result from the case of a single complex variable case we show that the inhomogeneous Cauchy-Riemann equation has a unique solution of class with compact support. In section 4, there is a short introduction to smooth approximate identities and cutoff functions. Finally, in section 5, we prove the general *Hartogs' phenomenon*.

For the sake of brevity, some of the proofs are presented for the functions of two complex variables. The proofs for the general case are essentially the same.

Chapter 1

Analyticity in One Complex Variable

In this chapter, some basic concepts and results for analytic functions in case of a single complex variable are presented which will be used as a tool for analytic functions in case of several complex variables.

1.1 Review of analyticity in one complex variable

This section is based on [7]

The Complex analysis deals with complex numbers so it will be wise to introduce complex numbers. To get the set of complex numbers which is denoted by $\mathbb{C}$, we add $\sqrt{-1}$ in the set of real numbers. We call this square root i. And we write the complex number z as

$$z = x + iy,$$

where $z \in \mathbb{C}$ and $(x, y) \in \mathbb{R}^2$.

An important transformation of a complex number is its conjugate and is defined as

$$\overline{z} = x - iy.$$

The size of z can be measured by taking its modulus

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

For a given complex number $z = x + iy$ for $x, y \in \mathbb{R}$, x is called the real part and y is called the imaginary part and these can be written as follows

$$Re(z) = Re(x + iy) = \frac{z + \overline{z}}{2} = x, Im(z) = Im(x + iy) = \frac{z - \overline{z}}{2} = y.$$

In order to describe holomorphic functions we use continuously differentiable function, which is a function $f : U \subset \mathbb{R}^n \to \mathbb{C}$ whose first(real) partial derivatives exist and are continuous. Such functions are denoted by C^1.

Holomorphic Functions : A function $f : U \to \mathbb{C}$ for an open set $U \subset \mathbb{C}$, is said to be holomorphic in U if it is complex differentiable at every point of U which means that

$$f'(z) = \lim_{\xi \in \mathbb{C} \to 0} \frac{f(z + \xi) - f(z)}{\xi} \quad \forall \quad z \in U.$$

Here it is important to note that ξ is complex.

Another way is to start with a continuously differentiable function $f = u + iy$,and we say that this function will be holomorphic if it will satisfy the following Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

In order to understand the above equations , we can take the help of following operators

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

The above operators are called Wirtinger operators and to determine these operators we assume that

$$\frac{\partial}{\partial z}z = 1, \qquad \frac{\partial}{\partial z}\overline{z} = 0 \qquad \frac{\partial}{\partial \overline{z}}\overline{z} = 1.$$

The function f will be holomorphic iff it will depend only on z, which can be describe as a single complex equation.

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

We can check it for a given function $f = u + iv$ as follows.

By applying one of the Wirtinger operator on f

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$$

$$= \frac{1}{2}(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y})$$

$$= \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + \frac{i}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}).$$

The above expression will be zero if

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

From the last above expression, we can see that Cauchy Riemann equations are satisfied.

Now it can be seen that,if the function will be holomorphic then the derivative in z will be standard complex derivative. By applying one of the Wirtinger operator and Cauchy Riemann equations we can show it as follows

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x},$$

$$= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial f}{\partial (iy)}.$$

Cauchy Integral Formula and some useful results

The following part is based on [7], [1].

It is one of the most important formulas in one variable.

Theorem 1.1.1. *Let $U \subset \mathbb{C}$ be a bounded domain and the boundary ∂U is piecewise smooth simple closed curve. Let $f : \overline{U} \to \mathbb{C}$ be a continuous and holomorphic function in U. Then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(\xi)}{\xi - z} d\xi \qquad \forall z \in U,$$

where ∂U is oriented positively.

Cauchy Integral Formula For Derivatives

Theorem 1.1.2. *If there is a function f which is analytic inside and on a simple closed positively oriented contour Γ and if z is any point inside Γ,then*

$$f^n(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi \quad , \quad n = 1, 2, ...$$

The above formula can also be written as

$$\int_\Gamma \frac{f(z)}{(z - z_0)^n}dz = \frac{2\pi i}{(n-1)!}f^{(n-1)}(z_0), \quad n = 1, 2, ...$$

where z_0 lies inside Γ.

Maximum Modulus Principle

It is another important result which follows from Cauchy integral.

Let there is a function $f(z)$, which is analytic in an open disc centered at z_0 and $|f(z_0)|$ is the maximum value of $|f(z)|$ over this disc, then $|f(z)|$ will be constant in that disc.

Theorem 1.1.3. *If a function f will be analytic in a domain S and $|f(z)|$ achieves its maximum value at a point z_0 in s, then f will be constant in S.*

Theorem 1.1.4. *If a function will be holomorphic in a bounded domain and also will be continuous up to and including its boundary, then it will attain its maximum modulus on the boundary.*

Theorem 1.1.5. (Cauchy - Goursat Theorem) If f is analytic in a simple connected domain D, and inside D there is a simple closed rectifiable contour γ, then

$$\int_\gamma f(z)dz = 0.$$

Theorem 1.1.6. (Morera's Theorem) If a function f is continuous in a domain D and $\int_\gamma f(z)dz = 0$ for every closed contour γ in D, then f is holomorphic in D.

Theorem 1.1.7. *Let there are two positively oriented simple closed contour γ_1 and γ_2 ,with γ_2 interior to γ_1. If a function f is analytic on the closed reigon containig $\{\gamma_1\}$ and $\{\gamma_2\}$ and the points between them, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Theorem 1.1.8. *Holomorphic functions can be represented by a uniformely convergent power series in z at each point a*

$$f(z) = \sum_{j=0}^{\infty} c_j (z-a)^j.$$

Such type of functions which can be represented by power series are called analytic functios. So we can use the terms holomorphic and analytic interchangeably and holomorphic functions are also called analytic functions.

Proof. We can prove it by using the Cauchy integral formula, for a disc of radius $\rho \geq 0$ around the centre $a \in \mathbb{C}$. The disc is defined as follows

$$\Delta_\rho(a) = \{z \in \mathbb{C}\}, |z-a| \leq \rho\}$$

Suppose that $f : U \to \mathbb{C}$ is holomorphic, U is open, $a \in U$ and $\overline{\Delta_\rho(a)} \subset U$ (which means that the boundary$\partial\Delta_\rho(a)$ is also in U).

For $z \in \Delta_\rho(a)$ and $\xi \in \partial\Delta_\rho(a)$

$$\left|\frac{z-a}{\xi-a}\right| = \frac{|z-a|}{\rho}.$$

Here $\left|\frac{z-a}{\xi-a}\right| \leq \frac{\rho'}{\rho} < 1$, if $|z-a| \leq \rho' < 1$,

then the geometric series

$$\sum_{j=0}^{\infty} \left(\frac{z-a}{\xi-a}\right)^j = \frac{1}{1-\frac{z-a}{\xi-a}} = \frac{\xi-a}{\xi-z}$$

will converge uniformly and absolutely for $(z,\xi) \in \overline{\Delta\rho'(a)} \times \partial\Delta_\rho(a)$. Let us compute the integral from Cauchy formula.

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi-a} d\xi$$

where γ is the path going around $\partial\Delta_\rho(a)$ in the positive direction.

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)(\xi-a)}{(\xi-z)(\xi-a)} d\xi$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi-a}\sum_{j=0}^{\infty}\left(\frac{z-a}{\xi-a}\right)^j d\xi$$

$$= \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - a)^{j+1}} d\xi \right) (z-a)^j,$$

whereas we can interchange the limit on sum because of the uniform convergence of the series. If z is fix and M is supremum of $\left|\frac{f(\xi)}{(\xi - a}\right| = \frac{|f(\xi)|}{\rho}$ on $\Delta_\rho(a)$ then

$$\left|\frac{f(\xi)}{\xi - a}\right| \left(\frac{z-a}{\xi - a}\right)^j \leq M \left(\frac{|z-a|}{\rho}\right)^j \quad and \quad \frac{|z-a|}{\rho} < 1.$$

So the function $f(z)$ can also be represented by the power series

$$f(z) = \sum_{j=0}^{\infty} c_j (z-a)^j.$$

We also have computed that the radius of convergence will be atleast ρ, where ρ is the maximum ρ such that$\Delta_\rho(a) \subset U$ and

$$c_j = \frac{1}{2\pi i} \int \frac{f(\xi)}{(\xi - a)^{j+1}} d\xi$$

is the formula for the coefficients c_j of series. From here we also obtain Cauchy inequalities

$$|c_j| \leq \frac{M}{\rho^j} \quad .$$

□

A function $f : \mathbb{C} \to \mathbb{C}$ is called entire function if f is a holomorphic function in $\mathbb{C}$.

Theorem 1.1.9. (Liouville) If a function f is entire and bounded then f is constant.

Theorem 1.1.10. *If there is a domain $U \subset \mathbb{C}$ and a holomorphic function $f : U \to \mathbb{C}$ such that the zero set $f^{-1}(0)$ has a limit point in U, then $f \equiv 0$.*

1.2 Inhomogeneous Cauchy-Riemann equation for single variable

This section is based on [6].

Let v be a function in $\mathbb{C}$. Consider the equation

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{i\partial y}\right) = v \quad on \quad \mathbb{C}. \tag{1.1}$$

This is the *inhomogeneous Cauchy-Riemann equation* for a single complex variable.

Support of a function: The support of a function f is the smallest closed set outside of which the function is equal to zero.

In equation (1.1), $v(z) = v(x+iy)$ is a function of class C^1 with compact support.

Pompeiu's Formula or Cauchy-Green Formula

Proposition 1.2.1. *Let D be a bounded domain in $\mathbb{C}$ whose boundary Γ consists of finitely many piecewise smooth curves and D lies to the left of Γ. If $f(z) = f(x+iy)$ be a function of class C^1 on $\overline{D}$ then*

$$f(a) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{(z-a)}dz - \frac{1}{\pi}\int_{D} \frac{\partial f}{\partial \overline{z}}\frac{1}{z-a}dxdy \quad \forall a \in D. \tag{1.2}$$

In the proof of this formula, the Cauchy Green's formula for integration by parts will be used ,which is

$$\int_{\partial D} Ldx + Mdy = \int_{D}\left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)dxdy$$

where L and M are the continuously differentiable functions over $\overline{D}$ and ∂D is the oriented boundary of D. In order to obtain a complex form of the Green's formula ,we put $L = F$ and $M = iF$ in the above formula with $F(z) = F(x+iy)$ in $C^1(\overline{D})$

$$\int_{\partial D} F(z)dz = \int_{\partial D} Fdx + iFdy = \int_{D}\left(i\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)dxdy. \tag{1.3}$$

$$= 2i\int_{D} \frac{\partial F}{\partial \overline{z}}dxdy.$$

Proof. In order to prove (1.2), we will apply (1.3) to the function

$$F(z) = \frac{f(z)}{(z-a)} \quad a \in D.$$

But the above function is not smooth at $z = a$. So we will apply Green's formula to F on

$$D_\epsilon = D - \overline{B}_\epsilon$$

where $\overline{B}_\epsilon = \overline{B}(a, \epsilon)$ is a closed disc of radius $\epsilon < d(a, \Gamma)$. The boundary ∂D_ϵ will consist of Γ and the circle $-C(a, \epsilon)$.

Since $\frac{1}{(z-a)}$ is holomorphic throughout the $\overline{D}_\epsilon$, so

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} + f(z) \frac{\partial}{\partial \overline{z}} \frac{1}{z-a} = \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} \quad , z \in \overline{D}_\epsilon. \tag{1.4}$$

Now by applying Green's formula (1.3) for $F(z)$

$$\int_\Gamma \frac{f(z)}{z-a} dz + \int_{-C(a,\epsilon)} \frac{f(z)}{(z-a)} dz = 2i \int_{D_\epsilon} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} dxdy. \tag{1.5}$$

Since, on $C(a, \epsilon)$, $z = a + \epsilon e^{it}$ and $dz = \epsilon i e^{it} dt$

so,

$$\int_{-C(a,\epsilon)} \frac{f(z)}{z-a} dz = -\int_0^{2\pi} \frac{f(a + \epsilon e^{it})}{a + \epsilon e^{it} - a} \epsilon i e^{it} dt$$

when $\epsilon \to 0$

$$\int_{-C(a,\epsilon)} \frac{f(z)}{z-a} dz = -2\pi i f(a). \tag{1.6}$$

Furthermore, since $\frac{\partial f}{\partial \overline{z}}$ is continuous on $\overline{D}$ and let M is a bound for $|\frac{\partial f}{\partial \overline{z}}|$ on $\overline{D}$, then

$$\left| \int_D \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} dxdy - \int_{D_\epsilon} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} dxdy \right| = \left| \int_{\overline{B}_\epsilon} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} dxdy \right| \leq M \int_{\overline{B}_\epsilon} \frac{1}{|z-a|}$$

$$= M \int_0^\epsilon \int_{-\pi}^{\pi} \frac{1}{r} r dr dt = M 2\pi\epsilon.$$

When $\epsilon \to 0$

$$\left| \int_D \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} dxdy - \int_{D\epsilon} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} dxdy \right| = 0. \tag{1.7}$$

From Equation (1.6) and (1.7), Equation (1.5) will become

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz - \frac{1}{\pi} \int_D \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} dxdy$$

which is the required result. □

Corollary 1.2.2. *Any C^1 function $f(z) = f(x+iy)$ on $\mathbb{C}$ of compact support can be represnted by the following*

$$f(z) = \frac{-1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}(\zeta)\frac{1}{\zeta - z}d\varepsilon d\eta \quad \textit{where} \quad \zeta = \varepsilon + i\eta.$$

Proof. Let there is a disc $D = B(0, R)$ which contains a fix $a \in \mathbb{C}$ and support of f. And let Γ be the boundary of B, then the integral over Γ in (1.2) will vanish and Pompeiu's formula will become

$$f(a) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}\frac{1}{z - a}dxdy.$$

Finally if we replace z by $\zeta = \varepsilon + i\eta$ and then a by z

$$f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}(\zeta)\frac{1}{\zeta - z}d\varepsilon d\eta.$$

□

Now we will show that the equation $\frac{\partial u}{\partial \overline{z}} = v$ has a solution and it will be given by the following

$$u(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{v(\zeta)}{\zeta - z}d\varepsilon d\eta \quad \forall z \in \mathbb{C}, \tag{1.8}$$

which is called the Cauchy-Green transform u of v.

Theorem 1.2.3. *Let v be a C^p function $(1 \leq p \leq \infty)$ on $\mathbb{C}$ of compact support. Then the Cauchy-Green transform u of v provides a C^p solution of the equation $\frac{\partial u}{\partial \overline{z}} = v$ on $\mathbb{C}$. This solution is unique and smooth and it tends to 0 as $|z| \to \infty$.*

Proof. If we replace ζ by $\zeta' + z$ in transformation (1.8), then

$$u(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{v(z + \zeta')}{\zeta'}d\varepsilon d\eta.$$

Rewrite the above expression without the prime

$$u(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{v(z + \zeta)}{\zeta}d\varepsilon d\eta. \tag{1.9}$$

In order to get the first order partial derivative of u with respect to x we will differentiate u under the integral sign. Let there is a fixed large disc $D(0, R)$, which contains a fixed a. We will vary $z = a + h$ over a small disc $D(0, r)$, since the function $v(z + \zeta)$ will be 0 for all ζ outside the $D(0, R)$. Here h is real and $h \neq 0$, So that

$$\frac{1}{h}\int_0^h \left\{\frac{\partial v}{\partial x}(a+t+\zeta) - \frac{\partial v}{\partial x}(a+\zeta)\right\}dt = \frac{v(a+h+\zeta) - v(a+\zeta)}{h} - \frac{\partial v}{\partial x}(a+\zeta).$$

The left hand side of above equation is a function of h and ζ and it will tends to zero as $h \to 0$ uniformly in ζ, because $\frac{\partial v}{\partial x}$ is a continuous and of compact support function. Now if we multiply the above expression by the absolutely integrable function $\frac{1}{\zeta}$ over $B(0, R)$ and integrate it over B then ,

$$0 = \int_B \frac{v(a+h+\zeta) - v(a+\zeta)}{h}\frac{1}{\zeta}d\varepsilon d\eta - \int_B \frac{\partial v}{\partial x}(a+\zeta)\frac{1}{\zeta}d\varepsilon d\eta.$$

By using (1.9), the above expression becomes

$$-\pi\frac{u(a+h) - u(a)}{h} = \int_B \frac{\partial v}{\partial x}(a+\zeta)\frac{1}{\zeta}d\varepsilon d\eta$$

and hence

$$-\frac{1}{\pi}\frac{\partial u}{\partial x}(a) = \int_B \frac{\partial u}{\partial x}(a+\zeta)\frac{1}{\zeta}d\varepsilon d\eta.$$

So the partial derivative of u with respect to x exists at a and it will be continuous because $\frac{\partial v}{\partial x}$ is uniformly continuous.

Similarly we can find the partial derivative with respect to y. So u is a continuously differentiable function and if we combine the both partial derivatives, then

$$\frac{\partial u}{\partial \overline{z}}(a) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial v}{\partial \overline{z}}(a+\zeta)\frac{1}{\zeta}d\varepsilon d\eta$$

which can also be written as

$$\frac{\partial u}{\partial \overline{z}} = \frac{-1}{\pi}\int_{\mathbb{C}} \frac{\partial v}{\partial \overline{z}}(\zeta)\frac{1}{\zeta - a}d\varepsilon d\eta.$$

As v is a continuously differentiable function of bounded support, so by corollary 1.2.2

$$\frac{\partial u}{\partial \overline{z}} = v(a).$$

Hence the Cauchy transform u of v satisfies the inhomogeneous Cauchy Reimann equation which means it is the solution of inhomogeneous Cauchy Reimann equation.

Likewise for $p \geq 2$,it can be proved that higher order partial derivatives of u exist and are continuous on $\mathbb{C}$. Hence Cauchy transform u of v provides a C^p solution of (1.1). It can also be seen from (1.8), that $u(z) \to 0$ as $|z| \to \infty$ and this solution is the unique smooth solution of (1.1).

□

1.3 Series

This section is based on [1]

Power Series

An infinite series of the form

$$\sum_{j=0}^{\infty} c_j(z-z_0)^j = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + ... + c_j(z-z_0)^j ...$$

is called a power series. Here z_0 is point of expansion and the constants c_j are called the coefficients of the power series. This series converges at z_0. If this series converges absolutely for$|z-z_0| < R$ and diverges for $|z-z_0| > R$,then R is called the radius of convergence. The following theorem determines the domain of convergence of the power series.

Theorem 1.3.1. *If the power series $\sum_{j=0}^{\infty} c_j(z-z_0)^j$ converges at $z = z_1(\neq z_0)$, then it converges uniformly and absolutely in the closed disc $\overline{\Delta}(z_0, r)$ where $|z_1 - z_0| > r$.*

Theorem 1.3.2. (Weierstrass's M-Test) Let $|f_j(z)|$ be a sequence of functions and $\sum_{j=0}^{\infty} M_j$ be a convergent series of positive numbers such that $|f_j(z)| \leq M_j$ for all z on a domain D and $j \geq 0$, then $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly and absolutely on D.

If R is the radius of convergence of the above power series then $g(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j$ is an analytic function on the disc $D(z_0, R)$ and by term by term differentiation

$$g'(z) = \frac{d}{dz}\sum_{j=0}^{\infty} c_j(z-z_0)^j = \sum_{j=0}^{\infty} jc_j(z-z_0)^{j-1}.$$

This series is infinitely differentiable for any $z \in D(z_0, R)$ and for any n

$$g^n(z) = \sum_{j=n}^{\infty} j(j-1)...(j-n+1)c_j(z-z_0)^{(j-n)}.$$

At $z = z_0$

$$c_n = \frac{g^n(z_0)}{n!}$$

for any $n = 0, 1, 2...$

Taylor Series

The series of the form

$$f(z) = f(z_0) + \frac{f^{'}(z_0)}{1!}(z-z_0) + \frac{f^{''}(z_0)}{2!}(z-z_0)^2 + ... = \sum_{j=0}^{\infty} \frac{f^j(z_0)}{j!}(z-z_0)^j$$

is called the Taylor series of f at z_0. It follows that if $f(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j$ for all $z \in D(z_0, R)$,then this series is the Taylor series of $f(z)$ at z_0.

Maclaurin Series

Taylor's series reduces to the following series at $z_0 = 0$

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2... = \sum_{j=0}^{\infty} \frac{f^j(0)}{j!}z^j$$

which is called the Maclaurin series of f.

Laurent Series Now we will see that a function that is analytic in an annulas domain can be expanded in a series, this series is called the Laurent series. We will use this series while proving the Hartogs' phenomenon.

Theorem 1.3.3. *If a function $f(z)$ is analytic in an annulus domain $D = \{z : r < |z-z_0| < R\}$, then this function can be represented by the Laurent series.*

$$f(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j + \sum_{j=1}^{\infty} \frac{d_j}{(z-z_0)^j} \quad \forall z \in D \tag{1.10}$$

where

$$c_j = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\xi)}{(\xi-z_0)^{j+1}}d\xi \quad j = 0, 1, 2...$$

and

$$d_j = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - z_0)^{j-1} d\xi \quad j = 1, 2, ...$$

and γ is positively oriented simple closed contour around z_0 and lying in domain D.

Proof. Since function is analytic in the domain $D = \{z : r < |z - z_0| < R\}$, so from Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\xi)}{\xi - z} d\xi \tag{1.11}$$

where γ_1 and γ_2 are circles with centre at z_0 and contained in D,with γ_1 lies interior to γ_1 and γ lies between γ_2 and γ_1.

It follows that

$$For \quad \xi \in \gamma_2, \quad \left|\frac{z - z_0}{\xi - z_0}\right| < 1$$

and

$$\begin{aligned} \frac{1}{\xi - z} &= \frac{1}{\xi - z - z_0 + z_0} \\ &= \frac{1}{\xi - z_0(1 - \frac{z - z_0}{\xi - z_0})} \\ &= \frac{1}{\xi - z_0}\sum_{j=0}^{\infty}\Big(\frac{z - z_0}{\xi - z_0}\Big)^j \end{aligned}$$

which converges uniformly and hence

$$\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\int_{\gamma_2} f(\xi)\sum_{j=0}^{\infty}\frac{(z - z_0)^j}{(\xi - z_0)^{j+1}} d\xi$$

$$= \sum_{j=0}^{\infty}\Big[\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0)^{j+1}} d\xi\Big](z - z_0)^j$$

$$\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\xi)}{(\xi - z)} d\xi = \sum_{j=0}^{\infty} c_j (z - z_0)^j. \tag{1.12}$$

For

$$\xi \in \gamma_1, \quad \left|\frac{\xi - z_0}{z - z_0}\right| < 1$$

and

$$
\begin{aligned}
\frac{1}{z-\xi} &= \frac{1}{z-z_0+z_0-\xi} \\
&= \frac{1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{\xi-z_0}{z-z_0}\right)^j,
\end{aligned}
$$

the above series converges uniformly and absolutely on γ_1. Hence we have

$$
\begin{aligned}
-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\xi)}{\xi-z}d\xi &= \frac{1}{2\pi i}\int_{\gamma_1} f(\xi)\sum_{j=0}^{\infty}\frac{(\xi-z_0)^j}{(z-z_0)^{j+1}}d\xi \\
&= \sum_{j=1}^{\infty}\left[\frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-z_0)^{j-1}\right]\frac{1}{(z-z_0)^j} \\
-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\xi)}{\xi-z}d\xi &= \sum_{j=0}^{\infty}\frac{d_j}{(z-z_0)^j}.
\end{aligned} \tag{1.13}
$$

By using Equations (1.12) and (1.13) into Equation (1.11), we get the required representation. □

If the function will be holomorphic in $D(z_0, R)$ then $\frac{f(\xi)}{(\xi-z_0)^{j-1}}$ will be holomorphic in $D(z_0, R)$ and the second sum in Equation (1.10) which is called the principal part of Laurent series will become 0 by Theorem 1.1.5 and hence the Laurent series will become the Taylor series of $f(z)$

Chapter 2

Basics on Harmonic and Subharmonic Functions

For working with analytic functions of several variables, we will need some tools from the theory of subharmonic functions, which we present in this chapter. This section is based on [2] and [10].

2.1 Harmonic Functions

2.1.1 Definition and Examples

Definition 2.1.1. *Let Ω be a nonempty open subset of $\mathbb{R}^m$. A real valued function u of class C^2, defined on Ω is said to be harmonic on Ω if*

$$\Delta u = 0$$

where $\Delta = D_1^2 + ...D_m^2$.

Examples

Let $x = (x_1, x_2, ...x_m)$ be a typical point in $\mathbb{R}^m$ and $|x| = (x_1^2 + ... + x_m^2)^{\frac{1}{2}}$ be the Euclidean norm of x

1. In $\mathbb{R}$, harmonic functions are those whose second derivative equals zero, so they are just linear functions $u(x) = ax + b$.

2. If $m = 2$, the function

$$u(x) = \ln |x|$$

is harmonic in $\mathbb{R}^2 \setminus \{0\}$.

3. $u(x) = |x|^{2-m}$ is harmonic on $\mathbb{R}^m \setminus \{0\}$ for $m > 2$.

2.1.2 Harmonic functions and Analytic functions in $\mathbb{R}^2 \sim \mathbb{C}$

For an analytic function $f(z) = u(x, y) + iv(x, y)$ on a region A, both u and v are harmonic functions on A. This is a consequence of Cauchy-Riemann equations. u and v are called the harmonic conjugates.

Theorem 2.1.2. *If there is a harmonic function $u(x, y)$ on a simply connected region A, then u is the real part of an analytic function $f(z) = u(x, y) + iv(x, y)$.*

From the above theorem, it follows that u will be infinitely differentiable.

2.1.3 Properties of Harmonic Functions

The Mean Value property

If u is harmonic on $\overline{B}(a, r)$ then u equals the average of u over $\partial B(a, r)$. More precisely

$$u(a) = \int_S u(a + r\zeta) d\sigma(\zeta). \tag{2.1}$$

In order to prove this property we will use Green's identity

$$\int_\Omega (u\Delta v - v\Delta u) dV = \int_{\partial\Omega} (uD_{\mathbf{n}} v - vD_{\mathbf{n}} u) ds. \tag{2.2}$$

Here Ω is a bounded open subset of $\mathbb{R}^m$, u and v are C^2 functions on a neighbourhood of $\overline{\Omega}$, V is Lebesgue volume measure on $\mathbb{R}^m$, s is the surface area measure on $\partial\Omega$ and $D_{\mathbf{n}}$ denotes the differentiation with respect to outward unit normal $\mathbf{n}$. For gradient of u,$\nabla u = (D_1 u, ...D_m u)$ and $\zeta \in \partial\Omega$

$$(D_{\mathbf{n}} u)(\zeta) = (\nabla u)(\zeta) \cdot \mathbf{n}(\zeta).$$

For a harmonic function u and $v \equiv 1$,the Green's identity becomes

$$\int_{\partial\Omega} D_{\mathbf{n}} u ds = 0. \tag{2.3}$$

In the proof of mean value property we will use the Green's identity for the unit ball B. The boundary of B is denoted by S which is the unit sphere and σ is the normalized surface area measure on S so that $\sigma(S) = 1$.

Proof. (Mean Value Property) Let $m > 2$.

let there is a unit ball B and a fix $\varepsilon \in (0,1)$. If we apply Green's identity with $\Omega = \{x \in \mathbb{R}^m : \varepsilon < |x| < 1\}$ and $v(x) = |x|^{2-m}$, we obtain

$$0 = (2-m)\int_S uds - (2-m)\varepsilon^{1-m}\int_{\varepsilon S} uds - \int_S D_{\mathbf{n}}uds - \varepsilon^{2-m}\int_{\varepsilon S} D_{\mathbf{n}}uds.$$

By (2.3), the last equation will become

$$\int_S uds = \varepsilon^{1-m}\int_{\varepsilon S} uds,$$

because $S = \partial B$ and $\epsilon S = \partial\epsilon B$ and $\zeta \in \partial B$, so the last equation can be written as

$$\int_S ud\sigma = \int_S u(\varepsilon\zeta)d\sigma(\zeta).$$

Since u is continuous at 0 and if we assume $\varepsilon \to 0$, then

$$u(0) = \int_S u(\zeta)d\sigma(\zeta). \tag{2.4}$$

For $m = 2$, the function $v(x) = \ln|x|$ should be chosen. □

Mean Value property, Volume Version

Harmonic functions also have mean value property with respect to volume,which states that, if u is a harmonic function on $\overline{B}(a,r)$ then $u(a)$ equals to the average of u over $B(a,r)$

$$u(a) = \frac{1}{V(B(a,r))}\int_{B(a,r)} udV. \tag{2.5}$$

Proof. We will prove it for the unit ball B. The polar co-ordinate formula for a Borel measurable integrable function f on $\mathbb{R}^m$ states that

$$\frac{1}{mV(B)}\int_{\mathbb{R}^m} fdV = \int_0^\infty r^{m-1}\int_S f(r\zeta)d\sigma(\zeta)dr. \tag{2.6}$$

By using (2.6) for u over B

$$\frac{1}{V(B)}\int_B udV = \int_S u(\zeta)d\sigma(\zeta).$$

And hence by (2.4)

$$\frac{1}{V(B)}\int_B udV = u(0). \tag{2.7}$$

□

The Maximum Principle

Following are the different versions of maximum principle

1. Let Ω be connected and u be a harmonic function on Ω. If $|u|$ has maximum in Ω, then u is constant.

2. Let Ω be connected and u be a function on $\overline{\Omega}$ that is harmonic on Ω. Then $|u|$ attains maximum value over $\overline{\Omega}$ on $\partial\Omega$.

3. Let u be a harmonic function on Ω and suppose that

$$\limsup_{k\to\infty} u(a_k) \leq M$$

for every sequence a_k in Ω converging either to point in $\partial\Omega$ or to ∞. Then $u \leq M$ on Ω.

The Poisson Kernel for the Ball

For every harmonic function u on $\overline{B}$ and $x \in B$, $u(x)$ is a weighted average of u over S. More precisely there exists a function P on $B \times S$ such that

$$u(x) = \int_S u(\zeta)P(x,\zeta)d\sigma(\zeta). \tag{2.8}$$

The function P in the above integral is called the Poisson Kernel for the ball and it is

$$P(x,\zeta) = \frac{1-|x|^2}{|x-\zeta|^n}.$$

Uniformly convergent sequence of harmonic functions

Theorem 2.1.3. *If a sequence $\{u_j\}$ of harmonic functions on Ω converges uniformly to a function u on each compact subset of Ω, then u is harmonic on Ω. And for every multi-index α, $D^\alpha\{u_j\}$ converges uniformly to $D^\alpha u$ on each compact subset of Ω.*

Converse of the Mean-Value Property

Theorem 2.1.4. *Let u is a locally integrable function on Ω such that*

$$u(a) = \frac{1}{V(B(a,r))} \int_{B(a,r)} u dV$$

whenever $\overline{B}(a,r) \subset \Omega$,then u is harmonic on Ω

From the above theorem it follows that the Mean Value property characterizes harmonic functions.

2.2 Subharmonic Functions

Most of the part of this section is based on [10]

2.2.1 Definition and Examples

Definition 2.2.1. *A function $f(x)$, defined on a set $S \subset \mathbb{R}^m$ with values in $[-\infty, \infty)$ is said to be upper semi continuous at a point $x_0 \in S$ if for every number $L > f(x_0)$ there exists a number $\delta = \delta(x_0, L)$ such that $f(x) < L$ whenever $|x - x_0| < \delta$ and $x \in S$. If f is continuous at each point of S then it is said to be upper semi continuous on S.*

Let for $x \in \mathbb{R}^m$ there is a function $u(x)$, with values in $[-\infty, \infty)$. Further suppose that $u(x)$ is measurable and bounded above on the sphere

$$S_r(x_0) = \{x : |x - x_0| = r\}.$$

The average of the function $u(x)$ on the sphere $S_r(x_0)$ is

$$Av_u(S(x_0, r)) = \frac{1}{\sigma_m} r^{m-1} \int_{S_r(x_0)} u(x) d\sigma \tag{2.9}$$

where σ_m is the area of the unit sphere in $\mathbb{R}^m$

$$\sigma_m = \frac{m\pi^{\frac{m}{r}}}{\Gamma(\frac{m}{r} + 1)}$$

and $d\sigma$ is the area element on the sphere $S_r(x_0)$.

The average of the measurable and bounded above function $u(x)$ in the ball $B_r(x_0)$ is

$$Av_u(B(x_0, r)) = \frac{1}{V_m r^m} \int_{B_r(x_0)} u(x) dV \tag{2.10}$$

where V_m is the volume of the unit ball and dV is the volume element in $\mathbb{R}^m$ and

$$V_m = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2} + 1)}.$$

By using the value of σ_m and V_m in the Equations (2.9) and (2.10) respectively, it can be seen that

$$Av_u(B(x_0, r)) = \frac{m}{r^m} \int_0^r Av_u(S(x_0, t)) t^{m-1} dt. \tag{2.11}$$

In the last section we have seen that the necessary and sufficient condition for a function $u(x)$ to be harmonic is

$$u(x) = Av_u(B(x, r)). \tag{2.12}$$

Or

$$u(x_0) = \frac{1}{V(B(x_0, r))} \int_{B(x_0,r)} u dV.$$

Definition 2.2.2. *A function $u(x)$ defined in a domain $D \subset \mathbb{R}^m$ is said to be subharmonic in D if is upper semi continuous in D and, for any point $x \in D$ and all sufficiently small positive r,*

$$u(x_0) \leq \frac{1}{V(B(x_0, r))} \int_B u dV. \tag{2.13}$$

So subharmonic functions are obtained by replacing the equality sign in (2.12) by an inequality sign. It follows from definition that every harmonic function is also subharmonic

Examples

1. For $m > 2$ the function $u(x) = -|x|^{2-m}$ is subharmonic. This is an upper semi continuous. It can also be noted that this function satisfies the inequality (2.13). Since $u(x)$ is harmonic everywhere except at the origin. Thus at every point $x \neq 0$ the inequality in (2.13) will become equality for all $r \in (0, |x|)$. The inequality (2.13) also holds clearly at $x = 0$.

2. For $m = 2$, the primary example of a subharmonic function is $u(z) = \log|f|$, where $f(z)$ is an analytic function. In this case also,u is an upper semi continuous function. In order to prove that u is a subharmonic function it will be sufficient to show that the inequality (2.13) holds for u. The case is trivial for all z such that $f(z) = 0$. When $f(z) \neq 0$ at some point z_0, then there is an analytic branch of $\log f$ around z such that $u(z) = Re(\log(f(z))$ is harmonic around z_0 and hence (2.13) holds with the sign of equality.

2.2.2 Properties and Application of Subharmonic functions

Simple properties of subharmonic functions

1. The product of a subharmonic function and a constant will also be a subharmonic function.

2. The sum of finitely many subharmonic functions will be a subharmonic function.

3. If the functions $u_1(x), ..., u_n(x)$ are subharmonic in a domain $D \subset \mathbb{R}^m$,then the function $u(x) = \max_{1\leq k\leq n} u_k(x)$ will also be subharmonic in D.

4. The limit of a uniformly convergent sequence of subharmonic functions will be a subharmonic function.

5. The limit of a monotonically decreasing sequence of subharmonic functions will be a subharmonic function.

The next result, known as Hartogs' lemma, will be used in Chapter 3 for proving Hartogs' theorem. This lemma follows [4] and [8]

Lemma 2.2.3. *(*Hartogs' lemma) Let $\{u_j\}_{j=1}^{\infty}$ be a sequence of subharmonic functions in a domain $\Omega \subset \mathbb{R}^m$, which are uniformly bounded above on every compact subset K of Ω and assume that $\limsup_{j\to\infty} u_j(x) \leq C$ for each $x \in \Omega$, then for any $\epsilon > 0$ and any compact set $K \in \Omega$, one can find a number j_0 such that

$$u_j(x) \leq C + \epsilon \quad \forall x \in K, \forall j > j_0.$$

Proof. let there is a closed ball of radius r in a compact set $K \in \mathbb{R}^m$. Since K is compact so in order to prove the lemma, it will be enough to show that for each point x_0 in the closed ball, there is a neighbourhood U of x_0 and a natural number j_0 such that $u_j(x) \leq C + \epsilon$ when $j \geq j_0$ and this j_0 will be independent of x.

Let δ be a fixed positive number such that $\delta < \frac{(C-r)}{3}$. By Fatou's lemma

$$\limsup_{j\to\infty} \int_{|x-x_0|<\delta} u_j(x)dV \leq \int_{|x-x_0|<\delta} \limsup_{j\to\infty} u_j(x)dV.$$

Since $\limsup_{j\to\infty} \leq C$,so

$$\limsup_{j\to\infty} \int_{|x-x_0|<\delta} u_j(x)dV \leq \int_{|x-x_0|<\delta} \lim_{j\to\infty} u_j(x)dV \leq C \times V$$

where V is the volume of ball $B \in \mathbb{R}^m$ and there is a natural number j_0 such that

$$\int_{|x-x_0|<\delta} u_j(x)dV < (C + \frac{\epsilon}{2})V \quad when \quad j \geq j_0.$$

Now let γ is a sufficiently small positive number less than δ and x_1 is a point such that $|x_1 - x_0| < \gamma$, then the ball of radius $\delta + \gamma$ centered at x_1 contains the ball of radius δ centered at x_0 with increased volume. The above last inequality will also hold for the ball centered at x_1 Because the given sequence of functions is bounded above so the above integral will be stable under small change of the center point. The sub mean-value-property of subharmonic function implies that

$$V_1 u_j(x_1) \leq \int_{|x-x_1|<\delta+\gamma} u_j(x)dV < (C + \frac{\epsilon}{2})V$$

where we have introduced a constant V_1 which is the volume of the ball of radius $\delta + \gamma$. When $j \geq j_0$, or

$$u_j(x_1) < \frac{(C + \frac{\epsilon}{2})V}{V_1}.$$

When $\lim \gamma \to 0$ then the right side of above will be $(C + \frac{\epsilon}{2})$. Hence there will be a small positive γ for which $u_j(x_1) < C + \epsilon$ when $j \geq j_0$ and x_1 is an arbitrary point of a ball of radius γ centered at x_0. Now for any $x_0 \in K$ there is a $U(x_0)$ where

$$u_j(x) \leq C + \epsilon \quad \forall j \geq j_0 \quad (j_0 = j(x_0)).$$

But $K \subset \cup_{x_0+K} U(x_0)$ which implies that

$$K \subset \bigcup_{x_0 \in K} U(x_0).$$

So there will be finitely many such open sets whose union contains K and the lemma will be true for compact set K. $\square$

Chapter 3

Holomorphy in Several Complex Variables

Here we present basic notions on holomorphic functions of several complex variables, including *the fundamental theorem of Hartogs on separate analyticity*. The presentation is based on [3], [6] [7] and [9].

3.1 The $\mathbb{C}^n$ Space

The n-dimensional Euclidean complex space is denoted by

$$\mathbb{C}^n = \mathbb{C} \times \ C \times ... \times \mathbb{C}$$

(n times) and its coordinates can be denoted by $z = (z_1, z_2, ..., z_n)$. The form of z_j will be $z_j = x_j + iy_j$ for every $j = 1, 2, ..., n$. In that way we can identify $\mathbb{C}^n$ by $\mathbb{R}^{2n}$.

Different types of domains in $\mathbb{C}^n$

This section is based on [7]

Definition 3.1.1. *For $\rho = (\rho_1, \rho_2, ..., \rho_n)$ where $\rho_j > 0$ and $a \in \mathbb{C}^n$ define a polydisc in $\mathbb{C}^n$*

$$\Delta_\rho(a) = \{z \in \mathbb{C}^n : |z_j - a_j| < \rho_j, j = 1, 2, ..., n\}.$$

We call a the centre and ρ the polyradius or simply the radius of the polydisc $\Delta_\rho(a)$.

The unit polydisc in several complex variable will actually be the product of n unit discs in one complex variable, that is

$$D_n = D \times D \times ... \times D = \Delta_1(0) = \{z \in \mathbb{C}^n : |z_j| < 1, \quad j = 1, 2, ..., n\}.$$

The set $\Gamma = \{z \in \mathbb{C}^n : |z_j| = 1; \forall j = 1, 2, ...n\}$ which is the product of n unit circles is called the distinguished boundary of the unit polydisc.

For $n = 2$ the polydisc $D = D_1 \times D_2$ is called the bidisc.

Definition 3.1.2. *As the Euclidean inner product on $\mathbb{C}^n$ is*

$$< z, w >= z_1, \overline{w}_1 + z_2\overline{w}_2 + ... + z_n\overline{w}_n.$$

The inner product gives us the standard Euclidean norm on $\mathbb{C}^n$

$$\|z\| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + |z_2|^2 + ... + |z_n|^2}.$$

And we define the balls in $\mathbb{C}^n$ as

$$B_\rho(a) = \{z \in \mathbb{C}^n : \|z - a\| < \rho\}$$

and define the unit ball as

$$B_1(0) = \{z \in \mathbb{C}^n : \|z\| < 1\}.$$

3.2 Holomorphic functions in$\mathbb{C}^n$

This section is based on [7]

Definition 3.2.1. *Let there is a domain Ω in $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$ is said to be holomorphic on Ω if it is complex differentiable at every point $z \in \Omega$ ie,*

$$f'(z) = \lim_{\xi \in \mathbb{C}^n \to 0} \frac{f(z + \xi) - f(z)}{\xi} \quad \textit{exists} \quad \forall z \in \Omega.$$

Definition 3.2.2. *A function $f : \Omega \to \mathbb{C}$ is said to be separately holomorphic if it is holomorphic in each variable, which means that f has complex derivative with respect to z_j when other variables are kept fixed ie,*

$$\lim_{\xi \to 0} \frac{f(z_1, ...z_j + \xi, ...z_n) - f(z)}{\xi} \quad \textit{exists} \quad \forall z \in \Omega \quad \textit{and} \quad j = 1, 2...n$$

where $\xi \in \mathbb{C}^n$.

We can see that a holomorphic function will be separately holomorphic,as it will be holomorphic for all $\xi = (0, 0, ..\xi_j, ...0)$. But it is nontrivial that separate holomorphy implies joint holomorphy. In the fourth section, we will show in several steps that separate holomorphy implies joint holomorphy.

Definition 3.2.3. *A function $f : \Omega \to \mathbb{C}$ that is continuously differentiable with respect to each pair of variables (x_j, y_j) on $\Omega \subset \mathbb{C}^n$ is said to be holomorphic if it satisfies the Cauchy Riemann equations in each variable.*

$$\frac{\partial f}{\partial \overline{z}} = 0$$

which means that

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}, \frac{\partial v}{\partial x_j} = -\frac{\partial u}{\partial y_j}$$

which is the necessary and sufficient condition for f to be complex differentiable at $z \in \mathbb{C}^n$.

3.3 Cauchy Integral Formula

This section is based on [9] and [3].

Any function which is separately holomorphic on a polydisc can be represented by the Cauchy integral of its values on the distinguished boundary of the polydisc.

Theorem 3.3.1. *Let f be a separately holomorphic function on the closed unit polydisc $\overline{D}^n$.Then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1|=1} \int_{|\xi_2|=1} ... \int_{|\xi|_n=1} \frac{f(\xi_1, \xi_2, ...\xi_n)}{\prod_{j=1}^{n} (\xi_j - z_j)} d\xi_1 d\xi_2 ... d\xi_n$$

for each $z = (z_1, z_2, ...z_n) \in D^n$.

Proof. To be simple we will prove it for $n = 2$. If we fix z_2 in the unit bidisc, the function $f(z_1, z_2)$ will be holomorphic in z_1, for$|z_1| < 1$. By applying Cauchy integral formula for one variable

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{|\xi_1|=1} \frac{f(\xi_1, z_2)}{\xi_1 - z_1} d\xi_1.$$

Now if we fix ξ_1, then $f(\xi_1, z_2)$ will be holomorphic in z_2 in closed unit disc and for $|z_2| < 1$

$$f(\xi_1, z_2) = \frac{1}{2\pi i} \int_{|\xi_2|=1} \frac{f(\xi_1, \xi_2)}{\xi_2 - z_2} d\xi_2.$$

By combining the above two expresions we will get

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{|\xi_1|=1} \int_{|\xi_2|=1} \frac{f(\xi_1, \xi_2)}{(\xi_1 - z_1)(\xi_2 - z_2)} d\xi_1 d\xi_2.$$

□

The above theorem which we stated for a unit polydisc, holds for polydiscs in general. It says that if a function will be holomorphic in the neighborhood of a closed polydisc then it can be expressed in the polydisc in the form of its Cauchy integral over the distinguised boundary Γ. Particularly, the values of a function in a polydisc can be determined completely by its values on the distinguished boundary.

Theorem 3.3.2. *If a function will be holomorphic in a polydisc $U = \{z : |z_v - a_v| < r_v\}$ and continuous in $\overline{U}$, then at any point $z \in U$, it can be represented by a multiple Cauchy integral*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\xi)}{(\xi_1 - z_1)(\xi_2 - z_2)...(\xi_n - z_n)} d\xi_1 d\xi_2 ... d\xi_n,$$

where Γ is the product of boundary circles $\gamma_v = \{|z_v - a_v| = r_v\}$ $\quad v = 1, 2, ...n.$

Proof. Since the given function is holomorphic, so it can be written as a repeated integral:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\gamma_1} \frac{d\xi_1}{\xi_1 - z_1} ... \int_{\gamma_n} \frac{f(\xi_1, ..., \xi_n)}{(\xi_n - z_n)} d\xi_n.$$

Because of the continuity of f in the closure of the polydisc, the repeated integral can be written as a multiple Cauchy integral over the product of boundary circles

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\xi_1, ..., \xi_n)}{(\xi_1 - z_1)...(\xi_n - z_n)} d\xi_1 ... d\xi_n.$$

It can also be written in abbreviated form:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\xi)}{(\xi - z)} d\xi \tag{3.1}$$

where $d\xi = d\xi_1 ... d\xi_n$ and $(\xi - z) = (\xi_1 - z_1)...(\xi_n - z_n)$. □

As a consequence of this, we can get the representation of such functions by multiple power series.

Theorem 3.3.3. *If a function f is separately holomorphic in U and continuous in $\overline{U}$, then at each point $z \in U$ it can be represented as multiple power series*

$$f(z) = \sum_{|k|=0}^{\infty} c_k (z-a)^k$$

where $(z-a)^k = (z_1 - a_1)^{k_1}, (z_2 - a_2)^{k_2}, ..., (z_n - a_n)^{k_n}$ and the coefficients c_k will be

$$c_k = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\xi)}{(\xi - a)^{k+1}} d\xi.$$

Proof. From the Cauchy integral representation of a function, we can also obtain its power series represenation as follows:

$$\frac{1}{(\xi - z)} = \frac{1}{(\xi - z + a - a)} = \frac{1}{(\xi - a)\Big(\big(1 - \frac{z_1 - a_1}{\xi_1 - a_1}\big)...\big(1 - \frac{z_n - a_n}{\xi_n - a_n}\big)\Big)}$$

$$= \frac{1}{\xi - a} \sum_{|k|=0}^{\infty} \left(\frac{z-a}{\xi - a}\right)^k.$$

The above expanision can also be written as follows:

$$\frac{1}{\xi - z} = \sum_{|k|=0}^{\infty} \frac{(z-a)^k}{(\xi - a)^{k+1}}$$

here $|k| = k_1 + k_2 + ... + k_n$ and $k = (k_1, ..., k_n)$ which is an integer vector, $k + 1 = (k_1 + 1, ..., k_n + 1)$. The above series will converge uniformly and absolutely in ξ on Γ for any $z \in U$.

And from the Cauchy integral representation we will obtain

$$f(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\xi)}{\xi - z} d\xi = \sum_{|k|=0}^{\infty} c_k (z-a)^k$$

where

$$c_k = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\xi)}{(\xi - a)^{k+1}} d\xi.$$

□

So we have shown that holomorphic functions are analytic because they can be represented as multiple power series, on the other hand we are going to show that multiple power series are holomorphic functions.

Theorem 3.3.4. (Abel's Lemma) *If there is a multiple power series $\sum_{|k|=0}^{\infty} c_k(z-a)^k$ and its terms are bounded at some point $\xi \in \mathbb{C}^n$. Then this series converges absolutely and uniformly on any compact subset K of the polydisc U of center a and vector radius ρ with $\rho_v = |\xi_v - a_v|$ and therefore be a holomorphic function.*

Proof. Since the terms of the multiple power series are bounded at some point $\xi \in \mathbb{C}^n$ so we can write

$$|c_k(\xi - a)^k| = |c_k|\rho^k \leq M$$

where

$$\rho^k = \rho_1^{k_1}...\rho_n^{k_n}.$$

If $K \subset\subset U$, then it follows

$$q_v = \max_{z \in K} \frac{1}{\rho_v}|z_v - a_v| < 1,$$

therefore we have

$$|c_k(z-a)^k| \leq M.q_k.$$

Since all $q_v < 1$, so the multiple geometric progression $\sum Mq_k$ will converge and hence the given series will converge absolutely and uniformly on any compact subset K of the polydisc. Since each term of this series is holomorphic so the uniform limit f of the holomorphic functions is also holomorphic. □

Theorem 3.3.5. *If a function f is given by a multiple power series in a polydisc U, then at any point $z \in U$ its partial derivatives of all orders will exist and will be holomorphic.*

Proof. Let

$$f(z) = \sum_{|k|=0}^{\infty} c_k(z-a)^k \quad at \quad any \quad point \quad z \in U.$$

From the case of one variable the partial derivative of f of all orders f will also exist. These derivatives can be obtained as follows:

$$f^k(z) = \sum_{|k|=0}^{\infty} c_k \frac{\partial^k}{\partial z^k}(z-a)^k. \tag{3.2}$$

From (3.2) we can see that these derivatives are also in the form of the power series and are obtained by the corresponding term by term differentiation of the power series $\sum_{|k|=0}^{\infty} c_k(z-a)^k$.

By Abel's lemma, this series converges uniformly on compact subsets of U. The terms of these series will be continuous with respect to all the variables, so any of the derivatives is $\mathbb{R}$ differentiable in U and as this derivative is holomorphic in each variable, therefore, it will be holomorphic in U. □

Theorem 3.3.6. *If a function $f(z)$ is separately holomorphic in a polydisc U and continuous in $\overline{U}$, then it is holomorphic in U.*

Proof. By Theorem 3.3.3, the function can be represented as a power series. And from Theorem 3.3.5 this function will be C differentiable and hence it will be holomorphic in the polydisc. □

Theorem 3.3.7. *For the power series expansion of a holomorphic function at a point a, the coefficients of this series are defined by Taylor's formula*

$$c_k = \frac{1}{k_1!...k_n!} \frac{\partial^{k_1+...+k_n} f}{\partial z_1^{k_1}...\partial z_n^{k_n}}\bigg|_{z=a} = \frac{1}{k!}\frac{\partial^{|k|} f}{\partial z^k}\bigg|_{z=a}.$$

Proof. The Cauchy integral formula for derivatives is

$$\frac{\partial^{|k|} f(z)}{\partial z^k} = \frac{1}{(2\pi i)^n}\int_\Gamma \frac{k! f(\xi)}{(\xi-a)^{k+1}} d\xi. \tag{3.3}$$

Or

$$\frac{1}{(2\pi i)^n}\int_\Gamma \frac{f(\xi)}{(\xi-a)^{k+1}} d\xi = \frac{1}{k!}\frac{\partial^{|k|} f(z)}{\partial z^k}$$

where Γ is the distinguished boundary and $|k| = k_1+k_2...k_n$ and $k! = k_1!...k_n!$.

Since

$$f(z) = \sum_{|k|=0}^{\infty} c_k(z-a)^k$$

and

$$c_k = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\xi)}{(\xi - a)^{k+1}} d\xi \tag{3.4}$$

hence

$$c_k = \frac{1}{k!} \frac{\partial^{|k|} f(z)}{\partial z^k}.$$

Cauchy's Inequalities

If the function f is separately holomorphic on U and continuous on $\overline{U}$ and $f \leq M$ on Γ ,then

$$c_k \leq \frac{M}{r^k}$$

where $r^k = r_1^{k_1}...r_n^{k_n}$.

□

Theorem 3.3.8. *For a rectifiable curve $\gamma_\mu : \xi_\mu = \xi_\mu(t)$ in the plane of ξ_μ ,where $\mu = 1, 2, ...k$ and $\gamma = \gamma_1 \times \gamma_2 \times ... \times \gamma_k$ and $\xi = \xi_1, \xi_2, ...\xi_k$ and let D be a domain in $\mathbb{C}^n$ and $\xi = \xi_1, \xi_2, ...\xi_k$ and $z \in \mathbb{C}^n$, if the function $f(\xi, z)$ is continuous on $\gamma \times D$ and holomorphic with respect to z for any $\xi \in \gamma$ and has continuous partial derivatives $\frac{\partial f}{\partial z_\nu}$ on $\gamma \times D$, then the integral*

$$F(z) = \int_{\gamma_1} d\xi_1 ... \int_{\gamma_k} f(\xi, z) d\xi_k = \int_\gamma f(\xi, z) d\xi$$

is holomorphic in D and

$$\frac{\partial F}{\partial z_\nu} = \int_\gamma \frac{\partial f(\xi, z)}{\partial z_\nu} d\xi \quad , \quad \nu = 1, 2, ...n.$$

Proof. Let for $z \in D$ and arbitrary $r > 0$,there is a polydisc $\Delta(z, r) \subset D$ and let there is a vector $u = (0, 0, ...u_\nu, , ...0) \in \mathbb{C}^n$ such thet $|u_\nu| < r$.

We have

$$\frac{1}{u_\nu}\{F(z+u) - F(z)\} = \frac{1}{u_\nu} \int_\gamma \{f(\xi, z+u) - f(\xi, z)\} d\xi$$

$$= \int_\gamma d\xi \int_0^1 \frac{\partial f(\xi, z+tu)}{\partial z_\nu} dt$$

and

$$\frac{1}{u_\nu}\{F(z+u)-F(z)\}-\int_\gamma \frac{\partial f(\xi,z)}{\partial z_\nu}d\xi = \int_\gamma d\xi \int_0^1 \left\{\frac{\partial f(\xi,z+tu)}{\partial z_\nu}-\frac{\partial f(\xi,z)}{\partial z_\nu}\right\}dt.$$

Since $\frac{\partial f(\xi,z+tu)}{\partial z_\nu}$ continuous uniformly for a fixed z on the compact set $\gamma \times [0,1]$, so for any $\epsilon > 0$ there exists a very small $\delta > 0$ such that for all $(\xi,t) \in \gamma \times [0,1]$ and for $|u| < \delta$, the following inequality will hold

$$\left|\frac{\partial f(\xi,z+tu)}{\partial z_\nu} - \frac{\partial f(\xi,z)}{\partial z_\nu}\right| < \epsilon$$

and hence

$$\frac{\partial F}{\partial z_\nu} \leq \epsilon|\gamma| \quad for \quad |u| < \delta$$

which means that at each point $z \in D$ the partial derivaties of F exist and it will be as follows

$$\frac{\partial F}{\partial z_\nu} = \int_\gamma \frac{\partial f(\xi,z)}{\partial z_\nu}d\xi$$

and hence F is holomorphic in D. □

Theorem 3.3.9. (Uniqueness Theorem) Let a function f is holomorphic in a domain $D \subset \mathbb{C}^n$. If f vanishes at some point in a nonempty open subset $S \subset D$ with all of its partial derivatives, then $f \equiv 0$ in the whole domain D.

This proof follows [7]

Proof. Let Z be the set of points where all derivatives are 0. Z is nonempty because $S \subset Z$. Since all derivatives are continuous, So Z is closed in D. Since at an arbitrary point $a \in Z$, f can be expanded in a power series, which will converge to f in a polydisc $\Delta_\rho(a) \subset D$. Now the coefficients of that power series which are given by derivatives of f become 0. So the power series becomes a zero series. Hence f is identically 0 in $\Delta_\rho(a) \in D$, which implies that Z is open. But we have seen that Z is also closed and nonempty and D is connected hence $Z \equiv D$.

□

There is another form of the uniqueness theorem which follows easily from the previous one.

Theorem 3.3.10. *Let f and ϕ be analytic on a connected domain $D \subset \mathbb{C}^n$ and suppose that $f = \phi$ throughout the nonempty open subset U of D. This will in particular be if f and ϕ have same power series at some point $a \in D$. Then $f = \phi$ throughout the D.*

3.4 The fundamental theorem of Hartogs

This section follows [3].

According to *Hartogs' theorem on separate analyticity*, if a function of several complex variables will be holomorphic with respect to each variable separately then it will be holomorphic with respect to all variables. In the last section, it has been proved for continuous functions, see Theorem 3.3.6. Therefore, in order to prove the general case, it will be sufficient to prove that if a function is holomorphic with respect to each variable then it is continuous with respect to the set of all the variables. In order to be simple, we will prove it for a function of two complex variables in a bidisc centered at 0.

In order to prove this theorem, we will take the help of a number of lemmas. In the proof of first lemma we will use the Schwarz lemma which says that, if a function $\phi : D \to \mathbb{C}$ is holomorphic and $\phi(0) = 0$, then

1. $|\phi(z)| \leq |z|$, and
2. $|\phi'(0)| \leq 1$.

We will use the general form of lemma given below.

Lemma 3.4.1. Schwarz Lemma: Suppose that the function ϕ is holomorphic in the disc $D_r = \{|z| < r\} \subset \mathbb{C}$ where $\phi(z_0) = 0$ at some point $z_0 \in D_r$ and $|\phi| \leq M$ everywhere in D_r. Then everywhere in D_r we have the estimate

$$|\phi(z)| \leq Mr \frac{|z - z_0|}{|r^2 - \overline{z}_0 z|}$$

Proof. In order to prove this lemma we can consider the following linear fractional mapping of D_r onto the unit disc D

$$\mu : z \to r \frac{z - z_0}{r^2 - \overline{z}_0 z}$$

where μ^{-1} is the inverse mapping $D \to D_r$.

Then the function

$$\Psi = \frac{\phi \circ \mu^{-1}(z)}{M}$$

vanishes at 0 and bounded by 1 on D. So it satisfies the hypothesis of usual Schwarz lemma and hence

$$|\Psi(z)| \leq |z|$$

and

$$|\frac{\phi \circ \mu^{-1}(z)}{M}| \leq |z|.$$

Replacing z by $\mu(z)$

$$|\phi(z)| \leq M|\mu|$$

which proves that

$$|\phi(z)| \leq Mr\frac{|z - z_0|}{|r^2 - \overline{z}_0 z|}.$$

□

If $r = M = 1$ and $z_0 = 0$ then we will obtain

$$|\phi(z)| \leq |z|$$

which is the usual statement of Schwarz lemma.

Lemma 3.4.2. *Let a function $f(z_1, z_2)$ be holomorphic with respect to each variable z_1 and z_2 separately in the polydisc $D = D(0, r)$ and it is also bounded in D, then it will be continuous at each point of D with respect to both variables.*

Proof. For any two points $z^o, z \in D$, we can write the increment of $f(z_1, z_2)$ as a sum of increments with respect to the individual coordinates as follows

$$f(z) - f(z^o) = \big(f(z_1, z_2) - f(z_1^o, z_2)\big) + \big(f(z_1^o, z_2) - f(z_1^o, z_2^o)\big). \tag{3.5}$$

Let the both terms of above are functions ϕ_1 and ϕ_2 with variables z_1 and z_2 respectively. Since the given function is bounded in D, then each of the functions ϕ_1 and ϕ_2 will satisfy the general form of the Schwarz lemma, and by applying this lemma on each term of the sum in Equation (3.5), we can find that

$$\phi_1 = |f(z_1, z_2) - f(z_1^o, z_2)| \leq M\frac{r_1|z_1 - z_1^o|}{|r_1^2 - \overline{z_1^o} z_1|},$$

$$\phi_2 = |f(z_1^o, z_2) - f(z_1^o - z_2^o| \leq M \frac{r_2|z_2 - z_2^o|}{|r_2^2 - \overline{z_2^o} z_2|}$$

and

$$|f(z) - f(z^o)| \leq M \left(\frac{r_1|z_1 - z_1^o|}{|r_1^2 - \overline{z_1^o} z_1|} + \frac{r_2|z_2 - z_2^o|}{r_2^2 - \overline{z_2^o}} \right).$$

In a unit bidisc at origin it will be

$$|f(z_1, z_2) - f(0, 0)| \leq M(|z_1| + |z_2|)$$

which proves that $f(z_1, z_2)$ is continuous with respect to the set of both variables. □

In the above lemma, we have proved that the boundedness of a function implies its continuity. So now we will prove the boundedness of a function in some bidisc, which follows from the continuity of f in each variable separately.

Lemma 3.4.3. (Osgood's lemma)

Let there is a bidisc $D = D_1 \times D_2 = \{z \in \mathbb{C}^2 : \|z\| < R\}$, where $D_1 = \{z_1 \in \mathbb{C} : |z_1| < R\}$ and $D_2 = \{z_2 \in \mathbb{C} : |z_2| < R\}$. If the function $f(z_1, z_2)$ is continuous with respect to z_1 in $\overline{D}_1$ for any $z_2 \in \overline{D}_2$ and is continuous with respect to z_2 in $\overline{D}_2$ for any $z_1 \in \overline{D}_1$,then there exists a smaller bidisc $W \subset D$ in which f is bounded.

Proof. For a fix $z_1 \in \overline{D}_1$ we can write

$$M(z_1) = \max |f(z_1, z_2)| \quad : \quad z_2 \in \overline{D}_2.$$

Now consider the sets S_m such that $S_m = \{z_1 \in \overline{D}_1 : M(z_1) \leq m\}$. These sets S_m will be closed , since if $z_1^{(\mu)} \in S_m$ for $(\mu = 1, 2...)$ and $z_1^{(\mu)} \to z_1$ then $z_1 \in S_m$ because in fact $|f(z_1^{(\mu)}, z_2)| \leq m$ for any $z_2 \in \overline{D}_2$ and it is also given that f is continuous in z_1, so $|f(z_1, z_2)| \leq m$ for any $z_2 \in \overline{D}_2$, which means that $M(z_1) \leq m$. The S_m will form an increasing sequence and any point $z_1 \in \overline{D}_1$ will also belong to all the S_m except for finitely many because of the boundedness.

Because S_m are closed and $D_1 \subset \overline{D_1} = \cup S_m$. So, if $S_1 = \overline{D_1} \Rightarrow D_1 \subset S_1$.

If $S_1 \neq \overline{D_1} \Rightarrow D_1 \backslash S_1$ is open and there would exist a ball B_1 in D_1:

$$B_1 \cap S_1 = \varnothing,$$

likewise, if $B_1 \not\subset S_2 \Rightarrow B_1 \backslash S_2$ is open

$$\Rightarrow B_{21} : B_2 \cap S_2 = \varnothing.$$

So there would exist a sequence of balls $\overline{B_1} \supset \overline{B_2} \supset ...\overline{B_m}$ such that

$$z_1^o \in \cap \overline{B_m} \neq \varnothing$$

$$\Rightarrow z_1^o \notin \cup S_m.$$

Which is a contradiction. So there will exist an S_M containing some domain $'G \subset D_1$ such that $|f(z_1, z_2)| \leq M$ for any $z_2 \in D_2$. Now if we choose a bidisc $'W = \{z_1 : |z_1 - z_1^o| < r\}$ in $'G$ then we will have $|f| \leq M$ in $W =' W \times D_2$ which means that f is bounded in the small bidisc W. □

Now, by using the holomorphy of f with respect to each variable separately we will prove the boundedness of in the whole bidisc. We will do it by the help of *Hartogs lemma* on subharmonic functions, and we will use the following notations in this lemma

$D_1 = D(a, R),' W = D(a, r)$ where $(r < R)$ and both are scalar)

$D_2 = \{|z_2| < R\}, V = D_1 \times D_2$ and $W =' W \times D_2$

Lemma 3.4.4. *If the fun $f(z_1, z_2)$ is holomorphic with respect to z_1 in $\overline{D_1}$ for any $z_2 \in D_2$ and jointly holomorphic in a smaller bidisc $\overline{W}$, then it is holomorphic in the entire bidisc $\overline{V}$.*

Proof. Without lose of generality we can assume that $a = 0$. Since the given function is holomorphic with respect to z_1, so it can be represented by a convergent power series for any fixed $z_2 \in D_2$ and any $z_1 \in D_1$.

$$f(z) = \sum_{|k|=0}^{\infty} c_k(z_2)(z_1)^k \tag{3.6}$$

for $k = (k_1, k_2, ...k_{n-1})$. The coefficients of this series will be as follows

$$c_k(z_2) = \frac{1}{k!} \frac{\partial^{|k|} f(0, z_2)}{(\partial z_1)^k}.$$

These coefficients will be holomorphic in the disc D_2 because these are the derivatives of a function which is holomorphic in z_2. So the functions $\frac{1}{|k|} \ln |c_k(z_2)|$ are subharmonic functions in D_2.

Let's choose an arbitrary number $\rho < R$ and since for any $z_2 \in D_2$

$$|c_k(z_2)|\rho^{|k|} \to 0 \quad as \quad |k| \to \infty,$$

then for any $z_2 \in D_2$ we will have a $|k|$ such that

$$\frac{1}{|k|} \ln |c_k(z_2) + \ln \rho \leq 0$$

$$\lim_{|k|\to\infty} \frac{1}{|k|} \ln |c_k(z_2)| \leq \ln (\rho)^{-1}$$

$$\lim_{|k|\to\infty} \frac{1}{|k|} \ln |c_k(z_2)| \leq \ln \frac{1}{\rho}.$$

Since the function is holomorphic and bounded in $\overline{W}$ and let $|f| \leq M$ and the Cauchy inequalities hold for any $z_2 \in D_2$,i.e.,

$$|c_k(z_2)|r^{|k|} \leq M.$$

So for any $z_2 \in D_2$ and any $|k|$ we will have

$$\frac{1}{|k|} \ln |c_k(z_2)| \leq \ln \frac{M^{\frac{1}{|k|}}}{r} \leq L.$$

These subharmonic functions satisfiy the hypothesis of Lemma 2.2.3. By using this lemma we can find a number k_0 for any $\sigma < \rho$, such that for all $|k| > k_0$ and for all z_2, $|z_2| < \sigma$, we will have

$$\frac{1}{|k| \ln |c_k(z_2)|} \leq \frac{1}{\sigma}$$

and it can also be written as

$$|c_k(z_2)|\sigma^{|k|} \leq 1.$$

So from the last expression it follows that the series in Equation 3.6 converges uniformly in any$\overline{D}(0, \sigma')$ and $\sigma' < \sigma$. But the terms of this series are continuous in z so the sum will be also continuous and hence it will be bounded in $D(0, \sigma')$. We can assume this bidisc arbitrarily close to V and f will be bounded in $\overline{V}$ and then by Lemma 3.4.2 it will be continuous in $\overline{V}$ and hence by Theorem 3.3.6 it will be holomorphic in $\overline{V}$. $\square$

Now we can prove the *fundamental theorem of Hartogs* by the help of the last three lemmas.

Hartogs' Theorem: If the function $f(z)$ where $z = z_1, z_2, ...z_j(j = 1, ..., n)$ is holomorphic at any point of the domain $U \in \mathbb{C}^n$ with respect to each of the variables z_j, then it is jointly holomorphic in U .

Proof. Again for the simplicity of presentation we will prove it for $n = 2$. It will be sufficient to show that if $f(z_1, z_2)$ is holomorphic separately in the bidisc $\overline{D}(0, R)$ then it will be holomorphic in some bidisc with center 0.

Let $D_1' = D(0, \frac{R}{3})$, then it follows that the function $f(z_1, z_2)$ is continuous with respect to z_1 in $\overline{D_1}$ for any z_2 in $D_2 = \{|z_2| < R\}$ and with respect to z_2 in $\overline{D_2}$ for any $z_1 \in D_1$.Then the function f will be bounded by Lemma 3.4.3 and hence it will be holomorphic in some bidisc $\overline{W} = \overline{D}' \times \overline{D}_2$ whereas $\overline{D}' = D(a, r) \subset D_1{}'$.

Let us now consider the bidisc $V = D_1 \times D_2$ where $D_1 = D(a, \frac{2R}{3})$. As, $\overline{V} \subset \overline{D(0, R)}$, so f is holomorphic with respect to z_1 in $\overline{D_1}$ for any $z_2 \in \overline{D_2}$ and it has been proved above that f is holomorphic with respect to z in $\overline{W}$. So, by Lemma 3.4.4 f is holomorphic with respect to both variables in V and V also contains the point $z = 0$. Hence proved that $f(z_1, z_2)$ will be holomorphic in a bidisc with center 0. $\square$

3.5 Multiple power series and multicircular domains

Most of the part of this section is based on [3] and [6]

Definition 3.5.1. *$M \subset \mathbb{C}^n$ is called a multicircular set or a Reinhardt set if*

$$a = (a_1, ...a_n) \in M \Rightarrow a' = (e^{i\theta_1}a_1, ...e^{i\theta_n}a_n) \in M$$

for all real $\theta_1, \theta_2, ...\theta_n$ and an open multicircular set is called the multicircular domain or Reinhardt domain.

Multicircular sets can be represented by their "trace" in the space $\mathbb{R}^n_+$, in which all coordinates are positive.

Definition 3.5.2. *If M is a multicircular set, then it's trace is given by*

$$trM = \{(|a_1|, ...|a_n|) \in \mathbb{R}^n_+ : (a_1, ...a_n) \in M\}.$$

Definition 3.5.3. *$M \subset \mathbb{C}^n$ is called a complete multicircular set or a complete Reinhardt set if*

$$a = (a_1, a_2, ...a_n) \in M \Rightarrow a' = (a'_1, a'_2...a'_n) \in M$$

when $|a'_j| \leq |a_j|$ for all $j = 1, 2, ...n$.

Since any function f that is holomorphic in the polydisc $D(a, r)$ can be expanded in a multiple power series with center at a. In the case of a single variable, the set of points of convergence of this series will be an open disc. But the situation will be different in case of several complex variables.

For example:

$$\frac{1}{1 - z_1 z_2} = \sum_{\alpha=0}^{\infty} z_1^{\alpha} z_2^{\alpha}.$$

The set of convergence of the above series in $\mathbb{C}^2$ is the complete Reinhardt domain $\{|z_1 z_2| < 1\}$.

Definition 3.5.4. *Let S be the set of the points $z \in \mathbb{C}^n$ where the power series*

$$\sum_{|\alpha|=0}^{\infty} c_{\alpha}(z - a)^{\alpha} \tag{3.7}$$

converges absolutely then the interior S^0 of S is called the domain of convergence of the series.

Theorem 3.5.5. *The closed polydisc $\overline{D} = \{z \in \mathbb{C}^n : |z_j - a_j| \leq |z_j^0 - a_j|$ belongs to S^o, for the point z^0 belongs to the domain of convergence of the power series (3.7) and this series converges uniformly and absolutely in $\overline{D}$.*

Proof. Since $z^0 \in S^o$ and the interior S^o is open, so there exists a point $\zeta \in S^o$ such that $|\zeta_j - a_j| \geq |z_j^0 - a_j|, j = 1, 2, ..., n$ and the power series converges at point ζ. As $D \subset\subset \{z \in \mathbb{C}^n : |z_j - a_j| < |\zeta_j - a_j|\}$, so by Abel's lemma the series converges uniformly and absolutely in $\overline{D}$. □

The above theorem can also be described as follows

Proposition 3.5.6. *The domain of convergence S^o of a multiple power series with centre 0 is a complete multicircular domain.*

Proof. Let S^o be nonempty and a is an arbitrary point in S^o. So S^o contains a ball $B(a, r)$, and this ball contains a point b such that $|b_j| > |a_j|$,$\forall j$. Since the multiple power series converges absolutely at b, so it will also be convergent throughout the polydisc $\Delta(0, ...0; |b_1|, ...|b_n|)$. So this polydisc in S^o will contain all points $|a'_j| \leq |a_j|$. Hence S^o is a complete multicircular set. $\square$

Theorem 3.5.7. *Any holomorphic function f in a complete Reinhardt domain $D \subset \mathbb{C}^n$ with center at a is represented by the Taylor series expansion in this domain.*

$$f(z) = \sum_{|\alpha|=0}^{\infty} (z - a)^{\alpha}. \tag{3.8}$$

Proof. For an arbitrary $z^0 \in D$, the polydisc

$$\overline{U} = \{|z_j - a_j| \leq |z_j^0 - a_j|\} \subset\subset D.$$

By Theorem 3.3.3 f is represented in U by series (3.8), with coefficients computed in Theorem 3.3.5. Hence f is represented as Taylor series expansion in D. $\square$

It means that complete Reinhardt domains play the same role for functions of several complex variables as discs do for functions of a single complex variable.

3.6 Multiple Laurent series on general multicircular domains

This section is based on [6].

The analog of Laurent series in a single variable for general $n \geq 2$ is a Laurent series in n variables.

Theorem 3.6.1. *Let f be holomorphic on a connected multicircular domain $D \subset \mathbb{C}^n (n \geq 2)$. Then there is a unique n variable Laurent series with center 0 and constant coefficients. This series converges to f at each point of D for some ordering of its terms.*

The series is

$$\sum_{\alpha_1 \in \mathbb{Z} ... \alpha_n \in \mathbb{Z}} c_{\alpha_1 ... \alpha_n} z_1^{\alpha_1} ... z_n^{\alpha_n} \tag{3.9}$$

and the coefficients are given by the formula

$$c_{\alpha_1...\alpha_n} = \frac{1}{(2\pi i)^n}\int_{\Gamma(0,r)} \frac{f(z)}{(z_1)^{\alpha_1+1}...(z_n)^{\alpha_n+1}} dz_1...dz_n \tag{3.10}$$

for any $r = r_1, r_2, ...r_n$ in trace of the D. The series will be convergent absolutely on D and will converge uniformly to f on any compact subset of D.

Proof. For the simplicity of presentation we will consider the case for $n = 2$. Let there is an annular domain

$$A_\delta(r) = \{(z_1, z_2) \in \mathbb{C}^2 : r_j - \delta_j < |z_j| < r_j + \delta_j, j = 1, 2\}.$$

For $r = (r_1, r_2) > 0$ and $0 < \delta = (\delta_1, \delta_2) < r$. Let there is a small $\epsilon < \frac{1}{2}r$ such that $\overline{A}_{2\epsilon}(r)$ also belongs to D. If there is a series as given in (3.9) and let it's terms form a bounded sequence at each point of $\overline{A}_{2\epsilon}(r) = r_j - 2\epsilon_j \leq |z_j| \leq r_j + 2\epsilon_j$. Then from the boundedness of the sequence $\{c_\alpha z^\alpha\}$ at the point $z = r + 2\epsilon$ it follows, that the power seris

$$\sum_{\alpha_1 \geq o, \alpha_2 \geq 0} c_\alpha z_1^{\alpha_1} z_2^{\alpha_2}$$

converges uniformly and absolutely on the polydisc $\Delta(0, r + \epsilon)$ and hence on $A_\epsilon(r)$. Now let us consider another point $z = (r_1 - 2\epsilon_1, r_2 + 2\epsilon_2)$ and by using the boundedness of the sequence at this point it follows that the power series in two variables converges uniformly and absolutely for $|z_1| > r_1 - \epsilon_1$ and $|z_2| < r_2 + \epsilon_2$ and hence on $A_\epsilon(r)$. Similarly by using the boundedness of the sequence at $z = (r - 2\epsilon)$ and at $z = (r_1 + 2\epsilon_1, r_2 - 2\epsilon_2)$, it follows that the given series converges uniformly and absolutely on $A_\epsilon(r)$ and the sum will be $f(z)$ for any arrangement of terms.

Now if we integrate termwise ,the absolutely and uniformly convergent series

$$f(z)z^{-\beta-1} = \sum_{\alpha \in \mathbb{Z}^2} c_\alpha z^{\alpha-\beta-1} \quad [-\beta - 1 = -\beta_1 - 1, -\beta_2 - 2]$$

over $\Gamma_(0, r)$ in $A_\epsilon(r)$,then we will get

$$\frac{1}{(2\pi i)^2}\int_{\Gamma_{(0,r)}} f(z)z^{-\beta-1}dz = \sum_\beta c_\beta \tag{3.11}$$

where we have used that

$$\frac{1}{2\pi i}\int_{C_{(0,r_j)}} z_j^{\alpha_j-\beta_j-1}dz_j = 1 \quad for \quad \alpha_j = \beta_j$$

and

$$= 0 \quad for \quad \alpha_j \neq \beta_j.$$

Hence the coefficient formula has been proved and if f is represented by the series (3.9) then the coefficient formula is given by (3.10) and this representation is unique. Now if K is a compact subset of D, then the Laurent series will converge uniformly and absolutely on K since this compact set can be covered by finitely many annular domains $A_\epsilon(r)$ such that $\overline{A}_{2\epsilon}(r) \subset D$.

Now we will see that such a series really exists for a function f on D and $\Gamma(0,r) \subset D$. By Theorem 1.1.7, the following will be true for $s_2 < r_2$

$$\int_{C(0,r_2)} f(z_1,z_2)z_2^{-\alpha_2-1}dz_2 = \int_{C(0,s_2)} f(z_1,z_2)z_2^{-\alpha_2-1}dz_2$$

where $z_1 \in C(0,r_1) = C(0,s_1)$. Multiplying above expression by $z_1^{-\alpha_1-1}$ and integrating with respect to z_1 we can see that

$$c_\alpha(r) = c_\alpha(s)$$

which means that coefficients are independent of r. So we can associate the coefficients in (3.10) with f. Now for a point w in D with $|w_j| = r_j, j = 1,2$,we can observe that

$$|c_{\alpha_1}c_{\alpha_2}w_1^{\alpha_1}w_2^{\alpha_2}| = \left|\frac{1}{(2\pi i)^2}\int_{\Gamma(0,r)} f(z)\Big(\frac{w_1}{z_1}\Big)^{\alpha_1}\Big(\frac{w_2}{z_2}\Big)^{\alpha_2}\frac{dz_1}{z_1}\frac{dz_2}{z_2}\right| \leq \sup_{\Gamma(0,r)}|f(z)|. \tag{3.12}$$

So with these coefficients each term of the series in (3.9) will form a bounded sequence at each point of D.

Now let for a fix $r > 0$ in trace of D, $A_\epsilon(r)$ be an annulus ,where $\epsilon < r$ such that $A_\epsilon(r)$ belong to D. If we fix z_1 in the annulus $r_1-\epsilon_1 < |z_1| < r_1+\epsilon_2$,then the function becomes holomorphic in z_2 on the annulus $r_2-\epsilon_2 < |z_2| < r_2+\epsilon_2$ and it can be represented by a one variable absolutely convergent Laurent series by Theorem 1.3.3

$$f(z_1,z_2) = \sum_{\alpha_2\in\mathbb{Z}} d_{\alpha_2}(z_1)z_2^{\alpha_2}; z \in A_\epsilon(r) \tag{3.13}$$

with coefficients

$$d_{\alpha_2}(z_1) = \frac{1}{2\pi i}\int_{C(0,r_2)} f(z_1, z_2) z_2^{-\alpha_2 - 1} dz_2. \tag{3.14}$$

These coefficients will be holomorphic function of z_1 on the annulus $r_1 - \epsilon_1 < |z_1| < r_1 + \epsilon_1$ by Theorem 3.3.8 and hence these coefficients can also be represented by absolute convergent Laurent series

$$d_{\alpha_2}(z_1) = \sum_{\alpha_1 \in \mathbb{Z}} d_{\alpha_1\alpha_2} z_1^{\alpha_1} \tag{3.15}$$

with

$$d_{\alpha_1\alpha_2} = \frac{1}{2\pi i}\int_{C(0,r_1)} d_{\alpha_2}(z_1) z_1^{-\alpha_1 - 1} dz_1. \tag{3.16}$$

Substituting the values from (3.15) into (3.13) we will get

$$f(z_1, z_2) = \sum_{\alpha_2} \Big\{ \sum_{\alpha_1} d_{1\alpha_2} z_1^{\alpha_2} \Big\} z_2^{\alpha_2} \in A_\epsilon(r) \tag{3.17}$$

and

$$d_{\alpha_1\alpha_2} = \frac{1}{(2\pi i)^2}\int_{C(0,r_1)} \Big\{ \int_{C(0,r_2)} f(z_1, z_2) z_2^{\alpha_2 - 1} dz_2 \Big\} z_1^{\alpha_1 - 1} dz_1. \tag{3.18}$$

Since f is continuous on $\Gamma(0, r)$, so the above equation can be written as

$$d_{\alpha_1\alpha_2} = \frac{1}{(2\pi i)^2}\int_{\Gamma(0,r)} f(z_1, z_2) z_1^{-\alpha_1 - 1} z_2^{-\alpha_2 - 1} dz_1 dz_2. \tag{3.19}$$

It can be noticed that

$$d_{\alpha_1\alpha_2} = c_{\alpha_1\alpha_2}(r) = c_{\alpha_1\alpha_2}.$$

Hence f can be represented locally and globally by a series on D as given in (3.9) and (3.10). And from (3.12), the terms form a bounded sequence at each point of D. Thus the series converges absolutely and hence converges to f on D for any arrangement of the terms. $\square$

Chapter 4

Hartogs' Extension Phenomenon

This chapter is based on [6] and [4].

The subject of analytic continuation reveals a very remarkable difference between several complex variables $n \geq 2$ and a single variable. For a domain D in $\mathbb{C}$ and any point $a \in \partial D$ there always exist an analytic function f in D which can not be continued analytically across the point a. By suitable distribution of singularities along ∂D, we can construct analytic functions on D which cannot be continued analytically across any boundary point and D is called the maximal domain of existence of these functions. However in $\mathbb{C}^n$ with $n \geq 2$ there are many domains D in which all holomorphic functions can be continued analytically across a certain part of the boundary. Several examples of this phenomenon that we are going to discuss in this chapter were discovered by Friedrich Hartogs around 1905 and known as Hartogs' Extension Phenomenon. A general form of the Hartogs extension theorem was obtained by means of the so-called $\bar{\partial}$-problem, a much more recent technique appeared in the 60s, see [5].

4.1 Analytic Extension for an analytic function on a punctured polydisc

We start with an original Hartogs' result showing that the boundary of a domain of holomorphy can not be arbitrary. The domain in the following theorem is known as Hartogs' figure.

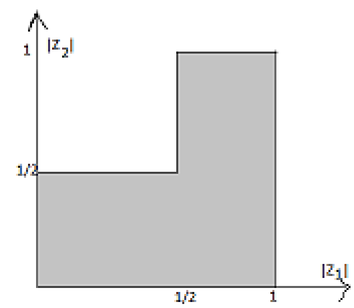

Figure 4.1: Hartogs figure

Theorem 4.1.1. *If f is an analytic function on a Hatrogs figure $D = D_1 \cup D_2$, where*

$$D_1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < \delta,\ |z_2| < 1\}$$

and

$$D_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1,\ 1 - \delta < |z_2| < 1\},$$

then f extends to be analytic on the whole unit bidisc

$$D = \{(z_1, z_2) : |z_1| < 1 \quad and \quad |z_2| < 1\}.$$

Proof. If we fix z_1 then the function $f(z_1, z_2)$ is analytic in an annulas $1-\delta < r < 1$ and hence can be expanded in Laurent series

$$\sum_{k=-\infty}^{\infty} c_k(z_1) z_2^k$$

and let

$$g(z_2) = f(z_1, z_2) = \sum_{k=-\infty}^{\infty} c_k(z_1) z_2^k$$

where the co-efficients are

$$c_k(z_1) = \frac{1}{(2\pi i)^n} \int_{|z_2|=r} \frac{f(z_1, z_2)}{z_2^{k+1}} dz_2 \tag{4.1}$$

where r is an arbitrary radius such that $1 - \delta < r < 1$.

This analytic function $g(z_2)$ can be extended analytically to the unit bidisc $D(0, 1)$ if all co-efficients with $k < 0$ becomes 0.

The function $f(z_1, z_2)$ is holomorphic and hence continuous jointly in both variables and it will be holomorphic in z_1 when $|z_1| < 1$ and $|z_2| = r$, therefore the itegral in equation (4.1) is a holomorphic function of z_1 by Theorem 3.3.8. It follows that $c_k(z_1)$ is a holomorphic function of z_1 in the unit disc. So for $|z_1| < \delta$, all coefficients of the Laurent series with $k < 0$ become 0 and by uniqueness Theorem 3.3.9 the coefficients $c_k(z_1)$ for all $k < 0$ becomes 0 in the whole unit disc.

So the Laurent series for $f(z_1, z_2)$ becomes the Maclaurin series

$$\sum_{k=0}^{\infty} c_k(z_1)(z_2)^k$$

for all values of z_1 in the unit disc.

If this power series with holomorphic coefficients converges uniformly and absolutely on every compact subset of $E = \{(z_1, z_2) : |z_1| < 1, |z_2| < r\}$ then by Abel's lemma the function $f(z_1, z_2)$ is holomorphic in the whole unit bidisc. Let $E_0 = \{|z_1| < m \quad and \quad |z_2| < r\}$ be a compact subset of E, where $m < 1$ is an arbitrary fix number. Since the function $f(z_1, z_2)$ is continuous so $|f(z_1, z_2)| \leq M$ on the compact set for $|z_1| < 1$ and $|z_2| = r$, where M is a finite number. And we will get the estimation of integral in (4.1) as follows

$$|c_k(z_1)| \leq \frac{M}{r^k}$$

and similarly for $|z_1| < m$ and $|z_2| < n$ where n an arbitrary positive number $< r$

$$|c_k(z_1)| \leq M\left(\frac{n}{r}\right)^k$$

and

$$\sum_{k=0}^{\infty} M\left(\frac{n}{r}\right)^k$$

is a convergent series. So by Comparison test

$$\sum_{k=0}^{\infty} c_k(z_1)(z_2)^k \leq \sum_{k=1}^{\infty} M\left(\frac{n}{r}\right)^k$$

converges uniformly and absolutely and hence by Abel's lemma $f(z_1, z_2)$ is a holomorphic function in the unit bidisc. □

4.2 Spherical Shell Theorem

Theorem 4.2.1. *Let f is a holomorphic function in a spherical shell $\{(z_1, z_2) \in \mathbb{C}^2 : \sigma^2 < |z_1|^2 + |z_2|^2 < 1\}$ for a positive $\sigma < 1$. Then f can be extended holomorphically to the whole unit ball.*

Proof. The given function is holomorphic in the spherical shell, if we fix z_1 then the function becomes holomorphic in an annulus $\sigma < |z_2| < 1$ and hence by Theorem 1.3.3 can be represented by one variable Laurent series

$$f(z_1, z_2) = \sum_{-\infty}^{\infty} c_k(z_1) z_2^k$$

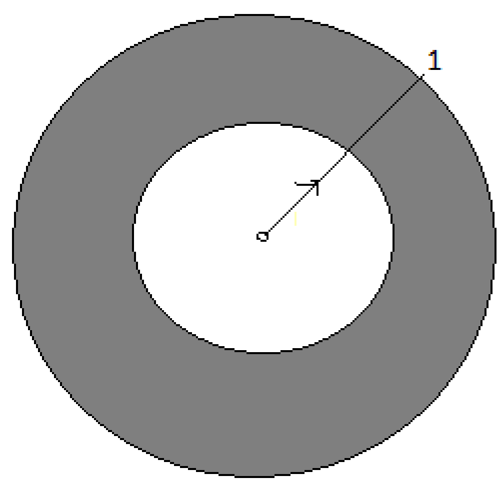

Figure 4.2: Spherical shell

$$c_k(z_1) = \frac{1}{2\pi i}\int_{|z_2|=r} \frac{f(z_1,z_2)}{z_2^{k+1}} dz_2 \tag{4.2}$$

where r is an arbitrary radius between 1 and $1-\sigma$. It follows from the Theorem 3.3.8 that the integral on the right hand side of Equation (4.2) is a holomorphic function of z_1 when $|z_1| < 1$ and $|z_2| = r$ and hence $c_k(z_1)$ is a holomorphic function of z_1 in the unit disc. Here it also is noted by Theorem 3.6.1 that for each fixed z_1 in the unit disc there is a neighborhood U of z_1 and a corresponding radius $s < r$ such that the cartesian product $U \times \{z_2 \in \mathbb{C} : |z_2| = s\}$ is contained in a compact subset of the spherical shell.

The above Laurent series is the Maclaurin series when $|z_1| < \sigma$. Which means that for $k < 0$ the coefficients will become zero when $|z_1| < \sigma$. But by Theorem 3.3.9 the holomorphic function $c_k(z_1)$ will be zero in the whole unit disc for $k < 0$ and hence

$$f(z_1, z_2) = \sum_0^\infty c_k(z_1) z_2^k.$$

So f has a power series representation and we can see from Theorem 4.1.1 that, this series converges uniformly on any compact subset of $\{(z_1, z_2), |z_1| < 1 \quad \textit{and} \quad |z_2| < r\}$. Hence f will be holomorphic in the unit ball. $\square$

4.3 Inhomogenous Cauchy-Riemann equation for $n \geq 2$ with compact support

The differential form

$$f = \sum_{j=1}^{n} (u_j dz_j + v_j d\overline{z}_j)$$

is said to be defined and of class C^p on $\Omega \subset \mathbb{C}^n$ if the coefficients u_j and v_j are defined and of class C^p on Ω as functions of real variables $x_1, y_1, ...x_n, y_n$. If all coefficients vanish on an open subset of Ω then the above form also vanishes on that open subset.

Support of f:Let there be a maximal open subset M of $\Omega \subset \mathbb{C}^n$ on which $f = 0$, then the complement of M is the support of f.

The 1-form:The above differential form is called a $(0,1)$ form or a 1-form if it does not contain any $u_j dz_j$.

The $\overline{\partial}$-Equation:

Let D be a domain in $\mathbb{C}^n$. Then the system of PDE's

$$\frac{\partial u}{\partial \overline{z}_j} = v_j \quad on \quad \mathbb{C}^n \quad for \quad j = 1, 2 \ldots n$$

subject to the integrability condition

$$\frac{\partial v_j}{\partial \overline{z}_k} = \frac{\partial v_k}{\partial \overline{z}_j} \quad \forall \quad j \neq k$$

are called the $\overline{\partial}$-*equation* or $\overline{\partial}$-*problem.*

Theorem 4.3.1. *Let M be a compact subset of $\mathbb{C}^n$, $n \geq 0$ with connected complement $M^c = \mathbb{C}^n - M$.*

Let $v = \sum_{j=1}^{n} v_j d\overline{z_j}$ be a $(0,1)$ form of class C^p $(1 \leq p \leq \infty)$ on $\mathbb{C}^n$ whose support belongs to M. Then the system of equations

$$\frac{\partial u}{\partial \overline{z}_j} = v_j \quad , j = 1, 2 \ldots, n$$

subject to the condition

$$\frac{\partial v_k}{\partial \overline{z}_j} = \frac{\partial v_j}{\partial \overline{z}_k}, \quad \forall \quad j, k$$

has a unique solution u of class C^p on $\mathbb{C}^n$ with support in M.

Proof. For simplicity of presentation we will prove it for $n = 2$. If we fix z_2 then the Cauchy-Green transform of v_1 with respect to z_1 is

$$u(z_1, z_2) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{v_1(\zeta, z_2)}{\zeta - z_1} d\varepsilon d\eta \tag{4.3}$$

$$= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{v_1(z_1 + \zeta, z_2)}{\zeta} d\varepsilon d\eta. \tag{4.4}$$

Since $v_1(z_1, z_2)$ which is the function of variable z_1 is smooth and has compact support so, by Theorem 1.2.3, the above transform (4.4) provides the solution of the equation

$$\frac{\partial u}{\partial \overline{z}_1} = v_1. \tag{4.5}$$

Now, in order to obtain an expression for $\frac{\partial u}{\partial \overline{z}_2}$ we will take (4.3) and

$$\frac{\partial u}{\partial \overline{z}_2} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial v_1(\zeta, z_2)}{\partial \overline{z}_2} \frac{1}{\zeta - z_1} d\varepsilon d\eta. \tag{4.6}$$

Since,

$$\frac{\partial v_1}{\partial \overline{z}_2} = \frac{\partial v_2}{\partial \overline{z}_1}.$$

Using above integrability condition in (4.6) we will get

$$\frac{\partial u}{\partial \overline{z}_2}(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial v_2}{\partial \overline{z}_1}(\zeta, z_2) \frac{1}{(\zeta - z_1)} d\varepsilon d\eta. \tag{4.7}$$

For fixed z_2, the smooth function $v_2(z_1, z_2)$ of one variable z_1 also has bounded support. So by corollary 1.2.2, the Equation (4.7) will be equal to $v_2(z_1, z_2)$

$$\frac{\partial u}{\partial \overline{z}_2} = v_2. \tag{4.8}$$

Hence from Equation (4.5) and (4.8),

$$\frac{\partial u}{\partial \overline{z}_j} = v_j \quad , \quad j = 1, 2$$

Since v is zero outside M, so it follows that $\overline{\partial} u = 0$ throughout M^c and hence u is holomorphic on the domain M^c. Infact $u = 0$ on M^c, Let us prove it.

Let there is a ball $B(0, R)$ of radius $R > 0$ such that the set M is contained in the ball. So $v_1(\zeta, z_2) = 0$ for $|z_2| > R$ and for arbitrary ζ. Thus for z_1 and all $|z_2| > R$, $u(z_1, z_2) = 0$. So that $u = 0$ on an open subset of M^c. But by uniqueness Theorem 3.3.9, $u = 0$ throughout the connected domain M^c □

4.4 Smooth approximate identities and cutoff functions

This section follows [6].

In the next section, we will prove the general result of the Hartogs' extension phenomenon. In that proof, we will use the smooth cutoff function and construct the smooth approximate solution. Cutoff functions can be constructed by the help of suitable C^∞ functions.

For example, the test function which is C^∞ on $\mathbb{R}$ and is defined as

$$\sigma(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \leq 0. \end{cases}$$

Now we define a C^∞ function τ on $\mathbb{R}$ with support $[-1, 1]$

$$\tau(x) = \sigma\{2(1+x)\}\sigma\{2(1-x)\} = \begin{cases} \exp\Big(-\frac{1}{1-x^2}\Big) & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \geq 1. \end{cases}$$

In case of $\mathbb{R}^n$ the function $\tau|x|$ provides a C^∞ function, where $|x|^2 = x_1^2 + ... + x_n^2$ and the support of τ is the closed unit ball $B(0,1)$. In order to make the integral over $\mathbb{R}^n$ equal to 1 we introduce $\rho(x)$

$$\rho(x) = c_n\tau(|x|) = \begin{cases} c_n \exp\Big(-\frac{1}{1-x^2}\Big) & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases} \quad , x \in \mathbb{R}^n$$

where $\int_{\mathbb{R}^n} \rho(x)dx = 1$ because of the choice of constant c_n.

From the last function, the important family of C^∞ functions can be derived as follows

$$\rho_\epsilon(x) = \frac{1}{\epsilon_n}\rho\Big(\frac{x}{\epsilon}\Big), \quad x \in \mathbb{R}^n, \quad \epsilon > 0$$

with support $\overline{B}(0,\epsilon)$. It can be noted that by change of scale, the following equality will hold

$$\int_{\mathbb{R}^n} \rho_\epsilon(x)dx = \int_{B(0,\epsilon)} \frac{1}{\epsilon^n}\rho\Big(\frac{x}{\epsilon}\Big)dx = \int_{B(0,1)} \rho(x)dx = 1.$$

Approximate Identities:

The standard example of C^∞ approximate identity on $\mathbb{R}^n$ relative to convolution is the above family of functions $\{\rho_\epsilon\}$,$\epsilon \to 0$ of $\rho(x)$ and $\rho_\epsilon(x)$. The family of functions $\{\rho_\epsilon\}$ is said to be an approximate identity for convolution if it possess the following properties

1. $\rho_\epsilon(x) \to 0$ as $\epsilon \to 0$, outside every neighbourhood of 0.

2. ρ_ϵ is integrable over $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} \rho_\epsilon(x)dx = 1$.

3. $\rho_\epsilon(x) \geq 0$ throughout $\mathbb{R}^n$.

From above three properties, it follows that for any continuous function f on $\mathbb{R}^n$ of compact support, when $\epsilon \to 0$

$$(f * \rho_\epsilon)(x) = \int_{\mathbb{R}^n} f(x-y)\rho_\epsilon(y)dy \to f(x)$$

and

$$f(x) = \int_{\mathbb{R}^n} f(x)\rho_\epsilon(y)dy.$$

Proposition 4.4.1. *For any set N in $\mathbb{R}^n$ and $\epsilon \to 0$, there is a cutoff function $\omega (0 \leq \omega \geq 1)$ on $\mathbb{R}^n$ which is equal to 1 on N and equal to zero at all points of $\mathbb{R}^n$ at a distance $\geq 2\epsilon$ from N.*

Proof. Let there is a set N in $\mathbb{R}^n$ and N_ϵ is the ϵ neighbourhood of N. Further, let χ_ϵ be the characteristic function of N_ϵ, which means that χ_ϵ is 1 on N_ϵ and is zero elsewhere. Now we will obtain the required ω as a convolution of the characterestic function with the C^∞ approximation ρ_ϵ to the identity

$$\omega(x) = (\chi_\epsilon * \rho_\epsilon)(x) = \int_{\mathbb{R}^n} \chi_\epsilon(x-y)\rho_\epsilon(y)dy = \int_{B(0,\epsilon)} \chi_\epsilon(x-y)\rho_\epsilon(y)dy \quad (4.9)$$

Or

$$\omega(x) = \int_{\mathbb{R}^n} \chi_\epsilon(y)\rho_\epsilon(x-y)dy = \int_{B(0,\epsilon)} \chi_\epsilon(y)\rho_\epsilon(x-y)dy. \quad (4.10)$$

If we take $x \in N$ then $|x-y|$ will be $< \epsilon$ for all $y \in B(0,\epsilon)$, So $(x-y) \in N_\epsilon$ and $\chi_\epsilon(x-y) = 1$ throughout $B(0,\epsilon)$ and then Equation (4.9) will become

$$\omega(x) = \int_{B(0,\epsilon)} \rho_\epsilon(y)dy$$

and hence

$$\omega(x) = 1.$$

Now if we take x outside $N_{2\epsilon}$ then $|x-y| > \epsilon$ for all y belongs to $B(0,\epsilon)$, so the point $(x-y)$ will lie outside N_ϵ and hence

$$\omega(x) = 0.$$

It can also be noted by definition of ρ_ϵ that ,$0 \leq \omega(x) \leq \int \rho_\epsilon = 1$ throughout $\mathbb{R}^n$.

By the method of Theorem 4.3.1, the continuous partial derivatives of ω exist and by repeating differentiation under the integral sign we can get the higher-order partial derivatives of ω. Hence ω be a smooth function.

□

4.5 Hartogs' Continuation Theorem

This is the general version of Hartogs' Phenomenon.

This theorem follows [6]

Theorem 4.5.1. *If K is a compact subset of an open set D in $\mathbb{C}^n (n \geq 2)$ and D/K is connected, then every analytic function f on D/K extends to an analytic function on D.*

Proof. Let K_ϵ be the ϵ neighborhood of K. Choose ϵ such that $0 < \epsilon < \frac{1}{3}d(k, \partial D)$ and $S = \mathbb{C}^n - \overline{K}_{3\epsilon}$, so that the the boundary ∂D is contained in S. And the unbounded component of S will be denoted by S_∞. By Proposition 4.4.1 there is a C^∞ cutoff function ω on $\mathbb{C}^n \sim \mathbb{R}^{2n}$ which is equal to 1 on S and equal to zero on K_ϵ.

Now we will find a holomorphic function F on D which will satisfy a certain condition and for that purpose first we construct a smooth approximate solution ϕ

$$\phi = \begin{cases} \omega f & \text{on} \quad D - K \\ 0 & \text{on} \quad K \end{cases}$$

and $\omega = 1$ on $D \cap S$ so $\phi = f$ on $D \cap S$. So ϕ provides a C^∞ continuation of f.

Now in order to obtain the required holomorphic extension F of f, we will subtract the non analytic part u from ϕ

$$F = \phi - u \quad . \tag{4.11}$$

From uniqueness Theorem 3.3.10, it will be sufficient to show that F be hlomorphic on D and F will be equal to f on a subdomain of $D - K$. If we take the subdomain $D \cap S_\infty$, then u will be 0 there, because $\phi = f$ on

$D \cap S_\infty$ and f is holomorphic and hence $\overline{\partial} F = 0$. So, it follows that u solves the $\overline{\partial}$ problem

$$\overline{\partial}\phi = \overline{\partial}u \quad on \quad D, \qquad u = 0 \quad on \quad D \cap S_\infty. \tag{4.12}$$

ϕ can be extended to a C^∞ $(0,1)$-form v on $\mathbb{C}^n$,

$$v = \begin{cases} \overline{\partial}\phi \quad on \quad D \\ 0 \quad on \quad \mathbb{C}^n - D \end{cases}.$$

The form v will satisfy the integrability conditions $\frac{\partial v_k}{\partial \bar{z} j} = \frac{\partial v_j}{\partial \bar{z}_k}$. It can also be noted that v has compact support, as support of $v \in \mathbb{C}^n - S$, which is the part of the compact set $M = \mathbb{C}^n - S_\infty$

Hence by theorem 4.3.1 there exists a unique u such that

$$\overline{\partial}u = v \quad on \quad \mathbb{C}^n, \quad u = 0 \quad on \quad M^c = S_\infty \quad .$$

Now by Equation (4.12) and the expression after that assures that $F = \phi - u$ is holomorphic on D. As $\phi = f$ on $D \cap S_\infty$, so $F = f$ throughout the connected domain $D - K$. Hence f can be extended analytically on D. $\square$

Conclusion

We proved the *fundamental theorem of Hartogs*, which states that separate analyticity implies joint analyticity. This is one of the most interesting features of analyticity in several complex variables. There is no analog of this feature in the case of a single complex variable or for the smooth functions of several variables.

For example the function

$$f(x, y) = \frac{xy}{x^2 + y^2}, \qquad f(0, 0) = 0$$

is differentiable with respect to the both variables separately, but is not continuous at the point $(0, 0) \subset \mathbb{R}^2$.

We also proved different results and general form of *Hartogs' extension phenomenon*, which reveals a basic difference between the analytic function of a single complex variable and analytic function of the several variables. For the proof, we used the $\bar{\partial}$ - technique which is one of the main tools of modern complex analysis.

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