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The Shapley value of coalitions to other coalitions

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The Shapley value for an n -person game is decomposed into a $2^n \times 2^n$ value matrix giving the value of every coalition to every other coalition. The cell $\phi_{IJ}(v, N)$ in the symmetric matrix is positive, zero, or negative, dependent on whether row coalition I is beneficial, neutral, or unbeneficial to column coalition J . This enables viewing the values of coalitions from multiple perspectives. The $n \times 1$ Shapley vector, replicated in the bottom row and right column of the $2^n \times 2^n$ matrix, follows from summing the elements in all columns or all rows in the $n \times n$ player value matrix replicated in the upper left part of the $2^n \times 2^n$ matrix. A proposition is developed, illustrated with an example, revealing desirable matrix properties, and applicable for weighted Shapley values. For example, the Shapley value of a coalition to another coalition equals the sum of the Shapley values of each player in the first coalition to each player in the second coalition.

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Introduction

Lloyd Shapley (1923–2016) is perhaps best known for his so-called Shapley value (Shapley, 1953b), interpreted by Roth (1988b, p. 6) as “player i ’s ‘fair share’ in the game.” Three other interpretations are a player’s expected marginal contribution, the weighted average of his marginal contributions to the coalition of all n players involved, and what player i can “reasonably” command to himself. The Shapley value influenced Shapley’s subsequent thinking causing the 2012 Nobel Memorial Prize in Economic Sciences (with Alvin E. Roth), “for the theory of stable allocations and the practice of market design.” See Weber (1988) for the well known Shapley value axioms and definitions, Serrano (2018) for a bibliography of Shapley’s contributions, and Yokote et al. (2017) for work relating the Shapley value to other solutions.

Hausken and Mohr (2001) decomposed the Shapley value into a value matrix. The sum of the elements of any row or column in the $n \times n$ matrix equals the Shapley value of the respective player in an n -person game. Towards the end of his work on multilinear extensions of games, as an aside at the end of the section labeled “Possible Further Applications,” Owen (1972, p. 76) proposed second order cross-derivatives which “can be thought of as measuring, in some sense, the value of player j to player i ,” as discussed by Hausken and Mohr (2001, p. 469). Owen (1972, pp. 77–78) thereafter presented three game examples. In the first three-person majority game he writes that players “1 and 2 are valuable to each other if” player “3 is unlikely to join,” “but get rather in each other’s way” otherwise.

Aside from Hausken and Mohr’s (2001) and Owen’s (1972) contributions, the authors are unaware of other work considering the value of a player or coalition to another player or coalition. The literature has not used this language, and has not approached the phenomenon from this angle. Whereas Hausken and Mohr (2001) present the value of a player to another player, this article generalizes to determine the value of row coalition I to column coalition J in a $2^n \times 2^n$ value matrix. The matrix is shown to have a variety of desirable properties. The usefulness of the new matrix is that any coalition can value any other coalition regardless of whether the coalitions are disjoint, overlap partly, or coincide. The values of coalitions can thus be conceptualized relative to each other from any imaginable perspective.

Two non-overlapping coalitions in a game may find it useful to know their values to each other. The values are shown to be equal due to symmetry. For example, if the value is negative, both coalitions may have an interest in excluding the other from the game, or ensuring that alternative coalitions are formed. Coalitions may or may not have formed in order to determine their value to each other. If two coalitions overlap, one may have been formed, and may consider its value to another coalition which may form by including or excluding members. Alternatively, a hypothetical coalition, i.e., not yet formed, may consider its value to another already formed coalition. Knowing this value may enable both the potential members of the hypothetical coalition and the members of the already formed coalition to determine whether the already formed coalition should alter its member structure.

Two natural settings for the application of the concept of the value of a coalition to another coalition are as follows. The first is a coalition formation environment, when in the status quo coalitions are already formed. Examples of coalition formation environments are changes and fluctuations in technology, economy, culture, laws, and players’ preferences and beliefs. The second is when there are restrictions in the set of feasible coalitions. Then each formed coalition might contemplate whether to merge with another in line with the concept developed in this article.

The section “Literature review” reviews the literature. The section “Basic definitions” presents basic definitions. The section “The Shapley value of coalition I to coalition J ” presents the Shapley value of a coalition to another coalition. The section “Example” illustrates with an example. The section “Interpreting $\phi_{IJ}(N, v)$ ” interprets the various Shapley values. The section “Usefulness, future research and applications” considers usefulness, future research and applications. The section “Applying the weighted Shapley value” applies the weighted Shapley value. The section “Conclusion” concludes.

Literature review

We suggest that the symmetry in the value of a coalition to another coalition has a weak indirect linkage to Myerson’s (1980) work on balanced contributions and Hart and Mas-Colell’s (1989) work on the preservation of differences for the potential function. Myerson (1977) adapted Shapley’s (1953b) axioms to games in partition function form. Myerson (1980) generalized to conferences of more than two players, and removed the side-payments assumptions. He showed that any characteristic function game has a unique fair allocation rule which satisfies a balanced contributions formula, related to Harsanyi’s (1963) generalized Shapley value. Hart and Mas-Colell (1989) showed that the potential, i.e., “a real-valued function defined on the space of cooperative games with transferable utility,” satisfying that the marginal contributions of all players are efficient, is unique, and that “the resulting payoff vector coincides with the Shapley value.” The potential yields a new internal consistency property. See Kongo (2018) for further work on balanced contributions.

An indirect linkage also exists between this article and Casajus and Huettner’s (2017) assignment to any player the difference between the worth of the grand coalition and its worth after this player leaves the game. They show that the Shapley value is a unique decomposable decomposer of this assignment.

Earlier work on coalitions has not considered the value of one coalition to another coalition. Maschler (1963) considered the power of a coalition, accounting for the players’ psychology, bargaining abilities, morality, etc., agreeing with Shapley that the Shapley value constitutes an a priori assessment. Aumann and Dreze (1974) developed theorems for the Shapley value, kernel, nucleolus, bargaining set, core, and the von Neumann–Morgenstern solution, “that connect a given solution notion, defined for a coalition structure \mathcal{B} with the same solution notion applied to appropriately defined games on each of the coalitions in \mathcal{B} .” Shenoy (1979) suggested two models of coalition formation, using only information in the characteristic function, and illustrating with the Shapley value, the core, the bargaining set, and individually rational payoffs. Kurz (1988) considered some ways in which the Shapley value may be used to determine how various coalition structures impact each player’s payoff. Aumann and Myerson (1988) used an extension of the Shapley value to specify how cooperation between players can be organized, where players choose whether and with whom to establish bilateral links. Hu and Li (2018) axiomatize the Shapley-solidarity value for games with a coalition structure. Skibski et al. (2018) consider the stochastic Shapley value for coalitional games with externalities.

Basic definitions

A cooperative game (N, v) is defined by a finite set of players N , called the grand coalition, and a characteristic function $v: 2^N \rightarrow \mathbb{R}$ from the set of all possible coalitions of players to a set of payments that satisfies $v(\emptyset) = 0$. The function v describes how

much collective payoff a set of players can gain by forming a coalition. Shapley (1953b) assigns a value

$$\phi_i(N, v) = \sum_{\substack{S \subseteq N, \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} (v(S) - v(S \setminus \{i\})), \phi_i(S \setminus \{i\}, v) \equiv 0 \tag{1}$$

where $s = |S|$ is the number of players in S , for each game (N, v) for each player $i, i \in N \subseteq U$.

Definition 1. The Shapley value of coalition $I, I \subseteq N$, equals the sum of the Shapley values for each player i in coalition $I, i \in I$, i.e.,

$$\phi_I(N, v) \equiv \sum_{i \in I} \phi_i(N, v) \tag{2}$$

Hausken and Mohr (2001, p. 469) assign a value $\phi_{ij}(N, v)$ for each game (N, v) for any two players i and j within the universe U of all possible players, i.e., $i \in N \subseteq U, j \in N \subseteq U, i, j = 0, 1, 2, \dots, N$.

Lemma 1. The Shapley value $\phi_i(N, v)$ for player $i \in N$ in a game of $n = |N|$ players, where $s = |S|$ is the number of players in S , is decomposed into n different values $\phi_{ij}(N, v), j \in N$, satisfying

$$\phi_i(N, v) = \sum_{j=1}^n \phi_{ij}(N, v) \tag{3}$$

where

$$\phi_{ij}(N, v) = \sum_{\substack{S \subseteq N, \\ i, j \in S}} \frac{(s-1)!(n-s)!}{n!} (\phi_j(S, v) - \phi_j(S \setminus \{i\}, v)), \phi_i(S \setminus \{i\}, v) \equiv 0 \tag{4}$$

and $s = |S|$ is the number of players in S .

Proof. See Hausken and Mohr’s (2001) Theorem 2.1. □

$\phi_{ij}(N, v)$ in Eq. (4) has the same structure as $\phi_i(N, v)$ in Eq. (1), though replacing $v(S)$ with $\phi_j(S, v)$ and replacing $v(S \setminus \{i\})$ with $\phi_j(S \setminus \{i\}, v)$, and summing over $S \subseteq N$ so that $i, j \in S$ in Eq. (4), while only $i \in S$ in $\phi_i(S, v)$.

Definition 2. For all $i \in N, j \in N$,

$$\phi_{\{i\}\{j\}}(N, v) \equiv \phi_{\{ij\}}(N, v) \equiv \phi_{\{i\}\{j\}}(N, v) \equiv \phi_{ij}(N, v) \tag{5}$$

Lemma 2. For all $i \in N, j \in N$, the following holds

$$\phi_{ij}(N, v) = \phi_{ji}(N, v) \tag{6}$$

Proof. Hausken and Mohr (2001) showed in Appendix 1 that

$$\phi_{ij}(N, v) = \sum_{R \subseteq N} \frac{(r-1)!(n-r)!}{n!} \times [\{v(R) - v(R \setminus \{j\})\} - \{v(R \setminus \{i\}) - v(R \setminus \{i, j\})\}] \sum_{s=r}^n \frac{1}{s} \tag{7}$$

Interchanging i and j does not impact Eq. (7), which implies Eq. (6). Lemma 2 also follows from Myerson’s (1980) axiom of balanced contributions. Segal (2003) refers to term within square brackets in Eq. (7) as “the second-order difference operator.” □

That the value of player i to player j is symmetric was also observed by Owen (1972, p. 76), due to continuity. The symmetry in Lemma 2 is indirectly linked to two earlier contributions, though without conceptualizing the phenomenon as the value of one player to another player. The first is Myerson’s (1980) notion of “balanced contributions,” mathematically related to the Frobenius integrability condition which involves symmetry of the cross partial derivatives. Myerson (1980, p. 173) stated that an allocation rule “has balanced contributions if j ’s contribution to i always equals i ’s contribution to j , in any conference structure.” The second is Hart and Mas-Colell’s (1989) section 3 labeled

“Preservation of Differences.” Hart and Mas-Colell (1989, p. 594) determined the difference to be preserved as “the difference between what i would get if j was not around and what j would get if i was not around.”

Lemma 3. The Shapley value $\phi_j(N, v)$ for player $j \in N$ in a game of $n = |N|$ players is decomposed into n different values $\phi_{ij}(N, v), i \in N$, satisfying

$$\phi_j(N, v) = \sum_{i=1}^n \phi_{ij}(N, v) \tag{8}$$

Proof. Inserting $\phi_{ij}(N, v) = \phi_{ji}(N, v)$ in Eq. (6) into the expression inside the summation sign in Lemma 1 gives

$$\phi_i(N, v) = \sum_{j=1}^n \phi_{ji}(N, v) \tag{9}$$

Interchanging i and j in Eq. (9) gives Eq. (8). □

The Shapley value of coalition I to coalition J

Analogously to the section “Basic definitions”, let us assign a value $\phi_{IJ}(N, v)$ for each game (N, v) for any two coalitions I and J within the universe U of all possible players, i.e., $I \subseteq N \subseteq U, J \subseteq N \subseteq U$, where U is finite or infinite. We posit three axioms following Shapley’s (1953b, p. 33) language on p. 33. For axiom 1 Shapley (1953b, p. 32) defines a function πv by $\pi v(\pi S) = v(S)$ where $S \subseteq N \subseteq U$ and πS is “the image of S under π .”

Axiom 1. Symmetry. For each π in $\Pi(U)$, where $\Pi(U)$ is the set of permutations of the universe U of all possible players, and $\pi v(\pi S) = v(S)$ for all $S \subseteq N \subseteq U$,

$$\phi_{\pi I \pi J}(\pi S, \pi v) = \phi_{IJ}(S, v) \tag{10}$$

For axiom 2 Shapley (1953b, p. 32) defines a carrier of v as any set $S \subseteq N \subseteq U$ with $v(S) = v(N \cap S)$ for all $S \subseteq U$. The Shapley value $\phi_i(N, v)$ of a null player i in a game (N, v) is zero. A player i is null in (N, v) if $v(S \cup \{i\}) = v(S)$ for all coalitions S that do not contain i . In accordance with e.g., Roth (1988a, p. 5), axiom 2 has two parts.

Axiom 2a. Efficiency carrier. For each carrier $S \subseteq N \subseteq U$ of v and any partitions p_I and p_J of N ,

$$\sum_{I \in p_I, J \in p_J} \phi_{IJ}(N, v) = v(N) \tag{11}$$

Axiom 2b. Null coalition carrier. If I is a null coalition in v defined as $v(S \cup I) = v(S)$ for all coalitions $S \subseteq N$, and/or J is a null coalition in v defined as $v(S \cup J) = v(S)$ for all coalitions $S \subseteq N$,

$$\phi_{IJ}(N, v) = 0 \tag{12}$$

Axiom 3. Additivity or law of aggregation. For any two games (N, v) and (N, w) with support equal to N ,

$$\phi_{IJ}(N, v) + \phi_{IJ}(N, w) = \phi_{IJ}(N, v + w) \forall I \subseteq N \subseteq U \text{ and } \forall J \subseteq N \subseteq U, \text{ i.e. } \phi(N, v) + \phi(N, w) = \phi(N, v + w) \tag{13}$$

Axiom 1 states that coalition names or identities are irrelevant when determining the value $\phi(N, v)$. Axiom 2a states that the value $\phi(N, v)$ fully distributes the yield of the game, thus excluding e.g., $\phi_{IJ}(N, v) = v(I \cup J)$ where any two coalitions I and J assume that all other players and coalitions cooperate against them. A partition p_I of a set N , and partition p_J of N , is a grouping of the set’s elements into non-empty subsets so that every element is included in one and only one of the subsets. Applying partitioning preserves the spirit of Shapley’s (1953b) efficiency axiom by ensuring that every individual player is included once (regardless of which subset the player is partitioned into), and also ensuring that no player is included twice (prevents double counting). Axiom 2b states that if at least one of the coalitions I

Table 1 The Shapley value of row coalition I to column coalition j when $\phi_{ii}(N, v) = \phi_i(N, v)$ and $\phi_{ij}(N, v) = 0$ for $i \neq j$ (alternative 1).

	{0}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{0}	0	0	0	0	0	0	0	0
{1}	0	1/3	0	0	1/3	1/3	0	1/3
{2}	0	0	1/3	0	1/3	0	1/3	1/3
{3}	0	0	0	1/3	0	1/3	1/3	1/3
{1,2}	0	1/3	1/3	0	2/3	1/3	1/3	2/3
{1,3}	0	1/3	0	1/3	1/3	2/3	1/3	2/3
{2,3}	0	0	1/3	1/3	1/3	1/3	2/3	2/3
{1,2,3}	0	1/3	1/3	1/3	2/3	2/3	2/3	1

Table 2 The Shapley value of row coalition I to column coalition J when $\phi_{ii}(N, v) = 0$ and $\phi_{ij}(N, v) = \phi_i(N, v)/(n - 1)$ for $i \neq j$ (alternative 2).

	{0}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{0}	0	0	0	0	0	0	0	0
{1}	0	0	1/6	1/6	1/6	1/6	1/3	1/3
{2}	0	1/6	0	1/6	1/6	1/3	1/6	1/3
{3}	0	1/6	1/6	0	1/3	1/6	1/6	1/3
{1,2}	0	1/6	1/6	1/3	1/3	1/2	1/2	2/3
{1,3}	0	1/6	1/3	1/6	1/2	1/3	1/2	2/3
{2,3}	0	1/3	1/6	1/6	1/2	1/2	1/3	2/3
{1,2,3}	0	1/3	1/3	1/3	2/3	2/3	2/3	1

and J is a null coalition, then the value $\phi_{IJ}(N, v)$ is zero. A null player or null coalition is a player or coalition which is null in every game (Casajus and Huettner, 2014; van Den Brink, 2007). Axiom 3 states that the values of independent games are added by considering only coalitions I and J in the two games.

These axioms, as formulated by Shapley (1953b, p. 32) for player i , gives a unique solution for player i , which Shapley (1953b, p. 33) finds remarkable. Determining how Lemmas 1–3 for coalitions follow from the axioms is similar to Shapley’s (1953b, p. 33) proof for individual players. However, the axioms for coalitions do not give a unique solution for the value of coalition I to coalition J . Hausken and Mohr (2001, p. 469) do not address the issue of uniqueness for the value of player i to player j . Let us illustrate with an example.

Example. Assume two real parameters a and b such that, for each $i \in S \subseteq N$ and $j \in S \subseteq N$, $i \neq j$, $\phi_{ij}(S, v) = a$, $\phi_{ii}(S, v) = b$, where S is a carrier of v , I is a null coalition in v defined as $v(S \cup I) = v(S)$, and $v(N) = 1$. According to the efficiency Axiom 2a for $s = |S|$ players, we must have

$$s(s - 1)a + sb = 1 \tag{14}$$

As alternative 1, $a = 0$ and $b = 1/s$ cause $\phi_{ii}(N, v) = \phi_i(N, v)$ and $\phi_{ij}(N, v) = 0$ for $i \neq j$ which satisfy the axioms. As alternative 2, $a = 1/(s(s - 1))$ and $b = 0$ cause $\phi_{ii}(N, v) = 0$ and $\phi_{ij}(N, v) = \phi_i(N, v)/(n - 1)$ for $i \neq j$. Both these two alternatives satisfy the axioms.¹ This contrasts with Shapley’s (1953b, p. 33) proof, where only one parameter is needed. That is, for each $i \in S \subseteq N$, $\phi_i(N, v) = a$, which can be computed from efficiency, causing a unique solution for player i . In other words, the Shapley value of coalitions assuming Axiom 1, Axiom 2a, Axiom 2b, and Axiom 3, is not unique, whereas the Shapley value of players is unique. To understand the phenomenon more thoroughly, Tables 1 and 2 present the Shapley value of row coalition I to column coalition J for the two alternatives assuming $n = |N| = 3$ players.

Based on the axioms it cannot be determined whether Tables 1 or 2 is correct. The bottom row where $I = \{1, 2, 3\} = N$, and right column where $J = \{1, 2, 3\} = N$, are equivalent in Tables 1 and 2. This suggests that an axiom that merely focuses on the set N of players may be insufficient to cause uniqueness. For example, an axiom such as $\phi_{IN}(N, v) = \phi_I(N, v)$ is insufficient, in addition to assuming the result in the Proposition developed below. An alternative axiom such as $\phi_{ij}(N, v) = \sum_{j \in I} \phi_{ij}(N, v)$ may be needed, but that also assumes the Proposition developed below. Since both Tables 1 and 2 seem realistic and plausible, it may also be possible that uniqueness is not desirable. That is, why would one choose axioms that might dictate either Tables 1 or 2 as correct, when both may be desirable? Because of these challenges, we leave the issue of one or several additional axioms to ensure uniqueness, or whether uniqueness may not be desirable,

as an open research question, and proceed with developing results. □

Definition 3. For all $I \subseteq N$, $J \subseteq N$,

$$\phi_{I \cup J}(N, v) \equiv \phi_{IJ}(N, v), \phi_{\{i\}J}(N, v) \equiv \phi_{ij}(N, v) \tag{15}$$

Proposition. The Shapley value of coalition I to coalition J , $I \subseteq N$, $J \subseteq N$, in an n -person game is

$$\phi_{IJ}(N, v) = \sum_{\substack{i \in I, \\ j \in J}} \phi_{ij}(N, v) = \phi_{II}(N, v) \tag{16}$$

Proof. If $I = \{i, k, \dots, m\} = \{i\} \cup \{k\} \cup \dots \cup \{m\}$, $\{i\} \cap \{k\} = \dots = \{i\} \cap \{m\} = \{k\} \cap \{m\} = \dots = \emptyset$, $J = \{j, q, \dots, u\} = \{j\} \cup \{q\} \cup \dots \cup \{u\}$, $\{j\} \cap \{q\} = \dots = \{j\} \cap \{u\} = \{q\} \cap \{u\} = \dots = \emptyset$, $i, k, \dots, m, j, q, \dots, u = 0, 1, \dots, n$, Eq. (16) becomes

$$\begin{aligned} \phi_{IJ}(N, v) &= \phi_{ij}(N, v) + \phi_{kj}(N, v) + \dots + \phi_{mj}(N, v) \\ &+ \phi_{iq}(N, v) + \phi_{kq}(N, v) + \dots + \phi_{mq}(N, v) \\ &+ \dots + \phi_{iu}(N, v) + \phi_{ku}(N, v) + \dots + \phi_{mu}(N, v) \end{aligned} \tag{17}$$

To illustrate consistency, for the three events $u = n$, $m = n$, $m = u = n$, Eq. (17) becomes $\phi_{IN}(N, v) = \phi_I(N, v)$, $\phi_{NJ}(N, v) = \phi_J(N, v)$, $\phi_{NN}(N, v) = \phi(N, v) = v(N) = v(1, 2, \dots, n)$, respectively. Proving that $\phi_{IJ}(N, v)$ in Eq. (16) satisfies Axiom 1, Axiom 2a, Axiom 2b, and Axiom 3, follows the same logic as Shapley’s (1953b) proof for $\phi_i(N, v)$, and for $\phi_{ij}(N, v)$ in Eq. (4) in Lemma 1. To prove that $\phi_{IJ}(N, v) = \phi_{II}(N, v)$, inserting Lemma 2, i.e., $\phi_{ij}(N, v) = \phi_{ji}(N, v)$ for all $i \in N$, $j \in N$, into each term in Eq. (17) gives

$$\begin{aligned} \phi_{IJ}(N, v) &= \phi_{ji}(N, v) + \phi_{jk}(N, v) + \dots + \phi_{jm}(N, v) \\ &+ \phi_{qi}(N, v) + \phi_{qk}(N, v) + \dots + \phi_{qm}(N, v) \\ &+ \dots + \phi_{ui}(N, v) + \phi_{uk}(N, v) + \dots + \phi_{um}(N, v) \\ &= \sum_{\substack{j \in I, \\ i \in I}} \phi_{ji}(N, v) = \phi_{II}(N, v) \end{aligned} \tag{18}$$

First, the Proposition determines the value of coalition I (to a player or coalition) by summing up the values of each player i in coalition I , $i \in I$. Second, the Proposition determines the value (of a player or coalition) to coalition J by summing up the values to each player j in coalition J , $j \in J$. Third, summing the value of coalition I and the value to coalition J gives the value of coalition I to coalition J . Fourth, the Proposition applies regardless of whether coalitions I and J overlap or not. Fifth, since an n -person game has 2^n possible coalitions, including the null coalition $\{0\}$ and the set $N = \{1, 2, \dots, n\}$ of all players, the Shapley value of

row coalition I to column coalition J is exhaustively expressed by a $2^n \times 2^n$ matrix. Sixth, the symmetry $\phi_{IJ}(N, \nu) = \phi_{JI}(N, \nu)$ in the Proposition corresponds to the symmetry in Lemma 2. The nature of the summation in the Proposition is such that the value is symmetric in the sense that the value of coalition I to coalition J equals the value of coalition J to coalition I . \square

Corollary 1. The Shapley value of coalition I to itself, $I \subseteq N$, in an n -person game equals the sum of the Shapley values of each player i , $i \in I$, in coalition I to itself and, due to symmetry, twice the Shapley values of player i to player j given that either $i < j$ or $i > j$, $j \in I$, i.e.,

$$\phi_{II}(N, \nu) = \sum_{i \in I} \phi_{ii}(N, \nu) + 2 \sum_{\substack{i \in I, \\ j \in I, \\ i < j}} \phi_{ij}(N, \nu) = \sum_{i \in I} \phi_{ii}(N, \nu) + 2 \sum_{\substack{i \in I, \\ j \in I, \\ i > j}} \phi_{ij}(N, \nu) \tag{19}$$

Proof. Inserting $J = I$ into Eq. (17) while replacing $J = \{j, q, \dots, u\}$ with $I = \{i, k, \dots, m\}$ gives

$$\begin{aligned} \phi_{II}(N, \nu) &= \phi_{ii}(N, \nu) + \phi_{ki}(N, \nu) + \dots + \phi_{mi}(N, \nu) \\ &\quad + \phi_{ik}(N, \nu) + \phi_{kk}(N, \nu) + \dots + \phi_{mk}(N, \nu) \\ &\quad + \dots + \phi_{im}(N, \nu) + \phi_{km}(N, \nu) + \dots + \phi_{mm}(N, \nu) \end{aligned} \tag{20}$$

Equation (20) contains the symmetric terms $\phi_{ki}(N, \nu)$ and $\phi_{ik}(N, \nu)$, ..., $\phi_{mi}(N, \nu)$ and $\phi_{im}(N, \nu)$, ..., and $\phi_{mk}(N, \nu)$ and $\phi_{km}(N, \nu)$. Using Lemma 2, we write these symmetric terms as $2\phi_{ik}(N, \nu)$, ..., $2\phi_{im}(N, \nu)$, ..., and $2\phi_{km}(N, \nu)$. Inserting into Eq. (20) gives

$$\begin{aligned} \phi_{II}(N, \nu) &= \phi_{ii}(N, \nu) + \phi_{kk}(N, \nu) + \dots + \phi_{mm}(N, \nu) + 2\phi_{ik}(N, \nu) \\ &\quad + \dots + 2\phi_{im}(N, \nu) + 2\phi_{km}(N, \nu) + \dots \end{aligned} \tag{21}$$

which is rewritten as Eq. (19). \square

Corollary 2. The Shapley value of coalition I to coalition J , the Shapley value of coalition I to the set N of all players, and the Shapley value of the set N of all players to coalition J , where I and J are both strict subsets of N , $I \subset N$, $J \subset N$, are all less than or equal to the characteristic function $\nu(N)$ of the set N of all players, i.e.,

$$\phi_{IJ}(N, \nu) \leq \nu(N), \phi_{IN}(N, \nu) \leq \nu(N), \phi_{NJ}(N, \nu) \leq \nu(N), I \subset N, J \subset N \tag{22}$$

Proof. Follows from the summations in the Proposition, which are all constrained from above by $\phi_{NN}(N, \nu) = \nu(N)$. \square

Corollary 3. For any partition p of N :

$$\sum_{I \in p} \phi_{IJ}(N, \nu) = \phi_j(N, \nu), I \subseteq N, j \in N \tag{23}$$

Proof. Follows from the Proposition. \square

Corollary 3 states that summing up the Shapley value of coalition I to player j , for any partition p of N , equals the Shapley value of player j .

Corollary 4. For any partition p of N :

$$\sum_{J \in p} \phi_{ij}(N, \nu) = \phi_i(N, \nu), J \subseteq N, i \in N \tag{24}$$

Proof. Follows from the Proposition. \square

Corollary 4 states that summing up the Shapley value of player i to coalition J , for any partition p of N , equals the Shapley value of player i .

Example

Assume that $S \subseteq N \subseteq U$ is a carrier of ν , and I is a null coalition in ν defined as $\nu(S \cup I) = \nu(S)$. Hausken and Mohr (2001, p. 468ff)

Table 3 The Shapley value of row coalition I to column coalition J .

	{0}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{0}	0	0	0	0	0	0	0	0
{1}	0	295	25	70	320	365	95	390
{2}	0	25	25	-20	50	5	5	30
{3}	0	70	-20	70	50	140	50	120
{1,2}	0	320	50	50	370	370	100	420
{1,3}	0	365	5	140	370	505	145	510
{2,3}	0	95	5	50	100	145	55	150
{1,2,3}	0	390	30	120	420	510	150	540

considered the game $N = \{1, 2, 3\}$, $\nu(1) = 180$, $\nu(2) = \nu(3) = \nu(2, 3) = 0$, $\nu(1, 2) = 360$, $\nu(1, 3) = \nu(1, 2, 3) = 540$. Inserting into the definition of $\phi_i(\nu, S)$ gives the Shapley values $\phi(S, \nu) = [\phi_1(N, \nu) \ \phi_2(N, \nu) \ \phi_3(N, \nu)]^T = [390 \ 30 \ 120]^T$, where T means transposed. The Shapley value of the eight coalitions of the three elements in $\phi(S, \nu)$ is given by Definition 1. The 3×3 value matrix $\phi_{ij}(N, \nu)$ giving the Shapley value of row player i to column player j is

$$\phi_{ij}(N, \nu) = \begin{bmatrix} 295 & 25 & 70 \\ 25 & 25 & -20 \\ 70 & -20 & 70 \end{bmatrix} \tag{25}$$

The $2^3 \times 2^3$ matrix in Table 3 gives the Shapley value of all possible coalitions I to all possible coalitions J according to the Proposition, $I \subseteq N$, $J \subseteq N$. The 3×3 matrix in Eq. (25) is replicated in the upper left part of Table 3, to the right of the 8×1 column of 0's giving the value of coalition I to the null coalition or null player $\{0\}$, and below the 1×8 row of 0's giving the value of the null coalition $\{0\}$ to coalition J . The lower right cell in Table 3 gives the value $\nu(1, 2, 3) = 540$ according to the Proposition, which is the Shapley value of the set of all players to the set of all players, which equals the Shapley value of the set of all players, which equals the characteristic function $\nu(N)$ of the set N of all players.²

The Proposition for $I = \{2, 3\}$ and $j = 3$ gives $\phi_{ij}(N, \nu) = 50$ (row 2 from the bottom and column 5 from the right). The Proposition gives $\phi_{ij}(N, \nu) = 50$ for $i = 3$ and $J = \{2, 3\}$ (column 2 from the right and row 5 from the bottom). The Proposition for $I = \{1, 3\}$ and $J = \{2, 3\}$ gives $\phi_{IJ}(N, \nu) = 145$ (column 2 from the right and row 3 from the bottom). The symmetry across the diagonal from top-left to bottom-right according to the Proposition is such that $\phi_{JI}(N, \nu) = 145$. The value 145 is found by summing four cells determined by the intersection of rows 1 and 3 and columns 2 and 3 in Eq. (25), i.e., $25 - 20 + 70 + 70 = 145$.

Interpreting $\phi_{IJ}(N, \nu)$

So far $\phi_{ij}(N, \nu)$ and $\phi_{IJ}(N, \nu)$ are mathematical expressions satisfying the Proposition, Lemmas 1–3, and Corollaries 1–4. We can think of the Shapley value $\phi_i(N, \nu)$ of player i as an element within an n -tuple, the Shapley value $\phi_{ij}(N, \nu)$ of player i to player j as an element within an $n \times n$ matrix, and the Shapley value $\phi_{IJ}(N, \nu)$ of coalition I to coalition J as an element within an $2^n \times 2^n$ matrix.

Hausken and Mohr (2001, p. 465) identified four interpretations of $\phi_i(N, \nu)$, i.e., player i 's expected marginal contribution to all n players, the weighted average of player i 's marginal contribution to all n players, what player i can reasonably command to himself, or player i 's fair share. See e.g., Roth (1988a) for some similar interpretations. Analogously, $\phi_{ij}(N, \nu)$ is interpreted as player i 's expected marginal contribution to player j in a game of

n players, the weighted average of player i 's marginal contribution to player j in a game of n players, what player i can reasonably command to himself when considering only players i and j in a game of n players, or player i 's fair share when considering only players i and j in a game of n players. Also analogously, $\phi_{ij}(N, v)$ is interpreted as coalition I 's expected marginal contribution to coalition J in a game of n players, the weighted average of coalition I 's marginal contribution to coalition J in a game of n players, what coalition I can reasonably command to itself when considering only coalitions I and J in a game of n players, or coalition I 's fair share when considering only coalitions I and J in a game of n players.

Furthermore, Hausken and Mohr (2001, p. 466) interpreted $\phi_{ij}(N, v)$ as player i 's power over player j , since player i contributes something player j values highly or is interested in. To the extent player i contributes something player j is interested in, player i has power over player j . This can also be interpreted so that player j depends on player i , since player i contributes something player j desires. Accordingly, $\phi_{ij}(N, v)$ can be interpreted as a matrix for the value of player i to player j , as a power matrix for player i 's power over player j , and as an interest matrix for player j 's interest in player i , and as a dependence matrix for how player j depends on player i .³

Disjoint coalitions I and J , $I \cap J = \emptyset$. Since one player exists, obviously two or n players also exist. That is, a team or group or collection of players, referred to as a coalition, exists. Hence mathematically, since $\phi_i(N, v)$ exists, $\phi_I(N, v)$ also exists, $I \subseteq N$. We proceed with $\phi_{ij}(N, v)$, first assuming $I \cap \{j\} = \emptyset$, $I \subseteq N$. If $I = \{i, k\}$, the Proposition gives

$$\phi_{ij}(N, v) = \phi_{ij}(N, v) + \phi_{kj}(N, v) \tag{26}$$

Hence, since $\phi_{ij}(N, v)$ and $\phi_{kj}(N, v)$ exist, $\phi_{ij}(N, v)$ exists. That is, since player i and player k individually have values $\phi_{ij}(N, v)$ and $\phi_{kj}(N, v)$ to player j , coalition $I = \{i, k\}$, which exists, has a value $\phi_{ij}(N, v)$ to player j . This argument applies so that $\phi_{ij}(N, v)$ exists as coalition I expands to $I = \{i, k, \dots, m\}$, which means that coalition I has maximally $n - 1$ members (players) since $I \cap \{j\} = \emptyset$.

We proceed with $\phi_{ij}(N, v)$, first assuming $J \cap \{i\} = \emptyset$, $J \subseteq N$. If $J = \{j, k\}$, the Proposition gives $\phi_{ij}(N, v) = \phi_{ij}(N, v) + \phi_{ik}(N, v)$. Hence, since $\phi_{ij}(N, v)$ and $\phi_{ik}(N, v)$ exist, $\phi_{ij}(N, v)$ exists. That is, since player i has a value $\phi_{ij}(N, v)$ to player j , and player i has a value $\phi_{ik}(N, v)$ to player k , player i has a value $\phi_{ij}(N, v)$ to coalition J , which exists. This argument applies so that $\phi_{ij}(N, v)$ exists as coalition J expands to $J = \{j, k, \dots, m\}$, which means that coalition J has maximally $n - 1$ members (players) since $J \cap \{i\} = \emptyset$.

We proceed with $\phi_{IJ}(N, v)$, first assuming $I \cap J = \emptyset$, $I \subseteq N$, $J \subseteq N$. If $I = \{i, k\}$ and $J = \{j, q\}$, the Proposition gives

$$\phi_{IJ}(N, v) = \phi_{ij}(N, v) + \phi_{iq}(N, v) + \phi_{kj}(N, v) + \phi_{kq}(N, v) \tag{27}$$

Hence, since $\phi_{ij}(N, v)$, $\phi_{iq}(N, v)$, $\phi_{kj}(N, v)$, and $\phi_{kq}(N, v)$ exist, $\phi_{IJ}(N, v)$ exists. That is, since player i has a value $\phi_{ij}(N, v)$ to player j , player i has a value $\phi_{iq}(N, v)$ to player q , player k has a value $\phi_{kj}(N, v)$ to player j , and player k has a value $\phi_{kq}(N, v)$ to player q , coalition I , which exists, has a value $\phi_{IJ}(N, v)$ to coalition J , which also exists. This argument applies so that $\phi_{IJ}(N, v)$ exists as coalition I expands to $I = \{i, k, \dots, m\}$, and coalition J expands to $J = \{j, q, \dots, u\}$, where $I \cap J = \emptyset$ means that the sum of the number of members (players) in coalitions I and J is equal to or less than n . This completes the interpretation of $\phi_{IJ}(N, v)$ for disjoint coalitions I and J , $I \cap J = \emptyset$.

One coalition is a subset of another coalition, $I \cup J = I$ or $I \cup J = J$. When one coalition is a subset of another coalition, $I \cup J = I$ or $I \cup J = J$, i.e., $I \cap J = I$ if $I \subseteq J$, and $I \cap J = J$ if $J \subseteq I$. Starting with $I = J$, $\phi_{ii}(N, v)$ is the value of player i to itself, which exists since $\phi_{ij}(N, v)$ exists. The extension from $\phi_i(N, v)$ to $\phi_I(N, v)$ and subsequent discussion above means that $\phi_{II}(N, v)$ is the value of coalition I to itself, which exists, $I \subseteq N$.

If $i \in I$, $\phi_{II}(N, v)$ is the value of coalition I to player i which is a member of coalition I . If $I = \{i, k, \dots, m\} \subseteq N$, the Proposition implies

$$\phi_{II}(N, v) = \phi_{ii}(N, v) + \phi_{ki}(N, v) + \dots + \phi_{mi}(N, v) \tag{28}$$

where $\phi_{ii}(N, v)$, $\phi_{ki}(N, v), \dots, \phi_{mi}(N, v)$ exist as discussed above, and thus $\phi_{II}(N, v)$ exists for $i \in I \subseteq N$.

If $j \in J$, $\phi_{JJ}(N, v)$ is the value of player j to coalition J , where player j is a member of coalition J . If $J = \{j, q, \dots, u\} \subseteq N$, the Proposition implies

$$\phi_{JJ}(N, v) = \phi_{jj}(N, v) + \phi_{jq}(N, v) + \dots + \phi_{ju}(N, v) \tag{29}$$

where $\phi_{jj}(N, v)$, $\phi_{jq}(N, v), \dots, \phi_{ju}(N, v)$ exist as discussed above, and thus $\phi_{JJ}(N, v)$ exists for $j \in J \subseteq N$.

If $I \subseteq J$, $\phi_{IJ}(N, v)$ is the value of coalition I to coalition J , where coalition I is a subcoalition of coalition J . If $I = \{i, \dots, j\} \subseteq N$ and $J = \{i, \dots, j, q, \dots, u\} \subseteq N$, the Proposition implies

$$\begin{aligned} \phi_{IJ}(N, v) &= \phi_{ii}(N, v) + \dots + \phi_{ji}(N, v) + \dots + \phi_{ij}(N, v) + \dots \\ &\quad + \phi_{jj}(N, v) + \phi_{iq}(N, v) \\ &\quad + \dots + \phi_{jq}(N, v) + \dots + \phi_{iu}(N, v) + \dots + \phi_{ju}(N, v) \end{aligned} \tag{30}$$

where $\phi_{ii}(N, v), \dots, \phi_{ji}(N, v), \dots, \phi_{ij}(N, v), \dots, \phi_{jj}(N, v)$, $\phi_{iq}(N, v), \dots, \phi_{jq}(N, v), \dots, \phi_{iu}(N, v), \dots, \phi_{ju}(N, v)$ exist as discussed above, and thus $\phi_{IJ}(N, v)$ exists for $I \subseteq J \subseteq N$.

If $J \subseteq I$, $\phi_{IJ}(N, v)$ is the value of coalition I to coalition J which is a subcoalition of coalition I . If $I = \{i, \dots, j, k, \dots, m\} \subseteq N$ and $J = \{i, \dots, j\} \subseteq N$, the Proposition implies

$$\begin{aligned} \phi_{IJ}(N, v) &= \phi_{ii}(N, v) + \dots + \phi_{ji}(N, v) + \phi_{ki}(N, v) + \dots + \phi_{mi}(N, v) \\ &\quad + \dots + \phi_{ij}(N, v) \\ &\quad + \dots + \phi_{jj}(N, v) + \phi_{kj}(N, v) + \dots + \phi_{mj}(N, v) \end{aligned} \tag{31}$$

where $\phi_{ii}(N, v), \dots, \phi_{ji}(N, v)$, $\phi_{ki}(N, v), \dots, \phi_{mi}(N, v), \dots, \phi_{ij}(N, v), \dots, \phi_{jj}(N, v)$, $\phi_{kj}(N, v), \dots, \phi_{mj}(N, v)$ exist as discussed above, and thus $\phi_{IJ}(N, v)$ exists for $J \subseteq I \subseteq N$.

Overlapping coalitions I and J , $I \cap J \neq \emptyset$. We finally consider $I \cap J \neq \emptyset$ where either $I \subset I \cup J$ or $J \subset I \cup J$, which means that coalition I and coalition J overlap partly. Assume first that $I = \{i, \dots, j, k, \dots, m\} \subseteq N$ and $J = \{i, \dots, j, q, \dots, u\} \subseteq N$, where $\{k, \dots, m\} \cap \{q, \dots, u\} = \emptyset$. The Proposition implies

$$\begin{aligned} \phi_{IJ}(N, v) &= \phi_{ii}(N, v) + \dots + \phi_{ji}(N, v) + \phi_{ki}(N, v) + \dots + \phi_{mi}(N, v) \\ &\quad + \dots + \phi_{ij}(N, v) + \dots + \phi_{jj}(N, v) + \phi_{kj}(N, v) + \dots + \phi_{mj}(N, v) \\ &\quad + \phi_{iq}(N, v) + \dots + \phi_{jq}(N, v) \\ &\quad + \phi_{kq}(N, v) + \dots + \phi_{mq}(N, v) + \dots + \phi_{iu}(N, v) + \dots + \phi_{ju}(N, v) \\ &\quad + \phi_{ku}(N, v) + \dots + \phi_{mu}(N, v) \end{aligned} \tag{32}$$

where $\phi_{ii}(N, v), \dots, \phi_{ji}(N, v)$, $\phi_{ki}(N, v), \dots, \phi_{mi}(N, v), \dots, \phi_{ij}(N, v), \dots, \phi_{jj}(N, v)$, $\phi_{kj}(N, v), \dots, \phi_{mj}(N, v)$, $\phi_{iq}(N, v), \dots, \phi_{jq}(N, v)$, $\phi_{kq}(N, v), \dots, \phi_{mq}(N, v), \dots, \phi_{iu}(N, v), \dots, \phi_{ju}(N, v)$, $\phi_{ku}(N, v), \dots, \phi_{mu}(N, v)$ exist as discussed above, and thus $\phi_{IJ}(N, v)$ exists for $I \subseteq N$, $J \subseteq N$.

Whereas two non-overlapping coalitions I and J , $I \cap J = \emptyset$, can form and coexist, two partly overlapping coalitions I and J , $I \cap J \neq \emptyset$, cannot both form and coexist at the same time. Whether no or one or two coalitions have formed or not is irrelevant in this article. The Shapley value $\phi_{IJ}(N, v)$ of coalition I to coalition J can always be calculated, even when coalition formation is hypothetical, i.e., regardless how $I \subseteq N$ and $J \subseteq N$. Each player $i \in N$, or any player not involved in the game, considers the hypothetical possibility that coalitions I and J are formed, and determines the value of the former to the latter. The section “Applying the weighted Shapley value” considers how coalitions emerge by assigning different weights to the players, as assumed by Shapley (1953a) and formulated by Dragan (2009) and Kalai and Samet (1987).

Usefulness, future research, and applications

The practical usefulness is especially evident for disjoint coalitions, since if two coalitions are both valuable to each other, they may merge. The conditions for the merger may depend on the different values they assign to each other. If one coalition values another coalition positively, while the other coalition values the first negatively, a merger may not occur, or may occur if external funding is acquired enabling side payments. If both coalitions value each other negatively, a merger cannot be expected, and the coalitions may be able to explain the non-merger to themselves.

If one coalition is contained within another coalition, as a subset or proper subset, the value of the former to the latter may help determine salaries and reimbursement, and the value of the latter to the former may aid the former in determining whether it should still belong to the latter coalition, e.g., compared against outside options such as external employment opportunities.

If two coalitions overlap, the issue rises of which coalitions have formed and which have not. This article provides Shapley values of coalitions to each other regardless of whether they overlap, have formed, or are hypothetical. First, if none have formed, the values may indicate which should form. Second, if one has formed while the other has not, the values may suggest, indirectly or through some deeper scrutiny, whether this coalition should continue to exist, or whether various alternatives should replace it. Third, two overlapping coalitions may jointly exist when certain conditions exist. For example, the two coalitions may be assigned two different tasks, and the overlapping members work on both tasks. Alternatively, the two coalitions may work on the same task, but the overlapping members keep it as a secret that they also belong to the other coalition. This means analyzing a game with incomplete information, suggested for future research.

The article enables interpreting existing results in innovative ways, recommended for future research. Examples are the various solution notions in cooperative game theory, and the properties for the linkages between these (Driessen, 1988), i.e., particularly the kernel, nucleolus, bargaining set, core, the von Neumann–Morgenstern solution (also known as the stable set), the Shapley value (Aumann and Dreze, 1974), the strong epsilon-core (Shapley and Shubik, 1966), and the core of a simple game with respect to preferences (Nakamura, 1979). For these known results, the value of each player and coalition to each other player and coalition should be determined.

Similar analyses can be conducted for theories of coalition formation. Examples are Myerson’s (1980) conference structures and fair allocation rules, Shenoy’s (1979) models, and work by Kurz (1988) and Aumann and Myerson (1988). Any theory of coalition building needs to account, directly or indirectly, for which values coalitions have to each other. Insights about

coalition formation impact which coalitions are likely to form and not form, and which coalitions can be expected to survive or not survive.

Exemplifying practical applications, Hausken and Mohr (2001) applied the analysis to determine the changing values of the members of the European Union in the European Union Council of Ministers during the enlargements in 1973, 1981, 1986, and 1995. The largest players lost voting power. It was shown how the ϕ_{ij} matrix is applicable to rank the importance of player i to player j . More generally, the ϕ_{IJ} matrix is applicable to rank the importance of coalition I to coalition J . The example can be extended to the subsequent enlargements since 1995, and Brexit January 31, 2020.

Applying the weighted Shapley value

One method for assuming different probabilities for which coalitions emerge is to assign different weights to the players, as assumed by Shapley (1953a) for the weighted Shapley value. Kalai and Samet (1987, p. 206) suggested that “bargaining ability, patience rates, or past experience” may impact weights. In addition, some players represent larger constituencies, possess more wealth, have higher competence, etc., which may impact weights. Kalai and Samet (1987, p. 211) assumed the following Axiom 4, required in addition to Axiom 1, Axiom 2a, Axiom 2b, and Axiom 3 for the unique weighted Shapley value ϕ_w :

Axiom 4. Partnership. If, in the game (N, v) , for each $T \subset S$ and each $R \subseteq N \setminus S$, $v(R \cup T) = v(R)$, then

$$\phi_i(v, S) = \phi_i\left(\sum_{k \in S} \phi_k(v) u_S, S\right) \forall i \in S \tag{33}$$

Applying Dragan’s (2009) formulation. When λ_i is the weight assigned to player i for the unanimity game u_S within coalition $S \subseteq N$, $S \neq \emptyset$, and $\lambda = [\lambda_1, \dots, \lambda_i, \dots, \lambda_n] \in R_{++}^n$ is the vector of weights across the n players, player i ’s weighted Shapley value is

$$\phi_{wi}(u_S, N, \lambda) = \begin{cases} \frac{\lambda_i}{\sum_{k \in S} \lambda_k} \forall i \in S \\ 0 \forall i \notin S \end{cases} \tag{34}$$

Among the many formulations of the weighted Shapley value, Dragan’s (2009, p. 2) Eq. (6) and Radzik’s (2012, p. 409) Eq. (12) retain $v(S) - v(S \setminus \{i\})$, which enables proving Lemma 1w below analogously to proving Lemma 1. Applying Dragan’s (2009) more compact notation, consistently with Shapley (1953a), the weighted Shapley value for player $i \in N$ for the game (N, v) is

$$\phi_{wi}(v, N, \lambda) = \lambda_i \sum_{\substack{S \subseteq N, \\ i \in S}} \gamma_S (v(S) - v(S \setminus \{i\})) \tag{35}$$

where

$$\gamma_S = \sum_{\substack{T \subseteq N, \\ T \cap S = \emptyset}} \frac{(-1)^t}{\sum_{k \in T \cup S} \lambda_k}, \forall S \subseteq N, S \neq \emptyset \tag{36}$$

Assuming the unanimity game where $u_R(S) = 1$ if $R \subseteq S$ and $u_R(S) = 0$ otherwise, Axiom 1, Axiom 2a, Axiom 2b, and Axiom 3 imply $\phi_i(u_R, S) = 1/|R|$ if $i \in R$ and $\phi_i(u_R, S) = 0$ otherwise.

Lemma 1w. The weighted Shapley value $\phi_{wi}(v, N, \lambda)$ for player $i \in N$ in a game of $n = |N|$ players is decomposed into n different

values $\phi_{wij}(v, N, \lambda), j \in N$, satisfying

$$\phi_{wi}(v, N, \lambda) = \sum_{j=1}^n \phi_{wij}(v, N, \lambda) \tag{37}$$

where

$$\phi_{wij}(v, N, \lambda) = \lambda_i \sum_{\substack{S \subseteq N, \\ i \in S}} \gamma_S \left(\phi_{wj}(v, S, \lambda) - \phi_{wj}(v, S \setminus \{i\}, \lambda) \right) \tag{38}$$

Proof. Using Axiom 2a for any subcoalition $S \subseteq N$, we rewrite Eq. (35) as

$$\phi_{wi}(v, N, \lambda) = \lambda_i \sum_{\substack{S \subseteq N, \\ i \in S}} \gamma_S \left(\sum_{j \in S} \phi_{wj}(v, S, \lambda) - \sum_{\substack{j \in S, \\ j \neq i}} \phi_{wj}(v, S, \lambda) \right) \tag{39}$$

For any player j outside subcoalition $S \subseteq N$, i.e., $j \notin S$, but among the set N of players, i.e., $j \in N$, Axiom 2b states that $\phi_j(v, S) = 0$. Hence Eq. (39) is rewritten as

$$\phi_{wi}(v, N, \lambda) = \lambda_i \sum_{\substack{S \subseteq N, \\ i \in S}} \gamma_S \left(\sum_{j=1}^n \phi_{wj}(v, S, \lambda) - \sum_{\substack{j=1, \\ j \neq i}}^n \phi_{wj}(v, S, \lambda) \right) \tag{40}$$

which is rewritten as

$$\phi_{wi}(v, N, \lambda) = \sum_{j=1}^n \lambda_i \sum_{\substack{S \subseteq N, \\ i \in S}} \gamma_S \left(\phi_{wj}(v, S, \lambda) - \phi_{wj}(v, S \setminus \{i\}, \lambda) \right) \tag{41}$$

where λ_i multiplied by the second summation sign equals $\phi_{wij}(v, N, \lambda)$ in Eq. (38). \square

Lemma 2w. For all $i \in N, j \in N$,

$$\phi_{wij}(v, N, \lambda) = \phi_{wji}(v, N, \lambda) \tag{42}$$

Proof. Analogous to the proof of Lemma 2. \square

Lemma 3w. The weighted Shapley value $\phi_{wj}(v, N, \lambda)$ for player $j \in N$ in a game of $n = |N|$ players is decomposed into n different values $\phi_{wij}(v, N, \lambda), i \in N$, satisfying

$$\phi_{wj}(v, N, \lambda) = \sum_{i=1}^n \phi_{wij}(v, N, \lambda) \tag{43}$$

Proof. Analogous to the proof of Lemma 3. \square

Proposition w. The weighted Shapley value of coalition I to coalition $J, I \subseteq N, J \subseteq N$, in an n -person game is

$$\phi_{wIJ}(v, N, \lambda) = \sum_{\substack{i \in I, \\ j \in J}} \phi_{wij}(v, N, \lambda) = \phi_{wJI}(v, N, \lambda) \tag{44}$$

Proof. Analogous to the proof of the Proposition. \square

Applying Kalai and Samet’s (1987) formulation. Kalai and Samet (1987) allowed players to have zero weight, assuming a lexicographic weight system. Their notation is as follows (Kalai and Samet, 1987, pp. 208–209): $\mathbb{R}(S)$ is the set of all orders R in coalition S . $B^{R,i}$ is the set of players preceding i in R in $\mathbb{R}(N)$. For an ordered partition $\Sigma = (S_1, \dots, S_m)$ of N , \mathbb{R}_Σ is the set of orders for N in which all the players of S_i precede those of $S_j, i = 1, \dots, m - 1$. Each R in \mathbb{R}_Σ is expressed as $R = (R_1, \dots, R_m), R_i \in \mathbb{R}(S_i), i = 1, \dots, m$. For $R = (i_1, \dots, i_s)$ in $\mathbb{R}(S), P_\lambda(R) =$

$\prod_{j=1}^s \frac{\lambda_{i_j}}{\sum_{k=1}^j \lambda_{i_k}}$ is a probability distribution associated with λ over

$\mathbb{R}(S), s = |S|, \lambda \in E_{++}^S$. $P_\lambda(R)$ is obtainable by arranging the players of S in an order, starting from the end. The probability of adding a player to the beginning of a partially created line is the ratio between the player’s weight and the sum of the weights of the players of S not yet in the line. A probability distribution P_ω over $\mathbb{R}(N)$ is associated with each weight system $\omega = (\lambda, \Sigma)$. $P_\omega(R) = \prod_{i=1}^m P_{\lambda_{S_i}}(R_i)$ for R in \mathbb{R}_Σ , where λ_{S_i} is the projection of λ on E^{S_i} , and P_ω vanishes outside \mathbb{R}_Σ . Player i ’s contribution is $C_i(v, R) = v(B^{R,i} \cup \{i\}) - v(B^{R,i})$ for game (N, v) and order R in $\mathbb{R}(N)$. Kalai and Samet (1987) proved that the weighted Shapley value of player $i \in N$ equals his expected contribution with respect to P_ω , i.e.,

$$\phi_{wi}(v, N, \omega) = E_{P_\omega}(C_i(v, \cdot)) = E_{P_\omega}(v(B^{i,i} \cup \{i\}) - v(B^{i,i})) \tag{45}$$

Lemma 1w. The weighted Shapley value $\phi_{wi}(v, N, \omega)$ for player $i \in N$ in a game of $n = |N|$ players is decomposed into n different values $\phi_{wij}(v, N, \omega), j \in N$, satisfying

$$\phi_{wi}(v, N, \omega) = \sum_{j=1}^n \phi_{wij}(v, N, \omega) \tag{46}$$

where

$$\phi_{wij}(v, N, \omega) = E_{P_\omega}(\phi_{wj}(v, \cdot, \omega) - \phi_{wj}(v, \cdot \setminus \{i\}, \omega)) \tag{47}$$

Proof. Since Kalai and Samet (1987) in Eq. (45) use \cdot in $B^{i,i}$ to denote an order, we do the same. Since $B^{i,i}$ is the set of players preceding i in \cdot, i is not included in $B^{i,i}$, whereas i is included in $B^{i,i} \cup \{i\}$. Thus, using Axiom 2a for any subcoalition $\cdot \subseteq N$, we rewrite Eq. (45) as

$$\phi_{wi}(v, N, \omega) = E_{P_\omega} \left(\sum_{j \in \cdot} \phi_{wj}(v, \cdot, \omega) - \sum_{\substack{j \in \cdot, \\ j \neq i}} \phi_{wj}(v, \cdot, \omega) \right) \tag{48}$$

For any player j outside order $\cdot \subseteq N$, i.e., $j \notin \cdot$, but among the set N of players, i.e., $j \in N$, Axiom 2b states that $\phi_{wj}(v, \cdot, \omega) = 0$. Hence Eq. (48) is rewritten as

$$\phi_{wi}(v, N, \omega) = E_{P_\omega} \left(\sum_{j=1}^n \phi_{wj}(v, \cdot, \omega) - \sum_{\substack{j=1, \\ j \neq i}}^n \phi_{wj}(v, \cdot, \omega) \right) \tag{49}$$

which, since the summation can be placed outside the expected value, is rewritten as

$$\phi_{wi}(v, N, \omega) = \sum_{j=1}^n E_{P_\omega}(\phi_{wj}(v, \cdot, \omega) - \phi_{wj}(v, \cdot \setminus \{i\}, \omega)) \tag{50}$$

where the expression inside the summation sign equals $\phi_{wij}(v, N, \omega)$ in Eq. (47). \square

Lemma 2w. For all $i \in N, j \in N$,

$$\phi_{wij}(v, N, \omega) = \phi_{wji}(v, N, \omega) \tag{51}$$

Proof. Analogous to the proof of Lemma 2. \square

Lemma 3w. The weighted Shapley value $\phi_{wj}(v, N, \omega)$ for player $j \in N$ in a game of $n = |N|$ players is decomposed into n different values $\phi_{wij}(v, N, \omega), i \in N$, satisfying

$$\phi_{wj}(v, N, \omega) = \sum_{i=1}^n \phi_{wij}(v, N, \omega) \tag{52}$$

Proof. Analogous to the proof of Lemma 3. \square

Proposition ω . The weighted Shapley value of coalition I to coalition J , $I \subseteq N$, $J \subseteq N$, in an n -person game is

$$\phi_{\omega IJ}(v, N, \omega) = \sum_{\substack{i \in I \\ j \in J}} \phi_{\omega ij}(v, N, \omega) = \phi_{\omega IJ}(v, N, \omega) \quad (53)$$

Proof. Analogous to the proof of the Proposition. \square

Conclusion

Shapley's (1953b) value has set its mark on science. Hausken and Mohr (2001) decomposed the value to a symmetric $n \times n$ matrix giving the value $\phi_{ij}(N, v)$ of row player i to column player j in a game (N, v) of a set N of players, $n = |N|$. This article generalizes to a symmetric $2^n \times 2^n$ matrix giving the value $\phi_{IJ}(N, v)$ of row coalition I to column coalition J , observing that an n -person game has 2^n coalitions including the empty coalition and the coalition of all n players. Values of coalitions to other coalitions can be positive, zero, or negative. The results also apply for weighted Shapley values.

The Shapley values for an n -person game are usually represented with an $n \times 1$ vector replicated in the bottom row of the $2^n \times 2^n$ matrix comprising only the columns of individual players numbered from 1 to n , and due to symmetry also replicated in the rightmost column of the $2^n \times 2^n$ matrix comprising only the rows of individual players numbered from 1 to n . The $n \times 1$ vector is determined by summing the elements in all columns or all rows in the $n \times n$ matrix which is replicated in the upper left part of the $2^n \times 2^n$ matrix comprising rows and columns for the n individual players.

The article presents results, illustrated with an example, demonstrating desirable properties of the matrix. First, the Shapley value of a coalition to a player equals the sum of the Shapley values of all players in the coalition to the given player. Due to symmetry, this equals the value of the given player to the coalition. Second, the Shapley value of a coalition to another coalition equals the sum of the Shapley values of each player in the first coalition to each player in the second coalition, regardless of whether the coalitions are disjoint, overlap partly, or coincide. Third, the Shapley value of a coalition to all players equals the Shapley value of the coalition, which equals the sum of the Shapley values of all players in the coalition. Fourth, the sum of the Shapley values to a player of disjoint coalitions comprising all players equals the Shapley value of the player. Fifth, the sum of the Shapley values of a player to a coalition, among multiple disjoint coalitions comprising all players, equals the Shapley value of the player. All these five values are specified in the corresponding cell in the $2^n \times 2^n$ matrix. These five values are not provided by specifying the characteristic function for the different coalitions, which merely gives a number for the collective payoff a set of players receives by forming a coalition. Hence the $2^n \times 2^n$ matrix provides substantially more information than merely calculating the characteristic function. In particular, the value of every player and coalition to every other player and coalition is specified for all possible coalitions. The advantages of the $2^n \times 2^n$ matrix outweigh the costs of developing it, which is low with today's computers.

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Notes

1 According to Shapley's (1953b, p. 33) proof, for each coalition $S \subseteq N$, the symmetry Axiom 1 implies that for each $i \in S, j \in S, k \in S, l \in S, i \neq j, k \neq l, \phi_{ij}(N, v) = \phi_{kl}(N, v)$ and $\phi_{ii}(N, v) = \phi_{jj}(N, v)$.

- 2 Table 3 illustrates that some players or coalitions may be a threat or no threat to the existence of various coalitions. If such players or coalitions make good on their latent threats, coalitions may dissolve. Then the issue arises which alternative coalitions emerge, and which players or coalitions may be a threat or no threat to the existence of these alternative coalitions.
- 3 Emerson (1962, p. 32) defined "The power of actor A over actor B <as> the amount of resistance on the part of B which can be potentially overcome by A," and "The dependence of actor A upon actor B <as> (1) directly proportional to A's motivational investment in goals mediated by B, and (2) inversely proportional to the availability of those goals to A outside of the A-B relation." Cook and Emerson (1978) generalized from dyads to exchange networks.

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Competing interests

The author declares no competing interests.

Additional information

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