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Wick-rotations of pseudo-Riemannian Lie groups

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ABSTRACT

We study Wick-rotations of left-invariant metrics on Lie groups, using results from real GIT (Helleland and Hervik, 2018; Helleland and Hervik, 2019). An invariant for Wick-rotation of Lie groups is given, and we describe when a pseudo-Riemannian Lie group (a Lie group with a left-invariant metric) can be Wick-rotated to a Riemannian Lie group. We define a Cartan involution of a general Lie algebra, and prove a general version of É. Cartan's result, namely the existence and conjugacy of Cartan involutions.

1. Introduction

This paper is motivated first of all by the study of Wick-rotations of pseudo-Riemannian manifolds defined in [3]. Given a pseudo-Riemannian manifold (M, g) of signature (p, q), it is interesting know whether it can be Wick-rotated to another space $(\tilde{M}, \tilde{g})$ (w.r.t. a fixed point $p \in M \cap \tilde{M}$) of signature $\tilde{p} + \tilde{q} = p + q$. In [2–4] the isometry action of the pseudo-orthogonal group $O(p, q)$ acting on tensors restricted to p is explored. For instance it is proved that if $\tilde{p} = 0$ (i.e. $\tilde{g}$ is Riemannian) then there is a Cartan involution of the metric $\theta \in O(p, q)$ (at p) which fixes the Riemann tensor R under the isometry action, i.e. $\theta \cdot R = R$. Thus (M, g) is *Riemann purely electric* (*RPE*) at p. More generally it is proved that for a space to be *purely electric* (respectively *purely magnetic*) or (RPE) (respectively *Riemann purely magnetic*) is preserved under a Wick-rotation at a common fixed point p.

A particular subclass of Wick-rotations which is of interest in its own right and deserves to be explored, is the class of *Lie groups G* equipped with left-invariant metrics, so called *pseudo-Riemannian Lie groups*. If we look at a semi-simple complex Lie group $G^{\mathbb{C}}$ equipped with the left-invariant Killing form: $-\kappa$, then there are natural examples of Wick-rotations to find at the identity point, simply because there exist real forms. Moreover by the theory of semi-simple Lie groups, one may always Wick-rotate a real form $(G, -\kappa) \subset (G^{\mathbb{C}}, -\kappa)$ to a Riemannian Lie group, simply because of the existence of a Cartan involution of the Lie algebra $\mathfrak{g}$. Thus motivated by this example, then for a general pseudo-Riemannian Lie group (G, g), an interesting question one may ask:

Given a pseudo-Riemannian Lie group (G, g)*, when can it be Wick-rotated to a Riemannian Lie group* $(\tilde{G}, \tilde{g})$*?*

Suppose (G, g) is Wick-rotated to a Riemannian Lie group $(\tilde{G}, \tilde{g})$, then in view of the results given in [2–4], then the so called Wick-rotatable tensors restricted to $\mathfrak{g}$ must be fixed by the isometry action (induced from the metric) of some (linear) Cartan involution $\theta \in O(p, q)$ of the metric. This could for instance be the Riemann tensor R (as mentioned above), and is related to the fact that R can be embedded into the same complex orbit as $\tilde{R}$ (the Riemann tensor of $(\tilde{G}, \tilde{g})$ restricted to $\tilde{\mathfrak{g}}$), i.e.

$$O(p + q, \mathbb{C}) \cdot R \ni \tilde{R}.$$

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However some tensors for a left-invariant metric (for instance the Levi-Civita connection, the Riemann tensor and so on) are very interlinked with the Lie bracket of the Lie algebra $\mathfrak{g}$. Moreover in the semi-simple case (equipped with the left-invariant Killing form) such tensors are naturally fixed by the Cartan involutions of the Lie algebra: $\theta \in Aut(\mathfrak{g})$. For example the Levi-Civita connection is given by: $\nabla_x y = \frac{1}{2}[x, y]$, thus naturally $\theta \cdot \nabla_x y = \nabla_x y$.

The author of this paper therefore pondered about the existence of a Cartan involution: $\theta \in Aut(\mathfrak{g})$, for a general left-invariant metric (on a general Lie group G) which can be Wick-rotated to a Riemannian Lie group $\tilde{G}$.

We prove an invariant for Wick-rotations of Lie groups, and give a complete answer to the question above, where we show that the answer is precisely related to the existence of a Cartan involution of the Lie algebra. Our main result of this paper is Theorem 3.1:

Theorem A. *Suppose (G, g) is a pseudo-Riemannian Lie group that can be Wick-rotated to another Lie group $(\tilde{G}, \tilde{g})$. Then there exists a Cartan involution of $\mathfrak{g}$ if and only if there exists a Cartan involution of $\tilde{\mathfrak{g}}$.*

We begin this paper by defining every notion we shall use throughout, and recall the definitions of Wick-rotations in [3]. Some new definitions are also given, in particular we define a *Wick-rotation of a Lie group*, and a *Cartan involution* of a general Lie algebra. We also state the results we use from [4], which makes the proofs easier to follow.

Remark 1.1. In this paper a Riemannian space shall always denote the signature: $(+, +, \ldots, +)$, and a Lorentzian space shall denote the signature: $(+, +, \ldots, +, -)$ and so on. The anti-isometry map $g \mapsto -g$ induces an isomorphism $O(p, q) \cong O(q, p)$. If we change signature via this anti-isometry map, then our results in this paper will be related precisely via this map as well. Moreover using a right-invariant metric instead of a left-invariant metric does not change the results of this paper.

Conventions: Throughout this paper κ shall denote the Killing form of a Lie algebra. A product of vector spaces $V \times V$ shall often be denoted by just V^2. A complex Lie group shall always be denoted by the symbol: $G^{\mathbb{C}}$.

2. Preliminaries

2.1. Real forms and left-invariant metrics

In this paper a real Lie group G shall be said to be an *immersive real form* of a complex Lie group $G^{\mathbb{C}}$, if there is a real immersion $G \to G^{\mathbb{C}}$ (of Lie groups) where $G^{\mathbb{C}}$ is viewed as a real Lie group, such that $\mathfrak{g}$ is embedded as a real form of $\mathfrak{g}^{\mathbb{C}}$ (the Lie algebra of $G^{\mathbb{C}}$). If the immersion is also injective then we shall call G a *virtual real form*. A virtual real form G which is also an embedding (i.e. the image of G is closed in $G^{\mathbb{C}}$), we shall say that the real form is an *embedded real form*. An embedded real form which also satisfies: $G^{\mathbb{C}} = G \cdot G_0^{\mathbb{C}}$ (abstract group product) shall be said to be a *real form*.

Note that a connected embedded real form is also a real form. All these specialised “complexifications” divide the Lie groups into different classes. For instance if G is a connected semi-simple Lie group, then it is a fact that G is a virtual real form if and only if G is linear.

One shall note that given any 1-connected real Lie group G, then we can complexify the Lie algebra via an inclusion i: $\mathfrak{g} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$. We can find a complex 1-connected Lie group: $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. One can find a smooth map (of real Lie groups): $G \to G^{\mathbb{C}}$ with differential i, thus G is an immersive real form of $G^{\mathbb{C}}$.

We shall abuse notation and write $G \subset G^{\mathbb{C}}$ for an immersive real form.

Example 2.1. Consider the complex orthogonal group: $O(4, \mathbb{C})$, then the map: $g \mapsto I_{3,1}\bar{g}I_{3,1}$, is a conjugation map (i.e. the differential is a conjugation map), where $(I_{3,1})_{ii} = +1$ for $1 \leq i \leq 2$, $(I_{3,1})_{33} = -1$ and zero otherwise. The fix points of this map are just $O(1, 3)$, which is an example of a real form of $O(4, \mathbb{C})$. Consider the universal covering group $G := \widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$, then it is a fact that G is not a virtual real form of any complex Lie group. However G is an immersive real form of $SL_2(\mathbb{C})$.

Let G be a real Lie group, then a left-invariant metric g on G is a pseudo-Riemannian metric satisfying:

$$g_{gh}(L_{gh^*}(x_h), L_{gh^*}(y_h)) = g_h(x_h, y_h), \forall g, h \in G, \ \forall x_h, y_h \in T_hG,$$

where L_{g^*} is the push-forward of the translation map: $G \xrightarrow{L_g} G$: $h \mapsto gh$. Instead of writing $g_e(-, -)$ for the metric at the identity point, we simply write just $g(-, -)$. A bi-invariant metric g on a real Lie group G is a left-invariant metric which is also right-invariant i.e. L_g above is replaced with $R_g : h \mapsto hg$.

On a real vector space V a symmetric non-degenerate bilinear form g shall be referred to as a *pseudo-inner product*, and an inner product in the case of positive definite. A pair (V, g) shall be referred to as a pseudo-inner product space (respectively inner product space). If we have a Lie algebra $\mathfrak{g}$ with a pseudo-inner product g which satisfies:

$$g([x, y], z) = g(x, [y, z]), \quad x, y, z \in \mathfrak{g},$$

then g shall be called *invariant*. Such a pair: $(\mathfrak{g}, g)$ is called a *quadratic Lie algebra*. For example the pair: $\left(\mathfrak{sl}_2(\mathbb{R}), -\kappa\right)$ is a quadratic Lie algebra, however the 3-dimensional Heisenberg Lie algebra: $\mathfrak{h}_3(\mathbb{R})$, is never a quadratic Lie algebra. We recall that an ideal $\mathfrak{I} \lhd \mathfrak{g}$ is called *non-degenerate* if $\mathfrak{g} = \mathfrak{I} \oplus \mathfrak{I}^{\perp}$ w.r.t. the invariant form g. In the case that $\mathfrak{g}$ is a reductive Lie algebra, then all ideals are in fact non-degenerate.

A *holomorphic inner product* $g^{\mathbb{C}}$ on a complex vector space $V^{\mathbb{C}}$ shall be a symmetric non-degenerate complex bilinear form. The definitions of left-invariance and so on above are analogous in the case of a complex Lie group equipped with a holomorphic metric.

Definition 2.1. A real Lie group G equipped with a left-invariant metric g, denoted (G, g) shall be called a *pseudo-Riemannian Lie group*. If g is also a Riemannian metric then the pair (G, g) shall be called a *Riemannian Lie group*. A complex Lie group $G^{\mathbb{C}}$ equipped with a left-invariant holomorphic metric, shall be called a *holomorphic Riemannian Lie group* (or a *complex Riemannian Lie group*).

Definition 2.2. Let (G, g_1) and (H, g_2) be two pseudo-Riemannian Lie groups. Then G is said to be isometric to H if there exists a Lie group isomorphism: $G \xrightarrow{F} H$, such that $F_* : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of pseudo-inner product spaces: $(\mathfrak{g}, g_1) \cong (\mathfrak{h}, g_2)$. The spaces are said to be locally isometric if there exists a local homomorphism $G \supset U \xrightarrow{F} V \subset H$ such that F_* is an isomorphism of pseudo-inner product spaces: $(\mathfrak{g}, g_1) \cong (\mathfrak{h}, g_2)$.

The left-invariant metrics on a real Lie group G are in bijections with the pseudo-inner products on the Lie algebra $\mathfrak{g}$. So we shall always work with a pseudo-inner product g on the Lie algebra and induce a left-invariant metric on the Lie group by:

$$g_h(x_h, y_h) := g\left(L_{h^{-1}_*}(x_h), L_{h^{-1}_*}(y_h)\right), \quad x_h, y_h \in T_hG.$$

We note that for a compact Lie group G, we can always complexify it to a complex Lie group: $G^{\mathbb{C}}$, such that $G \subset G^{\mathbb{C}}$ is a real form, by using the universal complexification group. In particular starting from a compact Lie group with a left-invariant metric we naturally have a candidate for a holomorphic Riemannian Lie group such that $G \subset G^{\mathbb{C}}$ is a real form. Recall that the universal complexification group of a real Lie group G, is a pair: $(G^{\mathbb{C}}, \eta)$, where η is a real Lie homomorphism: $G \to G^{\mathbb{C}}$, satisfying the universal property (see for instance [5]). For example the pseudo-orthogonal groups: $O(p, q)$ has universal complexification group $O(p + q, \mathbb{C})$.

2.2. Wick-rotations of pseudo-Riemannian manifolds

We recall some of the definitions of Wick-rotations given in [3], and define a Wick-rotation of a pseudo-Riemannian Lie group.

Definition 2.3. Given a holomorphic inner product space $(E, g^{\mathbb{C}})$. Then if $V \subset E$ is a real linear subspace for which $g := g^{\mathbb{C}}\big|_V$ is non-degenerate and real valued, i.e., $g(X, Y) \in \mathbb{R},\ \forall X, Y \in V$, we will call V a *real slice*.

Remark 2.1. In this paper we always assume $V \subset (E, g^{\mathbb{C}})$ has the same real dimension as the complex dimension of E. Thus V is also a real form of E, i.e. there is a conjugation map $E \xrightarrow{\sigma} E$ with fix points V. We shall simply refer to $V \subset (E, g^{\mathbb{C}})$ as a real form in such a case, to mean both a real slice and a real form.

Thus in the definition $(V, g := g^{\mathbb{C}}\big|_V)$ is a pseudo-inner product space, and if (p, q) denotes the signature of g, then the isometry group $O(p, q)$ of (V, g) is a real Lie group and is a real form of $O(p + q, \mathbb{C})$ (the isometries of $(E, g^{\mathbb{C}})$). Indeed if σ is the conjugation map of V in E then note the involution F of real Lie groups:

$$g \mapsto \sigma g \sigma, \ \ g \in O(p + q, \mathbb{C}).$$

The differential of this map is a conjugation map, and $O(p, q)$ is the fix points of F, i.e. is a real form. Such a map F is often called a *real structure*.

Definition 2.4. Given a complex (holomorphic) manifold $M^{\mathbb{C}}$ with complex (holomorphic) Riemannian metric $g^{\mathbb{C}}$. If a submanifold $M \subset M^{\mathbb{C}}$ for any point $p \in M$ we have that T_pM is a real slice of $(T_pM^{\mathbb{C}}, g^{\mathbb{C}})$ (in the sense of Definition 2.3), we will call M a real slice of $(M^{\mathbb{C}}, g^{\mathbb{C}})$.

This definition implies that the induced metric from $M^{\mathbb{C}}$ is real valued on M. M is therefore a pseudo-Riemannian manifold.

Definition 2.5 (*Wick-related Spaces*)**.** Two pseudo-Riemannian manifolds M and $\tilde{M}$ are said to be *Wick-related* if there exists a holomorphic Riemannian manifold $(M^{\mathbb{C}}, g^{\mathbb{C}})$ such that M and $\tilde{M}$ are embedded as real slices of $M^{\mathbb{C}}$.

Definition 2.6 (*Wick-rotation*)**.** If two Wick-related spaces (of the same real dimension) intersect at a point p in $M^{\mathbb{C}}$, then we will use the term *Wick-rotation*: the manifold M can be Wick-rotated to the manifold $\tilde{M}$ (with respect to the point p).

We now define a Wick-rotation of a pseudo-Riemannian Lie group:

Definition 2.7 (*Wick-rotation of a Pseudo-Riemannian Lie Group*)**.** Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ be two immersive real forms which are Wick-related in $(G^{\mathbb{C}}, g^{\mathbb{C}})$ for $g^{\mathbb{C}}$ a left-invariant holomorphic metric. Then we shall say that the pseudo-Riemannian Lie group (G, g) is Wick-rotated to $(\tilde{G}, \tilde{g})$.

Thus from the definition: $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ is a real slice of Lie groups, and shall write (p, q) for the signature of g. If there is another real slice $(\tilde{G}, \tilde{g}) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ of Lie groups, then we shall refer to the signature of $\tilde{g}$ as $(\tilde{p}, \tilde{q})$. We shall often just say a Wick-rotations of Lie groups. Note that two Lie groups which are Wick-related are also Wick-rotated at the identity point $p := 1$.

The definition implies that two Wick-rotatable metrics on real Lie groups are left-invariant themselves, and also note that a Wick-rotation of Lie groups induces in the obvious way a Wick-rotation of the identity components. Moreover the property of bi-invariance for connected groups is an invariant:

Proposition 2.1. *Suppose (G, g) is Wick-rotatable to $(\tilde{G}, \tilde{g})$ and they are both connected. Then $g(-, -)$ is bi-invariant if and only if $\tilde{g}(-, -)$ is bi-invariant.*

Proof. The proofs given in ([9], Lemma 7.1 and 7.2) also hold for pseudo-Riemannian left-invariant metrics, with $\mathfrak{o}(n)$ replaced with $\mathfrak{o}(p, q)$. Moreover if the metric $g(-, -)$ is bi-invariant, then because $ad(\mathfrak{g}) \subset \mathfrak{o}(p, q) \subset \mathfrak{o}(n, \mathbb{C})$, and $ad(\mathfrak{g})^{\mathbb{C}} = ad(\mathfrak{g}^{\mathbb{C}})$ it follows that the holomorphic metric must also be bi-invariant, thus also $\tilde{g}(-, -)$. The converse is identical. □

Note that the property of being connected or simply connected are not necessarily preserved under a Wick-rotation. However under a Wick-rotation of real forms, then being connected is conserved.

Example 2.2. Let $SL_2(\mathbb{R}) \subset SL_2(\mathbb{C}) \supset SU(2)$ be the natural inclusions. Then they are real forms, and Wick-rotated w.r.t. to the holomorphic Killing form κ on $\mathfrak{sl}_2(\mathbb{C})$. Note that $(SL_2(\mathbb{R}), \kappa)$ is Lorentzian and $(SU(2), \kappa)$ has signature: $(-, -, -)$.

We also define:

Definition 2.8. Let $V \subset (E, g^{\mathbb{C}})$ be a real slice. We say an involution $V \xrightarrow{\theta} V \in O(p, q)$, is a *Cartan involution* of $g := g^{\mathbb{C}}\big|_V$, if $g_\theta(\cdot, \cdot) := g^{\mathbb{C}}\big|_V(\cdot, \theta(\cdot))$, is an inner product on V. If $\theta = 1$ then V is said to be a *compact real slice*, or in the case that V is also a real form, then V shall be said to be a *compact real form*.

Note the resemblance (in the definition) with a compact real form of a complex semi-simple Lie algebra and its Killing form. In the case of Lie algebras: $(\mathfrak{g}, g) \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$, then a Cartan involution θ of g is not necessarily a homomorphism of Lie algebras, since we do not know it they exist. We do not even know if there exists a compact real form which is also a Lie subalgebra of $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$. But we know if $\mathfrak{g}$ is semi-simple, and $g^{\mathbb{C}} = -\kappa$, then there exists a Cartan involution θ which is also homomorphism of the Lie algebra.

But more generally we shall define:

Definition 2.9. Let $\mathfrak{g} \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a real form. A Cartan involution θ of $\mathfrak{g}$ is a Cartan involution of $g := g^{\mathbb{C}}_{|\mathfrak{g}}(-, -)$ which is also a homomorphism of Lie algebras.

Thus a Cartan involution of g is only a linear Cartan involution of the pseudo-inner product g, but a Cartan involution of $\mathfrak{g}$ is a Cartan involution of g which is also a homomorphism of Lie algebras. Currently at this point we only know that Cartan involutions of $\mathfrak{g}$ exist when $\mathfrak{g}$ is abelian or $\mathfrak{g}$ is semi-simple equipped with the Killing form: $-\kappa$. One shall note that there are examples where they do not exist, indeed by changing the sign to: κ, then it is straighforward to show that there are no Cartan involutions of $\mathfrak{g}$.

Definition 2.10. Two real forms V and $\widetilde{V}$ of E are said to be compatible if their conjugation maps commute, i.e. $[\sigma, \tilde{\sigma}] = 0$.

Often we shall refer to a pair $(V, \tilde{V})$ as a compatible pair, to mean that the spaces are compatible.

We recall from [3], that if $(E, g^{\mathbb{C}})$ is a holomorphic inner product space, and $V, \tilde{V}$ and W are real forms such that W is a compact real form (i.e. of Euclidean signature), then if they are pairwise compatible, the triple: $\Big(V, \tilde{V}, W\Big)$, is said to be a *compatible triple*. Note that Example 2.2 is an example of a compatible triple:

$$\Big(V := \mathfrak{sl}_2(\mathbb{R}), \tilde{V} := \mathfrak{su}(2), W := \mathfrak{su}(2)\Big).$$

We shall call the eigenspace decomposition of a Cartan involution: θ, for the Cartan decomposition.

Remark 2.2. By the uniqueness of a signature associated to a pseudo-inner product g then all Cartan involutions of g are conjugate in $O(p, q)$. In fact given two Cartan involutions: θ_j $(j = 1, 2)$ then $g \mapsto \theta_j g \theta_j$ is a global Cartan involution of $O(p, q)$. Thus if $g\theta_1 g^{-1} = \theta_2$ for some $g \in O(p, q)$, then writing $g = k_2 e^x$, where k_2 commutes with θ_2 and $x \in \mathfrak{o}(p, q)$, we obtain $\theta_1 = e^x \theta_2 e^{-x}$, and therefore θ_1, θ_2 are conjugate by an element $g \in O(p, q)_0$.

Suppose now we have a Wick-rotation of two real Lie groups: $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$. Let $\theta \in O(p, q)$ be a Cartan involution of the metric g, and let W denote the corresponding unique compact real form associated with θ, i.e. $W := V_+ \oplus iV_-$, where $\mathfrak{g} = V_+ \oplus V_-$ is the Cartan decomposition. Then by [4] it is possible to find a real form $\tilde{V} \subset \mathfrak{g}^{\mathbb{C}}$ (as vector spaces) and a linear isomorphism: $\tilde{V} \xrightarrow{\phi} \tilde{\mathfrak{g}}$ such that $\phi^{\mathbb{C}} \in O(n, \mathbb{C})$, and $(\mathfrak{g}, \tilde{V}, W)$ is a compatible triple. So consider the triple: $\Big(\mathfrak{o}(p, q), \mathfrak{o}(\tilde{p}, \tilde{q}), \mathfrak{o}(n)\Big)$, of Lie algebras of the isometry groups associated with the compatible triple $(\mathfrak{g}, \tilde{V}, W)$.

Then the following straightforward result is important to note:

Lemma 2.1 ([3], Lemma 3.6)**.** *The triple of real forms:* $\Big(\mathfrak{o}(p, q), \mathfrak{o}(\tilde{p}, \tilde{q}), \mathfrak{o}(n)\Big)$*, embedded into* $\mathfrak{o}(n, \mathbb{C})$ *is a compatible triple of Lie algebras.*

Thus we note that up to an isometry $g \in O(n, \mathbb{C})$ we may assume our two Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ (viewed as a vector space) form a compatible triple with a compact real form $W \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$.

2.3. Real GIT on compatible representations

In this section we recall some definitions and results of [4] that we shall use. We consider certain type of groups here. When considering a real form: $G \subset G^{\mathbb{C}}$, then $G^{\mathbb{C}}$ shall be of type linearly complex reductive, and G should either be linearly real reductive, or in the case where $G^{\mathbb{C}} \subset GL(V^{\mathbb{C}})$ is defined over $\mathbb{R}$, the real points: $G := GL(V) \cap G^{\mathbb{C}}$. This is the assumptions in the paper [4]. Thus we may for instance use the pseudo-orthogonal group $O(p, q) \subset O(n, \mathbb{C})$ defined as the isometry group of some pseudo-inner product space: $(V, g) \subset (V^{\mathbb{C}}, g^{\mathbb{C}})$. A compact real form of $G^{\mathbb{C}}$ shall always be denoted by U.

Let $G \subset GL(V)$ be such a group. A Cartan involution θ of $\mathfrak{g}$ is now a Cartan involution in the sense of a reductive Lie algebra. Recall that this means that θ is the restriction of a Cartan involution of $\mathfrak{gl}(V)$. In view of Definition 2.9, θ is a Cartan involution of $(\mathfrak{g}, g)$, with $g = \lambda\kappa \oplus B$ $(\lambda < 0)$, where κ is the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ and B a pseudo-inner product on $\mathfrak{z}(\mathfrak{g})$. We refer to for example [4] or [11] where such Cartan involutions are considered in more detail. A global Cartan involution Θ with $d\Theta = \theta$ of G always exists for such groups. For example the class of linear semisimple Lie groups of finitely many connected components (fcc) are one such class.

Definition 2.11. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ be two real Lie subgroups of a complex Lie group such that the real Lie algebras are real forms of $\mathfrak{g}^{\mathbb{C}}$. Then we say G and $\tilde{G}$ are *compatible* if the Lie algebras are compatible.

Definition 2.12. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ and $U \subset G^{\mathbb{C}}$ be real Lie subgroups of a complex Lie group such that the real Lie algebras are real forms of $\mathfrak{g}^{\mathbb{C}}$. Moreover assume U is compact. Then we say $\Big(G, \tilde{G}, U\Big)$ is a *compatible triple* if the Lie algebras are pairwise compatible.

If we use Lemma 2.1, in the context of Wick-rotations (see the previous section), then the triple of isometry groups: $\Big(O(p, q), O(\tilde{p}, \tilde{q}), O(n)\Big)$ form a compatible triple when the pseudo-inner product spaces they are isometries of, form a compatible triple.

Definition 2.13 ([11])**.** Let $G \xrightarrow{\rho_V^G} GL(V)$ be a real representation, then ρ_V^G is said to be a *balanced representation* if there exist an involution $V \xrightarrow{\theta} V$, and a global Cartan involution: $G \xrightarrow{\Theta} G$ such that:

$$\Big(\forall g \in G\Big)\Big(\rho_V^G(\Theta(g)) = \theta \circ \rho_V^G(g) \circ \theta\Big).$$

Thus if we have an involution θ of V balancing our action, then w.r.t. the global Cartan involution Θ of G with Cartan decomposition: $G = Ke^{\mathfrak{p}}$, there exists a pseudo-inner product $g(-, -)$ on V such that θ is a Cartan involution of $g(-, -)$, and the inner product $g_\theta(-, -) := g(-, \theta(-))$ is K-invariant. Let $\mathcal{M}(G, V)$ denote the minimal vectors of our action, i.e. those $v \in V$ satisfying: $\|g \cdot v\| \geq \|v\|$ for all $g \in G$, where $\|v\|^2 := g_\theta(v, v)$. Then if $V = V_+ \oplus V_-$ is the Cartan decomposition, we naturally have $V_+ \cup V_- \subset \mathcal{M}(G, V)$. The Cartan involutions of $g(-, -)$ which are conjugate by the action of G to θ are defined as the *inner Cartan involutions* of $g(-, -)$.

A complex action: $\rho^{\mathbb{C}}$ of $G^{\mathbb{C}}$ acting on $V^{\mathbb{C}}$ is said to be a complexified action of a real action ρ_V^G if $\rho^{\mathbb{C}}(g)(v) = \rho(g)(v)$ for all $g \in G$ and $v \in V$.

Definition 2.14. Let $G \subset G^{\mathbb{C}} \supset \tilde{G}$ be real forms, and $G \xrightarrow{\rho_V^G} GL(V)$ and $\tilde{G} \xrightarrow{\rho_{\tilde{V}}^{\tilde{G}}} GL(\tilde{V})$ be real representations of Lie groups. Suppose $G^{\mathbb{C}} \xrightarrow{\rho^{\mathbb{C}}} GL(V^{\mathbb{C}})$ is a complexified action of both ρ_V^G and $\rho_{\tilde{V}}^{\tilde{G}}$. Then we say that ρ_V^G is *compatible* with $\rho_{\tilde{V}}^{\tilde{G}}$, if the following two criteria are fulfilled:

(1) G and $\tilde{G}$ are compatible real forms of $G^{\mathbb{C}}$.

(2) V and $\tilde{V}$ are compatible real forms of $V^{\mathbb{C}}$.

Definition 2.15. Let ρ_V^G, $\rho_{\tilde{V}}^{\tilde{G}}$ and ρ_W^U be pairwise compatible representations, where $U \subset G^{\mathbb{C}}$, is a compact real form. Then the triple: $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$ is said to be a *compatible triple*.

If we have such a compatible triple, then all the real actions in the triple are balanced, and we can choose pseudo-inner products $g(-,-)$ and $\tilde{g}(-,-)$ on V and $\tilde{V}$ respectively, in such a way that they restrict from the same Hermitian form on $V^{\mathbb{C}}$. Moreover if τ denotes the conjugation map of W in $V^{\mathbb{C}}$ then it restricts to Cartan involutions: θ (of g) and $\tilde{\theta}$ (of $\tilde{g}$). The Cartan involutions also balance the real actions respectively. In particular the inner products g_θ and $\tilde{g}_{\tilde{\theta}}$ both restrict from the U-invariant Hermitian inner product $H(-, \tau(-))$. The minimal vectors satisfy:

$$\mathcal{M}(G, V) \subset \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}) \supset \mathcal{M}(\tilde{G}, \tilde{V}), \quad W \subset \mathcal{M}(G^{\mathbb{C}}, V^{\mathbb{C}}).$$

Denote the Cartan decompositions by $V = V_+ \oplus V_-$ and $\tilde{V} = \tilde{V}_+ \oplus \tilde{V}_-$ respectively.

Definition 2.16. Let $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}\right)$ be a compatible pair. Suppose $v \in V$ and $\tilde{v} \in \tilde{V}$ are such that $\tilde{v} \in G^{\mathbb{C}} v$, then we shall say that Gv is *compatible* with $\tilde{G}\tilde{v}$. We write $Gv \sim \tilde{G}\tilde{v}$.

It is important to note the following result:

Theorem 2.1 ([4]). *Let $\left(\rho_V^G, \rho_{\tilde{V}}^{\tilde{G}}, \rho_W^U\right)$ be a compatible triple. Suppose $v \in V$ and $\tilde{v} \in \tilde{V}$ are such that: $\tilde{G}\tilde{v} \sim Gv$. Then $Gv \cap V_+ \neq \emptyset$ (respectively $Gv \cap V_- \neq \emptyset$) if and only if $\tilde{G}\tilde{v} \cap \tilde{V}_+ \neq \emptyset$ (respectively $\tilde{G}\tilde{v} \cap \tilde{V}_- \neq \emptyset$).*

Observe that if there exists $v_+ \in Gv$, then $\theta(v_+) = v_+$, i.e. if $g \in G$ is such that $g \cdot v = v_+$, then there is an inner Cartan involution θ' of $g(-,-)$ such that $\theta'(v) = v$ using g.

We shall also state the following important result:

Theorem 2.2 ([4]). *Let (ρ_V^G, ρ_W^U) be a compatible pair. Let $v \in V$, then the following statements are equivalent:*

A There exists $w \in W$ such that $Uw \sim Gv$.
B There exists an inner Cartan involution $V \xrightarrow{\theta} V$ such that $\theta(v) = v$.
C There exists $w \in W$ such that $Uw \cap Gv \neq \emptyset$.

In fact if there is a $w \in W$ and $v \in V$ such that $Uw \sim Gv$ then:

$$\emptyset \neq Uw \cap Gv = Gv \cap \mathcal{M}(G, V) = Kv,$$

where $K = U \cap G$.

A worked out example of compatible representations is given in the next section in the context of Wick-rotations of Lie groups.

2.4. The isometry action on bilinear maps into the Lie algebra

In this section we shall consider the action that we are going to use to prove our main result of this paper. We shall explain in detail that under a Wick-rotation, the isometry groups of the pseudo-inner product spaces induces compatible representations (see Defn. Section 2.3).

Suppose we have a Wick-rotation of pseudo-Riemannian Lie groups:

$(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$. As we have seen we can choose a map $g \in O(n, \mathbb{C})$ such that we obtain a compatible triple: $(\mathfrak{g}, \tilde{V}, W)$, with $\tilde{V} := g(\tilde{\mathfrak{g}})$. We shall denote $\tilde{g}$ also for the pseudo-inner product on $\tilde{V}$ restricted from $g^{\mathbb{C}}$. We can choose a pseudo-orthonormal basis: $\{e_1, \dots, e_p, \dots, e_n\}$ (of g) and similarly $\{\tilde{e}_1, \dots, \tilde{e}_{\tilde{p}}, \dots, \tilde{e}_n\}$ (of $\tilde{g}$), such that W is the real span of both the sets: $Y := \{e_1, \dots, e_p, ie_{p+1} \dots ie_n\}$ and $\tilde{Y} := \{\tilde{e}_1, \dots, \tilde{e}_{\tilde{p}}, i\tilde{e}_{\tilde{p}+1}, \dots, i\tilde{e}_n\}$. Denote the corresponding Cartan involutions by θ (of g) and $\tilde{\theta}$ (of $\tilde{g}$). Note that Y and $\tilde{Y}$ are both an orthonormal basis of $g^{\mathbb{C}}$.

Consider the complex isometry action of $O(n, \mathbb{C})$ on $\mathfrak{g}^{\mathbb{C}}$ by $g \cdot x := g(x)$. This action restricts to the real isometry actions of $O(p, q)$ on $\mathfrak{g}$ and $O(\tilde{p}, \tilde{q})$ on V respectively. Let $\mathcal{V}$ and $\tilde{\mathcal{V}}$ denote the real vector spaces of bilinear maps: $\mathfrak{g}^2 \to \mathfrak{g}$ (respectively $\tilde{V}^2 \to \tilde{V}$). Thus $\mathcal{V} \subset \mathcal{V}^{\mathbb{C}} \supset \tilde{\mathcal{V}}$ are real forms, where $\mathcal{V}^{\mathbb{C}}$ is the complex vector space of complex bilinear maps: $(\mathfrak{g}^{\mathbb{C}})^2 \to \mathfrak{g}^{\mathbb{C}}$. The complex isometry action naturally extends to a complex action of $O(n, \mathbb{C})$ on $b \in \mathcal{V}^{\mathbb{C}}$, by

$$(g \cdot b)(x, y) := g\Big(b(g^{-1}(x), g^{-1}(y))\Big), \ x, y \in \mathfrak{g}^{\mathbb{C}}, \ g \in O(n, \mathbb{C}).$$

Note that the action again restricts to action of the real isometry groups on $\mathcal{V}$ and $\tilde{\mathcal{V}}$ respectively. Denote the real actions by ρ and $\tilde{\rho}$ respectively. The Cartan involution $\theta \in O(p, q)$ (respectively $\tilde{\theta} \in O(\tilde{p}, \tilde{q})$) naturally extends to an involution of $\mathcal{V}$ (respectively $\tilde{\mathcal{V}}$), by the action: $\rho(\theta)$ (respectively $\tilde{\rho}(\tilde{\theta})$). The holomorphic inner product $g^{\mathbb{C}}$ extends naturally to a holomorphic inner product: $\mathbf{g}^{\mathbb{C}}$, by defining:

$$\mathbf{g}^{\mathbb{C}}(b_1, b_2) := \sum_{j}^{n} g^{\mathbb{C}}\Big(b_1(y_j, y_j), b_2(y_j, y_j)\Big).$$

Observe that if we change basis w.r.t. to $\tilde{Y}$ instead then we obtain the same holomorphic inner product. Indeed this follows since we can find $g \in O(n,\mathbb{C})$ sending $Y \mapsto \tilde{Y}$. It is easy to check that $\mathcal{V} \subset (\mathcal{V}^{\mathbb{C}}, \mathbf{g}^{\mathbb{C}}) \supset \tilde{\mathcal{V}}$ are real slices. Similarly if we define $\mathcal{W}$ to be all bilinear maps: $W^2 \to W$, then by construction $\mathcal{W}$ is a compact real form of $(\mathcal{V}^{\mathbb{C}}, \mathbf{g}^{\mathbb{C}})$. Observe that the three real forms form a compatible triple in $\mathcal{V}^{\mathbb{C}}$. Therefore the actions form a compatible triple (see Section 2.3). There is a natural choice of $O(n)$-invariant Hermitian inner product on $\mathcal{V}^{\mathbb{C}}$, namely: $H := \mathbf{g}^{\mathbb{C}}(\cdot, \mathcal{T}(\cdot))$, where $\mathcal{T}$ is the conjugation map of $\mathcal{W}$. This Hermitian inner product restricts to inner products on $\mathcal{V}$, $\tilde{\mathcal{V}}$ and $\mathcal{W}$. Observe that the inner Cartan involutions of ρ (respectively $\tilde{\rho}$) are those conjugate to $\rho(\theta)$ (respectively $\tilde{\rho}(\tilde{\theta})$).

2.5. Wick-rotatable tensors of pseudo-Riemannian manifolds

For a Wick-rotation of Lie groups it is worth noting that the action in the previous section is just an example of a tensor action of $O(n,\mathbb{C})$ on a general tensor space of finite form:

$$\mathcal{V}^{\mathbb{C}} := \bigoplus_{k,m}\Big(\Big(\bigotimes_{i=1}^{k}\mathfrak{g}^{\mathbb{C}}\Big)\bigotimes\Big(\bigotimes_{i=1}^{m}(\mathfrak{g}^{\mathbb{C}})^*\Big)\Big),$$

induced from the isometry action of the holomorphic metric $g^{\mathbb{C}}$. Analogously we define:

$$\mathcal{V} := \bigoplus_{k,m}\Big(\Big(\bigotimes_{i=1}^{k}\mathfrak{g}\Big)\bigotimes\Big(\bigotimes_{i=1}^{m}\mathfrak{g}^*\Big)\Big), \quad \tilde{\mathcal{V}} := \bigoplus_{k,m}\Big(\Big(\bigotimes_{i=1}^{k}\tilde{\mathfrak{g}}\Big)\bigotimes\Big(\bigotimes_{i=1}^{m}\tilde{\mathfrak{g}}^*\Big)\Big).$$

The real isometry groups: $O(p,q)$ (respectively $O(\tilde{p},\tilde{q})$) restrict to acting on $\mathcal{V}$ (respectively $\tilde{\mathcal{V}}$).

More generally for a Wick-rotation of pseudo-Riemannian manifolds:

$$(M, g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g}),$$

at a common point $p \in M \cap \tilde{M}$, then by replacing $\mathfrak{g}$ with T_pM (respectively $\tilde{\mathfrak{g}}$ with $T_p\tilde{M}$), and $\mathfrak{g}^{\mathbb{C}}$ with $T_pM^{\mathbb{C}}$, we obtain the induced tensor action on real forms: $\mathcal{V} \subset \mathcal{V}^{\mathbb{C}} \supset \tilde{\mathcal{V}}$.

One shall note that the metrics, Cartan involutions all extend naturally to these spaces via the tangent spaces. Moreover if $g \in O(n,\mathbb{C})$ is such that T_pM and $g(T_p\tilde{M})$ form a compatible triple with a compact real form $W \subset T_pM^{\mathbb{C}}$, then naturally also $\mathcal{V}$ and $g \cdot \tilde{\mathcal{V}}$ form a compatible triple with $\mathcal{W} := \bigoplus_{k,m}\Big(\Big(\bigotimes_{i=1}^{k} W\Big)\bigotimes\Big(\bigotimes_{i=1}^{m} W^*\Big)\Big)$.

For example the induced action of $O(n,\mathbb{C})$ on $End(T_pM^{\mathbb{C}})$ given by conjugation: $g \cdot f := gfg^{-1}$ is just the tensor action: $g \cdot (v_1 \otimes v_2) := g(v_1) \otimes g(v_2)$, for an $O(n,\mathbb{C})$-module isomorphism: $End(T_pM^{\mathbb{C}}) \cong T_pM^{\mathbb{C}} \otimes T_pM^{\mathbb{C}}$. For a more detailed explanation of this example, and on the tensor action in general we refer to [4].

Consider the action in the previous section for instance, then one should observe that the complex Lie bracket $v := [-,-]$ of $\mathfrak{g}^{\mathbb{C}}$ is a vector in $\mathcal{V}$, but also there is a $g \in O(n,\mathbb{C})$ such that $\tilde{v} := g \cdot v \in g \cdot \tilde{\mathcal{V}}$, i.e. v and $\tilde{v}$ lie in the same complex orbit: $O(n,\mathbb{C})v \ni \tilde{v}$, in such a way that $O(p,q)v \sim O(\tilde{p},\tilde{q})\tilde{v}$ are compatible real orbits.

Thus it useful to define for general tensors $v \in \mathcal{V}$ and $\tilde{v} \in \tilde{\mathcal{V}}$:

Definition 2.17 (*[4]*)**.** Let (M, g) and $(\tilde{M}, \tilde{g})$ be two Wick-rotatable pseudo-Riemannian manifolds at a common point p. Then two tensors $v \in \mathcal{V}$ and $\tilde{v} \in \tilde{\mathcal{V}}$ are said to be *Wick-rotatable* at p, if they lie in the same $O(n,\mathbb{C})$-orbit, i.e.

$$O(n,\mathbb{C})v = O(n,\mathbb{C})\tilde{v}.$$

One should note the subset of Wick-rotatable tensors consisting of those in the intersection: $v \in \mathcal{V} \cap \tilde{\mathcal{V}}$. Then there is a map $g \in O(n,\mathbb{C})$, such that v and $g \cdot v \in g \cdot \tilde{\mathcal{V}}$ lie in the same complex orbit such that $O(p,q)v \sim O(\tilde{p},\tilde{q})\tilde{v}$ are compatible real orbits. More generally if v and $\tilde{v}$ are Wick-rotatable i.e. by definition $O(n,\mathbb{C})v = O(n,\mathbb{C})\tilde{v}$, then also $O(n,\mathbb{C})v = O(n,\mathbb{C})g \cdot \tilde{v}$. The main point is to be able to embed the vectors into the same complex orbit, such that we may apply the results of Section 2.3.

Let (M, g) be a pseudo-Riemannian manifold of signature (p,q), and $\theta \in O(p,q)$ be a Cartan involution of $g(-,-)$. Consider the isometry tensor action of $O(p,q)$ on $\mathcal{V}$ as above:

$$O(p,q) \xrightarrow{\rho_{\mathcal{V}}^{O(p,q)}} GL(\mathcal{V}).$$

Then θ naturally extends to an involution $\Theta := \rho_{\mathcal{V}}^{O(p,q)}(\theta)$ on $\mathcal{V}$, and the metric naturally induces a pseudo-inner product: $\mathbf{g}(-,-)$ on $\mathcal{V}$ such that Θ is a Cartan involution. Let now $R \in \mathcal{V}$ be the Riemann tensor of M at p for $\mathcal{V}$ some tensor space.

For example R may be considered as a multilinear form into T_pM: $T_pM^3 \to T_pM$, where the action is given by:

$$(g \cdot R)(x, y, z) := g\Big(R(g^{-1}(x), g^{-1}(y), g^{-1}(z))\Big),\ \ x, y, z \in T_pM,\ \ g \in O(p, q).$$

Another approach is to consider R as a map in $End(\mathfrak{o}(p, q)) \subset End\Big(End(T_pM)\Big)$ at the point p, where the action is given by:

$$(g \cdot R)(X) := gR(g^{-1}Xg)g^{-1},\ \ X \in \mathfrak{o}(p, q),\ \ g \in O(p, q).$$

The Riemann tensor at p is viewed in this way for instance in [2]. One may show that these two actions are equivalent up to an $O(p, q)$-module isomorphism, by identifying the spaces with the tensor space: $T_pM \otimes T_pM \otimes T_pM \otimes T_pM$.

We also recall the following definition:

Definition 2.18. If there exists a Cartan involution Θ such that $\Theta(R) = R$ (respectively $\Theta(R) = -R$), then the space (M, g) at p is called Riemann purely electric RPE (respectively Riemann purely magnetic (RPM)). If there is such a Θ for the Weyl tensor at p, then (M, g) at p is called purely electric (PE) (respectively purely magnetic (PM)).

Any Riemannian space (M, g) is RPE at any point $p \in M$, since the identity map $\theta := 1_{T_pM}$ is a Cartan involution of the metric g at any point, thus the Cartan involution extended to tensors: $\mathcal{V}$ is also the identity map, i.e. $\Theta(R) = R$.

The Levi-Civita connection ∇ of a real slice of a holomorphic Riemannian manifold $(M, g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}})$ at $p \in M$, restricts from the complex Levi-Civita connection: $\nabla^{\mathbb{C}}$ at p of the complex manifold $M^{\mathbb{C}}$. Thus the real Riemann curvature tensor R (of M) at p restricts from the complex Riemann curvature tensor $R^{\mathbb{C}}$ of $M^{\mathbb{C}}$ (at p). Moreover if ric_g denotes the real Ricci curvature: $T_pM^2 \to \mathbb{R}$, defined by:

$$ric_g(x, y) := Tr\Big(z \mapsto R(z, y)(x)\Big),$$

then using a real basis of T_pM also for $T_pM^{\mathbb{C}}$ we see that restricting the complex Ricci curvature: $ric_{g^{\mathbb{C}}}$ on $M^{\mathbb{C}}$ to T_pM we get ric_g. Similarly the real Ricci operator:

$$Ric_g \in End(T_pM),\quad g_p(Ric_g(x), y) = ric_g(x, y),$$

restricts form the complex Ricci curvature operator of $M^{\mathbb{C}}$ (at p).

This means that in terms of Wick-rotations of pseudo-Riemannian manifolds at a common point p: $(M, g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g})$, we see that the pairs of tensors:

$$(\nabla, \tilde{\nabla}),\ \ (R, \tilde{R}),\ \ (ric_g, ric_{\tilde{g}}),\ \ (Ric_g, Ric_{\tilde{g}}),$$

are examples of Wick-rotatable tensors (at p) in the intersection $\mathcal{V} \cap \tilde{\mathcal{V}}$. The induced isometry action of $O(n, \mathbb{C})$ on these tensors (induced from the isometry action of the metric) can be naturally seen as the actions:

$$(g \cdot \nabla)(x, y) := g(\nabla_{g^{-1}x} g^{-1}y),\quad (g \cdot R)(x, y, z) := g\Big(R(g^{-1}x, g^{-1}y, g^{-1}z)\Big)$$

and

$$(g \cdot ric_g)(x, y) := ric_g(gx, gy),\quad (g \cdot Ric_g)(x) := (g \circ Ric_g \circ g^{-1})(x).$$

An immediate new result is the following:

Theorem 2.3. *Let $(M, g) \subset (M^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{M}, \tilde{g})$ be a Wick-rotation at a common point $p \in M \cap \tilde{M}$. Assume $(\tilde{M}, \tilde{g})$ is a Riemannian space. Then the following statements hold:*

(1) There exists a Cartan involution θ of g such that $\nabla_{\theta(x)}\theta(y) = \theta(\nabla_x y)$ for all $x, y \in T_pM$.
(2) There exists a Cartan involution θ of g such that $ric_g\Big(\theta(x), \theta(y)\Big) = ric_g(x, y)$ for all $x, y \in T_pM$.
(3) There exists a Cartan involution θ of g such that $[\theta, Ric_g] = 0$.
(4) There exists a Cartan involution θ of g such that $R\Big(\theta(x), \theta(y)\Big)\Big(\theta(z)\Big) = \theta\Big(R(x, y)(z)\Big)$ for all $x, y, z \in T_pM$. Thus (M, g) is (RPE) at p.

Proof. It is enough to spell out the proof for the first case, as the other cases are identical. Let $v := \nabla \in \mathcal{V}$ and $\tilde{v} := \tilde{\nabla} \in \tilde{\mathcal{V}}$, and consider the isometry tensor action as above. The vectors v and $\tilde{v}$ are Wick-rotatable, thus up to a map $g \in O(n, \mathbb{C})$ we can assume the real actions are compatible, and that v and $\tilde{v}$ lie in the same complex orbit, such that the real orbits: $O(p, q)v \sim O(\tilde{p}, \tilde{q})$ are compatible. The result now follows from Theorem 2.2, since $O(\tilde{p}, \tilde{q}) = O(n)$ is a compact real form of $O(n, \mathbb{C})$. $\square$

One shall note that Case (4) of the theorem is proved in [2]. We shall strengthen Theorem 2.3 for Wick-rotations of pseudo-Riemannian Lie groups in the last section of the paper, by proving that a Cartan involution of g may be chosen to be a homomorphism of Lie algebras.

3. An invariant of Wick-rotation of Lie groups

In this section we shall prove the main theorem of the paper, which is an invariance result based on the existence of a Cartan involution of the Lie algebras (Definition 2.9).

Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Consider the action in Section 2.4 and following the notation there, then by our preparations, the main result is now easily deducible:

Theorem 3.1. *Suppose (G, g) is a pseudo-Riemannian Lie group that can be Wick-rotated to another Lie group $(\tilde{G}, \tilde{g})$. Then there exists a Cartan involution of $\mathfrak{g}$ if and only if there exists a Cartan involution of $\tilde{\mathfrak{g}}$.*

Proof. Consider the group action and the notation as in Section 2.4. Thus if $v := [-, -]$ is the Lie bracket of $\mathfrak{g}^{\mathbb{C}}$ then $v \in \mathcal{V}$ and restricts to the Lie bracket of $\tilde{\mathfrak{g}}$. We can choose $g \in O(n, \mathbb{C})$ such that $g \cdot v \in \tilde{\mathcal{V}}$, i.e. v and $\tilde{v} := g \cdot v$ lie in the same complex orbit, thus $O(p, q)v \sim O(\tilde{p}, \tilde{q})\tilde{v}$ are compatible real orbits. Suppose θ is a Cartan involution of $\mathfrak{g}$, and denote $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$ (respectively $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_+ \oplus \tilde{\mathcal{V}}_-$) for the Cartan decomposition w.r.t. to $\rho(\theta)$ (respectively $\tilde{\rho}(\tilde{\theta})$). Then the action of θ on v fixes v, i.e. $\rho(\theta)(v) := \theta \cdot v = v$, thus $v \in \mathcal{V}_+$. Hence the real orbit: $O(p, q)v$ intersects $\mathcal{V}_+$. But then by Theorem 2.1, it follows that there exists also $\tilde{v}' \in \tilde{\mathcal{V}}_+ \cap O(\tilde{p}, \tilde{q})\tilde{v}$. Therefore choose $h \in O(\tilde{p}, \tilde{q})$ such that $h \cdot \tilde{v} = \tilde{v}'$. By conjugating $\tilde{\rho}(\tilde{\theta})$ by h we obtain a Cartan involution $\tilde{\theta}'$ of $\tilde{g}$ such that $\tilde{\theta}' \cdot \tilde{v} = \tilde{v}$. Finally since $\tilde{V} := g(\tilde{\mathfrak{g}})$ for some $g \in O(n, \mathbb{C})$ then the Cartan involution $g^{-1}\tilde{\theta}'g$ fixes v, i.e. is a Cartan involution of $\tilde{g}$ and a homomorphism of Lie algebras. The converse is symmetric. The theorem is proved. □

We find it useful to define for future exploration:

Definition 3.1. A property of a pseudo-Riemannian Lie group (G, g) is said to be *Wick-rotatable* if it is an invariant under a Wick rotation of Lie groups.

Corollary 3.1. *The existence of a Cartan involution of $\mathfrak{g}$ is Wick-rotatable.*

Other Wick-rotatable properties include (see for example [1] on complexification of real Lie algebras): being semi-simple, abelian, nilpotent, solvable, reductive. Note that being simple, is not Wick-rotatable, indeed as an example consider the Lie group $O(1, 3)$ with the left-invariant metric being the Killing form. Then $\mathfrak{o}(1, 3)$ is simple, but we may Wick-rotate $O(1, 3)$ to $O(2, 2)$ which is semi-simple but not simple, as $\mathfrak{o}(2, 2) \cong \mathfrak{sl}_2(\mathbb{R})^2$ (two copies).

We can now answer the question for when an arbitrary left-invariant metric can be Wick-rotated to a Riemannian left-invariant metric. One should compare the result with semi-simple Lie groups equipped with the left-invariant Killing form: $g := -\kappa$.

Corollary 3.2. *Suppose $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ is a real slice of Lie groups. Then (G, g) can be Wick-rotated to a Riemannian Lie group $(\tilde{G}, \tilde{g})$ if and only if there exists a Cartan involution of $\mathfrak{g}$.*

Proof. ($\Rightarrow$). The identity map $\tilde{\mathfrak{g}} \xrightarrow{1} \tilde{\mathfrak{g}}$ is a Cartan involution of $\tilde{\mathfrak{g}}$. Thus by Theorem 3.1 the direction follows. Conversely suppose θ is a Cartan involution of $\mathfrak{g}$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, for the Cartan decomposition. Then is not difficult to show that $\tilde{\mathfrak{g}} := \mathfrak{k} \oplus i\mathfrak{p}$ is a Lie algebra and is a real form of $\mathfrak{g}^{\mathbb{C}}$. Moreover the complex metric $g^{\mathbb{C}}(-, -)$ restricts to an inner product on $\tilde{\mathfrak{g}}$ by construction. Thus if we let $\tilde{G}$ be the unique connected Lie subgroup of $G^{\mathbb{C}}$ (the real Lie group) with Lie algebra $\tilde{\mathfrak{g}}$, then the corollary follows. □

In view of Remark 1.1 with the signature change $g \mapsto -g$, if (G, g) can be Wick-rotated to a signature $(-, -, \dots, -)$, then $(G, -g)$ can be Wick-rotated to a Riemannian space, thus there would exist a Cartan involution of $\mathfrak{g}$ w.r.t. $-g$. We note in the Corollary that w.r.t. the existing Cartan involution, then the Wick-rotated Riemannian Lie group may be chosen to be a virtual real form. Moreover note that since a Wick-rotation is a local condition then on Lie algebra level we have proved:

Corollary 3.3. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, where $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra. Let $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ be a real form which is a real slice. Assume there exists a compact real form $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ which is also a real Lie subalgebra. Let σ be the conjugation map of $\mathfrak{g}$. Then there exists an automorphism $\phi \in Aut(\mathfrak{g}^{\mathbb{C}}) \cap O(n, \mathbb{C})$ such that: $\sigma(\phi(\mathfrak{u})) \subset \phi(\mathfrak{u})$.*

Note in the corollary that if τ denotes the conjugation map of the compact real form $\phi(\mathfrak{u}) \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$, then the map $\theta^{\mathbb{C}} := \sigma\tau$ restricts to a Cartan involution θ of $\mathfrak{g}$.

Thus we have proved a general version of É. Cartan's result: ([1], Thm 7.1). Note also that the proof given there for the semi-simple case w.r.t. to the Killing form is not valid for a general pair: $(\mathfrak{g}, g)$ as above, indeed following the notation of the proof, it is not obvious that $N := \sigma\tau \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$.

One shall note that it may be the case that a pseudo-Riemannian Lie group (G, g) can be Wick-rotated to more than one Riemannian Lie group, in such a case we have the following (again one should compare this to semi-simple compact real forms w.r.t. $-\kappa$):

Proposition 3.1. *Suppose there exist two Riemannian Wick-rotatable Lie groups: (G, g) and $(\tilde{G}, \tilde{g})$. Then (G, g) and $(\tilde{G}, \tilde{g})$ are locally isometric Lie groups. In particular if moreover G and $\tilde{G}$ are both simply connected then G and $\tilde{G}$ are isometric Lie groups.*

Proof. Choose a map $g \in O(n, \mathbb{C})$ mapping $\mathfrak{g} \mapsto \tilde{\mathfrak{g}}$. Consider the action and notation of Section 2.4. Using the map g the Lie bracket $v := [-, -]$ of $\mathfrak{g}^{\mathbb{C}}$ lies in $\mathcal{V}$, and also $\tilde{v} := g^{-1} \cdot v \in \mathcal{V}$. Thus $O(n, \mathbb{C})v \ni \tilde{v}$. But since $O(n)$ (the isometries of $(\mathfrak{g}, g)$) is compact, then we may choose $h \in O(n)$ such that the vectors v and $\tilde{v}$ lie in the same $O(n)$-orbit, i.e. $h \cdot v = \tilde{v}$. Or in other words:

$$gh \cdot v = v.$$

Now since h maps $\mathfrak{g}$ to $\mathfrak{g}$ by definition and gh fixes v, i.e. fixes the complex Lie bracket. Then $gh \in O(n, \mathbb{C})$ is an automorphism of complex Lie algebras, and it maps $\mathfrak{g} \mapsto \tilde{\mathfrak{g}}$. Therefore since the metrics are left-invariant we can conclude that (G, g) and $(\tilde{G}, \tilde{g})$ are locally isometric Lie groups as required. Finally if G and $\tilde{G}$ are both simply connected then since any local isometry is also an isometry, it follows that $(G, g) \cong (\tilde{G}, \tilde{g})$ are isometric Lie groups. The proposition is proved. □

Thus as a corollary for compact real forms:

Corollary 3.4. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, where $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra. Let $\mathfrak{u}_1 \subset \mathfrak{g}^{\mathbb{C}} \supset \mathfrak{u}_2$ be two real Lie subalgebras which are compact real forms. Then there exists a linear isomorphism: $\mathfrak{u}_1 \xrightarrow{\phi} \mathfrak{u}_2$, such that $\phi \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$.*

In the case of a complex semi-simple Lie group: $(G^{\mathbb{C}}, -\kappa)$, equipped with the left-invariant Killing form, then any compact real form: $\mathfrak{u} \subset \mathfrak{g}$, gives rise to a real form: $U \subset G^{\mathbb{C}}$ (thus is by definition a Riemannian real slice of Lie groups). It follows by the theory of semi-simple Lie groups that any two compact real forms of $G^{\mathbb{C}}$ are isomorphic Lie groups, and thus also isometric Lie groups.

Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice of Lie groups. Recall again the action of Section 2.4, and consider the Lie bracket $[-, -]$ of $\mathfrak{g}^{\mathbb{C}}$. Thus $[-, -] \in \mathcal{V}$ (the bilinear forms $\mathfrak{g}^2 \to \mathfrak{g}$.) Suppose as usual that the signature of g is (p, q). Then from real GIT there are a finite number of real $O(p, q)$-orbits in the complex orbit: $O(n, \mathbb{C}) \cdot [-, -]$, i.e.

$$O(n, \mathbb{C}) \cdot [-, -] \cap \mathcal{V} = \cup_{i=1}^{m} O(p, q)v_i,$$

for some $m \geq 1$. We shall put an equivalence relation on the real slices of Lie groups of $(G^{\mathbb{C}}, g^{\mathbb{C}})$ by the relation of local isometry of their identity components. Let $[(G, g)]$ denote an equivalence class, thus $[(G, g)] = [(\tilde{G}, \tilde{g})] \Leftrightarrow (G, g) \cong (\tilde{G}, \tilde{g})$ (locally).

We can thus generalise Proposition 3.1 in the following sense:

Theorem 3.2. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice of Lie groups, and (p, q) be the signature of g. Let $O(n, \mathbb{C}) \cdot [-, -] \cap \mathcal{V} = \cup_{i=1}^{m} O(p, q)v_i$. Then there are exactly m equivalence classes (up to a local isometry) of real slices of Lie groups in $(G^{\mathbb{C}}, g^{\mathbb{C}})$ with signature (p, q). In particular a Wick-rotation of two Lie groups (of the same signature) are locally isometric if and only if $m = 1$.*

Proof. Suppose $(\tilde{G}, \tilde{g})$ is Wick-rotated to (G, g) of the same signature. Let $h \in O(n, \mathbb{C})$ be such that $h(\mathfrak{g}) = \tilde{\mathfrak{g}}$, then $\tilde{v} := h^{-1} \cdot [-, -] \in \mathcal{V}$ is in the same complex orbit as $[-, -]$. Thus we have a mapping of an equivalence class:

$$[(\tilde{G}, \tilde{g})] \mapsto O(p, q)\tilde{v}.$$

The map does not depend on the choice of h, since if $h_1 \in O(n, \mathbb{C})$ also maps $h_1(\mathfrak{g}) = \tilde{\mathfrak{g}}$, then $h^{-1}h_1 \in O(p, q)$, and $h^{-1}h_1 \cdot \tilde{v} = h^{-1} \cdot [-, -]$. The map is well-defined. Indeed let (G_1, g_1) map to $O(p, q)v_1 := O(p, q) \cdot (h_1^{-1} \cdot [-, -])$ for some $h_1 \in O(n, \mathbb{C})$ with $h_1(\mathfrak{g}) = \mathfrak{g}_1$. Assume (G_1, g_1) is locally isometric to $(\tilde{G}, \tilde{g})$. Then there exists $g \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$ such that $g(\mathfrak{g}_1) = \tilde{\mathfrak{g}}$, therefore:

$$g_1 := h^{-1}gh_1 \in O(p, q), \quad g_1 \cdot v_1 = h^{-1}g \cdot [-, -] = h^{-1} \cdot [-, -] = \tilde{v},$$

using that g fixes the Lie bracket.

To see that the map is injective, then suppose $[(G_j, g_j)]$ maps to the same orbit for $j = 1, 2$. Then by definition: $[(G_j, g_j)] \mapsto O(p, q) \cdot (h_j^{-1} \cdot [-, -])$ for maps $h_j \in O(n, \mathbb{C})$ with $h_j(\mathfrak{g}) = \mathfrak{g}_j$. Thus since the orbits are the same, then choose $g \in O(p, q)$ such that $g \cdot (h_1^{-1} \cdot [-, -]) = h_2^{-1} \cdot [-, -]$, i.e. $h_2 g h_1^{-1} \cdot [-, -] = [-, -]$ so that $h_2 g h_1^{-1} \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$. Note that $h_2 g h_1^{-1}$ maps $\mathfrak{g}_1 \mapsto \mathfrak{g}_2$. It follows that $[(G_1, g_1)] = [(G_2, g_2)]$ as required.

It remains to show that the map is surjective. Indeed if $v_j \in \mathcal{V}$ is among the $v_1, \ldots, v_m$, then there exists $h \in O(n, \mathbb{C})$ such that $h \cdot v_j = [-, -]$. If $V_1 \subset \mathfrak{g}^{\mathbb{C}}$ denotes the real form (of vector spaces) $h(\mathfrak{g})$, then:

$$[V_1, V_1] = h\Big(v(h^{-1}(V_1), h^{-1}(V_1))\Big) \subset h(v(\mathfrak{g}, \mathfrak{g})) \subset h(\mathfrak{g}) := V_1.$$

Therefore V_1 is a real form of Lie algebras, thus redefine $V_1 := \mathfrak{g}_1$. Let G_1 be the virtual Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_1$, then G_1 is a real slice of Lie groups of signature (p, q). Thus $[(G_1, g_1)] \mapsto O(p, q)v_j$, which proves that the map is surjective. The theorem is proved. □

There are classes of Lie algebras with $m = 1$, for instance the trivial case of abelian Lie algebras. However in general $m \neq 1$. Indeed even a semi-simple Lie algebra is not determined by the signature of its Killing form: $-\kappa$. As an example consider the semi-simple real forms $\mathfrak{o}(1,4) \subset (\mathfrak{o}(5,\mathbb{C}), -\kappa) \supset \mathfrak{o}(2,3)$. Then the signatures are $(6,4)$ and $(4,6)$ respectively. Thus $\mathfrak{o}(1,4) \oplus \mathfrak{o}(2,3)$ is a real form of $(\mathfrak{o}(5,\mathbb{C})^2, -\kappa)$ of signature $(10,10)$. But also if $\mathfrak{o}(5,\mathbb{C})_{\mathbb{R}}$ denotes the real Lie algebra of $\mathfrak{o}(5,\mathbb{C})$, then it is also a real form of $\mathfrak{o}(5,\mathbb{C})^2$ which is simple, also of signature $(10,10)$, thus

$$\mathfrak{o}(5,\mathbb{C})_{\mathbb{R}} \ncong \mathfrak{o}(1,4) \oplus \mathfrak{o}(2,3),$$

and so $m \geq 2$ in this example.

We now give two examples, one where a Lie group is Wick-rotatable to a Riemannian Lie group, and the other where a Lie group is not Wick-rotatable to a Riemannian Lie group.

Example 3.1. Let $H_3(\mathbb{R}) \subset H_3(\mathbb{C})$ be the 3-dimensional real and complex Heisenberg groups. The Lie algebra of $H_3(\mathbb{R})$ denoted: $\mathfrak{h}_3(\mathbb{R})$, is the set of strictly upper triangular 3×3 matrices. A basis of the Lie algebra is given by $\{e_1, e_2, e_3\}$ with,

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

We may identify $\{e_j\}_j$ with the standard basis of $\mathbb{R}^3$. Let $g(-,-)$ be the standard Lorentzian pseudo-inner product on $\mathbb{R}^3$, i.e. of signature $(+,+,-)$. Thus $(H_3(\mathbb{R}), -g)$ is a real slice (of Lie groups) of $(H_3(\mathbb{C}), -g^{\mathbb{C}})$. Note that $g(-,-)$ is not bi-invariant, since $g([e_1,e_2],e_3) = -1 \neq g(e_1,[e_2,e_3]) = 0$. Define the linear map: $\theta \in End(\mathfrak{h}_3(\mathbb{R}))$ by:

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \mapsto -\lambda_1 e_1 - \lambda_2 e_2 + \lambda_3 e_3.$$

Then it is easy to show that this is an involution of Lie algebras, and moreover θ is a Cartan involution of $\mathfrak{h}_3(\mathbb{R})$ w.r.t. $-g(-,-)$, thus by Corollary 3.2 it follows that $H_3(\mathbb{R})$ can be Wick-rotated to a Riemannian Lie group $\tilde{G}$. Note that $\tilde{G}$ is the real form of $H_3(\mathbb{C})$ consisting of matrices of the form: $\begin{bmatrix} 1 & ix & iy \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$ for $x, y, z \in \mathbb{R}$.

Example 3.2. Consider the real form: $G := SL_2(\mathbb{R})^2 \subset G^{\mathbb{C}} := SL_2(\mathbb{C})^2$. Then we can equip G with a left-invariant metric $g(-,-)$ of signature $(3,3)$, by equipping one copy with $-\kappa$ and the other copy with κ. The real forms up to isomorphism of $\mathfrak{sl}_2(\mathbb{C})^2$ are:

$$\mathfrak{sl}_2(\mathbb{R})^2, \ \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{su}(2), \ \mathfrak{su}(2)^2, \ \mathfrak{o}(1,3).$$

Let $\tilde{\mathfrak{g}}$ be one of these real forms (except the last one), then we may Wick-rotate G to the corresponding real forms $\tilde{G}$ of $SL_2(\mathbb{C})^2$ of signature either: $(3,3)$ or $(1,5)$. In the case of Wick-rotating to $(SU(2)^2, \tilde{g})$ we get a signature of $(3,3)$. Now note that if G can be Wick-rotated to a signature: $(0,6)$ or Riemannian: $(6,0)$, then we can find (by Corollary 3.2) a Cartan involution of $\mathfrak{su}(2)^2$ w.r.t. $-\tilde{g}$ or $+\tilde{g}$ respectively:

$$\mathfrak{su}(2)^2 \xrightarrow{\theta} \mathfrak{su}(2)^2.$$

Suppose the Cartan involution is w.r.t $\tilde{g}$. Then if $\mathfrak{su}(2)^2 = \mathfrak{k} \oplus \mathfrak{p}$, is the Cartan decomposition w.r.t θ, we have $\mathfrak{g}_1 := \mathfrak{k} \oplus i\mathfrak{p} \cong \mathfrak{o}(1,3)$. Indeed $\theta^{\mathbb{C}}$ is a Cartan involution of $\mathfrak{g}_1$ (w.r.t $-\kappa$), thus $-\kappa$ has signature $(3,3)$, hence it must be the case that $\mathfrak{g}_1 \cong \mathfrak{o}(1,3)$. Now consider the copy $\mathfrak{o}(1,3)$ identified as the real form

$$\{(x, \tau(x)) | x \in \mathfrak{sl}_2(\mathbb{C})\} \subset \mathfrak{sl}_2(\mathbb{C})^2.$$

We can extend the inner product $g^{\mathbb{C}} := g_0$ on $\mathfrak{g}_1$ to $\mathfrak{o}(1,3)$ by using an isomorphism ϕ (of Lie algebras) $\mathfrak{g}_1 \cong \mathfrak{o}(1,3)$, thus by complexifying we get a holomorphic inner product b on $\mathfrak{sl}_2(\mathbb{C})^2$. On each copy of $\mathfrak{sl}_2(\mathbb{C})$ we get $b = \lambda\kappa$ where κ is the Killing form on $\mathfrak{sl}_2(\mathbb{C})$, thus we may assume the holomorphic inner product is of the form $b = \lambda_1\kappa + \lambda_2\kappa$. Using that $g^{\mathbb{C}} = -\kappa \oplus \kappa$, then necessarily λ_1, λ_2 are real. Now for $X := (x, \tau(x)) \in \mathfrak{o}(1,3)$ we get

$$\begin{aligned} b(X,X) &= \lambda_1\kappa(x,x) + \lambda_2\kappa(\tau(x),\tau(x)) \\ &= \lambda_1\kappa(x,x) + \lambda_2\overline{\kappa(x,x)} \\ &= (\lambda_1+\lambda_2)Re(\kappa(x,x)) + i(\lambda_1-\lambda_2)Im(\kappa(x,x)). \end{aligned}$$

Thus since $b(X,X)$ is real, then necessarily $\lambda_1 = \lambda_2$, so b is proportional to the Killing form on $\mathfrak{sl}_2(\mathbb{C})^2$, and so since $\phi^{\mathbb{C}}$ is an isomorphism:

$$\left(\mathfrak{sl}_2(\mathbb{C})^2, g^{\mathbb{C}}\right) \cong \left(\mathfrak{sl}_2(\mathbb{C})^2, b\right),$$

then we deduce that $g^{\mathbb{C}} = -\kappa \oplus \kappa$ is also proportional to the Killing form, this is a contradiction. The argument for the signature case: $(0,6)$, is identical with the change: $g \mapsto -g$. We conclude that (G,g) can not be Wick-rotated to a Riemannian Lie group nor to a Lie group of signature $(0,6)$.

One shall note that Proposition 3.1 does not hold for a general non-Riemannian signature. Indeed consider the previous example then $SL_2(\mathbb{R})^2$ has signature $(3,3)$ and can be Wick-rotated to $SU(2)^2$ also of signature $(3,3)$, but they are not locally isometric (since their Lie algebras are non-isomorphic). Thus $m \geq 2$ in the previous theorem.

We end this section by considering a result on semi-simple Lie groups.

Proposition 3.2. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice, and G be semi-simple. Then (G, g) can be Wick-rotated to a Riemannian compact Lie group if and only if there exists a Cartan involution θ of $\mathfrak{g}$ (w.r.t. g) which is also a Cartan involution of $\mathfrak{g}$ (w.r.t. $-\kappa$).*

Proof. $(\Rightarrow)$. If (G, g) is Wick-rotated to a Riemannian Lie group, then by Corollary 3.2, we can choose a Cartan involution θ of $\mathfrak{g}$. Denote $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition. Then following the proof of Corollary 3.2, then we can find a Riemannian Lie group $\tilde{G}$ with Lie algebra: $\tilde{\mathfrak{g}} := \mathfrak{k} \oplus i\mathfrak{p}$, which is Wick-rotated to G in $(G^{\mathbb{C}}, g^{\mathbb{C}})$. By Proposition 3.1, $\tilde{\mathfrak{g}}$ is compact, since we can Wick-rotate G to a Riemannian compact Lie group (by assumption). But since θ is a Cartan involution of $\mathfrak{g}$ (w.r.t. $-\kappa$) if and only if $\tilde{\mathfrak{g}}$ is compact, then the direction is proved. $(\Leftarrow)$. Suppose θ is a Cartan involution of $\mathfrak{g}$ w.r.t. $g(-,-)$ and $-\kappa$ simultaneously. Thus if $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$ associated with θ, then there exists a compact real form $U \subset G^{\mathbb{C}}$ with Lie algebra $\mathfrak{u}$. The proposition follows. □

Thus since we may lift a local Cartan involution: $\mathfrak{g} \xrightarrow{\theta} \mathfrak{g}$, to a global Cartan involution: $G \xrightarrow{\Theta} G$, then in view of the previous proposition, there is a Θ which is an isometry of (G, g), i.e. $\Theta \in Isom(G)$. Observe also that if there exists a real slice of Lie groups of $G^{\mathbb{C}}$ which is compact Riemannian, then the possible signatures (p, q) w.r.t. $g^{\mathbb{C}}$ is a subset of the possible signatures of $-\kappa$ (of $\mathfrak{g}^{\mathbb{C}}$).

It is tempting to think that if (G, g) can be Wick-rotated to a compact semi-simple Riemannian Lie group, then it would be locally isometric to $(G, -\lambda\kappa)$ $(\lambda > 0)$. However this is false, indeed consider $G := SL_2(\mathbb{R})^2$ equipped with the metric $g := -\kappa \oplus -2\kappa$. We can Wick-rotate to the compact Riemannian Lie group: $SU(2)^2$. Consider the real form $\cong \mathfrak{o}(1,3)$ identified with the set:

$$\{(x, \tau(x)) | x \in \mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}\} \subset \mathfrak{sl}_2(\mathbb{C})^2,$$

where τ is the conjugation map of $\mathfrak{sl}_2(\mathbb{C})$ with fix-points $\mathfrak{su}(2)$. If (G, g) is locally isometric to $\lambda\kappa$ for some $\lambda \in \mathbb{R}$, then we can Wick-rotate to a $\tilde{G} \subset G^{\mathbb{C}}$ with Lie algebra $\mathfrak{o}(1,3)$. However $\mathfrak{o}(1,3)$ on $g^{\mathbb{C}}$ is not a real slice. We thus conclude that (G, g) is not locally isometric to $(G, \lambda\kappa)$ for any $\lambda \in \mathbb{R}$.

Remark 3.1. One shall observe that if (G, g) and $(\tilde{G}, \tilde{g})$ are pseudo-Riemannian spaces, where G and $\tilde{G}$ are Lie groups, but the metrics are not assumed to be left-invariant, then the proof of Theorem 3.1 is still valid. The direction $(\Rightarrow)$ of Corollary 3.2 is also valid, however the direction $(\Leftarrow)$ does not necessarily hold.

4. Conjugacy of Cartan involutions

Given a pseudo-Riemannian Lie group (G, g), with two Cartan involutions θ_j $(j = 1, 2)$ of $\mathfrak{g}$, one may wonder if they are conjugate in $Aut(\mathfrak{g})$. This is in fact true as we will show here, and we note again the resemblance with semi-simple Lie groups G and Cartan involutions of $\mathfrak{g}$ (w.r.t. $-\kappa$).

Theorem 4.1. *Suppose $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ is a pseudo-Riemannian Lie group. Assume there exist two Cartan involutions: θ_1, θ_2 of $\mathfrak{g}$. Then θ_1 is conjugate to θ_2 in $Aut(\mathfrak{g})_0 \cap O(p,q)_0$.*

Proof. Write $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ for the Cartan decompositions w.r.t. θ_1 and θ_2 respectively. Denote also: $\mathfrak{k}_j \oplus i\mathfrak{p}_j := \mathfrak{u}_j$ $(j = 1, 2)$ for the real forms of $\mathfrak{g}^{\mathbb{C}}$. There exist Wick-rotations of G to connected virtual real forms: $U_j \subset G^{\mathbb{C}}$ with Lie algebras $\mathfrak{u}_j$ which are Riemannian (by Corollary 3.2). If σ denotes the conjugation map w.r.t. $\mathfrak{g}$, and τ_j denotes the conjugation map of $\mathfrak{u}_j$, then we have $\theta_j^{\mathbb{C}} = \sigma\tau_j$. Now since θ_1 is conjugate to θ_2 in $O(p,q)_0$ (see Remark 2.2), i.e. there is a $\phi \in O(p,q)_0$ such that

$$\phi\theta_1\phi^{-1} = \theta_2,$$

as linear maps, then $g := \phi^{\mathbb{C}}$ sends $\mathfrak{u}_1 \mapsto \mathfrak{u}_2$. Consider the action in Section 2.4 and the notation there. If $v := [-,-]$ is the complex Lie bracket of $\mathfrak{g}^{\mathbb{C}}$, then $v \in \mathcal{V}$ and $w := g^{-1} \cdot v \in \mathcal{W}$ (i.e. is a bilinear map $\mathfrak{u}_1^2 \to \mathfrak{u}_1$) lie in the same complex orbit. Note that $(\mathfrak{g}, \mathfrak{u}_1)$ is a compatible pair. Now w is a minimal vector since it belongs to $\mathcal{W}$, i.e.:

$$w \in O(n)_0 w \cap O(p,q)_0 v = K_0 v,$$

where

$$K := \{g \in O(p,q) | g\theta_1 = \theta_1 g\} \subset O(p,q),$$

is the maximal compact subgroup associated with the fixed global Cartan involution of $O(p,q)$: $g \mapsto \theta_1 g\theta_1$. Thus there is an element $k_0 \in K_0 \subset O(p,q)_0$ such that $k_0 v = w$, in other words: $gk_0 \cdot v = v$. Hence $gk_0 \in O(p,q)_0 \cap Aut(\mathfrak{g})$, and it follows that: $[\sigma, gk_0] = 0$, i.e.

$$\theta_2 = gk_0 \circ \theta_1 \circ k_0^{-1} g^{-1}.$$

The corollary is proved. $\square$

Thus on Lie algebras we get the following nice corollary:

Corollary 4.1. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, where $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra. Let $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ be a real form which is a real slice. Then any two Cartan involutions of $\mathfrak{g}$ are conjugate in $Aut(\mathfrak{g})_0 \cap O(p,q)_0$.*

Note that the corollary is a generalised version (for general pseudo-inner product spaces) of É. Cartan's result: ([1], Thm 7.2). Let us give an example (of a non-compact real form) where there is a unique Cartan involution of the Lie algebra:

Example 4.1. If we consider again the real Heisenberg group: $(H_3(\mathbb{R}), -g)$ and follow Example 3.1 with Cartan involution θ, then calculating the derivation algebra of $\mathfrak{h}_3(\mathbb{R})$ w.r.t. to the basis $\{e_j\}$, then the matrices have the form: $\begin{bmatrix} a & c & 0 \\ e & b & 0 \\ f & l & a+b \end{bmatrix}$, for $a, b, c, e, l, f \in \mathbb{R}$. Thus $Dim\Big(\mathfrak{der}(\mathfrak{h}_3(\mathbb{R}))\Big) = 6$. Now if such a derivation D belongs to $\mathfrak{o}(1,2)$, then an easy calculation shows that $D = \begin{bmatrix} 0 & c & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, i.e. the Lie algebra of $Aut(\mathfrak{h}_3(\mathbb{R})) \cap O(1,2)$ has dimension 1. Now a global Cartan involution Θ_1 of $O(1,2)$ is given by

$$f \mapsto \theta f \theta, \quad O(1,2) = Ke^{\mathfrak{p}},$$

and clearly since $\theta \in Aut(\mathfrak{h}_3(\mathbb{R}))$, then it leaves $H := Aut(\mathfrak{h}_3(\mathbb{R})) \cap O(1,2)$ invariant. But since the Lie algebra $\mathfrak{h}$ of H is fixed by the corresponding local Cartan involution of $\mathfrak{o}(1,2)$, then it follows that Θ_1 fixes pointwise all elements of H. Indeed this follows since H is algebraic (see for example [5]), so every $h \in H$ can be written as $h = ke^x$ for $k \in K \cap H$, $x \in \mathfrak{h} \cap \mathfrak{p}$. Thus we conclude that:

$$[\theta, f] = 0, \forall f \in H,$$

in other words by the previous theorem, there exists a unique Cartan involution of $\mathfrak{h}_3(\mathbb{R})$, namely θ.

Recall that for a real semi-simple Lie algebra $\mathfrak{g}$ equipped with the Killing form: $-\kappa$. Then it is proved in Helgason [1] that given any involution $\tilde{\theta}$ of $\mathfrak{g}$ there exists a Cartan involution of $\mathfrak{g}$ commuting with $\tilde{\theta}$. We can also prove a generalised version of this result for a general pseudo inner product space: $(\mathfrak{g}, g)$, by mimicking the proof given for semi-simple Lie algebras in [1] together with Corollary 3.3.

Corollary 4.2. *Let $(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ be a holomorphic inner product space, where $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra. Let $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ be a real form which is a real slice. Suppose there exists a compact real form of $\mathfrak{g}^{\mathbb{C}}$ which is also a real Lie subalgebra. Let $\tilde{\theta} \in Aut(g) \cap O(p,q)$ be an involution of $\mathfrak{g}$. Then there exists a Cartan involution θ of $\mathfrak{g}$ that commutes with $\tilde{\theta}$, i.e. $[\tilde{\theta}, \theta] = 0$.*

Proof. Let θ' be a Cartan involution of $\mathfrak{g}$ by Corollary 3.3. By mimicking the proof of ([1], Thm 7.1) in view of Exercise (4, Ch.3, [1]), we apply the proof given there to the inner product $g_{\theta'}(-,-) := g(-, \theta'(-))$, together with the symmetric operator: $N := \tilde{\theta}\theta'$. Thus there exists a $\psi \in O(p,q) \cap Aut(\mathfrak{g})$ such that $[\psi\theta'\psi^{-1}, \tilde{\theta}] = 0$, therefore let $\theta := \psi\theta'\psi^{-1}$. $\square$

Suppose now that $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ is a holomorphic inner product space on a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Let $O(n,\mathbb{C})$ be the isometry group. If $\phi \in O(n,\mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$ is an involution such that when $\mathfrak{g}^{\mathbb{C}} = V_+ \oplus V_-$, is the eigenspace decomposition then $\mathfrak{g} := V_+ \oplus iV_-$ is a real slice (i.e. $g^{\mathbb{C}}(\mathfrak{g}, \mathfrak{g}) \in \mathbb{R}$), then we shall write $\phi \in \mathcal{O}$ for such a map. We can put an equivalence relation on maps $\mathcal{O}$ by conjugacy in $O(n,\mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$.

The following theorem should be compared with a similar result of semi-simple Lie algebras equipped with their Killing form (see for example [10], Thm 1.3):

Theorem 4.2. *If $\left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ have a compact real form, then there is a bijection between isomorphism classes of real forms $\mathfrak{g} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ and conjugacy classes of $\mathcal{O}$.*

Proof. Let $\mathfrak{g} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$ be a real form, then we can choose a Cartan involution of $\mathfrak{g}$ say θ by (Corollary 3.3). Define the map $[\mathfrak{g}] \mapsto [\theta^{\mathbb{C}}]$. The map is well-defined (Theorem 4.1) since any two Cartan involutions are conjugate in $O(p,q) \cap Aut(\mathfrak{g})$, where (p, q) is the signature of the induced pseudo-inner product from $g^{\mathbb{C}}$. To see that the map is surjective, let $\phi \in \mathcal{O}$, and set $\mathfrak{g} := V_+ \oplus iV_-$ for the real form $\mathfrak{g} \subset \left(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}}\right)$. Then ϕ restricted to $\mathfrak{g}$ is an involution, and if σ denotes its conjugation map then $[\sigma, \phi] = 0$. But we may choose a Cartan involution θ of $\mathfrak{g}$ such that $[\theta, \phi] = 0$ by (Corollary 4.2). Thus $\theta^{\mathbb{C}} = \sigma\tau$ for τ a conjugation map of a compact real form $\mathfrak{u} \subset (\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$. Thus

$$\sigma\tau\phi = \phi\sigma\tau,$$

therefore

$$\sigma\tau\phi = \sigma\phi\tau,$$

or in other words by cancelling σ we obtain $[\tau, \phi] = 0$ so that ϕ is in fact a Cartan involution of $\mathfrak{g}$, and hence $[\mathfrak{g}] \mapsto [\phi]$.

Suppose now that $\mathfrak{g}_j$ are two real forms for $j = 1, 2$ such that the images are the same: $[\theta_1^{\mathbb{C}}] = [\theta_2^{\mathbb{C}}]$. Then if σ_j denotes the conjugation maps, and $\mathfrak{u}_j$ are the compact real forms compatible with $\mathfrak{g}_j$, then $\theta_j^{\mathbb{C}} = \sigma_j\tau_j$. But since the maps are conjugate in $O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$, say by ϕ, thus $\phi\theta_1^{\mathbb{C}}\phi^{-1} = \theta_2^{\mathbb{C}}$ then it is easy to see that: $\phi(\mathfrak{u}_1) = \mathfrak{u}_2$. Thus

$$\theta_2^{\mathbb{C}} = \phi\theta_1^{\mathbb{C}}\phi^{-1} = \phi\sigma_1\tau_1\phi^{-1} = \phi\sigma_1\phi^{-1}\phi\tau_1\phi^{-1} = \phi\sigma_1\phi^{-1}\tau_2,$$

thus cancelling τ_2 we obtain: $\phi\sigma_1\phi^{-1} = \sigma_2$, which proves that $[\mathfrak{g}_1] = [\mathfrak{g}_2]$, and hence the map is injective. The theorem is proved. □

5. Wick-rotating a Lorentzian signature

If we assume our left-invariant metric on our Lie group G is Lorentzian or of signature $(+, -, \dots, -)$, then being able to Wick-rotate to a Riemannian space puts some constraints on the structure of the Lie algebra (in view of Corollary 3.2). Now since a Wick-rotation is a local condition, it would be interesting to know what type of Lie algebra allows for a Wick-rotation to a Riemannian Lie group.

We recall by the fundamental *Levi-Malcev theorem* that our Lie algebra $\mathfrak{g}$ can be written as a semi-direct sum $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{h}$, where $\mathfrak{h}$ is the radical of $\mathfrak{g}$ and $\mathfrak{s} \subset \mathfrak{g}$ is either trivial or a semi-simple subalgebra of $\mathfrak{g}$ called the Levi-factor.

It is clear that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{s}^{\mathbb{C}} \ltimes \mathfrak{h}^{\mathbb{C}}$, and if $\tilde{\mathfrak{g}}$ is another real form of $\mathfrak{g}^{\mathbb{C}}$, then writing a Levi-decomposition: $\tilde{\mathfrak{g}} = \tilde{\mathfrak{s}} \ltimes \tilde{\mathfrak{h}}$, then $\tilde{\mathfrak{h}}$ is a real form of $\mathfrak{h}^{\mathbb{C}}$. To see that $\tilde{\mathfrak{s}}$ is a real form of $\mathfrak{s}^{\mathbb{C}}$, we note that there exists a $k \geq 1$ such that

$$\mathfrak{s}^{\mathbb{C}} = [\mathfrak{g}^{\mathbb{C}(k)}, \mathfrak{g}^{\mathbb{C}(k)}] \supset [\tilde{\mathfrak{g}}^{(k)}, \tilde{\mathfrak{g}}^{(k)}] = \tilde{\mathfrak{s}}.$$

In view of the existence of an involution of Lorentzian decomposition we can say the following:

Proposition 5.1. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice of Lie groups. Then the following statements hold:*

(1) Suppose $g(-, -)$ has Lorentzian signature. If (G, g) can be Wick-rotated to a Riemannian Lie group $(\tilde{G}, \tilde{g})$ then $\mathfrak{s} = 0$ or $\mathfrak{h} \neq 0$. Moreover if $\tilde{\mathfrak{s}}$ is a Levi-factor of $\tilde{\mathfrak{g}}$, then $\tilde{\mathfrak{s}} \cong \mathfrak{s}$.

(2) Suppose $g(-, -)$ has signature $(+, -, \dots, -)$. If (G, g) can be Wick-rotated to a Riemannian Lie group, then either $\mathfrak{s} = 0$ or $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$.

Proof. For case (1), assume $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{h}$ for $\mathfrak{s} \neq 0$, and choose a Cartan involution θ of $\mathfrak{g}$. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition. Then θ leaves $\mathfrak{s}$ invariant: $\theta(\mathfrak{s}) \subset \mathfrak{s}$. Indeed note that since $\mathfrak{h}$ is solvable, then there exists $k \geq 1$ such that the kth-derived algebra satisfies: $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{s}$, thus it follows that θ must leave $\mathfrak{s}$ invariant, and hence we can write,

$$\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{k}) \oplus (\mathfrak{s} \cap \mathfrak{p}).$$

We claim that $\mathfrak{s} \cap \mathfrak{p} = 0$, indeed suppose not, i.e. $\mathfrak{p} \subset \mathfrak{s}$ thus $[\mathfrak{s}, \mathfrak{p}] \subset \mathfrak{p}$ so $\mathfrak{p}$ is an abelian non-trivial ideal of $\mathfrak{s}$, contradicting the semi-simplicity of $\mathfrak{s}$. Thus θ fixes $\mathfrak{s}$ point wise. Moreover $\mathfrak{p} \lhd \mathfrak{g}$ is an abelian ideal, and so therefore $\mathfrak{p} \subset \mathfrak{h}$, i.e. $\mathfrak{h} \neq 0$. Finally since $g(-, -)$ restricted to $\mathfrak{s}$ and $\tilde{g}(-, -)$ restricted to $\tilde{\mathfrak{s}}$ is positive definite, then $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ give rise to a Wick-rotation of two Riemannian Lie groups, thus by Proposition 3.1 it follows that $\mathfrak{s} \cong \tilde{\mathfrak{s}}$, and case (1) is proved. For case (2) suppose $\mathfrak{g}$ is non-solvable (i.e. $\mathfrak{s} \neq 0$), then again w.r.t. θ we see that

$$\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{k}) \oplus (\mathfrak{s} \cap \mathfrak{p}),$$

where $\mathfrak{s} \cap \mathfrak{k} \neq 0$, since if not then $\mathfrak{s} \subset \mathfrak{p}$, i.e. $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$, which is a contradiction. Now since $\theta^{\mathbb{C}}$ is a Cartan involution of a real form $\tilde{\mathfrak{g}} \subset \mathfrak{s}^{\mathbb{C}}$, then $-\kappa$ on $\tilde{\mathfrak{g}}$ must also have the signature $(+, -, \dots, -)$, this follows since $\mathfrak{k}^{\mathbb{C}}$ is 1-dimensional. Now finally if $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ is a Cartan decomposition, then $\tilde{\mathfrak{k}}$ is abelian and 1-dimensional, thus it follows that $\tilde{\mathfrak{g}} \cong \mathfrak{sl}_2(\mathbb{R})$ see for example (Prop. 13.1.10, [5]). We conclude that $\mathfrak{s}^{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$, and hence also $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$. The proposition is proved. □

Thus restricting to the class of semi-simple Lie algebras it is impossible to Wick-rotate a Lorentzian metric to a Riemannian metric. However even for a nilpotent Lie algebra the converse of (1) is not necessarily true, indeed consider the nilpotent Lie algebra $\mathfrak{h}_3(\mathbb{R})$ of 3×3 strictly upper triangular matrices. Then if θ is an involution with $Dim(\mathfrak{p}) = 1$, we must be able to find a basis $\{x_1, x_2, x_3\}$ such that

$$[x_1, x_2] = C_{12}^1 x_1 + C_{12}^2 x_2, \quad [x_1, x_3] = C_{13}^3 x_3, \quad [x_2, x_3] = C_{23}^3 x_3.$$

But since $[\mathfrak{h}_3(\mathbb{R}), \mathfrak{h}_3(\mathbb{R})]$ has dimension 1, then it follows that $C^1_{12} = 0 = C^2_{12}$. Moreover since $\mathfrak{h}_3(\mathbb{R})$ is nilpotent of class 2, then we conclude also that $C^3_{12} = 0 = C^3_{23}$, i.e. $\mathfrak{h}_3(\mathbb{R})$ would have to be abelian, thus it is not possible to Wick-rotate a Lorentzian metric (i.e. of signature $(+, +, -)$) on $H_3(\mathbb{R})$ to a Riemannian metric. However Example 3.1, shows that $H_3(\mathbb{R})$ may possess a metric of signature $(-, -, +)$ where this is possible.

In view of case (2), there are examples of metrics (of signature $(+, -, \dots, -)$) that are Wick-rotatable to a Riemannian metric within: $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{h}_3(\mathbb{R})$ and even $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{h}_3(\mathbb{R})$.

If we impose the condition that the metric is bi-invariant, i.e. $(\mathfrak{g}, g)$ is a quadratic Lie algebra, then we have the following equivalence result:

Corollary 5.1. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}})$ be a real slice of Lie groups. Suppose g is bi-invariant. Then the following statements hold:*

(1) Suppose $g(-, -)$ has Lorentzian signature. Then (G, g) can be Wick-rotated to a Riemannian Lie group $(\tilde{G}, \tilde{g})$ if and only if $\mathfrak{g}$ is abelian or $\mathfrak{g}$ is a direct sum of $\mathfrak{h} \neq 0$ and $\mathfrak{s}$ is compact semi-simple. Moreover $\tilde{\mathfrak{g}} \cong \mathfrak{g}$.

(2) Suppose $g(-, -)$ has signature $(+, -, \dots, -)$. Then (G, g) can be Wick-rotated to a Riemannian Lie group, if and only if either $\mathfrak{g}$ is abelian or $\mathfrak{g}$ is a direct sum of $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{h}$ abelian.

Proof. Case (1). Since g is bi-invariant then $\tilde{\mathfrak{g}}$ must be reductive, because $\tilde{g}$ is bi-invariant and a Riemannian metric. Thus the complexification is also reductive, i.e. so are the real forms, thus $\mathfrak{g}$ is reductive. This means that either $\mathfrak{s} = 0$ or $\mathfrak{s}$ is semi-simple, and $\mathfrak{h}$ is abelian. So if $\mathfrak{g}$ is non-abelian then $\mathfrak{s}$ is semi-simple. Now by the proof of the previous proposition case (1), then given a Cartan involution of $\mathfrak{g}$ we must have that θ fixes point wise $\mathfrak{s}$. This means that g restricted to $\mathfrak{s}$ is an inner product. If $\mathfrak{s}^{\mathbb{C}}$ is simple, then $g_{|_{\mathfrak{s}}}$ must be proportional to the Killing form: $\lambda\kappa$ $(\lambda \in \mathbb{R})$. Now since g is positive definite on $\mathfrak{s}$ then $\lambda > 0$ i.e. $\mathfrak{s}$ is compact. If $\mathfrak{s}^{\mathbb{C}}$ is not simple then on each simple ideal, $g^{\mathbb{C}}$ is proportional to the Killing form. There are two cases to consider, either $\mathfrak{s}$ is simple (in which case $\mathfrak{s}$ has a complex structure) or each simple ideal $\mathfrak{J}$ of $\mathfrak{s}$ has a simple complexification $\mathfrak{J}^{\mathbb{C}} \lhd \mathfrak{s}^{\mathbb{C}}$ or has a complex structure. See for instance (Thm 6.94, [7]). Assume $\mathfrak{s}$ is simple, then $\mathfrak{s}^{\mathbb{C}} \cong \mathfrak{s} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a complex Lie algebra. Thus $g^{\mathbb{C}}$ restricted to $\mathfrak{s}$ is proportional to the complex Killing form on $\mathfrak{s}$, say $\lambda\kappa$. Thus viewing $\mathfrak{s}$ as a real Lie algebra, we get that $g^{\mathbb{C}}$ restricts to something proportional to the real part: $\lambda\frac{1}{2}Re(\kappa) = g$, which is positive definite by assumption. Therefore $\lambda \in \mathbb{R}$. But the real Killing form of $\mathfrak{s}$ is precisely $2Re(\kappa)$, so we conclude that either the Killing form is positive definite or negative definite, this is impossible. The argument for the other case is a combination of the previous two arguments. We conclude that $\mathfrak{s}$ is semi-simple compact. Now finally it follows that $\mathfrak{g} \cong \tilde{\mathfrak{g}}$ by the previous proposition and that $\mathfrak{h} \cong \tilde{\mathfrak{h}}$ (since they are abelian of the same dimension).

Conversely if $\mathfrak{g}$ is abelian then the statement is trivial, therefore assume $\mathfrak{s}$ is compact semi-simple. Then $\mathfrak{s} := [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$ forms an orthogonal sum w.r.t. g. Thus g restricted to $\mathfrak{s}$ must be positive definite, indeed restricting g on a compact simple ideal (which is a non-degenerate ideal) $\mathfrak{J} \lhd \mathfrak{s}$ we get something proportional to the Killing form on $\mathfrak{J}$: $\lambda\kappa$. Thus if $\lambda > 0$ then this would contradict g having Lorentzian signature. Therefore $\lambda < 0$. Hence g on $\mathfrak{h}$ must have Lorentzian signature, and so we can easily find a Cartan involution $\theta_{\mathfrak{h}}$ of $\mathfrak{h}$ such that $1_{\mathfrak{s}} \oplus \theta_{\mathfrak{h}}$ is a Cartan involution of $\mathfrak{g}$, now use Corollary 3.2.

Case (2). Again since $\mathfrak{g}$ must be reductive, then by the previous proposition case (2), if $\mathfrak{g}$ is not abelian then $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{h}$ is abelian. Conversely let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$. If $\mathfrak{s} = 0$, then the statements is obviously true. Suppose therefore that $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$. Note that $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$ is an orthogonal direct sum w.r.t. g, i.e. $[\mathfrak{g}, \mathfrak{g}]^{\perp} = \mathfrak{z}(\mathfrak{g})$. Thus g restricted to $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$ forms a quadratic Lie algebra, but since $\mathfrak{sl}_2(\mathbb{C})$ is simple, then g must be proportional to the Killing form: $\lambda\kappa$ $(\lambda \in \mathbb{R})$. Note that $\lambda < 0$ since other wise g would not be able to have signature: $(+, -, - \dots, -)$. Also note that g restricted to $\mathfrak{h}$ must be of signature: $(-, -, \dots, -)$. Thus choose any Cartan involution $\theta_{\mathfrak{s}}$ of $\mathfrak{s}$, and the Cartan involution $\theta_{\mathfrak{h}}$ of $\mathfrak{h}$ of the form: $\theta_{\mathfrak{h}}(x) := -x$. Then $\theta_{\mathfrak{s}} \oplus \theta_{\mathfrak{h}}$ is a Cartan involution of $\mathfrak{g}$, and the statement follows by Corollary 3.2. □

Thus a solvable Lie group (G, g) with a bi-invariant (non-Riemannian) metric is not Wick-rotatable to a Riemannian Lie group.

6. A remark on Wick-rotatable tensors of Lie groups

Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is Riemannian. By Corollary 3.2 there exists a Cartan involution θ of $\mathfrak{g}$. Recall the section on Wick-rotatable tensors. We prove in this section that if $\tilde{v} = v \in \mathcal{V} \cap \tilde{\mathcal{V}}$ are two tensors on the Lie algebras, then they are Wick-rotatable with respect to an embedding $\phi^{-1} \in H^{\mathbb{C}}$ into the same $H^{\mathbb{C}}$-orbit for

$$H^{\mathbb{C}} := Aut(\mathfrak{g}^{\mathbb{C}}) \cap O(n, \mathbb{C}) \subset O(n, \mathbb{C}),$$

such that $(\mathfrak{g}, \phi^{-1}(\tilde{\mathfrak{g}}))$ is a compatible pair (i.e. also a compatible triple). Denote $H := Aut(\mathfrak{g}) \cap O(p, q)$. Note that $H \subset H^{\mathbb{C}}$ is a real form. Indeed the real structure of $O(n, \mathbb{C})$ fixing $O(p, q)$ given by $A \mapsto \sigma A \sigma$ where σ is the conjugation map w.r.t. $\mathfrak{g}$, leaves $H^{\mathbb{C}}$ invariant, and thus fixes H. Note also that a global Cartan involution $\Theta\colon A \mapsto \theta A \theta$ of $O(p, q)$ where θ is a Cartan involution of $\mathfrak{g}$, also leave H invariant. Thus Θ is a global Cartan involution of H. The arguments above are analogous for the real from: $\tilde{H} := O(\tilde{p}, \tilde{q}) \cap Aut(\tilde{\mathfrak{g}})$.

Lemma 6.1. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is Riemannian. Then any two tensors $v = \tilde{v} \in \mathcal{V} \cap \tilde{\mathcal{V}}$ can be embedded into the same $H^{\mathbb{C}}$-orbit for some $\phi^{-1} \in H^{\mathbb{C}}$ such that $\left(\mathfrak{g}, \phi^{-1}(\tilde{\mathfrak{g}})\right)$ is a compatible pair.*

Proof. By Corollary 3.2, we can choose a Cartan involution θ of $\mathfrak{g}$. Moreover by Proposition 3.1 there is an isomorphism of Lie algebras: $\phi \in O(n, \mathbb{C}) \cap Aut(\mathfrak{g}^{\mathbb{C}})$ sending $\mathfrak{u} \mapsto \tilde{\mathfrak{g}}$, where $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ w.r.t. the Cartan decomposition of θ. Thus $\phi^{-1}(\tilde{\mathfrak{g}}) = \mathfrak{u}$ and $\mathfrak{g}$ are compatible. Let now $\tilde{v} = v \in \mathcal{V} \cap \tilde{\mathcal{V}}$ be two Wick-rotatable tensors, using the isometry tensor action of ϕ^{-1} on v, then $\phi^{-1} \cdot v$ and v lie in the same $H^{\mathbb{C}}$-orbit as required. □

Theorem 6.1. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is Riemannian. Let $\tilde{v} = v \in \mathcal{V} \cap \tilde{\mathcal{V}}$ be two tensors (i.e. they are also Wick-rotatable). There exists a Cartan involution θ of $\mathfrak{g}$ such that $\theta \cdot v = v$.*

Proof. Consider the real forms: $H \subset H^{\mathbb{C}} \supset \tilde{H}$ as above. Then one simply note that $H^{\mathbb{C}}$ is naturally algebraic, and moreover is a linearly complex reductive Lie group, simply because $\tilde{H}$ is a compact real form. Now since $O(n, \mathbb{C})$ and $Aut(\mathfrak{g}^{\mathbb{C}})$ are naturally algebraic groups defined over $\mathbb{R}$, then so is $H^{\mathbb{C}}$. Moreover H and $\tilde{H}$ are the real points of $H^{\mathbb{C}}$ (respectively). Thus the groups are naturally among the class of groups considered in Section 2.3. The theorem follows by Lemma 6.1 and Theorem 2.2. □

Thus we can restate a stronger version of Theorem 2.3 for Lie groups, (see also paragraph after Definition 2.18 for the tensors in question):

Theorem 6.2. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is a Riemannian Lie group. Then the following statements hold:*

(1) There exists a Cartan involution θ of $\mathfrak{g}$ such that $\nabla_{\theta(x)}\theta(y) = \theta(\nabla_x y)$ for all $x, y \in \mathfrak{g}$.
(2) There exists a Cartan involution θ of $\mathfrak{g}$ such that $ric_g\left(\theta(x), \theta(y)\right) = ric_g(x, y)$ for all $x, y \in \mathfrak{g}$.
(3) There exists a Cartan involution θ of $\mathfrak{g}$ such that $[\theta, Ric_g] = 0$.
(4) There exists a Cartan involution θ of $\mathfrak{g}$ such that $R\left(\theta(x), \theta(y)\right)\left(\theta(z)\right) = \theta\left(R(x, y)(z)\right)$ for all $x, y, z \in \mathfrak{g}$.

Any of the properties 1–4 of the previous theorem, are (using also Theorem 3.1) Wick-rotatable. Thus we state as a stronger result for Lie groups (compare with [2]):

Corollary 6.1. *Let (G, g) be a pseudo-Riemannian Lie group. Then the property of being Riemann purely electric (RPE) at 1 w.r.t. to a Cartan involution θ of $\mathfrak{g}$ is Wick-rotatable.*

Proof. Follows by Theorems 3.1 and 6.2. □

We end this section by also noting the following result:

Corollary 6.2. *Let $(G, g) \subset (G^{\mathbb{C}}, g^{\mathbb{C}}) \supset (\tilde{G}, \tilde{g})$ be a Wick-rotation of Lie groups. Assume $(\tilde{G}, \tilde{g})$ is Riemannian. Then*

$$\left(\forall x \in \mathfrak{g} \cap \tilde{\mathfrak{g}}\right)\left(\exists \theta \in Aut(\mathfrak{g})\right)\left(\theta(x) = x\right),$$

where θ is a Cartan involution of $\mathfrak{g}$.

Proof. Consider the isometry action of $O(n, \mathbb{C})$ (restricted to $H^{\mathbb{C}}$ defined above) on the complex Lie algebra: $\mathfrak{g}^{\mathbb{C}}$, i.e.

$$g \cdot x := g(x), \ g \in O(n, \mathbb{C}), \ x \in \mathfrak{g}^{\mathbb{C}}.$$

Let $x = \tilde{x} \in \mathfrak{g} \cap \tilde{\mathfrak{g}}$. Then $x = \tilde{x}$ are two Wick-rotatable tensors, thus w.r.t. a choice of $g \in H^{\mathbb{C}}$ we can assume that the real actions (of H and $\tilde{H}$) are compatible. Moreover x and $\tilde{x}$ lie in the same complex orbit, such that $O(p, q)x \sim O(n)\tilde{x}$ are compatible real orbits. We can now finish the proof by applying Theorem 6.1. □

7. Wick-rotating an algebraic soliton

A pseudo-Riemannian Lie group (G, g), such that the Ricci operator $Ric_g \in \mathfrak{gl}(\mathfrak{g})$ has the form:

$$Ric_g = \lambda \cdot 1_{\mathfrak{g}} + D,$$

where $\lambda \in \mathbb{R}$ and $D \in \mathfrak{der}(\mathfrak{g})$ (a derivation) is called an *algebraic soliton* (defined in [6]). If D can be taken to be $D = 0$, then (G, g) is said to be Einstein, and moreover if the Lie algebra is also nilpotent (resp. solvable), then an algebraic soliton (G, g), is said to be a *Ricci nilsoliton* (resp. *solsoliton*). For a discussion of Riemannian Ricci nilsolitons we refer to for example [8]. However we shall only be interested in Wick-rotating such a geometry.

We shall prove a result regarding the existence of a Wick-rotation of an algebraic soliton to a Riemannian Lie group, by using the results of the previous section.

Lemma 7.1. *The property of being an algebraic soliton is Wick-rotatable.*

Proof. Let (G, g) be Wick-rotatable to $(\tilde{G}, \tilde{g})$. Suppose (G, g) is an algebraic soliton. The Ricci operator: $\mathfrak{g} \xrightarrow{Ric_g} \mathfrak{g}$ on G is also a restriction of the Ricci operator on $G^{\mathbb{C}}$. So if $Ric_g = \lambda \cdot 1_{\mathfrak{g}} + D$ for some $\lambda \in \mathbb{R}$ and $D \in \mathfrak{der}(\mathfrak{g})$, then also

$$Ric_{g^{\mathbb{C}}} = (\lambda \cdot 1_{\mathfrak{g}})^{\mathbb{C}} + D^{\mathbb{C}} = \lambda 1_{\mathfrak{g}^{\mathbb{C}}} + D^{\mathbb{C}}.$$

Note that $D^{\mathbb{C}}$ is a derivation of $\mathfrak{g}^{\mathbb{C}}$, thus when restricting to $Ric_{\tilde{g}}$ we see that $D^{\mathbb{C}}$ must leave $\tilde{\mathfrak{g}}$ invariant and is thus a derivation $\tilde{D}$ of $\tilde{\mathfrak{g}}$ as required. The lemma follows. □

Note from the lemma that $D \in End(\mathfrak{g})$ and $\tilde{D} \in End(\tilde{\mathfrak{g}})$ are Wick-rotatable tensors, under the isometry action:

$$g \cdot f := gfg^{-1}, \; g \in O(n, \mathbb{C}), \; f \in End(\mathfrak{g}^{\mathbb{C}}).$$

Corollary 7.1. *The property of being a Ricci nilsoliton (resp. solsoliton) is Wick-rotatable.*

Corollary 7.2. *The property of being Einstein is Wick-rotatable.*

Applying the previous section, we get the following necessary condition for when an algebraic soliton can be Wick-rotated to a Riemannian algebraic soliton:

Theorem 7.1. *Suppose (G, g) is an algebraic soliton, with $Ric_g = \lambda \cdot 1_{\mathfrak{g}} + D$, which can be Wick-rotated to a Riemannian algebraic soliton: $(\tilde{G}, \tilde{g})$ with $Ric_{\tilde{g}} = \lambda \cdot 1_{\tilde{\mathfrak{g}}} + \tilde{D}$. Then there exists a Cartan involution θ of $\mathfrak{g}$ such that $[\theta, D] = 0$.*

Proof. The derivations: $D^{\mathbb{C}} = \tilde{D}^{\mathbb{C}} \in \mathcal{V} \cap \tilde{\mathcal{V}}$ are Wick-rotatable (see the proof of Lemma 7.1). Thus w.r.t. a choice of map $g \in H^{\mathbb{C}}$ we can assume w.l.o.g that D and $\tilde{D}$ lie in the same complex orbit under the conjugation action: $H^{\mathbb{C}} \cdot D \ni \tilde{D}$. By Theorem 6.1 there is a Cartan involution θ of $\mathfrak{g}$ such that $\theta \cdot D := \theta D \theta = D$ or in other words $[\theta, D] = 0$. The theorem is proved. □

Example 7.1. We follow Example 3.1. Thus consider the real 3-dimensional Heisenberg group $(H_3(\mathbb{R}), -g)$, then this is a Ricci nilsoliton of signature $(+, -, -)$. Indeed one can calculate with respect to the basis $\{e_1, e_2, e_3\}$, that the Ricci operator can be written uniquely as:

$$Ric_{-g} = -\frac{3}{2} \cdot I_3 + D,$$

where $D(e_1) = e_1, \; D(e_2) = e_2, \; D(e_3) = 2e_3$. Note that the Cartan involution θ commutes with D, i.e. $[\theta, D] = 0$. Thus when restricting to the Wick-rotated Riemannian Ricci nilsoliton $(\tilde{G}, \tilde{g})$ with Lie algebra: $\tilde{\mathfrak{g}} := \langle ie_1, ie_2, e_3 \rangle \subset \mathfrak{h}_3(\mathbb{C})$, we get the corresponding Ricci operator expressed as:

$$Ric_{\tilde{g}} = -\frac{3}{2} \cdot I_3 + \tilde{D},$$

with $\tilde{D}(ie_1) = ie_1, \; \tilde{D}(ie_2) = ie_2, \; \tilde{D}(e_3) = 2e_3$.

References

[1] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, AMS, 1978.
[2] C. Helleland, S. Hervik, A Wick-rotatable metric is purely electric, J. Geom. Phys. 123 (2018) 424–429, http://dx.doi.org/10.1016/j.geomphys.2017.09.015.
[3] C. Helleland, S. Hervik, Wick rotations and real GIT, J. Geom. Phys. 123 (2018) 343–361, http://dx.doi.org/10.1016/j.geomphys.2017.09.009.
[4] C. Helleland, S. Hervik, Real GIT with applications to compatible representations and Wick-rotations, J. Geom. Phys. 142 (2019) 92–110, http://dx.doi.org/10.1016/j.geomphys.2019.03.007.
[5] Joachim Hilgert, Karl-Hermann Neeb, Structure and Geometry of Lie Groups, in: Springer Monographs in Mathematics, 2012.
[6] M. Jablonski, Concerning the existence of Einstein and Ricci soliton metrics on solvable Lie groups, arXiv:1009.3283 [math.DG].
[7] Anthony W. Knapp, Lie Groups: Beyond and Introduction, second ed., Birkhauser Verlag, 2002.
[8] J. Lauret, Ricci soliton homogeneous nilmanifolds, Math. Ann. 319 (2001) 715.
[9] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. Math. 21 (1976) 293–329.
[10] A.L. Onishchik, E.B. Vinberg, Lie Groups and Lie Algebras III, in: Encyclopaedia of Mathematical Sciences, vol. 41.
[11] R.W. Richardson, P.J. Slodowy, Minimum vectors for real reductive algebraic groups, J. Lond. Math. Soc. 42 (1990) 409–429.